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ERROR ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF THE INSTATIONARY NAVIER–STOKES EQUATIONS

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Abstract. In this work we consider the two dimensional instationary Navier–Stokes equations with homogeneous Dirichlet/no-slip boundary conditions. We show error estimates for the fully discrete problem, where a discontinuous Galerkin method in time and inf-sup stable finite elements in space are used. Recently, best approximation type error estimates for the Stokes problem in the $L^\infty(I;L^2(\Omega))$, $L^2(I;H^1(\Omega))$ and $L^2(I;L^2(\Omega))$ norms have been shown. The main result of the present work extends the error estimate in the $L^\infty(I;L^2(\Omega))$ norm to the Navier–Stokes equations, by pursuing an error splitting approach and an appropriate duality argument. In order to discuss the stability of solutions to the discrete primal and dual equations, a specially tailored discrete Gronwall lemma is presented. The techniques developed towards showing the $L^\infty(I;L^2(\Omega))$ error estimate, also allow us to show best approximation type error estimates in the $L^2(I;H^1(\Omega))$ and $L^2(I;L^2(\Omega))$ norms, which complement this work.

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1. Introduction

In this paper, we consider the instationary Navier–Stokes equations in two space dimensions with homogeneous boundary conditions, *i.e.*,

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } I\times\Omega, \\ \nabla\cdot u = 0 & \text{in } I\times\Omega, \\ u(0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } I\times\partial\Omega. \end{cases} \tag{1}$$

Here $\nu > 0$ denotes the viscosity, $I = (0,T] \subset \mathbb{R}$ a bounded, half-open interval for some fixed finite endtime $T > 0$, and $\Omega \subset \mathbb{R}^2$ a bounded convex polygonal domain. The equations are discretized in time by a discontinuous Galerkin (dG) method, *i.e.*, the solution is approximated by piecewise polynomials in time, defined on subintervals of I, without any requirement of continuity at the time nodes, see, *e.g.*, [24, 49]. The parameter

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indicating the time discretization will be denoted by k and corresponds to the length of the largest subinterval in the partition of I. To discretize in space, we use inf-sup stable pairs of finite element spaces for the velocity and pressure components. The parameter indicating the spatial discretization will be denoted by h and corresponds to the largest diameter of cells in the mesh.

Due to the variational formulation of the dG time discretization, this discretization scheme is particularly suited for treating optimal control problems. See, *e.g.*, [13, 14] for optimal control problems governed by the Navier–Stokes equations, also [37,42] for optimal control of general parabolic problems. While in [13,14] the focus was put on low order schemes, recently the authors of [2] analyzed dG schemes of arbitrary order for the Navier–Stokes equations. Another advantage of the dG time discretization is the fact, that the maximal parabolic regularity, exhibited by parabolic problems, is preserved on the discrete level, and moreover can be extended to the limiting cases L^1 and L^∞ in time, at the expense of a logarithmic factor, see Theorems 11 and 12 of [35]. The natural energy norm for the Navier–Stokes equations is the norm of the space $L^\infty(I; L^2(\Omega)^2) \cap L^2(I; H^1(\Omega)^2)$. Indeed by formally testing (1) with the solution u, one obtains

$$\|u\|_{L^\infty(I;L^2(\Omega))} + \|u\|_{L^2(I;H^1(\Omega))} \le C\big(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^1(I;L^2(\Omega))}\big).$$

This bound is also preserved on the discrete level, *i.e.*, holds for the fully discrete solution u_{kh}, see Theorem 4.13. Our main goal in writing this paper, was the investigation of the discretization error in terms of the $L^\infty(I; L^2(\Omega))$-norm. In particular, such estimates are required for the analysis of optimal control problems, subject to state constraints pointwise in time, see [42], as the corresponding Lagrange multiplier in this case is a measure in time. There have been numerous approaches in the literature, deriving error estimates for the Navier–Stokes equations. In Theorem 4.7 of [13], the same combination of dG–cG discretization schemes was used, and for $f \in L^2(I; L^2(\Omega)^2)$ an estimate

$$\|u - u_{kh}\|_{L^\infty(I;L^2(\Omega))} \le C(\sqrt{k} + h) \tag{2}$$

was shown. Additional terms arise, when changes in the spatial mesh on different time intervals are permitted. Under the much stricter assumption $f \in W^{1,\infty}(I; L^2(\Omega)^2)$ and for the implicit Euler time discretization, in [31] the estimate

$$\|u - u_{kh}\|_{L^\infty(I;L^2(\Omega))} \le C\big(k + h^2\big) \tag{3}$$

was shown. It was extended to the Crank-Nicholson scheme in [32] under the assumption of $f \in W^{2,\infty}(I; L^2(\Omega)^2)$, yielding an error estimate at the time nodes of order $\mathcal{O}(k^2 + h^2)$. Beyond the references mentioned above, there are many works analyzing the stationary equations, *e.g.*, [27,45], semidiscrete equations in space, *e.g.*, [9,20], or semidiscrete equations in time, *e.g.*, [4,23,46]. Fully discrete error estimates for stabilized discretization schemes can be found *e.g.*, in [2, 5, 18, 34]. The error estimates are most often derived assuming the necessary regularity of u, such that the discretization schemes can exhibit their full approximative power. The main drawback of the results in the literature is the fact, that usually the errors in $L^\infty(I; L^2(\Omega))$ and $L^2(I; H^1(\Omega))$ are estimated in a combined fashion. Thus the estimate always has to account for the spatial error in the H^1 norm, yielding an order reduction for the estimate of the error in the spatial L^2 norm. The goal of this paper is to prove an error estimate which can be formulated as a best approximation type error estimate, which thus estimates the error in the $L^\infty(I; L^2(\Omega))$ norm in an isolated manner. Such an estimate for the instationary Stokes equations, has been derived recently in [7]. More specifically, for velocity and pressure fields (w, r), solving the instationary Stokes equations on the continuous level, and their fully discrete approximations (w_{kh}, r_{kh}), it holds

$$\|w - w_{kh}\|_{L^\infty(I;L^2(\Omega))} \le C \ln\left(\frac{T}{k}\right)\left(\inf_{\chi_{kh}\in V_{kh}} \|w - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} + \big\|w - R_h^S(w, r)\big\|_{L^\infty(I;L^2(\Omega))}\right),$$

where V_{kh} is the space of discretely divergence free space-time finite element functions. The operator R_h^S denotes a stationary Stokes projection, see [7] or Section 5 for a formal definition. Using this Stokes result, we will show that a best approximation type result also holds for the nonlinear Navier–Stokes equations. This main result is stated in Theorem 5.6. In Corollary 5.8, for $f \in L^\infty(I; L^2(\Omega)^2)$, we then obtain an estimate in terms of

$\mathcal{O}(l_k(k+h^2))$, where l_k denotes a logarithmic term depending on k. This result provides a better order of convergence compared to the estimate (2), shown in [13]. The order of convergence in the estimate (3), that was presented in [31] is comparable, but requires a much stronger regularity assumption. The main tools used in this paper for proving the proposed $L^\infty(I;L^2(\Omega))$ error estimate are an error splitting approach and a bootstrapping argument, to apply the corresponding error estimate for the Stokes equations to the first part of the error. In order to apply such an argument, understanding the precise regularity of the occurring nonlinear term $(u\cdot\nabla)u$ is crucial. The second part of the error will be estimated by a duality argument. This is possible due to the variational nature of the dG time discretization. We will derive a stability result for a discrete dual equation. For this result, a specially adapted version of a discrete Gronwall lemma, Lemma 4.11, will be presented. For the analysis of the discrete dual problem, we require an estimate in the $L^2(I;H^1(\Omega))$ norm, which is why we formulate this result first. We shall then show the error estimate in $L^\infty(I;L^2(\Omega))$ with the tools presented above. The proof of the error estimate in the $L^2(I;L^2(\Omega))$ norm is then straightforward and concludes our work. Summarizing, our main results read

$$
\begin{aligned}
\|u-u_{kh}\|_{L^2(I;H^1(\Omega))} &\le C\Big(\sqrt{k}+h\Big) && \text{if } f\in L^2(I;L^2(\Omega)^2) \text{ and } u_0\in V,\\
\|u-u_{kh}\|_{L^\infty(I;L^2(\Omega))} &\le C\ln(T/k)^2\big(k+h^2\big) && \text{if } f\in L^\infty(I;L^2(\Omega)^2) \text{ and } u_0\in V\cap H^2(\Omega)^2,\\
\|u-u_{kh}\|_{L^2(I;L^2(\Omega))} &\le C\big(k+h^2\big) && \text{if } f\in L^2(I;L^2(\Omega)^2) \text{ and } u_0\in V,
\end{aligned}
$$

and can be found in Theorems 4.16, 5.6, 5.7 and Corollaries 5.8, 5.9. All three results are, up to logarithmic terms, optimal in terms of order of convergence and in terms of required regularity. The structure of this paper will be as follows. First, we fix some notation and function spaces in Section 2. We proceed in Section 3 by stating the appropriate weak formulations of (1) with and without pressure and recall some known regularity results. We conclude the section with an analysis of the regularity properties of the nonlinear term $(u\cdot\nabla)u$. Section 4 will be devoted to the spatial and temporal discretizations. We present a version of a discrete Gronwall lemma, which allows us to treat problems with right hand sides that are only L^1 in time, and show stability results for fully discrete primal equations. With this result, we derive an error estimate for the Navier–Stokes equations in $L^2(I;H^1(\Omega))$. Lastly, in Section 5, we show stability results for a discrete dual problem, which in the end allows us to show the error estimates in the $L^\infty(I;L^2(\Omega))$ and $L^2(I;L^2(\Omega))$ norms.

2. Preliminary

For a convex, polygonal domain $\Omega\subset\mathbb{R}^2$, $1\le p\le\infty$ and $k\in\mathbb{N}$, we denote by $L^p(\Omega)$, $W^{k,p}(\Omega)$, $H^k(\Omega)$ and $H^1_0(\Omega)$ the usual Lebesgue and Sobolev spaces. The inner product on $L^2(\Omega)$ will be denoted by $(\cdot,\cdot)_\Omega$. The space $L^2_0(\Omega)$ is the subspace of $L^2(\Omega)$, consisting of all functions, that have zero mean. For $s\in\mathbb{R}\backslash\mathbb{N}$, $s>0$ the fractional order Sobolev(-Slobodeckij) space $W^{s,p}(\Omega)$ is defined, see, *e.g.*, [21], as

$$
W^{s,p}(\Omega):=\left\{v\in W^{\lfloor s\rfloor,p}(\Omega):\ \sum_{|\alpha|=\lfloor s\rfloor}\iint_{\Omega\times\Omega}\frac{|D^\alpha v(x)-D^\alpha v(y)|^p}{|x-y|^{(s-\lfloor s\rfloor)p+2}}\,\mathrm{d}x\,\mathrm{d}y<+\infty\right\}.
$$

In case $p=2$ we again use the notation $H^s(\Omega)$. Note that in this case $H^s(\Omega)$ can equivalently be obtained *via* real or complex interpolation of the integer degree spaces $H^k(\Omega)$. This is due to the fact, that in the Hilbert space setting, all resulting Bessel potential spaces $H^s_2(\Omega)$, Besov spaces $B^s_{2,2}(\Omega)$ and Sobolev-(Slobodeckij) spaces $H^s(\Omega)$ coincide, see [50], pp. 12, 39. For X being any function space over Ω, we denote by X^* its topological dual space, and abbreviate the duality pairing by $\langle\cdot,\cdot\rangle_\Omega$. We will also use the notation $H^1_0(\Omega)^*=H^{-1}(\Omega)$. The structure of the Stokes and Navier–Stokes equations requires also the definition of some vector valued spaces, consisting of divergence free vector fields. We denote by $\nabla\cdot$ the divergence operator and introduce the spaces

$$
V:=\overline{\{v\in C^\infty_0(\Omega)^2:\ \nabla\cdot v=0\}}^{H^1(\Omega)}\quad\text{and}\quad H:=\overline{\{v\in C^\infty_0(\Omega)^2:\ \nabla\cdot v=0\}}^{L^2(\Omega)}.
$$

Note that instead of the definition *via* closures, in the case of Ω being bounded and Lipschitz, these spaces are alternatively characterized in the following way, see Chapter 1, Theorems 1.4 and 1.6 of [48]:

$$V = \{v \in H_0^1(\Omega)^2 : \ \nabla \cdot v = 0\} \quad \text{and} \quad H = \{v \in L^2(\Omega)^2 : \ \nabla \cdot v = 0, \ u \cdot n = 0 \text{ on } \partial\Omega\},$$

where by $u \cdot n$ we denote the normal trace of the vector field u. To improve readability, whenever vectorial spaces like $H^1(\Omega)^2$ would arise in the subscript of some norm, we shall drop the outer superscript $(\cdot)^2$. For a Banach space X and $I = (0, T]$ we denote by $L^p(I; X)$ the Bochner space of X valued functions, for which the following norm is finite

$$\|v\|_{L^p(I;X)} = \left(\int_I \|v(t)\|_X^p \,\mathrm{d}t\right)^{1/p},$$

with the usual convention when $p = +\infty$. It holds $L^p(I; L^p(\Omega)) \cong L^p(I \times \Omega)$, and for $p = 2$, we denote the inner product by $(\cdot, \cdot)_{I\times\Omega}$. Whenever X is separable and $1 \le p < \infty$, it holds $(L^p(I; X))^* \cong L^{p^*}(I; X^*)$, where $1/p + 1/p^* = 1$. The duality pairing for such spaces will be denoted by $\langle \cdot, \cdot\rangle_{I\times\Omega}$. By $W^{k,p}(I; X)$ and $H^k(I; X)$ for $k \in \mathbb{N}$ we denote the spaces of functions v satisfying $\partial_t^j v \in L^p(I; X)$, $j = 0, \ldots, k$.

3. Navier–Stokes equations

We start by recalling some regularity results for the Navier–Stokes equations, and we are going to prove some additional results, especially adapted to the situation considered in this paper. Throughout this paper, we shall always assume the convexity of Ω. We will state explicitly, whenever results also hold in a more general setting. It is a well known result, that for $f \in L^2(I; V^*) + L^1(I; L^2(\Omega)^2)$ and $u_0 \in H$, there exists a unique weak solution, *i.e.*, the following Proposition holds, see Chapter 3, Theorem 3.1 and Remark 3.1 of [48]:

Proposition 3.1. *Let $f \in L^2(I; V^*) + L^1(I; L^2(\Omega)^2)$ and $u_0 \in H$. Then there exists a unique weak solution $u \in L^2(I; V) \cap C(\bar{I}; L^2(\Omega)^2)$ of (1), satisfying*

$$\langle \partial_t u, v\rangle_\Omega + \nu(\nabla u, \nabla v)_\Omega + ((u \cdot \nabla)u, v)_\Omega = (f, v)_\Omega \qquad \text{for all } v \in V \tag{4}$$

in the sense of distributions on I, and $u(0) = u_0$. Moreover, there holds an estimate

$$\|u\|_{L^\infty(I;L^2(\Omega))} + \sqrt{\nu}\|u\|_{L^2(I;V)} \le C\big(\|u_0\|_H + \|f\|_{L^2(I;V^*)+L^1(I;L^2(\Omega))}\big).$$

Note that the constant C above only depends on ν and Ω, but is independent of T, see Theorems V.1.4.2, V.1.5.3, V.3.1.1 of [47]. It is well known, that under the assumptions of Proposition 3.1, the nonlinearity satisfies for $1 \le s, q < 2$:

$$(u \cdot \nabla)u \in L^s\big(I; L^q(\Omega)^2\big) \quad \text{whenever} \quad \frac{1}{s} + \frac{1}{q} \ge \frac{3}{2}, \tag{5}$$

see Lemma V.1.2.1 of [47]. Equation (4) is the weak formulation of (1) in divergence free spaces. The proof of the above proposition relies heavily on the fact, that the trilinear form $c(\cdot, \cdot, \cdot)$ defined by

$$c : H_0^1(\Omega)^2 \times H_0^1(\Omega)^2 \times H_0^1(\Omega)^2 \to \mathbb{R}, \quad c(u, v, w) = ((u \cdot \nabla)v, w)_\Omega,$$

posesses the properties summarized in the following lemma.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open Lipschitz domain, then there holds the estimate*

$$\|v\|_{L^4(\Omega)} \le C\|v\|_{L^2(\Omega)}^{\frac{1}{2}}\|v\|_{H^1(\Omega)}^{\frac{1}{2}} \qquad \text{for all } v \in H^1(\Omega), \tag{6}$$

due to which, the trilinear form $c(\cdot, \cdot, \cdot)$ satisfies for all $u, v, w \in H_0^1(\Omega)^2$:

$$c(u, v, w) \le \|u\|_{L^4(\Omega)}\|\nabla v\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)},$$

$$c(u, v, u) \leq C \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.$$

Let further $\nabla \cdot u = 0$. Then it holds

$$c(u, v, w) = -c(u, w, v) \quad \text{and} \quad c(u, v, v) = 0.$$

Proof. The estimate for the $L^4(\Omega)$ norm can be found in Lemma II.3.2 from [26] and Theorem 3 from [1]. The properties of the trilinear forms are then consequences of Hölder's inequality and integration by parts, and are shown, *e.g.*, in Lemma IX.2.1 from [26]. □

In what follows, we often consider the trilinear form c integrated in time, which we denote by

$$c(\!(u, v, w)\!) := \int_I c(u, v, w)\, \mathrm{d}t.$$

In analyzing the Navier–Stokes equations, the instationary Stokes equations frequently arise as an auxiliary problem. For initial data $u_0 \in H$ and right hand side $f \in L^1(I; L^2(\Omega)^2) + L^2(I; V^*)$, there exists a unique solution $w \in L^2(I; V) \cap L^\infty(I; L^2(\Omega)^2)$ to the Stokes equations

$$\begin{cases} \langle \partial_t w, v\rangle_\Omega + \nu(\nabla w, \nabla v)_\Omega = (f, v)_\Omega \qquad \text{for all } v \in V, \\ \qquad\qquad w(0) = u_0, \end{cases} \tag{7}$$

where the first line of (7) is understood in the sense of distributions on I. We introduce the Stokes operator $A\colon D(A) \to H$ defined by

$$(Aw, v)_\Omega = \nu(\nabla w, \nabla v)_\Omega \qquad \text{for all } v \in V,$$

with domain $D(A) := \left\{v \in V : \Delta v \in L^2(\Omega)^2\right\}$ and the projection operator $\mathbb{P} : L^2(\Omega)^2 \to H$, defined by

$$(\mathbb{P}w, v)_\Omega = (w, v)_\Omega \qquad \text{for all } v \in H,$$

which is called the Helmholtz or Leray projection. Note that with the vector-valued Laplacian

$$-\Delta\colon D(\Delta) \to L^2(\Omega)^2,$$

with domain $D(\Delta) := \left\{v \in H_0^1(\Omega)^2 : \Delta v \in L^2(\Omega)^2\right\}$, the Stokes operator also satisfies the representation

$$A = -\mathbb{P}\Delta.$$

For convex Ω, the domains of the operators introduced above satisfy the representations

$$D(\Delta) = H_0^1(\Omega)^2 \cap H^2(\Omega)^2 \qquad \text{and} \qquad D(A) = V \cap H^2(\Omega)^2,$$

see [17] for the H^2 regularity of the Stokes operator. The Stokes operator A generates an analytic semigroup in H, see [7, 40], also [6] for a detailled general analysis. One important feature of the Stokes problem is the maximal parabolic regularity, which indicates, that both the time derivative $\partial_t w$ and the Stokes operator Aw individually inherit certain regularity properties of f, see Proposition 3.4 below. For homogeneous initial data, this consequence of the analyticity of the semigroup has been shown in [19], see also Chapter IV, Theorem 1.6.3 of [47]. Since our analysis should also treat inhomogeneous initial data u_0, we need to define the proper spaces for the initial data:

$$V_{1-1/s} := \left\{v \in H :\ \|v\|_{V_{1-1/s}} < \infty\right\},$$

see also [7] and Chapter 1, Section 3.3 of [6], where

$$\|v\|_{V_{1-1/s}} := \left(\int_I \|A\exp(-tA)v\|_H^s\, \mathrm{d}t\right)^{1/s} + \|v\|_H.$$

Remark 3.3. Instead of using the spaces $V_{1-1/s}$ explicitly as a requirement for the initial data, we can make use of the following imbedding results: For $1 < s \leq 2$ it holds $V \hookrightarrow V_{1-1/s}$ and for $1 < s < \infty$, it holds $V \cap D(\Delta) \hookrightarrow V_{1-1/s}$, see Remarks 2.8, 2.9 from [7].

