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ERROR ESTIMATES FOR A MIXED FINITE ELEMENT METHOD FOR THE MAXWELL'S TRANSMISSION EIGENVALUE PROBLEM

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Abstract. In this paper, we analyze a numerical method combining the Ciarlet-Raviart mixed finite element formulation and an iterative algorithm for the Maxwell's transmission eigenvalue problem. The eigenvalue problem is first written as a nonlinear quad-curl eigenvalue problem. Then the real transmission eigenvalues are proved to be the roots of a non-linear function. They are the generalized eigenvalues of a related linear self-adjoint quad-curl eigenvalue problem. These generalized eigenvalues are computed by a mixed finite element method. We derive the error estimates using the spectral approximation of compact operators, the theory of mixed finite element method for quad-curl problems, and the derivatives of eigenvalues.

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1. Introduction

The Maxwell's transmission eigenvalue problem arises from the inverse scattering theory for anisotropic media [3]. Let Ω be a bounded, simply connected region with piecewise smooth boundary $\partial\Omega$ and unit outer normal vector $\boldsymbol{n}$. The transmission eigenvalue problem for the Maxwell's equations in terms of the electric field is to find $k^2 \in \mathbb{C}$, $\boldsymbol{E} \in [L^2(\Omega)]^3$, $\boldsymbol{E}_0 \in [L^2(\Omega)]^3$, and $\boldsymbol{E} - \boldsymbol{E}_0 \in H_0(\mathrm{curl}^2, \Omega)$ such that

$$\nabla \times \nabla \times \boldsymbol{E} - k^2 N \boldsymbol{E} = 0 \qquad \text{in } \Omega, \tag{1.1}$$

$$\nabla \times \nabla \times \boldsymbol{E}_0 - k^2 \boldsymbol{E}_0 = 0 \qquad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{E}_0 \times \boldsymbol{n} \qquad \text{on } \partial\Omega, \tag{1.3}$$

$$(\nabla \times \boldsymbol{E}) \times \boldsymbol{n} = (\nabla \times \boldsymbol{E}_0) \times \boldsymbol{n} \qquad \text{on } \partial\Omega, \tag{1.4}$$

where the 3×3 matrix value function $N(x) \in L^\infty(\Omega, \mathbb{R}^{3\times3})$ is the index of refraction. The values $k^2 \in \mathbb{C}$ such that the above equation has non-trivial solutions $\boldsymbol{E}$ and $\boldsymbol{E}_0$ are called the Maxwell's transmission eigenvalues. This problem plays an important role in the inverse scattering theory. One important application is the derivation

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of uniqueness for the reconstruction of inhomogeneous non-absorbing media [4]. Another one is the estimation of the material properties, such as the index of refraction, since transmission eigenvalues can be reconstructed from either far-field or near-field scattering data [3, 22].

The non-selfadjoint and nonlinear nature of the transmission eigenvalue problem makes it a challenging topic as the standard theory of partial differential equations do not directly apply. It is often reformulated as a quadratic eigenvalue problem or non-standard mixed eigenvalue problem. Theoretical results in this area are indeed partial, and there are many open problems. Numerical evidence can provide valuable insights and guide theorists towards new directions and applications.

In the last decade, there have been considerable efforts to develop effective numerical methods for the Helmholtz transmission eigenvalue problem [7, 9, 12–15, 18, 23, 26, 28, 29]. However, there are only a few works addressing the computation and/or numerical analysis for the Maxwell's transmission eigenvalue problem due to its complexity [1, 11, 17, 25]. The first numerical treatment of the Maxwell's transmission eigenvalue problem appears in [17] where the curl-conforming and the mixed finite element method are studied. In [25], based on a quad-curl formulation, a combination of a mixed finite element method and an iterative method is proposed. However, no error estimates are given. In [11], a robust numerical algorithm is proposed for computing a few smallest positive eigenvalues. In addition, an efficient Fourier-spectral-element method is developed to treat spherically stratified media [1].

To the best of the authors' knowledge, there are no results on the theoretical analysis for Maxwell's transmission eigenvalue problem in the framework of finite element methods.

The aim of the paper is the error analysis for the numerical approach presented in [25] for the approximations of real Maxwell's transmission eigenvalues. We first rewrite the Maxwell's transmission eigenvalue problem as a quad-curl problem. Then the transmission eigenvalues are shown to be the roots of an algebraic equation related to a series of self-adjoint positive definite quad-curl eigenvalue problems. A mixed method similar to that in [24] is employed for solving the quad-curl eigenvalue problem. Finally, the secant method is used to compute the roots of the algebraic equation. The error estimate is based on the theory of spectral approximation of compact operators [2], the error estimate of the mixed finite element method for the quad-curl problem [27], and the convergence of the secant method.

The rest of the paper is organized as follows. In Section 2, we recall the associated generalized formulation for (1.1)–(1.4) in [25] and present some properties. In Section 3, we introduce the mixed finite element method and the secant method. The convergence analysis is given in Section 4. Section 5 contains conclusions and discussions.

2. The associated generalized eigenvalue problem

We denote by $(\cdot,\cdot)$ the $[L^2(\Omega)]^3$ scalar product and define

$$H(\mathrm{curl}^m,\Omega)=\{\boldsymbol{u}\in[L^2(\Omega)]^3:(\nabla\times)^m\boldsymbol{u}\in[L^2(\Omega)]^3, j=1,\cdots,m\},$$

where $m>0$ is an integer. The inner product and norm for $H(\mathrm{curl}^m,\Omega)$ are

$$(\boldsymbol{u},\boldsymbol{v})_{H(\mathrm{curl}^m,\Omega)}=(\boldsymbol{u},\boldsymbol{v})+\sum_{j=1}^{m}\left((\nabla\times)^j\boldsymbol{u},(\nabla\times)^j\boldsymbol{v}\right)$$

and $\|\boldsymbol{u}\|_{H(\mathrm{curl}^m,\Omega)}=\sqrt{(\boldsymbol{u},\boldsymbol{v})_{H(\mathrm{curl}^m,\Omega)}}$, respectively. Define

$$\begin{aligned} H_0(\mathrm{curl},\Omega)&:=\{\boldsymbol{u}\in H(\mathrm{curl},\Omega):\boldsymbol{u}\times\boldsymbol{n}=0\quad\text{on }\partial\Omega\},\\ H_0(\mathrm{curl}^2,\Omega)&:=\{\boldsymbol{u}\in H_0(\mathrm{curl},\Omega):\nabla\times\boldsymbol{u}\in H_0(\mathrm{curl},\Omega)\},\end{aligned}$$

which are equipped with the norms

$$\|\boldsymbol{v}\|_{H(\mathrm{curl},\Omega)} = \left(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla\times\boldsymbol{v}\|^2_{L^2(\Omega)}\right)^{1/2},$$
$$\|\boldsymbol{v}\|_{H(\mathrm{curl}^2,\Omega)} = \left(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla\times\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla\times\nabla\times\boldsymbol{v}\|^2_{L^2(\Omega)}\right)^{1/2},$$

respectively.

We assume that N is a bounded positive definite real matrix field on Ω, *i.e.*, $\overline{\boldsymbol{\xi}}\cdot N\boldsymbol{\xi} \geq \gamma|\boldsymbol{\xi}|^2,\ \forall\,\boldsymbol{\xi}\in\mathbb{C}^3$ *a.e.* in Ω with some $\gamma > 0$. The assumption also holds for N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$. For more discussions on these assumptions, we refer the readers to [6].

We first rewrite the transmission eigenvalue problem (1.1)–(1.4) as in [6, 21, 25]. Setting $\tau := k^2$ and $\boldsymbol{u} = \boldsymbol{E} - \boldsymbol{E}_0$, the variational formulation is to find $\tau\in\mathbb{C}$ and $\boldsymbol{u}\in H_0(\mathrm{curl}^2,\Omega)$ such that

$$A_\tau(\boldsymbol{u},\boldsymbol{v}) - \tau B(\boldsymbol{u},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v}\in H_0(\mathrm{curl}^2,\Omega), \tag{2.1}$$

where

$$A_\tau(\boldsymbol{u},\boldsymbol{v}) = \big((N-I)^{-1}(\nabla\times\nabla\times\boldsymbol{u} - \tau\boldsymbol{u}), (\nabla\times\nabla\times\boldsymbol{v} - \tau\boldsymbol{v})\big) + \tau^2(\boldsymbol{u},\boldsymbol{v}) \tag{2.2}$$

and

$$B(\boldsymbol{u},\boldsymbol{v}) = (\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}). \tag{2.3}$$

The eigenvalue problem (2.1) is a nonlinear non-selfadjoint quad-curl problem. It is shown in [6, 21] that, if $(N-I)^{-1}$ is a bounded positive definite matrix field on Ω, A_τ is a coercive Hermitian sesquilinear form on $H_0(\mathrm{curl}^2,\Omega)\times H_0(\mathrm{curl}^2,\Omega)$. Furthermore, the sesquilinear form B is Hermitian and non-negative. This leads us to consider the following auxiliary eigenvalue problem for fixed τ:

$$A_\tau(\boldsymbol{u},\boldsymbol{v}) - \lambda(\tau) B(\boldsymbol{u},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v}\in H_0(\mathrm{curl}^2,\Omega). \tag{2.4}$$

Note that the generalized eigenvalue $\lambda(\tau)$ depends on τ since A_τ depends on τ. Then a real transmission eigenvalue is the root of the function

$$f(\tau) := \lambda(\tau) - \tau, \tag{2.5}$$

where $\lambda(\tau)$ is a generalized eigenvalue of (2.4). Note that λ can be any eigenvalue of (2.4), which leads to different $f(\tau)$ and thus different transmission eigenvalues.

