ESAIM: M2AN 58 (2024) 1031–1052
https://doi.org/10.1051/m2an/2024018

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

EXISTENCE AND UNIQUENESS OF THE MOTION OF A PARTICLE SUBJECT TO A UNILATERAL CONSTRAINT AND FRICTION

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Abstract. We prove that there exists a unique solution to the initial value problem describing the motion of a particle subject to a unilateral constraint and Coulomb friction, if the external force acting on the particle is an analytic function of time and of the particle's position and velocity. Previous work claimed that this problem has a local series solution that corresponds to an analytic function, after any impacts have been resolved. However, a counterexample to that claim was recently discovered, involving a particle starting to slide, in which the series solution is divergent, and thus does not correspond to an analytic function. This paper corrects previous arguments by considering a general formal series solution for a particle that is starting to slide.

Mathematics Subject Classification. 70F40, 49J52, 34A60.

Received February 2, 2024. Accepted March 11, 2024.

1. Introduction

Models of systems subject to unilateral constraints and friction are foundational for the simulation, design and control of robots and other machines [8, 9, 12, 28]. When do initial value problems associated with such models have unique solutions? Answers to this question would help to resolve uniqueness and convergence concerns about simulation algorithms, and to guarantee that the full set of potentially relevant motions is considered, when designing mechanical systems or learning controllers [1, 27]. However, in spite of over a century of study [4, 13, 15, 22], today's answers are rather incomplete.

Even in the absence of friction, the behaviour of unilateral constraints can be surprising. Given a particle moving in one dimension subject to an external force and a unilateral constraint, one might imagine, based on the classical nature of the physics involved, that the system is deterministic and hence the motion is unique. But strikingly, Bressan [5] and Schatzman [26] showed that such systems can have multiple solutions, even if the external force is an infinitely differentiable function of time; and Ballard [2] showed the solution may be non-unique even if the impacts are perfectly inelastic. On the other hand, Percivale [23, 24] discovered that unique motions are guaranteed if the force is an *analytic* function of time. (Recall that a function $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^n$ is *(real) analytic* at point $\mathbf{a} \in \mathbb{R}^m$ if the Taylor series of $\mathbf{f}$ at $\mathbf{a}$ converges to $\mathbf{f}(\mathbf{x})$ for all $\mathbf{x}$ in some neighbourhood of $\mathbf{a}$.) Subsequently, Ballard [2] generalized Percivale's results from particles to frictionless multibody systems, acted on by forces that are analytic functions not only of time, but also of the bodies' positions and velocities.

Keywords and phrases. Unilateral dynamics with friction, frictional dynamical contact problems.

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Whereas both infinitely differentiable and analytic force functions allow an accumulation of impacts on the left of a given time (like a bouncing ball coming to rest), all known examples of nonuniqueness involve an accumulation of impacts on the right, which Ballard shows is impossible for an analytic force function.

Several authors have explored existence and uniqueness results for models with friction. Monteiro Marques [17] proved the existence of solutions to initial value problems involving a single unilateral constraint and dry friction, but only in situations with a perfectly inelastic impacts (zero restitution coefficient) and without addressing uniqueness. Later, Ballard and Basseville [3] presented arguments for the existence and uniqueness of solutions to initial value problems for a unilaterally constrained particle subject to dry friction and a force that is an analytic function of time and a linear function of the particle's position. Charles and Ballard [6] extended those arguments to finite collections of particles and to a more general class of force functions. In the case of a single particle, their problem setting is identical to that considered in this paper. A key step in the arguments of Ballard, Basseville and Charles is to derive a local solution given by a power series, and to claim that it corresponds to an analytic function. However, a simple counterexample to that claim was recently presented by Dance [7]. This counterexample involves a particle in $\mathbb{R}^3$ that starts to slide on a plane, under the influence of a tangential force whose Cartesian components in the plane grow with time t as $(1, t) \in \mathbb{R}^2$, and a friction force of unit magnitude. Although the tangential force is an analytic function of time, the particle's velocity is not: rather, the Taylor series of the velocity at $t = 0$ is divergent, and the magnitude of the coefficient of t^{2n} of the first component of the velocity is $\Omega(n!/3^n)$.

In this paper, we prove the existence and uniqueness of solutions to initial value problems describing the motion a particle subject to a unilateral constraint, dry friction, and a force given by an analytic function of time, position and velocity, which is also a Lipschitz continuous function of position and velocity at each time. Our approach is to repair the arguments of Ballard, Basseville and Charles [3,6], in the light of Dance's counterexample [7]. Given some initial conditions compatible with the unilateral constraint, we show that one can always find a local solution (that is, a motion on a time interval of positive length just after the initial time) that is an analytic function of time, *unless* the particle is just starting to slide. If the particle is just starting to slide, one can find a formal series solution. Although this formal series is in general divergent, we show there is an actual solution that is asymptotic to the formal series at the initial time. As the rest of the argument is similar to that given by Charles and Ballard, we only highlight the few changes necessary for that argument to work with the solution that is asymptotic to the formal series.

2. Problem Statement and Main Result

We consider the initial value problem for a particle moving in $\mathbb{R}^d$, acted on by an external force, and by a reaction that captures the effects of a unilateral constraint and dry friction. The unilateral constraint is an inequality constraint on the particle's position, which models an obstacle: the particle may collide with, stay in contact with or break contact with the surface of the obstacle; but it cannot enter the obstacle. We formulate the problem for an obstacle that occupies a half-space. This might sound restrictive, but in fact the problem has the same form for any obstacle defined by an analytic hypersurface, as shown by Charles and Ballard Section 5 from [6]; therefore our main result also holds for such obstacles. Due to collisions with the obstacle, the particle's velocity may be a discontinuous function of time, so we model its acceleration and the reaction of the obstacle on the particle as vector-valued measures. The times of discontinuity are the atoms of the acceleration and reaction measures.

The particle's position at time t in time interval $[0, T]$ is denoted by $\mathbf{u}(t)$, and the particle's left and right velocities are $\mathbf{v}^- = \dot{\mathbf{u}}^-$ and $\mathbf{v}^+ = \dot{\mathbf{u}}^+$. The particle's acceleration measure $\dot{\mathbf{v}}$ and the reaction measure $\mathbf{R}$ are both in the space $\mathcal{M}([0, T], \mathbb{R}^d)$ of $\mathbb{R}^d$-valued Radon measures on $[0, T]$. Thus the position function $\mathbf{u}$ is in the space of *motions with measure acceleration* [19,26] from $[0, T]$ to $\mathbb{R}^d$, denoted by $\mathrm{MMA}([0, T], \mathbb{R}^d)$. This space is defined as the set of functions $\mathbf{y} : [0, T] \to \mathbb{R}^d$, whose left and right derivatives $\dot{\mathbf{y}}^-$ and $\dot{\mathbf{y}}^+$ exist in the classical sense on $(0, T]$ and $[0, T)$ respectively, take defined values $\dot{\mathbf{y}}^-(0)$ and $\dot{\mathbf{y}}^+(T)$ at the endpoints, and have bounded variation on $[0, T]$. The external force $\mathbf{F} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function of time and of the particle's position and (left or right) velocity. We choose the coordinate system so that the first component, denoted by the superscript n, is parallel to the outward normal to the surface of the obstacle and the remaining components, denoted by the superscript t, are tangential to that surface: thus

$$\mathbf{u}(t) = (\mathrm{u}^{\mathrm{n}}(t), \mathbf{u}^{\mathrm{t}}(t)) \in \mathbb{R} \times \mathbb{R}^{d-1}.$$

The initial value problem and main result are as follows.

Problem $\mathscr{P}_{\mathrm{u}}$. Given initial conditions $(\mathbf{u}_0, \mathbf{v}_0^-) \in \mathbb{R}^d \times \mathbb{R}^d$ with $\mathrm{u}_0^{\mathrm{n}} \geq 0$, restitution coefficient $e \in [0,1]$, friction coefficient $\mu \geq 0$ and force function $\mathbf{F} : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, find a motion $\mathbf{u} \in \mathrm{MMA}([0,T], \mathbb{R}^d)$ and a reaction measure $\mathbf{R} \in \mathcal{M}([0,T], \mathbb{R}^d)$ that satisfy:

(i) the initial conditions $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{v}^-(0) = \mathbf{v}_0$;
(ii) the equation of motion $\dot{\mathbf{u}} = \mathbf{v}^-$ and $\dot{\mathbf{v}} = \mathbf{F}(t, \mathbf{u}, \mathbf{v}^-) + \mathbf{R}$ on $[0,T]$;
(iii) the contact constraints $\mathrm{u}^{\mathrm{n}} \geq 0$, $\mathrm{R}^{\mathrm{n}} \geq 0$ and $\mathrm{u}^{\mathrm{n}}\mathrm{R}^{\mathrm{n}} = 0$ on $[0,T]$;
(iv) the impact law $\mathrm{v}^{\mathrm{n}+}(t) = -e\mathrm{v}^{\mathrm{n}-}(t)$ for each t in $[0,T]$ at which $\mathrm{u}^{\mathrm{n}}(t) = 0$; and
(v) the friction law: for all continuous functions $\mathbf{w} : [0,T] \to \mathbb{R}^{d-1}$

$$\int_{[0,T]} (\mathbf{R}^{\mathrm{t}} \cdot \mathbf{v}^{\mathrm{t}+} + \mu \mathrm{R}^{\mathrm{n}} \|\mathbf{v}^{\mathrm{t}+}\|) \leq \int_{[0,T]} (\mathbf{R}^{\mathrm{t}} \cdot \mathbf{w} + \mu \mathrm{R}^{\mathrm{n}} \|\mathbf{w}\|).$$

Theorem 2.1. *Let $\mathscr{P}$ be an instance of Problem $\mathscr{P}_{\mathrm{u}}$ in which the force function is an analytic function of its arguments, and a Lipschitz function of position and velocity at each time. Then $\mathscr{P}$ has exactly one solution.*

A few remarks are in order concerning the formulation of Problem $\mathscr{P}_{\mathrm{u}}$.

Remark 2.2. In the equation of motion, it does not matter whether the left or right velocity is used on the right-hand side, as the left and right velocities agree for almost all times, and the force only appears as a density. The contact constraint $\mathrm{R}^{\mathrm{n}} \geq 0$ requires that the reaction is non-adhesive (it cannot pull the particle toward the obstacle). The constraint $\mathrm{u}^{\mathrm{n}}\mathrm{R}^{\mathrm{n}} = 0$ requires that the reaction vanishes when there is no contact: it may be stated more explictly as the requirement that the support of measure R^{n} is a subset of $\{t \in [0,T] : \mathrm{u}^{\mathrm{n}}(t) = 0\}$. In the impact law, if the restitution coefficient e is generalized to a function $e(t, \mathbf{u}, \mathbf{v}^-)$ from $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ to $[0,1]$, then Theorem 2.1 still holds.

Remark 2.3. As discussed by Ballard and Charles [4], the friction law generalizes the Amontons-Coulomb law of dry friction from reaction forces to reaction measures. Indeed, if $(\mathrm{r}^{\mathrm{n}}, \mathbf{r}^{\mathrm{t}})$ denotes the value of the reaction measure (with $\mathrm{r}^{\mathrm{n}} \geq 0$) at an atom of that measure, then the friction law implies that

$$\begin{cases} \mathbf{v}^{\mathrm{t}+} = \mathbf{0} & \Rightarrow \quad \|\mathbf{r}^{\mathrm{t}}\| \leq \mu \mathrm{r}^{\mathrm{n}} \\ \mathbf{v}^{\mathrm{t}+} \neq \mathbf{0} & \Rightarrow \quad \mathbf{r}^{\mathrm{t}} = -\mu \mathrm{r}^{\mathrm{n}} \mathbf{v}^{\mathrm{t}+}/\|\mathbf{v}^{\mathrm{t}+}\|. \end{cases} \tag{1}$$

The appearance of the right rather than the left velocity is essential here to ensure that friction does not increase kinetic energy. On the other hand, if the reaction measure is absolutely continuous with respect to Lebesgue measure and $(\mathrm{r}^{\mathrm{n}}, \mathbf{r}^{\mathrm{t}})$ instead denotes its density, then equation (1) is again satisfied for almost all times; moreover $\mathbf{v}^{\mathrm{t}+}$ may be replaced by the tangential velocity $\mathbf{v}^{\mathrm{t}}$, as the equation of motion implies that the velocity is continuous if the reaction measure has no atoms.