The Stokes problem exhibits the following regularity properties, see Proposition 2.6 of [7].

Proposition 3.4 (Maximal parabolic regularity)**.** *Let $1 < s < \infty$, $f \in L^s(I;L^2(\Omega)^2)$ and $u_0 \in V_{1-1/s}$. Then the solution w of the Stokes equations (7) satisfies*

$$\|\partial_t w\|_{L^s(I;L^2(\Omega))} + \|Aw\|_{L^s(I;L^2(\Omega))} \leq C\big(\|f\|_{L^s(I;L^2(\Omega))} + \|u_0\|_{V_{1-1/s}}\big).$$

In the setting of Proposition 3.4, the Stokes problem (7) formulated in divergence free spaces is equivalent to the following velocity-pressure formulation: Find $(w,r) \in \big[L^2(I;H_0^1(\Omega)^2) \cap C(\bar{I};L^2(\Omega)^2)\big] \times L^s(I;L_0^2(\Omega))$ satisfying

$$\langle \partial_t w, v\rangle_{I\times\Omega} + \nu(\nabla w, \nabla v)_{I\times\Omega} - (r, \nabla\cdot v)_{I\times\Omega} + (q, \nabla\cdot w)_{I\times\Omega} = (f,v)_{I\times\Omega} \tag{8}$$

for all $(v,q) \in \big[L^2(I;H_0^1(\Omega)^2) \cap L^\infty(I;L^2(\Omega)^2)\big] \times L^2(I;L_0^2(\Omega))$ and $w(0) = u_0$, see Theorem 2.10 of [7]. Furthermore, the maximal parabolic regularity results of Proposition 3.4 imply the following estimate for the pressure:

$$\|r\|_{L^s(I;L^2(\Omega))} \leq C\big(\|f\|_{L^s(I;L^2(\Omega))} + \|u_0\|_{V_{1-1/s}}\big).$$

Without additional smoothness assumptions, the maximal parabolic regularity for the Stokes problem does not immediately extend to the Navier–Stokes equations. A partial result can be obtained by considering that the maximal parabolic regularity of the Stokes operator was recently extended to the $L^p(\Omega)$ setting, where p in general depends on the smoothness of Ω. For a general Lipschitz domain Ω, Theorem 1.6 of [25] shows that for some $\varepsilon > 0$ and any p, such that $|1/p - 1/2| < 1/4 + \varepsilon$, the maximal parabolic regularity holds. With this, we obtain the following result for the Navier–Stokes equations.

Theorem 3.5. *Let $f \in L^s(I;L^2(\Omega)^2)$ and $u_0 \in V_{1-1/s}$ for some $s > 1$ and $u \in L^2(I;V) \cap L^\infty(I;L^2(\Omega)^2)$ the unique solution of (4). Then for $\gamma := \min\{s, 4/3\}$ it holds*

$$\partial_t u\ ,\ Au \in L^\gamma\Big(I;L^{4/3}(\Omega)^2\Big).$$

Further, there exists a unique $p \in L^\gamma(I;L_0^2(\Omega))$, such that

$$\langle \partial_t u, v\rangle_\Omega + \nu(\nabla u, \nabla v)_\Omega + ((u\cdot\nabla)u, v)_\Omega - (p, \nabla\cdot v)_\Omega + (\nabla\cdot u, q)_\Omega = (f,v)_\Omega$$

for all $v \in H_0^1(\Omega)^2, q \in L_0^2(\Omega)$ in the sense of distributions on I, and $u(0) = u_0$.

Proof. Due to (5) it holds $(u\cdot\nabla)u \in L^{4/3}(I;L^{4/3}(\Omega)^2)$. Since $f \in L^s(I;L^2(\Omega)^2)$, this implies $\tilde{f} := f - (u\cdot\nabla)u \in L^\gamma(I;L^{4/3}(\Omega)^2)$. Since $|3/4 - 1/2| = 1/4$, Theorem 1.6 of [25] shows that the Stokes problem posesses maximal parabolic regularity in $L^{4/3}(\Omega)$, which applied to u_0 and $\tilde{f}$ then yields $\partial_t u, Au \in L^\gamma(I;L^{4/3}(\Omega)^2)$.The existence of a pressure can then be proven as in Theorem 2.10 from [7]. □

The above result only recovers parts of the regularity available for the Stokes problem. Using H^2 regularity, we will now show improved regularity results for the Navier–Stokes equations. Note that from now on we explicitly require the convexity of Ω, whereas Propositions 3.1, 3.4 and Theorem 3.5 also hold for general Lipschitz domains.

Theorem 3.6 (H^2 regularity)**.** *Let $u_0 \in V$ and $f \in L^2(I;L^2(\Omega)^2)$. Then the weak solution u to the Navier–Stokes equations (4) satisfies the improved regularity*

$$u \in L^2\big(I;H^2(\Omega)^2\big) \cap H^1\big(I;L^2(\Omega)^2\big) \hookrightarrow C\big(\bar{I};H^1(\Omega)^2\big),$$

and there exist constants $C_1, C_2 > 0$, depending on ν, Ω but independent of T, such that there hold the bounds

$$\|u\|_{L^\infty(I;H^1(\Omega))} \leq C_1\big(\|u_0\|_V + \|f\|_{L^2(I\times\Omega)}\big)\exp\Big(C_2\Big(\|u_0\|_H^4 + \|f\|_{L^1(I;L^2(\Omega))}^4\Big)\Big),$$
$$\|u\|_{L^2(I;H^2(\Omega))} \leq C_1\big(\|u_0\|_V + \|f\|_{L^2(I\times\Omega)}\big)\Big(1 + \|u_0\|_H^2 + \|f\|_{L^1(I;L^2(\Omega))}^2\Big)\exp\Big(C_2\Big(\|u_0\|_H^4 + \|f\|_{L^1(I;L^2(\Omega))}^4\Big)\Big).$$

Proof. The proof of this result for C^2 domains can be found in Chapter 3, Theorem 3.10 from [48]. Instead of a C^2 boundary, we can also use the H^2 regularity for the Stokes operator on convex, polygonal domains, see, *e.g.*, Theorem 5.5 of [17] or Theorem 2 of [33], to obtain the claimed regularity. The norm bounds are obtained by the Gronwall lemma. □

Note that contrary to the H^2 regularity for the instationary Stokes problem, for the Navier–Stokes problem, even the proofs of the regularities $\partial_t u, Au \in L^2(I; L^2(\Omega)^2)$, that are contained in the above result, require convexity of the domain. We shall state a corresponding estimate for $\|\partial_t u\|_{L^2(I\times\Omega)}$ after discussing the regularity of the nonlinearity.

Remark 3.7. By Hölder's inequality, with this H^2 regularity result, and the imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, we immediately obtain

$$(u\cdot\nabla)u \in L^2\big(I; L^2(\Omega)^2\big).$$

The regularity $u \in C(\bar{I}; H^1(\Omega)^2)$ almost yields that u is uniformly bounded over the whole space time cylinder. However, since in two dimensions $H^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$, the boundedness in space has to be shown *via* an additional argument.

We first show, that the nonlinearity actually posesses more regularity, than what was claimed in Remark 3.7.

Theorem 3.8. *Let the assertions of Theorem 3.6 be satisfied, i.e., $u_0 \in V$ and $f \in L^2(I; L^2(\Omega)^2)$, and let u be the unique solution to the Navier–Stokes equations (4). Then the nonlinearity satisfies*

$$(u\cdot\nabla)u \in L^s\big(I; L^2(\Omega)^2\big) \qquad \text{for any } 1 \leq s < \infty,$$

and for any $0 < \delta \leq \min\{2, s\}$ there holds the estimate

$$\|(u\cdot\nabla)u\|_{L^s(I;L^2(\Omega))} \leq C\|u\|_{L^2(I;H^2(\Omega))}^{\frac{\delta}{s}}\|u\|_{L^\infty(I;H^1(\Omega))}^{2-\frac{\delta}{s}},$$

where the norms of u on the right hand side can be estimated by Theorem 3.6.

Proof. The proof follows the ideas of Theorem V.1.8.2 from [47], however we will use interpolation spaces instead of fractional powers of the Stokes operator. With Hölder's inequality and the Sobolev imbedding $H^{1+\frac{\delta}{s}}(\Omega) \hookrightarrow C(\overline{\Omega})$, it holds for any $s < \infty$ and $\delta > 0$:

$$\|(u\cdot\nabla)u\|_{L^2(\Omega)}^s \leq \|u\|_{L^\infty(\Omega)}^s\|\nabla u\|_{L^2(\Omega)}^s \leq C\|u\|_{H^{1+\frac{\delta}{s}}(\Omega)}^s\|\nabla u\|_{L^2(\Omega)}^s.$$

Since $\delta \leq s$, we can express the space $H^{1+\frac{\delta}{s}}(\Omega)$ as interpolation space $[H^1(\Omega), H^2(\Omega)]_{\delta/s}$, and obtain from ([11], Thm. 1) the estimate

$$\|u\|_{H^{1+\frac{\delta}{s}}(\Omega)} \leq C\|u\|_{H^1(\Omega)}^{1-\frac{\delta}{s}}\|u\|_{H^2(\Omega)}^{\frac{\delta}{s}}.$$

All in all, we see that

$$\|(u\cdot\nabla)u\|_{L^2(\Omega)}^s \leq C\|u\|_{H^2(\Omega)}^{\delta}\|u\|_{H^1(\Omega)}^{2s-\delta},$$

which is integrable in time, since $u \in L^\infty(I; H^1(\Omega)^2) \cap L^2(I; H^2(\Omega)^2)$, by Theorem 3.6 and $\delta \leq 2$. With Hölder's inequality, we obtain the proposed estimate, which concludes the proof. □

Remark 3.9. The previous result $(u \cdot \nabla)u \in L^s(I; L^2(\Omega)^2)$ shows, that the Navier–Stokes equations inherit the maximal parabolic regularity of the Stokes problem, in cases where $f \in L^s(I; L^2(\Omega)^2)$ and $u_0 \in V_{1-1/s}$ for some $2 \le s < \infty$. Especially it also holds $Au, \partial_t u \in L^s(I; L^2(\Omega)^2)$ with

$$\|\partial_t u\|_{L^s(I;L^2(\Omega))} + \|Au\|_{L^s(I;L^2(\Omega))} \le C\big(\|u_0\|_{V_{1-1/s}} + \|f\|_{L^s(I;L^2(\Omega))} + \|(u\cdot\nabla)u\|_{L^s(I;L^2(\Omega))}\big).$$

With Theorem 3.8, and slightly higher regularity of the data, we obtain the boundedness of u in the space-time cylinder:

Theorem 3.10. *Let $\varepsilon > 0$, $f \in L^{2+\varepsilon}(I; L^2(\Omega)^2)$ and $u_0 \in V_{1-1/(2+\varepsilon)}$. Then the unique weak solution u to (1) satisfies additionally*

$$u \in C(\bar{I} \times \bar{\Omega})^2,$$

and for ε sufficiently small, there holds the estimate

$$\|u\|_{L^\infty(I\times\Omega)} \le C\Big(\|u_0\|_{V_{1-1/(2+\varepsilon)}} + \|f\|_{L^{2+\varepsilon}(I;L^2(\Omega))} + \|u\|^{\varepsilon}_{L^2(I;H^2(\Omega))}\|u\|^{2-\varepsilon}_{L^\infty(I;H^1(\Omega))}\Big),$$

where the norms of u on the right hand side can be estimated by Theorem 3.6.

Proof. Applying Theorem 3.8, we observe, that it holds especially $(u \cdot \nabla)u \in L^{2+\varepsilon}(I; L^2(\Omega)^2)$. For ε small enough, such that $\delta := \varepsilon(2+\varepsilon) \le 2$, there moreover holds

$$\|(u\cdot\nabla)u\|_{L^{2+\varepsilon}(I;L^2(\Omega))} \le C\|u\|^{\varepsilon}_{L^2(I;H^2(\Omega))}\|u\|^{2-\varepsilon}_{L^\infty(I;H^1(\Omega))}.$$

By a bootstrapping argument, since $f \in L^{2+\varepsilon}(I; L^2(\Omega)^2)$ and $u_0 \in V_{1-1/\varepsilon}$, we can apply the maximal parabolic regularity of Proposition 3.4, and obtain together with H^2 regularity

$$\partial_t u \in L^{2+\varepsilon}\big(I; L^2(\Omega)^2\big), \qquad u \in L^{2+\varepsilon}\big(I; H^2(\Omega)^2\big).$$

Hence $u \in W^{1,2+\varepsilon}(I; L^2(\Omega)^2) \cap L^{2+\varepsilon}(I; H^2(\Omega)^2)$. With the definitions $p := 2+\varepsilon$, $s := \frac{1}{2+\varepsilon/2}$, $\theta := 1 - \frac{1}{2+\varepsilon/4} = \frac{1}{2} + \frac{2\varepsilon}{8+\varepsilon}$ it holds $1/p < s < 1/2$ and $1/2 < \theta < 1-s$. We can thus apply ([3], Thm. 3), see also Lemma 2.11b of [22], and obtain

$$u \in C^{s-1/p}\Big(\bar{I}; \big(L^2(\Omega), H^2(\Omega)\big)^2_{\theta,1}\Big).$$

Applying Theorems 3.4.1 and 6.4.5 of [8], and Theorem 4.57 of [21], we moreover have

$$u \in C^{s-1/p}\Big(\bar{I}; \big(L^2(\Omega), H^2(\Omega)\big)^2_{\theta,2}\Big) = C^{\frac{\varepsilon}{8+6\varepsilon+\varepsilon^2}}\Big(\bar{I}; H^{1+\frac{4\varepsilon}{8+\varepsilon}}(\Omega)^2\Big) \hookrightarrow C\big(\bar{I}\times\bar{\Omega}\big)^2.$$

Note that here we have used, that for the $L^2(\Omega)$ case, all corresponding Sobolev-Slobodeckij, Besov, Bessel-potential and interpolation spaces coincide, see, *e.g.*, [41]. For an overview over the topic of function spaces, see also [50]. From the used embeddings, we moreover have the proposed estimate. This concludes the proof. □

Corollary 3.11. *Let the assertions of Theorem 3.10 be satisfied, i.e., $f \in L^{2+\varepsilon}(I; L^2(\Omega)^2)$ and $u_0 \in V_{1-1/(2+\varepsilon)}$ for some $\varepsilon > 0$. Then the nonlinear term in the Navier–Stokes equations satisfies*

$$(u\cdot\nabla)u \in L^\infty\big(I; L^2(\Omega)^2\big),$$

and there holds the estimate

$$\|(u\cdot\nabla)u\|_{L^\infty(I;L^2(\Omega))} \le C\bigg(\|u_0\|_{V_{1-1/(2+\varepsilon)}} + \|f\|_{L^{2+\varepsilon}(I;L^2(\Omega))} + \|u\|^{\frac{2}{2+\varepsilon}}_{L^2(I;H^2(\Omega))}\|u\|^{2-\frac{2}{2+\varepsilon}}_{L^\infty(I;H^1(\Omega))}\bigg)\|u\|_{L^\infty(I;H^1(\Omega))}$$

where the norms of u on the right hand side can be estimated by Theorem 3.6.

Proof. This is a direct consequence of Theorem 3.6 and 3.10 and application of Hölder's inequality. □

In the formulation of equation (4), we have used divergence free test functions. Whenever we want to test the equation with functions, that are not divergence free, we have to consider an alternative, equivalent formulation, that includes the pressure. The following theorem guarantees, that we can freely switch between the two formulations, when $u_0 \in V$ and $f \in L^2(I; L^2(\Omega)^2)$.

