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A GENERALIZED FINITE ELEMENT θ-SCHEME FOR BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS AND ITS ERROR ESTIMATES

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Abstract. In this paper, we study numerical methods for solving a class of nonlinear backward stochastic partial differential equations. By utilizing finite element methods in space and θ-scheme in time, the proposed scheme forms a generalized spatio-temporal full discrete scheme, which can be solved in parallel. We rigorously prove the boundedness and error estimates, and obtain the optimal convergence rates in both time (first order/second order) and space ($k+1$, k in L^2 and H^1, respectively). Numerical results are finally provided to demonstrate the effectiveness of the proposed scheme and validate the theoretical analyses.

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1. Introduction

Let $D \subset \mathbb{R}^d (d = 1, 2, 3)$ be a bounded domain with smooth boundary, and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete, filtered probability space satisfying the usual conditions. The filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ (T the final time instant) is the natural filtration of a standard Brownian motion $W_t := (W_t^1, W_t^2, \ldots, W_t^q)^T$, $q \in \mathbb{N}^+$ with $\mathcal{F}_0$ containing all the P-null sets of $\mathcal{F}$. In this paper, we consider the following Itô-type nonlinear backward stochastic partial differential equations (BSPDEs) defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$\begin{cases} y(t,x) + \displaystyle\int_t^T \mathcal{L}y(s,x)\,\mathrm{d}s = \varphi(x,\mu(T)) + \int_t^T f(s,x,\mu(s),y(s,x),\nabla y(s,x),z(s,x))\,\mathrm{d}s \\ \qquad\qquad\qquad\qquad\qquad\qquad - \displaystyle\int_t^T z(s,x)\,\mathrm{d}W_s, \quad (t,x)\in[0,T]\times D, \\ y(t,x) = z(t,x) = 0, \qquad (t,x)\in[0,T)\times\partial D, \\ y(T,x) = \varphi(x,\mu(T)), \qquad x\in\partial D, \end{cases} \tag{1.1}$$

where the operator $\mathcal{L}$ is a second order parabolic operator, the terminal condition $\varphi : D \times \mathbb{R}^q \to \mathbb{R}$ and $f : [0,T] \times D \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^{1\times d} \times \mathbb{R}^{1\times q} \to \mathbb{R}$ are two given functions, $\mu(t)$ is a diffusion process defined by $\mu(t) = \mu_0 + W_t$ with $\mu_0 \in \mathbb{R}^q$, ∇y is the gradient of y with respect to the spatial variable $x = (x_1, \ldots, x_d)$.

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In addition, $\mathcal{L}y(s,x)$ is defined by

$$\mathcal{L}y := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(b_{jk}(s,x)\frac{\partial y}{\partial x_k}\right) + b_0(s,x)y, \tag{1.2}$$

where $(b_{jk}(s,x))_{d\times d}$ is a symmetric and uniformly positive definite matrix, $b_0(s,x) \geq 0$ is bounded and continuous function. A pair of random fields $(y,z) : [0,T]\times D\times\Omega \to \mathbb{R}\times\mathbb{R}^{1\times q}$ is called an adapted solution of (1.1), if it is $\mathbb{F}$-adapted, square integrable and satisfies the BSPDEs (1.1).

The existence and uniqueness of the solution for nonlinear backward stochastic differential equations (BSDEs) were first established by Pardoux and Peng [22] in 1990. Since then, the theories of BSDEs have been widely studied, see, *e.g.*, [12, 20, 24] and references therein. By incorporating the physical spatial variable into the system, BSPDEs as an extension of the BSDEs, are also of great study significance. In company with the stochastic partial differential equations (SPDEs), see [5, 21, 26], the theories of the BSPDEs are also developed. For some recent advance, the strong solution of BSPDEs was discussed in [9, 10, 16]. The adapted solutions of BSPDEs in infinite dimensions were studied in [13]. BSPDEs also have extensive application prospects, *e.g.*, nonlinear filtering [21], stochastic optimal control [2, 33] and mathematical finance [19], etc.

Since the analytical solutions of stochastic (partial) differential equations are seldom available in most cases, numerical methods become feasible approaches in those fields. In recent years, great efforts have been made for constructing and analyzing effective numerical schemes for BSDEs [14, 27, 28, 32] and stochastic partial differential equations (SPDEs) [3, 4, 7, 8, 15], etc. As a counterpart, BSPDEs are few numerically studied, but there are still some literature, among which are Euler method [11, 29], splitting-up method [18], and [17] for some recent developments.

In this paper, we focus on the numerical solution of a class of nonlinear BSPDEs, the main contribution is the development and analyses of an effective generalized finite element θ-scheme for solving BSPDEs (1.1). The stochastic mechanism of BSPDEs distinguishes them from deterministic PDEs and forward SPDEs. In order to address the analytical challenges associated with BSPDEs and to propose a generalized θ-scheme, we extensively incorporate stochastic analysis techniques such as conditional expectation and Feynman–Kac formula, etc., along with finite element methods for PDEs. Under proper regularity assumptions, we establish optimal convergence rates in both space and time. Our proposed numerical scheme also takes into consideration of computational efficiency, which is designed in parallel. This is particularly important since the computational complexity of BSPDEs is relatively higher compared to the forward SPDEs. The analytical techniques used in this work can be readily applied to handle more complex BSPDEs.

The rest of this paper is organized as follows. In Section 2, some useful spaces and notations are introduced, several assumptions and lemmas are given. In Section 3, we give the weak form of BSPDEs (1.1), and perform its full space-time discretization to propose a finite element θ-scheme for solving BSPDEs (1.1). Then we conduct theoretical analyses in Section 4. Finally, in Section 5, numerical tests are presented to verify our theoretical conclusions.

2. Preliminaries

Let $\mathcal{W}^{m,p}(D)$ with $m \geq 0$, $1 \leq p \leq \infty$ be standard Sobolev spaces with norm $\|\cdot\|_{\mathcal{W}^{m,p}}$ (see [6]). We set $H^m(D) = \mathcal{W}^{m,2}(D)$ with norm $\|\cdot\|_m$. Let $H_0^1(D)$ be the subspace of $H^1(D)$ with vanishing boundary condition, and $L^2(D)$ be the space of square integrable functions with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. For brevity, let H denote a general Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. To characterize the stochastic properties, given a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$, we use $L^2(\Omega,\mathcal{F}_t;H)$ to denote the space of $\mathcal{F}_t$-measurable and H-valued stochastic process φ satisfying $\mathbb{E}[\|\varphi\|_H^2] < \infty$, where $\mathbb{E}[\cdot]$ is the expectation operator. Let $L^p_{\mathcal{F}}([0,T];H)$ denote the space of all the $\mathbb{F}$-adapted H-valued processes $\varphi(t)$ satisfying $\mathbb{E}[\int_0^T \|\varphi(t)\|_H^p\,\mathrm{d}t] < \infty$. And we use $C_{\mathcal{F}}([0,T];L^2(\Omega,H))$ to represent the space of all the $\mathbb{F}$-adapted H-valued and mean square continuous processes φ satisfying $\sup_{0\leq t\leq T}\mathbb{E}[\|\varphi(t)\|_H^2] < \infty$.

Let $\mathcal{F}_t^{s,\xi}(s \le t \le T)$ be a σ-algebra generated by the diffusion process $\{\mu(\tau), s \le \tau \le t, \mu(s) = \xi\}$, and $\mathbb{E}_t^{\xi}[\phi]$ be the conditional mathematical expectation of random variable ϕ under $\mathcal{F}_t^{t,\xi}$.

Now we make assumptions for BSPDEs (1.1) that will be used in the sequel as follows. For the operator $\mathcal{L}$ in (1.2), with the integration by parts, for $u, v \in H_0^1(D)$, we have a bilinear form as

$$a(u,v) = \sum_{j,k=1}^{d} \left(\left(b_{jk}(s,x) \frac{\partial u}{\partial x_k} \right), \frac{\partial v}{\partial x_j} \right) + (b_0(s,x)u, v). \tag{2.1}$$

Assumption 2.1 (see [6]). *Assume $a(\cdot,\cdot)$ is bounded and coercive, that is, there exist constants $\alpha, \beta > 0$ such that*

$$|a(u,v)| \le \alpha \|u\|_1 \|v\|_1, \quad a(u,u) \ge \beta \|u\|_1^2.$$

For the function f in (1.1), we assume that f satisfies the following conditions.

Assumption 2.2. *We assume that function f is uniformly Lipschitz continuous and linear growth bounded, that is, there exist positive constants L' and C_f' such that for $y, y' \in \mathbb{R}$, $r, r' \in \mathbb{R}^{1\times d}$, and $z, z' \in \mathbb{R}^{1\times q}$,*

$$|f(t,x,\xi,y,r,z) - f(t,x,\xi,y',r',z')| \le L'(|y-y'| + |r-r'| + |z-z'|), \tag{2.2}$$
$$|f(t,x,\xi,y,r,z)| \le C_f'(1 + |y| + |r| + |z|). \tag{2.3}$$

Two lemmas that will be used for BSPDEs (1.1) are introduced as follows.

Lemma 2.1 (Feynman–Kac formula). *If $u(t,x,\xi) \in C_b^{1,2,2}$ satisfies the partial differential equation*

$$\frac{\partial u}{\partial t} - \mathcal{L}u + \frac{1}{2}\sum_{i=1}^{q} \frac{\partial^2 u}{\partial \xi_i^2} + f(t,x,\xi,u,\nabla u, \nabla_\xi u) = 0, \quad (t,x,\xi) \in [0,T) \times D \times \mathbb{R}^q, \tag{2.4}$$

with $u(t,x,\xi) = 0$ for $(t,x,\xi) \in [0,T) \times \partial D \times \mathbb{R}^q$ and $u(T,x,\xi) = \varphi(x,\xi)$, then solution (y,z) of (1.1) can be represented as

$$y(t,x) = u(t,x,\mu(t)), \quad z(t,x) = \nabla_\xi u(t,x,\mu(t)), \quad (t,x) \in [0,T) \times D, \tag{2.5}$$

where $\nabla_\xi u$ denotes the gradient of u with respect to the variable $\xi = (\xi_1, \xi_2, \ldots, \xi_q)$.

One can refer to the literature [23, 25] for the proof of above Feynman–Kac formula.

Lemma 2.2 (Backward discrete Gronwall's inequality). *Let $A_m, B_m, C_m \ge 0$ and $D_m > 0$ for $m = N, N-1, \ldots, 1$, and let the sequence A_m be nondecreasing. If $A_m + C_m \le D_m + \sum_{n=m}^{N-1} \Delta t B_{n+1} A_{n+1}$, then it holds*

$$A_m + C_m \le D_m \exp\left(\Delta t \prod_{n=m}^{N-1} B_{n+1} \right). \tag{2.6}$$

3. Generalized finite element θ-scheme

In this section, we introduce the weak solution of BSPDEs (1.1), and then propose its fully discrete generalized finite element θ-scheme. For simplicity, in the following context, we denote $y(t,\cdot)$, $z(t,\cdot)$, $\mu(t)$ and $f(t,x,\mu_t,y_t,\nabla y_t, z_t)$ by y_t, z_t, μ_t and f_t, respectively.

For the finite element analysis, we turn model (1.1) into its weak form by multiplying both sides of the first equation in (1.1) by a test function $v \in H_0^1(D)$ and integrating the obtained system over the spatial domain D. With the integration by parts and Fubini's theorem, we then have the weak form of model (1.1) as

$$(y_t, v) + \int_t^T a(y_s, v)\,\mathrm{d}s = (y_T, v) + \left(\int_t^T f_s\,\mathrm{d}s, v \right) - \left(\int_t^T z_s\,\mathrm{d}W_s, v \right). \tag{3.1}$$

Based on (3.1), we will introduce our numerical scheme for solving BSPDEs (1.1). To this end, we first introduce the time-space partition. For the time interval $[0,T]$, we consider a time partition $\pi_t : 0 = t_0 < t_1 < \cdots < t_N = T$ with $N \in \mathbb{N}^+$, which satisfies a regularity constraint $\frac{\Delta t}{\min_{0\leq n\leq N-1}\Delta t_n} \leq C_0$, with $\Delta t_n := t_{n+1} - t_n$, $\Delta t := \max_{0\leq n\leq N-1}\Delta t_n$, $C_0 > 0$ a positive constant. Further, let D be a polygonal domain and $\{\mathcal{T}_h\}_{h\geq 0}$ be a sequence of quasi-uniform regular triangulations of D with h the maximum mesh size. Given a regular partition $\mathcal{T}_h$, we define a finite element space S_h as a finite dimensional subspace of $H_0^1(D)$,

$$S_h := \big\{y_h \in C^0(D) : y_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h, y_h = 0 \text{ on } \partial D\big\} \subset H_0^1(D), \tag{3.2}$$

where $\mathcal{P}_k(K)$ denotes the set of polynomials of degree less than or equal to $k \in \mathbb{N}^+$ on element $K \in \mathcal{T}_h$.