It is necessary to show that the transmission eigenvalues exist as the roots of (2.5) with $\lambda(\tau)$ being a generalized eigenvalue of (2.4) under certain condition of $N(x)$. To this end, we denote

$$N^* = \sup_\Omega \eta_3(x), \qquad N_* = \inf_\Omega \eta_1(x), \tag{2.6}$$

where $0 \leq \eta_1(x) \leq \eta_2(x) \leq \eta_3(x)$ are the eigenvalues of the index of refraction $N(x)$. We also define

$$\Theta := 4\left(\frac{\kappa^{1/2}}{\beta_0} + \frac{\kappa}{\beta_0^2}\right),$$

where $\kappa > 0$ is an quad-curl eigenvalue studied in [24] and β_0 is the smallest Laplacian eigenvalue for Ω (*cf.* [16]).

Lemma 2.1. *The transmission eigenvalues exist as the roots of $f(\tau) := \lambda(\tau) - \tau$ where $\lambda(\tau)$ is the generalized eigenvalue of* (2.4) *provided*

$$1 + \Theta \leq N_* \leq \overline{\boldsymbol{\xi}}\cdot N(x)\boldsymbol{\xi} \leq N^* < \infty, \qquad \|\boldsymbol{\xi}\| = 1. \tag{2.7}$$

The condition (2.7) implies that

$$0 < \frac{1}{N^* - 1} \le \overline{\boldsymbol{\xi}} \cdot (N - I)^{-1} \boldsymbol{\xi} \le \frac{1}{N_* - 1} < \infty.$$

Under the above assumption, it is shown in [6] that if

$$0 < \tau_0 < \frac{\beta_0}{\sup_\Omega \|N\|_2}, \tag{2.8}$$

then $f(\tau_0) = \lambda(\tau_0) - \tau_0 > 0$. If

$$\tau_1 = \frac{\beta_0 - 2M\kappa^{1/2}}{2 + M}, \qquad M = \frac{1}{N_* - 1} \tag{2.9}$$

then $f(\tau_1) = \lambda(\tau_1) - \tau_1 < 0$. Since $f(\tau)$ is a continuous function of τ, there exists a τ^* such that $f(\tau^*) = 0$, which implies that τ^* is an transmission eigenvalue.

To compute the transmission eigenvalue, one can first determine two values of τ_0 and τ_1 as defined in equations (2.8) and (2.9). Then an iterative method such as the bisection method can be used to find the root of $f(\tau)$. However, the bisection method requires the computation of a generalized quad-curl problem at each step, which can be computationally intensive for 3D problems. In the following section, a secant method is employed, which is known to converge more quickly.

The condition (2.7) on $N(x)$ is rather strict and it can be relaxed significantly. The existence of transmission eigenvalues only requires $\|N(x)\|_2 \ge \alpha > 1$ for some positive α, for example, see Theorem 3.3 of [5]. Note that, in the inverse scattering theory, one often cares about the smallest transmission eigenvalue [25]. Note that the method in our paper can compute other real transmission eigenvalues. For example, to compute the second smallest eigenvalue, one computes the second smallest generalized eigenvalue of $A_\tau(\boldsymbol{u}, \boldsymbol{v}) - \tau B(\boldsymbol{u}, \boldsymbol{v}) = 0,\ \forall \boldsymbol{v} \in H_0(\mathrm{curl}^2, \Omega)$ and plug it in $f(\tau)$. The following analysis applies to all real eigenvalues.

3. Computation of transmission eigenvalues

Our numerical methods are based on finding the roots of a discrete version of (2.5). Since $\lambda(\tau)$ is the generalized eigenvalue of operator A_τ with respect to B, we need to compute an approximation $\lambda_h(\tau)$ for $\lambda(\tau)$. This is done by using a mixed finite element method for the generalized eigenvalue problem (2.4).

3.1. A mixed variational formulation

It is easy to see that the variational formulation (2.4) corresponds to the following quad-curl eigenvalue problem:

$$(\nabla \times \nabla \times -\tau)(N - I)^{-1}(\nabla \times \nabla \times -\tau)\boldsymbol{w} + \tau^2 \boldsymbol{w} = \lambda \nabla \times \nabla \times \boldsymbol{w}. \tag{3.1}$$

Similarly to the approach in [24, 27], we introduce the following auxiliary variables:

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{w}, \\ &\boldsymbol{v} = (N - I)^{-1}(\nabla \times \nabla \times -\tau)\boldsymbol{u}. \end{aligned}$$

Thus we obtain a mixed form for (3.1):

$$(\nabla \times \nabla \times -\tau)\boldsymbol{v} + \tau^2 \boldsymbol{u} = \lambda \nabla \times \nabla \times \boldsymbol{u}, \tag{3.2}$$

$$(\nabla \times \nabla \times -\tau)\boldsymbol{u} = (N - I)\boldsymbol{v}. \tag{3.3}$$

The weak formulation is to find $(\lambda, \boldsymbol{u}, \boldsymbol{v}) \in \big(\mathbb{R}, H_0(\text{curl}, \Omega), H(\text{curl}, \Omega)\big)$ such that

$$(\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{\xi}) - \tau(\boldsymbol{v}, \boldsymbol{\xi}) + \tau^2(\boldsymbol{u}, \boldsymbol{\xi}) = \lambda(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{\xi}), \tag{3.4}$$

$$(\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{\phi}) - \tau(\boldsymbol{u}, \boldsymbol{\phi}) = ((N - I)\boldsymbol{v}, \boldsymbol{\phi}), \tag{3.5}$$

for any $\boldsymbol{\xi} \in H_0(\text{curl}, \Omega)$ and $\boldsymbol{\phi} \in H(\text{curl}, \Omega)$.

For further analysis, we consider the corresponding source problem of the mixed formulation (3.4)–(3.5):

$$\begin{cases} (N - I)\boldsymbol{v} + (\tau - \nabla \times \nabla \times)\boldsymbol{u} = \boldsymbol{0}, \\ \qquad (\nabla \times \nabla \times -\tau)\boldsymbol{v} + \tau^2 \boldsymbol{u} = \nabla \times \nabla \times \boldsymbol{f}. \end{cases} \tag{3.6}$$

The weak formulation of (3.6) is to find $(\boldsymbol{u}, \boldsymbol{v}) \in H_0(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ such that

$$\begin{cases} a(\boldsymbol{v}, \boldsymbol{\phi}) + b(\boldsymbol{u}, \boldsymbol{\phi}) = \boldsymbol{0} & \forall \boldsymbol{\phi} \in H(\text{curl}, \Omega), \\ b(\boldsymbol{v}, \boldsymbol{\xi}) - c(\boldsymbol{u}, \boldsymbol{\xi}) = -(\nabla \times \boldsymbol{f}, \nabla \times \boldsymbol{\xi}) & \forall \boldsymbol{\xi} \in H_0(\text{curl}, \Omega), \end{cases} \tag{3.7}$$

where

$$a(\boldsymbol{v}, \boldsymbol{\phi}) = \big((N - I)\boldsymbol{v}, \boldsymbol{\phi}\big), \tag{3.8}$$

$$b(\boldsymbol{v}, \boldsymbol{\xi}) = \tau(\boldsymbol{v}, \boldsymbol{\xi}) - (\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{\xi}), \tag{3.9}$$

$$c(\boldsymbol{u}, \boldsymbol{\xi}) = \tau^2(\boldsymbol{u}, \boldsymbol{\xi}). \tag{3.10}$$

It is easy to see that the bilinear forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are bounded

$$a(\boldsymbol{v}, \boldsymbol{\phi}) \le C_1^* \|(N - I)\boldsymbol{v}\|_{L^2(\Omega)} \|\boldsymbol{\phi}\|_{L^2(\Omega)} \le C_1 \|\boldsymbol{v}\|_{H(\text{curl},\Omega)} \|\boldsymbol{\phi}\|_{H(\text{curl},\Omega)}, \tag{3.11}$$

$$\begin{aligned} b(\boldsymbol{v}, \boldsymbol{\xi}) &\le C_2^* \big(\|k\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\nabla \times \boldsymbol{v}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}} \big(\|k\boldsymbol{\xi}\|^2_{L^2(\Omega)} + \|\nabla \times \boldsymbol{\xi}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}} \\ &\le C_2 \|\boldsymbol{v}\|_{H(\text{curl},\Omega)} \|\boldsymbol{\xi}\|_{H(\text{curl},\Omega)}, \end{aligned} \tag{3.12}$$

$$c(\boldsymbol{u}, \boldsymbol{\xi}) \le C_3 \|\boldsymbol{u}\|_{H(\text{curl},\Omega)} \|\boldsymbol{\xi}\|_{H(\text{curl},\Omega)}, \tag{3.13}$$

since $(N - I)$ is a bounded real matrix.