Theorem 3.12. *Let the assertions of Theorem 3.6 be satisfied, i.e., $u_0 \in V$ and $f \in L^2(I; L^2(\Omega)^2)$. Then there exists a unique solution (u, p) with*

$$u \in L^2\big(I; H^2(\Omega)^2\big) \cap C\big(\bar{I}; V\big),\ \partial_t u \in L^2\big(I; L^2(\Omega)^2\big) \quad \text{and} \quad p \in L^2\big(I; H^1(\Omega) \cap L_0^2(\Omega)\big),$$

such that $u(0) = u_0$ and

$$(\partial_t u, v)_{I\times\Omega} + \nu(\nabla u, \nabla v)_{I\times\Omega} + ((u\cdot\nabla)u, v)_{I\times\Omega} - (p, \nabla\cdot v)_{I\times\Omega} + (q, \nabla\cdot u)_{I\times\Omega} = (f, v)_{I\times\Omega}, \tag{9}$$

for all $(v, q) \in L^2(I; H_0^1(\Omega)^2) \times L^2(I; L_0^2(\Omega))$. Further it holds

$$\|p\|_{L^2(I;H^1(\Omega))} \le C\big(\|u_0\|_V + \|f\|_{L^2(I;L^2(\Omega))} + \|u\|_{L^2(I;H^2(\Omega))}\|u\|_{L^\infty(I;H^1(\Omega))}\big),$$

where the norms of u on the right hand side can be estimated by Theorem 3.6.

Proof. This result can be shown using Theorem 3.8 with the choice $s = 2$, together with a bootstrapping argument. Moving $(u\cdot\nabla)u \in L^2(I; L^2(\Omega)^2)$ to the right hand side and applying Proposition 3.4 yields $\partial_t u, Au \in L^2(I; L^2(\Omega)^2)$. The pressure is then obtained by following the same steps as Theorem 2.10, Corollary 2.11 of [7], where its uniqueness is given by its construction. □

4. Discretization

After discussing the continuous formulation of the Navier–Stokes equations, we now turn towards their discretization.

4.1. Spatial discretization

Let $\{\mathcal{T}_h\}$ denote a family of quasi-uniform triangulations of $\overline{\Omega}$ consisting of closed simplices. The index h denotes the maximum meshsize. We discretize the velocity u by a discrete function space $U_h \subset H_0^1(\Omega)^2$ and the pressure p by the discrete space $M_h \subset L_0^2(\Omega)$, where (U_h, M_h) satisfy the discrete, uniform LBB-condition

$$\sup_{v_h \in U_h} \frac{(\nabla\cdot v_h, q_h)_\Omega}{\|\nabla v_h\|_{L^2(\Omega)}} \ge \beta\|q_h\|_{L^2(\Omega)} \qquad \text{for all } q_h \in M_h,$$

with a constant $\beta > 0$ independent of h. Throughout this work, we will assume the following approximation properties of the spaces U_h and M_h. This assumption is valid, *e.g.*, for Taylor-Hood and MINI finite elements, even on shape regular meshes, see Assumption 7.2 of [7].

Assumption 4.1. *There exist interpolation operators $i_h\colon H^2(\Omega)^2 \cap H_0^1(\Omega)^2 \to U_h$ and $r_h\colon L^2(\Omega) \to M_h$, such that*

$$\begin{aligned} \|\nabla(v - i_h v)\|_{L^2(\Omega)} &\le ch\|\nabla^2 v\|_{L^2(\Omega)} \quad &&\text{for all } v \in H^2(\Omega)^2 \cap H_0^1(\Omega)^2,\\ \|q - r_h q\|_{L^2(\Omega)} &\le ch\|\nabla q\|_{L^2(\Omega)} &&\text{for all } q \in H^1(\Omega). \end{aligned}$$

The (vector valued) discrete Laplacian $\Delta_h : U_h \to U_h$ is defined by

$$(\nabla u_h, \nabla v_h)_\Omega = -(\Delta_h u_h, v_h)_\Omega \qquad \text{for all } v_h \in U_h.$$

We introduce the space V_h of discretely divergence free functions as

$$V_h := \{v_h \in U_h : \ (\nabla \cdot v_h, q_h)_\Omega = 0 \quad \text{for all } q_h \in M_h\}.$$

The L^2 projection onto this space will be denoted by $\mathbb{P}_h \colon L^2(\Omega)^2 \to V_h$, satisfying

$$(\mathbb{P}_h v, \phi_h)_\Omega = (v, \phi_h)_\Omega \qquad \text{for all } \phi_h \in V_h,$$

and allows us to introduce the discrete Stokes operator $A_h \colon V_h \to V_h$, $A_h v_h = -\mathbb{P}_h \Delta_h v_h$. Having defined these discrete spaces and operators, we can now consider the Ritz projection for the stationary Stokes problem. For any $(w, r) \in H_0^1(\Omega)^2 \times L^2(\Omega)$, the projection $(R_h^S(w, r), R_h^{S,p}(w, r)) \in U_h \times M_h$ is defined by

$$\begin{aligned} \big(\nabla\big(w - R_h^S(w,r)\big), \nabla \phi_h\big)_\Omega - \Big(r - R_h^{S,p}(w,r), \nabla \cdot \phi_h\Big)_\Omega &= 0 \qquad \text{for all } \phi_h \in U_h \\ \big(\nabla \cdot \big(w - R_h^S(w,r)\big), \psi_h\big)_\Omega &= 0 \qquad \text{for all } \psi_h \in M_h. \end{aligned} \tag{10}$$

In case that w is discretely divergence free, *i.e.*, $(\nabla \cdot w, \psi_h)_\Omega = 0$ for all $\psi_h \in M_h$, it holds $R_h^S(w, r) \in V_h$. Note that the space V_h is in general not a subspace of the space V of pointwise divergence free functions. This means, that on the discrete space V_h, the form $c(\cdot,\cdot,\cdot)$ does not posess the anti symmetry properties shown in Lemma 3.2. Hence we define, as in [16, 31], an anti symmetric variant:

$$\hat{c} : H_0^1(\Omega)^2 \times H_0^1(\Omega)^2 \times H_0^1(\Omega)^2 \to \mathbb{R}, \quad \hat{c}(u, v, w) = \frac{1}{2} c(u, v, w) - \frac{1}{2} c(u, w, v). \tag{11}$$

Analogously to the trilinear form $c(\!(\cdot,\cdot,\cdot)\!)$, we will use the notation

$$\hat{c}(\!(u, v, w)\!) := \int_I \hat{c}(u, v, w)\, \mathrm{d}t.$$

Remark 4.2. Note that, due to Lemma 3.2, we can equivalently replace $c(\cdot,\cdot,\cdot)$ in the continuous formulation of the Navier–Stokes equations (4) by $\hat{c}(\cdot,\cdot,\cdot)$, as the two forms coincide on V.

By its definition, $\hat{c}(\cdot,\cdot,\cdot)$ now has the following antisymmetric properties on the space V_h, which will later allow us to show the stability of the fully discrete solutions.

Lemma 4.3. *The trilinear form $\hat{c}(\cdot,\cdot,\cdot)$ satisfies*

$$\begin{aligned} &\hat{c}(u_h, v_h, w_h) \le C \|\nabla u_h\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} \|\nabla w_h\|_{L^2(\Omega)} && \text{for all } u_h, v_h, w_h \in U_h, \\ &\hat{c}(u_h, v_h, w_h) = -\hat{c}(u_h, w_h, v_h) && \text{for all } u_h, v_h, w_h \in U_h, \\ &\hat{c}(u_h, v_h, v_h) = 0 && \text{for all } u_h, v_h \in U_h. \end{aligned}$$

Proof. The last two identities are a direct consequence of the definition of $\hat{c}(\cdot,\cdot,\cdot)$. The first estimate follows from Lemma 3.2 and the imbedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. □

Note that due to the above lemma, formally we are still allowed to switch the second and third argument of $\hat{c}(\cdot,\cdot,\cdot)$. The original form $c(\cdot,\cdot,\cdot)$ however had a strict disctinction between the two arguments, as it contains the gradient of the second argument, but only the function values of the third argument. This is of importance when estimating the form in terms of its arguments. In $\hat{c}(\cdot,\cdot,\cdot)$ gradients occur in the second and third argument, thus switching the arguments does not allow us to obtain improved estimates. For this reason, we state the following lemma, which allows us to switch the second and third arguments of $c(\cdot,\cdot,\cdot)$ by introducing an additional term, even if the first argument is not (pointwise) divergence free.

Lemma 4.4. *Let $u, v, w \in H_0^1(\Omega)^2$. Then it holds*

$$c(u, v, w) = -c(u, w, v) - (\nabla \cdot u, v \cdot w)_\Omega.$$

Proof. The proof is simply an application of integration by parts, and can also be seen, *e.g.*, from equation (2.9) of [2]. □

We conclude this subsection on the space discretization by recalling some important interpolation estimates. On the continuous level, applying (6) to the first order derivatives, and using H^2 regularity, yields

$$\|\nabla w\|_{L^4(\Omega)} \le C\|\nabla w\|_{L^2(\Omega)}^{\frac{1}{2}}\|Aw\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } w \in H^2(\Omega)^2. \tag{12}$$

Using the Stokes operator on the right hand side, instead of second order derivatives, allows us to translate this result to the discrete setting. This is facilitated by the following result, showing that for discretely divergence free functions, the discrete Laplacian Δ_h can be bounded in terms of the discrete Stokes operator A_h:

$$\|\Delta_h w_h\|_{L^2(\Omega)} \le C\|A_h w_h\|_{L^2(\Omega)} \qquad \text{for all } w_h \in V_h, \tag{13}$$

see Corollary 4.4 of [30] or Lemma 4.1 of [28]. With this, we can translate (12) to the discrete setting, by considering for some fixed $w_h \in V_h$ the solution $w \in H_0^1(\Omega)^2$ to the continuous problem

$$(\nabla w, \nabla v)_\Omega = (-\Delta_h w_h, v)_\Omega \qquad \text{for all } v \in H_0^1(\Omega)^2.$$

By the stability of the Poisson Ritz projection in $W^{1,4}(\Omega)$, (12) and (13), we then obtain the discrete version of (12).

$$\|\nabla w_h\|_{L^4(\Omega)} \le C\|\nabla w_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|A_h w_h\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } w_h \in V_h. \tag{14}$$

Analogously, the Gagliardo–Nirenberg inequality,

$$\|w\|_{L^\infty(\Omega)} \le C\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\|Aw\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{15}$$

which is a consequence of Theorem 3 from [1] together with H^2 regularity, has a discrete analogon. It can be shown using the standard discrete Gagliardo–Nirenberg inequality

$$\|w_h\|_{L^\infty(\Omega)} \le C\|w_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|\Delta_h w_h\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } w_h \in U_h,$$

which was proven in Lemma 3.3 of [29]. The proof stated there for smooth domains remains the same for convex domains. The discrete version of (15) is then again obtained by applying (13) and reads

$$\|w_h\|_{L^\infty(\Omega)} \le C\|w_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|A_h w_h\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } w_h \in V_h. \tag{16}$$

Straightforward calculations, using the definition of A_h, also give

$$\|\nabla w_h\|_{L^2(\Omega)} \le C\|w_h\|_{L^2(\Omega)}^{\frac{1}{2}}\|A_h w_h\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } w_h \in V_h. \tag{17}$$

With these considerations regarding the spatial discretization, we can now consider the fully discrete Navier–Stokes equations by also discretizing in time.

4.2. Temporal discretization

For discretization in time, we employ the discontinuous Galerkin method of order q (dG(q)), which is also used, *e.g.*, in [7,16,24]. The time interval $I=(0,T]$ is partitioned into M half-open sub-intervals $I_m=(t_{m-1},t_m]$ with $0=t_0<t_1<t_2<\ldots<t_M=T$. We denote each timestep by $k_m=t_m-t_{m-1}$ and for fixed M the maximal timestep by $k:=\max_{1\le m\le M}k_m$, as well as the minimal one by $k_{\min}:=\min_{1\le m\le M}k_m$. If we want to emphasize that I_m belongs to a discretization level k, we denote it by $I_{m,k}$. We make some standard assumptions on the properties on the time discretization:

(1) There are constants $C,\beta>0$ independent of k, such that

$$k_{\min}\ge Ck^{\beta}.$$

(2) There is a constant $\kappa>0$ independent of k, such that for all $m=1,2,\ldots,M-1$

$$\kappa^{-1}\le\frac{k_m}{k_{m-1}}\le\kappa.$$

(3) It holds $k\le\frac{T}{4}$.

A dG(q) function with values in a given Banach space $\mathcal{B}$ is then given as a function in the space

$$X_k^q(\mathcal{B}):=\left\{v\in L^2(I;\mathcal{B}):\ v|_{I_m}\in\mathcal{P}_q(I_m;\mathcal{B})\quad\text{for all } 1\le m\le M\right\},$$

where on each I_m the space $\mathcal{P}_q(I_m;\mathcal{B})$ is given as the space of polynomials in time up to degree q with values in $\mathcal{B}$:

$$\mathcal{P}_q(I_m;\mathcal{B})=\left\{v\in L^2(I_m;\mathcal{B}):\ \exists\, v_0,\ldots,v_q\in\mathcal{B}\text{ s.t. } v=\sum_{j=0}^{q}v_jt^j\right\}.$$

Note that no continuity is required at the time nodes t_m, which is why we use the following standard notations for one sided limits and jump terms:

$$v_m^+:=\lim_{\varepsilon\to0^+}v(t_m+\varepsilon),\qquad v_m^-:=\lim_{\varepsilon\to0^+}v(t_m-\varepsilon),\qquad [v]_m:=v_m^+-v_m^-.$$

We introduce the compact notations

$$V_{kh}:=X_k^q(V_h),\qquad U_{kh}:=X_k^q(U_h),\qquad M_{kh}:=X_k^q(M_h),\qquad Y_{kh}:=U_{kh}\times M_{kh}.$$

Having defined these dG spaces, we introduce the following projection operator in time: $\pi_\tau\colon C(I;L^2(\Omega))\to X_k^q(L^2(\Omega))$ defined by

$$\begin{cases}(\pi_\tau v-v,\varphi)_{I_m\times\Omega}=0 & \text{for all }\varphi\in\mathcal{P}_{q-1}(I_m;L^2(\Omega)),\ \text{if } q>0,\\ \pi_\tau v(t_m^-)=v(t_m^-), & \end{cases}\tag{18}$$

for all $m=1,2,\ldots,M$. In case $q=0$, the projection operator is defined solely by the second condition.

Remark 4.5. In this paper we will restrict ourselves to the two lowest cases $q=0$ and $q=1$. Since we work in a setting of low regularity of the right hand side f, the error estimates would not benefit from higher order schemes. Also, since the Navier–Stokes equations already pose a challenging large system to solve, higher order schemes in many applications are not feasible from the standpoint of computational cost.

We can now introduce the time-discretized formulation of the Navier–Stokes equations. We define the time-discrete bilinear form for the transient Stokes equations as in [7] by

$$\mathfrak{B}(u,v) := \sum_{m=1}^{M} (\partial_t u, v)_{I_m\times\Omega} + \nu(\nabla u, \nabla v)_{I\times\Omega} + \sum_{m=2}^{M} \left([u]_{m-1}, v_{m-1}^+\right)_\Omega + \left(u_0^+, v_0^+\right)_\Omega.$$

Since we frequently will test some discrete equations with their respective solutions, let us recall the following lemma.

Lemma 4.6. *For any $v_k \in X_k^q(L^2(\Omega))$ it holds*

$$\begin{aligned}(\partial_t v_k, v_k)_{I_m\times\Omega} + \Big([v_k]_{m-1}, v_{kh,m-1}^+\Big)_\Omega &= \frac{1}{2}\Big(\|v_{k,m}^-\|_{L^2(\Omega)}^2 + \|[v_k]_{m-1}\|_{L^2(\Omega)}^2 - \|v_{k,m-1}^-\|_{L^2(\Omega)}^2\Big),\\ -(v_k, \partial_t v_k)_{I_m\times\Omega} - \Big(v_{k,m}^-, [v_k]_m\Big)_\Omega &= \frac{1}{2}\Big(\|v_{k,m-1}^+\|_{L^2(\Omega)}^2 + \|[v_k]_m\|_{L^2(\Omega)}^2 - \|v_{k,m}^+\|_{L^2(\Omega)}^2\Big).\end{aligned}$$

Proof. For the first equality, we can express the integral over the time derivatives *via*

$$(\partial_t v_k, v_k)_{I_m\times\Omega} = \frac{1}{2}\|v_{k,m}^-\|^2 - \frac{1}{2}\|v_{k,m-1}^+\|^2.$$

Writing $v_{k,m-1}^+ = [v]_{k,m-1} + v_{k,m-1}^-$ and recombining terms gives the first identity. The proof of the second equality works completely anologous. □

The fully discrete formulation of the transient Navier–Stokes equations, using the anti symmetrized trilinear form $\hat{c}(\cdot,\cdot,\cdot)$, introduced in (11), is now given as: Find $u_{kh} \in V_{kh}$, such that

$$\mathfrak{B}(u_{kh}, v_{kh}) + \hat{c}(\!(u_{kh}, u_{kh}, v_{kh})\!) = (f, v_{kh})_{I\times\Omega} + \Big(u_0, v_{kh,0}^+\Big)_\Omega \qquad \text{for all } v_{kh} \in V_{kh}. \tag{19}$$

Since in general $V_h \not\subset V$, the discrete solution $u_{kh} \in V_{kh}$ is not divergence free, and thus, we are not allowed to use it as a test function for the divergence-free continuous formulation (4). Because of this, we introduce an equivalent formulation with pressure: Find $(u_{kh}, p_{kh}) \in Y_{kh}$, such that for all $(v_{kh}, q_{kh}) \in Y_{kh}$ it holds

$$B((u_{kh}, p_{kh}), (v_{kh}, q_{kh})) + \hat{c}(\!(u_{kh}, u_{kh}, v_{kh})\!) = (f, v_{kh})_{I\times\Omega} + \Big(u_0, v_{kh,0}^+\Big)_\Omega, \tag{20}$$

where the mixed bilinear form B is defined by

$$\begin{aligned}B((u,p),(v,q)) := &\sum_{m=1}^{M} (\partial_t u, v)_{I_m\times\Omega} + \nu(\nabla u, \nabla v)_{I\times\Omega} + \sum_{m=2}^{M} \left([u]_{m-1}, v_{m-1}^+\right)_\Omega + \left(u_0^+, v_0^+\right)_\Omega\\ &- (\nabla\cdot v, p)_{I\times\Omega} + (\nabla\cdot u, q)_{I\times\Omega}.\end{aligned}$$

Theorem 4.7. *The two formulations (19) and (20) are equivalent in the sense that, if $u_{kh} \in V_{kh}$ satisfies (19), then there exists $p_{kh} \in M_{kh}$ such that (u_{kh}, p_{kh}) solves (20). Conversely, if (u_{kh}, p_{kh}) satisfies (20), then u_{kh} is an element of V_{kh} and satisfies (19).*

Proof. This can be shown by using the same arguments as Proposition 4.3 from [7]. □

4.3. Stokes error estimates

Before further analyzing the fully discrete Navier–Stokes equations, let us recall the discrete formulation of the Stokes equations (8), and the available error estimates that we wish to extend to the Navier–Stokes equations in this work. The fully discrete formulation of the instationary Stokes equation reads: Find $(w_{kh}, r_{kh}) \in Y_{kh}$ satisfying

$$B((w_{kh}, r_{kh}), (\phi_{kh}, \psi_{kh})) = \left(u_0, \phi_{kh,0}^+\right)_\Omega + (f, \phi_{kh})_{I\times\Omega} \qquad \text{for all } (\phi_{kh}, \psi_{kh}) \in Y_{kh}. \tag{21}$$

In the recent contributions [7, 36], best approximation type error estimates for the Stokes problem were shown in the norms of $L^\infty(I; L^2(\Omega))$, $L^2(I; L^2(\Omega))$ and $L^2(I; H^1(\Omega))$. There hold the following results, formulated in terms of best approximation error terms and the errors of the projection in time π_τ, defined in (18), and the Stokes Ritz projection R_h^S, defined in (10).