Now we are ready to achieve the generalized finite element θ-scheme. From the weak solution (3.1), given a time interval $[t_n, t_{n+1}]$, for $0 \leq n \leq N-1$, $\forall v \in H_0^1(D)$, the $\mathcal{F}_{t_n}$-measurable random field (y_{t_n}, z_{t_n}) satisfies

$$(y_{t_n}, v) + \int_{t_n}^{t_{n+1}} a(y_s, v)\,\mathrm{d}s = \big(y_{t_{n+1}}, v\big) + \Big(\int_{t_n}^{t_{n+1}} f_s\,\mathrm{d}s, v\Big) - \Big(\int_{t_n}^{t_{n+1}} z_s\,\mathrm{d}W_s, v\Big). \tag{3.3}$$

Taking the conditional mathematical expectation $\mathbb{E}_{t_n}^{\xi}[\cdot]$ on both sides of (3.3), we have

$$(y_{t_n}, v) + \int_{t_n}^{t_{n+1}} a\Big(\mathbb{E}_{t_n}^{\xi}[y_s], v\Big)\,\mathrm{d}s = \Big(\mathbb{E}_{t_n}^{\xi}\big[y_{t_{n+1}}\big], v\Big) + \Big(\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}[f_s]\,\mathrm{d}s, v\Big). \tag{3.4}$$

Considering the θ-scheme of numerical integration, the integrals in equation (3.4) can be represented as, with $\theta_1, \theta_2 \in [0,1]$,

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}[y_s]\,\mathrm{d}s = \theta_1 \Delta t_n y_{t_n} + (1-\theta_1)\Delta t_n \mathbb{E}_{t_n}^{\xi}\big[y_{t_{n+1}}\big] - R_{y1}^n, \tag{3.5}$$

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}[f_s]\,\mathrm{d}s = \theta_2 \Delta t_n f_{t_n} + (1-\theta_2)\Delta t_n \mathbb{E}_{t_n}^{\xi}\big[f_{t_{n+1}}\big] + R_{y2}^n, \tag{3.6}$$

where R_{y1}^n, R_{y2}^n are the associated truncation error terms. Setting

$$R_y^n = R_{y1}^n + R_{y2}^n, \tag{3.7}$$

then the equation (3.4) can be rewritten in the following form:

$$\begin{aligned}(y_{t_n}, v) &+ \theta_1 \Delta t_n a(y_{t_n}, v) + (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[y_{t_{n+1}}\big], v\Big) \\ &= \Big(\mathbb{E}_{t_n}^{\xi}\big[y_{t_{n+1}}\big], v\Big) + \theta_2 \Delta t_n (f_{t_n}, v) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[f_{t_{n+1}}\big], v\Big) + \big(R_y^n, v\big).\end{aligned} \tag{3.8}$$

For standard Brownian motion W_t, in the following sequel, we denote the increments $W_{t_{n+1}} - W_{t_n}$, $W_t - W_s$ by $\Delta W_{t_{n+1}}$ and $\Delta W_{s,t}$, respectively. We multiply (3.3) by $\Delta W_{t_{n+1}}^{\top}$ and replace v by $w \in (H_0^1(D))^{1\times q}$ to obtain

$$\begin{aligned}\Big(y_{t_n}\Delta W_{t_{n+1}}^{\top}, w\Big) &+ \int_{t_n}^{t_{n+1}} a\Big(y_s \Delta W_{t_{n+1}}^{\top}, w\Big)\,\mathrm{d}s \\ &= \Big(y_{t_{n+1}}\Delta W_{t_{n+1}}^{\top}, w\Big) + \Big(\int_{t_n}^{t_{n+1}} f_s\,\mathrm{d}s \Delta W_{t_{n+1}}^{\top}, w\Big) - \Big(\int_{t_n}^{t_{n+1}} z_s\,\mathrm{d}W_s \Delta W_{t_{n+1}}^{\top}, w\Big).\end{aligned} \tag{3.9}$$

Taking the conditional mathematical expectation $\mathbb{E}_{t_n}^{\xi}[\cdot]$ on both sides of (3.9), and considering the Itô isometry, we obtain

$$\begin{aligned}\int_{t_n}^{t_{n+1}} a\Big(\mathbb{E}_{t_n}^{\xi}\big[y_s \Delta W_{t_n,s}^{\top}\big], w\Big)\,\mathrm{d}s &= \Big(\mathbb{E}_{t_n}^{\xi}\Big[y_{t_{n+1}}\Delta W_{t_{n+1}}^{\top}\Big], w\Big) - \Big(\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}[z_s]\,\mathrm{d}s, w\Big) \\ &\quad + \Big(\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}\big[f_s \Delta W_{t_n,s}^{\top}\big]\,\mathrm{d}s, w\Big).\end{aligned} \tag{3.10}$$

Similarly, for the terms of integrals in (3.10), we have

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}\left[y_s \Delta W_{t_n,s}^{\top}\right] \mathrm{d}s = (1-\theta_3)\Delta t_n \mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right] - R_{z1}^n, \tag{3.11}$$

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}\left[f_s \Delta W_{t_n,s}^{\top}\right] \mathrm{d}s = (1-\theta_4)\Delta t_n \mathbb{E}_{t_n}^{\xi}\left[f_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right] + R_{z2}^n, \tag{3.12}$$

$$\int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\xi}\left[z_s\right] \mathrm{d}s = \theta_5 \Delta t_n z_{t_n} + (1-\theta_5)\Delta t_n \mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right] - R_{z3}^n, \tag{3.13}$$

where R_{z1}^n, R_{z2}^n and R_{z3}^n are the associated truncation error terms and θ_3, θ_4, θ_5 are parameters in $[0,1]$. We set

$$R_z^n = R_{z1}^n + R_{z2}^n + R_{z3}^n. \tag{3.14}$$

Then we substitute (3.14) into (3.10) to obtain

$$\begin{aligned}\theta_5 \Delta t_n (z_{t_n}, w) &+ (1-\theta_3)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) \\ &= \Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) - (1-\theta_5)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big) \\ &\quad + (1-\theta_4)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[f_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) + (R_z^n, w).\end{aligned} \tag{3.15}$$

By Lemma 2.1 and the integration by parts, we have the identity

$$\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right] = \Delta t_n \mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right]. \tag{3.16}$$

Applying the relation (3.16), we have the following identities

$$a\Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) = \Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big), \tag{3.17}$$

$$\Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) = \theta_6' \Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) + (1-\theta_6')\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big), \tag{3.18}$$

and

$$\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big) = \theta_7' \Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big) + (1-\theta_7')\Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big), \tag{3.19}$$

with the auxiliary parameters θ_6', $\theta_7' \in [0,1]$. Substituting the above three equations into (3.15) leads to

$$\begin{aligned}\theta_5 \Delta t_n (z_{t_n}, w) &= \theta_6 \Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big) + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\left[y_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) + (R_z^n, w) \\ &\quad + (1-\theta_4)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\left[f_{t_{n+1}} \Delta W_{t_{n+1}}^{\top}\right], w\Big) - (1-\theta_3)(\Delta t_n)^2 a\Big(\mathbb{E}_{t_n}^{\xi}\left[z_{t_{n+1}}\right], w\Big).\end{aligned} \tag{3.20}$$

Here, we set $\theta_6 = (1-\theta_6') - \theta_7'(1-\theta_5)$, $\theta_5 - \theta_6 = \theta_6' - (1-\theta_7')(1-\theta_5)$.

Now we are in a position to propose our numerical scheme. Let (y_h^n, z_h^n) be the discrete approximation of analytical solution (y_t, z_t) of equation (1.1) at time instant $t = t_n$, $n = N, N-1, \cdots, 0$. We denote $f(t_n, x, \mu_{t_n}, y_h^n, \nabla y_h^n, z_h^n)$ by f_h^n. Based on (3.8), (3.20), omitting the truncation error terms, we propose Scheme 3.1 for solving the BSPDEs (1.1).

Scheme 3.1 (Generalized finite element θ-scheme)**.** *Given terminal conditions (y_h^N, z_h^N), solve for a pair of $\mathbb{F}$-adapted $S_h \times S_h^{1\times q}$-valued random processes (y_h^n, z_h^n), $n = N-1, \ldots, 0$ with boundary conditions $y_h^n = 0$,*

$z_h^n = 0$ *on* ∂D *such that for all* $v_h \in S_h$, $w_h \in (S_h)^{1\times q}$,

$$(y_h^n, v_h) + \theta_1 \Delta t_n a(y_h^n, v_h) = \Big(\mathbb{E}_{t_n}^{\xi}\big[y_h^{n+1}\big], v_h\Big) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[f_h^{n+1}\big], v_h\Big) + \theta_2 \Delta t_n (f_h^n, v_h) - (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[y_h^{n+1}\big], v_h\Big), \tag{3.21}$$

$$\theta_5 \Delta t_n (z_h^n, w_h) = \theta_6 \Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[z_h^{n+1}\big], w_h\Big) + (1-\theta_4)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[f_h^{n+1}\Delta W_{t_{n+1}}^{\top}\Big], w_h\Big) + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\Big[y_h^{n+1}\Delta W_{t_{n+1}}^{\top}\Big], w_h\Big) - (1-\theta_3)(\Delta t_n)^2 a\Big(\mathbb{E}_{t_n}^{\xi}\big[z_h^{n+1}\big], w_h\Big), \tag{3.22}$$

where θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 *are parameters satisfying* $\{\theta_i\}_{1\le i\le 4} \in [0,1]$, $\theta_5 \in (0,1]$, $\theta_6 \in (0,1)$ *and* $\theta_6 < \theta_5$.

Remark 3.1. The finite element θ-scheme 3.1 is general and different parameters θ_i $(i = 1, \dots, 6)$ can be chosen in the applications. In most cases, Scheme 3.1 will form a sequence of nonlinear discrete systems which can be efficiently solved in parallel with the MPI-RMA parallel techniques.

4. Numerical analyses

For notational simplicity, we consider the Brownian motion W_t with $q = 1$ in the following. The model (1.1) with multi-dimensional Brownian motions, *i.e.*, $q > 1$, can be similarly studied through multi-dimensional Itô's formula and Itô isometry. The numerical analyses of the boundedness and error estimates of Scheme 3.1 are performed in this section.

4.1. Boundedness

We first introduce the inverse inequality for the functions in the finite element space S_h, which will be used in our analyses.

Lemma 4.1 (Inverse inequality [6])**.** *Given a regular triangulation* $\mathcal{T}_h$, *then there exists a positive constant* C_I *such that for any* $v_h \in S_h$

$$\|v_h\|_1^2 \le \frac{C_I}{h^2}\|v_h\|^2. \tag{4.1}$$

We state the boundedness of (3.1) in the following theorem.

Theorem 4.1 (Boundedness)**.** *Under Assumptions 2.1 and 2.2, let* (y_h^n, z_h^n), $0 \le n \le N-1$, *be the solution of Scheme 3.1. Then for sufficiently small time step* $\Delta t > 0$ *and* $\frac{\Delta t}{h^2} \le \frac{\eta_0}{\alpha^2 C_I}$ *with* $0 < \eta_0 \le \theta_6 \alpha$, *it holds that*

$$\mathbb{E}\Big[\|y_h^n\|^2\Big] + \sum_{i=n}^{N-1}\Delta t \mathbb{E}\Big[\|y_h^i\|_1^2\Big] + \sum_{i=n}^{N-1}\Delta t \mathbb{E}\Big[\|z_h^i\|^2\Big] + \sum_{i=n}^{N-1}(\Delta t)^2 \mathbb{E}\Big[\|z_h^i\|_1^2\Big] \le C\Big(\mathbb{E}\Big[\|y_h^N\|^2\Big] + \Delta t \mathbb{E}\Big[\|y_h^N\|_1^2\Big] + \Delta t \mathbb{E}\Big[\|z_h^N\|^2\Big] + T\Big), \tag{4.2}$$

where C *is a positive constant independent of* Δt *and* h.

Proof. We divide the proof into three steps. In Step 1, the estimates are carried out on (3.21) in Scheme 3.1. Then it was followed by Step 2 focusing on (3.22). Finally we combine the obtained results in Step 3.

Step 1. We set $v_h = y_h^n$ in (3.21) and get

$$\|y_h^n\|^2 + \theta_1 \Delta t_n a(y_h^n, y_h^n) = \Big(\mathbb{E}_{t_n}^{\xi}\big[y_h^{n+1}\big], y_h^n\Big) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[f_h^{n+1}\big], y_h^n\Big) + \theta_2 \Delta t_n (f_h^n, y_h^n) - (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[y_h^{n+1}\big], y_h^n\Big). \tag{4.3}$$

With the identity $2(a,b) = (a,a) + (b,b) - (a-b, a-b)$ for $(\mathbb{E}^{\xi}_{t_n}[y_h^{n+1}], y_h^n)$, according to Assumption 2.1, inverse inequality (4.1) and Cauchy–Schwarz inequality, we derive that

$$\begin{aligned}\frac{1}{2}\|y_h^n\|^2 + \Delta t_n \beta \|y_h^n\|_1^2 + \frac{1}{2}\Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big] - y_h^n\Big\|^2 \\ \le \frac{1}{2}\Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big]\Big\|^2 + (1-\theta_2)\Delta t_n \Big\|\mathbb{E}^{\xi}_{t_n}\big[f_h^{n+1}\big]\Big\| \cdot \|y_h^n\| + \theta_2 \Delta t_n \|f_h^n\| \cdot \|y_h^n\| \\ + (1-\theta_1)\Delta t_n \alpha \Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big] - y_h^n\Big\|_1 \cdot \|y_h^n\|_1.\end{aligned} \tag{4.4}$$

With Assumption 2.2, there is a constant $C_f > 0$ such that

$$\|f_h^n\|^2 \le C_f\Big(1 + \|y_h^n\|^2 + \|y_h^n\|_1^2 + \|z_h^n\|^2\Big). \tag{4.5}$$

Then by (4.5) and elementary inequality $ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$, $\epsilon > 0$, we obtain

$$\Big\|\mathbb{E}^{\xi}_{t_n}\big[f_h^{n+1}\big]\Big\| \cdot \|y_h^n\| \le \frac{2C_0C_f}{\eta}\|y_h^n\|^2 + \frac{\eta}{8C_0}\Big(1 + \mathbb{E}^{\xi}_{t_n}\Big[\big\|y_h^{n+1}\big\|^2 + \big\|y_h^{n+1}\big\|_1^2 + \big\|z_h^{n+1}\big\|^2\Big]\Big), \tag{4.6}$$

$$\|f_h^n\|\|y_h^n\| \le \frac{2C_0C_f}{\eta}\|y_h^n\|^2 + \frac{\eta}{8C_0}\Big(1 + \|y_h^n\|^2 + \|y_h^n\|_1^2 + \|z_h^n\|^2\Big), \tag{4.7}$$

where η is a positive constant, such that $0 < \eta_0 \le \eta \le \min\{1, \frac{\beta}{2}\}$. By inverse inequality (4.1), the last term in (4.4) is estimated as

$$\alpha\Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big] - y_h^n\Big\|_1 \cdot \|y_h^n\|_1 \le \frac{C_I\alpha^2}{2\eta h^2}\Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big] - y_h^n\Big\|^2 + \frac{\eta}{2}\|y_h^n\|_1^2. \tag{4.8}$$