Lemma 3.1. *For each $\boldsymbol{f} \in H(curl, \Omega)$, the problem (3.7) is well-posed.*

Proof. Define two closed subspaces of $H(\text{curl}, \Omega)$ by

$$K = \{\boldsymbol{v} \in H(\text{curl}, \Omega) : b(\boldsymbol{v}, \boldsymbol{\xi}) = 0 \quad \forall \boldsymbol{\xi} \in H_0(\text{curl}, \Omega)\},$$
$$H = \{\boldsymbol{\xi} \in H_0(\text{curl}, \Omega) : b(\boldsymbol{v}, \boldsymbol{\xi}) = 0 \quad \forall \boldsymbol{v} \in H(\text{curl}, \Omega)\}.$$

First, $a(\cdot, \cdot)$ is coercive on K. More precisely, if we take γ to be the smallest eigenvalue of $(N - I)$, then we have that

$$a(\boldsymbol{v}, \boldsymbol{v}) \ge \gamma \|\boldsymbol{v}\|^2_{L^2(\Omega)} \ge C_4 \|\boldsymbol{v}\|^2_{H(\text{curl},\Omega)} \quad \forall \boldsymbol{v} \in K, \tag{3.14}$$

where C_4 is a positive constant satisfying $C_4 \le \frac{\gamma}{1+\tau}$. Moreover, by taking $\boldsymbol{v} = \boldsymbol{\xi} \in H$, we have that

$$c(\boldsymbol{\xi}, \boldsymbol{\xi}) \ge C_5 \|\boldsymbol{\xi}\|^2_{H(\text{curl},\Omega)} \quad \forall \boldsymbol{\xi} \in H, \tag{3.15}$$

where C_5 is a positive constant satisfying $C_5 \le \frac{\tau^2}{1+\tau}$. Let $K^\perp$ be the orthogonal complement of K in $H(\text{curl}, \Omega)$. For given $\boldsymbol{v} \in K^\perp$, we can take $\boldsymbol{\xi} = \boldsymbol{v} \in H_0(\text{curl}, \Omega)$ to obtain

$$\sup_{\boldsymbol{\xi} \in H(\text{curl},\Omega)} \frac{b(\boldsymbol{v}, \boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_{H(\text{curl},\Omega)}} \ge \frac{\|\nabla \times \boldsymbol{v}\|^2_{L^2(\Omega)}}{\|\boldsymbol{v}\|_{H(\text{curl},\Omega)}} \ge C_6 \|\boldsymbol{v}\|_{H(\text{curl},\Omega)}, \tag{3.16}$$

where C_6 is the constant in the Friedrichs inequality. Similarly, for any $\boldsymbol{\xi} \in H^{\perp}$, we take $\boldsymbol{v} = \boldsymbol{\xi} \in H(\mathrm{curl}, \Omega)$ to obtain

$$\sup_{\boldsymbol{v}\in H(\mathrm{curl},\Omega)} \frac{b(\boldsymbol{\xi}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{H(\mathrm{curl},\Omega)}} \geq \frac{\|\nabla \times \boldsymbol{\xi}\|^2_{L^2(\Omega)}}{\|\boldsymbol{\xi}\|_{H(\mathrm{curl},\Omega)}} \geq C_6 \|\boldsymbol{\xi}\|_{H(\mathrm{curl},\Omega)}. \tag{3.17}$$

The well-posedness of (3.7) directly follows from (3.14), (3.15), (3.16) and (3.17). □

Due to the well-posedness of the weak problem (3.7) from Lemma 3.1, we can define two operators for the spectral approximation of problems (3.4)–(3.5):

$$\begin{aligned} &\mathbb{A} : H(\mathrm{curl}, \Omega) \to H(\mathrm{curl}, \Omega), \qquad && \boldsymbol{f} \to \mathbb{A}\boldsymbol{f} := \boldsymbol{v}, \\ &\mathbb{B} : H(\mathrm{curl}, \Omega) \to H_0(\mathrm{curl}, \Omega), \qquad && \boldsymbol{f} \to \mathbb{B}\boldsymbol{f} := \boldsymbol{u}, \end{aligned}$$

where $(\boldsymbol{u}, \boldsymbol{v})$ is the solution of the source problem (3.6). Then the eigenpair $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ can be characterized by these operators. It is also known that the problem (3.4)–(3.5) does not admit the null eigenvalue. Hence λ is an eigenvalue of (3.4)–(3.5) if and only if $\frac{1}{\lambda}$ is an eigenvalue of $\mathbb{B}$ associated with the same eigenvector $(\mathbb{B}(\lambda \boldsymbol{u}), \mathbb{A}(\lambda \boldsymbol{v}))$. We can easily check that the operator $\mathbb{B}$ is a bounded linear operator. Furthermore, the operator $\mathbb{B}$ is compact due to the compact imbedding of $H_0(\mathrm{curl}, \Omega)$ into $H(\mathrm{curl}, \Omega)$. Thus the spectrum $\sigma(\mathbb{B})$ has 0 as the only possible accumulation point.

3.2. Mixed finite element and iterative algorithm

In this subsection we introduce the computation method for the Maxwell's transmission eigenvalue problem combining a mixed finite element method and an iterative algorithm, which follows the idea from [25].

We use the curl conforming edge elements of Nédélec [19,20] to discretize the generalized eigenvalue problem. Let S_h be the space of linear edge element and S_h^0 be the space of functions in S_h that have vanishing DoF on $\partial\Omega$, *i.e.*, $S_h^0 = S_h \cap H_0(\mathrm{curl}, \Omega)$. The discrete formulation for (3.4)–(3.5) is to find $(\lambda, \boldsymbol{u}_h, \boldsymbol{v}_h) \in (\mathbb{R}, S_h^0, S_h)$ such that

$$(\nabla \times \boldsymbol{v}_h, \nabla \times \boldsymbol{\xi}_h) - \tau(\boldsymbol{v}_h, \boldsymbol{\xi}_h) + \tau^2(\boldsymbol{u}_h, \boldsymbol{\xi}_h) = \lambda_h(\nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{\xi}_h) \quad \forall \boldsymbol{\xi}_h \in S_h^0, \tag{3.18}$$

$$(\nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{\phi}_h) - \tau(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = ((N - I)\boldsymbol{v}_h, \boldsymbol{\phi}_h) \qquad \forall \boldsymbol{\phi}_h \in S_h. \tag{3.19}$$

Let $\psi_1, \cdots, \psi_K$ be a basis for S_h^0 and $\psi_1, \cdots, \psi_K, \psi_{K+1}, \cdots, \psi_T$ be a basis for S_h. Let $\boldsymbol{u}_h = \sum_{i=1}^{K} u_i \psi_i$ and $\boldsymbol{v}_h = \sum_{i=1}^{T} v_i \psi_i$. Furthermore, let $\overrightarrow{\boldsymbol{u}} = (u_1, \cdots, u_K)^T$ and $\overrightarrow{\boldsymbol{v}} = (v_1, \cdots, v_T)^T$. Then the matrix form is

$$S_{K\times T}\overrightarrow{\boldsymbol{v}} - \tau M_{K\times T}\overrightarrow{\boldsymbol{v}} + \tau^2 M_{K\times K}\overrightarrow{\boldsymbol{u}} = \lambda_h S_{K\times K}\overrightarrow{\boldsymbol{u}}, \tag{3.20}$$

$$S_{T\times K}\overrightarrow{\boldsymbol{u}} - \tau M_{T\times K}\overrightarrow{\boldsymbol{u}} = M_{T\times T}^{N-1}\overrightarrow{\boldsymbol{v}}, \tag{3.21}$$

where the matrices are defined in Table 1.

From (3.21) we get

$$\overrightarrow{\boldsymbol{v}} = (M_{T\times T}^{N-I})^{-1}(S_{T\times K} - \tau M_{T\times K})\overrightarrow{\boldsymbol{u}}.$$

Substituting $\overrightarrow{\boldsymbol{v}}$ in (3.20), we obtain a generalized matrix eigenvalue problem

$$A\overrightarrow{\boldsymbol{u}} = \lambda S_{K\times K}\overrightarrow{\boldsymbol{u}}, \tag{3.22}$$

where

$$A = \big((S_{K\times T} - \tau M_{K\times T})(M_{T\times T}^{N-I})^{-1}(S_{T\times K} - \tau M_{T\times K}) + \tau^2 M_{K\times K}\big).$$

For a mesh $\mathcal{T}_h$ for Ω, assume $\lambda_h(\tau)$ is the jth eigenvalue of (3.22) $(j = 1, 2, \ldots)$. Note that $\lambda_h(\tau)$ depends on τ continuously. Then we use $\lambda_h(\tau)$ to compute the root of (2.5). The following lemma follows the analysis in the next section (*cf.* [2,8] and Thm. 4.5 in Sect. 4.).