Remark 2.4. The *subdifferential* $\partial f(\mathbf{x})$ of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x} \in \mathbb{R}^n$ is the set of vectors $\mathbf{a} \in \mathbb{R}^n$ such that $f(\mathbf{x}') - f(\mathbf{x}) \geq \mathbf{a} \cdot (\mathbf{x}' - \mathbf{x})$ for all $\mathbf{x}' \in \mathbb{R}^n$. Thus $\partial\|\mathbf{x}\|$ is the closed unit ball $\{\mathbf{a} \in \mathbb{R}^n : \|\mathbf{a}\| \leq 1\}$ if $\mathbf{x} = \mathbf{0}$, and the singleton set $\{\mathbf{x}/\|\mathbf{x}\|\}$ otherwise. Using this subdifferential, law (1) takes the succinct form

$$\mathbf{r}^{\mathrm{t}} \in -\mu \mathrm{r}^{\mathrm{n}} \partial \|\mathbf{v}^{\mathrm{t}+}\|.$$

Integrating the equation of motion over an atom of the reaction measure thus gives $\mathbf{v}^{t+} - \mathbf{v}^{t-} \in -\mu \mathrm{r}^{\mathrm{n}} \partial \|\mathbf{v}^{t+}\|$. It follows that $\mathbf{v}^{t+}$ must be a minimizer of $\mathbf{x} \mapsto \frac{1}{2}\|\mathbf{x} - \mathbf{v}^{t-}\|^2 + \mu \mathrm{r}^{\mathrm{n}}\|\mathbf{x}\|$. As this is a strictly convex function, the right velocity $\mathbf{v}^{t+}$ is unique given $\mathbf{v}^{t-}$ and $\mu \mathrm{r}_{\mathrm{n}} \geq 0$. Clearly, the normal reaction impulse r_{n} is also uniquely determined by the equation of motion and the impact law.

Remark 2.5. Our formulation is the same as that of Charles and Ballard [6], with two exceptions. First, we work with the outward normal to the obstacle rather than the inward normal, so we require the normal position and normal reaction measure to be nonnegative. Although closely related work requires nonpositive normal positions [3, 6], the convention of nonnegative unilateral constraint functions is also common [4, 8, 10]. Second, our initial conditions specify the left velocity rather than the right velocity. Other related work also specifies the left velocity [19]: this seems natural if the acceleration is a measure on $[0, T]$ rather than $(0, T)$, and it avoids the need for a condition ruling out an impact at the initial time.

The rest of the paper proves Theorem 2.1. Like previous authors [3, 6], we look for a series solution to the initial value problem after any initial impact has been resolved. The series solutions determined in previous work are correct, except in the case of a particle that is just starting to slide, and Section 3 focuses on that case. We prove the existence of a formal power series solution to the initial value problem for a particle that is starting to slide (Prop. 3.4). Although this formal series is in general divergent, we show that there is an actual local solution that is asymptotic to it; and although the normal reaction force corresponding to this local solution may not be analytic at the initial time, we show that its time derivative satisfies an inequality which plays a key role in proving uniqueness (Prop. 3.6). Section 4 presents the few changes necessary for the arguments of Charles and Ballard to apply to this actual local solution. We show that there is always a local solution to the unilateral problem (Prop. 4.1) and that this is the unique local solution to the unilateral problem in the space of motions with measure acceleration (Prop. 4.2). Theorem 2.1 follows immediately from these local existence and uniqueness results, by the arguments of Charles and Ballard Corollary 4.3 from [6]. As no changes are required to those arguments, they are not repeated here.

3. The Starting-to-Slide Problem

This section focuses on the case of a particle that is initially at rest, in contact with an obstacle, and is acted on by a tangential force that exceeds the friction limit, making the particle start to slide. After recalling relevant results on formal power series, we define an algebraically constrained initial value problem, the *starting-to-slide problem*, and prove it has a formal power series solution (Prop. 3.4). Although this formal series is in general divergent [7], we show that there is an actual local solution that is asymptotic to it (Prop. 3.6). Our proof relies upon an extension of the classic Cauchy-Lipschitz (Picard-Lindelöf) existence argument for differential equations. The density of the normal reaction measure corresponding to this local solution can be taken to be a differentiable function. Although this function is not in general analytic at the initial time, we show that it satisfies an inequality which also holds for analytic functions, and which plays a key role in proving uniqueness. Specifically, if $f : \mathbb{R} \to \mathbb{R}$ is analytic at 0, then there exist real numbers a and b such that

$$t\left|\frac{\mathrm{d}f}{\mathrm{d}t}\right| \leq (a + bt)|f(t)| \tag{2}$$

on $[0, \eta)$ for some $\eta > 0$. This inequality does not hold in general if f is analytic at t for $t > 0$ *but not at* $t = 0$: for instance, if $f(t) = \exp(-1/t)$ then $\mathrm{d}f/\mathrm{d}t = f(t)/t^2$.

3.1. Reminder on Formal Series

A *formal power series* over the reals is an infinite sequence $a = (a_0, a_1, \dots)$ whose elements are real numbers known as coefficients [11, 21]. A formal power series is often written as a sum $a(t) = \sum_{i \geq 0} a_i t^i$ involving an *indeterminate* t: this differs from the usual notion of a power series, as this sum is not required to converge for some nonzero value of t. The *order of vanishing* $\mathrm{ord}(a)$ of formal power series a is the least index i such that $a_i \neq 0$, with the convention that $\mathrm{ord}(a) = \infty$ if all coefficients of a vanish. If $\mathrm{ord}(a) < \infty$ then the *leading coefficient* of a is $\mathrm{lcf}(a) \coloneqq a_{\mathrm{ord}(a)}$. The *derivative* of a is the formal power series $\dot{a}(t) \coloneqq \sum_{i\geq 0}(i+1)a_{i+1}t^i$.

The result of truncating formal power series a at exponent n is denoted by $a_{\leq n}(t) \coloneqq \sum_{0\leq i\leq n} a_i t^i$ or $a_{\leq n}$. We use the convention that $a_{\leq i}$ and a_i vanish if i is negative. Clearly, if t is taken to be a real variable, then $a_{\leq n}(t)$ is a polynomial with degree at most n. We use square brackets to group truncations or extraction of coefficients. For instance, if formal power series a, b and c satisfy $ab = c$, then $[ab]_{\leq i} = c_{\leq i}$ and $[ab]_i = c_i$; and if x_k is a formal power series then its coefficients are $[x_k]_0, [x_k]_1, \ldots$. Also, we use the following two results repeatedly.

Lemma 3.1. *Let a and b be formal power series, and let i be an integer. Then*

$$[ab]_i = [a_{\leq i-\mathrm{ord}(b)} b_{\leq i-\mathrm{ord}(a)}]_i.$$

Proof. By the Cauchy product formula, $[ab]_i = \sum_{j=0}^{i} a_j b_{i-j} = \sum_{j=\mathrm{ord}(a)}^{i-\mathrm{ord}(b)} a_j b_{i-j}$. As this sum only depends on $a_{\leq i-\mathrm{ord}(b)}$ and $b_{\leq i-\mathrm{ord}(a)}$, the result follows. □

Lemma 3.2. *Let $x_1, \ldots, x_n$ be formal power series over the reals in indeterminate t and $f : \mathbb{R}^n \to \mathbb{R}$ be real analytic at $([x_1]_0, \ldots, [x_n]_0)$. Then the composition $f(x_1, \ldots, x_n)$ is a well-defined formal power series in t with*

$$[f(x_1, \ldots, x_n)]_i = [f([x_1]_{\leq i}, \ldots, [x_n]_{\leq i})]_i \qquad \text{for } i \in \mathbb{Z}_{\geq 0}.$$

Proof. As f is analytic at $([x_1]_0, \ldots, [x_n]_0)$, we may define the composition in terms of the Taylor series

$$f(x_1, \ldots, x_n) = \textstyle\sum_{m\geq 0} \sum_{j:|j|=m} g_j\, y_1^{j_1} \cdots y_n^{j_n},$$

where j is a multi-index with $|j| \coloneqq j_1 + \cdots + j_n$, where each g_j is a real number involving factorials and a partial derivative, and where $y_k \coloneqq x_k - [x_k]_0$ for $k = 1, \ldots, n$. As $\mathrm{ord}(y_k) \geq 1$ for each k, we have $[y_1^{j_1} \cdots y_n^{j_n}]_i = 0$ if $|j| > i$. Thus Lemma 3.1 gives

$$[f(x_1, \ldots, x_n)]_i = \textstyle\sum_{m=0}^{i} \sum_{j:|j|=m} g_j\, [([y_1]_{\leq i})^{j_1} \cdots ([y_n]_{\leq i})^{j_n}]_i.$$

As this is a finite sum, it is a real number, so the composition is well-defined. Now let $z_k :- [x_k]_{\leq i}$. Then $[y_k]_{\leq i} = [z_k - [z_k]_0]_{\leq i}$. Therefore $[f(x_1, \ldots, x_n)]_i = [f(z_1, \ldots, z_n)]_i$. □

The above definitions and results extend readily to formal power series with vector-valued coefficients. In particular, given a formal power series $\mathbf{a} = \sum_{i=0}^{\infty} \mathbf{a}_i t^i$ with coefficients $\mathbf{a}_i \in \mathbb{R}^n$, we define

$$\mathrm{ord}(\mathbf{a}) \coloneqq \inf\{i \geq 0 : \mathbf{a}_i \neq \mathbf{0}\}.$$

By considering two formal power series $\mathbf{a}$ and $\mathbf{b}$ with coefficients in $\mathbb{R}^n$ as tuples $(\mathrm{a}_1, \ldots, \mathrm{a}_n)$ and $(\mathrm{b}_1, \ldots, \mathrm{b}_n)$ of formal power series over the reals, we define the sum, dot product, squared norm and derivative:

$$\mathbf{a} + \mathbf{b} \coloneqq (\mathrm{a}_1 + \mathrm{b}_1, \ldots, \mathrm{a}_n + \mathrm{b}_n), \qquad \mathbf{a} \cdot \mathbf{b} \coloneqq \mathrm{a}_1\mathrm{b}_1 + \cdots + \mathrm{a}_n\mathrm{b}_n, \qquad \|\mathbf{a}\|^2 \coloneqq \mathbf{a} \cdot \mathbf{a}, \qquad \dot{\mathbf{a}} \coloneqq (\dot{\mathrm{a}}_1, \ldots, \dot{\mathrm{a}}_n).$$

Consider a function $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that the composition $\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}})$ is a well-defined formal power series for a given formal power series $\mathbf{x}$ with coefficients in $\mathbb{R}^n$ and its derivative $\dot{\mathbf{x}}$. A *formal power series solution* of $\mathbf{g} = \mathbf{0}$ is a formal power series $\mathbf{x}$ such that $[\mathbf{g}(\mathbf{x}, \dot{\mathbf{x}})]_i = \mathbf{0}$ for all nonnegative integers i.

3.2. Formal Series Solution to the Starting-to-Slide Problem

The starting-to-slide problem is a special case of Problem $\mathscr{P}_{\mathrm{u}}$, in which the particle starts from rest, and attains a nonzero tangential velocity, while remaining in contact with the obstacle. Any solution to this problem corresponds to a local solution to Problem $\mathscr{P}_{\mathrm{u}}$, *provided* the normal component of the force is nonpositive on a right neighbourhood of the time origin (that is, an interval of the form $(0, \eta)$ or $[0, \eta)$ for some $\eta > 0$). The problem is written so as to be amenable to formal series solution, as we now explain.