Proposition 4.8 ([36], Thms. 6.1 and 6.3)**.** *Let $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$, let (w, r) and (w_{kh}, r_{kh}) be the continuous and fully discrete solutions to the Stokes problems (8) and (21). Then for any $\chi_{kh} \in V_{kh}$, there holds*

$$\|w - w_{kh}\|_{L^2(I\times\Omega))} \le C\Big(\|w - \chi_{kh}\|_{L^2(I\times\Omega)} + \|w - \pi_\tau w\|_{L^2(I\times\Omega)} + \left\|w - R_h^S(w, r)\right\|_{L^2(I\times\Omega)}\Big)$$
$$\|\nabla(w - w_{kh})\|_{L^2(I\times\Omega))} \le C\Big(\|\nabla(w - \chi_{kh})\|_{L^2(I\times\Omega)} + \|\nabla(w - \pi_\tau w)\|_{L^2(I\times\Omega)} + \left\|\nabla\big(w - R_h^S(w, r)\big)\right\|_{L^2(I\times\Omega)}\Big).$$

Proposition 4.9 ([7], Cor. 6.4)**.** *Let $f \in L^s(I; L^2(\Omega)^2)$ and $u_0 \in V_{1-1/s}$ for some $s > 1$, and let (w, r) and (w_{kh}, r_{kh}) be the continuous and fully discrete solutions to the Stokes equations (8) and (21). Then for any $\chi_{kh} \in V_{kh}$, there holds*

$$\|w - w_{kh}\|_{L^\infty(I;L^2(\Omega))} \le C\left(\ln\frac{T}{k}\right)\Big(\|w - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} + \left\|w - R_h^S(w, r)\right\|_{L^\infty(I;L^2(\Omega))}\Big).$$

Having shown these error estimates for the Stokes equations, the natural question arises, whether these results can be extended to the Navier–Stokes equations. We first give a positive answer for the $L^2(I; H^1(\Omega))$ error in Theorem 4.16. The main result of this work is the error estimate in the $L^\infty(I; L^2(\Omega))$ norm, presented in Theorem 5.6. With the same techniques, the proof of the $L^2(I; L^2(\Omega))$ error estimate is then straightforward, and we state the result in Theorem 5.7.

4.4. Existence and stability results

We begin our analysis of the fully discrete Navier–Stokes equations by presenting some stability results, followed by an existence result of fully discrete solutions. To show these, let us first recall the following version of a discrete Gronwall lemma, which is stated in Lemma 5.1 of [32] for a constant timestep k, but its proof can easily be adapted to the setting of variable timesteps k_m.

Lemma 4.10. *Let $\{k_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{\gamma_n\}$ be sequences of nonnegative numbers and $B \ge 0$ a constant, such that for each $n \in \mathbb{N}_0$ it holds*

$$a_n + \sum_{m=0}^{n} k_m b_m \le \sum_{m=0}^{n} k_m \gamma_m a_m + \sum_{m=0}^{n} k_m c_m + B,$$

and $k_m\gamma_m < 1$ for all $m \in \{0, \dots, n\}$. Then with $\sigma_m := (1 - k_m\gamma_m)^{-1}$ it holds

$$a_n + \sum_{m=0}^{n} k_m b_m \le \exp\left(\sum_{m=0}^{n} k_m \sigma_m \gamma_m\right) \cdot \left(\sum_{m=0}^{n} k_m c_m + B\right).$$

Proof. By choosing $\widetilde{k} = 1$, $\widetilde{\gamma}_m = k_m\gamma_m$ and $\widetilde{c}_m = k_m c_m$, the quantities with $\widetilde{\cdot}$ correspond to the notation of Lemma 5.1 from [32]. The result then directly follows from this redefinition. $\square$

Continuous and discrete Gronwall lemmas are stated in many different forms in the literature, see, *e.g.*, [38] and the references therein for an overview over different generalizations of the original lemma. Note that, since the sum on the right hand side of the assumed bound goes up to n, the additional assumption $k_m\gamma_m < 1$ is needed. This is not the case in explicit forms of discrete Gronwall lemmas, where the sum on the right goes only up to $n-1$, which is the form most often considered in the literature, *e.g.*, [43]. In the context of dG timestepping schemes, the sequences $\{a_n\}$ and $\{b_n\}$ often correspond to squared norms at time nodes or over subintervals. It therefore seems natural to also state the following version of a Gronwall lemma, where we also include a sum over non-squared contributions. To the best of the authors knowledge, a result like this has not been explicitly used in the literature. In later sections of this work, this lemma will facilitate the analysis of discrete problems with right hand sides that are L^1 in time. Here the norms of the solution occur in a non squared contribution. The following lemma shows, that the weights of the squared sum enter exponentially into the estimate, whereas the weights of the linear sum enter linearly in the estimate for x_n. It can be understood as an adaptation of Bihari's inequality to the discrete setting, see [10, 39]. To improve readability, we drop the explicit mentioning of the timesteps k_m, as we can always apply a transformation as done in the proof of Lemma 4.10.

Lemma 4.11. *Let $\{x_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\gamma_n\}$ be sequences of nonnegative numbers and $B \geq 0$ a constant, such that for each $n \in \mathbb{N}_0$ it holds*

$$x_n^2 + \sum_{m=0}^{n} b_m \leq \sum_{m=0}^{n} \gamma_m x_m^2 + \sum_{m=0}^{n} d_m x_m + \sum_{m=0}^{n} c_m + B,$$

and $\gamma_m < 1$ for all $m \in \{0, \dots, n\}$. Then with $\sigma_m := (1-\gamma_m)^{-1}$ it holds

$$x_n^2 + \sum_{m=0}^{n} b_m \leq 2\exp\left(2\sum_{m=0}^{n} \sigma_m\gamma_m\right) \cdot \left[\left(\sum_{m=0}^{n} d_m\right)^2 + \sum_{m=0}^{n} c_m + B\right].$$

Proof. For $n \in \mathbb{N}_0$ define $\delta_n \geq 0$ such that

$$x_n^2 + \sum_{m=0}^{n} b_m + \delta_n = \sum_{m=0}^{n} \gamma_m x_m^2 + \sum_{m=0}^{n} d_m x_m + \sum_{m=0}^{n} c_m + B, \tag{22}$$

and set $X_n := \sqrt{x_n^2 + \sum_{m=0}^{n} b_m + \delta_n}$. We will show

$$X_n^2 \leq 2\exp\left(2\sum_{m=0}^{n} \sigma_m\gamma_m\right) \cdot \left[\left(\sum_{m=0}^{n} d_m\right)^2 + \sum_{m=0}^{n} c_m + B\right] \qquad \text{for all } n \in \mathbb{N}_0,$$

from which the assertion follows by the definition of X_n. Note that by assumption it holds

$$X_n^2 - X_{n-1}^2 = \gamma_n x_n^2 + d_n x_n + c_n \geq 0 \qquad \text{for all } n \in \mathbb{N},$$

and thus the sequences $\{X_n^2\}$ and $\{X_n\}$ are monotonically increasing. By $X_n \geq x_n$ and the monotonicity of X_n, from (22) we obtain

$$X_n^2 \leq \sum_{m=0}^{n} \gamma_m X_m^2 + \sum_{m=0}^{n} d_m X_m + \sum_{m=0}^{n} c_m + B \quad \leq \left(\sum_{m=0}^{n} \gamma_m X_m + \sum_{m=0}^{n} d_m\right) X_n + \sum_{m=0}^{n} c_m + B. \tag{23}$$

Let us now recall, that for $a, b > 0$, an estimate $x^2 \leq ax + b$ implies $x \leq a + \sqrt{b}$. To see this, we start by computing the roots of the quadratic polynomial, yielding $x \leq \frac{a}{2} + \sqrt{\frac{a^2}{4} + b}$. The root of the polynomial can then be estimated by $\frac{a}{2} + \sqrt{\frac{a^2}{4} + b} \leq a + \sqrt{b}$, which can be shown by subtracting $a/2$ and squaring both sides. Applied to equation (23), we have thus for every $n \in \mathbb{N}_0$ the estimate

$$X_n \leq \sum_{m=0}^{n} \gamma_m X_m + \sum_{m=0}^{n} d_m + \sqrt{\sum_{m=0}^{n} c_m + B} = \sum_{m=0}^{n} \gamma_m X_m + \sum_{m=0}^{n} \tilde{c}_m + \tilde{B},$$

where we have defined $\tilde{B} := 0$ as well as

$$\tilde{c}_0 := \left(d_0 + \sqrt{c_0 + B}\right) \geq 0, \quad \text{and} \quad \tilde{c}_m := \left(d_m + \sqrt{\sum_{j=0}^{m} c_j + B} - \sqrt{\sum_{j=0}^{m-1} c_j + B}\right) \geq 0 \quad \text{for } m > 0.$$

We can thus apply Lemma 4.10 to X_n and obtain after resubstituting the tilded quantities:

$$X_n \leq \exp\left(\sum_{m=0}^{n} \gamma_m \sigma_m\right) \cdot \left(\sum_{m=0}^{n} \tilde{c}_m + \tilde{B}\right) \leq \exp\left(\sum_{m=0}^{n} \gamma_m \sigma_m\right) \cdot \left(\sum_{m=0}^{n} d_m + \sqrt{\sum_{m=0}^{n} c_m + B}\right).$$

Squaring and estimating the square of the sum yields the result. $\square$

With these technical lemmas, we can show stability of the discrete solutions in different norms under different assumptions on f. Solutions to the discrete problem also satisfy the same energy bounds as the weak solutions, *i.e.*, there holds the following proposition, see Lemma 5.1, Theorem A.1 of [16].

Proposition 4.12 (Stability of discrete Navier–Stokes)**.** *Let $f \in L^2(I; H^{-1}(\Omega)^2)$, $u_0 \in H$ and $u_{kh} \in V_{kh}$ satisfy (19). Then there hold the bounds*

$$\|u_{kh}\|_{L^\infty(I;L^2(\Omega))} + \|u_{kh}\|_{L^2(I;H^1(\Omega))} \leq C\big(\|u_0\|_H + \|f\|_{L^2(I;H^{-1}(\Omega))}\big), \qquad \text{if } q = 0,$$
$$\big(\|u_{kh}\|_{L^\infty(I;L^2(\Omega))}\big)^{\frac{1}{2}} + \|u_{kh}\|_{L^2(I;H^1(\Omega))} \leq C\big(\|u_0\|_H + \|f\|_{L^2(I;H^{-1}(\Omega))}\big), \qquad \text{if } q \geq 1,$$

with a constant C depending on the domain Ω and the viscosity ν.

Note that for the case $q \geq 1$, contrary to the continuous setting of Proposition 3.1, due to the exponent $\frac{1}{2}$ on the left hand side, the above proposition states a bound for $\|u_{kh}\|_{L^\infty(I;L^2(\Omega))}$, which depends on the squared norms of the data u_0 and f. For the two low order cases $q = 0$ and $q = 1$, and $f \in L^2(I; L^2(\Omega)^2)$, Lemma 5.1 of [16] also shows an estimate of the form

$$\max_{1 \leq m \leq M} \|u_{kh,m}^-\|_{L^2(\Omega)} + \|u_{kh}\|_{L^2(I;H^1(\Omega))} \leq C_1 e^{C_2 T}\big(\|u_0\|_H + \|f\|_{L^2(I;L^2(\Omega))}\big).$$

With our discrete Gronwall estimate Lemma 4.11, we can generalize this result to $f \in L^1(I; L^2(\Omega)^2)$. Furthermore, by using the version of Gronwall's lemma presented here, contrary to [16], the bound does not grow exponentially in T. This is in agreement with the result of Proposition 3.1 in the continuous setting. We obtain the following result, which now yields an estimate for the norms of u_{kh}, that is linear in terms of the data.

Theorem 4.13. *Let $f \in L^1(I; L^2(\Omega)^2) + L^2(I; H^{-1}(\Omega)^2)$, $u_0 \in H$ and $u_{kh} \in V_{kh}$ satisfy (19) for either $q = 0$ or $q = 1$. Then there holds the bound*

$$\|u_{kh}\|_{L^\infty(I;L^2(\Omega))} + \sqrt{\nu}\|u_{kh}\|_{L^2(I;H^1(\Omega))} \leq C\big(\|u_0\|_H + \|f\|_{L^1(I;L^2(\Omega))+L^2(I;H^{-1}(\Omega))}\big),$$

with a constant C only depending on Ω, ν. If $f \in L^1(I; L^2(\Omega)^2)$, then C only depends on Ω.

Proof. We first prove the result for $f \in L^1(I; L^2(\Omega)^2)$ and remark at the end, which modifications are needed to also cover the case, where f is partly in $L^2(I; H^{-1}(\Omega)^2)$. Note that in the special case of $f \in L^2(I; H^{-1}(\Omega)^2)$ and $q = 0$, this theorem is precisely the first statement of Lemma 5.1 from [16]. For notational simplicity, we use the convention $u_{kh,0}^- := u_0$ and accordingly $[u_{kh}]_0 = u_{kh,0}^+ - u_0$. By testing (19) with $u_{kh}|_{I_m}$ and using Lemma 4.3 we arrive on each time interval at:

$$(\partial_t u_{kh}, u_{kh})_{I_m \times \Omega} + \nu(\nabla u_{kh}, \nabla u_{kh})_{I_m \times \Omega} + \Big([u_{kh}]_{m-1}, u_{kh,m-1}^+\Big)_\Omega = (f, u_{kh})_{I_m \times \Omega}.$$

Applying Lemma 4.6 gives

$$\frac{1}{2}\Big\|u_{kh,m}^-\Big\|_{L^2(\Omega)}^2 + \frac{1}{2}\|[u_{kh}]_{m-1}\|_{L^2(\Omega)}^2 + \nu\|\nabla u_{kh}\|_{L^2(I_m \times \Omega)}^2 = \frac{1}{2}\Big\|u_{kh,m-1}^-\Big\|_{L^2(\Omega)}^2 + (f, u_{kh})_{I_m \times \Omega}.$$

Multiplying by two and summing up the identity over the intervals $1, \dots, n$, we obtain

$$\Big\|u_{kh,n}^-\Big\|_{L^2(\Omega)}^2 + \sum_{m=1}^n \|[u_{kh}]_{m-1}\|_{L^2(\Omega)}^2 + 2\nu\|\nabla u_{kh}\|_{L^2(I_m \times \Omega)}^2 = \Big\|u_{kh,0}^-\Big\|_{L^2(\Omega)}^2 + 2\sum_{m=1}^n (f, u_{kh})_{I_m \times \Omega}. \tag{24}$$

From this point on, we have to treat the two cases $q = 0$ and $q = 1$ separately.

Case 1. If $q = 0$, then it holds $\|u_{kh}\|_{L^\infty(I_m; L^2(\Omega))} = \|u_{kh,m}^-\|_{L^2(\Omega)}$. Thus with Hölder's inequality, the terms in the last sum of (24) can be estimated as

$$(f, u_{kh})_{I_m \times \Omega} \le \|f\|_{L^1(I_m; L^2(\Omega))}\|u_{kh}\|_{L^\infty(I_m; L^2(\Omega))} \le \|f\|_{L^1(I_m; L^2(\Omega))}\Big\|u_{kh,m}^-\Big\|_{L^2(\Omega)}.$$

Hence, from (24) we obtain the following

$$\Big\|u_{kh,n}^-\Big\|_{L^2(\Omega)}^2 + \sum_{m=1}^n \|[u_{kh}]_{m-1}\|_{L^2(\Omega)}^2 + 2\nu\|\nabla u_{kh}\|_{L^2(I_m \times \Omega)}^2 \le \|u_0\|_{L^2(\Omega)}^2 + 2\sum_{m=1}^n \|f\|_{L^1(I_m; L^2(\Omega))}\Big\|u_{kh,m}^-\Big\|_{L^2(\Omega)}.$$

An application of Lemma 4.11 proves the assertion.