Combining (4.4), (4.6) and (4.7) related to (3.21), we deduce

$$\begin{aligned}\Big[1 - \Big(\frac{4C_0C_f}{\eta} + \frac{\eta\theta_2}{4C_0}\Big)\Delta t_n\Big]\|y_h^n\|^2 + \Big[2\beta - \frac{\eta\theta_2}{4C_0} - \eta(1-\theta_1)\Big]\Delta t_n \|y_h^n\|_1^2 \\ + \Big(1 - \frac{\alpha^2 C_I(1-\theta_1)\Delta t_n}{\eta h^2}\Big)\Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big] - y_h^n\Big\|^2 - \frac{\eta\theta_2}{4C_0}\Delta t_n \|z_h^n\|^2 \\ \le \Big\|\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\big]\Big\|^2 + \frac{\eta(1-\theta_2)}{4C_0}\Delta t_n \mathbb{E}^{\xi}_{t_n}\Big[\big\|y_h^{n+1}\big\|^2 + \big\|y_h^{n+1}\big\|_1^2 + \big\|z_h^{n+1}\big\|^2\Big] + \frac{\eta\Delta t_n}{4C_0}.\end{aligned} \tag{4.9}$$

Step 2. Setting $w_h = z_h^n$ in (3.22) gives

$$\begin{aligned}\theta_5 \Delta t_n (z_h^n, z_h^n) = \theta_6 \Delta t_n \Big(\mathbb{E}^{\xi}_{t_n}\big[z_h^{n+1}\big], z_h^n\Big) + (1-\theta_4)\Delta t_n \Big(\mathbb{E}^{\xi}_{t_n}\big[f_h^{n+1}\Delta W_{t_{n+1}}\big], z_h^n\Big) \\ + (\theta_5 - \theta_6)\Big(\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\Delta W_{t_{n+1}}\big], z_h^n\Big) - (1-\theta_3)(\Delta t_n)^2 a\Big(\mathbb{E}^{\xi}_{t_n}\big[z_h^{n+1}\big], z_h^n\Big).\end{aligned} \tag{4.10}$$

As is the procedure for Step 1, we have the following inequality,

$$\begin{aligned}\theta_5 \Delta t_n \|z_h^n\|^2 + \frac{(1-\theta_3)\beta}{2}(\Delta t_n)^2\Big(\|z_h^n\|_1^2 + \Big\|\mathbb{E}^{\xi}_{t_n}\big[z_h^{n+1}\big]\Big\|_1^2\Big) \\ + \Big(\frac{\theta_6}{2} - \frac{\alpha C_I(1-\theta_3)\Delta t_n}{2h^2}\Big)\Delta t_n \Big\|\mathbb{E}^{\xi}_{t_n}\big[z_h^{n+1}\big] - z_h^n\Big\|^2 \\ \le \frac{\theta_6}{2}\Delta t_n\Big(\Big\|\mathbb{E}^{\xi}_{t_n}\big[z_h^{n+1}\big]\Big\|^2 + \|z_h^n\|^2\Big) + (1-\theta_4)\Delta t_n\Big(\mathbb{E}^{\xi}_{t_n}\big[f_h^{n+1}\Delta W_{t_{n+1}}\big], z_h^n\Big) \\ + (\theta_5 - \theta_6)\Big(\mathbb{E}^{\xi}_{t_n}\big[y_h^{n+1}\Delta W_{t_{n+1}}\big], z_h^n\Big).\end{aligned} \tag{4.11}$$

By $\|\mathbb{E}_{t_n}^{\xi}[y_h^{n+1}\Delta W_{t_{n+1}}]\|^2 \leq \Delta t_n(\mathbb{E}_{t_n}^{\xi}[\|y_h^{n+1}\|^2] - \|\mathbb{E}_{t_n}^{\xi}[y_h^{n+1}]\|^2)$ and (4.5), the last two terms related to $\Delta W_{t_{n+1}}$ in (4.11) are further bounded as follows, respectively,

$$\begin{aligned}
&\left(\mathbb{E}_{t_n}^{\xi}\left[f_h^{n+1}\Delta W_{t_{n+1}}\right], z_h^n\right) \\
&\quad \leq \frac{1}{2}\left(\frac{(\theta_5-\theta_6)\|z_h^n\|^2}{4} + \frac{4\left\|\mathbb{E}_{t_n}^{\xi}\left[f_h^{n+1}\Delta W_{t_{n+1}}\right]\right\|^2}{\theta_5-\theta_6}\right) \\
&\quad \leq \frac{(\theta_5-\theta_6)\|z_h^n\|^2}{8} + \frac{2C_f\Delta t_n\left(1+\mathbb{E}_{t_n}^{\xi}\left[\left\|y_h^{n+1}\right\|_1^2\right]+\mathbb{E}_{t_n}^{\xi}\left[\left\|y_h^{n+1}\right\|^2\right]+\mathbb{E}_{t_n}^{\xi}\left[\left\|z_h^{n+1}\right\|^2\right]\right)}{\theta_5-\theta_6}
\end{aligned} \tag{4.12}$$

and

$$\left(\mathbb{E}_{t_n}^{\xi}\left[y_h^{n+1}\Delta W_{t_{n+1}}\right], z_h^n\right) \leq \frac{\Delta t_n}{2}\|z_h^n\|^2 + \frac{1}{2}\left(\mathbb{E}_{t_n}^{\xi}\left[\left\|y_h^{n+1}\right\|^2\right] - \left\|\mathbb{E}_{t_n}^{\xi}\left[y_h^{n+1}\right]\right\|^2\right). \tag{4.13}$$

Considering (4.11)–(4.13), we obtained

$$\begin{aligned}
&\left(\theta_5 - \frac{(\theta_5-\theta_6)(1-\theta_4)}{4}\right)\Delta t_n\|z_h^n\|^2 + (1-\theta_3)\beta(\Delta t_n)^2\left(\|z_h^n\|_1^2 + \left\|\mathbb{E}_{t_n}^{\xi}\left[z_h^{n+1}\right]\right\|_1^2\right) \\
&\quad + \left(\theta_6 - \frac{\alpha C_I(1-\theta_3)\Delta t_n}{h^2}\right)\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[z_h^{n+1}\right] - z_h^n\right\|^2 \\
&\quad \leq \left[\theta_6\Delta t_n + \frac{4(1-\theta_4)C_f(\Delta t_n)^2}{\theta_5-\theta_6}\right]\left\|\mathbb{E}_{t_n}^{\xi}\left[z_h^{n+1}\right]\right\|^2 + (\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[\left\|y_h^{n+1}\right\|^2\right] - \left\|\mathbb{E}_{t_n}^{\xi}\left[y_h^{n+1}\right]\right\|^2\right) \\
&\quad\quad + \left(1+\frac{C_I}{h^2}\right)\frac{4(1-\theta_4)C_f(\Delta t_n)^2}{\theta_5-\theta_6}\left\|\mathbb{E}_{t_n}^{\xi}\left[y_h^{n+1}\right]\right\|^2 + \frac{4(1-\theta_4)C_f}{\theta_5-\theta_6}(\Delta t_n)^2.
\end{aligned} \tag{4.14}$$

Step 3. In the following context, we combine the estimates (4.9) for y and (4.14) for z, together. Multiplying (4.9) by C_0, dividing both sides of (4.14) by $\frac{(\theta_5-\theta_6)\Delta t_n}{\Delta t}$ with $\theta_5 - \theta_6 > 0$, and adding the two obtained inequalities give

$$\begin{aligned}
&C_0\left[1 - \left(\frac{4C_0C_f}{\eta} + \frac{\eta\theta_2}{4C_0}\right)\Delta t\right]\|y_h^n\|^2 + \left[2\beta - \frac{\eta\theta_2}{4C_0} - \eta(1-\theta_1)\right]\Delta t\|y_h^n\|_1^2 \\
&\quad + \left(\frac{\theta_5}{\theta_5-\theta_6} - \frac{1-\theta_4}{4} - \frac{\eta\theta_2}{4}\right)\Delta t\|z_h^n\|^2 + \frac{(1-\theta_3)\beta(\Delta t)^2}{C_0(\theta_5-\theta_6)}\left(\|z_h^n\|_1^2 + \left\|\mathbb{E}_{t_n}^{\xi}\left[z_h^{n+1}\right]\right\|_1^2\right) \\
&\quad + \left[\frac{\theta_6}{\theta_5-\theta_6} - \frac{\alpha C_I(1-\theta_3)}{h^2(\theta_5-\theta_6)}\Delta t\right]\Delta t\left\|\mathbb{E}_{t_n}^{\xi}\left[z_h^{n+1}\right] - z_h^n\right\|^2 \\
&\quad + C_0\left[1 - \frac{\alpha^2C_I(1-\theta_1)}{\eta h^2}\Delta t\right]\left\|\mathbb{E}_{t_n}^{\xi}\left[y_h^{n+1}\right] - y_h^n\right\|^2 \\
&\quad \leq C_0\left[1 + \frac{\eta(1-\theta_2)}{4C_0}\Delta t + \left(1+\frac{C_I}{h^2}\right)\frac{4(1-\theta_4)C_f}{C_0(\theta_5-\theta_6)^2}(\Delta t)^2\right]\mathbb{E}_{t_n}^{\xi}\left[\left\|y_h^{n+1}\right\|^2\right] \\
&\quad\quad + \left[\frac{\eta(1-\theta_2)}{4}\Delta t + \frac{\theta_6}{\theta_5-\theta_6}\Delta t + \frac{4(1-\theta_4)}{(\theta_5-\theta_6)^2}C_f(\Delta t)^2\right]\mathbb{E}_{t_n}^{\xi}\left[\left\|z_h^{n+1}\right\|^2\right] \\
&\quad\quad + \frac{\eta(1-\theta_2)}{4}\Delta t\mathbb{E}_{t_n}^{\xi}\left[\left\|y_h^{n+1}\right\|_1^2\right] + \frac{4(1-\theta_4)}{(\theta_5-\theta_6)^2}C_f(\Delta t)^2 + \frac{\eta\Delta t}{4}\cdot
\end{aligned} \tag{4.15}$$

With $1-\theta_1 \geq 0$, $1-\theta_3 \geq 0$, $\theta_5-\theta_6 > 0$, for sufficiently small Δt and $\frac{\Delta t}{h^2} \leq \frac{\eta_0}{\alpha^2 C_I}$ with $0 < \eta_0 \leq \theta_6\alpha$, it holds that $\frac{\theta_6}{\theta_5-\theta_6} - \frac{\alpha C_I(1-\theta_3)}{h^2(\theta_5-\theta_6)}\Delta t \geq 0$ and $1-\frac{\alpha^2 C_I(1-\theta_1)}{\eta h^2}\Delta t \geq 0$. Then we take mathematical expectation $\mathbb{E}[\cdot]$ on both sides of (4.15) and get

$$\begin{aligned}
&C_0\left[1-\left(\frac{4C_0C_f}{\eta}+\frac{\eta\theta_2}{4C_0}\right)\Delta t\right]\mathbb{E}\left[\|y_h^n\|^2\right]+\left[2\beta-\frac{\eta\theta_2}{4C_0}-\eta(1-\theta_1)\right]\Delta t\mathbb{E}\left[\|y_h^n\|_1^2\right]\\
&\quad+\left(\frac{\theta_5}{\theta_5-\theta_6}-\frac{1-\theta_4}{4}-\frac{\eta\theta_2}{4}\right)\Delta t\mathbb{E}\left[\|z_h^n\|^2\right]+\frac{1-\theta_3}{C_0(\theta_5-\theta_6)}\beta(\Delta t)^2\mathbb{E}\left[\|z_h^n\|_1^2\right]\\
&\quad\leq C_0\left[1+\frac{\eta(1-\theta_2)}{4C_0}\Delta t+\left(1+\frac{C_I}{h^2}\right)\frac{4(1-\theta_4)C_f}{C_0(\theta_5-\theta_6)^2}(\Delta t)^2\right]\mathbb{E}\left[\|y_h^{n+1}\|^2\right]\\
&\qquad+\left[\frac{\eta(1-\theta_2)}{4}\Delta t+\frac{\theta_6}{\theta_5-\theta_6}\Delta t+\frac{4(1-\theta_4)}{(\theta_5-\theta_6)^2}C_f(\Delta t)^2\right]\mathbb{E}\left[\|z_h^{n+1}\|^2\right]\\
&\qquad+\frac{\eta(1-\theta_2)}{4}\Delta t\mathbb{E}\left[\|y_h^{n+1}\|_1^2\right]+\frac{4(1-\theta_4)}{(\theta_5-\theta_6)^2}C_f(\Delta t)^2+\frac{\eta\Delta t}{4}\cdot
\end{aligned}\tag{4.16}$$

For $0 < \Delta t_n \leq \Delta t \leq \Delta\bar{t}$, $n \in \mathbb{N}$, $C_0 \geq 1$, we set $\Delta\bar{t}$ sufficient small such that the following inequalities holds $C_a = 1-(\frac{4C_0C_f}{\eta}+\frac{\eta\theta_2}{4C_0})\Delta t \geq 1-(\frac{4C_0C_f}{\eta_0}+\frac{1}{4C_0})\Delta\bar{t} > 0$, $C_d = 2\beta-\frac{\theta_2\eta}{4C_0}-\frac{\eta(1-\theta_2)}{4}-\eta(1-\theta_1) > 0$, $C_z = 1-\frac{1-\theta_4}{4}-\frac{\eta}{4}-\frac{4(1-\theta_4)}{(\theta_5-\theta_6)^2}C_f\Delta t \geq \frac{1}{2}-\frac{4C_f\Delta\bar{t}}{(\theta_5-\theta_6)^2} > 0$, $C_y = \frac{4C_0C_f}{\eta}+\frac{\eta}{4C_0}+(1+\frac{C_I}{h^2})\frac{4(1-\theta_4)C_f}{(\theta_5-\theta_6)^2}\Delta t \leq \frac{4C_0C_f}{\eta_0}+\frac{1}{4C_0}+\frac{4C_I\eta_0C_f}{\alpha^2(\theta_5-\theta_6)^2C_I}+\frac{4C_f\Delta\bar{t}}{(\theta_5-\theta_6)^2}$, $C_b = \frac{\eta(1-\theta_2)}{4}+\frac{\theta_6}{\theta_5-\theta_6}+\frac{4(1-\theta_4)}{(\theta_5-\theta_6)^2}C_f\Delta t \leq \frac{1}{4}+\frac{\theta_6}{\theta_5-\theta_6}+\frac{4C_f\Delta\bar{t}}{(\theta_5-\theta_6)^2}$. Summing (4.16) over $i = n, n+1, \ldots, N-1$ with the above given constraints, it follows

$$\begin{aligned}
&\mathbb{E}\left[\|y_h^n\|^2\right]+\frac{C_d}{C_0C_a}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\left[\|y_h^i\|_1^2\right]+\frac{C_z}{C_0C_a}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\left[\|z_h^i\|^2\right]+\frac{1}{C_aC_0^2}\frac{1-\theta_3}{\theta_5-\theta_6}\sum_{i=n}^{N-1}(\Delta t)^2\beta\mathbb{E}\left[\|z_h^i\|_1^2\right]\\
&\quad\leq\mathbb{E}\left[\|y_h^N\|^2\right]+\frac{C_y}{C_a}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\left[\|y_h^{i+1}\|^2\right]+\frac{\Delta t}{4C_0C_a}\mathbb{E}\left[\|y_h^N\|_1^2\right]+\frac{C_b\Delta t}{C_0C_a}\mathbb{E}\left[\|z_h^N\|^2\right]\\
&\qquad+\frac{4}{(\theta_5-\theta_6)^2}\frac{C_f}{C_0C_a}\sum_{i=n}^{N-1}(\Delta t)^2+\frac{1}{C_0C_a}\sum_{i=n}^{N-1}\frac{\Delta t}{4}\cdot
\end{aligned}\tag{4.17}$$

