QUANTIFYING AND ELIMINATING THE TIME DELAY IN STABILIZATION EXPONENTIAL TIME DIFFERENCING RUNGE–KUTTA SCHEMES FOR THE ALLEN–CAHN EQUATION

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Abstract. Although the stabilization technique is favorable in designing unconditionally energy stable or maximum-principle-preserving schemes for gradient flow systems, the induced time delay is intractable in computations. In this paper, we propose a class of delay-free stabilization schemes for the Allen–Cahn gradient flow system. Considering the Fourier pseudo-spectral spatial discretization for the Allen–Cahn equation with either the polynomial or the logarithmic potential, we establish a semi-discrete, mesh-dependent maximum principle by adopting a stabilization technique. To unconditionally preserve the mesh-dependent maximum principle and energy stability, we investigate a family of exponential time differencing Runge–Kutta (ETDRK) integrators up to the second-order. After reformulating the ETDRK schemes as a class of parametric Runge–Kutta integrators, we quantify the lagging effect brought by stabilization, and eliminate delayed convergence using a relaxation technique. The temporal error estimate of the relaxation ETDRK integrators in the maximum norm topology is analyzed under a fixed spatial mesh. Numerical experiments demonstrate the delay-free and structure-preserving properties of the proposed schemes.

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1. Introduction

Many differential equations in physics, chemistry, fluid dynamics, and material science are naturally equipped with stiff and nonlinear terms. Studies have demonstrated that the effective treatment of the stiff or nonlinear terms is not only important for the stability of time integration methods, but is relevant for efficient simulations. In the 1950s, Douglas and Rachford [9] introduced the stabilization (also known as alternating direction [8], splitting and stabilizing correction in Ref. [22]) technique that consists of adding and subtracting a term (e.g., \( \kappa(f(u) - f(u)) \) [22], \( \kappa(u - u) \) [34], \( \kappa(\Delta u - \Delta u) \) [18], \( \kappa(\Delta^2 u - \Delta^2 u) \) [38]) to and from a system, and then treating the two components differently. This has been widely developed and applied to construct efficient and stable methods for a variety of models with strong stiffness and nonlinearity, such as reaction-diffusion equations [1,46], fluid dynamics [21, 47], gradient flow models [13, 18, 43], and image processing [33]. However, as pointed by Xu et al. [45], the standard first-order convex splitting scheme for the Allen–Cahn (AC) gradient flow equation is
exactly the same as the first-order fully implicit scheme but with a smaller time step size. To understand the unconditional stability brought by stabilization exponential time differencing (ETD) integrators, we focus on the classical AC equation, and take a step toward large time-stepping, delay-free integrators.

Consider the free energy functional

$$E(u) = \int_{\Omega} \frac{1}{2} \epsilon^2 |\nabla u|^2 + F(u) \, dx,$$

(1.1)

where $x \in \Omega = \prod_{k=1}^{d} (a_k, b_k)$, $u$ is the difference between the concentrations of two mixtures, $\epsilon$ characterizes the width of the diffuse interface. If we let $f(u) = -F'(u)$ be the negative derivative of the potential function, then the AC equation is given by the $L^2$ gradient flow associated with the energy functional (1.1),

$$\begin{cases} u_t = \epsilon^2 \Delta u + f(u), & x \in \Omega, t \in (0, T], \\ u(x, 0) = u^0(x), & x \in \Omega. \end{cases}$$

(1.2)

In (1.2), the function $f(u)$ can be given by either the polynomial Ginzburg–Landau (GL) potential function,

$$F(u) = \frac{1}{4}(1 - u^2)^2, \quad f(u) = u - u^3,$$

(1.3)

or the logarithmic Flory–Huggins (FH) potential function,

$$F(u) = \frac{\theta}{2} [(1 + u) \ln(1 + u) + (1 - u) \ln(1 - u)] - \frac{\theta_c}{2} u^2, \quad f(u) = -\frac{\theta}{2} \ln \frac{1 + u}{1 - u} + \theta_c u,$$

(1.4)

where $\theta$ and $\theta_c$ are positive constants satisfying $\theta < \theta_c$, which guarantees a double-well shape for the potential.

Assuming that $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$ is periodic over the domain, simple calculations lead to the energy dissipation law,

$$\frac{d}{dt} E(u) = \langle \frac{\delta E(u)}{\delta u}, u_t \rangle = -\| \frac{\partial u}{\partial t} \|^2 \leq 0, \quad \forall t > 0,$$

where the symbols $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the standard $L^2$ inner product and norm, respectively, defined as

$$\langle u, v \rangle = \int_{\Omega} u v \, dx, \quad \| u \| = \left( \int_{\Omega} |u|^2 \, dx \right)^{\frac{1}{2}}, \quad \forall u, v \in L^2(\Omega).$$

In addition to the dissipation of energy, an important feature of the AC equation is the maximum principle [11,41] (also known as the maximum bound principle [12]): if the initial and boundary conditions are bounded by a certain constant $\beta$ ($\beta \geq 1$ for the GL potential, and $\beta \in [0.9575, 1)$ for the FH potential with $\theta = 0.8, \theta_c = 1.6$ [12]), then the solution to (1.2) is uniformly bounded by $\beta$, that is, $\| u(t) \|_{L^\infty} \leq \beta$ for all $t > 0$.

In studies on gradient flows, the guarantees of energy stability and the maximum principle for any time steps are appealing, and have been frequently used as fidelity checks of algorithms. Consequently, various integrators have been designed to meet such criteria, including convex splitting schemes [4,13], stabilization schemes [11,18,43,50,54], implicit energy quadratization schemes [48], and scalar auxiliary variable (SAV) schemes [37]. Among them, stabilization exponential time differencing (ETD) Runge–Kutta (RK) schemes have attracted much attention [7,10,19]. The ETD schemes have become popular in recent years. Du et al. [11] utilized the stabilization technique to develop unconditional maximum-principle-preserving (MPP) ETD1 and ETD2 schemes for the nonlocal AC equation. Fu and Yang [14] and Ju et al. [23] proved that stabilization ETD and SAV-ETD schemes up to the second order are unconditionally energy dissipative for a large family of gradient flow models, respectively.

However, as noted by Gomez and Hughes [15], Li, Qiao, and Tang [27], and Xu et al. [44,45], the gain of energy stability or maximum-principle-preservation of convex splitting and stabilization comes at the expense
of a possible loss of accuracy. Theoretical and numerical evidence \cite{5, 51, 52} demonstrates that such schemes would slow down the convergence rate \cite{27} or cause delayed convergence \cite{45}, and could be inaccurate when using large time steps \cite{14}. Xu et al. \cite{45} pointed out that convex splitting and stabilization approaches for the AC equation introduce time-step rescaling, while Laplacian stabilization for the Cahn–Hilliard (CH) equation leads to a modified/convexified CH equation with a time-step-dependent viscosity parameter (see also Ref. \cite{25}), which thus introduces a time delay.

To increase the accuracy of numerical simulations, much effort has been devoted to avoiding the potential time delays brought by stabilization. Gomez and Hughes \cite{15} developed and analyzed a second-order accurate and unconditional stable variational method by using a second-order perturbation of the midpoint rule for the CH equation. Xu et al. \cite{45} suggested nonlinear implicit schemes, since they are workable for large time-step sizes. By considering an unconditionally stable linear convex splitting scheme that includes Laplacian and biharmonic stabilization terms, Cheng and Warren \cite{3} established the relationship between the analytic and algorithmic time steps for the CH equation based on a scaling hypothesis. Lee et al. \cite{24} estimated mode-dependent effective time steps of nonlinear convex splitting schemes for the CH equation using eigenfunction decomposition and eigenvalue estimation. Still, there is scant analysis of the time delay in stabilization ETD integrators.

The second barrier in the development of MPP schemes for AC-type equations lies in spatial discretizations. It is well known that the maximum principle can be inherited by the central finite difference method or the mass lumped finite element method of the second-order \cite{12}, because of the diagonal dominance of the differentiation matrix. Yang et al. \cite{49} proved that the Fourier spectral Galerkin and spectral collocation (pseudo-spectral) methods preserve the $L^2$ bound of the solution to the AC equation. To preserve the maximum principle in high-order spatial discretization, some specifically designed temporal integrators have been proposed \cite{28, 35}. It is worth noting that Li \cite{26} developed a framework of effective maximum principles that allow the numerical solutions to deviate from the maximum bound of the continuous equation by a controllable discretization error.

To eliminate the lagging effect brought by stabilization, we utilize the parametric Runge–Kutta (pRK) framework in our previous study \cite{53}. Consequently, we propose delay-free, mesh-dependent maximum-principle-preserving and energy-stable schemes for the AC equation with periodic boundary conditions. The distinguishing aspect and novelty of the present work lie in two ways:

- The lagging effect of stabilization ETDRK integrators is quantified, and the time delay eliminated by a relaxation technique;
- Unconditionally mesh-dependent maximum-principle-preserving and energy-stable schemes are developed and proved on a fixed spatial mesh.

The rest of this paper is organized as follows. Section 2 presents the Fourier pseudo-spectral discretization for the AC equation, and proposes a stabilization technique to derive the maximum principle for the semi-discrete system. In Section 3, we illustrate the main ideas to construct MPP schemes for the semi-discrete system, quantify the lagging effect of stabilization, and develop a time-step relaxation technique that eliminates delayed convergence. Unconditional maximum principle preservation, energy stability and convergence estimate are analyzed in Section 4. In Section 5, various experiments demonstrate the high-order accuracy and structure-preserving properties of the proposed schemes. Some concluding remarks are presented in Section 6.

2. Preliminaries

2.1. Fourier pseudo-spectral discretization

Consider the one-dimensional (1D) interval $(a_1, b_1)$ with periodic boundary conditions. Letting $N_1$ be an even integer denoting the number of subintervals, and denoting space step $h = \frac{b_1 - a_1}{N_1}$, we define the mesh as $\Omega_h = \{x_j | x_j = a_1 + jh, j = 0, 1, \ldots, N_1 - 1\}$, and the space of periodic grid functions on $\Omega_h$ as $V_{N_1} = \{v | v = [v(x_0), \ldots, v(x_{N_1-1})]^T, x_j \in \Omega_h \} \subset \mathbb{R}^{N_1}$. 
We consider the interpolation space \( S_{N_1} = \text{span}\{h_j(x), j = 0, 1, \ldots, N_1 - 1\} \), where \( h_j(x) = \frac{1}{N_1} \sum_{\ell = -N_1/2}^{N_1/2} \frac{1}{c_\ell} e^{i\ell \mu_1 (x - x_j)} \),

where, \( c_\ell = 1(|\ell| \neq N_1/2), c_{-N_1/2} = c_{N_1/2} = 2 \), and \( \mu_1 = \frac{2\pi}{b_1 - a_1} \).

Defining the interpolation operator \( I_{N_1} : C(\Omega) \rightarrow S_{N_1} \) as \( I_{N_1} u(x) = \sum_{j=0}^{N_1-1} u(x_j) h_j(x) \), by taking the \( m \)-th derivative, we get

\[
\frac{\partial^m}{\partial x^m} I_{N_1} u(x_i) = \sum_{j=0}^{N_1-1} u(x_j) h_j^{(m)}(x_i) = \sum_{j=0}^{N_1-1} D_{1,i,j}^{(m)} u(x_j),
\]

where \( D_{1,i,j}^{(m)} \) is the \( m \)-th-order differentiation matrix approximating \( \partial_x^m \). The entries of the second-order Fourier differentiation matrix \( D_{1}^{(2)} = (d_{1,i,j}) \in \mathbb{R}^{N_1 \times N_1} \) are determined by

\[
d_{1,i,j} = \begin{cases} 
\frac{1}{2} \mu_1^2 (-1)^{i+j+1} \sin^2 \left( \frac{(j-i)\pi}{N_1} \right), & i \neq j, \\
-\mu_1^2 N_1^2 + 2 \pi^2, & i = j,
\end{cases}
\]

for \( i, j = 0, 1, \ldots, N_1 - 1 \). (2.1)

For a \( d \)-dimensional (\( d \leq 3 \)) hypercube domain \( \Omega = \prod_{k=1}^{d} (a_k, b_k) \), we denote an even number \( N_k \) as the grid number in the \( k \)-th dimension, set \( N = \prod_{k=1}^{d} N_k \) as the total grid number, and let \( h_k = \frac{b_k - a_k}{N_k}, \mu_k = \frac{2\pi}{b_k - a_k} \). We reshape the grid function \( u \) as a vector in \( \mathbb{R}^N \) in the order of the first dimension, the second dimension, and the third dimension. The space of grid functions \( \mathbb{V}_N \) is equipped with the \( \ell^2 \)-inner product, and \( \ell^\infty \)-norm defined by

\[
\langle u, v \rangle := \prod_{k=1}^{d} h_k v^T u = \prod_{k=1}^{d} h_k \sum_{j=0}^{N_k-1} u_j v_j, \quad ||u||_{\ell^\infty} = \max_j |u_j|, \quad \forall u, v \in \mathbb{V}_N.
\]

The induced \( \infty \)-norm of the matrix is then given by \( ||A||_{\ell^\infty} = \sup_{||u||_{\ell^\infty} = 1} ||Au||_{\ell^\infty} \), for any \( A \in \mathbb{R}^{N \times N}, u \in \mathbb{R}^N \).