Case 2. If $q = 1$, then the $L^\infty(I_m)$ norm can be estimated by the evaluation at the two endpoints of the interval:

$$\|u_{kh}\|_{L^\infty(I_m; L^2(\Omega))} \le \max\bigg\{\Big\|u_{kh,m}^-\Big\|_{L^2(\Omega)}, \Big\|u_{kh,m-1}^+\Big\|_{L^2(\Omega)}\bigg\}.$$

With triangle inequality we can estimate the right sided limit in terms of a left sided limit and a jump:

$$\|u_{kh}\|_{L^\infty(I_m; L^2(\Omega))} \le \max\bigg\{\Big\|u_{kh,m}^-\Big\|_{L^2(\Omega)}, \Big\|u_{kh,m-1}^-\Big\|_{L^2(\Omega)} + \|[u_{kh}]_{m-1}\|_{L^2(\Omega)}\bigg\}.$$

Defining $x_n^2 := \|u_{kh,n}^-\|_{L^2(\Omega)}^2 + \sum_{m=1}^n \|[u_{kh}]_{m-1}\|_{L^2(\Omega)}^2$ for $n = 1, \dots, M$ yields from (24) the estimate

$$x_n^2 + 2\nu\sum_{m=1}^n \|\nabla u_{kh}\|_{L^2(I_m; L^2(\Omega))}^2 \le \|u_0\|_{L^2(\Omega)}^2 + 2\sum_{m=1}^n \|f\|_{L^1(I_m; L^2(\Omega))}(x_m + x_{m-1}),$$

or after an index shift

$$x_n^2 + 2\nu\sum_{m=1}^n \|\nabla u_{kh}\|_{L^2(I_m; L^2(\Omega))}^2 \le \|u_0\|_{L^2(\Omega)}^2 + 2\|f\|_{L^1(I_1; L^2(\Omega))}\|u_0\| + 2\sum_{m=1}^n \|f\|_{L^1(I_m \cup I_{m+1}; L^2(\Omega))} \cdot x_m.$$

In order to shorten the notation, we have added here a term $\|f\|_{L^1(I_{n+1};L^2(\Omega))}x_n$, where in case $n = M$ we use the convention $I_{M+1} = \emptyset$. Again we can apply Lemma 4.11, which yields the estimate

$$x_n^2 + 2\nu \sum_{m=1}^{n} \|\nabla u_{kh}\|^2_{L^2(I_m;L^2(\Omega))} \leq 2\left[\left(2\sum_{m=1}^{n} \|f\|_{L^1(I_m \cup I_{m+1};L^2(\Omega))}\right)^2 + \|u_0\|^2_{L^2(\Omega)} + 2\|f\|_{L^1(I_1;L^2(\Omega))}\|u_0\|\right].$$

Using Young's inequality, to estimate the term $2\|f\|_{L^1(I_1;L^2(\Omega))}\|u_0\|$, concludes the proof. In case $f = f_1 + f_2$, with $f_1 \in L^1(I;L^2(\Omega)^2)$, $f_2 \in L^2(I;H^{-1}(\Omega)^2)$, the contribution of f_1 can be treated as above. For the f_2 contribution, in (24) one applies Hölder's and Young's inequalities, and absorbs the $\|\nabla u_{kh}\|_{L^2(I_m;L^2(\Omega))}$ term to the left. Note that in both cases $q = 0, 1$, in the estimates before the application of Lemma 4.11, no x_m^2 terms are summed on the right hand side, thus no exponential dependency is introduced in the final estimate.

□

The techniques presented above now allow us to show the existence of solutions to the discrete equations (19).

Theorem 4.14. *Let $u_0 \in H$ and $f \in L^1(I;L^2(\Omega)^2) + L^2(I;H^{-1}(\Omega)^2)$. Let further either $q = 0$ or $q = 1$. Then there exists a solution $u_{kh} \in V_{kh}$ of (19).*

Proof. The existence can be shown by applying a standard fixpoint argument, using the stability result of Theorem 4.13. □

Remark 4.15. Note that uniqueness of solutions u_{kh} to (19) is not known at this point. We shall show this later in Theorem 4.19, as we first need to obtain a first convergence result, that allows us to apply Lemma 4.11.

4.5. $L^2(I;H^1(\Omega))$ error estimates

With the stability results presented thus far, we can show a first error estimate in the norm of $L^2(I;H^1(\Omega))$. For this norm, Theorem 5.2 of [16] shows a result, estimating the error for the Navier–Stokes equations in terms of the error of a Stokes problem. The discrete Stokes problem there is defined with right hand side $\partial_t u - \nu\Delta u$, which corresponds to $f - \nabla p - (u\cdot\nabla)u$, *i.e.*, the pressure is included on the right. This means, that when applying the corresponding orthogonality relations, a pressure term remains. In the following result, we use a different right hand side for the discrete Stokes problem, yielding the following error estimate for the Navier–Stokes equations in $L^2(I;H^1(\Omega))$.

Theorem 4.16. *Let $f \in L^2(I;L^2(\Omega)^2)$, $u_0 \in V$ and let (u,p), (u_{kh}, p_{kh}) solve the continuous and fully discrete Navier–Stokes equations (9) and (20) respectively. Further let k be small enough. Then for any $\chi_{kh} \in V_{kh}$, there holds*

$$\|\nabla(u - u_{kh})\|_{L^2(I\times\Omega)} \leq C\Big(\|\nabla(u-\chi_{kh})\|_{L^2(I\times\Omega)} + \|\nabla(u - \pi_\tau u)\|_{L^2(I\times\Omega)} + \big\|\nabla\big(u - R_h^S(u,p)\big)\big\|_{L^2(I\times\Omega)}\Big).$$

Proof. The proof uses the same arguments as the one of Theorem 5.2 from [16]. We repeat the main steps, in order to motivate, why due to the choice of our instationary Stokes projection, no error term for the pressures arises on the right hand side. We denote by $e := u - u_{kh}$ the error, which we want to estimate, and introduce $(\tilde{u}_{kh}, \tilde{p}_{kh}) \in Y_{kh}$ as the instationary Stokes-projection of (u,p), *i.e.*, the solution to

$$B((\tilde{u}_{kh}, \tilde{p}_{kh}), (\phi_{kh}, \psi_{kh})) = (f, \phi_{kh})_{I\times\Omega} - c((u, u, \phi_{kh})) + \Big(u_0, \phi^+_{kh,0}\Big)_\Omega \qquad \text{for all } (\phi_{kh}, \psi_{kh}) \in Y_{kh}. \tag{25}$$

Note that for the definition of this equation, we can equivalently choose c or $\hat{c}$, as we have u with $\nabla \cdot u = 0$ in its first argument. There exists a solution $(\tilde{u}_{kh}, \tilde{p}_{kh})$ to this discrete problem, since the assumptions on f and

u_0 yield $(u\cdot\nabla)u \in L^2(I;L^2(\Omega)^2)$, see Theorem 3.8. Furthermore, due to Theorem 4.10 of [16], the solution $\tilde{u}_{kh}$ to the above discrete Stokes problem satisfies a bound

$$\|\tilde{u}_{kh}\|_{L^\infty(I;H^1(\Omega))} \le C, \tag{26}$$

with C depending on f, u_0, but independent of k, h. We split the velocity error into two components $e = \xi + \eta_{kh}$ where $\xi := u - \tilde{u}_{kh}$ and $\eta_{kh} := \tilde{u}_{kh} - u_{kh}$. The error in the pressure will be denoted by $r := p - p_{kh}$, and we will split it into the same contributions, $r = \omega + \kappa_{kh}$, with $\omega := p - \tilde{p}_{kh}$ and $\kappa_{kh} := \tilde{p}_{kh} - p_{kh}$. The error of the Stokes projection ξ is then bounded by Proposition 4.8 since by Theorem 3.8 it holds $f - (u\cdot\nabla)u \in L^2(I;L^2(\Omega)^2)$. It remains to estimate η_{kh}. By definition, we have the following Galerkin orthogonalities: for all $(\phi_{kh},\psi_{kh}) \in Y_{kh}$ it holds

$$\begin{aligned} B((u-u_{kh}, p-p_{kh}),(\phi_{kh},\psi_{kh})) + \hat{c}((u,u,\phi_{kh})) - \hat{c}((u_{kh},u_{kh},\phi_{kh})) &= 0,\\ B((\xi,\omega),(\phi_{kh},\psi_{kh})) &= 0. \end{aligned}$$

Subtracting the two and choosing $(\phi_{kh},\psi_{kh}) = (\eta_{kh},\kappa_{kh})$ yields

$$B((\eta_{kh},\kappa_{kh}),(\eta_{kh},\kappa_{kh})) + \hat{c}((u,u,\eta_{kh})) - \hat{c}((u_{kh},u_{kh},\eta_{kh})) = 0. \tag{27}$$

The two trilinear forms in (27) satisfy the relation

$$\hat{c}((u,u,\eta_{kh})) - \hat{c}((u_{kh},u_{kh},\eta_{kh})) = \hat{c}((\xi,u,\eta_{kh})) + \hat{c}((\tilde{u}_{kh},\xi,\eta_{kh})) + \hat{c}((\eta_{kh},\tilde{u}_{kh},\eta_{kh})) + \hat{c}((u_{kh},\eta_{kh},\eta_{kh})).$$

Due to Lemma 4.3, the last term on the right hand side vanishes, and for the remaining ones, we obtain by Hölder's and Young's inequalities

$$\begin{aligned}
\hat{c}((\xi,u,\eta_{kh})) &\le \sum_{m=1}^M \|\xi\|_{L^2(I_m;H^1(\Omega))}\|u\|_{L^\infty(I;H^1(\Omega))}\|\eta_{kh}\|_{L^2(I_m;H^1(\Omega))}\\
&\le \sum_{m=1}^M C\|\xi\|^2_{L^2(I_m;H^1(\Omega))}\|u\|^2_{L^\infty(I;H^1(\Omega))} + \frac{\nu}{4}\|\eta_{kh}\|^2_{L^2(I_m;H^1(\Omega))},\\
\hat{c}((\xi,\tilde{u}_{kh},\eta_{kh})) &\le \sum_{m=1}^M \|\xi\|_{L^2(I_m;H^1(\Omega))}\|\tilde{u}_{kh}\|_{L^\infty(I;H^1(\Omega))}\|\eta_{kh}\|_{L^2(I_m;H^1(\Omega))}\\
&\le \sum_{m=1}^M C\|\xi\|^2_{L^2(I_m;H^1(\Omega))}\|\tilde{u}_{kh}\|^2_{L^\infty(I;H^1(\Omega))} + \frac{\nu}{4}\|\eta_{kh}\|_{L^2(I_m;H^1(\Omega))},\\
\hat{c}((\eta_{kh},\tilde{u}_{kh},\eta_{kh})) &\le \sum_{m=1}^M \int_{I_m} \|\eta_{kh}\|^{\frac{3}{2}}_{H^1(\Omega)}\|\eta_{kh}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\tilde{u}_{kh}\|_{H^1(\Omega)}\,\mathrm{d}t\\
&\le \sum_{m=1}^M C\|\tilde{u}_{kh}\|^4_{L^\infty(I;H^1(\Omega))}k_m\|\eta_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))} + \frac{\nu}{4}\|\eta_{kh}\|^2_{L^2(I_m;H^1(\Omega))}.
\end{aligned}$$

Due to u_{kh} and $\tilde{u}_{kh}$ both being discretely divergence free, all pressure contributions in (27) vanish, and with the above estimates, after absorbing all $\frac{\nu}{4}\|\eta_{kh}\|_{L^2(I_m;H^1(\Omega))}$ terms to the left, we obtain

$$\begin{aligned}
\sum_{m=1}^M (\partial_t\eta_{kh},\eta_{kh})_{I_m\times\Omega} + \frac{\nu}{4}(\nabla\eta_{kh},\nabla\eta_{kh})_{I_m\times\Omega} + \sum_{m=1}^M \Big([\eta_{kh}]_{m-1},\eta^+_{kh,m-1}\Big)_\Omega&\\
\le C\Big(\|u\|^2_{L^\infty(I;H^1(\Omega))} + \|\tilde{u}_{kh}\|^2_{L^\infty(I;H^1(\Omega))}\Big)\|\xi\|^2_{L^2(I;H^1(\Omega))}&\\
+ C\|\tilde{u}_{kh}\|^4_{L^\infty(I;H^1(\Omega))}\sum_{m=1}^M k_m\|\eta_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}.&
\end{aligned}$$

To abbreviate notation we have used here $[\eta_{kh}]_0 := \eta_{kh,0}^+$. Choosing k small enough, and using the boundedness of $\|\tilde{u}_{kh}\|_{L^\infty(I;H^1(\Omega))}$, due to (26) as well as of $\|u\|_{L^\infty(I;H^1(\Omega))}$, shown in Theorem 3.6, allows us to apply the discrete Gronwall Lemma 4.11 as in Theorem 4.13, which concludes the proof. □

4.6. Uniqueness and strong stability of discrete solutions

We next show the uniqueness and stability of the discrete solution u_{kh} in stronger norms, comparable to the continuous result of Theorem 3.6. These results will be obtained, by applying the discrete Gronwall Lemma 4.11, to which end we need the two following technical lemmas, guaranteeing, that the coefficients in the Gronwall lemma become arbitrarily small, uniformly in m, as $(k, h) \to 0$.

Lemma 4.17. *Let $I \subset \mathbb{R}$ a bounded interval, $\varepsilon > 0$ and $\zeta \in L^1(I)$. Then*

$$\sup_{x\in \bar{I}} \int_x^{x+\varepsilon} |\zeta(s)| \,\mathrm{d}s \xrightarrow{\varepsilon\to 0} 0.$$

To keep notation simple, we formally extend ζ to $[0, T + \varepsilon]$ by values 0 such that the integration is well defined.

Proof. The proof relies on dominated convergence and Dini's theorem, see, *e.g.*, [12], p. 125. We first define for fixed ε the function $\sigma_\varepsilon : \bar{I} \to \mathbb{R}, x \mapsto \int_x^{x+\varepsilon} |\zeta(s)| \,\mathrm{d}s$. Note that for each $\varepsilon > 0$ this function σ_ε is continuous. To see this, let $x \in I$ be fixed and let $y \to x$. W.l.o.g. let $y > x$:

$$|\sigma_\varepsilon(x) - \sigma_\varepsilon(y)| \le \left| \int_x^y |\zeta(s)| \,\mathrm{d}s + \int_{x+\varepsilon}^{y+\varepsilon} |\zeta(s)| \,\mathrm{d}s \right| = \left| \int_I \chi_{x,y}(s)|\zeta(s)| \,\mathrm{d}s \right|$$

where $\chi_{x,y}(s) = 1$ if $s \in (x, y) \cup (x + \varepsilon, y + \varepsilon)$, and 0 otherwise. For a fixed s, the integrand $\chi_{x,y}(s)|\zeta(s)| \to 0$ as $y \to x$, thus the integrand converges pointwise to 0. By the dominated convergence theorem, this means that the integral converges to 0. For fixed $\varepsilon > 0$ this shows the continuity of σ_ε. Moreover, by the same argument we can show, that for fixed x and $\varepsilon \to 0$ it holds $\sigma_\varepsilon(x) \to 0$. Thus the sequence σ_ε converges pointwise to 0. Note moreover, that for $\delta > \varepsilon$, and fixed x, it holds $\sigma_\delta(x) \ge \sigma_\varepsilon(x)$, as the integration covers a larger interval. We thus have shown, that when $\varepsilon \to 0$ monotonically, also this pointwise convergence of $\sigma_\varepsilon(x)$ is monotone. Hence we can apply Dini's theorem, see [12], p. 125, to obtain

$$\|\sigma_\varepsilon\|_{L^\infty(I)} \to 0, \qquad \text{as } \varepsilon \to 0.$$

□

Lemma 4.18. *Let $u \in L^2(I; H_0^1(\Omega)^2)$, and $u_{kh} \in U_{kh}$ such that $\|u - u_{kh}\|_{L^2(I;H^1(\Omega))} \to 0$ as $(k, h) \to 0$. Then it holds*

$$\sup_{1\le m\le M} \int_{I_{m,k}} \|\nabla u\|_{L^2(\Omega)}^2 \,\mathrm{d}t \xrightarrow{(k,h)\to 0} 0 \quad \text{and} \quad \sup_{1\le m\le M} \int_{I_{m,k}} \|\nabla u_{kh}\|_{L^2(\Omega)}^2 \,\mathrm{d}t \xrightarrow{(k,h)\to 0} 0.$$

Proof. We first show the statement for u. To this end, note that for $k = \max_{1\le m\le M} |I_{m,k}|$

$$\sup_{1\le m\le M} \int_{I_{m,k}} \|\nabla u\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le \sup_{x\in\bar{I}} \int_x^{x+k} \|\nabla u\|_{L^2(\Omega)}^2 \,\mathrm{d}t \xrightarrow{k\to 0} 0$$

by Lemma 4.17. In order to show the result for the discrete solution u_{kh}, we cannot directly apply the previous lemma, as u_{kh} depends on k. We thus insert insert $\pm u$ and apply the triangle inequality to obtain

$$\sup_{1\le m\le M} \int_{I_{m,k}} \|\nabla u_{kh}\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le 2 \sup_{1\le m\le M} \int_{I_{m,k}} \|\nabla u\|_{L^2(\Omega)}^2 \,\mathrm{d}t + 2 \int_I \|\nabla(u - u_{kh})\|_{L^2(\Omega)}^2 \,\mathrm{d}t.$$

With the claim for u and $u_{kh} \to u$ in $L^2(I; H^1(\Omega))$, we have shown the claim for u_{kh}. □

With these technical lemmas, we can show two results, the uniqueness of solutions u_{kh} and their boundedness in stronger norms. We first obtain the following uniqueness result.