We denote the normal and tangential components of the force $\mathbf{F} = (\mathrm{F}^{\mathrm{n}}, \mathbf{F}^{\mathrm{t}})$ for a particle in contact with the obstacle and with zero normal velocity by

$$\begin{aligned} \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) &:= \mathrm{F}^{\mathrm{n}}(t, (0, \mathbf{u}^{\mathrm{t}}), (0, \mathbf{v}^{\mathrm{t}})), \\ \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) &:= \mathbf{F}^{\mathrm{t}}(t, (0, \mathbf{u}^{\mathrm{t}}), (0, \mathbf{v}^{\mathrm{t}})). \end{aligned} \tag{3}$$

As noted in Remark 2.3, if contact is maintained and the reaction measure is absolutely continuous with respect to Lebesgue measure, then the tangential and normal reaction forces $\mathbf{r}^{\mathrm{t}}$ and r^{n} are related for almost all times by the familiar friction law

$$\mathbf{r}^{\mathrm{t}} = -\mu \mathrm{r}^{\mathrm{n}} \frac{\mathbf{v}^{\mathrm{t}}}{\|\mathbf{v}^{\mathrm{t}}\|} = \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) \frac{\mathbf{v}^{\mathrm{t}}}{\|\mathbf{v}^{\mathrm{t}}\|} \qquad \text{if } \mathbf{v}^{\mathrm{t}} \neq \mathbf{0}, \tag{4}$$

where the second equality follows as the normal force and normal reaction force are equal and opposite for a particle that remains in contact. Since we wish to determine a formal series solution to an equation of motion of a particle that is starting to slide, we must find the coefficients of t^k of the right-hand side of (4), given formal series for $\mathbf{u}^{\mathrm{t}}$ and $\mathbf{v}^{\mathrm{t}}$, but we are hampered by the fact that the function $\mathbf{v}^{\mathrm{t}} \mapsto \frac{\mathbf{v}^{\mathrm{t}}}{\|\mathbf{v}^{\mathrm{t}}\|}$ is not analytic at $\mathbf{v}^{\mathrm{t}}(0) = \mathbf{0}$. To overcome this impediment, we shall introduce a new variable $\mathbf{v}^{\div}$ in terms of which this function is analytic. First, define the unit vector

$$\hat{\mathbf{a}} = \lim_{t \to 0^+} \frac{\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})}{\|\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})\|}. \tag{5}$$

(This limit exists under the hypotheses of Proposition 3.4, which require that $\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})$ is analytic at $t = 0$ and nonzero for $t \in (0, \eta)$ for some $\eta > 0$.) We will look for solutions in which the particle starts to slide in this direction. Decomposing the tangential velocity into components parallel and perpendicular to this unit vector, we have

$$\mathbf{v}^{\mathrm{t}} = \mathrm{v}^{\parallel} \hat{\mathbf{a}} + \mathbf{v}^{\perp} \qquad \text{where} \qquad \mathbf{v}^{\perp} \cdot \hat{\mathbf{a}} = 0, \tag{6}$$

and $\mathrm{v}^{\parallel} = \hat{\mathbf{a}} \cdot \mathbf{v}^{\mathrm{t}}$. We introduce the variable

$$\mathbf{v}^{\div} = \frac{\mathbf{v}^{\perp}}{\mathrm{v}^{\parallel}},$$

which is well defined for $\mathrm{v}^{\parallel} \neq 0$. In terms of this variable, we have

$$\frac{\mathbf{v}^{\mathrm{t}}}{\|\mathbf{v}^{\mathrm{t}}\|} = \frac{\hat{\mathbf{a}} + \mathbf{v}^{\div}}{\|\hat{\mathbf{a}} + \mathbf{v}^{\div}\|} = \frac{\hat{\mathbf{a}} + \mathbf{v}^{\div}}{(1 + \|\mathbf{v}^{\div}\|^2)^{1/2}}, \tag{7}$$

in which the right-hand side is indeed analytic in $\mathbf{v}^{\div}$ at $\mathbf{0}$. The condition $\mathbf{v}^{\perp} \cdot \hat{\mathbf{a}} = 0$ reads $\mathrm{v}^{\parallel} \mathbf{v}^{\div} \cdot \hat{\mathbf{a}} = 0$, but we will apply this condition when $\mathrm{v}^{\parallel}$ is a formal series other than the all-zero formal series, so we write decomposition (6) in the form

$$\mathbf{v}^{\mathrm{t}} = (\hat{\mathbf{a}} \cdot \mathbf{v}^{\mathrm{t}})(\hat{\mathbf{a}} + \mathbf{v}^{\div}) \qquad \text{where} \qquad \hat{\mathbf{a}} \cdot \mathbf{v}^{\div} = 0. \tag{8}$$

We assume without loss of generality that the initial tangential position is $\mathbf{u}^{\mathrm{t}}(0) = \mathbf{0}$. As the particle is starting to slide we have $\mathbf{v}^{\mathrm{t}}(0) = \mathbf{0}$. Furthermore, as we look for motions where the particle starts to slide in direction $\hat{\mathbf{a}}$, it follows from equations (7) and (8) that $\mathbf{v}^{\div}(0) = \mathbf{0}$. The problem may now be stated as the combination of these initial conditions, with the equation of motion entailed by equations (4) and (7), and with the constraints (8).

Starting-to-Slide Problem. Given functions $\mathbf{f}^{\mathrm{t}} : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ and $\mathrm{f}^{\mathrm{n}} : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$, find a time $\eta > 0$, and functions $\mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}$ and $\mathbf{v}^{\div}$ from time interval $[0, \eta)$ to $\mathbb{R}^{d-1}$ that satisfy:

$$\mathbf{u}^{\mathrm{t}}(0) = \mathbf{0}, \qquad \mathbf{v}^{\mathrm{t}}(0) = \mathbf{0}, \qquad \mathbf{v}^{\div}(0) = \mathbf{0}, \tag{9a}$$

$$\dot{\mathbf{u}}^{\mathrm{t}} = \mathbf{v}^{\mathrm{t}} \qquad \text{on } [0, \eta), \tag{9b}$$

$$\dot{\mathbf{v}}^{\mathrm{t}} = \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) + \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}})(\hat{\mathbf{a}} + \mathbf{v}^{\div})(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2} \qquad \text{on } [0, \eta), \tag{9c}$$

$$\mathbf{v}^{\mathrm{t}} = (\hat{\mathbf{a}} \cdot \mathbf{v}^{\mathrm{t}})(\hat{\mathbf{a}} + \mathbf{v}^{\div}) \qquad \text{and} \qquad \hat{\mathbf{a}} \cdot \mathbf{v}^{\div} = 0 \qquad \text{on } (0, \eta), \tag{9d}$$

where $\hat{\mathbf{a}}$ is defined by equation (5).

Remark 3.3. In the following proposition, the hypothesis on the leading coefficient of $\mu \mathrm{f}^{\mathrm{n}}$ implies that $\mu \neq 0$. The case $\mu = 0$ is not addressed until Proposition 4.1.

Proposition 3.4. *Let the functions $\mathbf{f}^{\mathrm{t}}$ and f^{n} be analytic at $(0, \mathbf{0}, \mathbf{0})$ and satisfy*

$$\|\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})\| > |\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{0}, \mathbf{0})| \tag{10}$$

on $(0, \eta)$ for some $\eta > 0$. Let $\mathrm{lcf}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{f}}, \tilde{\mathbf{v}}^{\mathrm{f}})) < 0$, where $\tilde{\mathbf{u}}^{\mathrm{f}}$ and $\tilde{\mathbf{v}}^{\mathrm{f}}$ are the formal series with coefficients

$$\begin{aligned} \tilde{\mathbf{u}}^{\mathrm{f}}_0 &= \mathbf{0}, & \tilde{\mathbf{v}}^{\mathrm{f}}_0 &= \mathbf{0}, \\ i\tilde{\mathbf{u}}^{\mathrm{f}}_i &= \tilde{\mathbf{v}}^{\mathrm{f}}_{i-1}, & i\tilde{\mathbf{v}}^{\mathrm{f}}_i &= [\mathbf{f}^{\mathrm{t}}(t, \tilde{\mathbf{u}}^{\mathrm{f}}_{\le i-1}, \quad \tilde{\mathbf{v}}^{\mathrm{f}}_{\le i-1})]_{i-1}, \qquad i \geq 1. \end{aligned} \tag{11}$$

Then there exists a formal series solution $(\tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}}, \mathbf{v}^{\div})$ to the starting-to-slide problem (9) such that

$$\mathrm{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) < \infty \qquad \textit{and} \qquad \mathrm{lcf}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})) < 0. \tag{12}$$

Furthermore, for any positive integer m there exist real numbers c_m and $\eta_m > 0$ such that the polynomials $\mathbf{u}^{\mathrm{t}} := [\tilde{\mathbf{u}}^{\mathrm{t}}]_{m+\mathrm{ord}(\tilde{\mathbf{u}}^{\mathrm{t}})}$ and $\mathbf{v}^{\mathrm{t}} := [\tilde{\mathbf{v}}^{\mathrm{t}}]_{m+\mathrm{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})}$ satisfy

$$\left\| -\dot{\mathbf{v}}^{\mathrm{t}} + \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) + \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) \frac{\mathbf{v}^{\mathrm{t}}}{\|\mathbf{v}^{\mathrm{t}}\|} \right\| \leq c_m t^m \qquad \textit{on } (0, \eta_m). \tag{13}$$

Proof. We claim that the following algorithm generates coefficients $\tilde{\mathbf{u}}^{\mathrm{t}}_0, \ldots, \tilde{\mathbf{u}}^{\mathrm{t}}_{i_{\max}}$ and $\tilde{\mathbf{v}}^{\mathrm{t}}_0, \ldots, \tilde{\mathbf{v}}^{\mathrm{t}}_{i_{\max}}$ of a formal series solution to the starting-to-slide problem, given a nonnegative integer $i_{\max}$ as input.

$p \leftarrow \mathrm{ord}(\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})) + 1$ ▷ p will turn out to be a lower bound on $\mathrm{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})$
$\tilde{\mathbf{u}}^{\mathrm{t}}_0 \leftarrow \mathbf{0}$, $\tilde{\mathbf{v}}^{\mathrm{t}}_0 \leftarrow \mathbf{0}$ and $\mathbf{v}^{\div}_0 \leftarrow \mathbf{0}$
for $i \leftarrow 1, 2, \ldots, i_{\max}$ **do**
  $\tilde{\mathbf{u}}^{\mathrm{t}}_i \leftarrow \tilde{\mathbf{v}}^{\mathrm{t}}_{i-1}/i$
  if $i \leq p$ **then** solve equation (14) for $\tilde{\mathbf{v}}^{\mathrm{t}}_i$; **else** solve (14), (15) and (16) simultaneously for $\tilde{\mathbf{v}}^{\mathrm{t}}_i$ and $\mathbf{v}^{\div}_{i-p}$

$$[-\dot{\tilde{\mathbf{v}}}^{\mathrm{t}} + \mathbf{f}^{\mathrm{t}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}}) + \mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})(\hat{\mathbf{a}} + \mathbf{v}^{\div})(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}]_{i-1} = \mathbf{0} \tag{14}$$

$$[(\hat{\mathbf{a}} \cdot \tilde{\mathbf{v}}^{\mathrm{t}})(\hat{\mathbf{a}} + \mathbf{v}^{\div}) - \tilde{\mathbf{v}}^{\mathrm{t}}]_i = \mathbf{0} \tag{15}$$

$$\hat{\mathbf{a}} \cdot \mathbf{v}^{\div}_{i-p} = 0 \tag{16}$$

return $(\tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}}, \mathbf{v}^{\div})$

By the hypothesis on $\|\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})\|$ and the hypothesis that $\mathbf{f}^{\mathrm{t}}$ is analytic, Line 1 sets p to a finite value, and the unit vector $\hat{\mathbf{a}}$ of equation (5) is well defined. Line 2 ensures that the algorithm's output satisfies the initial conditions (9a). Line 4 ensures that the coefficients of t^{i-1} in differential equation (9b) match. Line 5 *would* ensure that coefficients of the remaining equations (9c) and (9d) match to any desired order, *provided* that:

I. The left-hand side of (14) is the coefficient of a well-defined formal series.
II. Equation (15) is automatically satisfied for $i \le p$.
III. Equations (14), (15) and (16) may be written as equations for unknowns $\tilde{\mathbf{v}}^{\mathrm{t}}_i$ and $\mathbf{v}^{\div}_{i-p}$, given the values of previously determined coefficients.
IV. These equations may indeed be solved.

Thus to prove the existence of a formal series solution, it remains to prove that these four conditions hold.

I. Showing the left-hand side of equation (14) is a well-defined formal series. Line 2 sets $\tilde{\mathbf{u}}^{\mathrm{t}}_0 = \mathbf{0}$, $\tilde{\mathbf{v}}^{\mathrm{t}}_0 = \mathbf{0}$, and $\mathbf{v}^{\div}_0 = \mathbf{0}$. The function $\mathbf{x} \mapsto (1+\|\mathbf{x}\|^2)^{-1/2}$ is analytic at $\mathbf{0}$, and the functions $\mathbf{f}^{\mathrm{t}}$ and f^{n} are analytic at $(0,\mathbf{0},\mathbf{0})$. As sums and products of analytic functions are analytic, it follows from Lemma 3.2 that the expression

$$\mathbf{f}^{\mathrm{t}}(t,\tilde{\mathbf{u}}^{\mathrm{t}},\tilde{\mathbf{v}}^{\mathrm{t}}) + \mu \mathrm{f}^{\mathrm{n}}(t,\tilde{\mathbf{u}}^{\mathrm{t}},\tilde{\mathbf{v}}^{\mathrm{t}})(\hat{\mathbf{a}}+\mathbf{v}^{\div})(1+\|\mathbf{v}^{\div}\|^2)^{-1/2}$$

is a well-defined formal series, irrespective of the values of $\tilde{\mathbf{u}}^{\mathrm{t}}_i, \tilde{\mathbf{v}}^{\mathrm{t}}_i$ and $\mathbf{v}^{\div}_i$ for $i > 0$. As the time derivative $\dot{\tilde{\mathbf{v}}}^{\mathrm{t}}$ is a valid formal power series, we conclude that the left-hand side of equation (14) is well defined.