**Lemma 2.1.** [16] Letting

\[
\Lambda_k^{(m)} = \begin{cases} 
[i \mu_k \text{diag}(0, 1, \ldots, N_k/2 - 1, 0, -N_k/2 + 1, \ldots, -1)]^m, & m \text{ is odd}, \\
[i \mu_k \text{diag}(0, 1, \ldots, N_k/2, -N_k/2 + 1, \ldots, -1)]^m, & m \text{ is even},
\end{cases}
\]

then \( D_k^{(m)} = F_{N_k}^{H} \Lambda_k^{(m)} F_{N_k} \), where \( F_{N_k} \) is the discrete Fourier transform with elements \( F_{N_k,j,\ell} = \frac{1}{\sqrt{N_k}} e^{-i \frac{2\pi j \ell}{N_k}} \), and \( F_{N_k}^{H} \) is the conjugate transpose of \( F_{N_k} \).

The discretization of the Laplacian in 2D and 3D can be implemented using the Kronecker products as

\[
\text{2D: } \Delta_{2,h} := I_2 \otimes D_1^{(2)} + D_2^{(2)} \otimes I_1,
\]

\[
\text{3D: } \Delta_{3,h} := I_3 \otimes I_2 \otimes D_1^{(2)} + I_3 \otimes D_2^{(2)} \otimes I_1 + D_3^{(2)} \otimes I_2 \otimes I_1,
\]

where \( D_k^{(2)} \) and \( I_k \) are the corresponding differentiation and identity matrices, respectively, in the \( k \)-th dimension. Denoting \( \Delta_{d,h} = (d_{i,j}) \in \mathbb{R}^{N \times N} \), since the eigenvalues of the differentiation matrix are non-positive, it can be concluded that \( \Delta_{d,h} \) is negative semi-definite on \( \mathbb{R}^N \).

The semi-discretization of the AC equation (1.2) on a \( d \)-dimensional domain has the form

\[
\begin{aligned}
\dot{u}_t &= Lu + N(u), \\
\dot{u}(0) &= u^0(x),
\end{aligned}
\]

(2.2)
where \( L = e^2 \Delta_{d,h} \), and \( N(u) \) is the discretization of the nonlinear term \( f(u) \). Then it can be derived that the semi-discrete energy is dissipative, i.e., \( E(u(s)) \leq E(u(t)) \) for all \( s \geq t \geq 0 \), where \( E(u) = -\frac{1}{2} \langle u, Lu \rangle + \langle F(u), 1 \rangle \).

### 2.2. Mesh-dependent maximum principle of the Fourier pseudo-spectral discretization

To derive a maximum bound for the Fourier pseudo-spectral semi-discrete system, we consider adding and subtracting a linear term \( \kappa_0 u, \kappa_0 \geq 0 \) to and from (2.2) to obtain a reformulation

\[
\dot{u}_t = \left( L - \kappa_0 I \right) u + N(u) + \kappa_0 u =: \hat{L} u + \hat{N}(u).
\]

**Lemma 2.2** (Lemmas 1a, 1c in [39]). For any matrix \( A = (a_{i,j}) \in \mathbb{R}^{N \times N} \), define the logarithmic norm by

\[
\mu_\infty(A) = \lim_{\tau \to 0^+} \frac{\ln \| e^{\tau A} \|_\infty}{\tau}.
\]

Then it holds that \( \mu_\infty(A) = \max_i \{a_{i,i} + \sum_{j \neq i} |a_{i,j}| \} \) and \( \| e^{\tau A} \|_\infty \leq e^{\tau \mu_\infty(A)}, \forall \tau > 0 \).

**Lemma 2.3.** Assume that

\[
\kappa_0 \geq \sum_{k=1}^d \epsilon^2 \mu_k^2 \frac{N_k^2 - 4}{12},
\]

then the matrix \( \hat{L} = L - \kappa_0 I \) is diagonally dominant, and it holds that

\[
\| e^\tau \hat{L} \|_\infty \leq 1, \quad \forall \tau > 0;
\]

\[
\exists \bar{\tau}_{FE} = \frac{1}{\kappa_0 + \epsilon^2 \sum_{k=1}^d \mu_k^2 \frac{N_k^2 + 2}{12}},
\]

such that \( \| u + \tau \hat{L} u \|_\infty \leq \| u \|_\infty, \quad \forall \tau \in (0, \bar{\tau}_{FE}] \). (2.6)

**Proof.** Note that \( \sum_{k=1}^{N_k-1} \sin^{-2}(\frac{j \pi}{N_k}) = \frac{N_k^2 - 1}{3}, \quad k = 1, \ldots, d \). [40]. Substituting the elements in (2.1) to \( \hat{L}_{i,i} + \sum_{j \neq i} |\hat{L}_{i,j}| \) gives

\[
-\kappa_0 + \epsilon^2 \hat{d}_{i,i} + \epsilon^2 \sum_{j \neq i} |\hat{d}_{i,j}| \leq \sum_{k=1}^d \epsilon^2 \mu_k^2 \left( - \frac{N_k^2 - 4}{12} - \frac{N_k^2 + 2}{12} + \frac{1}{2} \sum_{j=1}^{N_j-1} \sin^{-2}(\frac{j \pi}{N_j}) \right) = 0, \quad i = 0, \ldots, N - 1.
\]

Thus \( \mu_\infty(\hat{L}) \leq 0 \). Applying Lemma 2.2 yields \( \| e^\tau \hat{L} \|_\infty \leq 1 \) for any \( \tau > 0 \).

When \( \tau \in (0, \bar{\tau}_{FE}] \), we have \( 1 + \tau(\epsilon^2 \hat{d}_{i,i} - \kappa_0) \geq 0 \). By applying (2.7), the summation of absolute values of the \( i \)-th row of \( I + \tau \hat{L} \) satisfies

\[
|1 + \tau(\epsilon^2 \hat{d}_{i,i} - \kappa_0)| + \tau \epsilon^2 \sum_{j \neq i} |\hat{d}_{i,j}| = 1 + \tau(\epsilon^2 \hat{d}_{i,i} - \kappa_0) + \tau \epsilon^2 \sum_{j \neq i} |\hat{d}_{i,j}| \leq 1, \quad i = 1, \ldots, N.
\]

Thus \( \| I + \tau \hat{L} \|_\infty \leq 1 \) for any \( \tau \in (0, \bar{\tau}_{FE}] \), and (2.6) holds.

This completes the proof. \( \square \)

**Lemma 2.4.** For the GL function (1.3) and FH function (1.4), letting \( \hat{N}_{\kappa_1}(\xi) := \hat{N}(\xi) + \kappa_1 \xi \), assuming that

\[
\kappa_1 \geq \max_{|\xi| \leq \beta} |\hat{N}'(\xi)|,
\]

(2.8)
then
\[
|\dot{N}(\xi) + \kappa_1 \xi| \leq \kappa_1 \beta, \quad \forall |\xi| \leq \beta,
\]
\[
|\ddot{N}_{\kappa_1}(\xi_1) - \dot{N}_{\kappa_1}(\xi_2)| \leq 2\kappa_1 |\xi_1 - \xi_2|, \quad \forall |\xi_i| \leq \beta, i = 1, 2.
\] (2.9)

Here, the values of $\beta$ and $\kappa_1$ are given by
\[
\text{GL: } \forall \beta \geq \sqrt{1 + \kappa_0}, \text{ and } \kappa_1 \geq \max_{|\xi| \leq \beta} |\dot{N}'(\xi)| = 3\beta^2 - (1 + \kappa_0),
\] (2.11)
\[
\text{FH: } \forall \beta \in [M_+, 1), \text{ and } \kappa_1 \geq \max_{|\xi| \leq \beta} |\dot{N}'(\xi)| = \frac{\theta}{1 - \beta^2} - \theta_c - \kappa_0,
\] (2.12)

where $M_+$ in (2.12) is the positive root of the FH function $\dot{N}(\xi) = 0$.

**Proof.** (1) Consider the GL function. We have $N(\xi) = \xi - \xi^3$ and $\dot{N}(\xi) = (1 + \kappa_0)\xi - \xi^3$. Letting $M_+$ be the positive root of $\dot{N}(\xi) = 0$, we obtain $M_+ = \sqrt{1 + \kappa_0}$.

For any $\beta \geq M_+$ and $\kappa_1 \geq \max_{|\xi| \leq \beta} |\dot{N}'(\xi)| = 3\beta^2 - (1 + \kappa_0)$, it can be verified that
\[
\dot{N}'_{\kappa_1}(\xi) = (1 + \kappa_0) - 3\xi^2 + \kappa_1 \geq 0, \quad \forall |\xi| \leq \beta.
\]

Note that $\beta \geq M_+$ implies $\dot{N}(\beta) \leq 0$, since $\dot{N}_{\kappa_1}(\xi)$ is an odd function, we obtain
\[
|\dot{N}_{\kappa_1}(\xi)| \leq \dot{N}_{\kappa_1}(\beta) = (1 + \kappa_0)\beta - 3\xi^2 + \kappa_1 \beta \leq \kappa_1 \beta, \quad \forall |\xi| \leq \beta.
\]

(2) For the FH function, we have $N(\xi) = \frac{\theta}{2} \ln \frac{1 - \xi}{1 + \xi} + \theta_c \xi$ and $\dot{N}(\xi) = \frac{\theta}{2} \ln \frac{1 - \xi}{1 + \xi} + (\theta_c + \kappa_0)\xi$. Let $M_+$ be the positive root of $\dot{N}(\xi) = 0$. For any $\beta \in [M_+, 1)$ and $\kappa_1 \geq \max_{|\xi| \leq \beta} |\dot{N}'(\xi)| = \frac{\theta}{1 - \beta^2} - \theta_c - \kappa_0$, it holds that
\[
\dot{N}'_{\kappa_1}(\xi) = -\frac{\theta}{1 - \xi^2} + \theta_c + \kappa_0 + \kappa_1 \geq 0, \quad \forall |\xi| \leq \beta.
\]

Since $\dot{N}_{\kappa_1}(\xi)$ is an odd function and $\beta \geq M_+$ implies $\dot{N}(\beta) \leq 0$, we obtain
\[
|\dot{N}_{\kappa_1}(\xi)| \leq \dot{N}_{\kappa_1}(\beta) = \frac{\theta}{2} \ln \frac{1 - \beta}{1 + \beta} + (\theta_c + \kappa_0)\beta + \kappa_1 \beta \leq \kappa_1 \beta, \quad \forall |\xi| \leq \beta.
\]

For both potential functions, the Lipschitz conditions can be proved as below.
\[
|\dot{N}_{\kappa_1}(\xi_1) - \dot{N}_{\kappa_1}(\xi_2)| \leq (\max_{|\xi| \leq \beta} |\dot{N}'(\xi)| + \kappa_1) |\xi_1 - \xi_2| \leq 2\kappa_1 |\xi_1 - \xi_2|, \quad \forall |\xi_i| \leq \beta, i = 1, 2.
\]

This completes the proof.

\[\square\]

**Lemma 2.5.** Assume $\kappa_0$ and $\kappa_1$ satisfy the conditions (2.4) and (2.8), respectively. Then the summation $\hat{L}u + \hat{N}(u)$ satisfies the following forward Euler condition,
\[
\exists \tau_{FE} = \frac{\hat{\tau}_{FE} \hat{\tau}_{FE}}{\tau_{FE} + \hat{\tau}_{FE}}, \quad \text{such that } \|u + \tau \hat{L}u + \hat{N}(u)\|_{L^\infty} \leq \beta, \quad \forall 0 < \tau \leq \tau_{FE}, \forall \|u\|_{L^\infty} \leq \beta,
\] (2.13)

where $\tau_{FE}$ is the forward Euler time step in (2.6), and $\hat{\tau}_{FE} = \frac{1}{\max_{|\xi| \leq \beta} |\dot{N}'(\xi)|}$.

**Proof.** The result (2.13) can be directly proved by combining (2.6) with (2.9), therefore, we omit it. \[\square\]
Since $\tilde{N}(u)$ is Lipschitz continuous when $\|u\|_{\ell^{\infty}} \leq \beta$ for the GL and FH functions, by adopting the analytical framework proposed by Du et al. [12], applying the contraction property (2.5) and the circle condition (2.9), we present the semi-discrete, mesh-dependent maximum principle on a fixed spatial mesh as below.

**Theorem 2.6.** The semi-discrete system (2.2) has a unique solution $u(t) \in C([0,T], \mathbb{V}_N)$, and admits the semi-discrete, mesh-dependent maximum principle:

\[
\begin{align*}
\text{GL: } & \exists \beta = \max\{\sqrt{1 + \kappa_0}, \|u(0)\|_{\ell^{\infty}}\}, \quad \text{such that } \|u(t)\|_{\ell^{\infty}} \leq \beta, \quad \forall t \in [0,T], \\
\text{FH: } & \exists \beta = \max\{M_{+}, \|u(0)\|_{\ell^{\infty}}\} < 1, \quad \text{such that } \|u(t)\|_{\ell^{\infty}} \leq \beta, \quad \forall t \in [0,T],
\end{align*}
\]

where $M_{+}$ in (2.15) is given by Lemma 2.4.