Theorem 4.19. *Let $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$. Then for (k,h) small enough, the solution u_{kh} to (19) is unique.*

Proof. To show uniqueness, we assume two solutions u^1_{kh}, u^2_{kh} and define $e_{kh} := u^1_{kh} - u^2_{kh}$. Testing the equations for $m = 1, \dots, M$ with $e_{kh}\chi_m$, where χ_m denotes the characteristic function of the interval I_m, and subtracting them leads to

$$\mathfrak{B}(e_{kh}, e_{kh}\chi_m) + \hat{c}\big((u^1_{kh}, u^1_{kh}, e_{kh}\chi_m)\big) - \hat{c}\big((u^2_{kh}, u^2_{kh}, e_{kh}\chi_m)\big) = 0.$$

Adding and subtracting $\hat{c}\big((u^1_{kh}, u^2_{kh}, e_{kh}\chi_m)\big)$ and applying Lemma 4.3 yields

$$\mathfrak{B}(e_{kh}, e_{kh}\chi_m) + \hat{c}\big((e_{kh}, u^2_{kh}, e_{kh}\chi_m)\big) = 0.$$

Applying Lemma 4.6, the bilinear form can be written as

$$\mathfrak{B}(e_{kh}, e_{kh}\chi_m) = \frac{1}{2}\left(\left\|e^-_{kh,m}\right\|^2_{L^2(\Omega)} + \|[e_{kh}]_{m-1}\|^2_{L^2(\Omega)} - \left\|e^-_{kh,m-1}\right\|^2_{L^2(\Omega)}\right) + \nu\|\nabla e_{kh}\|^2_{L^2(I_m;L^2(\Omega))}.$$

From the definition of $\hat{c}$, the estimates of Lemma 3.2, and Hölder's inequality in time, we obtain

$$\begin{aligned}
\hat{c}\big((e_{kh}, u^2_{kh}, e_{kh}\chi_m)\big) &= \frac{1}{2}\Big[\big((e_{kh}\cdot\nabla)u^2_{kh}, e_{kh}\big)_{I_m\times\Omega} - \big((e_{kh}\cdot\nabla)e_{kh}, u^2_{kh}\big)_{I_m\times\Omega}\Big] \\
&\le C\|e_{kh}\|_{L^\infty(I_m;L^2(\Omega))}\|\nabla e_{kh}\|_{L^2(I_m;L^2(\Omega))}\left\|\nabla u^2_{kh}\right\|_{L^2(I_m;L^2(\Omega))} \\
&\quad + C\|e_{kh}\|^{\frac{1}{2}}_{L^\infty(I_m;L^2(\Omega))}\|\nabla e_{kh}\|^{\frac{3}{2}}_{L^2(I_m;L^2(\Omega))}\left\|u^2_{kh}\right\|^{\frac{1}{2}}_{L^\infty(I_m;L^2(\Omega))}\left\|\nabla u^2_{kh}\right\|^{\frac{1}{2}}_{L^2(I_m;L^2(\Omega))}.
\end{aligned}$$

We can apply Young's inequality and obtain

$$\hat{c}\big((e_{kh}, u^2_{kh}, e_{kh}\chi_m)\big) \le \frac{\nu}{2}\|\nabla e_{kh}\|^2_{L^2(I_m;L^2(\Omega))} + C\|e_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}\left(1 + \left\|u^2_{kh}\right\|^2_{L^\infty(I_m;L^2(\Omega))}\right)\left\|\nabla u^2_{kh}\right\|^2_{L^2(I_m;L^2(\Omega))}$$

where the first summand can be absorbed into $\mathfrak{B}(e_{kh}, e_{kh}\chi_m)$. Thus on each time interval, there holds the bound

$$\begin{aligned}
&\frac{1}{2}\left(\left\|e^-_{kh,m}\right\|^2_{L^2(\Omega)} + \|[e_{kh}]_{m-1}\|^2_{L^2(\Omega)} - \left\|e^-_{kh,m-1}\right\|^2_{L^2(\Omega)} + \nu\|\nabla e_{kh}\|^2_{L^2(I_m;L^2(\Omega))}\right) \\
&\qquad \le C\|e_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}\left(1 + \left\|u^2_{kh}\right\|^2_{L^\infty(I_m;L^2(\Omega))}\right)\left\|\nabla u^2_{kh}\right\|^2_{L^2(I_m;L^2(\Omega))}.
\end{aligned}$$

Multiplying the derived inequality by 2 and summing up the inequalities for $m = 1, \dots, n$ yields for any $n \in \{1, \dots, M\}$

$$\begin{aligned}
&\left\|e^-_{kh,n}\right\|^2_{L^2(\Omega)} + \sum_{m=1}^{n}\left(\|[e_{kh}]_{m-1}\|^2_{L^2(\Omega)} + \nu\|\nabla e_{kh}\|^2_{L^2(I_m;L^2(\Omega))}\right) \\
&\qquad \le C\sum_{m=1}^{n}\|e_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}\left(1 + \left\|u^2_{kh}\right\|^2_{L^\infty(I_m;L^2(\Omega))}\right)\left\|\nabla u^2_{kh}\right\|^2_{L^2(I_m;L^2(\Omega))}.
\end{aligned}$$

Here we have used $e^-_{kh,0} = 0$ as u^1_{kh} and u^2_{kh} satisfy the same initial condition. As in the proof of Theorem 4.13, $\|e_{kh}\|_{L^\infty(I_m;L^2(\Omega))}$ can be bounded by evaluations of one-sided limits at the time nodes. Due to $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$, Theorem 4.16 holds, and thus Lemma 4.18 yields $\|\nabla u^2_{kh}\|^2_{L^2(I_m;L^2(\Omega))} \to 0$ uniformly in m for $(k,h) \to 0$. Together with Theorem 4.13, bounding $\|u^2_{kh}\|_{L^\infty(I_m;L^2(\Omega))}$, we can choose the discretization fine enough, such that $C(1 + \|u^2_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))})\|\nabla u^2_{kh}\|^2_{L^2(I_m;L^2(\Omega))} < \frac{1}{2}$ for all m. All in all, as in Theorem 4.13, an application of the Gronwall Lemma 4.11 shows that $e_{kh} = 0$, concluding the proof of uniqueness. ∎

We now turn toward proving the stability of u_{kh} in stronger norms. The proof follows the steps of the continuous result, shown in Chapter 3, Theorem 3.10 from [48], using the discrete analogons of the inequalities presented in Section 4.1, and the discrete Gronwall Lemma 4.11. For the application of the lemma, we need coefficients, that become small, as $k \to 0$. These coefficients depend on u_{kh}, and thus on k, h, hence we apply Theorem 4.16 together with Lemma 4.18, in order to have coefficients, that converge to 0 uniformly in m. There holds the following.

Theorem 4.20. *Let $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$, then for (k,h) small enough, the unique solution $u_{kh} \in V_{kh}$ to (19) satisfies*

$$\|u_{kh}\|_{L^\infty(I;H^1(\Omega))} + \|A_h u_{kh}\|_{L^2(I;L^2(\Omega))} \leq C_1 \exp\left(C_2 \|u_{kh}\|^2_{L^\infty(I;L^2(\Omega))} \|\nabla u_{kh}\|^2_{L^2(I;L^2(\Omega))}\right) \times \left(\|f\|_{L^2(I;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)}\right),$$

with constants C_1, C_2 independent of k, h. The $L^\infty(I; L^2(\Omega))$ and $L^2(I; H^1(\Omega))$ norms of u_{kh} can be estimated by the results of Theorem 4.13.

Proof. For $m = 1, \ldots, M$, we test (19) with $A_h u_{kh}|_{I_m}$, which yields with the convention $[u_{kh}]_0 := u^+_{kh,0} - u_0$

$$(\partial_t u_{kh}, A_h u_{kh})_{I_m \times \Omega} + \nu(\nabla u_{kh}, \nabla A_h u_{kh})_{I_m \times \Omega} + \left([u_{kh}]_{m-1}, A_h u^+_{kh,m-1}\right)_\Omega + \hat{c}(\!(u_{kh}, u_{kh}, A_h u_{kh} \chi_{I_m})\!) = (f, A_h u_{kh})_{I_m \times \Omega}.$$

From the above identity, for $m = 2, \ldots, M$, the definition of A_h gives

$$(\partial_t \nabla u_{kh}, \nabla u_{kh})_{I_m \times \Omega} + \nu \|A_h u_{kh}\|^2_{L^2(I_m;L^2(\Omega))} + \left([\nabla u_{kh}]_{m-1}, \nabla u^+_{kh,m-1}\right)_\Omega + \hat{c}(\!(u_{kh}, u_{kh}, A_h u_{kh} \chi_{I_m})\!) = (f, A_h u_{kh})_{I_m \times \Omega}. \tag{28}$$

Here the terms containing time derivatives and jumps can be combined according to Lemma 4.6. Some care has to be taken on the first time interval. It holds

$$\left(u_0, A_h u^+_{kh,0}\right)_\Omega = -\left(u_0, \mathbb{P}_h \Delta_h u^+_{kh,0}\right)_\Omega = -\left(\mathbb{P}_h u_0, \Delta_h u^+_{kh,0}\right)_\Omega = \left(\nabla \mathbb{P}_h u_0, \nabla u^+_{kh,0}\right)_\Omega.$$

Thus on the first time interval, with the same arguments as in the proof of Lemma 4.6, it holds

$$(\partial_t u_{kh}, A_h u_{kh})_{I_1 \times \Omega} + (u^+_{kh,0} - u_0, A_h u^+_{kh,0})_\Omega = \frac{1}{2}\left(\|\nabla u^-_{kh,1}\|^2_{L^2(\Omega)} + \|\nabla(u^+_{kh,0} - \mathbb{P}_h u_0)\|^2_{L^2(\Omega)} - \|\nabla \mathbb{P}_h u_0\|^2_{L^2(\Omega)}\right).$$

For a more compact notation, we shall write $[\nabla u_{kh}]_0 := \nabla(u^+_{kh,0} - \mathbb{P}_h u_0)$. Note the need to include the projection on the right hand side. With this slight abuse of notation, (28) also holds on the first time interval. Since $u_0 \in V$ we can use the stability of $\mathbb{P}_h$ in H^1 for continuously divergence free functions, see Lemma 5.4 of [36], yielding an estimate

$$\|\nabla \mathbb{P}_h u_0\|_{L^2(\Omega)} \leq C \|\nabla u_0\|_{L^2(\Omega)}. \tag{29}$$

For the terms of (28) involving the right hand side f, it holds

$$(f, A_h u_{kh})_{I_m \times \Omega} \leq \|f\|_{L^2(I_m;L^2(\Omega))} \|A_h u_{kh}\|_{L^2(I_m;L^2(\Omega))} \leq C \|f\|^2_{L^2(I_m;L^2(\Omega))} + \frac{\nu}{4} \|A_h u_{kh}\|^2_{L^2(I_m;L^2(\Omega))},$$

where we can absorb the $A_h u_{kh}$ term. We now turn towards estimating the trilinear form remaining in (28), which poses the main difficulty of this proof. Due to Lemma 4.4, the trilinear term satisfies the expression

$$\hat{c}(\!(u_{kh}, u_{kh}, A_h u_{kh} \chi_{I_m})\!) = ((u_{kh} \cdot \nabla) u_{kh}, A_h u_{kh})_{I_m \times \Omega} + \frac{1}{2}(\nabla \cdot u_{kh}, u_{kh} \cdot A_h u_{kh})_{I_m \times \Omega},$$

which we can estimate with Hölder's inequality by

$$\hat{c}((u_{kh}, u_{kh}, A_h u_{kh} \chi_{I_m})) \leq C \int_{I_m} \|u_{kh}\|_{L^\infty(\Omega)} \|\nabla u_{kh}\|_{L^2(\Omega)} \|A_h u_{kh}\|_{L^2(\Omega)} \,\mathrm{d}t.$$

Using (16) and Young's inequality, we obtain the estimate

$$\hat{c}((u_{kh}, u_{kh}, A_h u_{kh} \chi_{I_m})) \leq \int_{I_m} C \|u_{kh}\|^2_{L^2(\Omega)} \|\nabla u_{kh}\|^4_{L^2(\Omega)} + \frac{\nu}{4} \|A_h u_{kh}\|^2_{L^2(\Omega)} \,\mathrm{d}t.$$

The latter term can be absorbed, hence we proceed by discussing the first one. It holds

$$\int_{I_m} C \|u_{kh}\|^2_{L^2(\Omega)} \|\nabla u_{kh}\|^4_{L^2(\Omega)} \,\mathrm{d}t \leq C \|u_{kh}\|^2_{L^\infty(I;L^2(\Omega))} \|\nabla u_{kh}\|^2_{L^2(I_m;L^2(\Omega))} \|\nabla u_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}.$$

Let us introduce $\gamma_m := C \|u_{kh}\|^2_{L^\infty(I;L^2(\Omega))} \|\nabla u_{kh}\|^2_{L^2(I_m;L^2(\Omega))}$. All in all, we have derived from (28) an estimate of the form

$$\begin{aligned} \frac{1}{2}\left(\left\| \nabla u^-_{kh,m} \right\|^2_{L^2(\Omega)} + \|[\nabla u_{kh}]_{m-1}\|^2_{L^2(\Omega)} - \left\| \nabla u^-_{kh,m-1} \right\|^2_{L^2(\Omega)} \right) + \frac{1}{2}\nu \|A_h u_{kh}\|^2_{L^2(I_m;L^2(\Omega))} \\ \leq \|f\|^2_{L^2(I_m;L^2(\Omega))} + \gamma_m \|\nabla u_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}, \end{aligned}$$

with the usual modifications on the first time interval. Multiplying this by 2 and summing up from $m = 1, \ldots, n$ yields for any $n \in \{1, \ldots, M\}$ an estimate of the form

$$\begin{aligned} \left\| \nabla u^-_{kh,m} \right\|^2_{L^2(\Omega)} + \sum_{m=1}^{n} \left(\|[\nabla u_{kh}]_{m-1}\|^2_{L^2(\Omega)} + \nu \|A_h u_{kh}\|^2_{L^2(I_m;L^2(\Omega))} \right) \\ \leq C \|\nabla u_0\|^2_{L^2(\Omega)} + \sum_{m=1}^{n} \left(\|f\|^2_{L^2(I_m;L^2(\Omega))} + 2\gamma_m \|\nabla u_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))} \right), \end{aligned}$$

where due to (29) we have dropped the projection $\mathbb{P}_h$ of the initial data on the right hand side. Since $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$, Theorem 4.16 shows that $u_{kh} \to u$ in $L^2(I; H^1(\Omega))$ as $(k, h) \to 0$. Hence, due to Lemma 4.18, and Theorem 4.13, it holds $\gamma_m \to 0$ uniformly, as $(k, h) \to 0$. This implies that by following the same steps as the proof of Theorem 4.13, we obtain the result as a consequence of the discrete Gronwall Lemma 4.11. □

5. Error estimates

5.1. Duality based best approximation type estimates

In this concluding section of our work, we will show the main results, which are the $L^\infty(I; L^2(\Omega))$ and $L^2(I; L^2(\Omega))$ error estimates. Their proofs are based on duality arguments, and hence we begin this section by showing two stability results for discrete dual equations. Let us first motivate the specific dual problem, that we will consider. Due to the nonlinear structure of the Navier–Stokes equations, there does not hold a Galerkin orthogonality with respect to the bilinear form B, *i.e.*, for solutions (u, p) of (9) and their discrete counterparts (u_{kh}, p_{kh}) of (20), it holds for test functions $(v_{kh}, q_{kh}) \in Y_{kh}$:

$$B((u - u_{kh}, p - p_{kh}), (v_{kh}, q_{kh})) = -\hat{c}((u, u, v_{kh})) + \hat{c}((u_{kh}, u_{kh}, v_{kh})),$$

where the right hand side is nonzero in general. Since we want to use this orthogonality relation after testing the dual equation with $u - u_{kh}$, we need to reformulate the trilinear terms, such that $u - u_{kh}$ occurs linearly. To this end, we use the identity

$$\hat{c}((u, u, v_{kh})) - \hat{c}((u_{kh}, u_{kh}, v_{kh})) = \hat{c}((\overline{uu_{kh}}, u - u_{kh}, v_{kh})) + \hat{c}((u - u_{kh}, \overline{uu_{kh}}, v_{kh})), \tag{30}$$

where we linearize around the average of continuous and discrete solutions to the Navier–Stokes equations

$$\overline{uu_{kh}} := \frac{1}{2}(u + u_{kh}). \tag{31}$$

With these considerations, we have the following lemma:

Lemma 5.1. *Let (u, p) solve the Navier-Stokes equations (9), and let (u_{kh}, p_{kh}) be their discrete approximation solving (20). Let further $\overline{uu_{kh}}$ denote the average of u and u_{kh} as defined by (31). Then for any $(v_{kh}, q_{kh}) \in Y_{kh}$, it holds*

$$B((u - u_{kh}, p - p_{kh}), (v_{kh}, q_{kh})) + \hat{c}(\!(\overline{uu_{kh}}, u - u_{kh}, v_{kh})\!) + \hat{c}(\!(u - u_{kh}, \overline{uu_{kh}}, v_{kh})\!) = 0.$$

Proof. This result is an immediate consequence of the definitions of solutions to (9) and (20), together with the identity (30). □

This motivates the choice of $\overline{uu_{kh}}$ as linearization point for setting up a dual equation:
Find $(z_{kh}, \varrho_{kh}) \in Y_{kh}$ such that for all $(\phi_{kh}, \psi_{kh}) \in Y_{kh}$ it holds

$$B((\phi_{kh}, \psi_{kh}), (z_{kh}, \varrho_{kh})) + \hat{c}(\!(\overline{uu_{kh}}, \phi_{kh}, z_{kh})\!) + \hat{c}(\!(\phi_{kh}, \overline{uu_{kh}}, z_{kh})\!) = \langle g, \phi_{kh} \rangle_{I\times\Omega}, \tag{32}$$

where the right hand side g will be chosen appropriately, see the proofs of Theorems 5.6 and 5.7. Note that the right hand side of (32) implicitly prescribes the final data $z^+_{kh,M} = 0$. This dual equation will help us in deriving the sought error estimates, which we will do in the following section.

Remark 5.2. To analyze this dual problem, it will be convenient, to have dual represenations of $\mathfrak{B}$ and B at hand, which are obtained by partial integration on each I_m and rearranging the terms. Note that in this representation, the time derivative is applied to the second argument. It holds

$$\mathfrak{B}(u, v) = -\sum_{m=1}^{M} (u, \partial_t v)_{I_m\times\Omega} + \nu(\nabla u, \nabla v)_{I\times\Omega} - \sum_{m=1}^{M-1} \left(u_m^-, [v]_m\right)_\Omega + \left(u_M^-, v_M^-\right)_\Omega, \tag{33}$$

$$\begin{aligned} B((u,p),(v,q)) = &-\sum_{m=1}^{M} (u, \partial_t v)_{I_m\times\Omega} + \nu(\nabla u, \nabla v)_{I\times\Omega} - \sum_{m=1}^{M-1} \left(u_m^-, [v]_m\right)_\Omega + \left(u_M^-, v_M^-\right)_\Omega \\ &+ (\nabla\cdot u, q)_{I\times\Omega} - (\nabla\cdot v, p)_{I\times\Omega}, \end{aligned} \tag{34}$$

see also [7, 35].