By applying the backward Gronwall's inequality in Lemma 2.2, we have that

$$\begin{aligned}
&\mathbb{E}\left[\|y_h^n\|^2\right]+\frac{C_d}{C_0C_a}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\left[\|y_h^i\|_1^2\right]+\frac{C_z}{C_0C_a}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\left[\|z_h^i\|^2\right]+\frac{1}{C_aC_0^2}\frac{1-\theta_3}{\theta_5-\theta_6}\sum_{i=n}^{N-1}(\Delta t)^2\beta\mathbb{E}\left[\|z_h^i\|_1^2\right]\\
&\quad\leq\exp\left(\frac{C_0C_yT}{C_a}\right)\left[\mathbb{E}\left[\|y_h^N\|^2\right]+\frac{\Delta t}{4C_0C_a}\mathbb{E}\left[\|y_h^N\|_1^2\right]+\frac{C_b\Delta t}{C_0C_a}\mathbb{E}\left[\|z_h^N\|^2\right]\right.\\
&\qquad\left.+\frac{T}{4C_a}+\frac{4C_fT\Delta t}{C_a(\theta_5-\theta_6)^2}\right],
\end{aligned}\tag{4.18}$$

from which the proof ends.

□

Remark 4.1. For the deterministic PDEs, the constraint $\frac{\Delta t}{h^2} \leq C$ is always existing in the explicit/semi-explicit finite element schemes. Similarly to the deterministic PDEs, the constraint of time-space partition also can not always be avoided in the explicit/semi-explicit scheme for the BSPDE (1.1), while, for $\theta_1 = 1$ and $\theta_3 = 1$, Theorem 4.1 holds without the condition of $\frac{\Delta t}{h^2} \leq \frac{\eta_0}{\alpha^2 C_I}$.

4.2. Error estimates

In this subsection, we devote to give the error estimates of Scheme 3.1 for solving the BSPDEs (1.1). It is worth pointing out that in our proof, we assume that the conditional expectation $\mathbb{E}_{t_n}^{\xi}[\cdot]$ is exactly obtained and do not consider its numerical discretization. In the sequel, analogous to numerical analysis for the forward SPDEs, we only consider the spatial and temporal discretization in physical space and time direction, respectively.

To begin with, we introduce the elliptic projection operator $\mathcal{R}_h : H^1(D) \to S_h$ defined by, for $v \in H^1(D)$

$$a(\mathcal{R}_h v, \varphi_h) = a(v, \varphi_h), \quad \forall \varphi_h \in S_h. \tag{4.19}$$

We provide the following lemmas that will be used in error estimates.

Lemma 4.2 (see [6])**.** *For the operator $\mathcal{R}_h$, there exists a constant C independent of h such that, for any $v \in H^{k+1}(D) \cap \mathcal{W}^{k+1,4}(D)$,*

$$\|v - \mathcal{R}_h v\| + h\|v - \mathcal{R}_h v\|_1 \le Ch^{k+1}\|v\|_{k+1}, \tag{4.20}$$

$$\|v - \mathcal{R}_h v\|_{\mathcal{W}^{1,4}} \le Ch^{k}\|v\|_{\mathcal{W}^{k+1,4}}. \tag{4.21}$$

Lemma 4.3. *We assume that the second partial derivatives of f is uniformly bounded. Then for $v \in H_0^1(D)$, there are two positive constants $\bar{C}_f$ and $\tilde{C}_f$ such that*

$$\int_D (f(t,x,\xi,y,\nabla g_1,z) - f(t,x,\xi,y,\nabla g_2,z)) \cdot v\mathrm{d}x \le \bar{C}_f\|g_1 - g_2\|\|v\|_1 + \tilde{C}_f\Big|\Big(|\nabla(g_1 - g_2)|^2, v\Big)\Big|, \tag{4.22}$$

where $t \in [0,T]$, $(\xi, y, z) \in \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}$ and g_1, $g_2 : D \to \mathbb{R}$.

Proof. It follows from the Taylor expansion and the integration by parts. □

With the Martingale property of the Brownian motion, the following lemma can be obtained by the Cauchy–Schwarz inequality and (4.5).

Lemma 4.4. *For BSPDEs (1.1), we assume that $(y,z) \in L^2_{\mathcal{F}}((0,T); H^{k+3}(D)) \times L^2_{\mathcal{F}}((0,T); H^{k+1}(D))$ and Assumption 2.2 holds, then for any s, $t \in [0,T]$ with $s < t$, there exists a constant C such that*

$$\big\|\mathbb{E}_s^{\xi}[y_t - y_s]\big\|_{k+1}^2 \le C(t-s)\int_s^t \mathbb{E}_s^{\xi}\Big[\|y_\tau\|_{k+3}^2 + \|z_\tau\|_{k+1}^2\Big]\,\mathrm{d}\tau, \tag{4.23}$$

$$\big\|\mathbb{E}_s^{\xi}[(y_t - y_s)\Delta W_{s,t}]\big\|_{k+1}^2 \le C(t-s)\int_s^t \mathbb{E}_s^{\xi}\Big[\|y_\tau\|_{k+3}^2 + \|z_\tau\|_{k+1}^2\Big]\,\mathrm{d}\tau. \tag{4.24}$$

Lemma 4.5. *Let R_y^n and R_z^n be the truncation errors defined in (3.7) and (3.14), respectively. If f and φ are smooth and bounded with their derivatives, we have the following conclusions. For sufficiently small time step $\Delta t_n > 0$, $n = 1, 2, \ldots, N$, and parameters $\theta_i \in [0,1], i = 1,2,3,4$, $\theta_5 \in (0,1]$, $\theta_6 \in (0,1)$ with $\theta_6 < \theta_5$, it holds*

$$\|R_y^n\| + \|R_z^n\| \le C(\Delta t_n)^2. \tag{4.25}$$

In particular, for $\theta_i = \frac{1}{2}, i = 1,2,3,4,5$, and $\theta_6 \in (0,1)$ with $\theta_6 < \theta_5$, we have the estimates

$$\|R_y^n\| + \|R_z^n\| \le C(\Delta t_n)^3. \tag{4.26}$$

Proof. Readers can refer to [31] for details. □

In the sequel, we present the main results of error estimates for Scheme 3.1.

4.2.1. The main results

Since the error estimates between the continuous solutions y_{t_n}, z_{t_n} and their projections $\mathcal{R}_h y_{t_n}, \mathcal{R}_h z_{t_n}$ are known in Lemma 4.2, what will be done is deriving the error estimates between the projections $\mathcal{R}_h y_{t_n}, \mathcal{R}_h z_{t_n}$ and the finite element approximates y_h^n, z_h^n. To be brief, we set

$$\begin{aligned}
e_y^n &= y_{t_n} - y_h^n = y_{t_n} - \mathcal{R}_h y_{t_n} + \mathcal{R}_h y_{t_n} - y_h^n =: \rho_y^n + \zeta_y^n,\\
e_z^n &= z_{t_n} - z_h^n = z_{t_n} - \mathcal{R}_h z_{t_n} + \mathcal{R}_h z_{t_n} - z_h^n =: \rho_z^n + \zeta_z^n,\\
e_f^n &= f_{t_n} - f_h^n = f(t_n, x, \mu_{t_n}, y_{t_n}, \nabla y_{t_n}, z_{t_n}) - f(t_n, x, \mu_{t_n}, y_h^n, \nabla y_h^n, z_h^n),\\
\tilde{e}_f^n &= f(t_n, x, \mu_{t_n}, y_{t_n}, \nabla \mathcal{R}_h y_{t_n}, z_{t_n}) - f(t_n, x, \mu_{t_n}, y_h^n, \nabla y_h^n, z_h^n).
\end{aligned}$$

Theorem 4.2. *Under the conditions of Theorem 4.1 and by Lemma 4.3, for sufficient small Δt and $0 \le n \le N-1$, there exists a positive constant C such that*

$$\begin{aligned}
&\mathbb{E}\Big[\|\zeta_y^n\|^2\Big] + \sum_{i=n}^{N-1} \Delta t \mathbb{E}\Big[\|\zeta_y^i\|_1^2\Big] + \sum_{i=n}^{N-1} \Delta t \mathbb{E}\Big[\|\zeta_z^i\|^2\Big] + \sum_{i=n}^{N-1} (\Delta t)^2 \mathbb{E}\Big[\|\zeta_z^i\|_1^2\Big]\\
&\quad \le C \exp(CT) \Bigg[\mathbb{E}\Big[\|\zeta_y^N\|^2\Big] + \Delta t \mathbb{E}\Big[\|\zeta_z^N\|^2\Big] + \Delta t \mathbb{E}\Big[\|\zeta_y^N\|_1^2\Big] + \sum_{i=n}^{N} \Delta t \mathbb{E}\Big[\|\rho_y^i\|^2\Big]\\
&\qquad + \sum_{i=n}^{N} \Delta t \mathbb{E}\Big[\|\nabla \rho_y^i\|_{L^4}^4\Big] + \sum_{i=n}^{N-1} \frac{2C_0}{\Delta t} \mathbb{E}\Big[\|R_y^i\|^2\Big] + \sum_{i=n}^{N-1} \frac{4C_0 \mathbb{E}\Big[\|R_z^i\|^2\Big]}{(\theta_5 - \theta_6)^2 \Delta t}\\
&\qquad + \sum_{i=n}^{N} \Delta t \mathbb{E}\Big[\|\rho_z^i\|^2\Big] + \sum_{i=n}^{N-1} \frac{2C_0}{\Delta t} \mathbb{E}\Big[\Big\|\mathbb{E}_{t_n}^{\xi}[\rho_y^{i+1} - \rho_y^i]\Big\|^2\Big]\\
&\qquad + \sum_{i=n}^{N-1} \frac{4C_0 \mathbb{E}\Big[\Big\|\mathbb{E}_{t_n}^{\xi}\big[(\rho_y^{i+1} - \rho_y^i)\Delta W_{t_{i+1}}\big]\Big\|^2\Big]}{(\theta_5 - \theta_6)\Delta t} \Bigg],
\end{aligned} \tag{4.27}$$

where C is a constant independent of Δt and h.

Theorem 4.3. *Under the conditions in Lemmas 4.2–4.5 and Theorem 4.2, by setting $y_h^N = y_T$ and $z_h^N = z_T$, we assume that $(y, z) \in \big(L^2_{\mathcal{F}}((0,T); H_0^1(D) \cap H^{k+3}(D)) \cap C_{\mathcal{F}}([0,T]; L^2(\Omega; \mathcal{W}^{k+1,4}(D)))\big) \times L^2_{\mathcal{F}}((0,T); H_0^1(D) \cap H^{k+1}(D))$. Then for $0 \le n \le N-1$, $k \ge 1$, we have the following conclusions.*

(1) *For parameters $\theta_i \in [0,1] (i = 1,2,3,4)$, $\theta_5 \in (0,1]$ and $\theta_6 \in (0,1)$ with constraint $\theta_6 < \theta_5$, we have*

$$\mathbb{E}\big[\|e_y^n\|^2\big] + \sum_{i=n}^{N-1} \Delta t \mathbb{E}\Big[\|e_z^i\|^2 + h^2 \|e_y^i\|_1^2 + h^2 \|e_z^i\|_1^2\Big] \le C\Big(h^{2(k+1)} + (\Delta t)^2\Big). \tag{4.28}$$

(2) *For parameters $\theta_i = \frac{1}{2} (i = 1,2,3,4,5)$ and $\theta_6 \in (0, \frac{1}{2})$, then we have*

$$\mathbb{E}[\|e_y^n\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E}\Big[\|e_z^i\|^2 + h^2 \|e_y^i\|_1^2 + h^2 \|e_z^i\|_1^2\Big] \le C\Big(h^{2(k+1)} + (\Delta t)^4\Big). \tag{4.29}$$

4.2.2. Proof of Theorem 4.2

The detailed procedure is split into four main steps. We first derive the error equations, then give the estimates of y, followed by the estimates of z. Finally, we conclude the main result in (4.27).