**Remark 2.7.** For the second-order central finite difference discretization and the continuous AC equation, the maximum bound $\beta$ is obtained by setting $\kappa_0 = 0$ in Theorem 2.6. When considering higher-than-second-order finite difference or mass-lumped finite element method, this stabilization technique could be directly applied to derive a mesh-dependent maximum principle. By introducing a suitable stabilization term $-\kappa_0I$ to the differentiation matrix $L$, the resulting matrix $L = L - \kappa_0I$ becomes diagonally dominant. This ensures the contraction property $\|e^{\tau L}\|_{\ell^{\infty}} \leq 1$ for $\tau > 0$. Although the mesh-dependent maximum principle is universal, there is a byproduct associated with this stabilization. The magnitude of the stabilization parameter $\kappa_0$ is typically of $O(\epsilon^2 h^{-2})$, where $h$ is the minimum space step. Thus the maximum bound for the polynomial potential becomes $\beta = O(\epsilon h^{-1})$, which can be very large when $h$ approaches zero. This poses a challenge in estimating the temporal and spatial errors, as the temporal truncation error now depends on the space step $h$. Fortunately, it is common practice to choose a space step $h$ of $O(\epsilon)$, because $\epsilon$ characterizes the interface width. Therefore, the stabilization parameter $\kappa_0$ and the maximum bound $\beta$ are both of $O(1)$, which makes the proposed stabilization technique practical in computations. In addition, the maximum bound of AC equation with the FH potential (1.4) strictly satisfies $\beta < 1$, with the distance $1 - \beta$ depending on the spatial step $h$. In the literature, other more effective maximum principles for Fourier spectral Galerkin and pseudo-spectral discretizations have been investigated using different approaches; interested readers should refer to the work of Li [26].

3. Large time-stepping, delay-free relaxation ETDRK integrators

Although the existing stabilization ETD1 and ETD2 schemes preserve the maximum principle and dissipate the energy for any time step, they may encounter a time delay issue in simulations where the numerical solutions lag behind the exact solutions. In this section, we develop a family of relaxation ETDRK schemes that preserve the aforementioned structures while simultaneously eliminating the delay effect for large time-step sizes.

### 3.1. Stabilization ETDRK schemes

To define ETDRK methods, we introduce the functions [19]

\[
\varphi_k(z) = \int_0^1 e^{(1-s)z} \frac{s^{k-1}}{(k-1)!} \, ds > 0, \quad k \geq 1,
\]

which satisfy the recursion

\[
\varphi_k(z) = \frac{\varphi_{k-1}(z) - \frac{1}{(k-1)!}}{z}, \quad k \geq 1, \quad \text{with } \varphi_0(z) = e^z. \tag{3.1}
\]

To construct unconditionally MPP schemes, we first consider a reformulation of (2.3) in the form

\[
u_t = g(t, \nu) - \kappa_1 \nu + \kappa_3 \nu, \tag{3.2}
\]
where \( g(t, u) := \hat{L} u + \hat{N}(u) \), and the stabilization parameter is required to satisfy \( \kappa_1 \geq \frac{1}{\tau_{FE}} \), where \( \tau_{FE} \) is the forward Euler time step of \( \hat{L} u + \hat{N}(u) \) in (2.13).

We divide the time interval \([0,T]\) using a uniform time step \( \tau \), let \( t_n = n\tau \), and assume \( u^n \) is a suitable approximation to \( u(t_n) \). Considering exponential functions of the parameter \(-\tau \kappa_1\), by discretizing \( g(t, u) := \hat{L} u + \hat{N}(u) \) explicitly, we outline an \( s \)-stage ETDRK formulation for (3.2) in the form

\[
\begin{align*}
\mathbf{u}_{n,0} &= \mathbf{u}^n, \\
\mathbf{u}_{n,i} &= \varphi_0(-c_i \tau \kappa_1) \mathbf{u}^n + \tau \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1) [g(t_n + c_j \tau, \mathbf{u}_{n,j}) + \kappa_1 \mathbf{u}_{n,j}], & i = 1, \ldots, s, \\
\mathbf{u}^{n+1} &= \mathbf{u}_{n,s}.
\end{align*}
\]

(3.3)

Some first- (I) and second-order (II–IV) explicit ETDRK schemes [19] are presented using the Butcher tableaux:

\[
\begin{array}{c|ccc}
\vdots & \cdots & \cdots & \cdots \\
\cdots & a_{s,j} & \cdots & \vdots \\
\hline
1 & \cdots & \cdots & \cdots
\end{array}
\quad
\begin{array}{c|ccc}
I: 0 & 0 & 0 & 0 \\
\hline
1 & \varphi_1 & \varphi_1 & \varphi_1
\end{array}
\quad
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\hline
1 & \varphi_1 - \frac{1}{c_1} \varphi_2 & \frac{1}{c_1} \varphi_2 & \frac{1}{c_1} \varphi_2
\end{array}
\quad
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\hline
1 & \varphi_1 - \frac{1}{c_1} \varphi_2 & \frac{1}{c_1} \varphi_2 & \frac{1}{c_1} \varphi_2
\end{array}
\quad
\begin{array}{c|ccc}
0 & 0 & 0 & 0 \\
\hline
1 & \varphi_1 - \frac{1}{c_1} \varphi_2 & \frac{1}{c_1} \varphi_2 & \frac{1}{c_1} \varphi_2
\end{array}
\]

(3.4)

where \( \varphi_{k,i}(-\tau \kappa_1) := \varphi_k(-c_i \tau \kappa_1) \), and the abscissas satisfy \( c_1 \geq 1 \) (II, IV), \( c_1 \geq \frac{1}{2} \) (III).

One feature of the stabilization ETDRK schemes in (3.4) is that, all coefficients \( a_{i,j}(-\tau \kappa_1) \) are nonnegative, hence, they unconditionally preserve the maximum principle, positivity, strong stability, and contractivity for a large class of problems, providing that the term \( g(t, u) \) satisfies the corresponding forward Euler conditions and the stabilization parameter is suitably chosen [53]. In the literature, such schemes are referred to as positive ETDRK schemes due to their ability to preserve positivity as proven by Ostermann and van Daele[31]. Considering the maximum principle, we have the following result.

**Theorem 3.1.** For the system (3.2), suppose \( \kappa_0 \) satisfy (2.4) and \( \kappa_1 \geq \frac{1}{\tau_{FE}} \), where \( \tau_{FE} \) is defined in Lemma 2.5. Then the stabilization ETDRK formulation (3.3) with Butcher tableaux (I–IV) (3.4) preserves the mesh-dependent maximum principle for any time step \( \tau > 0 \), that is, \( \| u^{n+1} \|_{\ell^\infty} \leq \beta, \) for all \( n \geq 0 \) and \( \tau > 0 \), where \( \beta \) is given by Theorem 2.6.

**Proof.** We prove the theorem by induction. Assuming that \( \| u_{n,j} \|_{\ell^\infty} \leq \beta, j = 0, \ldots, i - 1, \) since \( \kappa_1 \geq \frac{1}{\tau_{FE}} \), by applying the forward Euler condition (2.13) and the nonnegativity of \( a_{i,j}(-\tau \kappa_1) \), we obtain

\[
\| u_{n,i} \|_{\ell^\infty} \leq \varphi_0(-c_i \tau \kappa_1) \| u^n \|_{\ell^\infty} + \tau \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1) [g(t_n + c_j \tau, u_{n,j}) + \kappa_1 u_{n,j}] \\
\leq \varphi_0(-c_i \tau \kappa_1) + \kappa_1 \| u^n \|_{\ell^\infty} + \tau \| g(t_n + c_j \tau, u_{n,j}) \|_{\ell^\infty} \\
= \beta, \quad i \leq s.
\]

Thus \( \| u^{n+1} \|_{\ell^\infty} = \| u_{n,s} \|_{\ell^\infty} \leq \beta. \) \hfill \Box

Unfortunately, the solutions of stabilization schemes have been observed to have lagging phenomena and less accurate solutions, especially when large values of stabilization parameters and time steps are considered.
3.2. Rescaled time-step size and the relaxation technique

To quantify the time-delay, we propose reformulating the stabilization ETDRK formulation into a parametric Runge–Kutta (pRK) framework with RK coefficients depending on the stabilization parameter and time step.

By substituting stage solutions into the term \( \kappa_i u_{n,j} \), using order conditions \( \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1) = c_i \varphi_1(-c_i \tau \kappa_1) \), \( i = 1, \ldots, s \) [19] and equality \( \varphi_0(z) - z \varphi_1(z) = 1 \), we rewrite the ETDRK formulation (3.3) as

\[
\begin{align*}
 u_{n,i} &= \varphi_0(-c_i \tau \kappa_1) u^n + \tau \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1) [g(t_n + c_j \tau, u_{n,j}) + \kappa_1 u_{n,j}] \\
&= [\varphi_0(-c_i \tau \kappa_1) + c_i \tau \kappa_1 \varphi_1(-c_i \tau \kappa_1)] u^n + \tau \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1) g(t_n + c_j \tau, u_{n,j}) \\
&= \underbrace{u^n + \tau \sum_{j=0}^{i-1} \hat{a}_{i,j}(-\tau \kappa_1) g(t_n + c_j \tau, u_{n,j})}_{=: u^n} \quad i = 1, \ldots, s,
\end{align*}
\]

(3.5)

where \( \hat{a}_{i,j}(-\tau \kappa_1) \) are explicitly calculated by the recursive relationship

\[
\hat{a}_{i,j}(-\tau \kappa_1) = a_{i,j}(-\tau \kappa_1) + \tau \kappa_1 \sum_{k=j+1}^{i-1} a_{i,k}(-\tau \kappa_1) \hat{a}_{k,j}(-\tau \kappa_1), \quad i = 1, \ldots, s, \quad j = 0, \ldots i - 1.
\]

(3.6)

Letting \( z := -\tau \kappa_1 \), \( \hat{c}_i(z) = \sum_{j=0}^{i-1} \hat{a}_{i,j}(z) \), we present the pRK Butcher tableaux of (3.5) corresponding to (3.4) as follows:

\[
\begin{array}{ccc}
\text{I:} & 0 & 0 \\
& \hat{c}_1 & \varphi_1 \\
\text{II:} & \hat{c}_2 & \varphi_1 - \frac{1}{c_1} \varphi_2 - z \varphi_2 \varphi_1,1 - \frac{1}{c_1} \varphi_2 \\
& c_1 \varphi_1,1 & 0 \\
\text{III:} & \hat{c}_1 & \varphi_1,1 \\
& c_1 \varphi_1,1 - \frac{1}{2 c_1} \varphi_1 - \frac{1}{2 c_1} \varphi_1,1 & 0 \\
\text{IV:} & \hat{c}_3 & \varphi_1 - \frac{1}{c_1} \varphi_2 - z \varphi_2 \varphi_1,1 - \frac{1}{c_1} \varphi_2,1 \\
& c_1 \varphi_1,1,1 & 0 \\
\end{array}
\]

For each tableau, we have the estimate of the last abscissa \( \hat{c}_s \) and the limitation of the rescaled time-step size \( \hat{\tau} = \hat{c}_s(-\tau \kappa_1) \tau \), as below:

\[
\begin{align*}
\text{I:} \quad \hat{c}_1(-\tau \kappa_1) &= \varphi_1(-\tau \kappa_1) = \frac{1 - e^{-\tau \kappa_1}}{\tau \kappa_1} + \mathcal{O}(\tau \kappa_1), & \lim_{\tau \to +\infty} \hat{c}_1(-\tau \kappa_1) \tau &= \frac{1}{\kappa_1}; \\
\text{II:} \quad \hat{c}_2(-\tau \kappa_1) &= \varphi_1(-\tau \kappa_1) + \tau \kappa_1 \varphi_1(-c_1 \tau \kappa_1) \varphi_2(-\tau \kappa_1) = 1 + \mathcal{O}((\tau \kappa_1)^2), & \lim_{\tau \to +\infty} \hat{c}_2(-\tau \kappa_1) \tau &= \frac{1 + c_1}{c_1 \kappa_1}; \\
\text{III:} \quad \hat{c}_3(-\tau \kappa_1) &= \varphi_1(-\tau \kappa_1) + \tau \kappa_1 \varphi_1(-\tau \kappa_1) \varphi_1(-c_1 \tau \kappa_1) = 1 + \mathcal{O}((\tau \kappa_1)^2), & \lim_{\tau \to +\infty} \hat{c}_3(-\tau \kappa_1) \tau &= \frac{1 + 2 c_1}{2 c_1 \kappa_1}; \\
\text{IV:} \quad \hat{c}_s(-\tau \kappa_1) &= \varphi_1(-\tau \kappa_1) + \tau \kappa_1 \varphi_1(-\tau \kappa_1) \varphi_2(-\tau \kappa_1) + (\tau \kappa_1)^2 \varphi_2(-\tau \kappa_1) \varphi_1(-c_1 \tau \kappa_1) & \lim_{\tau \to +\infty} \hat{c}_s(-\tau \kappa_1) \tau &= \frac{1 + 2 c_1}{c_1 \kappa_1}.
\end{align*}
\]

Since the ETDRK schemes with Butcher tableaux (3.4) preserve the maximum principle for any \( \kappa_1 \geq \frac{1}{\tau \kappa_1} \), we borrow the terminology strong-stability-preserving (SSP) [17], and define the rescaled SSP coefficient by
\( \hat{\tau} \) and the rescaled time step \( \hat{\tau} = \hat{\tau}_s(\tau \kappa_1) \tau \) with \( \kappa_1 = \frac{1}{\tau_{FE}} = 1 \) (right) for stabilization ETDRK schemes. (a) \( \hat{\tau}s(\tau \kappa_1) \). (b) Rescaled time step.