TABLE 1. Definitions of the finite element matrices.

Matrix	Dimension	Definition
$S_{K\times K}$	$K\times K$	$S^{i,j}_{K\times K}=(\nabla\times\psi_i,\nabla\times\psi_j), 1\le i\le K, 1\le j\le K$
$S_{K\times T}$	$K\times T$	$S^{i,j}_{K\times T}=(\nabla\times\psi_i,\nabla\times\psi_j), 1\le i\le K, 1\le j\le T$
$S_{T\times K}$	$T\times K$	$S^{i,j}_{T\times K}=(\nabla\times\psi_i,\nabla\times\psi_j), 1\le i\le T, 1\le j\le K$
$M_{K\times K}$	$K\times K$	$M^{i,j}_{K\times K}=(\psi_i,\psi_j), 1\le i\le K, 1\le j\le K$
$M_{K\times T}$	$K\times T$	$M^{i,j}_{K\times T}=(\psi_i,\psi_j), 1\le i\le K, 1\le j\le T$
$M_{T\times K}$	$T\times K$	$M^{i,j}_{T\times K}=(\psi_i,\psi_j), 1\le i\le T, 1\le j\le K$
$M^{N-I}_{T\times T}$	$T\times T$	$(M^{N-I}_{T\times T})^{i,j}=((N-I)\psi_i,\psi_j), 1\le i\le T, 1\le j\le T$

Lemma 3.2. *Assume that we apply the mixed finite element method for* (2.4) *on a Lipschitz domain* Ω *and the index of refraction* $N(x)$ *satisfies* (2.7). *Let* $\lambda_h(\tau)$ *be the finite element approximation of a generalized eigenvalue* $\lambda(\tau)$ *on a triangular mesh* $\mathcal{T}_h$ *with mesh size* h. *Then for any* $\varepsilon>0$, *there exists an* h_0 *such that if* $h\le h_0$ *then*

$$|\lambda_h(\tau)-\lambda(\tau)|\le\varepsilon.$$

Now we are in the position to present a secant method to compute the root of

$$f_h(\tau):=\lambda_h(\tau)-\tau. \tag{3.23}$$

Algorithm (Secant Method): $\tau = secantTE(x_0, x_1, N(x), tol, maxit)$
generate a regular tetrahedra method for D
set $it=1$ and $\delta=\text{abs}(x_1-x_0)$
compute the generalized eigenvalue $\lambda_{h,A}$ of (2.4) for $\tau=x_0$
compute the generalized eigenvalue $\lambda_{h,B}$ of (2.4) for $\tau=x_1$
while $\delta>tol$ and $it<maxit$
$\tau=x_1-\lambda_{h,B}\frac{x_1-x_0}{\lambda_{h,B}-\lambda_{h,A}}$
compute the eigenvalue λ_τ of $A\boldsymbol{x}=\lambda B\boldsymbol{x}$
$\delta=\text{abs}(\lambda_\tau-\tau)$
$x_0=x_1$, $x_1=\tau$, $\lambda_{h,A}=\lambda_{h,B}$, $\lambda_{h,B}=\lambda_\tau$, $it=it+1$.
end

Here x_0 and x_1 are initial values which are chosen close to zero and $x_0<x_1<\frac{\lambda_0}{\sup_D\|N\|_2}$. This is due to the fact that $f(\tau)$ is positive in an interval I right to zero. The parameters $maxit$ and tol are the maximum number of iterations and precision, respectively.

4. Convergence analysis

In this section, we first prove the convergence of the mixed method for the eigenvalue problem (3.1). Then we establish the error estimate of the iterative algorithm for the transmission eigenvalue, *i.e.*, the root of (2.5).

4.1. Error analysis for the mixed finite element method

To analyze the convergence of the mixed finite element method (3.18)–(3.19), we first define some discrete solution operators for the associated source problem and then prove the convergence using the abstract spectral approximation theory for compact operators [2]. To this end, we introduce the following discrete solution operators:

$$\mathbb{A}_h: H(\text{curl},\Omega)\to S_h, \qquad \boldsymbol{f}\to\mathbb{A}_h\boldsymbol{f}:=\boldsymbol{v}_h,$$
$$\mathbb{B}_h: H(\text{curl},\Omega)\to S_h^0, \qquad \boldsymbol{f}\to\mathbb{B}_h\boldsymbol{f}:=\boldsymbol{u}_h,$$

with $(\boldsymbol{u}_h, \boldsymbol{v}_h)$ satisfying

$$\begin{cases} a(\boldsymbol{v}_h, \boldsymbol{\phi}_h) + b(\boldsymbol{u}_h, \boldsymbol{\phi}_h) = \mathbf{0}, & \forall\, \boldsymbol{\phi}_h \in S_h, \\ b(\boldsymbol{v}_h, \boldsymbol{\xi}_h) - c(\boldsymbol{u}_h, \boldsymbol{\xi}_h) = -(\nabla \times \boldsymbol{f}, \nabla \times \boldsymbol{\xi}_h), & \forall\, \boldsymbol{\xi}_h \in S_h^0. \end{cases} \tag{4.1}$$

Let

$$K_h = \{\boldsymbol{v}_h \in S_h : b(\boldsymbol{v}_h, \boldsymbol{\xi}_h) = 0 \quad \forall\, \boldsymbol{\xi}_h \in S_h^0\}, \tag{4.2}$$

$$H_h = \{\boldsymbol{\xi}_h \in S_h^0 : b(\boldsymbol{v}_h, \boldsymbol{\xi}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in S_h\}. \tag{4.3}$$

The form $a(\cdot,\cdot)$ is coercive on K_h, *i.e.*, there exists a constant $C_1^h > 0$ such that

$$a(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_1^h \|\boldsymbol{v}_h\|^2_{H(\mathrm{curl},\Omega)} \qquad \forall\, \boldsymbol{v}_h \in K_h. \tag{4.4}$$

Analogous to the continuous problem, we have

$$b(\boldsymbol{\xi}_h, \boldsymbol{\xi}_h) = \tau \|\boldsymbol{\xi}_h\|^2_{L^2(\Omega)} - \|\nabla \times \boldsymbol{\xi}_h\|^2_{L^2(\Omega)} = 0 \quad \text{for } \boldsymbol{\xi}_h \in S_h.$$

Hence, the choice of $C_3^h \leq \frac{\tau^2}{1+\tau}$ implies

$$c(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq C_3^h \|\boldsymbol{u}_h\|^2_{H(\mathrm{curl},\Omega)} \qquad \forall\, \boldsymbol{u}_h \in S_h^0. \tag{4.5}$$

Let $K_h^\perp$ be the complement of K_h in S_h^0. Then for arbitrary $\boldsymbol{v}_h \in K_h^\perp$, we take $\boldsymbol{\xi}_h = \boldsymbol{v}_h \in S_h$ to get

$$\sup_{\boldsymbol{\xi}_h \in S_h^0} \frac{b(\boldsymbol{v}_h, \boldsymbol{\xi}_h)}{\|\boldsymbol{\xi}_h\|_{H(\mathrm{curl},\Omega)}} \geq \frac{\|\nabla \times \boldsymbol{v}_h\|^2_{L^2(\Omega)}}{\|\boldsymbol{v}_h\|_{H(\mathrm{curl},\Omega)}} \geq C_2^h \|\boldsymbol{v}_h\|_{H(\mathrm{curl},\Omega)} \tag{4.6}$$

for some $C_2^h > 0$. Similarly, for any $\boldsymbol{\xi}_h \in H_h^\perp$, we have

$$\sup_{\boldsymbol{v}_h \in S_h} \frac{b(\boldsymbol{\xi}_h, \boldsymbol{v}_h)}{\|\boldsymbol{v}_h\|_{H(\mathrm{curl},\Omega)}} \geq \frac{\|\nabla \times \boldsymbol{\xi}\|^2_{L^2(\Omega)}}{\|\boldsymbol{\xi}_h\|_{H(\mathrm{curl},\Omega)}} \geq C_2^h \|\boldsymbol{\xi}_h\|_{H(\mathrm{curl},\Omega)}. \tag{4.7}$$

The well-posedness of the discrete scheme (4.1) then follows from (4.4), (4.5), (4.6) and (4.7) (see Thm. 3.1 of [27]).

Next, we study the convergence of $\mathbb{B}_h$ and $\mathbb{A}_h$ to the continuous operators $\mathbb{B}$ and $\mathbb{A}$, respectively. The first step is to derive an auxiliary error estimate.