II. Showing equation (15) is automatically satisfied for $i \le p$. Equation (14) simplifies to

$$i\tilde{\mathbf{v}}^{\mathrm{t}}_i = [\mathbf{f}^{\mathrm{t}}(t,\tilde{\mathbf{u}}^{\mathrm{t}},\tilde{\mathbf{v}}^{\mathrm{t}})]_{i-1} = [\mathbf{f}^{\mathrm{t}}(t,\tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1},\tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1})]_{i-1} \qquad \text{for } 1 \le i \le \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t,\tilde{\mathbf{u}}^{\mathrm{t}},\tilde{\mathbf{v}}^{\mathrm{t}})), \tag{17}$$

using Lemma 3.2. An induction coupling this result, the initial conditions and the fact that $i\tilde{\mathbf{u}}^{\mathrm{t}}_i = \tilde{\mathbf{v}}^{\mathrm{t}}_{i-1}$ gives

$$\tilde{\mathbf{u}}^{\mathrm{t}}_i = \mathbf{0}, \qquad \tilde{\mathbf{v}}^{\mathrm{t}}_i = \mathbf{0} \qquad \text{for } 0 \le i \le \min\{\mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0})), \mathrm{ord}(\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0}))\}.$$

But the hypothesis on $\|\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})\|$ implies that $\mathrm{ord}(\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})) \le \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0}))$ and Line 1 of the algorithm set $p = \mathrm{ord}(\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})) + 1$, and it follows that

$$\tilde{\mathbf{u}}^{\mathrm{t}}_i = \mathbf{0}, \qquad \tilde{\mathbf{v}}^{\mathrm{t}}_i = \mathbf{0} \qquad \text{for } i < p, \tag{18}$$

which clearly implies that

$$\mathrm{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) \ge p. \tag{19}$$

Writing equation (14) for $i = p$, and applying Lemma 3.2, the fact that $\mathbf{v}^{\div}_0 = \mathbf{0}$ and equation (18) gives

$$p\tilde{\mathbf{v}}^{\mathrm{t}}_p = [\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0}) + \mu \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0})\hat{\mathbf{a}}]_{p-1}. \tag{20}$$

But by definition of $\hat{\mathbf{a}}$ and p, coefficient $[\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})]_{p-1}$ is parallel to $\hat{\mathbf{a}}$, and it follows that

$$\tilde{\mathbf{v}}^{\mathrm{t}}_p = (\tilde{\mathbf{v}}^{\mathrm{t}}_p \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}. \tag{21}$$

Finally, applying Lemma 3.1 (using Eq. (19) and the fact that $\mathbf{v}^{\div}_0 = \mathbf{0}$) to the left-hand side of equation (15) gives

$$[(\hat{\mathbf{a}} \cdot \tilde{\mathbf{v}}^{\mathrm{t}})(\hat{\mathbf{a}}+\mathbf{v}^{\div}) - \tilde{\mathbf{v}}^{\mathrm{t}}]_i = [(\hat{\mathbf{a}} \cdot \tilde{\mathbf{v}}^{\mathrm{t}}_{\le p})\hat{\mathbf{a}} - \tilde{\mathbf{v}}^{\mathrm{t}}_{\le p}]_i = \mathbf{0} \qquad \text{for } i \le p,$$

where the second equality follows from equations (18) and (21).

Observations about $n := \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}}))$. Before showing that condition III holds, we present some facts about the order n. Let $n^{\mathrm{f}} := \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{f}}, \tilde{\mathbf{v}}^{\mathrm{f}}))$, noting that this is finite by the hypothesis that $\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{f}}, \tilde{\mathbf{v}}^{\mathrm{f}})$ has a leading coefficient. Comparing equation (17) with the definition of the hypothesized series $\tilde{\mathbf{u}}^{\mathrm{f}}$ and $\tilde{\mathbf{v}}^{\mathrm{f}}$, we see that

$$\tilde{\mathbf{u}}^{\mathrm{t}}_{\le n^{\mathrm{f}}} = \tilde{\mathbf{u}}^{\mathrm{f}}_{\le n^{\mathrm{f}}}, \qquad \tilde{\mathbf{v}}^{\mathrm{t}}_{\le n^{\mathrm{f}}} = \tilde{\mathbf{v}}^{\mathrm{f}}_{\le n^{\mathrm{f}}} \qquad \text{and} \qquad [\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})]_{\le n^{\mathrm{f}}} = [\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{f}}, \tilde{\mathbf{v}}^{\mathrm{f}})]_{\le n^{\mathrm{f}}}.$$

Hence,

$$n :- \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})) = n^{\mathrm{f}} < \infty. \tag{22}$$

Moreover, from the hypothesis on the leading coefficient of $\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{f}}, \tilde{\mathbf{v}}^{\mathrm{f}})$, it follows that

$$[\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})]_n < 0. \tag{23}$$

We now prove by contradiction that

$$n + 1 \ge p. \tag{24}$$

Assume $n < p-1$. By equation (18), this assumption gives $\tilde{\mathbf{u}}^{\mathrm{t}}_{\le n} = \mathbf{0}$ and $\tilde{\mathbf{v}}^{\mathrm{t}}_{\le n} = \mathbf{0}$. Thus, $n = \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})) = \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{0}, \mathbf{0}))$. But the hypothesis on $\|\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})\|$ implies that $\mathrm{ord}(\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})) \le \mathrm{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{0}, \mathbf{0}))$, and based on the previous sentence this is equivalent to $p - 1 \le n$, contradicting the assumption. Therefore equation (24) holds.

III. Rewriting equations (14), (15) and (16). By the definition of n, Lemma 3.1 gives

$$\Big[\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})\, (\hat{\mathbf{a}} + \mathbf{v}^{\div})(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}\Big]_{i-1} = \Big[\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})\, (\hat{\mathbf{a}} + \mathbf{v}^{\div}_{\le i-n-1})\, [(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}]_{\le i-n-1}\Big]_{i-1}.$$

As $\mathbf{v}^{\div}_0 = \mathbf{0}$, Lemma 3.1 also gives $[\|\mathbf{v}^{\div}\|^2]_{\le i-n-1} = \|\mathbf{v}^{\div}_{\le i-n-2}\|^2$. Therefore, applying Lemma 3.2 we may reformulate equation (14) as

$$i\tilde{\mathbf{v}}^{\mathrm{t}}_i = \Big[\mathbf{f}^{\mathrm{t}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1}) + \mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1})(\hat{\mathbf{a}} + \mathbf{v}^{\div}_{\le i-n-1})(1 + \|\mathbf{v}^{\div}_{\le i-n-2}\|^2)^{-1/2}\Big]_{i-1}. \tag{25}$$

We define the decompositions of $\tilde{\mathbf{v}}^{\mathrm{t}}$ and function $\mathbf{f}^{\mathrm{t}}$ parallel and perpendicular to unit vector $\hat{\mathbf{a}}$ by

$$\mathrm{v}^{\parallel} := \hat{\mathbf{a}} \cdot \tilde{\mathbf{v}}^{\mathrm{t}}, \qquad \mathbf{v}^{\perp} := \tilde{\mathbf{v}}^{\mathrm{t}} - \mathrm{v}^{\parallel}\hat{\mathbf{a}}$$
$$\mathrm{f}^{\parallel} := \hat{\mathbf{a}} \cdot \mathbf{f}^{\mathrm{t}}, \qquad \mathbf{f}^{\perp} := \mathbf{f}^{\mathrm{t}} - \mathrm{f}^{\parallel}\hat{\mathbf{a}}.$$

As the initial conditions require $\mathbf{v}^{\div}_0 = \mathbf{0}$, as the algorithm requires equation (16) to hold for $i > p$, and as $n + 1 \ge p$ by equation (24), it follows that

$$\hat{\mathbf{a}} \cdot \mathbf{v}^{\div}_{\le i-n-1} = 0.$$

Thus, taking the dot product of each side of equation (25) with $\hat{\mathbf{a}}$ gives

$$i\mathrm{v}^{\parallel}_i = \Big[\mathrm{f}^{\parallel}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1}) + \mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1})(1 + \|\mathbf{v}^{\div}_{\le i-n-2}\|^2)^{-1/2}\Big]_{i-1}. \tag{26}$$

Multiplying each side of equation (26) by $\hat{\mathbf{a}}$ and subtracting from equation (25) gives

$$i\mathbf{v}^{\perp}_i = \Big[\mathbf{f}^{\perp}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1}) + \mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\le i-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\le i-1})\mathbf{v}^{\div}_{\le i-n-1}(1 + \|\mathbf{v}^{\div}_{\le i-n-2}\|^2)^{-1/2}\Big]_{i-1}. \tag{27}$$

Finally, in equation (19) we showed that $\operatorname{ord}(\mathrm{v}^{\parallel}) \geq p$. Therefore, we may reformulate equation (15) as

$$\mathbf{v}_i^{\perp} = [\mathrm{v}_{\leq i-1}^{\parallel}\mathbf{v}_{\leq i-p}^{\div}]_i. \tag{28}$$

An observation about $\mathrm{v}_p^{\parallel}$. Before showing that condition IV holds, we show that

$$\mathrm{v}_p^{\parallel} \geq 0 \qquad \text{and} \qquad \text{if } n \geq p \text{ then } \mathrm{v}_p^{\parallel} > 0. \tag{29}$$

Taking the dot product of both sides of equation (20) with $\hat{\mathbf{a}}$ gives

$$p\mathrm{v}_p^{\parallel} = [\mathrm{f}^{\parallel}(t, \mathbf{0}, \mathbf{0}) + \mu\mathrm{f}^{\mathrm{n}}(t, \mathbf{0}, \mathbf{0})]_{p-1}.$$

By definition of $\hat{\mathbf{a}}$ and p, we have $[\mathrm{f}^{\parallel}(t, \mathbf{0}, \mathbf{0})]_{p-1} = \|[\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})]_{p-1}\| > 0$; by the hypothesis on $\|\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})\|$ we have $\|[\mathbf{f}^{\mathrm{t}}(t, \mathbf{0}, \mathbf{0})]_{p-1}\| \geq |[\mu\mathrm{f}^{\mathrm{n}}(t, \mathbf{0}, \mathbf{0})]_{p-1}|$; and it follows that $\mathrm{v}_p^{\parallel} \geq 0$. Moreover, if $n \geq p$ then equation (18) gives $0 = [\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})]_{p-1} = [\mathrm{f}^{\mathrm{n}}(t, \mathbf{0}, \mathbf{0})]_{p-1}$; and it follows that $\mathrm{v}_p^{\parallel} = [\mathrm{f}^{\parallel}(t, \mathbf{0}, \mathbf{0}) + 0]_{p-1}/p > 0$.

IV. Solving equations (14), (15) and (16). Equation (17) shows we may solve equation (14) when $i \leq n$. Equation (20) shows we may solve equation (14) when $i = p$. But we showed in equation (24) that $n + 1 \geq p$. Therefore, it only remains to show that we can solve the equations of Line 5 when $i > p$. We actually show that we can solve the reformulated equations (26), (27) and (28) along with the original (16) for $\mathrm{v}_i^{\parallel}, \mathbf{v}_i^{\perp}$ and $\mathbf{v}_{i-p}^{\div}$. Clearly, given $\mathrm{v}_i^{\parallel}$ and $\mathbf{v}_i^{\perp}$, we may reconstruct $\tilde{\mathbf{v}}_i^{\mathrm{t}}$ as $\tilde{\mathbf{v}}_i^{\mathrm{t}} = \mathrm{v}_i^{\parallel}\hat{\mathbf{a}} + \mathbf{v}_i^{\perp}$.

As the right-hand side of equation (26) is independent of the variables $\mathrm{v}_i^{\parallel}, \mathbf{v}_i^{\perp}$ and $\mathbf{v}_{i-p}^{\div}$, we can solve it for $\mathrm{v}_i^{\parallel}$ simply by dividing by i. To solve equations (27) and (28) simultaneously, we consider two cases: $n \geq p$ and $n + 1 = p$. In the light of equation (24), these cases cover all possibilities.