\[
\hat{C}(\kappa_1) = \lim_{\tau \to +\infty} \frac{\hat{\tau}_s(\tau \kappa_1 \tau)}{\tau_{FE}},
\]

\[
\hat{C}_I(\kappa_1) = \frac{1 + 2c_1}{2c_1 \kappa_1 \tau_{FE}} \leq 1,
\]

\[
\hat{C}_{II}(\kappa_1) = \frac{1 + 2c_1}{2c_1 \kappa_1 \tau_{FE}} \leq 1 + 2c_1 \leq 2 \ (c_1 \geq \frac{1}{2})
\]

We present the profiles of \( \hat{\tau}_s(\tau \kappa_1) \) and \( \hat{\tau} = \hat{\tau}_s(\tau \kappa_1) \tau \) for \( \kappa_1 = \frac{1}{\tau_{FE}} = 1 \) in Figure 1. It can be observed that the rescaling effect of \( \hat{\tau}_s \) in Figure 1a becomes stronger with increasing \( \tau \kappa_1 \) or decreasing stage number, and the rescaled time step \( \hat{\tau} \) in Figure 1b approaches \( \hat{\tau}(\kappa_1) \tau_{FE} \) as \( \tau \) goes to infinity. Since \( \hat{\tau}_s(\tau \kappa_1) \neq 1 \) for any \( \tau \kappa_1 > 0 \), the stabilization ETDRK schemes are not exactly consistent. This inconsistency is caused by the different treatment of the stabilization terms \( \kappa(u - u) \), which introduces additional truncation errors. Consequently, the numerical solution \( u^n + 1 = u_{n, i} \), obtained by (3.5) lags behind the exact solution \( u(t_n + \tau) \). This phenomenon is referred to as the time delay. However, it is important to note that the coefficient functions \( \hat{a}_{i,j} \) depend on \( \tau \kappa_1 \); thus, high-order accuracy can still be obtained if the order conditions are approximately satisfied. This is indeed the case for the stabilization ETDRK method.

To eliminate the time-delay and make the stabilization method consistent, we denote \( \hat{t}_n = n \hat{\tau} \), assume \( u^n \) is an approximation to \( u(\hat{t}_n) \), and propose replacing \( t_n + c_j \tau \) with \( \hat{t}_n + \hat{c}_j \tau \), to obtain

\[
u_{n,i} = \varphi_0(-c_i \tau \kappa_1)u^n + \tau \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1)[g(\hat{t}_n + \hat{c}_j \tau, u_{n,j}) + \kappa_1 u_{n,j}]
\]

\[
u_{n,i} = u^n + \tau \sum_{j=0}^{i-1} \hat{a}_{i,j}(-\tau \kappa_1)g(\hat{t}_n + \hat{c}_j \tau, u_{n,j}), \quad i = 1, \ldots, s.
\]

Here, the solutions \( u_{n,i} \) are considered as approximations of the solutions at relaxed time-points \( \hat{t}_n + \hat{c}_i \tau, i = 1, \ldots, s \), that is \( u(\hat{t}_n + \hat{c}_i \tau) \approx u_{n,i} \). This approach, which eliminates the time delay, is referred to as the relaxation ETDRK (RETDRK) method.

Remark 3.2. In the relaxation framework, we utilized the original time step \( \tau \) to compute \( u^n + 1 \), while the rescaled (actual) time step \( \hat{\tau} \) is used for time-stepping. It is worth noting that the rescaled SSP coefficient \( \hat{C}_j \) for
Heun’s second-order scheme (obtained by setting \( \alpha \) does not bring any benefit to actual calculations since the underlying forward Euler scheme (obtained by setting the first-order ETD I scheme is limited to a maximum value of 1. This implies that the stabilization technique proves beneficial as it increases the SSP coefficients. For instance, the stabilization ETD II scheme with \( c_1 = 1 \) and \( \kappa_1 = \frac{1}{\tau_{\max}} \) exhibits an SSP coefficient \( \hat{c}_{II} = 2 \), which is twice that of the underlying Heun’s second-order scheme (obtained by setting \( c_1 = 1, \kappa_1 = 0 \) in ETD II) with an SSP coefficient \( \hat{c} = 1 \) (p 15 of Ref. [17]).

3.3. Accuracy of RETDRK schemes

Ostermann and van Daele [31] demonstrated that positive ETDRK schemes possess a second-order barrier, thus limiting the preservation of the maximum principle to second order. In Table 1, we present the \( p \)th-order \((p \leq 2)\) conditions for the Runge–Kutta-type schemes [2] using the parametric coefficients \( \bar{a}_{i,j} \) (3.6).

**Theorem 3.3.** Assume that (3.2) has an exact solution \( u \in C^{p+1}([0,T];\mathbb{V}_N) \), and \( u^n = u(\hat{t}_n) \). If the ETDRK meets the \( p \)th-order conditions \((p \leq 2)\) in Table 1, then the RETDRK formulation (3.8) has a truncation error of \( O(\tau^{p+1}) \) when interpreted as an approximation to \( u(\hat{t}_{n+\tau}) \), i.e.,

\[
\mathbf{u}(\hat{t}_{n+\tau}) - \mathbf{u}^{n+1} = O(\tau^{p+1}\sum_{k=0}^{p-1} \kappa_1^k) = O(\tau^{p+1}).
\]

**Proof.** To derive order conditions for the RETDRK schemes, we set \( \tilde{g}(t) = g(t, u(t)) \). Note that the exact solutions at \( \hat{t}_n + \hat{c}_i \tau \) can be given by

\[
\mathbf{u}(\hat{t}_{n+\hat{c}_i \tau}) = \mathbf{u}(\hat{t}_n) + \int_0^{\hat{c}_i \tau} \tilde{g}(\hat{t}_n + s)ds, \quad i = 1, \ldots, s.
\]

The Taylor expansion of \( \tilde{g}(\hat{t}_n + s) \) has the form

\[
\tilde{g}(\hat{t}_n + s) = \sum_{k=1}^{q} \frac{s^{k-1}}{(k-1)!} \tilde{g}^{(k-1)}(\hat{t}_n) + \int_0^{\hat{c}_i \tau} \frac{(s - \sigma)^{q-1}}{(q-1)!} \tilde{g}^{(q)}(\hat{t}_n + \sigma)d\sigma.
\]

Substituting (3.11) in the right-hand side of (3.10) yields

\[
\mathbf{u}(\hat{t}_{n+\hat{c}_i \tau}) = \mathbf{u}(\hat{t}_n) + \sum_{k=1}^{q_i} \frac{\hat{c}_i \tau}{k!} \tilde{g}^{(k-1)}(\hat{t}_n) + \int_0^{\hat{c}_i \tau} \frac{(s - \sigma)^{q_i-1}}{(q_i-1)!} \tilde{g}^{(q_i)}(\hat{t}_n + \sigma)d\sigma, \quad i = 1, \ldots, s - 1,
\]

\[
\mathbf{u}(\hat{t}_{n+\hat{c}_s \tau}) = \mathbf{u}(\hat{t}_n) + \sum_{k=1}^{p} \frac{\hat{c}_s \tau}{k!} \tilde{g}^{(k-1)}(\hat{t}_n) + \int_0^{\hat{c}_s \tau} \frac{(s - \sigma)^{p-1}}{(p-1)!} \tilde{g}^{(p)}(\hat{t}_n + \sigma)d\sigma.
\]
Inserting the exact solution in the numerical scheme (3.9) gives
\[
\mathbf{u}(\hat{t}_n + \hat{c}_i \tau) = \mathbf{u}(\hat{t}_n) + \tau \sum_{j=0}^{i-1} \hat{a}_{i,j}(-\tau \kappa_1) \hat{g}(\hat{t}_n + \hat{c}_j \tau) + \Delta_{n,i}, \quad i = 1, \ldots, s - 1,
\]
with defects \(\Delta_{n,i}, i = 1, \ldots, s\). Substituting (3.11) in (3.14) gives
\[
\mathbf{u}(\hat{t}_n + \hat{c}_s \tau) = \mathbf{u}(\hat{t}_n) + \tau \sum_{j=0}^{s-1} \hat{a}_{s,j}(-\tau \kappa_1) \hat{g}(\hat{t}_n + \hat{c}_j \tau) + \Delta_{n,s},
\]
(3.15)

Subtracting (3.12) from (3.16) gives the expression of the defects,
\[
\Delta_{n,i} = \sum_{k=1}^{q_i} \tau^k \frac{\hat{c}_k}{k!} - \sum_{j=0}^{i-1} \hat{a}_{i,j}(-\tau \kappa_1) \frac{\hat{c}_j^{k-1}}{(k-1)!} \hat{g}^{(k-1)}(\hat{t}_n) + \Delta_{n,i}^{[q_i]}, \quad i = 1, \ldots, s - 1,
\]
(3.17)

with remainders \(\Delta_{n,i}^{[q_i]}\).

Similarly, the defect at time \(t_n + \hat{c}_s \tau\) is calculated as
\[
\Delta_{n,s} = \sum_{k=1}^{p} \tau^k \frac{\hat{c}_k}{k!} - \sum_{j=0}^{s-1} \hat{a}_{s,j}(-\tau \kappa_1) \frac{\hat{c}_j^{k-1}}{(k-1)!} \hat{g}^{(k-1)}(\hat{t}_n) + \Delta_{n,s}^{[p]}.
\]
(3.18)

Recalling that \(\hat{c}_i(-\tau \kappa_1) = \sum_{j=0}^{i-1} \hat{a}_{i,j}(-\tau \kappa_1)\), the stage defects (3.17) are at least of \(\mathcal{O}(\tau^2)\). To have \(p\)th-order \((p \leq 2)\) accuracy, for defect (3.18), it is required that
\[
\sum_{j=0}^{s-1} \hat{a}_{s,j}(-\tau \kappa_1) \hat{c}_j^{k-1} - \frac{\hat{c}_k}{k} = \mathcal{O}(\tau^{p+1-k} \kappa_1 p^{p-k}), \quad 2 \leq k \leq p.
\]
(3.19)

When \(k = 1\), the definition of \(\hat{c}_s\) ensures the first condition in (3.19).

When \(p = 2\) and \(k = 2\), noting that \(\hat{c}_i(-\tau \kappa_1) = \sum_{j=0}^{i-1} \hat{a}_{i,j}(-\tau \kappa_1) = 1 + \mathcal{O}(\tau \kappa_1^2)\), using the order conditions in Table 1, it can be verified that the conditions in (3.19) for \(k = 2\) are fulfilled. In the error expansion, each of these residuals is multiplied by its corresponding \(\mathcal{O}(\tau^k)\) term, and the overall truncation error is \(\mathcal{O}(\tau^{p+1} \sum_{k=0}^{p} \kappa_1^k)\), which is still \(\mathcal{O}(\tau^{p+1})\) after absorbing \(\kappa_1\) into the constant.

\[\square\]

3.4. Relaxation ETDRK for stiff system

As the linear term \(\hat{L}\mathbf{u}\) obtained via the spatial discretization is stiff, explicit treatment within the relaxation framework (3.8) requires a large stabilization parameter \(\kappa_1 \geq \frac{1}{\tau \beta} \) to maintain the mesh-dependent maximum principle. Unfortunately, this leads to a small rescaled time-step size \(\hat{c}_s \tau\), resulting in increased computational cost and truncation error within the relaxation framework. Therefore we consider employing a stabilization formulation of (2.3) in the form
\[
\mathbf{u}_t = \frac{(\hat{L} - \kappa_1 I) \mathbf{u}}{\hat{L}_1} + \frac{\hat{N}(\mathbf{u}) + \kappa_1 \mathbf{u}}{\hat{N}_1} =: \hat{L}_\kappa \mathbf{u} + \hat{N}_\kappa \mathbf{u},
\]
\[
\hat{L}_1 = \frac{L_1}{\kappa_1}, \quad \hat{N}_1 = \frac{N_1}{\kappa_1},
\]
to develop MPP schemes which require a relatively small stabilization parameter for the stiff system.

The idea is to treat \( L_{\kappa_1} \) instead of \(-\kappa_1\) as the parameter of the exponential functions, then we obtain

\[
\mathbf{u}_{n,i} = \varphi_0(c_i \tau \hat{L}_{\kappa_1}) \mathbf{u}^n + \tau \sum_{j=0}^{i-1} \hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) \hat{N}_{\kappa_1}(\mathbf{u}_{n,j}) \tag{3.20}
\]

\[
= \mathbf{u}^n + \tau \sum_{j=0}^{i-1} \hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) [N(\mathbf{u}_{n,j}) + L \mathbf{u}_{n,j}], \quad i = 1, \ldots, s, \tag{3.21}
\]

where \( \hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) \) are explicitly defined by the recursive relationship

\[
\hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) = a_{i,j}(\tau \hat{L}_{\kappa_1}) - \tau \sum_{k=0}^{i-1} a_{i,k}(\tau \hat{L}_{\kappa_1}) \hat{a}_{k,j}(\tau \hat{L}_{\kappa_1}) \hat{L}_{\kappa_1}, \quad i = 1, \ldots, s, \quad j = 0, \ldots i - 1.
\]

Letting \( \kappa = \kappa_0 + \kappa_1 \), we interpret the solutions \( \mathbf{u}_{n,i} \) obtained by (3.20) as approximations to \( \mathbf{u}(\hat{t}_n + \hat{c}_i(-\tau \kappa) \tau) \).