Theorem 4.1. *Given $\boldsymbol{f} \in H(curl,\Omega)$, let $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{u}_h, \boldsymbol{v}_h)$ be the solutions of (3.7) and (4.1), respectively. Then there exists a positive constant C independent of h and $\boldsymbol{f}$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} + \|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(\Omega)} \leq C\Bigg(\inf_{\substack{\boldsymbol{\xi} \in S_h^0 \\ \boldsymbol{\phi} \in S_h}} \left(\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)} + \|\boldsymbol{v} - \boldsymbol{\phi}_h\|_{L^2(\Omega)} \right) + \inf_{\boldsymbol{\mu}_h \in S_h} \|\boldsymbol{v} - \boldsymbol{\mu}_h\|_{H(curl,\Omega)} \Bigg), \tag{4.8}$$

where $\boldsymbol{\xi}_h$ and $\boldsymbol{\phi}_h$ satisfy the constraint

$$(\nabla \times \boldsymbol{\xi}_h, \nabla \times \boldsymbol{\mu}_h) - \tau(\boldsymbol{\xi}_h, \boldsymbol{\mu}_h) = ((N - I)\boldsymbol{\phi}_h, \boldsymbol{\mu}_h) \quad \forall\, \boldsymbol{\mu}_h \in S_h^0. \tag{4.9}$$

Proof. Assume $\boldsymbol{u} \in H(\mathrm{curl}^3, \Omega)$ such that

$$(\nabla \times \nabla \times \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) = (\nabla \times \nabla \times \boldsymbol{u}, \nabla \times \nabla \times \boldsymbol{v}) \tag{4.10}$$

for all $\boldsymbol{v} \in [\mathcal{D}(\Omega)]^3$. By density argument, it holds that for all $\boldsymbol{\xi} \in H_0(\mathrm{curl}, \Omega)$,

$$\begin{aligned}&\big(\nabla \times (N-I)^{-1}(\nabla \times \nabla \times -\tau)\boldsymbol{u}, \nabla \times \boldsymbol{\xi}\big)\\ &\quad - \tau\big((N-I)^{-1}(\nabla \times \nabla \times -\tau)\boldsymbol{u}, \boldsymbol{\xi}\big) + \tau^2(\boldsymbol{u}, \boldsymbol{\xi}) = (\nabla \times \boldsymbol{f}, \nabla \times \boldsymbol{\xi}).\end{aligned} \tag{4.11}$$

Assume that $(\boldsymbol{\xi}_h, \boldsymbol{\phi}_h) \in S_h^0 \times S_h$ satisfies the following relation

$$(\nabla \times \boldsymbol{\xi}_h, \nabla \times \boldsymbol{\mu}_h) - \tau(\boldsymbol{\xi}_h, \boldsymbol{\mu}_h) = ((N-I)\boldsymbol{\phi}_h, \boldsymbol{\mu}_h) \quad \forall\, \boldsymbol{\mu}_h \in S_h^0. \tag{4.12}$$

Note that the solution $(\boldsymbol{u}_h, \boldsymbol{v}_h)$ of the discrete problem (4.1) also satisfies

$$(\nabla \times \boldsymbol{u}_h, \nabla \times \boldsymbol{\mu}_h) - \tau(\boldsymbol{u}_h, \boldsymbol{\mu}_h) = ((N-I)\boldsymbol{v}_h, \boldsymbol{\mu}_h) \quad \forall\, \boldsymbol{\mu}_h \in S_h^0. \tag{4.13}$$

Subtracting (4.12) from (4.13) and letting $\boldsymbol{\mu}_h = \boldsymbol{u}_h - \boldsymbol{\xi}_h$, we have

$$\begin{aligned}&\big(\nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h), \nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h)\big) - \tau(\boldsymbol{u}_h - \boldsymbol{\xi}_h, \boldsymbol{u}_h - \boldsymbol{\xi}_h)\\ &\qquad = \big((N-I)(\boldsymbol{v}_h - \boldsymbol{\phi}_h), \boldsymbol{u}_h - \boldsymbol{\xi}_h\big).\end{aligned} \tag{4.14}$$

The above equation implies that

$$\begin{aligned}(1-\tau C)\|\nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h)\|^2_{L^2(\Omega)} &\le \|\nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h)\|^2_{L^2(\Omega)} - \tau\|\boldsymbol{u}_h - \boldsymbol{\xi}_h\|^2_{L^2(\Omega)}\\ &\le \gamma^*\|\boldsymbol{v}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{u}_h - \boldsymbol{\xi}_h\|_{L^2(\Omega)},\end{aligned} \tag{4.15}$$

where C is the constant in the Friedrichs inequality and $\gamma^* = \|N - I\|_2$. Note that $\gamma^* = \max_k \sigma_k$ with σ_k being the singular value of the matrix $(N - I)$. Here, we assume $1 - \tau C > 0$ in (4.15). Again by the Friderichs inequality, we have

$$\|\nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h)\|_{L^2(\Omega)} \le \frac{\gamma^*}{C(1-\tau C)}\|\boldsymbol{v}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}, \tag{4.16}$$

$$\|\boldsymbol{u}_h - \boldsymbol{\xi}_h\|_{L^2(\Omega)} \le \frac{\gamma^*}{C^2(1-\tau C)}\|\boldsymbol{v}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}. \tag{4.17}$$

It is easy to see that for any function pair $(\boldsymbol{\xi}_h, \boldsymbol{\phi}_h)$ that satisfies (4.12), it holds

$$\begin{aligned}(\nabla \times \boldsymbol{\xi}_h, \nabla \times \boldsymbol{v}_h) - \tau(\boldsymbol{\xi}_h, \boldsymbol{v}_h) &= ((N-I)\boldsymbol{\phi}_h, \boldsymbol{v}_h),\\ (\nabla \times \boldsymbol{v}_h, \nabla \times \boldsymbol{\xi}_h) - \tau(\boldsymbol{v}_h, \boldsymbol{\xi}_h) + \tau^2(\boldsymbol{u}_h, \boldsymbol{\xi}_h) &= (\nabla \times \boldsymbol{f}, \nabla \times \boldsymbol{\xi}_h),\end{aligned}$$

which implies

$$((N-I)\boldsymbol{\phi}_h, \boldsymbol{v}_h) =(\nabla \times \boldsymbol{f}, \nabla \times \boldsymbol{\xi}_h) - \tau^2(\boldsymbol{u}_h, \boldsymbol{\xi}_h), \tag{4.18}$$

$$((N-I)\boldsymbol{v}_h, \boldsymbol{v}_h) =(\nabla \times \boldsymbol{f}, \nabla \times \boldsymbol{u}_h) - \tau^2(\boldsymbol{u}_h, \boldsymbol{u}_h). \tag{4.19}$$

For any $\boldsymbol{\mu}_h \in S_h^0$, it follows from (4.11), (4.14), (4.18) and (4.19) that

$$\begin{aligned}&\big(\nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h), \nabla \times (\boldsymbol{v} - \boldsymbol{\mu}_h)\big) - \tau(\boldsymbol{u}_h - \boldsymbol{\xi}_h, \boldsymbol{v} - \boldsymbol{\mu}_h) - \big((N-I)(\boldsymbol{v}_h - \boldsymbol{\phi}_h), \boldsymbol{v} - \boldsymbol{\mu}_h\big)\\ &= \big(\nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h), \nabla \times \boldsymbol{v}\big) - \tau(\boldsymbol{u}_h - \boldsymbol{\xi}_h, \boldsymbol{v}) - \big((N-I)(\boldsymbol{v}_h - \boldsymbol{\phi}_h), \boldsymbol{v}\big)\\ &= (\nabla \times \boldsymbol{f}, \nabla \times (\boldsymbol{u}_h - \boldsymbol{\xi}_h)) - \tau^2(\boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{\xi}_h) - \big((N-I)(\boldsymbol{v}_h - \boldsymbol{\phi}_h), \boldsymbol{v}\big)\\ &= -\tau^2(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \boldsymbol{\xi}_h) - \big((\boldsymbol{v} - \boldsymbol{v}_h), (N-I)(\boldsymbol{v}_h - \boldsymbol{\phi}_h)\big).\end{aligned} \tag{4.20}$$