- Say $n \geq p$. Then the right-hand side of equation (27) is independent of the variables $\mathbf{v}_i^{\perp}$ and $\mathbf{v}_{i-p}^{\div}$, so we can solve for $\mathbf{v}_i^{\perp}$ simply by dividing by i. Equation (28) rearranges to give

  $$\mathrm{v}_p^{\parallel}\mathbf{v}_{i-p}^{\div} = \mathbf{v}_i^{\perp} - [\mathrm{v}_{\leq i-1}^{\parallel}\mathbf{v}_{\leq i-p-1}^{\div}]_i,$$

  which is readily solved for $\mathbf{v}_{i-p}^{\div}$, as $\mathrm{v}_p^{\parallel} > 0$ when $n \geq p$ by equation (29).
- Say $n + 1 = p$. In this case, we have $\mathbf{v}_{\leq i-n-2}^{\div} = \mathbf{v}_{\leq i-p-1}^{\div}$. As $\mathbf{v}_0^{\div} = \mathbf{0}$, the constant term of the formal series $(1 + \|\mathbf{v}_{\leq i-n-2}^{\div}\|^2)^{-1/2}$ is 1; as $i > p = n + 1$, the series $\mu\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}_{\leq i-1}^{\mathrm{t}}, \tilde{\mathbf{v}}_{\leq i-1}^{\mathrm{t}})$ is of order n; and it follows that the only term on the right-hand side of equation (27) involving $\mathbf{v}_{i-p}^{\div}$ is

  $$\left[\mu\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}_{\leq i-1}^{\mathrm{t}}, \tilde{\mathbf{v}}_{\leq i-1}^{\mathrm{t}})\right]_n \mathbf{v}_{i-p}^{\div}.$$

  Substituting for $\mathbf{v}_i^{\perp}$ from equation (28) into equation (27) thus gives

  $$i\mathrm{v}_p^{\parallel}\mathbf{v}_{i-p}^{\div} + i[\mathrm{v}_{\leq i-1}^{\parallel}\mathbf{v}_{\leq i-p-1}^{\div}]_i = \left[\mu\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}_{\leq i-1}^{\mathrm{t}}, \tilde{\mathbf{v}}_{\leq i-1}^{\mathrm{t}})\right]_n \mathbf{v}_{i-p}^{\div} + [\mathbf{S}(t, \tilde{\mathbf{u}}_{\leq i-1}^{\mathrm{t}}, \tilde{\mathbf{v}}_{\leq i-1}^{\mathrm{t}}, \mathbf{v}_{\leq i-p-1}^{\div})]_{i-1}$$

  for an appropriate function $\mathbf{S}$. This equation is readily solved for $\mathbf{v}_{i-p}^{\div}$, as $i\mathrm{v}_p^{\parallel} \geq 0$ by equation (29), and

  $$\left[\mu\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}_{\leq i-1}^{\mathrm{t}}, \tilde{\mathbf{v}}_{\leq i-1}^{\mathrm{t}})\right]_n < 0$$

  by equation (23). Substituting the resulting value of $\mathbf{v}_{i-p}^{\div}$ into equation (28) gives the value of the remaining variable $\mathbf{v}_i^{\perp}$.

This completes the proof of existence of a formal series solution.

Proof of conclusion (12) of Proposition 3.4. First, we prove that $\operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) < \infty$, arguing by contradiction. Assume $\operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) = \infty$. Equation (14) then simplifies to

$$[\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})]_i = -[\mu \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0})(\hat{\mathbf{a}} + \mathbf{v}^{\div})(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}]_i$$

for $i \geq 0$, and thus

$$[\|\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})\|^2]_i = [(\mu \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0})^2 \|\hat{\mathbf{a}} + \mathbf{v}^{\div}\|^2 (1 + \|\mathbf{v}^{\div}\|^2)^{-1}]_i$$

for $i \geq 0$. But as the initial conditions and equation (16) imply that $\hat{\mathbf{a}} \cdot \mathbf{v}^{\div}_j = 0$ for $j \geq 0$, and as $\hat{\mathbf{a}}$ is a unit vector, we have $\|\hat{\mathbf{a}} + \mathbf{v}^{\div}\|^2 = 1 + \|\mathbf{v}^{\div}\|^2$. It follows that

$$\|\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})\|^2 = (\mu \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0}))^2,$$

which contradicts the hypothesis about $\|\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})\|$. Therefore $\operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) < \infty$.

The conclusion that $\operatorname{lcf}(\mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})) < 0$ is immediate from equations (22) and (23).

Proof of conclusion (13) of Proposition 3.4. We begin by showing that

$$\operatorname{lcf}(\mathrm{v}^{\parallel}) > 0. \tag{30}$$

If $n \geq p$ then equations (18) and (29) immediately give $\operatorname{lcf}(\mathrm{v}^{\parallel}) > 0$. Otherwise, say $p = n + 1$. Let

$$\mathrm{f}^{\parallel 0} \coloneqq \mathrm{f}^{\parallel}(t,\mathbf{0},\mathbf{0}), \qquad \mathbf{f}^{\perp 0} \coloneqq \mathbf{f}^{\perp}(t,\mathbf{0},\mathbf{0}), \qquad \mathrm{f}^{\mathrm{n}0} \coloneqq \mathrm{f}^{\mathrm{n}}(t,\mathbf{0},\mathbf{0}).$$

Let $q \coloneqq \operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})$. Then equation (27) gives

$$\mathbf{0} = [\mathbf{f}^{\perp 0} + \mu \mathrm{f}^{\mathrm{n}0} \mathbf{v}^{\div}(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}]_{\leq q-2}. \tag{31}$$

As $n = p - 1$, equation (18) gives $n = \operatorname{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}_{\leq p-1}, \tilde{\mathbf{v}}^{\mathrm{t}}_{\leq p-1})) = \operatorname{ord}(\mu \mathrm{f}^{\mathrm{n}0})$, so equation (31) may be solved for $\mathbf{v}^{\div}_{\leq q-2-n}$. From this solution, we may determine

$$[(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}]_{\leq q-1-n} = [(1 - \|(\mathbf{f}^{\perp 0}/t^n)/(\mu \mathrm{f}^{\mathrm{n}0}/t^n)\|^2)^{1/2}]_{\leq q-1-n},$$

in which: the left-hand side only depends on $\mathbf{v}^{\div}_{\leq q-2-n}$ since $\mathbf{v}^{\div}_0 = \mathbf{0}$, as argued just before equation (25); we divide by t^n on the right to highlight the validity of dividing these formal series; and we observe that $\operatorname{ord}(\mathbf{f}^{\mathrm{t}}(t,\mathbf{0},\mathbf{0})) = n$ as $p = n + 1$, so that $[(\mathbf{f}^{\perp 0}/t^n)/(\mu \mathrm{f}^{\mathrm{n}0}/t^n)]_0 = 0$ and thus the composition with the function $\mathbf{x} \mapsto (1 - \|\mathbf{x}\|^2)^{1/2}$, which is analytic at $\mathbf{0}$, is well defined. As $q = \operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})$, equation (26) gives

$$\text{for } i \leq q \qquad i\mathrm{v}^{\parallel}_i = [h]_{i-1} \qquad \text{where} \qquad h \coloneqq \mathrm{f}^{\parallel 0} + \mu \mathrm{f}^{\mathrm{n}0}(1 - \|(\mathbf{f}^{\perp 0}/t^n)/(\mu \mathrm{f}^{\mathrm{n}0}/t^n)\|^2)^{1/2}.$$

As $\mathbf{v}^{\perp} = \mathrm{v}^{\parallel}\mathbf{v}^{\div}$ and $\mathbf{v}^{\div}_0 = \mathbf{0}$, we have $\operatorname{ord}(\mathbf{v}^{\perp}) \geq \operatorname{ord}(\mathrm{v}^{\parallel}) + 1$. Thus $q = \operatorname{ord}(\mathrm{v}^{\parallel})$, so that

$$q\mathrm{v}^{\parallel}_q = \operatorname{lcf}(h).$$

By hypothesis, $(\mathrm{f}^{\parallel 0})^2 + \|\mathbf{f}^{\perp 0}\|^2 \geq (\mu \mathrm{f}^{\mathrm{n}0})^2$ on a right neighbourhood of $t = 0$; by definition of $\hat{\mathbf{a}}$, we have $\operatorname{lcf}(\mathrm{f}^{\parallel 0}) > 0$; and as $p = n + 1$, equations (18) and (23) give $\operatorname{lcf}(\mathrm{f}^{\mathrm{n}0}) < 0$. It follows that

$$\mathrm{f}^{\parallel 0} \geq ((\mu \mathrm{f}^{\mathrm{n}0})^2 - \|\mathbf{f}^{\perp 0}\|^2)^{1/2} = -\mu \mathrm{f}^{\mathrm{n}0}(1 - \|(\mathbf{f}^{\perp 0}/t^n)/(\mu \mathrm{f}^{\mathrm{n}0}/t^n)\|^2)^{1/2} \qquad \text{for } t \in [0, \eta)$$

for some $\eta > 0$. Therefore, $\operatorname{lcf}(h) > 0$ and we conclude that $\operatorname{lcf}(\mathrm{v}^{\parallel}) = \mathrm{v}^{\parallel}_q > 0$.

Consider the positive integer m and polynomials $\mathbf{u}^{\mathrm{t}}$ and $\mathbf{v}^{\mathrm{t}}$ appearing in conclusion (13). As $\operatorname{ord}(\mathbf{v}^{\mathrm{t}}) = q = \operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) < \infty$, the ratio $(\mathbf{v}^{\mathrm{t}}/t^q)/\|\mathbf{v}^{\mathrm{t}}/t^q\|$ is analytic at 0. Thus the function

$$\mathbf{g}(t) \coloneqq -\dot{\mathbf{v}}^{\mathrm{t}} + \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) + \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) \frac{\mathbf{v}^{\mathrm{t}}/t^q}{\|\mathbf{v}^{\mathrm{t}}/t^q\|}$$

is analytic at 0. Moreover, as $\tilde{\mathbf{v}}^{\mathrm{t}} = \mathrm{v}^{\parallel}(\hat{\mathbf{a}} + \mathbf{v}^{\div})$, Lemmas 3.1 and 3.2 give

$$\left[\frac{\mathbf{v}^{\mathrm{t}}/t^q}{\|\mathbf{v}^{\mathrm{t}}/t^q\|}\right]_{\le m-1} = \left[\frac{(\mathrm{v}^{\parallel}_{\le m+q-1}/t^q)\ [\hat{\mathbf{a}} + \mathbf{v}^{\div}]_{\le m+q-1}}{\|(\mathrm{v}^{\parallel}_{\le m+q-1}/t^q)\ [\hat{\mathbf{a}} + \mathbf{v}^{\div}]_{\le m+q-1}\|}\right]_{\le m-1}$$
$$= [(\hat{\mathbf{a}} + \mathbf{v}^{\div})(1 + \|\mathbf{v}^{\div}\|^2)^{-1/2}]_{\le m-1},$$

where the second line follows as $\mathrm{lcf}(\mathrm{v}^{\parallel}) > 0$ by equation (30), and as $\hat{\mathbf{a}} \cdot \mathbf{v}^{\div} = 0$. Thus, equation (25) gives

$$\mathbf{g}_{\le m-1} = \mathbf{0}.$$

Because $\mathbf{g}$ is analytic at 0, there is a positive convergence radius $\rho > 0$ such that the Taylor series of $\mathbf{g}$ converges absolutely for all $|t| < \rho$. Thus, for any real numbers t and η_m with $0 \le t < \eta_m < \rho$,

$$\|\mathbf{g}(t)\| \le \sum_{k=m}^{\infty} \|\mathbf{g}_k\| t^k \le (t/\eta_m)^m \sum_{k=m}^{\infty} \|\mathbf{g}_k\| \eta_m^k \le c_m t^m$$

for some real number c_m. Therefore, conclusion (13) holds. This completes the proof. □

3.3. Local Solution to the Starting-to-Slide Problem

As the formal series solution to the starting-to-slide problem is not always convergent, it is not obvious how it relates to motions satisfying the unilateral problem. We now relate them by showing that the solution to a *bilateral problem* (defined just below) is asymptotic to the formal series, and that the bilateral problem reduces to the unilateral problem, since the normal force is nonpositive for this solution. The bilateral problem is so-called as it can be interpreted as a model of a particle subject to the two unilateral constraints $\mathrm{u}^{\mathrm{n}} \ge 0$ and $\mathrm{u}^{\mathrm{n}} \le 0$.

Bilateral Problem. Given functions $\mathbf{f}^{\mathrm{t}} : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ and $\mathrm{f}^{\mathrm{n}} : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \to \mathbb{R}$, find tangential position and velocity functions $\mathbf{u}^{\mathrm{t}}$ and $\mathbf{v}^{\mathrm{t}}$ from a time interval $[0, T]$ to $\mathbb{R}^{d-1}$ that satisfy:

$$\mathbf{u}^{\mathrm{t}}(0) = \mathbf{0}, \qquad \mathbf{v}^{\mathrm{t}}(0) = \mathbf{0}, \tag{32a}$$
$$\dot{\mathbf{u}}^{\mathrm{t}} = \mathbf{v}^{\mathrm{t}} \qquad \text{on } [0, T], \tag{32b}$$
$$\dot{\mathbf{v}}^{\mathrm{t}} - \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) \in -|\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}})|\, \partial \|\mathbf{v}^{\mathrm{t}}\| \qquad \text{for almost all } t \text{ in } [0, T]. \tag{32c}$$

Remark 3.5. In bilateral problems, as the absolute value $|\mathrm{f}^{\mathrm{n}}|$ may be interpreted as the density of a normal reaction measure, we only require the differential inclusion to hold for almost all t in $[0, T]$. Thus, any solution $\mathbf{u}^{\mathrm{t}}$ is naturally interpreted as an element of the Sobolev space $W^{2,1}$.