Remark 5.3. Similar to the discrete Navier–Stokes equations, we can consider an equivalent formulation for the discrete dual equation in discretely divergence free spaces: Find $z_{kh} \in V_{kh}$ satisfying

$$\mathfrak{B}(\phi_{kh}, z_{kh}) + \hat{c}(\!(\overline{uu_{kh}}, \phi_{kh}, z_{kh})\!) + \hat{c}(\!(\phi_{kh}, \overline{uu_{kh}}, z_{kh})\!) = \langle g, \phi_{kh} \rangle_{I\times\Omega} \qquad \text{for all } \phi_{kh} \in V_{kh}. \tag{35}$$

With the same argument as Proposition 4.3 from [7], there holds: If $z_{kh} \in V_{kh}$ solves (35), then there exists $\varrho_{kh} \in M_{kh}$ such that $(z_{kh}, \varrho_{kh}) \in Y_{kh}$ solves (32). If $(z_{kh}, \varrho_{kh}) \in Y_{kh}$ solves (32), then $z_{kh} \in V_{kh}$ and it solves (35).

We first show unique solvability of the discrete dual problem, and the stability in $L^\infty(I; L^2(\Omega)) \cap L^2(I; H^1(\Omega))$. Since both u and u_{kh} occur in the formulation of the discrete problem, we need both results of Lemma 4.18 to hold true.

Theorem 5.4. *Let $f \in L^2(I; L^2(\Omega)^2)$, $u_0 \in V$ and u, $u_{kh} \in L^\infty(I; L^2(\Omega)^2) \cap L^2(I; H^1(\Omega)^2)$ be solutions to the weak and fully discretized Navier–Stokes equations (4) and (19) for either $q = 0$ or $q = 1$. Then for (k, h) small enough, problem (35) has a unique solution $z_{kh} \in V_{kh}$ for any $g \in L^1(I; L^2(\Omega)^2)$, and there holds the bound*

$$\|z_{kh}\|_{L^\infty(I;L^2(\Omega))} + \|z_{kh}\|_{L^2(I;H^1(\Omega))} \leq K\big(\|\overline{uu_{kh}}\|_{L^2(I;H^1(\Omega))\cap L^\infty(I;L^2(\Omega))}\big)\|g\|_{L^1(I;L^2(\Omega))},$$

where $K : [0, +\infty) \to [0, +\infty)$ is a strictly monotonically increasing, continuous nonlinear function, independent of k, h.

Proof. On the continuous level and for $g \in L^2(I; H^{-1}(\Omega)^2)$, the proof of a corresponding estimate can be found in Proposition 2.7 of [15]. We adapt it to the discrete setting and to $g \in L^1(I; L^2(\Omega)^2)$, making use of the previously derived discrete Gronwall Lemma 4.11. We only have to prove the norm bound, since problem (32) is a quadratic system of linear equations, thus is solvable, if it is injective. The norm bound yields, that for right hand side $g = 0$, $z_{kh} = 0$ is the only solution, thus the norm bound implies existence and uniqueness. For ease of notation, we use the convention $[z_{kh}]_M = -z^-_{kh,M}$. Testing equation (35) with $z_{kh}\chi_{I_m}$, $m = 1, \ldots, M$, where by χ_{I_m} we denote the indicator function of the subinterval I_m, yields:

$$\begin{aligned}
-(z_{kh}\chi_{I_m}, \partial_t z_{kh})_{I_m\times\Omega} + \nu(\nabla z_{kh}\chi_{I_m}, \nabla z_{kh})_{I\times\Omega} - \Big(z^-_{kh,m}, [z_{kh}]_m\Big)_\Omega + \hat{c}(\!(\overline{uu_{kh}}, z_{kh}\chi_{I_m}, z_{kh})\!) + \hat{c}(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!)\\
= (g, z_{kh}\chi_{I_m})_{I\times\Omega}.
\end{aligned}$$

Applying Lemma 4.6 and writing the inner product as norm yields

$$\begin{aligned}
\frac{1}{2}\Big\|z^+_{kh,m-1}\Big\|^2_{L^2(\Omega)} + \frac{1}{2}\|[z_{kh}]_m\|^2_{L^2(\Omega)} + \nu\|\nabla z_{kh}\|^2_{L^2(I_m;L^2(\Omega))} + \hat{c}(\!(\overline{uu_{kh}}, z_{kh}\chi_{I_m}, z_{kh})\!) + \hat{c}(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!)\\
= \frac{1}{2}\Big\|z^+_{kh,m}\Big\|^2_{L^2(\Omega)} + (g, z_{kh})_{I_m\times\Omega}.
\end{aligned} \tag{36}$$

We proceed by estimating the trilinear forms. According to its definition (11), the first term vanishes, and for the second one, it holds

$$\hat{c}(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!) = \frac{1}{2}c(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!) - \frac{1}{2}c(\!(z_{kh}\chi_{I_m}, z_{kh}, \overline{uu_{kh}})\!).$$

After applying Hölder's inequality in space, we obtain

$$\hat{c}(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!) \leq \frac{1}{2}\int_{I_m}\Big(\|z_{kh}\|^2_{L^4(\Omega)}\|\nabla\overline{uu_{kh}}\|_{L^2(\Omega)} + \|z_{kh}\|_{L^4(\Omega)}\|\overline{uu_{kh}}\|_{L^4(\Omega)}\|\nabla z_{kh}\|_{L^2(\Omega)}\Big)\,\mathrm{d}t.$$

After estimating the L^4 norms by Lemma 3.2 and applying Hölder's inequality in time, we arrive at

$$\begin{aligned}
\hat{c}(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!) \leq C\Big(&\|\nabla z_{kh}\|_{L^2(I_m\times\Omega)}\|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))}\|\nabla\overline{uu_{kh}}\|_{L^2(I_m\times\Omega)}\\
&+ \|z_{kh}\|^{\frac{1}{2}}_{L^\infty(I_m;L^2(\Omega))}\|\nabla z_{kh}\|^{\frac{3}{2}}_{L^2(I_m\times\Omega)}\|\overline{uu_{kh}}\|^{\frac{1}{2}}_{L^\infty(I_m;L^2(\Omega))}\|\nabla\overline{uu_{kh}}\|^{\frac{1}{2}}_{L^2(I_m\times\Omega)}\Big).
\end{aligned}$$

An application of Young's inequality yields

$$\hat{c}(\!(z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})\!) \leq \frac{\nu}{2}\|\nabla z_{kh}\|^2_{L^2(I_m\times\Omega)} + C\|z_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))}\Big(1 + \|\overline{uu_{kh}}\|^2_{L^\infty(I_m;L^2(\Omega))}\Big)\|\nabla\overline{uu_{kh}}\|^2_{L^2(I_m\times\Omega)}.$$

In order to abbreviate the notation, we introduce $\gamma_m := C\|\nabla\overline{uu_{kh}}\|^2_{L^2(I_m\times\Omega)}(1+\|\overline{uu_{kh}}\|^2_{L^\infty(I_m;L^2(\Omega))})$. We insert the above estimate into (36), absorb terms, and multiply by 2, which yields

$$\Big\|z^+_{kh,m-1}\Big\|^2_{L^2(\Omega)} + \|[z_{kh}]_m\|^2_{L^2(\Omega)} + \nu\|\nabla z_{kh}\|^2_{L^2(I_m;L^2(\Omega))} \leq \gamma_m\|z_{kh}\|^2_{L^\infty(I_m;L^2(\Omega))} + \Big\|z^+_{kh,m}\Big\|^2_{L^2(\Omega)} + 2(g, z_{kh})_{I_m\times\Omega}.$$

After an application of Hölder's inequality, we obtain

$$\left\|z_{kh,m-1}^{+}\right\|_{L^2(\Omega)}^2 + \|[z_{kh}]_m\|_{L^2(\Omega)}^2 + \nu\|\nabla z_{kh}\|_{L^2(I_m;L^2(\Omega))}^2 \leq \gamma_m\|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))}^2 + \left\|z_{kh,m}^{+}\right\|_{L^2(\Omega)}^2 + 2\|g\|_{L^1(I_m;L^2(\Omega))}\|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))}. \tag{37}$$

Since we have assumed $q = 0$ or $q = 1$, we can estimate the $\|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))}$ terms by evaluations at the right and left endpoints: $\|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))} = \max\{\|z_{kh,m-1}^{+}\|_{L^2(\Omega)}, \|z_{kh,m}^{-}\|_{L^2(\Omega}\}$. With triangle inequality there hold the following estimates:

$$\begin{aligned}\|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))} &\leq \max\left\{\left\|z_{kh,m-1}^{+}\right\|_{L^2(\Omega)}, \left\|z_{kh,m}^{+}\right\|_{L^2(\Omega)} + \|[z_{kh}]_m\|_{L^2(\Omega)}\right\},\\ \|z_{kh}\|_{L^\infty(I_m;L^2(\Omega))}^2 &\leq \max\left\{\left\|z_{kh,m-1}^{+}\right\|_{L^2(\Omega)}^2, 2\left\|z_{kh,m}^{+}\right\|_{L^2(\Omega)}^2 + 2\|[z_{kh}]_m\|_{L^2(\Omega)}^2\right\}.\end{aligned}$$

We introduce $x_n^2 := \|z_{kh,n-1}^{+}\|_{L^2(\Omega)}^2 + \sum_{m=n}^{M}\|[z_{kh}]_m\|_{L^2(\Omega)}^2 + \nu\|\nabla z_{kh}\|_{L^2(I_m;L^2(\Omega))}^2$. Note that by the terminal condition for z_{kh}, it holds $x_{M+1} = 0$. Then after summing up (37) from $m = n$ to $m = M$, we have:

$$x_n^2 \leq \sum_{m=n}^{M} 2\gamma_m\big(x_m^2 + x_{m+1}^2\big) + 2\|g\|_{L^1(I_m;L^2(\Omega))}(x_m + x_{m+1}).$$

Shifting indices, we arrive at

$$x_n^2 \leq \sum_{m=n}^{M} 2(\gamma_m + \gamma_{m-1})x_m^2 + 2\|g\|_{L^1(I_m\cup I_{m-1};L^2(\Omega))}x_m,$$

where for $n = 1$ we use the convention $I_0 = \emptyset$. Hence we are in the setting of Lemma 4.11, where formally we have to introduce an index transformation $\tilde{n} = M-n$. In order to apply the lemma, we need to verify $2(\gamma_m+\gamma_{m-1}) < 1$ for all m. Due to $f \in L^2(I;L^2(\Omega)^2)$ and $u_0 \in V$, Theorem 4.16 shows $\|u - u_{kh}\|_{L^2(I;H^1(\Omega))} \to 0$ as $(k,h) \to 0$ and we can apply Lemma 4.18 and obtain with triangle inequality, that $\sup_{1\leq m\leq M}\|\nabla\overline{uu_{kh}}\|_{L^2(I_m;L^2(\Omega))}^2 \to 0$ as $(k,h) \to 0$. Together with the bounds from Proposition 3.1 and Theorem 4.13 for $\|u\|_{L^\infty(I;L^2(\Omega))}$ and $\|u_{kh}\|_{L^\infty(I;L^2(\Omega))}$, we obtain that $\gamma_m \to 0$ uniformly in n for $(k,h) \to 0$. Thus we can choose the discretization fine enough, such that $\gamma_m < 1/8$, and thus we obtain from Lemma 4.11

$$\begin{aligned}&\left\|z_{kh,n-1}^{+}\right\|_{L^2(\Omega)}^2 + \sum_{m=n}^{M}\|[z_{kh}]_m\|_{L^2(\Omega)}^2 + \nu\|\nabla z_{kh}\|_{L^2(I_m;L^2(\Omega))}^2\\ &\qquad\leq C_1\exp\Big(C_2\|\nabla\overline{uu_{kh}}\|_{L^2(I\times\Omega)}^2\Big(1 + \|\overline{uu_{kh}}\|_{L^\infty(I,L^2(\Omega))}^2\Big)\Big)\|g\|_{L^1(I;L^2(\Omega))}^2.\end{aligned} \tag{38}$$

□

The next theorem, similar to Theorem 4.20, states a stability result for the discrete dual solution in stronger norms, whenever the right hand side posesses more regularity.

Theorem 5.5. *Let the assumptions of Theorem 5.4 hold true, i.e., let $f \in L^2(I;L^2(\Omega)^2)$, $u_0 \in V$ and u, u_{kh} be solutions to the weak and fully discretized Navier–Stokes equations (4) and (19) for either $q = 0$ or $q = 1$. Then for (k,h) small enough, for any $g \in L^2(I;L^2(\Omega)^2)$, the unique solution $z_{kh} \in V_{kh}$ to (35) satisfies the bound*

$$\|z_{kh}\|_{L^\infty(I;H^1(\Omega))} + \|A_h z_{kh}\|_{L^2(I;L^2(\Omega))} \leq C_{u,u_{kh}}\|g\|_{L^2(I\times\Omega)},$$

with a constant $C_{u,u_{kh}}$ depending on u, u_{kh} in the form

$$C_{u,u_{kh}} = C_1 \exp\Big(C_2\Big(\|Au\|^2_{L^2(I;L^2(\Omega))}\|u\|^2_{L^\infty(I;L^2(\Omega))} + \|A_h u_{kh}\|^2_{L^2(I;L^2(\Omega))}\|u_{kh}\|^2_{L^\infty(I;L^2(\Omega))}\Big)\Big),$$

where the norms of u, u_{kh} are bounded by Proposition 3.1 and Theorems 3.6, 4.20 and 4.13, due to the assumptions on f and u_0.

Proof. The proof follows the same steps as the proof the stability of u_{kh} in stronger norms, presented in Theorem 4.20. We begin by testing the dual equation with $A_h z_{kh}\chi_{I_m}$ for arbitrary $m = 1,\dots,M$, where χ_{I_m} denotes the characteristic function of the time interval I_m. The nonlinear terms that occur for the discrete dual equation are of the form

$$\hat{c}((\overline{uu_{kh}}, A_h z_{kh}\chi_{I_m}, z_{kh})) + \hat{c}((A_h z_{kh}\chi_{I_m}, \overline{uu_{kh}}, z_{kh})).$$

By Hölder's inequality in space, (17) and the discrete Gagliardo–Nirenberg inequality (16), these terms can be estimated by

$$C\int_{I_m} \|z_{kh}\|^{\frac12}_{L^2(\Omega)}\|A_h z_{kh}\|^{\frac32}_{L^2(\Omega)}\Big(\|\nabla\overline{uu_{kh}}\|_{L^2(\Omega)} + \|\overline{uu_{kh}}\|_{L^\infty(\Omega)}\Big)\,\mathrm{d}t.$$

Applying the continuous and discrete Gagliardo–Nirenberg inequalities (15) & (16), and Equation (17) to $\overline{uu_{kh}}$, it remains

$$C\int_{I_m} \|z_{kh}\|^{\frac12}_{L^2(\Omega)}\|A_h z_{kh}\|^{\frac32}_{L^2(\Omega)}\Big(\|u\|^{\frac12}_{L^2(\Omega)}\|Au\|^{\frac12}_{L^2(\Omega)} + \|u_{kh}\|^{\frac12}_{L^2(\Omega)}\|A_h u_{kh}\|^{\frac12}_{L^2(\Omega)}\Big)\,\mathrm{d}t.$$

An application of Young's inequality, absorbing terms and summing over the subintervals allows us to conclude the proof. By Theorem 5.4 it holds $z_{kh} \in L^\infty(I; L^2(\Omega)^2)$ with a bound independent on k, h and linear in $\|g\|_{L^1(I;L^2(\Omega))}$. Further, the terms involving u, u_{kh} are summable, since the $L^\infty(I; L^2(\Omega))$ norms of u, u_{kh} remain bounded *via* Proposition 3.1 and Theorem 4.13, and the Au, $A_h u_{kh}$ terms are summable by Theorems 3.6 and 4.20, where the latter holds true for (k, h) small enough, due to $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$. □

We now turn towards showing the main result of our work, *i.e.*, the error estimate for the Navier–Stokes equations in the $L^\infty(I; L^2(\Omega))$ norm. As in the proof of Theorem 4.16, we will split the error $u - u_{kh}$ into an error for a Stokes problem, and a remainder term, which we will estimate using the discrete dual equation (35), to which we can apply the results of Theorem 5.4. Similar to the result for the Stokes equations of Proposition 4.9, the error estimate will consist of two terms with the first one being a best approximation error, and the second one being the error of the stationary Stokes Ritz projection introduced in (10). This result estimates the $L^\infty(I; L^2(\Omega))$ norm in an isolated fashion, and thus does not suffer from an order reduction, which is observed in results that estimate the error norm combined with the $L^2(I; H^1(\Omega))$ norm.