Step 1. Error equations. By setting $v = v_h \in S_h$ in (3.8) and $w = w_h \in S_h$ in (3.20), subtracting (3.21) and (3.22) from the resulted equations, respectively, we get the error equations of y and z as

$$\begin{aligned} \big(e_y^n, v_h\big) + \theta_1 \Delta t_n a\big(e_y^n, v_h\big) &= \Big(\mathbb{E}_{t_n}^{\xi}\big[e_y^{n+1}\big], v_h\Big) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Big], v_h\Big) \\ &\quad + \theta_2 \Delta t_n \big(e_f^n, v_h\big) - (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[e_y^{n+1}\big], v_h\Big) + \big(R_y^n, v_h\big), \end{aligned} \tag{4.30}$$

$$\begin{aligned} \theta_5 \Delta t_n (e_z^n, w_h) &= \theta_6 \Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[e_z^{n+1}\big], w_h\Big) + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\big[e_y^{n+1}\Delta W_{t_{n+1}}\big], w_h\Big) \\ &\quad - (1-\theta_3)(\Delta t_n)^2 a\Big(\mathbb{E}_{t_n}^{\xi}\big[e_z^{n+1}\big], w_h\Big) + (R_z^n, w_h) \\ &\quad + (1-\theta_4)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Delta W_{t_{n+1}}\Big], w_h\Big). \end{aligned} \tag{4.31}$$

With the orthogonal property of the elliptic projection in (4.19), the above two equations are rewritten as follows,

$$\begin{aligned} \big(\rho_y^n + \zeta_y^n, v_h\big) + \theta_1 \Delta t_n a\big(\zeta_y^n, v_h\big) &= \Big(\mathbb{E}_{t_n}^{\xi}\big[\rho_y^{n+1} + \zeta_y^{n+1}\big], v_h\Big) + \theta_2 \Delta t_n \big(e_f^n, v_h\big) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Big], v_h\Big) \\ &\quad - (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_y^{n+1}\big], v_h\Big) + \big(R_y^n, v_h\big), \end{aligned} \tag{4.32}$$

$$\begin{aligned} \theta_5 \Delta t_n (\rho_z^n + \zeta_z^n, w_h) &= \theta_6 \Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[\rho_z^{n+1} + \zeta_z^{n+1}\big], w_h\Big) - (1-\theta_3)(\Delta t_n)^2 a\Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_z^{n+1}\big], w_h\Big) \\ &\quad + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\big[(\rho_y^{n+1} + \zeta_y^{n+1})\Delta W_{t_{n+1}}\big], w_h\Big) + (R_z^n, w_h) \\ &\quad + (1-\theta_4)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Delta W_{t_{n+1}}\Big], w_h\Big). \end{aligned} \tag{4.33}$$

Using the property of the condition expectation $\mathbb{E}_{t_n}^{\xi}[\cdot]$, we will have an equivalent form

$$\begin{aligned} \big(\zeta_y^n, v_h\big) + \theta_1 \Delta t_n a\big(\zeta_y^n, v_h\big) &= \Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_y^{n+1}\big], v_h\Big) + \Big(\mathbb{E}_{t_n}^{\xi}\big[\rho_y^{n+1} - \rho_y^n\big], v_h\Big) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Big], v_h\Big) \\ &\quad + \theta_2 \Delta t_n \big(e_f^n, v_h\big) - (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_y^{n+1}\big], v_h\Big) + \big(R_y^n, v_h\big), \end{aligned} \tag{4.34}$$

$$\begin{aligned} \theta_5 \Delta t_n (\zeta_z^n, w_h) &= -\theta_5 \Delta t_n (\rho_z^n, w_h) + \theta_6 \Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\big[\rho_z^{n+1} + \zeta_z^{n+1}\big], w_h\Big) \\ &\quad + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\big[(\rho_y^{n+1} + \zeta_y^{n+1})\Delta W_{t_{n+1}}\big], w_h\Big) \\ &\quad - (1-\theta_3)(\Delta t_n)^2 a\Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_z^{n+1}\big], w_h\Big) + (R_z^n, w_h) \\ &\quad + (1-\theta_4)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Delta W_{t_{n+1}}\Big], w_h\Big). \end{aligned} \tag{4.35}$$

With the help of the error equations (4.34) and (4.35), we further give the error estimates of y, z in Steps 2 and 3, respectively.

Step 2. Estimates of y. By setting $v_h = \zeta_y^n$ in (4.34), we derive that

$$\begin{aligned} \big(\zeta_y^n, \zeta_y^n\big) + \theta_1 \Delta t_n a\big(\zeta_y^n, \zeta_y^n\big) &= \Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_y^{n+1}\big], \zeta_y^n\Big) + \Big(\mathbb{E}_{t_n}^{\xi}\big[\rho_y^{n+1} - \rho_y^n\big], \zeta_y^n\Big) + (1-\theta_2)\Delta t_n \Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Big], \zeta_y^n\Big) \\ &\quad + \theta_2 \Delta t_n \big(e_f^n, \zeta_y^n\big) - (1-\theta_1)\Delta t_n a\Big(\mathbb{E}_{t_n}^{\xi}\big[\zeta_y^{n+1}\big], \zeta_y^n\Big) + \big(R_y^n, \zeta_y^n\big). \end{aligned} \tag{4.36}$$

Considering Assumption 2.1 for the bilinear form $a(\cdot,\cdot)$, Lemma 4.3 for f, the identity $2(a,b)=(a,a)+(b,b)-(a-b,a-b)$ for $(\mathbb{E}_{t_n}^{\xi}[\zeta_y^{n+1}],\zeta_y^n)$, and using the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}
&\frac{\|\zeta_y^n\|^2}{2}+\beta\Delta t_n\|\zeta_y^n\|_1^2+\frac{\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}-\zeta_y^n\right]\right\|^2}{2}\\
&\quad\le\frac{\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\right]\right\|^2}{2}+\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}-\rho_y^n\right]+R_y^n\right\|\cdot\|\zeta_y^n\|+\theta_2\Delta t_n\|\tilde e_f^n\|\|\zeta_y^n\|\\
&\qquad+(1-\theta_2)\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[\tilde e_f^{n+1}\right]\right\|\|\zeta_y^n\|+(1-\theta_1)\Delta t_n\alpha\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}-\zeta_y^n\right]\right\|_1\|\zeta_y^n\|_1\\
&\qquad+\bar C_f\Delta t_n\|\zeta_y^n\|_1\Big(\theta_2\|\rho_y^n\|+(1-\theta_2)\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\right]\right\|\Big)+\tilde C_f\Delta t_n\Big[\theta_2\Big|\Big(|\nabla\rho_y^n|^2,\zeta_y^n\Big)\Big|\\
&\qquad+(1-\theta_2)\Big|\Big(\mathbb{E}_{t_n}^{\xi}\Big[|\nabla\rho_y^{n+1}|^2\Big],\zeta_y^n\Big)\Big|\Big]\\
&\quad=\mathbb{I}_y^1+\mathbb{I}_y^2+\mathbb{I}_y^3+\mathbb{I}_y^4+\mathbb{I}_y^5+\mathbb{I}_y^6+\mathbb{I}_y^7.
\end{aligned}\tag{4.37}$$

In the following, the terms on the right side of (4.37) are estimated, separately. The term $\mathbb{I}_y^1$ is directly given as follows,

$$\mathbb{I}_y^1=\frac{\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\right]\right\|^2}{2}.\tag{4.38}$$

With the elementary inequality of $ab\le\frac{a^2}{2\epsilon}+\frac{\epsilon b^2}{2}$, $\epsilon>0$, for $\mathbb{I}_y^2$, we have

$$\mathbb{I}_y^2=\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}-\rho_y^n\right]+R_y^n\right\|\cdot\|\zeta_y^n\|\le\frac{\Delta t_n\|\zeta_y^n\|^2}{2}+\frac{1}{\Delta t_n}\Big(\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}-\rho_y^n\right]\right\|^2+\|R_y^n\|^2\Big).\tag{4.39}$$

With Assumption 2.2, there is a constant L satisfying the following inequality,

$$\|\tilde e_f^n\|\le L\Big(\|\zeta_y^n\|_1+\|e_y^n\|+\|e_z^n\|\Big).\tag{4.40}$$

For terms $\mathbb{I}_y^3$ and $\mathbb{I}_y^4$ related to $\|\tilde e_f^n\|$ and $\|\tilde e_f^{n+1}\|$, by above inequality (4.40), we get

$$\begin{aligned}
\mathbb{I}_y^3&=\theta_2\Delta t_n\|\tilde e_f^n\|\cdot\|\zeta_y^n\|\le\frac{\theta_2\Delta t_n}{2}\left(\frac{36C_0L^2}{\eta}\|\zeta_y^n\|^2+\frac{\eta\|\tilde e_f^n\|^2}{36C_0L^2}\right)\\
&\le\frac{\theta_2\Delta t_n}{2}\left[\frac{36C_0L^2}{\eta}\|\zeta_y^n\|^2+\frac{\eta}{6C_0}\left(\frac{\|\zeta_y^n\|_1^2}{2}+\|\rho_y^n\|^2+\|\zeta_y^n\|^2+\|\rho_z^n\|^2+\|\zeta_z^n\|^2\right)\right],
\end{aligned}\tag{4.41}$$

and

$$\begin{aligned}
\mathbb{I}_y^4&=(1-\theta_2)\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[\tilde e_f^{n+1}\right]\right\|\cdot\|\zeta_y^n\|\\
&\le\frac{(1-\theta_2)\Delta t_n}{2}\left[\frac{36C_0L^2}{\eta}\|\zeta_y^n\|^2+\frac{\eta}{6C_0}\left(\frac{\mathbb{E}_{t_n}^{\xi}\left[\|\zeta_y^{n+1}\|_1^2\right]}{2}+\mathbb{E}_{t_n}^{\xi}\left[\|\rho_y^{n+1}\|^2\right]\right.\right.\\
&\qquad\left.\left.+\mathbb{E}_{t_n}^{\xi}\left[\|\zeta_y^{n+1}\|^2\right]+\mathbb{E}_{t_n}^{\xi}\left[\|\rho_z^{n+1}\|^2\right]+\mathbb{E}_{t_n}^{\xi}\left[\|\zeta_z^{n+1}\|^2\right]\right)\right],
\end{aligned}\tag{4.42}$$

where η is a constant, such that $0 < \eta_0 \le \eta \le \min\{1, \frac{\beta}{2}\}$. For $\mathbb{I}_y^5$, by the inverse inequality (4.1), it follows that

$$\begin{aligned}\mathbb{I}_y^5 &= (1-\theta_1)\Delta t_n \alpha \Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1} - \zeta_y^n\right]\Big\|_1 \|\zeta_y^n\|_1 \\ &\le \frac{(1-\theta_1)\alpha^2 C_I \Delta t_n}{2\eta h^2}\Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1} - \zeta_y^n\right]\Big\|^2 + \frac{(1-\theta_1)\eta\Delta t_n}{2}\|\zeta_y^n\|_1^2.\end{aligned} \tag{4.43}$$

For the last two terms $\mathbb{I}_y^6$ and $\mathbb{I}_y^7$, it is estimated as

$$\begin{aligned}\mathbb{I}_y^6 &= \bar{C}_f \Delta t_n \|\zeta_y^n\|_1 \Big(\theta_2\|\rho_y^n\| + (1-\theta_2)\Big\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\right]\Big\|\Big) \\ &\le \bar{C}_f \Delta t_n \left[\frac{\beta\|\zeta_y^n\|_1^2}{2\bar{C}_f} + \frac{\bar{C}_f}{\beta}\Big(\theta_2^2\|\rho_y^n\|^2 + (1-\theta_2)^2\Big\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\right]\Big\|^2\Big)\right].\end{aligned} \tag{4.44}$$

$$\begin{aligned}\mathbb{I}_y^7 &= \tilde{C}_f\theta_2\Delta t_n \Big|\Big(|\nabla\rho_y^n|^2, \zeta_y^n\Big)\Big| + \tilde{C}_f(1-\theta_2)\Delta t_n \Big|\Big(\mathbb{E}_{t_n}^{\xi}\Big[|\nabla\rho_y^{n+1}|^2\Big], \zeta_y^n\Big)\Big| \\ &\le \tilde{C}_f\Delta t_n \|\zeta_y^n\| \Big(\theta_2\|\nabla\rho_y^n\|_{L^4}^2 + (1-\theta_2)\mathbb{E}_{t_n}^{\xi}\Big[\|\nabla\rho_y^{n+1}\|_{L^4}^2\Big]\Big) \\ &\le \tilde{C}_f\Delta t_n \left(\frac{\|\zeta_y^n\|^2}{2} + \theta_2^2\|\nabla\rho_y^n\|_{L^4}^4 + (1-\theta_2)^2\mathbb{E}_{t_n}^{\xi}\Big[\|\nabla\rho_y^{n+1}\|_{L^4}^4\Big]\right).\end{aligned} \tag{4.45}$$

Combining the estimates (4.37) and (4.38)–(4.45) and the resulted inequalities, we can get the following estimates,

$$\begin{aligned}&\left[1 - \left(1 + \frac{36C_0L^2}{\eta} + \tilde{C}_f + \frac{\theta_2\eta}{6C_0}\right)\Delta t_n\right]\|\zeta_y^n\|^2 + \left(\beta - (1-\theta_1)\eta - \frac{\theta_2\eta}{12C_0}\right)\Delta t_n\|\zeta_y^n\|_1^2 \\ &\quad + \left(1 - \frac{(1-\theta_1)\alpha^2C_I\Delta t_n}{\eta h^2}\right)\Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1} - \zeta_y^n\right]\Big\|^2 - \frac{\theta_2\eta}{6C_0}\Delta t_n\|\zeta_z^n\|^2 \\ &\quad \le \Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\right]\Big\|^2 + \frac{(1-\theta_2)\eta}{6C_0}\Delta t_n\mathbb{E}_{t_n}^{\xi}\Big[\|\zeta_y^{n+1}\|^2\Big] + \frac{2}{\Delta t_n}\Big(\Big\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1} - \rho_y^n\right]\Big\|^2 + \|R_y^n\|^2\Big) \\ &\quad + \frac{2\bar{C}_f^2}{\beta}\Delta t_n\Big(\theta_2^2\|\rho_y^n\|^2 + (1-\theta_2)^2\Big\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\right]\Big\|^2\Big) + \frac{\theta_2\eta\Delta t_n}{6C_0}\Big(\|\rho_y^n\|^2 + \|\rho_z^n\|^2\Big) \\ &\quad + 2\tilde{C}_f\Delta t_n\Big(\theta_2^2\|\nabla\rho_y^n\|_{L^4}^4 + (1-\theta_2)^2\mathbb{E}_{t_n}^{\xi}\Big[\|\nabla\rho_y^{n+1}\|_{L^4}^4\Big]\Big) + \frac{(1-\theta_2)\eta\Delta t_n}{12C_0}\mathbb{E}_{t_n}^{\xi}\Big[\|\zeta_y^{n+1}\|_1^2\Big] \\ &\quad + \frac{(1-\theta_2)\eta\Delta t_n}{6C_0}\Big[\mathbb{E}_{t_n}^{\xi}\Big[\|\rho_y^{n+1}\|^2\Big] + \mathbb{E}_{t_n}^{\xi}\Big[\|\rho_z^{n+1}\|^2\Big] + \mathbb{E}_{t_n}^{\xi}\Big[\|\zeta_z^{n+1}\|^2\Big]\Big].\end{aligned} \tag{4.46}$$