Charles and Ballard Proposition 3.1 from [6] proved the existence and uniqueness of a solution to bilateral problems, for a broader range of initial conditions than our all-zero initial conditions, under the hypothesis that $\mathbf{f}^{\mathrm{t}}$ and f^{n} are Lipschitz in their position and velocity arguments for each time t. We reuse a result from Step 3 of their proof, which considers an iterative process for solving the bilateral problem, that generates a sequence of functions $\mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0}, \mathbf{u}^{\mathrm{t},1}, \mathbf{v}^{\mathrm{t},1}, \ldots, \mathbf{u}^{\mathrm{t},k}, \mathbf{v}^{\mathrm{t},k}, \ldots$. Charles and Ballard set the first elements ($k = 0$) of this sequence, which we call the *ansatz*, based on the initial conditions of the bilateral problem. Subsequent elements ($k > 0$) are uniquely defined by

$$\begin{gathered} \mathbf{u}^{\mathrm{t},k}(0) = \mathbf{0}, \qquad \mathbf{v}^{\mathrm{t},k}(0) = \mathbf{0}, \\ \dot{\mathbf{u}}^{\mathrm{t},k} = \mathbf{v}^{\mathrm{t},k}, \qquad \dot{\mathbf{v}}^{\mathrm{t},k} - \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},k}, \mathbf{v}^{\mathrm{t},k}) \in -|\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t},k-1}, \mathbf{v}^{\mathrm{t},k-1})|\, \partial \|\mathbf{v}^{\mathrm{t},k}\| \end{gathered} \tag{33}$$

for almost all t in $[0, T]$. It is shown Line 7 of page 11 from [6] that there exists a real number $\beta > 0$, depending only on the time T and the Lipschitz constants of $\mathbf{f}^{\mathrm{t}}$ and f^{n}, such that the difference between consecutive iterates is bounded by

$$\|\mathbf{v}^{\mathrm{t},k+1}(t) - \mathbf{v}^{\mathrm{t},k}(t)\| \leq \beta \int_0^t \|\mathbf{v}^{\mathrm{t},k}(s) - \mathbf{v}^{\mathrm{t},k-1}(s)\| \,\mathrm{d}s \tag{34}$$

for all t in $[0, T]$ and for all $k = 1, 2, \ldots$. Although Charles and Ballard prove this for a specific ansatz, with the particle's position varying linearly with time, examining their proof, one sees that equation (34) holds for *any* ansatz in which $\mathbf{u}^{\mathrm{t},0}$ and $\mathbf{v}^{\mathrm{t},0}$ are continuous and satisfy the initial conditions of the bilateral problem. Moreover, for any such ansatz, one sees that iteration (33) converges to a limit which is the unique solution of the bilateral problem. The idea of the proof of the following proposition is to use truncated formal series as ansatz in this iteration.

Proposition 3.6. *Let the functions $\mathbf{f}^{\mathrm{t}}$ and f^{n} be analytic in all their arguments, and Lipschitz in their position and velocity arguments for all times. Let $(\tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})$ be a formal series solution to the starting-to-slide problem (9) for functions $\mathbf{f}^{\mathrm{t}}$ and f^{n}, which satisfies the conclusions of Proposition 3.4. Let $(\mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}})$ be a solution to the bilateral problem (32) also for functions $\mathbf{f}^{\mathrm{t}}$ and f^{n}. Then the solution to the bilateral problem is asymptotic to the formal series solution: for each nonnegative integer m there exist real numbers C_m and $\eta_m > 0$ such that*

$$\left\| \begin{pmatrix} \mathbf{u}^{\mathrm{t}}(t) \\ \mathbf{v}^{\mathrm{t}}(t) \end{pmatrix} - \sum_{k=0}^{m} \begin{pmatrix} \tilde{\mathbf{u}}^{\mathrm{t}}_k \\ \tilde{\mathbf{v}}^{\mathrm{t}}_k \end{pmatrix} t^k \right\| \leq C_m t^{m+1} \tag{35}$$

for all $t \in [0, \eta_m)$. Furthermore, there exist real numbers n, α and $\eta > 0$ such that

$$\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}(t), \mathbf{v}^{\mathrm{t}}(t)) < 0 \qquad \text{on } (0, \eta) \tag{36}$$

$$\text{and} \qquad t \left| \frac{\mathrm{d}}{\mathrm{d}t} \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}(t), \mathbf{v}^{\mathrm{t}}(t)) \right| \leq (n + \alpha t) \left| \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}(t), \mathbf{v}^{\mathrm{t}}(t)) \right| \qquad \text{on } [0, \eta). \tag{37}$$

Remark 3.7. Given a formal series solution to the starting-to-slide problem, one might apply results on formal series solutions of differential equations [16] to guarantee the existence of a potentially complex-valued function that is asymptotic to that formal series solution, and analytic on a sector of the complex plane, treating time as a complex number. However, those results do not guarantee that the imaginary part of this function vanishes when time is on the positive real axis. For this reason, the following proof uses a Cauchy-Lipschitz argument instead.

Proof of inequality (35). First note that if inequality (35) holds for some positive integer m, then it holds for all nonnegative integers $m' < m$, for appropriate choices of $C_{m'}$ and $\eta_{m'}$. Therefore, in the following, let m be any positive integer with

$$m > \operatorname{ord}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})) =: n, \tag{38}$$

noting that n is finite by the hypothesis that the conclusions of Proposition 3.4 hold. Consider iteration (33) with an ansatz given by the truncated formal series

$$\mathbf{u}^{\mathrm{t},0} := [\tilde{\mathbf{u}}^{\mathrm{t}}]_{\leq m + \operatorname{ord}(\tilde{\mathbf{u}}^{\mathrm{t}})}, \qquad \mathbf{v}^{\mathrm{t},0} := [\tilde{\mathbf{v}}^{\mathrm{t}}]_{\leq m + \operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})}. \tag{39}$$

We draw three other inferences from the fact that the formal series satisfy the conclusions of Proposition 3.4. First, by the conclusion that $\operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}}) < \infty$, it follows that the ansatz consists of well-defined polynomials. Second, as $n < m + \operatorname{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})$, Lemma 3.1 gives

$$[\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})]_i = [\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0})]_i \qquad \text{for } i \leq n,$$

so by the conclusion that $\operatorname{lcf}(\mu \mathrm{f}^{\mathrm{n}}(t, \tilde{\mathbf{u}}^{\mathrm{t}}, \tilde{\mathbf{v}}^{\mathrm{t}})) < 0$, there exist positive numbers η_1 and α_1 such that

$$\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0}) < -\alpha_1 t^n \qquad \text{for } t \in (0, \eta_1). \tag{40}$$

Third, we may define a time $T > 0$, a continuous function $\boldsymbol{\rho}^0 : [0, T] \to \mathbb{R}^{d-1}$ that we call the *residual*, and a real number c_m such that

$$\boldsymbol{\rho}^0 \in -\dot{\mathbf{v}}^{\mathrm{t},0} + \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0}) - |\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0})|\, \partial \|\mathbf{v}^{\mathrm{t},0}\| \qquad \text{and} \qquad \|\boldsymbol{\rho}^0\| \le c_m t^m \qquad \text{on } [0, T], \tag{41}$$

in which the use of the absolute value $|\mu \mathrm{f}^{\mathrm{n}}|$ is justified by inequality (40).

Consider the function $\mathbf{v}^{\mathrm{t},1}$ resulting from the first iteration of equation (33), for the time $T > 0$ just defined. By monotonicity of the subdifferential,

$$(\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}) \cdot (\partial \|\mathbf{v}^{\mathrm{t},1}\| - \partial \|\mathbf{v}^{\mathrm{t},0}\|) \ge 0.$$

Equivalently, using equations (33) and (41),

$$(\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}) \cdot (\dot{\mathbf{v}}^{\mathrm{t},1} - \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},1}, \mathbf{v}^{\mathrm{t},1}) - \dot{\mathbf{v}}^{\mathrm{t},0} + \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0}) - \boldsymbol{\rho}^0) \le 0.$$

Integrating, noting that $\mathbf{v}^{\mathrm{t},0}(0) = \mathbf{v}^{\mathrm{t},1}(0)$, and using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
\frac{1}{2}\|\mathbf{v}^{\mathrm{t},1}(t) - \mathbf{v}^{\mathrm{t},0}(t)\|^2 &\le \int_0^t (\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}) \cdot (\mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},1}, \mathbf{v}^{\mathrm{t},1}) - \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0}) + \boldsymbol{\rho}^0)\, dt \\
&\le \int_0^t \|\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}\| (\|\mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},1}, \mathbf{v}^{\mathrm{t},1}) - \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0})\| + \|\boldsymbol{\rho}^0\|)\, dt
\end{aligned}$$

for $t \in [0, T]$, in which $\boldsymbol{\rho}^0$ is integrable as it is continuous. Thus the Ou-Yang inequality[1] gives

$$\|\mathbf{v}^{\mathrm{t},1}(t) - \mathbf{v}^{\mathrm{t},0}(t)\| \le \int_0^t (\|\mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},1}, \mathbf{v}^{\mathrm{t},1}) - \mathbf{f}^{\mathrm{t}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0})\| + \|\boldsymbol{\rho}^0\|)\, dt.$$

By hypothesis, $\mathbf{f}^{\mathrm{t}}$ has a Lipschitz constant L, and by the bound (41) on residual $\boldsymbol{\rho}^0$, we find

$$\begin{aligned}
\|\mathbf{v}^{\mathrm{t},1}(t) - \mathbf{v}^{\mathrm{t},0}(t)\| &\le \int_0^t (L\|\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}\| + L\|\mathbf{u}^{\mathrm{t},1} - \mathbf{u}^{\mathrm{t},0}\| + c_m t^m)\, dt \\
&\le \frac{c_m}{m+1} t^{m+1} + \int_0^t L(1+T)\|\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}\|\, dt,
\end{aligned}$$

having observed that as $\tilde{\mathbf{v}}^{\mathrm{t}}$ is the formal derivative of $\tilde{\mathbf{u}}^{\mathrm{t}}$, the ansatz (39) satisfies $\dot{\mathbf{u}}^{\mathrm{t},0} = \mathbf{v}^{\mathrm{t},0}$, thus

$$\|\mathbf{u}^{\mathrm{t},1}(t) - \mathbf{u}^{\mathrm{t},0}(t)\| = \|\int_0^t (\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0})\, dt\| \le \int_0^t \|\mathbf{v}^{\mathrm{t},1} - \mathbf{v}^{\mathrm{t},0}\|\, dt \qquad \text{on } [0, T].$$

As $t \mapsto \|\mathbf{v}^{\mathrm{t},1}(t) - \mathbf{v}^{\mathrm{t},0}(t)\|$ is continuous and $c_m t^{m+1}/(m+1)$ is nondecreasing, Grönwall's inequality gives

$$\|\mathbf{v}^{\mathrm{t},1}(t) - \mathbf{v}^{\mathrm{t},0}(t)\| \le \frac{c_m}{m+1} t^{m+1} \exp\left(\int_0^t L(1+T)\, dt\right) \le \alpha_2 t^{m+1} \qquad \text{on } [0, T] \tag{42}$$

for some $\alpha_2 > 0$.

[1]A corollary of the Ou-Yang inequality Corollary 1.2.1 from [25] is as follows. Let $f \in L^1(0, T)$ with $f(t) \ge 0$ for almost all t in $[0, T]$. For some constant $a \ge 0$, let the continuous function $w : [0, T] \to \mathbb{R}$ satisfy $\frac{w(t)^2}{2} \le \frac{a^2}{2} + \int_0^t f(s) w(s)\, \mathrm{d}s$. Then for all t in $[0, T]$, we have $|w(t)| \le a + \int_0^t f(s)\, \mathrm{d}s$.