This interpretation is again denoted as RETDRK. When the solution \( \mathbf{u}_{n,s} \) of (3.20) is considered as an approximation to \( \mathbf{u}(t_n + \tau) \), we denote it as ETDRK.

**Theorem 3.4.** Assume that (2.2) has an exact solution \( \mathbf{u} \in C^{p+1}([0,T]; \mathbb{V}_N) \) and \( \mathbf{u}^n = \mathbf{u}(\hat{t}_n) \). If the underlying Butcher tableau meets the \( p \)-th order conditions (\( p \leq 2 \)) in Table 1, then the solution \( \mathbf{u}^{n+1} = \mathbf{u}_{n,s} \) in formulation (3.20) has a local truncation error of \( \mathcal{O}(\tau^{p+1}) \) when interpreted as an approximation to \( \mathbf{u}(\hat{t}_n + \hat{\tau}) \), where \( \hat{\tau} = \hat{c}_s \tau \).

**Proof.** Letting \( \hat{g}(t) = L \mathbf{u}(t) + N(\mathbf{u}(t)) \) and \( \mathbf{1} = [1, \ldots, 1]^T \in \mathbb{R}^N \), following the proof of Theorem 3.3, we present the defects for the RETDRK formulation (3.21) as follows:

\[
\Delta_{n,i} = \sum_{k=1}^{q_i} \tau^k \frac{c_i^k (-\tau \kappa)}{k!} 1 - \sum_{j=0}^{i-1} \hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) \frac{c_j^{k-1} (-\tau \kappa)}{(k-1)!} \hat{g}^{(k-1)}(\hat{t}_n) + \Delta_{n,i}^{[q_i]}, \quad i = 1, \ldots, s - 1
\]

\[
\Delta_{n,s} = \sum_{k=1}^{p} \tau^k \frac{c_s^k (-\tau \kappa)}{k!} 1 - \sum_{j=0}^{s-1} \hat{a}_{s,j}(\tau \hat{L}_{\kappa_1}) \frac{c_j^{k-1} (-\tau \kappa)}{(k-1)!} \hat{g}^{(k-1)}(\hat{t}_n) + \Delta_{n,s}^{[p]},
\]

Consider the second-order accuracy (\( p = 2 \)) as an example. It is required that

\[
\sum_{j=0}^{i-1} \hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) - c_i 1 = \mathcal{O}(\tau), \quad i = 1, \ldots, s - 1
\]

\[
\sum_{j=0}^{s-1} \hat{a}_{s,j}(\tau \hat{L}_{\kappa_1}) c_j^{k-1} (-\tau \kappa) \frac{1}{k!} 1 = \mathcal{O}(\tau^{p+1-k}), \quad 1 \leq k \leq p.
\]

Recall that the pRK solution (3.21), when interpreted as an approximation to \( \mathbf{u}(t_n + \tau) \), satisfies the second-order conditions:

\[
\sum_{j=0}^{i-1} \hat{a}_{i,j}(\tau \hat{L}_{\kappa_1}) - c_i 1 = \mathcal{O}(\tau), \quad i = 1, \ldots, s;
\]

\[
\sum_{j=0}^{s-1} \hat{a}_{s,j}(\tau \hat{L}_{\kappa_1}) [c_j^k c_j^{k-1} 1] - \frac{1}{k} 1 = \mathcal{O}(\tau^{p+1-k}), \quad k = 1, \ldots, p, \quad i = 0, \ldots, k - 1.
\]

Since the parametric abscissas satisfy \( \hat{c}_i(-\tau \kappa) = c_i + \mathcal{O}(\tau \kappa), i = 1, \ldots, s - 1 \), and \( \hat{c}_s(-\tau \kappa) = 1 + \mathcal{O}(\tau \kappa \kappa) \), by substituting (3.23) in (3.22), the order conditions can be directly verified. For the RETD I scheme, the first-order accuracy can be similarly derived since \( \hat{c}_1 = 1 + \mathcal{O}(\tau \kappa) \). \( \square \)
Lemma 4.1. For any reaction-diffusion equations in the form \( u_t = \epsilon^2 \Delta u + \mathcal{N}(u) =: \mathcal{L}(u) + \mathcal{N}(u) \), the stabilization schemes can also be employed. For a given solution \( u^n \) in the state space, the stabilization equation is

\[
u_t = \mathcal{L}u + \mathcal{J}_nu + \mathcal{N}(u) - \mathcal{J}_nu, \quad \text{with} \quad \mathcal{J}_n = \frac{\partial \mathcal{N}}{\partial u}(u^n),\]

where \( \mathcal{J}_n \) denotes the derivative of \( \mathcal{N}(u) \) evaluated at \( u^n \). Then the stabilization ETDRK schemes can be written as

\[
u_{n,i} = \varphi_0(\tau(\mathcal{L} + \mathcal{J}_n))\nu^n + \tau \sum_{j=0}^{i-1} a_{i,j}(\tau(\mathcal{L} + \mathcal{J}_n)) (\mathcal{N}(u_{n,j}) - \mathcal{J}_nu_{n,j}), \quad i = 1, \ldots, s,
\]

with the functions \( a_{i,j}(\cdot) \) giving by the Butcher tableaux (3.4). In the literature, this type of method is also known as the exponential Rosenbrock-type method [20, 30]. However, analyzing the rescaled time step and eliminating the time delay become more challenging due to the non-scalar nature of \( \mathcal{J}_n \).

4. Numerical analysis

4.1. Preservation of the maximum principle

The proofs of maximum-principle-preservation of ETD1 and ETD2 schemes for AC-type equations discretized with second-order finite difference method have been extensively studied by Du et al. [11, 12]. Below, we briefly present a unified framework to derive the preservation of the mesh-dependent maximum principle for the (R)ETDRK integrators (3.20) incorporated with the Fourier pseudo-spectral discretization.

Remark 3.5. For general reaction-diffusion equations in the form \( u_t = \epsilon^2 \Delta u + \mathcal{N}(u) =: \mathcal{L}(u) + \mathcal{N}(u) \), the stabilization schemes can also be employed. For a given solution \( u^n \) in the state space, the stabilization equation is

\[
u_t = \mathcal{L}u + \mathcal{J}_nu + \mathcal{N}(u) - \mathcal{J}_nu, \quad \text{with} \quad \mathcal{J}_n = \frac{\partial \mathcal{N}}{\partial u}(u^n),\]

This completes the proof. \( \square \)
\textbf{Theorem 4.2.} For the Fourier pseudo-spectral discretization system (2.3), suppose the stabilization parameters \( \kappa_0 \) and \( \kappa_1 \) satisfy (2.4) and (2.8), respectively. Then the (R)ETDRK formulation (3.20) with Butcher tableaux (I-IV) (3.4) preserves the mesh-dependent maximum principle for any time step \( \tau > 0 \),

\[
\|u^{n+1}\|_{\infty} \leq \beta, \quad \forall n \geq 0, \tau > 0,
\]

where \( \beta \) depending on the space step \( h \) is given by Theorem 2.6.

\textbf{Proof.} Note that every entry \( a_{i,j}(\tau \hat{L}_{\kappa_1}) \) of formulation (3.20) consists of the functions \( \varphi_k(c_i \tau \hat{L}_{\kappa_1}), k = 1, 2, \) and \( \varphi_1(\tau \hat{L}_{\kappa_1}) - \frac{1}{c_i} \varphi_2(\tau \hat{L}_{\kappa_1}) \). Applying Lemma 4.1 to these entries gives

\[
\|a_{i,j}(\tau \hat{L}_{\kappa_1})\|_{\infty} \leq a_{i,j}(-\tau \kappa_1), \quad i = 1, \ldots, s, \quad j = 0, \ldots, i - 1.
\]

(4.3)

We prove the theorem by induction. Assuming that \( \|u_{n,j}\|_{\infty} \leq \beta \) for \( j = 0, \ldots, i - 1 \), by taking the \( \ell^\infty \)-norm on both sides of (3.20), using the circle condition (2.9), the properties (4.3), the order conditions \( \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1) = c_i \varphi_1(-c_i \tau \kappa_1) \), and equality \( \varphi_0(z) - z \varphi_1(z) = 1 \), we obtain

\[
\|u_{n,i}\|_{\infty} \leq \|\varphi_0(c_i \tau \hat{L}_{\kappa_1})\|_{\infty} \|u^n\|_{\infty} + \tau \sum_{j=0}^{i-1} \|a_{i,j}(\tau \hat{L}_{\kappa_1})\|_{\infty} \|\hat{N}_{\kappa_1}(u_{n,j})\|_{\infty}
\]

\[
\leq \|\varphi_0(-c_i \tau \kappa_1) + \tau \kappa_1 \sum_{j=0}^{i-1} a_{i,j}(-\tau \kappa_1)\|_1
\]

\[
= [\varphi_0(-c_i \tau \kappa_1) + c_i \tau \kappa_1 \varphi_1(-c_i \tau \kappa_1)] \beta = \beta, \quad i \leq s.
\]

Thus \( \|u^{n+1}\|_{\infty} = \|u_{n,s}\|_{\infty} \leq \beta \). \hfill \Box

\textbf{4.2. Dissipation of Energy}

In addition to the preservation of maximum principle, we show that the (R)ETDRK formulations are energy stable for any time step size on a fixed spatial mesh.

\textbf{Theorem 4.3.} Suppose the stabilization parameters \( \kappa_0 \) and \( \kappa_1 \) satisfy (2.4) and (2.8), respectively. Then the (R)ETDRK formulation (3.20) with Butcher tableau I, II (\( c_1 = 1 \)), III (\( c_1 = 1 \)), and IV (\( c_1 = 1 \)) preserves the energy dissipation of (2.2), i.e., \( E(u^{n+1}) \leq E(u^n), \forall \tau > 0, n \geq 0 \).

\textbf{Proof.} The proofs of the energy dissipation for ETD I and II (\( c_1 = 1 \)) schemes can be found in [11] and [14]. Below, we show that ETD III (\( c_1 = 1 \)) and IV (\( c_1 = 1 \)) schemes are unconditionally energy dissipative.

(1) Proof for ETD III (\( c_1 = 1 \)).

Noting that \( \kappa = \kappa_0 + \kappa_1 \geq \max_{|\xi| \leq \beta} |f'(\xi)| \), applying a Taylor expansion to the difference of nonlinear parts of the energy, and using the identity \( \hat{N}_{\kappa_1}(u) = f(u) + \kappa u \) give

\[
\langle F(v) - F(u), 1 \rangle = \langle v - u, -f(u) \rangle + \langle v - u, -\frac{1}{2} f'(\xi) \cdot (v - u) \rangle
\]

\[
\leq \langle v - u, -f(u) \rangle + \frac{1}{2} \kappa \langle v - u, v - u \rangle
\]

\[
= -\langle v - u, \hat{N}_{\kappa_1}(u) \rangle + \kappa (v - u, u) + \frac{1}{2} \kappa (v - u, v - u).
\]

(4.4)

For the difference of the linear parts, it holds that

\[
-\frac{1}{2} \langle (v, L v) - \langle u, L u \rangle \rangle = -\langle v - u, L v \rangle + \frac{1}{2} \langle v - u, L (v - u) \rangle
\]

\[
= -\langle v - u, \hat{L}_{\kappa_1} v \rangle - \kappa (v - u, v) + \frac{1}{2} (v - u, L (v - u)).
\]

(4.5)
Adding (4.4) to (4.5) and using the negative semi-definite property of $L$ yields

$$E(v) - E(u) \leq \langle v - u, -[\hat{L}_{\kappa_1} v + \hat{N}_{\kappa_1}(u)] \rangle - \frac{1}{2} \kappa \langle v - u, v - u \rangle + \frac{1}{2} \langle v - u, L(v - u) \rangle$$

(4.6)

Applying (4.6) to the first-stage solution $u_{n,1} = \varphi_0(\tau \hat{L}_{\kappa_1})u^n + \tau \varphi_1(\tau \hat{L}_{\kappa_1})\hat{N}_{\kappa_1}(u^n)$, using the equality $\varphi_1(\tau \hat{L}_{\kappa_1}) = \frac{\varphi_0(\tau \hat{L}_{\kappa_1}) - I}{\tau \hat{L}_{\kappa_1}}$, and denoting $\varphi_k(\tau \hat{L}_{\kappa_1})$ as $\varphi_k$, we obtain

$$E(u_{n,1}) - E(u^n) \leq \langle u_{n,1} - u^n, -[\hat{L}_{\kappa_1} u_{n,1} + \hat{N}_{\kappa_1}(u^n)] \rangle$$

$$= \langle u_{n,1} - u^n, -[\hat{L}_{\kappa_1} u_{n,1} + (\tau \varphi_1)^{-1}(u_{n,1} - u^n + (I - \varphi_0)u^n)] \rangle$$

$$= \langle u_{n,1} - u^n, -[\hat{L}_{\kappa_1} u_{n,1} + (\tau \varphi_1)^{-1}(u_{n,1} - u^n) - \hat{L}_{\kappa_1} u^n] \rangle$$

$$= \langle u_{n,1} - u^n, -[\hat{L}_{\kappa_1} + (\tau \varphi_1)^{-1}](u_{n,1} - u^n) \rangle$$

$$= \langle u_{n,1} - u^n, \Delta_1(u_{n,1} - u^n) \rangle.$$  

(4.7)

Here, $\Delta_1 = -\tau^{-1}h_1(\tau \hat{L}_{\kappa_1})$, where $h_1(z) = z + \varphi_1^{-1}(z) = \frac{ze^z - 1}{e^z - 1}$. Noting that $\hat{L}_{\kappa_1}$ is negative definite and $h_1(z) > 0$ for any $z \neq 0$, $\Delta_1$ is negative definite and $E(u_{n,1}) - E(u^n) \leq 0$.