Theorem 5.6. *Let $f \in L^2(I; L^2(\Omega)^2)$ and $u_0 \in V$. Let (u, p) be the unique solution to the Navier–Stokes equations (9), and (u_{kh}, p_{kh}) the corresponding solution to the discretized equations (20) for a discontinuous Galerkin method in time with order $q = 0$ or $q = 1$, for sufficiently small discretization parameters (k, h). Then there holds*

$$\|u - u_{kh}\|_{L^\infty(I;L^2(\Omega))} \le C\Big(\ln\frac{T}{k}\Big)\Big(\inf_{\chi_{kh}\in V_{kh}}\|u - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} + \big\|u - R_h^S(u,p)\big\|_{L^\infty(I;L^2(\Omega))}\Big).$$

Proof. We use the same notation as in the proof of Theorem 4.16 and denote the velocity error by $e := u - u_{kh}$ and the pressure error by $r := p - p_{kh}$. We consider the Stokes projection $(\tilde{u}_{kh}, \tilde{p}_{kh})$ of (u, p), solving (25), and the induced splitting of the error into $e = \xi + \eta_{kh}$, $r = \omega + \kappa_{kh}$, where $\xi = u - \tilde{u}_{kh}$, $\eta_{kh} = \tilde{u}_{kh} - u_{kh}$, $\omega = p - \tilde{p}_{kh}$ and $\kappa_{kh} = \tilde{p}_{kh} - p_{kh}$. We immediately obtain the estimate

$$\|\xi\|_{L^\infty(I;L^2(\Omega))} \le C\Big(\ln\frac{T}{k}\Big)\Big(\inf_{\chi_{kh}\in V_{kh}}\|u - \chi_{kh}\|_{L^\infty(I;L^2(\Omega))} + \big\|u - R_h^S(u,p)\big\|_{L^\infty(I;L^2(\Omega))}\Big),$$

by Proposition 4.9, since by Theorem 3.8 it holds $f-(u\cdot\nabla)u \in L^2(I;L^2(\Omega)^2)$. Thus to finalize the proof, we have to estimate the $L^2(\Omega)$ norm of η_{kh} pointwise in time. To this end, as in Theorem 6.2 of [7], we fix $\tilde t \in I$ and construct $\theta \in C_0^\infty(I)$ in such a way, that $\operatorname{supp}\theta \subset I_m$ where m is chosen such that $\tilde t \in I_m$, and

$$\big(\eta_{kh}(\tilde t)\theta, \phi_{kh}\big)_{I\times\Omega} = \big(\eta_{kh}(\tilde t), \phi_{kh}(\tilde t)\big)_\Omega \qquad \text{for all } \phi_{kh}\in U_{kh}, \quad \|\theta\|_{L^1(I)} \le C \text{ independent of } \tilde t \text{ and } k.$$

For the construction of such a function θ serving the purpose of a regularized Dirac measure, we refer to Appendix A.5 of [44]. We then define the dual solution $z_{kh}\in V_{kh}$ such that for all $\phi_{kh}\in V_{kh}$, it satisfies

$$\mathfrak{B}(\phi_{kh}, z_{kh}) + \hat c(\!(\overline{uu_{kh}}, \phi_{kh}, z_{kh})\!) + \hat c(\!(\phi_{kh}, \overline{uu_{kh}}, z_{kh})\!) = \big(\eta_{kh}(\tilde t)\theta, \phi_{kh}\big)_{I\times\Omega}.$$

By Theorem 5.4 we have the existence, uniqueness and regularity of z_{kh}, satisfying the bound

$$\|z_{kh}\|_{L^\infty(I;L^2(\Omega))} + \|\nabla z_{kh}\|_{L^2(I\times\Omega)} \le K\big(\|\overline{uu_{kh}}\|_{L^2(I;H^1(\Omega))\cap L^\infty(I;L^2(\Omega))}\big)\big\|\eta_{kh}(\tilde t)\theta\big\|_{L^1(I;L^2(\Omega))}, \tag{39}$$

for (k,h) small enough. From Remark 5.3, we obtain the existence of an associated pressure $\varrho_{kh}\in M_{kh}$, such that for all $(\phi_{kh},\psi_{kh})\in Y_{kh}$, it holds

$$B((\phi_{kh},\psi_{kh}),(z_{kh},\varrho_{kh})) + \hat c(\!(\overline{uu_{kh}}, \phi_{kh}, z_{kh})\!) + \hat c(\!(\phi_{kh}, \overline{uu_{kh}}, z_{kh})\!) = \big(\eta_{kh}(\tilde t)\theta, \phi_{kh}\big)_{I\times\Omega}.$$

Choosing the specific test functions $(\phi_{kh},\psi_{kh}) = (\eta_{kh},\kappa_{kh})\in Y_{kh}$, we have

$$\begin{aligned}
\big\|\eta_{kh}(\tilde t)\big\|^2_{L^2(\Omega)} &= \big(\eta_{kh}(\tilde t)\theta, \eta_{kh}\big)_{I\times\Omega}\\
&= B((\eta_{kh},\kappa_{kh}),(z_{kh},\varrho_{kh})) + \hat c(\!(\overline{uu_{kh}}, \eta_{kh}, z_{kh})\!) + \hat c(\!(\eta_{kh}, \overline{uu_{kh}}, z_{kh})\!)\\
&= B((e-\xi, r-\omega),(z_{kh},\varrho_{kh})) + \hat c(\!(\overline{uu_{kh}}, e-\xi, z_{kh})\!) + \hat c(\!(e-\xi, \overline{uu_{kh}}, z_{kh})\!)\\
&= B((e,r),(z_{kh},\varrho_{kh})) + \hat c(\!(\overline{uu_{kh}}, e, z_{kh})\!) + \hat c(\!(e, \overline{uu_{kh}}, z_{kh})\!)\\
&\quad - B((\xi,\omega),(z_{kh},\varrho_{kh})) - \hat c(\!(\overline{uu_{kh}}, \xi, z_{kh})\!) - \hat c(\!(\xi, \overline{uu_{kh}}, z_{kh})\!).
\end{aligned}$$

For ξ we can use the Galerkin orthogonality with respect to B, *i.e.*, from (25), we obtain

$$B((\xi,\omega),(\phi_{kh},\psi_{kh})) = 0 \qquad \text{for all } (\phi_{kh},\psi_{kh})\in Y_{kh}.$$

Furthermore, due to Lemma 5.1 it holds

$$B((e,r),(z_{kh},\varrho_{kh})) + \hat c(\!(\overline{uu_{kh}}, e, z_{kh})\!) + \hat c(\!(e, \overline{uu_{kh}}, z_{kh})\!) = 0.$$

Thus we see directly

$$\big\|\eta_{kh}(\tilde t)\big\|^2_{L^2(\Omega)} = -\hat c(\!(\overline{uu_{kh}}, \xi, z_{kh})\!) - \hat c(\!(\xi, \overline{uu_{kh}}, z_{kh})\!).$$

We want to make use of the $L^\infty(I;L^2(\Omega))$ estimate of ξ and thus have to move it to an argument of $\hat c$ that has no spatial gradient applied to it. Since $\hat c$ was obtained by anti-symmetrizing c, there are gradients in the second and third argument of $\hat c$, which is why we revert to the original trilinear form c. The first argument has no gradient, and thus

$$\big\|\eta_{kh}(\tilde t)\big\|^2_{L^2(\Omega)} = -\frac12 c(\!(\overline{uu_{kh}}, \xi, z_{kh})\!) + \frac12 c(\!(\overline{uu_{kh}}, z_{kh}, \xi)\!) - \hat c(\!(\xi, \overline{uu_{kh}}, z_{kh})\!). \tag{40}$$

Lemma 4.4 allows us to switch ξ to the third argument, and we thus have

$$\big\|\eta_{kh}(\tilde t)\big\|^2_{L^2(\Omega)} = \frac12(\nabla\cdot\overline{uu_{kh}}, z_{kh}\cdot\xi)_{I\times\Omega} + c(\!(\overline{uu_{kh}}, z_{kh}, \xi)\!) - \hat c(\!(\xi, \overline{uu_{kh}}, z_{kh})\!).$$

After application of Hölder's inequality, we obtain

$$\begin{aligned}\left\|\eta_{kh}(\tilde{t})\right\|_{L^2(\Omega)}^2 \leq C\|\xi\|_{L^\infty(I;L^2(\Omega))}\|1\|_{L^4(I;L^\infty(\Omega))}& \\ \times \left(\|z_{kh}\|_{L^4(I;L^4(\Omega))}\|\nabla\overline{uu_{kh}}\|_{L^2(I;L^4(\Omega))} + \|\nabla z_{kh}\|_{L^2(I\times\Omega)}\|\overline{uu_{kh}}\|_{L^4(I;L^\infty(\Omega))}\right).&\end{aligned} \tag{41}$$

By Theorems 3.6 and 4.20, the solutions u,u_{kh} satisfy the bound

$$\|\overline{uu_{kh}}\|_{L^\infty(I;H^1(\Omega))} + \|u\|_{L^2(I;H^2(\Omega))} + \|A_h u_{kh}\|_{L^2(I;L^2(\Omega))} \leq C,$$

with a constant depending on the data. Hölder's inequality and the continuous and discrete Gagliardo–Nirenberg inequalities (15) and (16) yield moreover

$$\|\overline{uu_{kh}}\|_{L^4(I;L^\infty(\Omega))} \leq C\Big(\|u\|^{\frac{1}{2}}_{L^\infty(I;L^2(\Omega))}\|Au\|^{\frac{1}{2}}_{L^2(I;L^2(\Omega))} + \|u_{kh}\|^{\frac{1}{2}}_{L^\infty(I;L^2(\Omega))}\|A_h u_{kh}\|^{\frac{1}{2}}_{L^2(I;L^2(\Omega))}\Big).$$

From (12) and (14), we further obtain

$$\|\nabla\overline{uu_{kh}}\|_{L^2(I;L^4(\Omega))} \leq C\Big(\|\nabla u\|^{\frac{1}{2}}_{L^\infty(I;L^2(\Omega))}\|Au\|^{\frac{1}{2}}_{L^2(I;L^2(\Omega))} + \|\nabla u_{kh}\|^{\frac{1}{2}}_{L^\infty(I;L^2(\Omega))}\|A_h u_{kh}\|^{\frac{1}{2}}_{L^2(I;L^2(\Omega))}\Big).$$

With the above estimates, we thus obtain from (41)

$$\left\|\eta_{kh}(\tilde{t})\right\|_{L^2(\Omega)}^2 \leq CT^{\frac{1}{4}}\|\xi\|_{L^\infty(I;L^2(\Omega))}\Big(\|z_{kh}\|_{L^4(I;L^4(\Omega))} + \|\nabla z_{kh}\|_{L^2(I\times\Omega)}\Big).$$

With Lemma 3.2, we further obtain the estimate

$$\|z_{kh}\|_{L^4(I;L^4(\Omega))} \leq \|z_{kh}\|^{\frac{1}{2}}_{L^\infty(I;L^2(\Omega))}\|\nabla z_{kh}\|^{\frac{1}{2}}_{L^2(I\times\Omega)}.$$

Hence, applying (39) yields

$$\left\|\eta_{kh}(\tilde{t})\right\|_{L^2(\Omega)}^2 \leq CT^{\frac{1}{4}}\|\xi\|_{L^\infty(I;L^2(\Omega))}\left\|\eta_{kh}(\tilde{t})\theta\right\|_{L^1(I;L^2(\Omega))}. \tag{42}$$

By definition of θ we can estimate $\|\eta_{kh}(\tilde{t})\theta\|_{L^1(I;L^2(\Omega))}$ by $C\|\eta_{kh}(\tilde{t})\|_{L^2(\Omega)}$. This allows us to divide (42) by $\|\eta_{kh}(\tilde{t})\|_{L^2(\Omega)}$ which shows the bound

$$\left\|\eta_{kh}(\tilde{t})\right\|_{L^2(\Omega)} \leq CT^{\frac{1}{4}}\|\xi\|_{L^\infty(I;L^2(\Omega))}.$$

As $\tilde{t} \in I$ was arbitrary, this shows

$$\|\eta_{kh}\|_{L^\infty(I;L^2(\Omega))} \leq CT^{\frac{1}{4}}\|\xi\|_{L^\infty(I;L^2(\Omega))}.$$

Making use of the previously derived bound for $\|\xi\|_{L^\infty(I;L^2(\Omega))}$ and triangle inequality for $e = \xi + \eta_{kh}$ concludes the proof. □

The above theorem is the main result of this work, and the development of the discrete Gronwall lemma and analysis of the dual problem were the key ingredients, in order to prove it. With these techniques established, it is now straightforward, to also prove an error estimate in the $L^2(I;L^2(\Omega))$ norm. If we follow the steps of the proof of Theorem 5.6 up to (40), we then have to estimate the occuring trilinear form in terms of $\|\xi\|_{L^2(I;L^2(\Omega))}$, *i.e.*, the $L^2(I;L^2(\Omega))$ norm of the Stokes error. This implies that we need to estimate the occuring trilinear terms by stronger norms of the dual state z_{kh}, which are bounded by the results presented in Theorem 5.5. With these considerations, we can show the following theorem.

Theorem 5.7. *Let $f \in L^2(I;L^2(\Omega)^2)$ and $u_0 \in V$. Further let (u,p) and (u_{kh},p_{kh}) be the continuous and fully discrete solutions to the Navier–Stokes equations (9) and (20), for sufficiently small discretization parameters (k,h). Then for any $\chi_{kh} \in V_{kh}$, there holds*

$$\|u-u_{kh}\|_{L^2(I\times\Omega)} \le C\Big(\|u-\chi_{kh}\|_{L^2(I\times\Omega)} + \|u-\pi_\tau u\|_{L^2(I\times\Omega)} + \big\|u-R_h^S(u,p)\big\|_{L^2(I\times\Omega)}\Big).$$

Proof. To deduce the error estimate for the Navier–Stokes equations from the corresponding Stokes result of Proposition 4.8, we follow a duality argument similar to the proof of Theorem 5.6. We split the error in the same fashion $u-u_{kh} = u-\tilde u_{kh} + \tilde u_{kh} - u_{kh}$ where $\xi := u-\tilde u_{kh}$ is estimated by the Stokes result, and for $\eta_{kh} = \tilde u_{kh} - u_{kh}$ we consider (z_{kh},ϱ_{kh}) a solution to the dual problem

$$B((\phi_{kh},\psi_{kh}),(z_{kh},\varrho_{kh})) + \hat c(\!(\overline{uu_{kh}},\phi_{kh},z_{kh})\!) + \hat c(\!(\phi_{kh},\overline{uu_{kh}},z_{kh})\!) = (\eta_{kh},\phi_{kh})_{I\times\Omega}\, \forall(\phi_{kh},\psi_{kh}) \in Y_{kh}.$$

As before, we test the dual equation with η_{kh} and obtain after elimination of terms by applying Galerkin orthogonalities and Lemma 4.4:

$$\|\eta_{kh}\|^2_{L^2(I\times\Omega)} = \frac{1}{2}(\nabla\cdot\overline{uu_{kh}}, z_{kh}\cdot\xi)_{I\times\Omega} + c(\!(\overline{uu_{kh}},z_{kh},\xi)\!) - \hat c(\!(\xi,\overline{uu_{kh}},z_{kh})\!).$$

Applying Hölder's inequality yields

$$\begin{aligned}\|\eta_{kh}\|^2_{L^2(I\times\Omega)} \le{}& C\|\xi\|_{L^2(I\times\Omega)}\|1\|_{L^4(I;L^\infty(\Omega))}\\ &\times\big(\|\overline{uu_{kh}}\|_{L^4(I;L^\infty(\Omega))}\|\nabla z_{kh}\|_{L^\infty(I;L^2(\Omega))} + \|z_{kh}\|_{L^4(I;L^\infty(\Omega))}\|\nabla\overline{uu_{kh}}\|_{L^\infty(I;L^2(\Omega))}\big).\end{aligned}$$

By Proposition 3.1, Theorems 3.6, 4.13 and 4.20 all terms containing u, u_{kh} are bounded. Together with the discrete Gagliardo–Nirenberg inequality (16) and Young's inequality, we obtain

$$\|\eta_{kh}\|^2_{L^2(I\times\Omega)} \le C(u,u_{kh})\|\xi\|_{L^2(I\times\Omega)}T^{\frac14}\Big(\|A_h z_{kh}\|_{L^2(I;L^2(\Omega))} + \|\nabla z_{kh}\|_{L^\infty(I;L^2(\Omega))} + \|z_{kh}\|_{L^2(I;L^2(\Omega))}\Big).$$

Using Theorem 5.5, we can bound the norms of z_{kh} and obtain

$$\|\eta_{kh}\|^2_{L^2(I\times\Omega)} \le C(u,u_{kh})T^{\frac14}\|\xi\|_{L^2(I\times\Omega)}\|\eta_{kh}\|_{L^2(I\times\Omega)}.$$

Canceling terms concludes the proof. ☐

5.2. Explicit orders of convergence

Using the same arguments from before, instead of the best approximation type estimate, we can also directly use the error estimate for the Stokes projection, shown in Theorem 7.4 of [7] and Corollaries 6.2 and 6.4 of [36] to obtain the following corollaries, yielding explicit orders of convergence.

Corollary 5.8. *Let Assumption 4.1 be fulfilled. Further let $f \in L^\infty(I;L^2(\Omega)^2)$, $u_0 \in V\cap H^2(\Omega)^2$ and let u, u_{kh} be the continuous and fully discrete solutions to the Navier–Stokes equations (4) and (19) respectively. Then there holds*

$$\|u-u_{kh}\|_{L^\infty(I;L^2(\Omega)))} \le C\left(\ln\frac{T}{k}\right)^2(k+h^2)\Big(\|f\|_{L^\infty(I;L^2(\Omega))} + \|u_0\|_{V\cap H^2(\Omega)^2} + \|(u\cdot\nabla)u\|_{L^\infty(I;L^2(\Omega))}\Big),$$

where the constants C depend continuously on $\|f\|_{L^2(I;L^2(\Omega))}$ and $\|u_0\|_V$. The last term $\|(u\cdot\nabla)u\|_{L^\infty(I;L^2(\Omega))}$ can be bounded in terms of $\|f\|_{L^\infty(I;L^2(\Omega))}$ and $\|u_0\|_{V\cap H^2(\Omega)^2}$ by Corollary 3.11.

Proof. From Corollary 3.11 and Remark 3.3, we obtain $(u\cdot\nabla)u \in L^\infty(I;L^2(\Omega)^2)$. Hence this result is a direct consequence of Theorem 5.6 and Theorem 7.4 of [7]. ☐

Corollary 5.9. *Let Assumption 4.1 hold true. Further let* $f \in L^2(I; L^2(\Omega)^2)$, $u_0 \in V$ *and let* u, u_{kh} *be the continuous and fully discrete solutions to the Navier–Stokes equations* (4) *and* (19) *respectively. Then there hold the estimates*

$$\|\nabla(u - u_{kh})\|_{L^2(I;L^2(\Omega))} \le C\Big(k^{\frac{1}{2}} + h\Big), \quad \text{and}$$
$$\|(u - u_{kh})\|_{L^2(I;L^2(\Omega))} \le C\big(k + h^2\big),$$

where the constants C *depend continuously on* $\|f\|_{L^2(I;L^2(\Omega))}$ *and* $\|u_0\|_V$.

Proof. This result is a direct consequence of Theorems 4.16, 5.7 and Corollaries 6.2, 6.4 of [36]. □

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