As discussed above, Charles and Ballard Step 3 of the proof of Proposition 3.1 from [6] showed that the following pointwise limits exist and are the unique solutions to the bilateral problem:

$$\mathbf{u}^{\mathrm{t}} := \lim_{k\to\infty} \mathbf{u}^{\mathrm{t},k} \qquad \text{and} \qquad \mathbf{v}^{\mathrm{t}} := \lim_{k\to\infty} \mathbf{v}^{\mathrm{t},k};$$

and they showed that inequality (34) holds for some real number $\beta > 0$, from which we get

$$\|\mathbf{v}^{\mathrm{t},k+1}(t) - \mathbf{v}^{\mathrm{t},k}(t)\| \le \beta^k \int_0^t ds_k \cdots \int_0^{s_2} ds_1 \sup_{0\le s_1\le t} \|\mathbf{v}^{\mathrm{t},1}(s_1) - \mathbf{v}^{\mathrm{t},0}(s_1)\| \le \frac{\beta^k t^k}{k!} \sup_{0\le s\le t} \|\mathbf{v}^{\mathrm{t},1}(s) - \mathbf{v}^{\mathrm{t},0}(s)\|.$$

Thus, using inequality (42), we may bound the distance of the ansatz from the solution by

$$\|\mathbf{v}^{\mathrm{t}}(t) - \mathbf{v}^{\mathrm{t},0}(t)\| \le \sum_{k=0}^{\infty} \|\mathbf{v}^{\mathrm{t},k+1} - \mathbf{v}^{\mathrm{t},k}\| \le \sum_{k=0}^{\infty} \frac{\beta^k t^k}{k!} \sup_{0\le s\le t} \|\mathbf{v}^{\mathrm{t},1}(s) - \mathbf{v}^{\mathrm{t},0}(s)\| \le e^{\beta t}\alpha_2 t^{m+1}. \tag{43}$$

Hence,

$$\|\mathbf{u}^{\mathrm{t}}(t) - \mathbf{u}^{\mathrm{t},0}(t)\| \le \int_0^t \|\mathbf{v}^{\mathrm{t}}(t) - \mathbf{v}^{\mathrm{t},0}(t)\|\, dt \le e^{\beta t}\alpha_2 \frac{t^{m+2}}{m+2}. \tag{44}$$

Therefore, the solution to the bilateral problem is asymptotic to the formal series solution:

$$\begin{aligned}\left\| \begin{pmatrix} \mathbf{u}^{\mathrm{t}}(t) \\ \mathbf{v}^{\mathrm{t}}(t) \end{pmatrix} - \sum_{k=0}^{m} \begin{pmatrix} \tilde{\mathbf{u}}^{\mathrm{t}}_k \\ \tilde{\mathbf{v}}^{\mathrm{t}}_k \end{pmatrix} t^k \right\| &\le \|\mathbf{u}^{\mathrm{t}}(t) - \mathbf{u}^{\mathrm{t},0}(t)\| + \|\mathbf{v}^{\mathrm{t}}(t) - \mathbf{v}^{\mathrm{t},0}(t)\| + \left\| \sum_{k=m+1}^{m+\mathrm{ord}(\tilde{\mathbf{u}}^{\mathrm{t}})} \tilde{\mathbf{u}}^{\mathrm{t}}_k t^k \right\| + \left\| \sum_{k=m+1}^{m+\mathrm{ord}(\tilde{\mathbf{v}}^{\mathrm{t}})} \tilde{\mathbf{v}}^{\mathrm{t}}_k t^k \right\| \\ &\le C_m t^{m+1}\end{aligned}$$

on $[0, T]$, for some real number C_m. □

Proof of inequality (36). Let L be a Lipschitz constant for both f^{n} and $\mathbf{f}^{\mathrm{t}}$. Let $m > n$ as in definition (38), and let the ansatz $\mathbf{u}^{\mathrm{t},0}$ and $\mathbf{v}^{\mathrm{t},0}$ be as in definition (39). Then

$$\begin{aligned}\mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t}}, \mathbf{v}^{\mathrm{t}}) &\le \mu \mathrm{f}^{\mathrm{n}}(t, \mathbf{u}^{\mathrm{t},0}, \mathbf{v}^{\mathrm{t},0}) + \mu L(\|\mathbf{v}^{\mathrm{t}}(t) - \mathbf{v}^{\mathrm{t},0}(t)\| + \|\mathbf{u}^{\mathrm{t}}(t) - \mathbf{u}^{\mathrm{t},0}(t)\|) \\ &\le -\alpha_1 t^n + C t^{m+1} < 0 \qquad \text{on } (0, \eta)\end{aligned}$$

for some real numbers C and $\eta > 0$, where the second line follows from inequalities (35) and (40). □

Before proving inequality (37), we demonstrate a Lipschitz-like inequality for the non-Lipschitz function $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$.

Lemma 3.8. *Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ with $\|\mathbf{x} - \mathbf{y}\| < \|\mathbf{y}\| \ne 0$. Then*

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| \le \sqrt{2} \frac{\|\mathbf{x} - \mathbf{y}\|}{\|\mathbf{y}\|}.$$

Proof. The angle subtended at the origin by a ball of radius $r < 1$ centred at unit distance from the origin is $2 \arcsin r$. As the vector $\mathbf{x}$ lies in a ball of radius $\|\mathbf{x} - \mathbf{y}\|$ centred at distance $\|\mathbf{y}\|$ from the origin, the angle between the unit vectors in directions $\mathbf{x}$ and $\mathbf{y}$ is at most $\arcsin r$ where $r = \|\mathbf{x} - \mathbf{y}\|/\|\mathbf{y}\| < 1$. Thus

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| \le \sqrt{2 - 2\cos \arcsin r} = \sqrt{2 - 2\sqrt{1 - r^2}} \le \sqrt{2} r,$$

where the last inequality follows as $\sqrt{1 - r^2} \ge 1 - r^2$. □

Proof of inequality (37). In the following, we denote the value of a function g for arguments $(t, \mathbf{u}^{\rm t}(t), \mathbf{v}^{\rm t}(t))$ by $g(\cdot^{\rm t})$ and its value for arguments $(t, \mathbf{u}^{\rm t,0}(t), \mathbf{v}^{\rm t,0}(t))$ by $g(\cdot^0)$, where $\mathbf{u}^{\rm t,0}$, $\mathbf{v}^{\rm t,0}$ and the associated integer m are as in definition (39). As the function ${\rm f}^{\rm n}(\cdot^0)$ is analytic with ${\rm ord}({\rm f}^{\rm n}(\cdot^0)) = n < \infty$, by considering its Taylor expansion, we see that there exist real numbers c_1, c_2, c_3 and $\eta > 0$ such that

$$\frac{t|({\rm d}/{\rm d}t){\rm f}^{\rm n}(\cdot^0)|}{|{\rm f}^{\rm n}(\cdot^0)|} \le \frac{n|[{\rm f}^{\rm n}(\cdot^0)]_n| + c_1 t}{|[{\rm f}^{\rm n}(\cdot^0)]_n| - c_2 t} \le n + c_3 t \qquad \text{on } [0, \eta). \tag{45}$$

Let $q :- m + {\rm ord}(\mathbf{v}^{\rm t,0}) + 1$ so that $\mathbf{v}^{\rm t,0}(t) = \tilde{\mathbf{v}}^{\rm t}_{\le q-1}(t)$. As ${\rm f}^{\rm n}$ has Lipschitz constant L, we have

$$\begin{aligned} |{\rm f}^{\rm n}(\cdot^0)| &\le |{\rm f}^{\rm n}(\cdot^{\rm t})| + |{\rm f}^{\rm n}(\cdot^0) - {\rm f}^{\rm n}(\cdot^{\rm t})| \\ &\le |{\rm f}^{\rm n}(\cdot^{\rm t})| + L\|\mathbf{u}^{\rm t,0}(t) - \mathbf{u}^{\rm t}(t)\| + L\|\mathbf{v}^{\rm t,0}(t) - \mathbf{v}^{\rm t}(t)\|. \end{aligned}$$

Thus, by inequality (35), there exists a real number C such that

$$|{\rm f}^{\rm n}(\cdot^0)| \le |{\rm f}^{\rm n}(\cdot^{\rm t})| + LCt^q \qquad \text{on } [0, \eta), \tag{46}$$

taking a smaller $\eta > 0$ if necessary. Now, using the chain rule, equation of motion (32c), definition (41) of residual $\boldsymbol{\rho}^0$, and inequalities (36) and (40), the total time derivative of the normal force may be written in the form

$$\frac{{\rm d}{\rm f}^{\rm n}}{{\rm d}t}(\cdot^{\rm t}) = a(\cdot^{\rm t}) + \mathbf{b}(\cdot^{\rm t}) \cdot \frac{\mathbf{v}^{\rm t}(t)}{\|\mathbf{v}^{\rm t}(t)\|} \tag{47}$$

$$\frac{{\rm d}{\rm f}^{\rm n}}{{\rm d}t}(\cdot^0) = a(\cdot^0) + \mathbf{b}(\cdot^0) \cdot \frac{\mathbf{v}^{\rm t,0}(t)}{\|\mathbf{v}^{\rm t,0}(t)\|} - \boldsymbol{\rho}^0(t) \cdot \mathbf{c}(\cdot^0) \tag{48}$$

on $(0, \eta)$ taking a smaller $\eta > 0$ if necessary, where

$$\begin{aligned} a(t, \mathbf{x}, \mathbf{y}) &\coloneqq \frac{\partial {\rm f}^{\rm n}}{\partial t}(t, \mathbf{x}, \mathbf{y}) + \mathbf{y} \cdot \frac{\partial {\rm f}^{\rm n}}{\partial \mathbf{x}}(t, \mathbf{x}, \mathbf{y}) + \mathbf{f}^{\rm t}(t, \mathbf{x}, \mathbf{y}) \cdot \frac{\partial {\rm f}^{\rm n}}{\partial \mathbf{y}}(t, \mathbf{x}, \mathbf{y}) \\ \mathbf{b}(t, \mathbf{x}, \mathbf{y}) &\coloneqq \mu {\rm f}^{\rm n}(t, \mathbf{x}, \mathbf{y}) \frac{\partial {\rm f}^{\rm n}}{\partial \mathbf{y}}(t, \mathbf{x}, \mathbf{y}) \\ \mathbf{c}(t, \mathbf{x}, \mathbf{y}) &\coloneqq \frac{\partial {\rm f}^{\rm n}}{\partial \mathbf{y}}(t, \mathbf{x}, \mathbf{y}). \end{aligned}$$

By inequality (35), the possible arguments of the functions $a, \mathbf{b}, \mathbf{c}$ appearing in the total time derivatives (47) and (48) are in the set

$$\mathcal{S} \coloneqq \left\{ (s, \mathbf{x}, \mathbf{y}) \ : \ 0 \le s \le \eta, \quad \|\mathbf{x}\| \le C\eta^q + \sup_{0 \le t \le \eta} \|\mathbf{u}^{\rm t,0}(t)\|, \quad \|\mathbf{y}\| \le C\eta^q + \sup_{0 \le t \le \eta} \|\mathbf{v}^{\rm t,0}(t)\| \right\},$$

taking a larger real number C if necessary. As $\mathbf{u}^{\rm t,0}$ and $\mathbf{v}^{\rm t,0}$ are polynomials, they are bounded on $[0, \eta]$. Hence, set $\mathcal{S}$ is compact. As a, $\mathbf{b}$ and $\mathbf{c}$ are analytic functions, it follows that there exists a real number B such they are bounded by B and B-Lipschitz on $\mathcal{S}$:

$$\left. \begin{aligned} |a(s, \mathbf{x}, \mathbf{y})| &< B \\ \|\mathbf{b}(s, \mathbf{x}, \mathbf{y})\| &< B \\ \|\mathbf{c}(s, \mathbf{x}, \mathbf{y})\| &< B \\ |a(s, \mathbf{x}, \mathbf{y}) - a(s, \mathbf{x}', \mathbf{y}')| &< B\|\mathbf{x} - \mathbf{x}'\| + B\|\mathbf{y} - \mathbf{y}'\| \\ \|\mathbf{b}(s, \mathbf{x}, \mathbf{y}) - \mathbf{b}(s, \mathbf{x}', \mathbf{y}')\| &< B\|\mathbf{x} - \mathbf{x}'\| + B\|\mathbf{y} - \mathbf{y}'\| \end{aligned} \right\} \qquad \text{for all } (s, \mathbf{x}, \mathbf{y}) \text{ and } (s', \mathbf{x}', \mathbf{y}') \text{ in } \mathcal{S}.$$

In the next lines, we take a larger $C < \infty$ and smaller $\eta > 0$ if necessary. Inequality (35) implies that

$$|a(\cdot^{\mathrm{t}}) - a(\cdot^{0})| \le 2BCt^{q} \qquad \text{on } [0,\eta). \tag{49}$$

Inequality (35) and Lemma 3.8 imply that

$$\begin{aligned}\left|\mathbf{b}(\cdot^{\mathrm{t}})\cdot\frac{\mathbf{v}^{\mathrm{t}}(t)}{\|\mathbf{v}^{\mathrm{t}}(t)\|} - \mathbf{b}(\cdot^{0})\cdot\frac{\mathbf{v}^{\mathrm{t},0}(t)}{\|\mathbf{v}^{\mathrm{t},0}(t)\|}\right| &\le \left|(\mathbf{b}(\cdot^{\mathrm{t}}) - \mathbf{b}(\cdot^{0}))\cdot\frac{\mathbf{v}^{\mathrm{t}}(t)}{\|\mathbf{v}^{\mathrm{t}}(t)\|}\right| + \left|\mathbf{b}(\cdot^{0})\cdot\left(\frac{\mathbf{v}^{\mathrm{t}}(t)}{\|\mathbf{v}^{\mathrm{t}}(t)\|} - \frac{\mathbf{v}^{\mathrm{t},0}(t)}{\|\mathbf{v}^{\mathrm{t},0}(t)\|}\right)\right| \\ &\le 2BC't^{q} + B\sqrt{2}\frac{\|\mathbf{v}^{\mathrm{t}}(t) - \mathbf{v}^{\mathrm{t},0}(t)\|}{\|\mathbf{v}^{\mathrm{t},0}(t)\|} \qquad \text{for some real number } C' \\ &\le Ct^{m+1} \qquad \text{on } (0,\eta).\end{aligned} \tag{50}$$