Step 3. Estimates of z. Similarly, let $w_h = \zeta_z^n$ in (4.35), then we use inverse inequality and the identity $2(a,b) = (a,a) + (b,b) - (a-b, a-b)$ for $(\mathbb{E}_{t_n}^{\xi}[\zeta_z^{n+1}], \zeta_z^n)$, $a(\mathbb{E}_{t_n}^{\xi}[\zeta_z^{n+1}], \zeta_z^n)$, we obtain

$$\begin{aligned}&\left(\theta_5 - \frac{\theta_6}{2}\right)\Delta t_n\|\zeta_z^n\|^2 + \frac{1-\theta_3}{2}(\Delta t_n)^2\beta\Big(\|\zeta_z^n\|_1^2 + \Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\Big\|_1^2\Big) \\ &\quad + \left(\frac{\theta_6}{2}\Delta t_n - \frac{(1-\theta_3)\alpha C_I}{2h^2}(\Delta t_n)^2\right)\Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right] - \zeta_z^n\Big\|^2 \\ &\quad \le \frac{\theta_6}{2}\Delta t_n\Big\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\Big\|^2 + \theta_6\Delta t_n\Big(\mathbb{E}_{t_n}^{\xi}\left[\rho_z^{n+1}\right], \zeta_z^n\Big) + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\Delta W_{t_{n+1}}\right], \zeta_z^n\Big) \\ &\quad - \theta_5\Delta t_n(\rho_z^n, \zeta_z^n) + (\theta_5 - \theta_6)\Big(\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\Delta W_{t_{n+1}}\right], \zeta_z^n\Big) + (R_z^n, \zeta_z^n) \\ &\quad + (1-\theta_4)\Delta t_n\Big(\mathbb{E}_{t_n}^{\xi}\Big[e_f^{n+1}\Delta W_{t_{n+1}}\Big], \zeta_z^n\Big).\end{aligned} \tag{4.47}$$

By Lemma 4.3, for the term related to f, we have

$$\left(\mathbb{E}_{t_n}^{\xi}\left[e_f^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right) \leq \bar{C}_f\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\Delta W_{t_{n+1}}\right]\right\|\|\nabla\zeta_z^n\| + \left(\mathbb{E}_{t_n}^{\xi}\left[\tilde{e}_f^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right) + \tilde{C}_f\left|\left(\mathbb{E}_{t_n}^{\xi}\left[\left|\nabla\rho_y^{n+1}\right|^2\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\right|. \tag{4.48}$$

Combining (4.47) and (4.48), we obtain the following inequality

$$\begin{aligned}
&\left(\theta_5-\frac{\theta_6}{2}\right)\Delta t_n\|\zeta_z^n\|^2+\frac{1-\theta_3}{2}\beta(\Delta t_n)^2\left(\|\zeta_z^n\|_1^2+\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\right\|_1^2\right)\\
&\quad+\left[\frac{\theta_6}{2}\Delta t_n-\frac{(1-\theta_3)\alpha C_I}{2h^2}(\Delta t_n)^2\right]\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]-\zeta_z^n\right\|^2\\
&\quad\leq\frac{\theta_6}{2}\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\right\|^2+\theta_6\Delta t_n\left(\mathbb{E}_{t_n}^{\xi}\left[\rho_z^{n+1}\right],\zeta_z^n\right)+(\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\\
&\qquad+(\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right)+(R_z^n,\zeta_z^n)\\
&\qquad+\bar{C}_f(1-\theta_4)\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\Delta W_{t_{n+1}}\right]\right\|\|\nabla\zeta_z^n\|+(1-\theta_4)\Delta t_n\left(\mathbb{E}_{t_n}^{\xi}\left[\tilde{e}_f^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\\
&\qquad+\tilde{C}_f(1-\theta_4)\Delta t_n\left|\left(\mathbb{E}_{t_n}^{\xi}\left[\left|\nabla\rho_y^{n+1}\right|^2\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\right|+\theta_5\Delta t_n\|\rho_z^n\|\|\zeta_z^n\|\\
&\quad\leq\mathbb{I}_z^1+\mathbb{I}_z^2+\mathbb{I}_z^3+\mathbb{I}_z^4+\mathbb{I}_z^5+\mathbb{I}_z^6+\mathbb{I}_z^7+\mathbb{I}_z^8+\mathbb{I}_z^9.
\end{aligned}\tag{4.49}$$

As is the case of the estimates of y, we separately approximate each term in the right hand sides of (4.49). The term $\mathbb{I}_z^1$ is directly written as

$$\mathbb{I}_z^1=\frac{\theta_6}{2}\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\right\|^2. \tag{4.50}$$

Noting that $\theta_5-\theta_6>0$, for term $\mathbb{I}_z^2$, we get

$$\mathbb{I}_z^2=\theta_6\Delta t_n\left(\mathbb{E}_{t_n}^{\xi}\left[\rho_z^{n+1}\right],\zeta_z^n\right)\leq\frac{\theta_6\Delta t_n}{2}\left(\frac{\theta_5-\theta_6}{4}\|\zeta_z^n\|^2+\frac{4}{\theta_5-\theta_6}\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_z^{n+1}\right]\right\|^2\right). \tag{4.51}$$

With the martingale property of Brownian motion, $\mathbb{E}_{t_n}^{\xi}[\rho_y^n\Delta W_{t_{n+1}}]=0$, we analyze $\mathbb{I}_z^3$ and $\mathbb{I}_z^4$ as follows,

$$\begin{aligned}
\mathbb{I}_z^3&=(\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\\
&=(\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[(\rho_y^{n+1}-\rho_y^n)\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\\
&\leq\frac{\theta_5-\theta_6}{2}\left(\frac{\theta_5-\theta_6}{4}\Delta t_n\|\zeta_z^n\|^2+\frac{4}{(\theta_5-\theta_6)\Delta t_n}\left\|\mathbb{E}_{t_n}^{\xi}\left[(\rho_y^{n+1}-\rho_y^n)\Delta W_{t_{n+1}}\right]\right\|^2\right),
\end{aligned}\tag{4.52}$$

$$\mathbb{I}_z^4=(\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\Delta W_{t_{n+1}}\right],\zeta_z^n\right)\leq\frac{\theta_5-\theta_6}{2}\left(\mathbb{E}_{t_n}^{\xi}\left[\left\|\zeta_y^{n+1}\right\|^2\right]-\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\right]\right\|^2+\Delta t_n\|\zeta_z^n\|^2\right). \tag{4.53}$$

Then, for the term related to truncation error $\|R_z^n\|$, we have

$$\mathbb{I}_z^5=(R_z^n,\zeta_z^n)\leq\frac{(\theta_5-\theta_6)\Delta t_n}{8}\|\zeta_z^n\|^2+\frac{2\|R_z^n\|^2}{(\theta_5-\theta_6)\Delta t_n}. \tag{4.54}$$

By the Itô isometry formula, the terms $\mathbb{I}_z^6$ and $\mathbb{I}_z^7$ are estimated as

$$\begin{aligned}\mathbb{I}_z^6 &= (1-\theta_4)\Delta t_n \bar{C}_f \left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_y^{n+1}\Delta W_{t_{n+1}}\right]\right\| \|\nabla \zeta_z^n\| \\ &\leq \frac{\bar{C}_f C_I (1-\theta_4)(\Delta t_n)^2}{2h^2}\|\zeta_z^n\|^2 + \frac{\bar{C}_f(1-\theta_4)\Delta t_n}{2}\mathbb{E}_{t_n}^{\xi}\left[\left\|\rho_y^{n+1}\right\|^2\right], \quad (4.55)\\ \mathbb{I}_z^7 &= (1-\theta_4)\Delta t_n \tilde{C}_f \left|\left(\mathbb{E}_{t_n}^{\xi}\left[\left|\nabla\rho_y^{n+1}\right|^2\Delta W_{t_{n+1}}\right], \zeta_z^n\right)\right| \\ &\leq \tilde{C}_f(1-\theta_4)\Delta t_n \|\zeta_z^n\| \cdot \mathbb{E}_{t_n}^{\xi}\left[\||\nabla\rho_y^{n+1}|^2\Delta W_{t_{n+1}}\|\right] \\ &\leq \frac{\tilde{C}_f(1-\theta_4)(\Delta t_n)^2}{2}\|\zeta_z^n\|^2 + \frac{\tilde{C}_f(1-\theta_4)\Delta t_n}{2}\mathbb{E}_{t_n}^{\xi}\left[\left\|\nabla\rho_y^{n+1}\right\|_{L^4}^4\right]. \quad (4.56)\end{aligned}$$

Then, by (4.40) in Assumption 2.2 and inverse inequality, we have

$$\begin{aligned}\mathbb{I}_z^8 &= (1-\theta_4)\Delta t_n\left(\mathbb{E}_{t_n}^{\xi}\left[\tilde{e}_f^{n+1}\Delta W_{t_{n+1}}\right], \zeta_z^n\right) \\ &\leq \frac{(1-\theta_4)\Delta t_n}{2}\left(\frac{\theta_5-\theta_6}{4}\|\zeta_z^n\|^2 + \frac{4\Delta t_n}{\theta_5-\theta_6}\mathbb{E}_{t_n}^{\xi}\left[\left\|\tilde{e}_f^{n+1}\right\|^2\right]\right) \\ &\leq \frac{(\theta_5-\theta_6)(1-\theta_4)\Delta t_n}{8}\|\zeta_z^n\|^2 + \frac{12(1-\theta_4)(\Delta t_n)^2L^2}{\theta_5-\theta_6}\left(\left(1+\frac{C_I}{2h^2}\right)\mathbb{E}_{t_n}^{\xi}\left[\left\|\zeta_y^{n+1}\right\|^2\right]\right. \\ &\quad \left. + \mathbb{E}_{t_n}^{\xi}\left[\left\|\rho_y^{n+1}\right\|^2\right] + \mathbb{E}_{t_n}^{\xi}\left[\left\|\rho_z^{n+1}\right\|^2\right] + \mathbb{E}_{t_n}^{\xi}\left[\left\|\zeta_z^{n+1}\right\|^2\right]\right). \quad (4.57)\end{aligned}$$

With the Cauchy–Schwarz inequality, the last term $\mathbb{I}_z^9$ in (4.49) is directly approximated as

$$\mathbb{I}_z^9 = \theta_5\Delta t_n\|\rho_z^n\|\|\zeta_z^n\| \leq \frac{12\theta_5\Delta t_n}{\theta_5-\theta_6}\|\rho_z^n\|^2 + \frac{\theta_5(\theta_5-\theta_6)\Delta t_n}{48}\|\zeta_z^n\|^2. \quad (4.58)$$

Combining the above estimates (4.50)–(4.58), we have the following estimates for z as

$$\begin{aligned}&\left[\theta_5 - \frac{\theta_5-\theta_6}{4}(2+\theta_5-\theta_4) - \frac{\theta_5(\theta_5-\theta_6)}{24} - \left(\frac{C_I\bar{C}_f}{h^2}+\tilde{C}_f\right)(1-\theta_4)\Delta t_n\right]\Delta t_n\|\zeta_z^n\|^2 \\ &\quad + \left(\theta_6\Delta t_n - \frac{C_I\alpha(1-\theta_3)}{h^2}(\Delta t_n)^2\right)\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]-\zeta_z^n\right\|^2 + (1-\theta_3)\beta(\Delta t_n)^2\left(\|\zeta_z^n\|_1^2 + \left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\right\|_1^2\right) \\ &\leq \theta_6\Delta t_n\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_z^{n+1}\right]\right\|^2 + (\theta_5-\theta_6)\left(\mathbb{E}_{t_n}^{\xi}\left[\left\|\zeta_y^{n+1}\right\|^2\right] - \left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\right]\right\|^2\right) + \frac{24\theta_5\Delta t_n}{\theta_5-\theta_6}\|\rho_z^n\|^2 \\ &\quad + \frac{4}{\Delta t_n}\left\|\mathbb{E}_{t_n}^{\xi}\left[(\rho_y^{n+1}-\rho_y^n)\Delta W_{t_{n+1}}\right]\right\|^2 + \frac{4\theta_6\Delta t_n}{\theta_5-\theta_6}\left\|\mathbb{E}_{t_n}^{\xi}\left[\rho_z^{n+1}\right]\right\|^2 + \frac{4\|R_z^n\|^2}{(\theta_5-\theta_6)\Delta t_n} \\ &\quad + \bar{C}_f(1-\theta_4)\Delta t_n\mathbb{E}_{t_n}^{\xi}\left[\left\|\rho_y^{n+1}\right\|^2\right] + \tilde{C}_f(1-\theta_4)\Delta t_n\mathbb{E}_{t_n}^{\xi}\left[\left\|\nabla\rho_y^{n+1}\right\|_{L^4}^4\right] + \frac{24(1-\theta_4)L^2}{\theta_5-\theta_6}(\Delta t_n)^2 \\ &\quad \cdot\left[\left(1+\frac{C_I}{2h^2}\right)\mathbb{E}_{t_n}^{\xi}\left[\left\|\zeta_y^{n+1}\right\|^2\right] + \mathbb{E}_{t_n}^{\xi}\left[\left\|\rho_y^{n+1}\right\|^2\right] + \mathbb{E}_{t_n}^{\xi}\left[\left\|\rho_z^{n+1}\right\|^2\right] + \mathbb{E}_{t_n}^{\xi}\left[\left\|\zeta_z^{n+1}\right\|^2\right]\right]. \quad (4.59)\end{aligned}$$

In the final step, we will integrate the estimates (4.46) for y and (4.59) for z, and give the final error estimates.