For the solution $u^{n+1} = u_{n,1} - \frac{1}{2} \tau \varphi_1 \hat{N}_{\kappa_1}(u^n) + \frac{1}{2} \tau \varphi_1 \hat{N}_{\kappa_1}(u_{n,1})$, it holds that

$$E(u^{n+1}) - E(u^n) \leq \langle u^{n+1} - u^n, -[\hat{L}_{\kappa_1} u^{n+1} + \hat{N}_{\kappa_1}(u^n)] \rangle$$

$$= \langle u^{n+1} - u^n, -[\hat{L}_{\kappa_1} u^{n+1} - \hat{L}_{\kappa_1} u_{n,1} + \tau \varphi_1^{-1}(u^{n+1} - u_{n,1})] \rangle$$

$$= \langle u^{n+1} - u^n, -[\hat{L}_{\kappa_1} + \tau \varphi_1^{-1}](u^{n+1} - u_{n,1}) \rangle + \langle u^{n+1} - u_{n,1}, \Delta_1(u_{n,1} - u^n) \rangle$$

$$= \langle u^{n+1} - u_{n,1}, \Delta_2(u^{n+1} - u_{n,1}) \rangle + \langle u^{n+1} - u_{n,1} - \Delta_1(u_{n,1} - u^n) \rangle.$$  

(4.8)

Here, $\Delta_2 = -\tau^{-1}h_2(\tau \hat{L}_{\kappa_1})$, where $h_2(z) = z + 2\varphi_1^{-1}(z) = \frac{ze^{\frac{z}{2}} + 2e^\frac{z}{2} - 1}{e^{\frac{z}{2}} - 1}$. Since $h_2(z) > 0$ for any $z \neq 0$, $\Delta_2$ is negative definite.

Adding (4.7) to (4.8) gives

$$E(u^{n+1}) - E(u^n) \leq A \langle u_{n,1} - u^n, \Delta_1(u_{n,1} - u^n) \rangle + B \langle u^{n+1} - u_{n,1}, \Delta_2(u^{n+1} - u_{n,1}) \rangle + C \langle u^{n+1} - u_{n,1} - \Delta_1(u_{n,1} - u^n) \rangle.$$  

(4.9)

where

$$B = \frac{1}{2} \langle u^{n+1} - u_{n,1}, \Delta_1(u^{n+1} - u_{n,1}) \rangle + \langle u^{n+1} - u_{n,1}, \Delta_1(u^{n+1} - u_{n,1}) \rangle.$$  

Here, $\Delta_2 = -\frac{1}{2}h_3(\tau \hat{L}_{\kappa_1})$, where $h_3 = h_2 - \frac{1}{2} = \frac{1}{2} + \frac{3}{2} \tau \varphi_1^{-1}(z) = \frac{ze^{\frac{z}{2}} + 2e^\frac{z}{2} - 1}{e^{\frac{z}{2}} - 1}$. It can be verified that $h_3(z) > 0, \forall z \neq 0$. Thus, $\Delta_2 = -\frac{1}{2}h_3$ is negative definite and $B_2 \leq 0$.

Noting that

$$\frac{1}{2} A + \frac{1}{2} C = \frac{1}{2} \langle u^{n+1} - u^n, \Delta_1(u_{n,1} - u^n) \rangle, \quad B_1 + \frac{1}{2} C = \frac{1}{2} \langle u^{n+1} - u_{n,1}, \Delta_1(u^{n+1} - u_{n,1}) \rangle,$$

then $\frac{1}{2} A + B_1 + C = \frac{1}{2} \langle u^{n+1} - u^n, \Delta_1(u_{n,1} - u^n) \rangle \leq 0$. Thus, $E(u^{n+1}) - E(u^n) \leq \frac{1}{2} A + \frac{1}{2} A + B_1 + C + B_2 \leq 0$.

(2) Proof for ETD IV ($c_1 = 1$).

The first-stage solution $u_{n,1}$ satisfies inequality (4.7).
For \( u_{n,2} = u_{n,1} - \tau \varphi_2 \tilde{N}_{\kappa_1}(u^n) + \tau \varphi_2 \tilde{N}_{\kappa_1}(u_{n,1}) \), it holds that

\[
\begin{align*}
E(u_{n,2}) - E(u_{n,1}) &\leq \langle u_{n,2} - u_{n,1}, -[\tilde{L}_{\kappa_1} u_{n,2} + \tilde{N}_{\kappa_1}(u_{n,1})] \rangle \\
&= \langle u_{n,2} - u_{n,1}, -[\tilde{L}_{\kappa_1} u_{n,2} - \tilde{L}_{\kappa_1} u_{n,1}, + (\tau \varphi_2)^{-1}(u_{n,2} - u_{n,1}) + \tilde{L}_{\kappa_1} u_{n,1} + \tilde{N}_{\kappa_1}(u^n)] \rangle \\
&= \langle u_{n,2} - u_{n,1}, -[\tilde{L}_{\kappa_1} + (\tau \varphi_2)^{-1}](u_{n,2} - u_{n,1}) \rangle + \langle u_{n,2} - u_{n,1}, \Delta_1(u_{n,1} - u^n) \rangle \\
&= \langle u_{n,2} - u_{n,1}, \Delta_2(u_{n,2} - u_{n,1}) \rangle + \langle u_{n,2} - u_{n,1}, \Delta_1(u_{n,1} - u^n) \rangle.
\end{align*}
\]

(4.10)

Here, \( \Delta_2 = -\tau^{-1}\tilde{h}_2(\tau \hat{L}_{\kappa_1}) \), where \( \tilde{h}_2(z) = z + \varphi_2^{-1}(z) = \frac{z(e^z - 1)}{e^z - 1} \). Since \( \tilde{h}_2(z) > 0 \) for any \( z \neq 0 \), then \( \Delta_2 \) is negative definite.

For the final solution \( u^{n+1} = u_{n,2} - \tau \varphi_2 \tilde{N}_{\kappa_1}(u_{n,1}) + \tau \varphi_2 \tilde{N}_{\kappa_1}(u_{n,2}) \), we derive

\[
\begin{align*}
E(u^{n+1}) - E(u^n) &\leq \langle u_{n,1} - u^n, \Delta_1(u_{n,1} - u^n) \rangle + \langle u_{n,2} - u_{n,1}, \Delta_2(u_{n,2} - u_{n,1}) \rangle + \langle u^{n+1} - u_{n,2}, \Delta_2(u^{n+1} - u_{n,2}) \rangle \\
&\quad + \langle u^{n+1} - u^n, \Delta_1(u^{n+1} - u^n) \rangle + \langle u^{n+1} - u^n, \Delta_2(u^{n+1} - u^n) \rangle \\
&\quad + \langle u^{n+1} - u^n, \Delta_2(u^{n+1} - u^n) \rangle.
\end{align*}
\]

(4.11)

Adding (4.7) and (4.10) to (4.11) gives

\[
\begin{align*}
E(u^{n+1}) - E(u^n) &\leq \langle (u_{n,2} - u^n, \Delta_1(u_{n,2} - u^n) \rangle + \langle u_{n,2} - u_{n,1}, \Delta_2(u_{n,2} - u_{n,1}) \rangle + \langle u^{n+1} - u_{n,2}, \Delta_2(u^{n+1} - u_{n,2}) \rangle \\
&\quad + \langle u^{n+1} - u^n, \Delta_1(u^{n+1} - u^n) \rangle + \langle u^{n+1} - u_{n,2}, \Delta_2(u^{n+1} - u_{n,2}) \rangle + \langle u^{n+1} - u^n, \Delta_2(u^{n+1} - u^n) \rangle,
\end{align*}
\]

where

\[
B = \frac{1}{2}(u_{n,2} - u_{n,1}, \Delta_1(u_{n,2} - u_{n,1})), \quad C = \frac{1}{2}(u_{n,2} - u_{n,1}, \Delta_2(u_{n,2} - u_{n,1})), \quad C = \frac{1}{2}(u_{n,2} - u_{n,1}, \Delta_2(u_{n,2} - u_{n,1})),
\]

\[
\begin{align*}
\frac{1}{2} D + B_1 &\leq \frac{1}{2}(u_{n,2} - u_{n,1}, \Delta_1(u_{n,2} - u_{n,1})), \quad \frac{1}{2} D + \frac{1}{2} A = \langle u_{n,2} - u^n, \Delta_1(u_{n,1} - u^n) \rangle, \\
\frac{1}{2} F + B_2 &\leq \frac{1}{2}(u^{n+1} - u_{n,2}, \Delta_2(u^{n+1} - u_{n,2})), \quad \frac{1}{2} F + C_2 = \langle u^{n+1} - u_{n,2}, \Delta_2(u^{n+1} - u_{n,1}) \rangle, \\
\frac{1}{2} E + C_1 &\leq \frac{1}{2}(u^{n+1} - u_{n,2}, \Delta_1(u^{n+1} - u_{n,2} + u_{n,1} - u^n)), \quad \frac{1}{2} E + \frac{1}{2} A = \langle u^{n+1} - u_{n,2} + u_{n,1} - u^n, \Delta_1(u_{n,1} - u^n) \rangle.
\end{align*}
\]

We apply the symmetry and negative definiteness of \( \Delta_1, \Delta_2 \) to obtain

\[
\begin{align*}
D + B_1 + \frac{1}{2} A &\leq \frac{1}{2}(u_{n,2} - u^n, \Delta_1(u_{n,2} - u^n)) \leq 0, \\
F + B_2 + C_2 &\leq \frac{1}{2}(u^{n+1} - u_{n,1}, \Delta_2(u^{n+1} - u_{n,1})) \leq 0, \\
E + C_1 + \frac{1}{2} A &\leq \frac{1}{2}(u^{n+1} - u_{n,2} + u_{n,1} - u^n, \Delta_1(u^{n+1} - u_{n,2} + u_{n,1} - u^n)) \leq 0.
\end{align*}
\]
Finally, we obtain

$$E(u^{n+1}) - E(u^n) \leq (D + B_1 + \frac{1}{2}A) + (E + C_1 + \frac{1}{2}A) + (F + B_2 + C_2) + B_3 + C_3 \leq 0.$$ 

\[\square\]

4.3. Convergence estimate

As we pointed out in Remark 2.7, the mesh-dependent maximum principle poses a challenge in estimating the temporal and spatial errors, as the temporal truncation error depends on the space step \(h\). For convenience, we only provide the temporal error estimate under a fixed spatial mesh.

**Theorem 4.4.** Assume the semi-discrete AC equation (2.2) has an exact solution \(u \in C^{p+1}([0,T];V_N)\), and \(u^n\) is computed by a RETDRK scheme that meets the \(p\)-th order conditions in Table 1. When \(\kappa_0\) and \(\kappa_1\) satisfy the conditions (2.4) and (2.8), respectively, scheme (3.20) has error estimate

$$
\|u(\hat{t}_n) - u^n\|_{L^\infty} \leq \frac{e^{-\gamma_0 \tau} C_{\tau}^{p}}{C_{\tau}^p + e^{-(\gamma_0 + 4C_{\tau}^p) \tau}} + \frac{2^p \beta C_{\tau}^{p}}{C_{\tau}^p + e^{-(\gamma_0 + 4C_{\tau}^p) \tau}}, \quad \forall \hat{t}_n \leq T, \tau \leq \tau^*,
$$

where \(\hat{t}_n = n\hat{\tau}\), \(\hat{\tau} = \hat{\epsilon}_s(\tau^0)\tau\), \(\gamma_0 > 0\) is arbitrary, \(\tau^*\) depends on \(\kappa\), \(C_0 = \max\{c_0, \ldots, c_{s-1}, c_s\}\), the constant \(C > 0\) depends on the \(C^p([0,T];V_N)\) norm of \(u\), \(C^p[-\beta, \beta]\)-norm of \(f\), \(\|L\|_{\infty}\), \(s\), \(\kappa_0\), and \(\kappa_1\), but is independent of \(\tau\).

**Proof.** We introduce reference functions [42] \(U_{n,i}\) for \(0 \leq i \leq s\), where \(U_{n,0} = u(\hat{t}_n)\) and \(U_{n,s} = u(\hat{t}_n + \hat{\tau})\), satisfying

$$
U_{n,i} = \varphi_0(c_i \tau \hat{L}_{\kappa_1})u(\hat{t}_n) + \tau \sum_{j=0}^{i-1} a_{i,j}(\tau \hat{L}_{\kappa_1})\hat{N}_{\kappa_1}(U_{n,j}), \quad i = 1, \ldots, s - 1,
$$

$$
U_{n,s} = \varphi_0(\tau \hat{L}_{\kappa_1})u(\hat{t}_n) + \tau \sum_{j=0}^{s-1} a_{s,j}(\tau \hat{L}_{\kappa_1})\hat{N}_{\kappa_1}(U_{n,j}) + R^n_s.
$$

For an \(s\)-stage, \(p\)-th order \((p \leq 2)\) RETDRK scheme, the defect of the solution satisfies

$$
\max_{0 \leq n \leq [T/\tau] - 1} \|R^n_s\|_{L^\infty} \leq C_s \tau^{p+1},
$$

where the constant \(C_s\) depends on the \(C^p([0,T];V_N)\)-norm of \(u\), \(C^p[-\beta, \beta]\)-norm of \(f\), \(\|L\|_{\infty}\), \(s\), \(\kappa_0\), and \(\kappa_1\), but is independent of \(\tau\).