From (4.17), (4.16) and (4.20), we have

$$
\begin{aligned}
&\left|\tau^2(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{\xi}_h)+\big((\boldsymbol{v}-\boldsymbol{v}_h),(N-I)(\boldsymbol{v}_h-\boldsymbol{\phi}_h)\big)\right| \\
&\quad\le \|\nabla\times(\boldsymbol{u}_h-\boldsymbol{\xi}_h)\|_{L^2(\Omega)}\|\nabla\times(\boldsymbol{v}-\boldsymbol{\mu}_h)\|_{L^2(\Omega)}+\tau\|\boldsymbol{u}_h-\boldsymbol{\xi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{L^2(\Omega)} \\
&\qquad\qquad+\gamma^*\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{L^2(\Omega)} \\
&\quad\le \frac{\gamma^*}{C(1-\tau C)}\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\nabla\times(\boldsymbol{v}-\boldsymbol{\mu}_h)\|_{L^2(\Omega)} \\
&\qquad+\frac{\tau\gamma^*}{C^2(1-\tau C)}\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{L^2(\Omega)} \\
&\qquad\qquad+\gamma^*\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{L^2(\Omega)} \\
&\le C_1\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{H(\mathrm{curl},\Omega)},
\end{aligned}
\tag{4.21}
$$

where $C_1=\max\{\frac{\gamma^*}{C(1-\tau C)},\frac{\tau\gamma^*}{C^2(1-\tau C)},\gamma^*\}$. It then directly follows from (4.21) that

$$
\begin{aligned}
&(N_*-1)\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|^2_{L^2(\Omega)}+\tau^2\|\boldsymbol{u}_h-\boldsymbol{\xi}_h\|^2_{L^2(\Omega)} \\
&\quad\le \big((N-I)(\boldsymbol{v}_h-\boldsymbol{\phi}_h),\boldsymbol{v}_h-\boldsymbol{\phi}_h\big)+\tau^2(\boldsymbol{u}_h-\boldsymbol{\xi}_h,\boldsymbol{u}_h-\boldsymbol{\xi}_h) \\
&\quad= -\big((N-I)(\boldsymbol{v}_h-\boldsymbol{\phi}_h),\boldsymbol{v}-\boldsymbol{v}_h\big)+\big((N-I)(\boldsymbol{v}_h-\boldsymbol{\phi}_h),\boldsymbol{v}-\boldsymbol{\phi}_h\big) \\
&\qquad-\tau^2(\boldsymbol{u}_h-\boldsymbol{\xi}_h,\boldsymbol{u}-\boldsymbol{u}_h)+\tau^2(\boldsymbol{u}_h-\boldsymbol{\xi}_h,\boldsymbol{u}-\boldsymbol{\xi}_h) \\
&\quad\le \left|\big((N-I)(\boldsymbol{v}_h-\boldsymbol{\phi}_h),(\boldsymbol{v}-\boldsymbol{v}_h)\big)+\tau^2(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_h-\boldsymbol{\xi}_h)\right| \\
&\qquad+\big((N-I)(\boldsymbol{v}_h-\boldsymbol{\phi}_h),\boldsymbol{v}-\boldsymbol{\phi}_h\big)+\tau^2(\boldsymbol{u}_h-\boldsymbol{\xi}_h,\boldsymbol{u}-\boldsymbol{\xi}_h) \\
&\quad\le C_1\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{H(\mathrm{curl},\Omega)} \\
&\qquad+\gamma^*\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v}-\boldsymbol{\phi}_h\|_{L^2(\Omega)}+\tau^2\|\boldsymbol{u}_h-\boldsymbol{\xi}_h\|_{L^2(\Omega)}\|\boldsymbol{u}-\boldsymbol{\xi}_h\|_{L^2(\Omega)},
\end{aligned}
\tag{4.22}
$$

where N_* is given in (2.7). Therefore, by (4.17) and (4.22), we get

$$
\begin{aligned}
\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)}\le \frac{C_1}{N_*-1}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{H(\mathrm{curl},\Omega)}+\frac{1}{N_*-1}\|\boldsymbol{v}-\boldsymbol{\phi}_h\|_{L^2(\Omega)} \\
+\frac{\tau^2\gamma^*}{C^2(1-\tau C)(N_*-1)}\|\boldsymbol{u}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.23}
$$

Moreover, by using (4.17) and (4.23), we have

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)}+\|\boldsymbol{v}-\boldsymbol{v}_h\|_{L^2(\Omega)} \\
&\quad\le \|\boldsymbol{u}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}+\|\boldsymbol{\xi}_h-\boldsymbol{u}_h\|_{L^2(\Omega)}+\|\boldsymbol{v}-\boldsymbol{\phi}_h\|_{L^2(\Omega)}+\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)} \\
&\quad\le \|\boldsymbol{u}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}+\|\boldsymbol{v}-\boldsymbol{\phi}_h\|_{L^2(\Omega)}+\left(1+\frac{\gamma^*}{C^2(1-\tau C)}\right)\|\boldsymbol{v}_h-\boldsymbol{\phi}_h\|_{L^2(\Omega)} \\
&\quad\le C_1^*\|\boldsymbol{u}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}+C_2^*\|\boldsymbol{v}-\boldsymbol{\phi}_h\|_{L^2(\Omega)}+C_3^*\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{H(\mathrm{curl},\Omega)},
\end{aligned}
\tag{4.24}
$$

where C_1^*, C_2^* and C_3^* are positive constants that depend on C_1, τ, γ^* and C in (4.23). The proof is then completed by taking the infimum of the right hand side of (4.24).

□

Theorem 4.2. *Given $\boldsymbol{f}\in H(curl,\Omega)$, let $(\boldsymbol{u},\boldsymbol{v})$ and $(\boldsymbol{u}_h,\boldsymbol{v}_h)$ be the solutions of (3.7) and (4.1), respectively. Then there exists a positive constant C independent of the mesh size h and $\boldsymbol{f}$ such that*

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)}+\|\boldsymbol{v}-\boldsymbol{v}_h\|_{L^2(\Omega)}\le \\
&C\left(\inf_{\boldsymbol{\xi}_h\in S_h^0}\big(\|\boldsymbol{u}-\boldsymbol{\xi}_h\|_{L^2(\Omega)}+(1+\alpha(h))\|\nabla\times(\boldsymbol{u}-\boldsymbol{\xi}_h)\|_{L^2(\Omega)}\big)+\inf_{\boldsymbol{\mu}_h\in S_h}\|\boldsymbol{v}-\boldsymbol{\mu}_h\|_{L^2(\Omega)}\right),
\end{aligned}
\tag{4.25}
$$

where $\alpha(h)=C_2/h$ with C_2 being a positive constant.

Proof. Taking $\boldsymbol{\mu}_h = \boldsymbol{\mu}_h - \boldsymbol{\phi}_h$ in (4.9), we have

$$\big(\nabla \times \boldsymbol{\xi}_h, \nabla \times (\boldsymbol{\mu}_h - \boldsymbol{\phi}_h)\big) - \tau(\boldsymbol{\xi}_h, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h) = \big((N-I)\boldsymbol{\phi}_h, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h\big). \tag{4.26}$$

Since $\boldsymbol{n} \times (\nabla \times \boldsymbol{u}) = \mathbf{0}$ on $\partial\Omega$, it holds that

$$\begin{aligned}(\nabla \times \nabla \times \boldsymbol{u}, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h) - \tau(\boldsymbol{u}, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h)&\\ =\big(\nabla \times \boldsymbol{u}, \nabla \times (\boldsymbol{\mu}_h - \boldsymbol{\phi}_h)\big) - \tau(\boldsymbol{u}, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h).&\end{aligned}$$

Subtracting the above two equations, we have

$$\begin{aligned}&\big((\nabla \times \nabla \times \boldsymbol{u} - \tau\boldsymbol{u}) - (N-I)\boldsymbol{\phi}_h, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h\big)\\ &\quad = \big(\nabla \times (\boldsymbol{u} - \boldsymbol{\xi}_h), \nabla \times (\boldsymbol{\mu}_h - \boldsymbol{\phi}_h)\big) - \tau(\boldsymbol{u} - \boldsymbol{\xi}_h, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h).\end{aligned} \tag{4.27}$$

For regular meshes, we have the following inverse inequality [10]

$$\|\nabla \times \boldsymbol{v}_h\|_{L^2(\Omega)} \le Ch^{-1}\|\boldsymbol{v}_h\|_{L^2(\Omega)} \qquad \forall\, \boldsymbol{v}_h \in S_h,$$

where C is a positive constant independent of h. Therefore,

$$\begin{aligned}&\big|\big((\nabla \times \nabla \times \boldsymbol{u} - \tau\boldsymbol{u}) - (N-I)\boldsymbol{\phi}_h, \boldsymbol{\mu}_h - \boldsymbol{\phi}_h\big)\big|\\ &\quad \le \|\nabla \times (\boldsymbol{u} - \boldsymbol{\xi}_h)\|_{L^2(\Omega)}\|\nabla \times (\boldsymbol{\mu}_h - \boldsymbol{\phi}_h)\|_{L^2(\Omega)}\\ &\qquad + \tau\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)}\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}\\ &\quad \le \alpha(h)\|\nabla \times (\boldsymbol{u} - \boldsymbol{\xi}_h)\|_{L^2(\Omega)}\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)} + \tau\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)}\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}.\end{aligned} \tag{4.28}$$