Step 4. Estimates of z and y. Dividing both sides of (4.59) by $\frac{(\theta_5-\theta_6)\Delta t_n}{\Delta t}$, multiplying (4.46) by C_0, and adding this two associated inequalities together, for $\frac{\Delta t}{h^2} \leq \frac{\eta_0}{\alpha^2 C_I}$, $0 < \eta_0 \leq \theta_6\alpha$, we obtain the following estimates as

$$C_0(1-C_1\Delta t)\left\|\zeta_y^n\right\|^2 + C_2\Delta t\left\|\zeta_y^n\right\|_1^2 + C_3\Delta t\|\zeta_z^n\|^2 + C_4\Delta t\left\|\mathbb{E}_{t_n}^{\xi}\left[\zeta_y^{n+1}\right]-\zeta_y^n\right\|^2$$

$$
\begin{aligned}
&+ C_5\Delta t\Big\|\mathbb{E}^{\xi}_{t_n}\left[\zeta_z^{n+1}\right]-\zeta_z^n\Big\|^2+\frac{1-\theta_3}{\theta_5-\theta_6}\beta\Delta t_n\Delta t\Big(\|\zeta_z^n\|_1^2+\Big\|\mathbb{E}^{\xi}_{t_n}\left[\zeta_z^{n+1}\right]\Big\|_1^2\Big)\\
&\le C_0(1+C_6\Delta t)\mathbb{E}^{\xi}_{t_n}\Big[\big\|\zeta_y^{n+1}\big\|^2\Big]+C_7\Delta t\mathbb{E}^{\xi}_{t_n}\Big[\big\|\zeta_z^{n+1}\big\|^2\Big]+\bigg(\frac{2C_0\bar{C}_f^2\theta_2^2}{\beta}+\frac{\theta_2\eta}{6}\bigg)\Delta t\big\|\rho_y^n\big\|^2\\
&\quad+C_8\Delta t\mathbb{E}^{\xi}_{t_n}\Big[\big\|\rho_y^{n+1}\big\|^2\Big]+2C_0\tilde{C}_f\theta_2^2\Delta t\big\|\nabla\rho_y^n\big\|_{L^4}^4+C_9\Delta t\mathbb{E}^{\xi}_{t_n}\Big[\big\|\nabla\rho_y^{n+1}\big\|_{L^4}^4\Big]\\
&\quad+\frac{2C_0^2}{\Delta t}\big\|R_y^n\big\|^2+\frac{4C_0^2}{(\theta_5-\theta_6)^2\Delta t}\|R_z^n\|^2+\frac{2C_0^2}{\Delta t}\Big\|\mathbb{E}^{\xi}_{t_n}\left[\rho_y^{n+1}-\rho_y^n\right]\Big\|^2\\
&\quad+C_{10}\Delta t\mathbb{E}^{\xi}_{t_n}\Big[\big\|\rho_z^{n+1}\big\|^2\Big]+\frac{4C_0^2}{(\theta_5-\theta_6)\Delta t}\Big\|\mathbb{E}^{\xi}_{t_n}\left(\rho_y^{n+1}-\rho_y^n\right)\Delta W_{t_{n+1}}\Big\|^2\\
&\quad+C_{11}\Delta t\|\rho_z^n\|^2+C_{12}\Delta t\mathbb{E}^{\xi}_{t_n}\Big[\big\|\zeta_y^{n+1}\big\|_1^2\Big],
\end{aligned}
\tag{4.60}
$$

where

$$
\begin{aligned}
&C_1=1+\frac{36C_0L^2}{\eta}+\tilde{C}_f+\frac{\theta_2\eta}{6C_0},\quad C_2=\beta-(1-\theta_1)\eta-\frac{\theta_2\eta}{12C_0},\\
&C_3=\frac{\theta_5}{\theta_5-\theta_6}-\frac{2+\theta_5-\theta_4}{4}-\frac{\theta_5}{24}-\frac{\theta_2\eta}{6}-\bigg(\frac{C_I\bar{C}_f}{h^2}+\tilde{C}_f\bigg)\frac{1-\theta_4}{\theta_5-\theta_6}\Delta t,\\
&C_4=C_0\bigg(1-\frac{(1-\theta_1)\alpha^2C_I\Delta t}{\eta h^2}\bigg),\quad C_5=\frac{\theta_6}{\theta_5-\theta_6}-\frac{C_I\alpha(1-\theta_3)}{(\theta_5-\theta_6)h^2}\Delta t,\\
&C_6=\frac{(1-\theta_2)\eta}{6C_0}+\bigg(1+\frac{C_I}{2h^2}\bigg)\frac{24(1-\theta_4)L^2\Delta t}{C_0(\theta_5-\theta_6)^2},\\
&C_7=\frac{(1-\theta_2)\eta}{6}+\frac{\theta_6}{\theta_5-\theta_6}+\frac{24(1-\theta_4)L^2}{(\theta_5-\theta_6)^2}\Delta t,\\
&C_8=\frac{2C_0\bar{C}_f^2}{\beta}(1-\theta_2)^2+\frac{(1-\theta_2)\eta}{6}+\frac{\bar{C}_f(1-\theta_4)}{\theta_5-\theta_6}+\frac{24(1-\theta_4)L^2}{(\theta_5-\theta_6)^2}\Delta t,\\
&C_9=2C_0\tilde{C}_f(1-\theta_2)^2+\frac{\tilde{C}_f(1-\theta_4)}{\theta_5-\theta_6},\\
&C_{10}=\frac{(1-\theta_2)\eta}{6}+\frac{4\theta_6}{(\theta_5-\theta_6)^2}+\frac{24L^2(1-\theta_4)}{(\theta_5-\theta_6)^2}\Delta t,\\
&C_{11}=\frac{\theta_2\eta}{6}+\frac{24\theta_5}{(\theta_5-\theta_6)^2},\quad C_{12}=\frac{(1-\theta_2)\eta}{12}.
\end{aligned}
$$

It can be checked that there exists a small $\Delta\bar{t}$ with $0<\Delta t_n\le\Delta t\le\Delta\bar{t}$ such that all the coefficients in (4.60) are positive and bounded. Moreover, there are also constants C_1', C_2' and C_3' such that $1-C_1\Delta t\ge C_1'>0$, $C_2-C_{12}\ge C_2'>0$, $C_3-C_7\ge C_3'>0$.

Summing (4.60) over $i=n,n+1,\cdots,N-1$, re-arranging the resulted estimates and omitting the positive terms related to $\|\mathbb{E}^{\xi}_{t_n}[\zeta_z^{n+1}]-\zeta_z^n\|^2$ deduce that

$$
\begin{aligned}
&\mathbb{E}\Big[\big\|\zeta_y^n\big\|^2\Big]+\frac{C_2-C_{12}}{C_0(1-C_1\Delta t)}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\zeta_y^i\big\|_1^2\Big]+\frac{C_3-C_7}{C_0(1-C_1\Delta t)}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\zeta_z^i\big\|^2\Big]\\
&\quad+\frac{(1-\theta_3)\beta}{C_0^2(\theta_5-\theta_6)(1-C_1\Delta t)}\sum_{i=n}^{N-1}(\Delta t)^2\mathbb{E}\Big[\big\|\zeta_z^i\big\|_1^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\le \mathbb{E}\Big[\big\|\zeta_y^N\big\|^2\Big] + \sum_{i=n}^{N-1}\frac{(C_1+C_6)\Delta t}{1-C_1\Delta t}\big\|\zeta_y^{i+1}\big\|^2 + \frac{C_7\Delta t}{C_0(1-C_1\Delta t)}\mathbb{E}\Big[\big\|\zeta_z^N\big\|^2\Big]\\
&\quad + \frac{C_{12}\Delta t}{C_0(1-C_1\Delta t)}\mathbb{E}\Big[\big\|\zeta_y^N\big\|_1^2\Big] + \sum_{i=n}^{N-1}\frac{1}{C_0(1-C_1\Delta t)}\Bigg[\bigg(\frac{2C_0\bar{C}_f^2\theta_2^2}{\beta}+\frac{\theta_2\eta}{6}\bigg)\Delta t\mathbb{E}\Big[\big\|\rho_y^i\big\|^2\Big]\\
&\quad + C_8\Delta t\mathbb{E}\Big[\big\|\rho_y^{i+1}\big\|^2\Big] + 2C_0\tilde{C}_f\theta_2^2\Delta t\mathbb{E}\Big[\big\|\nabla\rho_y^i\big\|_{L^4}^4\Big] + C_9\Delta t\mathbb{E}\Big[\big\|\nabla\rho_y^{i+1}\big\|_{L^4}^4\Big]\\
&\quad + \frac{2C_0^2}{\Delta t}\mathbb{E}\Big[\big\|R_y^i\big\|^2\Big] + \frac{4C_0^2}{(\theta_5-\theta_6)^2\Delta t}\mathbb{E}\Big[\big\|R_z^i\big\|^2\Big] + \frac{2C_0^2}{\Delta t}\mathbb{E}\Big[\Big\|\mathbb{E}_{t_i}^{\xi}\big[\rho_y^{i+1}-\rho_y^i\big]\Big\|^2\Big] + C_{10}\Delta t\mathbb{E}\\
&\quad \times\Big[\big\|\rho_z^{i+1}\big\|^2\Big] + \frac{4C_0^2}{(\theta_5-\theta_6)\Delta t}\mathbb{E}\Big[\Big\|\mathbb{E}_{t_i}^{\xi}\big[(\rho_y^{i+1}-\rho_y^i)\Delta W_{t_{i+1}}\big]\Big\|^2\Big] + C_{11}\Delta t\mathbb{E}\Big[\big\|\rho_z^i\big\|^2\Big]\Bigg].
\end{aligned}
\tag{4.61}
$$

Since $1-C_1\Delta t$ has upper and lower bounds for sufficient small Δt, we introduce an auxiliary constant C^* such that $C^*>C_1$, $C^*>C_6$ and $1>1-C_1\Delta t\ge\frac{1}{C^*}>0$. Under Lemma 2.2, we obtain the following error estimates,

$$
\begin{aligned}
&\mathbb{E}\Big[\big\|\zeta_y^n\big\|^2\Big] + \frac{C_2-C_{12}}{C_0}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\zeta_y^i\big\|_1^2\Big] + \frac{C_3-C_7}{C_0}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\zeta_z^i\big\|^2\Big] + \frac{1-\theta_3}{(\theta_5-\theta_6)C_0^2}\beta\sum_{i=n}^{N-1}(\Delta t)^2\mathbb{E}\Big[\big\|\zeta_z^i\big\|_1^2\Big]\\
&\le C^*\exp\Big(2C_0(C^*)^2T\Big)\Bigg[\mathbb{E}\Big[\big\|\zeta_y^N\big\|^2\Big] + \frac{C_7}{C_0}\Delta t\mathbb{E}\Big[\big\|\zeta_z^N\big\|^2\Big] + \frac{C_{12}}{C_0}\Delta t\mathbb{E}\Big[\big\|\zeta_y^N\big\|_1^2\Big]\\
&\quad + \bigg(\frac{2\bar{C}_f\theta_2^2}{\beta}+\frac{\theta_2\eta}{6C_0}\bigg)\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\rho_y^i\big\|^2\Big] + \frac{C_8}{C_0}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\rho_y^{i+1}\big\|^2\Big]\\
&\quad + 2\tilde{C}_f\theta_2^2\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\nabla\rho_y^i\big\|_{L^4}^4\Big] + \frac{C_9}{C_0}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\nabla\rho_y^{i+1}\big\|_{L^4}^4\Big]\\
&\quad + \sum_{i=n}^{N-1}\frac{2C_0}{\Delta t}\mathbb{E}\Big[\big\|R_y^i\big\|^2\Big] + \sum_{i=n}^{N-1}\frac{4C_0\mathbb{E}\Big[\big\|R_z^i\big\|^2\Big]}{(\theta_5-\theta_6)^2\Delta t} + \sum_{i=n}^{N-1}\frac{2C_0}{\Delta t}\mathbb{E}\Big[\Big\|\mathbb{E}_{t_i}^{\xi}\big[\rho_y^{i+1}-\rho_y^i\big]\Big\|^2\Big]\\
&\quad + \frac{C_{10}}{C_0}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\rho_z^{i+1}\big\|^2\Big] + \frac{C_{11}}{C_0}\sum_{i=n}^{N-1}\Delta t\mathbb{E}\Big[\big\|\rho_z^i\big\|^2\Big]\\
&\quad + \sum_{i=n}^{N-1}\frac{4C_0\mathbb{E}\Big[\Big\|\mathbb{E}_{t_i}^{\xi}\big[(\rho_y^{i+1}-\rho_y^i)\Delta W_{t_{i+1}}\big]\Big\|^2\Big]}{(\theta_5-\theta_6)\Delta t}\Bigg].
\end{aligned}
\tag{4.62}
$$

Dividing (4.62) by $\min\{1,\frac{C_2-C_{12}}{C_0},\frac{C_3-C_7}{C_0},\frac{(1-\theta_3)\beta}{(\theta_5-\theta_6)C_0^2}\}$ and considering the lower bound and upper bound of the coefficients of left and right terms of the resulted inequality, we end the proof. □

4.2.3. Proof of Theorem 4.3

According to the definitions of the errors e_y and e_z, with the triangle inequalities, we have

$$
\begin{aligned}
\|e_y\|^2+\|e_z\|^2 &\le 2\Big(\|\zeta_y\|^2+\|\rho_y\|^2+\|\zeta_z\|^2+\|\rho_z\|^2\Big),\\
h^2\Big(\|e_y\|_1^2+\|e_z\|_1^2\Big) &\le 2h^2\Big(\|\zeta_y\|_1^2+\|\rho_y\|_1^2+\|\zeta_z\|_1^2+\|\rho_z\|_1^2\Big).
\end{aligned}
\tag{4.63}
$$

Following Theorem 4.2, with the estimations of $\mathbb{E}[\|\rho_y^i\|^2]$, $\mathbb{E}[\|\rho_z^i\|^2]$ and $\mathbb{E}[\|\nabla\rho_y^i\|_{L^4}^4]$ by using Lemma 4.2, $\mathbb{E}[\|\mathbb{E}_{t_i}^{\xi}[\rho_y^{i+1}-\rho_y^i]\|^2]$ and $\mathbb{E}[\|\mathbb{E}_{t_i}^{\xi}[(\rho_y^{i+1}-\rho_y^i)\Delta W_{t_{i+1}}]\|^2]$ by using Lemma 4.4, $\mathbb{E}[\|R_y^i\|^2]$ and $\mathbb{E}[\|R_z^i\|^2]$ by using Lemma 4.5, then we conclude Theorem 4.3. □

5. Numerical results

In this section, the numerical simulations are conducted to demonstrate the effectiveness of Scheme 3.1 and verify the theoretical results in Section 4. The random mechanism for the BSPDEs is different from forward SPDEs, which determines that a computer with large memory is required to do the numerical simulations. In our tests, all the computations are performed on HPC with 128 processors and 512G memory, and the numerical examples are done in parallel with 128 MPI processes. The resulted nonlinear discrete systems are efficiently solved by nonlinear multigrid newton methods with multifrontal massively parallel sparse (MUMPS) solver [1].

In order to carry out the finite element θ-scheme 3.1, the approximation of $\mathbb{E}_{t_n}^{\xi}[\cdot]$ is needed. In this paper, we refer to [30,32] for the detailed implementations. In the calculations of the conditional mathematical expectation $\mathbb{E}_{t_n}^{\xi}[\cdot]$, the Gauss–Hermite quadrature rule is used, and the values of the integrands of the conditional mathematical expectations at non-grid points are approximated by local cubic interpolations.