Letting \(e^n = u(\hat{t}_n) - u^n\) and \(e_{n,i} = U_{n,i} - u_{n,i}\) for \(0 \leq i \leq s\), then \(e_{n,0} = e^n\) and \(e_{n,s} = e^{n+1}\). Subtracting (3.20) from (4.13) yields

$$
e_{n,i} = \varphi_0(c_i \tau \hat{L}_{\kappa_1})e^n + \tau \sum_{j=0}^{i-1} a_{i,j}(\tau \hat{L}_{\kappa_1})[\hat{N}_{\kappa_1}(U_{n,j}) - \hat{N}_{\kappa_1}(u_{n,j})],
$$

$$
e_{n,s} = \varphi_0(\tau \hat{L}_{\kappa_1})e^n + \tau \sum_{j=0}^{s-1} a_{s,j}(\tau \hat{L}_{\kappa_1})[\hat{N}_{\kappa_1}(U_{n,j}) - \hat{N}_{\kappa_1}(u_{n,j})] + R^n_s.$$
Since Theorem 4.2 guarantees \( \| U_{n,i} \|_{\ell^\infty} \leq \beta, \| u_{n,i} \|_{\ell^\infty} \leq \beta, i = 0, \ldots, s, \) denoting \( C_0 = \max \{ c_1, \ldots, c_{s-1}, c_s \}, \) by applying Lipschitz condition (2.10) and the inequalities in Lemma 4.1, we derive

\[
\| e_{n,i} \|_{\ell^\infty} \leq \varphi_0(-c_i \tau \kappa_1) \| e^n \|_{\ell^\infty} + \tau \sum_{j=0}^{i-1} a_{i,j} \varphi_1(-c_j \tau \kappa_1) \| \hat{N}_{\kappa_1}(U_{n,j}) - \hat{N}_{\kappa_1}(u_{n,j}) \|_{\ell^\infty} \\
\leq \varphi_0(-c_i \tau \kappa_1) \| e^n \|_{\ell^\infty} + 2 \tau \kappa_1 c_i \varphi_1(-c_i \tau \kappa_1) \sum_{j=0}^{i-1} \| e_{n,j} \|_{\ell^\infty} < 1 \\
\leq (1 + 2C_0 \tau \kappa_1)^i \| e^n \|_{\ell^\infty}, \quad i = 1, \ldots, s - 1,
\]

\[
\| e_{n,s} \|_{\ell^\infty} \leq \varphi_0(-c_s \tau \kappa_1) \| e^n \|_{\ell^\infty} + \tau \sum_{j=0}^{s-1} a_{s,j} \varphi_1(-c_j \tau \kappa_1) \| \hat{N}_{\kappa_1}(U_{n,j}) - \hat{N}_{\kappa_1}(u_{n,j}) \|_{\ell^\infty} + \| R^n_s \|_{\ell^\infty} \\
\leq \varphi_0(-c_s \tau \kappa_1) \| e^n \|_{\ell^\infty} + 2 \tau \kappa_1 \sum_{j=0}^{s-1} a_{s,j} \varphi_1(-c_j \tau \kappa_1) \| e_{n,j} \|_{\ell^\infty} + \| R^n_s \|_{\ell^\infty} \\
= \varphi_1(-\kappa_1) < 1
\]

\[
\| e^n \|_{\ell^\infty} \leq (1 + 2C_0 \tau \kappa_1)^s \| e^n \|_{\ell^\infty} + C_s \tau^{p+1} \\
\leq (1 + 2C_0 \tau \kappa_1)^s \| e^n \|_{\ell^\infty} + C_s \tau^{p+1}.
\]

By induction, we obtain

\[
\| e^n \|_{\ell^\infty} \leq (1 + 2C_0 \tau \kappa_1)^n \| e^0 \|_{\ell^\infty} + C_s \tau^{p+1} \sum_{i=0}^{n-1} (1 + 2C_0 \tau \kappa_1)^i \\
\leq e^{2C_0 \kappa_1 n \tau} \| e^0 \|_{\ell^\infty} + \frac{C_s}{2C_0 \kappa_1 \theta} (e^{2C_0 \kappa_1 n \tau} - 1) \tau^p.
\]

Noting that \( \hat{t}_n = n \hat{\varepsilon} \tau, \) and the parametric abscissa \( \hat{\varepsilon}_r(-\tau \kappa) \) of RETDRK schemes is monotonically decreasing for \( \tau \kappa \geq 0. \) For each \( \kappa > 0, \) we can find \( \tau^* \) such that \( \hat{\varepsilon}_r \geq \frac{1}{2} \) when \( \tau \leq \tau^*, \) then \( \frac{\hat{t}_n}{\hat{\varepsilon}_r} \leq 2T. \) Since \( \| e^0 \|_{\ell^\infty} = 0, \) letting \( C = \frac{C_s}{2C_0 \kappa_1 \theta}, \) we obtain

\[
\| e^n \|_{\ell^\infty} \leq e^{2C_0 \kappa_1 \tau \hat{t}_n / \hat{\varepsilon}_r} \tau^p \leq e^{2C_0 \kappa_1 \tau (\hat{t}_n / \hat{\varepsilon}_r)} \tau^p.
\]

(4.14)

Note that

\[
\| e^n \|_{\ell^\infty} = \| u(\hat{t}_n) - u^n \|_{\ell^\infty} \leq 2\beta.
\]

(4.15)

Let \( \gamma = \frac{C_0 \tau^p}{e^{-\gamma_0 + 4C_0 \kappa_1 \tau T} + e^{-2C_0 \kappa_1 \tau T}} \) when \( \gamma_0 > 0, \) is arbitrary. Multiplying (4.15) by \( \gamma \) and (4.14) by \( 1 - \gamma = \frac{e^{-\gamma_0 + 4C_0 \kappa_1 \tau T}}{e^{-\gamma_0 + 4C_0 \kappa_1 \tau T} + e^{-2C_0 \kappa_1 \tau T}}, \) and doing a summation gives the desired result.

The inequality (4.12) implies that the \( p \)-th order temporal convergence of the RETDRK schemes, and shows that, as \( \hat{t}_n \to +\infty \) with \( h \) and \( \tau \) fixed, the \( \ell^\infty \) error does not exceed \( 2\beta. \) It should be noted that, when using the second-order finite difference method for spatial discretization, the constant \( C \) and maximum bound \( \beta \) will not depend on the space step \( h. \)
We present benchmark examples that illustrate the delay-free and structure-preserving properties of proposed relaxation ETDRK integrators (3.20). We compute the maximum bounds and stabilization parameters for the second-order finite difference (FD) discretization and Fourier pseudo-spectral (FP) discretization of AC equation in Table 2 by using the formulas in Lemma 2.4. To accelerate simulations, we compute products of matrix exponentials with vectors via the fast Fourier transform [11].

5. Numerical experiments

We first verify the temporal convergence of the proposed RETDRK schemes (3.20).

Example 5.1. Consider the Ginzburg–Landau potential function (1.3) and a modified AC equation,

$$u_t - \epsilon^2 \Delta u - f(u) = \tilde{f}(x, t), \quad x \in \Omega = (0, 2\pi),$$

such that the analytical solution to (5.1) has the form

$$u_{ex}(x, t) = \sin(x) \cos(t), \quad t \in [0, \infty).$$

The artificial, time-dependent forcing term $\tilde{f}(x, t)$ is computed from the left-hand side using the exact solution. We chose the parameter $\epsilon = 0.1$.

By fixing the uniform spatial mesh with grid number $N = 256$, we computed the numerical solutions using the ETDRK at $T = 2$, and RETDRK schemes (3.20) at $T \approx 2.0$ with different time steps. Note that this large grid number will lead to a large stabilization parameter according to (2.4) and (2.8), thus introducing more truncation errors. Since the time-step sizes used for accuracy tests were at most $\tau = 1$, we chose a fixed stabilization parameter $\kappa = \kappa_0 + \kappa_1 = 2$ in all tests.

The $\ell^\infty$-errors between the numerical solutions and analytical solution are presented in Figure 2. It is observed that each of the ETDRK and RETDRK schemes converges with the theoretical order of accuracy. For the two-stage, second-order (R)ETD II and (R)ETD III schemes, the obtained errors are very close, while the three-stage (R)ETD IV scheme gives more accurate solutions. This shows the advantage of using more stages in simulations. In addition, because of the elimination of the time delay, the RETDRK schemes in Figure 2b effectively reduce the numerical errors of the stabilization ETDRK schemes in Figure 2a. This demonstrates the superiority of introducing the relaxation step in the stabilization ETDRK framework.

We then show the efficiency of the proposed relaxation schemes by comparing the CPU times between the ETDRK and RETDRK schemes with $c_1 = 1$. We chose the spatial grid number $N = 2^{12}$. Taking time steps $\tau = 2^{-i}, i = 2, \ldots, 11$, we present the $\ell^\infty$-errors versus CPU times at $T = 2$ with $\kappa = 2.0$ in Figure 3. It can be clearly observed that, to reach a given accuracy, the RETDRK schemes (dashed curves) took less CPU times than ETDRK schemes (solid curves), and the three-stage (R)ETD IV schemes are more efficient.
Example 5.2. To test the convergence for stiff problems, we considered the well known example from Prothero and Robinson \[32\]

\[ u_t = \lambda (g(t) - u) + g'(t), \quad u(0) = g(0), \tag{5.2} \]

with \( g(t) = \sin(\frac{\pi}{4} + t) \).

We solved this ODE to final time \( t = 0.1 \) using equidistant step size \( \tau = \frac{1}{10^{k+2}}, k = 1, \ldots, 12 \) by the ETDRK and RETDRK schemes with stabilization parameter \( \kappa = \lambda \). In Figure 4, we present the numerical results for three cases: (a) \( \lambda = 1 \), (b) \( \lambda = 5000 \) computed using ETDRK schemes, and (c) \( \lambda = 5000 \) computed using RETDRK schemes. For the case of \( \lambda = 1 \), all methods converge with the expected order, and the curves obtained by ETD II and ETD IV overlap. However, for the case of \( \lambda = 5000 \), the errors of all schemes increase, and the second-order ETDRK schemes suffer from order reductions, dropping down to first-order in the time-step interval of \([10^{-3}, 10^{-1}]\). It is worth noting that despite suffering from order reduction, the ETD II and ETD IV schemes still exhibit better accuracy than the ETD I and ETD III schemes. After performing the relaxation
Example 5.3. To illustrate the time-delay issues, we considered an exact traveling wave (TW) solution, 

$$u(x, t) = \frac{1}{2}(1 - \tanh \frac{x - st}{2\sqrt{2}\epsilon})$$

$$\Omega = (-1, 3), \quad s = \frac{3\epsilon}{\sqrt{2}},$$

to the 1D AC equation (1.2) with Dirichlet boundary conditions and diffusion parameter $\epsilon = 0.01$.

We chose the second-order finite difference discretization to deal with the Dirichlet boundary conditions, and computed the TW solutions to final time $T = 105$ using the ETDRK and RETDRK schemes (3.20) with spatial grid number $N = 256$, time-step size $\tau = 1.5$ and stabilization parameter $\kappa = 2.0$. The numerical solutions are presented in Figure 5. From Figure 5a, it can be observed that the solution of each ETDRK scheme is better interpreted as an approximation to the TW solution at an earlier time $\hat{t} = \hat{c} \hat{s} T$, and the time delay becomes weaker with increasing order and stage numbers. After introducing the relaxation step, the RETDRK schemes in Figure 5b significantly eliminate the lagging phenomenon. Since the rescaled time step $\hat{\tau}$ is always smaller than the original time step $\tau$, the RETDRK schemes need $\frac{1}{\hat{c}}$ times of the calculations in ETDRK schemes to reach the final time.