Then by using (4.28), we obtain that

$$\begin{aligned}(N_* - 1)\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|^2_{L^2(\Omega)} &\le \big(\boldsymbol{\mu}_h - \boldsymbol{\phi}_h, (N-I)(\boldsymbol{\mu}_h - \boldsymbol{\phi}_h)\big)\\ &= \big(\boldsymbol{\mu}_h - \boldsymbol{\phi}_h, (N-I)\boldsymbol{\mu}_h - (\nabla \times \nabla \times -\tau)\boldsymbol{u}\big)\\ &\quad + \big(\boldsymbol{\mu}_h - \boldsymbol{\phi}_h, (\nabla \times \nabla \times -\tau)\boldsymbol{u} - (N-I)\boldsymbol{\phi}_h\big)\\ &\le \gamma^*\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{v} - \boldsymbol{\mu}_h\|_{L^2(\Omega)}\\ &\quad + \alpha(h)\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\nabla \times (\boldsymbol{u} - \boldsymbol{\xi}_h)\|_{L^2(\Omega)}\\ &\quad + \tau\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)},\end{aligned} \tag{4.29}$$

where $\gamma^* = \|N - I\|_2$ is the maximum of the singular value for matrix $(N - I)$. This indicates that

$$\begin{aligned}\|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)} \le \frac{\gamma^*}{N_* - 1}\|\boldsymbol{v} - \boldsymbol{\mu}_h\|_{L^2(\Omega)} &+ \frac{\alpha(h)}{N^* - 1}\|\nabla \times (\boldsymbol{u} - \boldsymbol{\xi}_h)\|_{L^2(\Omega)}\\ &+ \frac{\tau}{N^* - 1}\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)}.\end{aligned} \tag{4.30}$$

By (4.30) and the triangle inequality, we have that

$$\begin{aligned}\|\boldsymbol{v} - \boldsymbol{\phi}_h\|_{L^2(\Omega)} &\le \|\boldsymbol{v} - \boldsymbol{\mu}_h\|_{L^2(\Omega)} + \|\boldsymbol{\mu}_h - \boldsymbol{\phi}_h\|_{L^2(\Omega)}\\ &\le \left(1 + \frac{\gamma^*}{N_* - 1}\right)\|\boldsymbol{v} - \boldsymbol{\mu}_h\|_{L^2(\Omega)} + \frac{\alpha(h)}{N^* - 1}\|\nabla \times (\boldsymbol{u} - \boldsymbol{\xi}_h)\|_{L^2(\Omega)}\\ &\quad + \frac{\tau}{N^* - 1}\|\boldsymbol{u} - \boldsymbol{\xi}_h\|_{L^2(\Omega)}.\end{aligned} \tag{4.31}$$

Finally, the error estimate (4.25) can be obtained by combining (4.24) and (4.31). ☐

Theorem 4.3. *Given $\boldsymbol{f} \in H(curl,\Omega)$, let $(\boldsymbol{u},\boldsymbol{v})$ be the solution of (3.7) and assume that $\boldsymbol{u} \in [H^s(\Omega)]^3$, $(\nabla\times)^j\boldsymbol{u} \in [H^s(\Omega)]^3$ with $s > 1/2, j = 1,2,3$. Let $(\boldsymbol{u}_h,\boldsymbol{v}_h)$ be the solution of the discrete problem (4.1). Then there exists a positive constant C independent of the mesh size h and $\boldsymbol{f}$ such that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)} + \|\boldsymbol{v}-\boldsymbol{v}_h\|_{L^2(\Omega)} \le Ch^{s-1}\big(\|\boldsymbol{u}\|_{H^s(\Omega)} + \|\nabla\times\boldsymbol{u}\|_{H^s(\Omega)}\big). \tag{4.32}$$

Proof. The result follows by applying the property of the interpolation operator of Nédélec element and Theorem 4.2. The proof is similar to that Theorem 4.3 of [27] and thus is omitted here. □

We turn to the convergence of the mixed method for the quad-curl eigenvalue problem using Theorem 4.3. The quad-curl eigenvalue problem (3.1) can be written as follows. Find $\lambda \in \mathbb{R}$ and $(\boldsymbol{u},\boldsymbol{v}) \in H_0(\mathrm{curl},\Omega)\times H(\mathrm{curl},\Omega)$ such that

$$\begin{cases} a(\boldsymbol{v},\boldsymbol{\phi}) + b(\boldsymbol{\phi},\boldsymbol{u}) = 0 & \forall\,\boldsymbol{\phi} \in H(\mathrm{curl},\Omega),\\ b(\boldsymbol{v},\boldsymbol{\xi}) - c(\boldsymbol{u},\boldsymbol{\xi}) = -\lambda(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{\xi}) & \forall\,\boldsymbol{\xi} \in H_0(\mathrm{curl},\Omega), \end{cases} \tag{4.33}$$

where $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are defined by (3.8), (3.9) and (3.10), respectively.

The mixed finite element method is to find $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{u}_h,\boldsymbol{v}_h) \in S_h^0 \times S_h$ such that

$$\begin{cases} a(\boldsymbol{v}_h,\boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h,\boldsymbol{u}_h) = 0 & \forall\,\boldsymbol{\phi}_h \in S_h,\\ b(\boldsymbol{v}_h,\boldsymbol{\xi}_h) - c(\boldsymbol{u}_h,\boldsymbol{\xi}_h) = -\lambda_h(\nabla\times\boldsymbol{u}_h,\nabla\times\boldsymbol{\xi}_h) & \forall\,\boldsymbol{\xi}_h \in S_h^0. \end{cases} \tag{4.34}$$

It is easy to see that $(\lambda,(\boldsymbol{u},\boldsymbol{v}))$ is an eigenpair of (4.33) if and only if $\frac{1}{\lambda}$ is an eigenvalue of $\mathbb{B}$ with the same eigenvector. The following result is an immediate consequence of Theorem 4.3.

Lemma 4.4. *The operators $\mathbb{A}_h : H(curl,\Omega) \to S_h$ and $\mathbb{B}_h : H(curl,\Omega) \to S_h^0$ converge in norm to $\mathbb{A} : H(curl,\Omega) \to H(curl,\Omega)$ and $\mathbb{B} : H(curl,\Omega) \to H_0(curl,\Omega)$, respectively, i.e.,*

$$\lim_{h\to 0}\|\mathbb{A}-\mathbb{A}_h\| = 0 \qquad \text{and} \qquad \lim_{h\to 0}\|\mathbb{B}-\mathbb{B}_h\| = 0.$$

We denote the spectrum of the compact operator $\mathbb{B}$ by $\sigma(\mathbb{B})$ and let $\mu \in \sigma(\mathbb{B})$ be an eigenvalue with multiplicity m. Then, for h small enough, there exist m eigenvalues $\mu_h^1,\cdots,\mu_h^m$ of $\mathbb{B}_h$ such that

$$\lim_{h\to 0}\mu_h^j = \mu, \quad \text{for } j = 1,\cdots,m. \tag{4.35}$$

In summary, the following error estimate for the quad-curl eigenvalues holds.

Theorem 4.5. *Let λ be an eigenvalue of (4.33) with multiplicity m. For h small enough, there exist m eigenvalues $\lambda_{h,j}, j = 1,\ldots,m$, of(4.34) such that*

$$|\lambda-\lambda_{h,j}| \le Ch^{2s-2}, \quad j = 1,\cdots,m, \tag{4.36}$$

for some constant C.

Proof. Note that $\lambda = 1/\mu$ and $\lambda_{h,j} = 1/\mu_h^j$. The result follows Theorem 4.3, Lemma 4.1 and Theorem 11.1 of [2]. □

4.2. Error estimate for the iterative method

Now we establish the error estimate of the secant method. The idea is to obtain the convergence of the root of (2.5) to that of the discrete form (3.23) provided the mesh size is small enough. It follows the framework in [23] for the Helmholtz transmission eigenvalue problem, where the derivatives of the eigenvalues are utilized. The key is to verify that $f_h(\tau)$ is a strictly monotonic function on some interval containing the transmission eigenvalue.

Lemma 4.6. *Let $f(\tau)$ and $f_h(\tau)$ be two continuous functions on (a,b). Let $\tau_0 \in (a,b)$ be some root of $f_h(\tau)$, i.e., $f_h(\tau_0)=0$. For h small enough, there exists a τ_* such that $f(\tau_*)=0$. Furthermore, if $f_h'(\tau_*) \leq -\delta < 0$ for some $\delta > 0$, then it holds that*

$$|\tau_0 - \tau_*| \leq \frac{|f_h(\tau_*) - f(\tau_*)|}{\delta}.$$

The first part of the lemma follows the convergence of λ_h to λ. The assumption $f_h'(\tau_*) \leq -\delta < 0$ implies that $f_h(\tau)$ is a strictly monotonic function in a neighborhood of $\tau_* \in (a,b)$ such that the bound holds. We now verify this as follows.