In the following numerical tests, two numerical examples in 1 and 2-dimensional space domain are provided. The linear and quadratic finite element spaces are considered, respectively. To simplify the notations, the hextuple $\Theta=(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6)$ are used to represent discretization in time. Referring to the numerical tests in [32] for the generalized θ-scheme for BSDEs, four sets of parameters are chosen as $\Theta_1=(1,1,1,1,1,1/2)$, $\Theta_2=(1,1,1/2,1/2,1,1/2)$, which form the temporal first-order schemes and $\Theta_3=(1/2,1/2,1/2,1/2,1/2,1/4)$, $\Theta_4=(1/2,1/2,1/2,1/2,1/2,1/5)$ for temporal second-order schemes.

Example 1 (1-dimensional case)**.** We first consider the following 1-dimensional nonlinear BSPDE:

$$-\mathrm{d}y_t-\kappa\partial_x y_t\,\mathrm{d}t=f(t,x,W_t,y_t,\nabla y_t,z_t)\,\mathrm{d}t-z_t\,\mathrm{d}W_t,\ \text{ in } [0,T)\times[0,1]. \tag{5.1}$$

Here, W_t is a standard Brownian motion, $\kappa=1$ and $f(t,x,W_t,y_t,\nabla y_t,z_t)=y_tz_t-\nabla y_t+3.5y_t-2\sin(t+W_t+x)-\sin(t+W_t+x)\cos(t+W_t+x)$. With Itô's formula, it can verify that (5.1) has an analytical solution as

$$(y_t,z_t)=(\sin(t+W_t+x),\cos(t+W_t+x)). \tag{5.2}$$

In the following numerical experiments, the computational results at $t=0$ are given. For the sake of brevity, the notations $\|e_y^0\|$, $\|e_z^0\|$, $\|e_y^0\|_1$, $\|e_z^0\|_1$ are used to represent $\|y_0-y_h^0\|$, $\|y_0-y_h^0\|_1$, $\|z_0-z_h^0\|$, $\|z_0-z_h^0\|_1$, which are the errors between the exact solution (y_0,z_0) of (1.1) and the discrete solution (y_h^0,z_h^0) of Scheme 3.1.

For the time convergence tests, we set the terminal time $T=1$ and choose the P_2 finite element space with a fixed space mesh size $h=1/160$. A sequence of tests are then performed with $\Delta t=\frac{1}{2^i}, i=5,6,7,8$. The computational results with four sets of parameters $\{\Theta_i\}_{i=1}^4$ are presented in Table 1. The upper two tables in Table 1 show that the numerical Scheme 3.1 with parameters $\Theta_1=(1,1,1,1,1,1/2)$, $\Theta_2=(1,1,1/2,1/2,1,1/2)$, for y and z are all first-order convergence in time. The lower two tables in Table 1 illustrate that for parameters $\Theta_3=(1/2,1/2,1/2,1/2,1/2,1/4)$, $\Theta_4=(1/2,1/2,1/2,1/2,1/2,1/5)$, the second-order convergence rates are obtained in time for y and z.

To check the space convergence of Scheme 3.1, we consider two finite element spaces with P_1 and P_2 basis functions, respectively. With a sufficient small $\Delta t=10^{-4}$ and the second order time scheme with parameter Θ_3, the numerical tests are carried out on a sequence spatial triangulations with $h_i=\frac{1}{2^i}$, $i=2,3,4,5$. The terminal time instant T is set as $10^3\Delta t$. The upper table in Table 2 shows the computational results with P_1 finite element space, from which it can be seen that the optimal convergence rates 2 and 1 in space are both obtained in L^2 and H^1 norms, respectively. The computational results in lower one are gotten with P_2 finite element space. It shows that the convergence rates 3 and 2 in space are also obtained in L^2 and H^1 norms. The computational results in Tables 1 and 2 show that all the convergence rates in space and time are consistent with the theoretical analysis results in Theorem 4.3.

TABLE 1. Errors and convergence rates in time for Example 1.

		$\theta_1 = 1, \theta_2 = 1, \theta_3 = 1,$ $\theta_4 = 1, \theta_5 = 1, \theta_6 = 1/2$		
$1/\Delta t$	$\|y_0 - y^0\|$	$\|z_0 - z^0\|$	$\|y_0 - y^0\|_1$	$\|z_0 - z^0\|_1$
32	8.4889E−04	5.8388E−02	3.0020E−03	1.8350
64	3.9177E−04	2.9198E−02	1.3934E−03	9.2036E−01
128	3.9177E−04	1.4593E−02	6.66461E−04	4.6074E−01
256	9.0866E−05	7.2940E−03	3.2522E−04	2.3049E−01
TCR	1.0597	0.9871	1.0540	0.9845
		$\theta_1 = 1, \theta_2 = 1, \theta_3 = 1/2,$ $\theta_4 = 1/2, \theta_5 = 1, \theta_6 = 1/2$		
32	9.8036E−04	2.9114E−02	3.2819E−03	9.2117E−01
64	5.0322E−04	1.4572E−02	1.6832E−03	4.6090E−01
128	2.5497E−04	7.2900E−03	8.5251E−04	2.3052E−01
256	1.2834E−04	3.6459E−03	4.2902E−04	1.1528E−01
TCR	0.9652	0.9859	0.9659	0.9862
32	3.7794E−05	4.0674E−04	1.2767E−04	1.9769E−02
64	1.0050E−05	1.0682E−04	3.3947E−05	5.0288E−03
128	2.5893E−06	2.7385E−05	8.8247E−06	1.2686E−03
256	6.5703E−07	6.9334E−06	2.5405E−06	3.1862E−04
TCR	1.9238	1.9328	1.8643	1.9590
		$\theta_1 = 1/2, \theta_2 = 1/2, \theta_3 = 1/2,$ $\theta_4 = 1/2, \theta_5 = 1/2, \theta_6 = 1/5$		
32	3.3347E−05	3.4479E−04	1.1234E−04	1.6609E−02
64	8.7318E−06	8.9613E−05	2.9429E−05	4.2084E−03
128	2.2331E−06	2.2847E−05	7.6199E−06	1.0594E−03
256	5.6461E−07	5.7683E−06	2.2702E−06	2.6580E−04
TCR	1.9361	1.9416	1.8582	1.9623

TABLE 2. Errors and convergence rates in space for Example 1.

		P_1 finite element space		
$1/h$	$\|y_0 - y^0\|$	$\|z_0 - z^0\|$	$\|y_0 - y^0\|_1$	$\|z_0 - z^0\|_1$
4	3.5761E−03	5.8982E−03	3.7766E−02	6.1971E−02
8	9.0349E−04	1.4838E−03	1.8853E−02	3.0827E−02
16	2.2648E−04	3.7156E−04	9.4227E−03	1.5393E−02
32	5.6681E−05	9.2970E−05	4.7108E−03	7.6943E−03
SCR	1.9934	1.9960	1.0010	1.0031
		P_2 finite element space		
4	7.6695E−05	4.6752E−05	1.9891E−03	1.2112E−03
8	9.5814E−06	5.8633E−06	4.9677E−04	3.0375E−04
16	1.1983E−06	7.3761E−07	1.2416E−04	7.5998E−05
32	1.5473E−07	1.1350E−07	3.1039E−05	1.9004E−05
SCR	2.9859	2.9049	2.0006	1.9981

TABLE 3. Errors and convergence rates in time for Example 2.

$1/\Delta t$	$\|y_0 - y^0\|$	$\|z_0 - z^0\|$	$\|y_0 - y^0\|_1$	$\|z_0 - z^0\|_1$
		$\theta_1 = 1, \theta_2 = 1, \theta_3 = 1,$ $\theta_4 = 1, \theta_5 = 1, \theta_6 = 1/2$		
10	3.7847E−04	7.7300E−03	1.7783E−03	2.4768E−01
15	2.5626E−04	5.2816E−03	1.2057E−03	1.7009E−01
20	1.9357E−04	4.0111E−03	9.1150E−04	1.2955E−01
25	1.5549E−04	3.2335E−03	7.3262E−04	1.0463E−01
TCR	0.9706	0.9508	0.9676	0.9400
		$\theta_1 = 1, \theta_2 = 1, \theta_3 = 1/2,$ $\theta_4 = 1/2, \theta_5 = 1, \theta_6 = 1/2$		
10	1.8366E−04	4.0495E−03	8.7058E−04	1.3147E−01
15	1.2275E−04	2.7242E−03	5.8241E−04	8.8714E−02
20	9.2111E−05	2.0518E−03	4.3727E−04	6.6927E−02
25	7.3704E−05	1.6456E−03	3.5002E−04	5.3731E−02
TCR	0.9965	0.9826	0.9944	0.9764
		$\theta_1 = 1/2, \theta_2 = 1/2, \theta_3 = 1/2,$ $\theta_4 = 1/2, \theta_5 = 1/2, \theta_6 = 1/4$		
10	7.2486E−06	2.4293E−04	3.9301E−05	8.8675E−03
15	3.2891E−06	1.1427E−04	1.8298E−05	4.2065E−03
20	1.8676E−06	6.6291E−05	1.0726E−05	2.4465E−03
25	1.2017E−06	4.3247E−05	7.2444E−06	1.6058E−03
TCR	1.9609	1.8828	1.8499	1.8651
		$\theta_1 = 1/2, \theta_2 = 1/2, \theta_3 = 1/2,$ $\theta_4 = 1/2, \theta_5 = 1/2, \theta_6 = 1/5$		
10	6.0245E−06	2.0991E−04	3.3193E−05	7.6969E−03
15	2.7157E−06	9.7771E−05	1.5394E−05	3.6127E−03
20	1.5378E−06	5.6405E−05	9.0714E−06	2.0876E−03
25	9.8811E−07	3.6665E−05	6.2114E−06	1.3659E−03
TCR	1.9728	1.9036	1.8353	1.8875

Example 2 (2-dimensional case)**.** In this test, we consider a 2-dimensional case with space domain $D = [0,1] \times [0,1]$. Given an analytical solution with $(x_1, x_2) \in D$

$$(y_t, z_t) = \left(\frac{\exp\left(\frac{t}{4} + W_t + x_1 + x_2\right)}{1 + \exp\left(\frac{t}{4} + W_t + x_1 + x_2\right)}, \frac{\exp\left(\frac{t}{4} + W_t + x_1 + x_2\right)}{\left(1 + \exp\left(\frac{t}{4} + W_t + x_1 + x_2\right)\right)^2} \right), \tag{5.3}$$

a 2-dimensional nonlinear BSPDE is written as

$$-\mathrm{d}y_t - \kappa \Delta y_t \,\mathrm{d}t = f(t, x, W_t, y_t, \nabla y_t, z_t)\,\mathrm{d}t - z_t \,\mathrm{d}W_t, \tag{5.4}$$

from which the associated boundary conditions and the terminal conditions can be consequently induced. Here, $\kappa = 0.01$ and $f(t, x, W_t, y_t, \nabla y_t, z_t) = (1 + 2\kappa) y_t z_t - \frac{3}{4} z_t - 2\kappa \frac{\exp(\frac{t}{4} + W_t + x_1 + x_2)}{(1 + \exp(\frac{t}{4} + W_t + x_1 + x_2))^3}$.

In this experiment, the domain D is first uniformly constructed by partitioning the square domain D into $n \times n$ uniform subrectangles and then dividing each square element into two triangles along the diagonal lines. The mesh size of $\mathcal{T}_h$ is denoted by $h = 1/n$. The configurations of time discretization parameters Θ and the

TABLE 4. Errors and convergence rates in space for Example 2.

$1/h$	$\|y_0 - y^0\|$	$\|z_0 - z^0\|$	$\|y_0 - y^0\|_1$	$\|z_0 - z^0\|_1$
		P_1 finite element space		
2	5.8109E−03	3.6118E−03	3.8333E−02	2.5322E−02
4	1.4935E−03	9.4530E−04	1.9079E−02	1.3080E−02
8	3.7626E−04	2.3959E−04	9.5289E−03	6.5892E−03
16	9.4253E−05	6.0121E−05	4.7631E−03	3.3006E−03
SCR	1.9827	1.9706	1.0027	0.9808
		P_2 finite element space		
2	1.7558E−04	3.3237E−04	2.5681E−03	4.9953E−03
4	2.2445E−05	4.1309E−05	6.7493E−04	1.2475E−03
8	2.8259E−06	5.1524E−06	1.7104E−04	3.1214E−04
16	3.5427E−07	6.4383E−07	4.2916E−05	7.8065E−05
SCR	2.9849	3.0039	1.9690	1.9998

terminal time instants are the same with those in Example 1 and the computational results at time instant $t = 0$ are then given in Tables 3 and 4.

Table 3 lists the time convergence information at $t = 0$. With fixed mesh size $h = 1/2^6$, a sequence of simulations are done with $\Delta t = 1/10, 1/15, 1/20, 1/25$. The upper two tables in Table 3 are computed with Θ_1, Θ_2, from which they show that the rates of convergence 1 in time are obtained. The lower two tables in Table 3 illustrate that with Θ_3 and Θ_4, we gain two order of convergence rate in time with both L^2 and H^1 norms, respectively.

The spatial convergence results are presented in Table 4, which are computed with a sequence spatial triangulations of $h_i = \frac{1}{2^i}$, $i = 1, 2, 3, 4$, and fixed $\Delta t = 10^{-4}$. The upper table in Table 4 lists the convergence errors and rates with P_1 finite element space which shows that the optimal 2 and 1 order of convergence rates in space with L^2 and H^1, respectively, are gained. The computational results with P_2 finite element space are also given in lower table in Table 4. We see that the optimal space convergence rates are 3, 2 in L^2 and H^1 norms, which is what we expected.

The above numerical tests demonstrate that the accuracy of the Scheme 3.1 for solving the nonlinear BSPDEs (1.1) with one and two-dimensional physical space. All the computational results are consistent with our theoretical results.

6. Conclusions

In this paper, we proposed an effectively generalized finite element θ-scheme for solving nonlinear BSPDEs and rigorously derived its error estimates. The process of calculating the proposed scheme can efficiently occur in parallel to handle the large-scale computational complexity of the BSPDEs. The numerical results demonstrate the feasibility and effectiveness of proposed finite element θ-scheme and are consistent with the theoretical results. The analytical and simulation techniques we used can be directly applied to more complicated BSPDEs.

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