5.2. Structure-preserving tests

Example 5.4. Considering both the GL and FH potentials, we tested the maximum-principle-preservation and energy stability of the RETDRK schemes on the 1D AC equation with initial condition

$$u(x, 0) = 0.1 \times \sin(2\pi x), \quad \Omega = (0, 2).$$

We chose $\epsilon = 0.01$, $N = 128$. For FP discretization, Table 2 gives the corresponding maximum bound $\beta \approx 1.5321$ and $\kappa \approx 6.0416$. To avoid the singularity of the exponential functions at $\kappa = 0$, we computed the reference solutions with a refined time step $\tau = 0.01$ and $\kappa = 0.20$. The profiles of the GL potential computed by ETDRK and RETDRK with $c_1 = 1$ and different choices of $\tau$ and $\kappa$ are presented in column (a) of Figure 6. When $\kappa = 0.20$, with increasing of time step $\tau$ from 0.01 to 1.50, the energy profile (orange triangle line) computed by ETD II oscillates, and the solution deviates from the reference solution. After adopting a larger
Figure 5. Example 5.3: TW solutions computed by ETDRK and RETDRK schemes at $T = 105$. Rescaled time steps are: RETD I: $\hat{\tau} \approx 0.4751$, RETD II: $\hat{\tau} \approx 0.7997$, RETD III: $\hat{\tau} \approx 0.7008$, RETD IV: $\hat{\tau} \approx 1.0215$. Dashed curves denote the exact TW solutions at $T = 105$, and the color dotted lines denote TW solutions at $t = c_\ast T$, where $c_\ast \approx 0.3167, 0.5332, 0.4672, 0.6810$ are the rescaled last abscissas of the RETDRK schemes. Parameters: $\epsilon = 0.01, N = 256, \kappa = 2.0, \tau = 1.5$. (a) ETDRK. (b) RETDRK.

$k \approx 6.0416$, the oscillations in the energy profile (blue dotted line) disappear, and the solution becomes stable and satisfies the maximum principle. However, the time delay of the ETD II scheme ($k \approx 6.0416$) is obvious, and the solution (top row) at $t \approx 12$ is no longer accurate. By adopting the relaxation technique, the RETD II scheme gives a very stable and accurate solution (green curve) with no time delay.

Realizing that the large stabilization parameter $k \approx 6.0416$ of FP discretization leads to a strong delay effect and a small rescaled time step $\hat{\tau} \approx 0.3292$ for the RETD II ($c_1 = 1$) scheme, thus the relaxation step introduces more computational costs. In column (b) of Figure 6, we considered RETD III scheme for the GL potential with stabilization parameter $k = 2.00$, which is the same as that required by FD discretization. Clearly, the time delay of ETD III ($k = 2.00$, blue square curve) becomes weaker, and the rescaled time step of RETD III is increased to $\hat{\tau} \approx 0.7500$. In addition, it demonstrated that the solution obtained by RETD II with $k = 2.00$ has good performance in preserving the maximum principle and decreasing the energy.

We consider the FH potential in column (c) of Figure 6. To preserve the maximum principle of the FH potential, a very large stabilization parameter $k \approx 312.73$ will be theoretically needed for FP discretization. To avoid large splitting errors, by setting $k \approx 8.02$, it can be observed that the RETD IV scheme (green curve) eliminates the time delay of the ETD IV scheme (blue square curve), and keeps the energy stable. The stabilization parameters should theoretically satisfy conditions (2.4) and (2.8) to preserve the structures, but in practice, this example demonstrates that a smaller stabilization parameter $k$ (the same as that of FD discretization) can be used to accelerate computations while having good ability in preserving the maximum principle and maintaining stability.

Example 5.5. Consider the 2D mean curvature problem [45] with periodic boundary conditions and disk-type initial value

$$u(x, y, 0) = \tanh\left(\frac{\sqrt{x^2 + y^2} - 0.6}{\sqrt{2}\epsilon}\right), \quad \Omega = (-1, 1)^2.$$ 

It is known that the circular interface is unstable: the radius of the disk will shrink at the rate of the curvature of the circle and eventually disappear. The analytical relationship is given by $R(t) = \sqrt{R_0^2 - 2\epsilon^2 t}$ asymptotically, with initial radius $R_0 = 0.6$. At $t = \frac{0.18}{\epsilon^2}$, the singularity occurs and the disk disappears.
Figure 6. Example 5.4: Profiles of $u$ at $t \approx 12$ (first row), evolutions of $\|u\|_{\ell^\infty}$ (second row), and energy (third row) computed with different combinations of $\tau$ and $\kappa$ using ETDRK and RETDRK. Black dashed lines in the first and second rows denote the levels of $\pm 1$ (columns a, b), and $\pm 0.9575, \pm 0.9987$ (column c). Parameters: $\epsilon = 0.01, N = 128$. (a) ETD II ($c_1 = 1$), $\hat{\tau} \approx 0.3292$, GL. (b) ETD III ($c_1 = 1$), $\hat{\tau} \approx 0.7500$, GL. (c) ETD IV ($c_1 = 1$), $\hat{\tau} \approx 0.3587$, FH.

We set the parameters $\epsilon = 0.02, \kappa = 2.0$, spatial grid number $N = 256^2$, and the abscissa $c_1 = 1$. The reference solution is computed using RETD IV with a refined time step of $\tau = 0.1$. The evolutions of the circle radius and the energy profiles are shown in Figure 7. Reassuringly, the radius of the reference solution (red dotted curve) closely follows the analytical curve. Because of the large splitting errors, the radius curve of ETD II (orange triangle curve) falls behind that of the ETD IV scheme (blue square curve). After the relaxation step, the radius curves of both RETD II and RETD IV improve and closely follow the analytical curve. These improvements can also be observed in the energy profiles in Figure 7b. In addition, all energy curves are decreasing. Figure 8 presents the snapshots of the solution profiles computed by RETD IV with $\tau = 0.5, \kappa = 2.0$ and $\hat{\tau} \approx 0.4751$. 
Figure 7. Example 5.5: Evolutions of the radius (left) and energy (right) computed by ETDRK and RETDRK schemes with $c_1 = 1$. (a) Radius. (b) Energy.

Figure 8. Example 5.5: Solutions computed by RETD IV ($\tau = 0.5, \kappa = 2.0, \hat{\tau} \approx 0.4751$). (a) $t = 0$. (b) $t \approx 300$. (c) $t \approx 420$. (d) $t \approx 460$.

Table 3. Centers $(x_i, y_i)$ and radii of Example 5.6.

<table>
<thead>
<tr>
<th>$i$</th>
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<th>$y_i$</th>
<th>$r_i$</th>
</tr>
</thead>
<tbody>
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<td>$\pi/2$</td>
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<tr>
<td>2</td>
<td>$\pi/4$</td>
<td>$3\pi/4$</td>
<td>$2\pi/15$</td>
</tr>
<tr>
<td>3</td>
<td>$\pi/2$</td>
<td>$5\pi/4$</td>
<td>$\pi/10$</td>
</tr>
<tr>
<td>4</td>
<td>$\pi$</td>
<td>$\pi/4$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>5</td>
<td>$3\pi/2$</td>
<td>$\pi$</td>
<td>$3\pi/2$</td>
</tr>
<tr>
<td>6</td>
<td>$\pi$</td>
<td>$\pi/4$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>7</td>
<td>$3\pi/2$</td>
<td>$\pi$</td>
<td>$\pi/4$</td>
</tr>
</tbody>
</table>

As can be seen, the solutions never cross the maximum bound $\beta = 1.0$ of the continuous AC equation, and the circle disappears at $t \approx 460$, which is only a little later comparing with the exact disappearing time.

Example 5.6. Consider the seven circles benchmark problem [6] with initial condition

$$u(x, y, 0) = -1 + \sum_{i=1}^{\infty} f_0(\sqrt{(x - x_i)^2 + (y - y_i)^2 - r_i}), \quad \Omega = (0, 2\pi)^2, \quad f_0(s) = \begin{cases} 2e^{-s^2}, & \text{if } s < 0, \\ 0, & \text{otherwise}. \end{cases}$$

This initial condition consists of seven circles, with centers and radii given in Table 3.
Computations were done with the GL potential (1.3), the parameter $\epsilon = 0.05$, $N = 256^2$, and final time $T = 250$. Evolutions of the maximum bounds and energy profiles obtained by the ETD II and RETD II schemes with $c_1 = 1$ and different choices of $\tau$, $\kappa$ are recorded in Figure 9. When $\kappa = 0.2$, as the time step increases from $\tau = 0.1$ to 1.5, the solution (orange triangle curve) obtained by ETD II breaks the maximum bound ($\beta = 1.0$) of the continuous AC equation, and the energy no longer decreases. Although the introduction of $\kappa = 2.0$ stabilizes the simulation, the obtained energy profile (blue square curve) deviates greatly from the reference. Adopting the relaxation step, the energy profile of the RETD II scheme (green curve) improves and follows the reference energy curve closely. Figure 10 shows the reference solutions ($\tau = 0.1, \kappa = 0.2$) and large-time-step solutions ($\tau = 1.5, \kappa = 2.0, \hat{\tau} \approx 0.7997$) obtained by RETD II scheme. The annihilation of the circles takes place gradually, and the large-time-step solutions computed by RETD II are in good agreement with the reference.

**Example 5.7.** Consider FH potential (1.4) and the 2D profile given by

$$u(x, y, 0) = 0.1 \times \sin(2\pi x) \sin(2\pi y), \quad \Omega = (-0.5, 0.5)^2,$$

with periodic boundary conditions.
Choosing \( \epsilon = 0.01 \) and discretizing the domain using \( N = 128^2 \) grid points, we solved the AC equation to final time \( T = 100 \) using the ETD II and RETD II schemes with \( c_1 = 1 \). The reference solution was computed by RETD II with \( \tau = 0.01 \) and \( \kappa = 0.20 \). Figure 11 illustrates that RETD II with \( \tau = 0.40 \) and \( \kappa = 8.02 \) (green curves) outperforms the ETD II schemes by stabilizing the solutions (\( \kappa = 0.20 \), orange triangle curve) and eliminating the time-delay issue (\( \kappa = 8.02 \), blue square curve). In addition, the RETD II scheme (green curves) maintains the maximum principle \( \beta \approx 0.9575 \) and dissipates the energy of the AC equation. Figure 12 presents the solutions of different schemes at \( t \approx 10 \). It can be seen that the RETD II scheme in Figure 12d removes the oscillations of the ETD II scheme with small stabilization parameter in Figure 12b and the time delay in Figure 12c.

**Example 5.8.** Finally, we considered the 3D AC equation with the GL potential and a uniformly random distributed phase field as the initial condition:

\[
u(x, y, z, t = 0) = 1.8 \times \text{rand}(x, y, z) - 0.9, \quad \Omega = (-0.5, 0.5)^3,
\]

where rand\((x, y)\) is uniformly distributed in \((0, 1)\).

We set \( \epsilon = 0.01 \), and discretized the 3D domain using \( N = 128^3 \) grid points. The isosurfaces computed by the RETD II scheme (\( c_1 = 1, \tau = 1.5, \kappa = 2.0, \hat{\tau} \approx 0.7997 \)) at \( t = 3.20, 30.39, 120.76, \) and \( 360.68 \) are presented in Figure 13. Figure 14 shows the evolutions of the maximum bounds and energy computed by ETD II and
Figure 13. Example 5.8: Snapshots of solutions obtained by RETD II with $\tau = 1.5, \kappa = 2.0$, and $\hat{\tau} \approx 0.7997$ using GL potential. (a) $t \approx 3.20$. (b) $t \approx 30.39$. (c) $t \approx 120.76$. (d) $t \approx 360.68$.

Figure 14. Example 5.8: Evolutions of $\|u\|_{\ell^\infty}$ (left) and energy $E_h$ (right) computed by ETD II and RETD II. (a) $\|u\|_{\ell^\infty}$. (b) $E_h$.

RETII. It can be seen that the maximum bounds computed by ETD II with $\tau = 1.5$ and $\kappa = 0.2$ oscillate, and the energy profile deviates from the reference curve computed with $\tau = 0.5$ and $\kappa = 0.2$. In contrast, the RETD II scheme with $\tau = 1.5$ and $\kappa = 2.0$ well preserves the maximum principle, and the decreasing energy closely follows the reference curve. This again demonstrates the superiority of the relaxation technique.

6. Concluding remarks

We developed a paradigm to quantify the time delay of the stabilization ETDRK schemes for the AC equation, and proposed a relaxation technique to eliminate the lagging phenomenon in simulations. The mesh-dependent maximum principle for the Fourier pseudo-spectral discretization was established for the first time on a fixed mesh for the AC equation equipped with either the polynomial or logarithmic potential by adopting a stabilization technique. With a suitable choice of stabilization parameter, the maximum-principle-preservation of the proposed relaxation ETDRK integrators was proven in a unified approach, and the energy stability was derived for some particular Butcher tableaux. Ample numerical examples validated the rationality, efficiency, and accuracy of the relaxation approach.

One distinctive feature of the work is the quantification and elimination of the time delay of stabilization ETDRK integrators constructed by introducing $\kappa(u - u)$. We derived that the solution of this stabilization
approach is better interpreted as an approximation to an earlier time that is determined by the rescaled time step. This not only reduces the splitting error but provides a new strategy to study the time delay of other stabilization schemes. It is worth mentioning that the analysis of general stabilization techniques such as Laplacian stabilization $\Delta(u - u)$ or biharmonic stabilization $\Delta^2(u - u)$ has not been fully solved. For further study, it would be interesting to quantify the lagging effects for different stabilization formulations.

Note that the proposed relaxation integrators exhibit a second-order barrier in the temporal direction to preserve the maximum principle, and an inevitable fact of such integrators is that the rescaled time step $\hat{\tau}$ is always smaller than $\tau_{FE}$. To accelerate the computations, it is meaningful to develop high-order temporal integrators that can increase the SSP coefficient $\hat{C}$ and preserve the structures for large time steps. Furthermore, it should be acknowledged that the maximum principle derived using the stabilization technique is dependent on the mesh. Although this presents challenge in estimating spatial and temporal errors, this type of stabilization has demonstrated effectiveness in practical applications.

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REFERENCES


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