Let λ_h be a generalized eigenvalue of (3.22) and U be a matrix of eigenvectors associated with λ_h such that $U^T S_{K\times K} U = I$. Thus we have

$$AU = S_{K\times K} U \Lambda_h,$$

where $\Lambda_h = \lambda_h I$. It is well-known that the choice of U is not unique. In general, a repeated eigenvalue λ_h can separate as τ changes and the derivative of the eigenvalue λ_h with multiplicity $m > 1$ is not a scalar. We will denote it by $\Lambda_h' = diag(\lambda_{1,h}', \cdots, \lambda_{m,h}')$. There exists a matrix $\Gamma \in \mathbb{R}^{m\times m}$ such that $\Gamma^T\Gamma = I$ and the columns of orthogonal transformation $Z = U\Gamma$ are the eigenvectors for which a derivative can be defined. Differentiating $AZ = S_{K\times K} Z \Lambda_h$, we obtain

$$A'Z + AZ' = S_{K\times K}' Z\Lambda_h + S_{K\times K} Z'\Lambda_h + S_{K\times K} Z \Lambda_h'.$$

Collecting similar terms we get

$$(A - \lambda_h S_{K\times K}) Z' = (\lambda_h S_{K\times K}' - A') Z + S_{K\times K} Z \Lambda_h'.$$

Multiplying U^T, substituting $Z = U\Gamma$ and using the fact that $U^T(A - \lambda_h S_{K\times K}) = 0$, we have

$$\Gamma \Lambda_h' = U^T (A' - \lambda_h S_{K\times K}') U \Gamma.$$

Since $S_{K\times K}$ does not depend on τ, we have $S_{K\times K}' = 0$ and thus

$$\Lambda_h' = (U\Gamma)^T A' (U\Gamma). \tag{4.37}$$

If λ_h is a simple eigenvalue, we have that

$$\lambda_h' = \boldsymbol{u}^T A' \boldsymbol{u},$$

where $\boldsymbol{u}$ is the associated eigenvector such that $\boldsymbol{u}^T S_{K\times K} \boldsymbol{u} = 1$.

Now we show that $f_h'(\tau)$ is negative on an interval to the right of τ_*.

Lemma 4.7. *Let A_h', B_h' and C_h' represent the derivatives of A_h, B_h and C_h, respectively. We have $f_h'(\tau) < 0$ when*

$$\tau < \left(\frac{\lambda_0(\Omega)}{2} + \frac{1}{N^* - 1} \right) \frac{N^* - 1}{N^*}. \tag{4.38}$$

Proof. By simple calculations, we have

$$a'(\boldsymbol{v}, \boldsymbol{\phi}) = 0 \quad \text{and} \quad b'(\boldsymbol{u}, \boldsymbol{\phi}) = (\boldsymbol{u}, \boldsymbol{\phi}) \quad \text{and} \quad c'(\boldsymbol{u}, \boldsymbol{\xi}) = 2\tau(\boldsymbol{u}, \boldsymbol{\xi}).$$

Letting $\boldsymbol{\phi} = \boldsymbol{v}$ and $\boldsymbol{\xi} = \boldsymbol{u}$, we have

$$a'(\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \text{and} \quad b'(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}) \quad \text{and} \quad c'(\boldsymbol{u}, \boldsymbol{u}) = 2\tau(\boldsymbol{u}, \boldsymbol{u}).$$

Let A'_h, B'_h and C'_h be the matrices corresponding to a', b' and c', respectively. Note that $A = B_h A_h^{-1} B_h^T + C_h$ and, as a consequence,

$$\begin{aligned} A' &= B'_h A_h^{-1} B_h^T + B_h (A_h^{-1})' B_h^T + B_h A_h^{-1} (B_h^T)' + C'_h, \\ &= B'_h A_h^{-1} B_h^T + B_h A_h^{-1} (B_h^T)' + C'_h. \end{aligned}$$

Assume that the operator A_h^{-1} corresponds to the bilinear form $a^{inv}(\boldsymbol{v}, \boldsymbol{v})$, *i.e.*, $a^{inv}(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{v}^T A_h^{-1} \boldsymbol{v}$. We choose $\boldsymbol{v}$ such that $\boldsymbol{v}^T \cdot \boldsymbol{v} = 1$, thus

$$a^{inv}(\boldsymbol{v}, \boldsymbol{v}) a(\boldsymbol{v}, \boldsymbol{v}) = \boldsymbol{v}^T A_h^{-1} \boldsymbol{v} \boldsymbol{v}^T A_h \boldsymbol{v} = 1.$$

In addition, we have

$$a(\boldsymbol{v}, \boldsymbol{v}) = \big((N - I)\boldsymbol{v}, \boldsymbol{v}\big) = \boldsymbol{v}^T (N - I) \boldsymbol{v} \le (N^* - 1),$$

which indicates that

$$a^{inv}(\boldsymbol{v}, \boldsymbol{v}) \le \frac{1}{N^* - 1}.$$

Let $\boldsymbol{u}$ be a column of Z in S_h^0. Then

$$\begin{aligned} \lambda'_h &= Z^T A' Z \\ &= (\boldsymbol{u}, \boldsymbol{v}) a^{inv}(\boldsymbol{v}, \boldsymbol{v}) \big[\tau(\boldsymbol{v}, \boldsymbol{u}) - (\nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{u})\big] \\ &\quad + \big[\tau(\boldsymbol{u}, \boldsymbol{v}) - (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v})\big] a^{inv}(\boldsymbol{v}, \boldsymbol{v})(\boldsymbol{v}, \boldsymbol{u}) + 2\tau(\boldsymbol{u}, \boldsymbol{u}) \\ &\le \frac{1}{N^* - 1} \|\boldsymbol{u}\| \|\boldsymbol{v}\| (\tau \|\boldsymbol{v}\| \|\boldsymbol{u}\| - \|\nabla \times \boldsymbol{v}\| \|\nabla \times \boldsymbol{u}\|) \\ &\quad + \frac{1}{N^* - 1} (\tau \|\boldsymbol{u}\| \|\boldsymbol{v}\| - \|\nabla \times \boldsymbol{u}\| \|\nabla \times \boldsymbol{v}\|) \|\boldsymbol{v}\| \|\boldsymbol{u}\| + 2\tau \|\boldsymbol{u}\|^2 \\ &= \frac{2\tau}{N^* - 1} \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - \frac{2}{N^* - 1} \|\boldsymbol{u}\| \|\nabla \times \boldsymbol{u}\| \|\boldsymbol{v}\| \|\nabla \times \boldsymbol{v}\| + 2\tau \|\boldsymbol{u}\|^2 \\ &\le \frac{2(\tau - 1)}{N^* - 1} \|\boldsymbol{u}\|^2 + 2\tau \|\boldsymbol{u}\|^2 \\ &\le \frac{2(\tau - 1)}{N^* - 1} \frac{1}{\lambda_0(\Omega)} + \frac{2\tau}{\lambda_0(\Omega)}, \end{aligned}$$

where we have used the Poincaré inequality and, for simplicity, denoted $\|\cdot\|_{L^\infty}$ by $\|\cdot\|$. Thus if

$$\frac{2(\tau - 1)}{N^* - 1} \frac{1}{\lambda_0(\Omega)} + \frac{2\tau}{\lambda_0(\Omega)} < 1,$$

i.e.,

$$\tau < \left(\frac{\lambda_0(\Omega)}{2} + \frac{1}{N^* - 1}\right) \frac{N^* - 1}{N^*}, \tag{4.39}$$

we have $f'_h = \lambda'_h(\tau) - 1 < 0$, which implies that $f_h(\tau)$ is monotonically decreasing. □

The combination of Lemmas 4.6, 4.7 and the triangle inequality leads to the following theorem.

Theorem 4.8. *Assume we apply the mixed finite element method for* (2.4) *on a regular mesh* $\mathcal{T}_h$*. Under the assumption of Lemma 4.7, let* τ_* *be the root of* $f(\tau)$ *and* τ_h *be the approximation of* τ_* *computed by the secant method. Then for any* $\varepsilon > 0$ *such that* $|f_h(\tau_*) - f(\tau_*)| < \varepsilon$*, there exists an* h_0 *such that, for* $h < h_0$*, it holds that*

$$|\tau_h - \tau_*| \le \varepsilon/\delta + tol, \tag{4.40}$$

where $\delta > 0$ *is indepedent of* h *and* tol *is given in the secant method.*

Proof. Let τ_0 be the root of $f_h(\tau)$, *i.e.*, $\lambda_h(\tau_0) - \tau_0 = 0$. If τ satisfies the condition in Lemma 4.7, there exists $\delta > 0$ such that $f_h'(\tau_*) < -\delta$. By Lemma 4.6, the triangle inequality and the fact that $|\tau_h - \tau_0| < tol$, we obtain

$$\begin{aligned}|\tau_h - \tau_*| &< |\tau_h - \tau_0| + |\tau_0 - \tau_*| \\ &< tol + \frac{|f_h(\tau_*) - f(\tau_*)|}{\delta},\end{aligned}$$

which gives (4.40). The proof is complete. □

Our paper aims to provide effective theoretical analysis for an existing mixed finite element method. Numerical experiments have already been carried out in our previous work [17] for the unit ball with tetrahedra meshes. The second order convergence of the lowest Maxwell's transmission eigenvalues is obtained, which verifies our analysis. Numerical examples using a similar mixed method can also be found in [25], where the same convergence rate is observed.

5. Conclusion

In this work, the convergence analysis for a numerical method to compute real Maxwell's transmission eigenvalues is carried out. The error analysis of a mixed finite element for the associated generalized eigenvalue problem is obtained using the quad-curl formulation using the framework of spectral approximation of compact operators. Then the convergence of the secant method is established by using the derivatives of eigenvalues.

The method proposed in this paper does not apply to complex transmission eigenvalues, which are known to exist. It is certainly interesting to develop effective finite element methods to compute the complex eigenvalues and carry out the error analysis.

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