Also, inequality (41) implies that

$$|\boldsymbol{\rho}^{0}(t)\cdot\mathbf{c}(\cdot^{0})| \le Ct^{m} \qquad \text{on } [0,\eta). \tag{51}$$

Combining equations (45)–(51) and recalling the assumption that $m > n = \mathrm{ord}(\mathrm{f}^{\mathrm{n}}(\cdot^{\mathrm{t}}))$ gives

$$\begin{aligned} t\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f}^{\mathrm{n}}(\cdot^{\mathrm{t}})\right| &\le t\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f}^{\mathrm{n}}(\cdot^{0})\right| + t\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f}^{\mathrm{n}}(\cdot^{\mathrm{t}}) - \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f}^{\mathrm{n}}(\cdot^{0})\right| \\ &\le (n + c_3 t)\,|\mathrm{f}^{\mathrm{n}}(\cdot^{0})| + t|a(\cdot^{\mathrm{t}}) - a(\cdot^{0})| + t\left|\mathbf{b}(\cdot^{\mathrm{t}})\cdot\frac{\mathbf{v}^{\mathrm{t}}(t)}{\|\mathbf{v}^{\mathrm{t}}(t)\|} - \mathbf{b}(\cdot^{0})\cdot\frac{\mathbf{v}^{\mathrm{t},0}(t)}{\|\mathbf{v}^{\mathrm{t},0}(t)\|}\right| + t|\boldsymbol{\rho}^{0}(t)\cdot\mathbf{c}(\cdot^{0})| \\ &\le (n + c_3 t)\left(|\mathrm{f}^{\mathrm{n}}(\cdot^{\mathrm{t}})| + LCt^{q}\right) + t\left(2BCt^{q} + Ct^{m+1} + Ct^{m}\right) \\ &\le (n + \alpha t)|\mathrm{f}^{\mathrm{n}}(\cdot^{\mathrm{t}})| \qquad \text{on } (0,\eta)\end{aligned}$$

for some $\eta > 0$ and $\alpha > 0$. This shows that inequality (37) holds on $(0,\eta)$, and as inequality (37) clearly holds for $t = 0$, this completes the proof. □

4. Local Solution to the Unilateral Problem

In this section, we show that there is always a local solution to the unilateral problem and that this solution is unique in the space of motions with measure acceleration. Theorem 2.1 follows immediately from these results. Our arguments correct analogous arguments presented by Charles and Ballard [6], by accounting for the fact that the motion of a particle that is starting to slide may not be analytic at the initial time. Specifically, our Propositions 4.1 and 4.2 correspond to Proposition 4.1 and Theorem 4.2 of Charles and Ballard; and the proof of our Theorem 2.1 is identical to the proof of their Corollary 4.3, so we do not repeat it.

4.1. Existence of a Local Solution

We now show that there is always a local solution to the unilateral problem (with an analytic and Lipschitz force), in which the normal reaction measure can be described as a reaction force on a time interval $(0, T_{\mathrm{as}})$ for some $T_{\mathrm{as}} > 0$, and this reaction force satisfies an inequality of the form (2). The subscript "as" emphasizes that the solution need only be *asymptotic* to the corresponding formal series solution. Since there might be an impact at time $t = 0$, we use the right velocity $\mathbf{v}_0^{+}$ as initial condition. There is a unique such right velocity for a given unilateral problem, as discussed in Remark 2.4. We formulate this version of the unilateral problem as follows.

Unilateral Reaction Force Problem. Given an instance of Problem $\mathscr{P}_{\mathrm{u}}$, with data $\mathbf{u}_0, \mathbf{v}_0^{-}, e, \mu$ and $\mathbf{F}$, let $\mathbf{v}_0^{+}$ denote the right velocity resulting from resolving any initial impact. Find a time $T_{\mathrm{as}} > 0$ and functions $\mathbf{u}_{\mathrm{as}} : [0, T_{\mathrm{as}}) \to \mathbb{R}^d$, $\mathbf{v}_{\mathrm{as}} : [0, T_{\mathrm{as}}) \to \mathbb{R}^d$ and $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}} : [0, T_{\mathrm{as}}) \to \mathbb{R}$ that satisfy:

$$\mathbf{u}_{\mathrm{as}}(0) = \mathbf{u}_0, \qquad \mathbf{v}_{\mathrm{as}}(0) = \mathbf{v}_0^{+}; \tag{52a}$$

$$\dot{\mathbf{u}}_{\mathrm{as}} = \mathbf{v}_{\mathrm{as}}, \qquad \dot{\mathrm{v}}^{\mathrm{n}}_{\mathrm{as}} = \mathrm{F}^{\mathrm{n}}(t, \mathbf{u}_{\mathrm{as}}, \mathbf{v}_{\mathrm{as}}) + \mathrm{R}^{\mathrm{n}}_{\mathrm{as}}, \qquad \dot{\mathbf{v}}^{\mathrm{t}}_{\mathrm{as}} - \mathbf{F}^{\mathrm{t}}(t, \mathbf{u}_{\mathrm{as}}, \mathbf{v}_{\mathrm{as}}) \in -\mu\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}\partial\|\mathbf{v}^{\mathrm{t}}_{\mathrm{as}}\| \qquad \text{on } [0, T_{\mathrm{as}}); \tag{52b}$$

$$\mathrm{u}^{\mathrm{n}}_{\mathrm{as}} \ge 0, \qquad \mathrm{R}^{\mathrm{n}}_{\mathrm{as}} \ge 0, \qquad \mathrm{u}^{\mathrm{n}}_{\mathrm{as}}\mathrm{R}^{\mathrm{n}}_{\mathrm{as}} = 0 \qquad \text{on } [0, T_{\mathrm{as}}). \tag{52c}$$

Proposition 4.1. *Let the force function be an analytic function of its arguments and a Lipschitz function of its position and velocity arguments at each time. Then there exists a time $T_{\mathrm{as}} > 0$, functions $\mathbf{u}_{\mathrm{as}}$ and $\mathbf{v}_{\mathrm{as}}$, and a differentiable function $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ that solve the unilateral reaction force problem (52). Furthermore, if $\mathrm{u}^{\mathrm{n}}_0 = 0$ then the time $T_{\mathrm{as}} > 0$ and real numbers $m \geq 0$ and $\beta \geq 0$ may be chosen such that*

$$\mathrm{v}^{\mathrm{n}}_{\mathrm{as}}(t) \geq 0 \qquad \textit{and} \qquad t\,|\dot{\mathrm{R}}^{\mathrm{n}}_{\mathrm{as}}(t)| \leq (m + \alpha t)\,|\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}(t)| \qquad \textit{for } t \in [0, T_{\mathrm{as}}). \tag{53}$$

Proof. Recall the Cauchy-Kovalevskaya theorem [14]: let $\mathcal{V} \subseteq \mathbb{R}^n$ be an open set, let $\mathbf{x}_0 \in \mathcal{V}$, and let $\mathbf{f} : \mathcal{V} \to \mathbb{R}^n$ be analytic on $\mathcal{V}$; then the initial value problem $\mathbf{x}(0) = \mathbf{x}_0$ and $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a unique solution $\mathbf{x} : \mathcal{I} \to \mathbb{R}^n$ that is analytic on some open interval $\mathcal{I}$ containing zero. For the problem at hand, we say the particle is in *firm contact* when the normal reaction force is positive.

We argue that there exists a solution corresponding to a particle in one of the following states:

(i) not in firm contact, in which case the conclusions follow from the Cauchy-Kovalevskaya theorem;
(ii) already sliding, in which case the conclusions also follow from the Cauchy-Kovalevskaya theorem;
(iii) stuck, in which case the conclusions follow from the fact that F^{n} is analytic; or
(iv) starting to slide, in which case the conclusions follow from Propositions 3.4 and 3.6.

The proof addresses these possibilities in numerical order, with two additional steps: Step (ii$^-$) derives some facts that must hold if there is no solution of type (i); and Step (ii$^+$) considers the case of vanishing friction coefficient.

Step (i). Consider the initial value problem obtained by setting $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}} = 0$ and ignoring inequalities (52c):

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \mathbf{v}(0) = \mathbf{v}^+_0 \qquad \text{and} \qquad \dot{\mathbf{u}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{F}(t, \mathbf{u}, \mathbf{v}) \qquad \text{on } [0, T_{\mathrm{as}}). \tag{54}$$

As $\mathbf{F}$ is analytic, the Cauchy-Kovalevskaya theorem implies that problem (54) has an analytic solution on $[0, T_{\mathrm{as}})$ for some $T_{\mathrm{as}} > 0$. If this solution satisfies $\mathrm{u}^{\mathrm{n}} \geq 0$ on $[0, T_{\mathrm{as}})$ for some $T_{\mathrm{as}} > 0$, then setting

$$(\mathbf{u}_{\mathrm{as}}, \mathbf{v}_{\mathrm{as}}, \mathrm{R}^{\mathrm{n}}_{\mathrm{as}}) = (\mathbf{u}, \mathbf{v}, 0) \tag{55}$$

solves problem (52), and the function $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ is differentiable. Furthermore, as v^{n} is analytic, for some $T_{\mathrm{as}} > 0$ either

$$\mathrm{v}^{\mathrm{n}} < 0 \text{ on } (0, T_{\mathrm{as}}) \qquad \text{or} \qquad \mathrm{v}^{\mathrm{n}} \geq 0 \text{ on } (0, T_{\mathrm{as}}).$$

But, if $\mathrm{u}^{\mathrm{n}}_0 = 0$, $\dot{\mathrm{u}}^{\mathrm{n}} = \mathrm{v}^{\mathrm{n}}$ and $\mathrm{u}^{\mathrm{n}} \geq 0$, then we cannot have $\mathrm{v}^{\mathrm{n}} < 0$ on $(0, T_{\mathrm{as}})$. Also, as v^{n} is continuous, if $\mathrm{v}^{\mathrm{n}} \geq 0$ on $(0, T_{\mathrm{as}})$ then $\mathrm{v}^{\mathrm{n}}(0) \geq 0$. Therefore, solution (55) satisfies

$$\mathrm{v}^{\mathrm{n}}_{\mathrm{as}} \geq 0 \qquad \text{on } [0, T_{\mathrm{as}}),$$

which is the first part of conclusion (53). The second part of conclusion (53) holds trivially when $\dot{\mathrm{R}}^{\mathrm{n}}_{\mathrm{as}} = \mathrm{R}^{\mathrm{n}}_{\mathrm{as}} = 0$.

Step (ii$^-$). Now say there is no $T_{\mathrm{as}} > 0$ such that the solution to problem (54) satisfies $\mathrm{u}^{\mathrm{n}} \geq 0$ on $[0, T_{\mathrm{as}})$. We work under this assumption for the rest of the proof. As u^{n} is continuous and the initial conditions of the unilateral reaction force problem satisfy $\mathrm{u}^{\mathrm{n}}_0 \geq 0$, it follows that

$$\mathrm{u}^{\mathrm{n}}_0 = 0. \tag{56a}$$

As v^{n} is analytic and $\dot{\mathrm{u}}^{\mathrm{n}} = \mathrm{v}^{\mathrm{n}}$, it follows that $\mathrm{v}^{\mathrm{n}} < 0$ on $(0, T_{\mathrm{as}})$ for some $T_{\mathrm{as}} > 0$. But v^{n} is continuous, and by definition of the unilateral reaction force problem, $\mathrm{u}^{\mathrm{n}}_0 = 0$ implies that $\mathrm{v}^{\mathrm{n}+}_0 \geq 0$. Hence,

$$\mathrm{v}^{\mathrm{n}+}_0 = 0. \tag{56b}$$

As $\mathrm{v}^\mathrm{n} < 0$ on $(0, T_\mathrm{as})$, it follows that the leading coefficient of the Taylor series of v^n at $t = 0$ satisfies $\mathrm{lcf}(\mathrm{v}^\mathrm{n}) < 0$. And as $\mathrm{v}^\mathrm{n}(0) = 0$, we must have $\mathrm{lcf}(\mathrm{F}^\mathrm{n}(t, \mathbf{u}, \mathbf{v})) < 0$. In particular, defining $n \coloneqq \mathrm{ord}(\mathrm{F}^\mathrm{n}(t, \mathbf{u}, \mathbf{v}))$, in the tangential part of the the series solution to (54), coefficients $\mathbf{u}_0^\mathrm{t}$ and $\mathbf{v}_0^\mathrm{t}$ are given by the initial conditions (52a), the next few coefficients are

$$i\mathbf{u}_i^\mathrm{t} = \mathbf{v}_{i-1}^\mathrm{t}, \qquad i\mathbf{v}_i^\mathrm{t} = [\mathbf{F}^\mathrm{t}(t, (0, \mathbf{u}_{\le i-1}^\mathrm{t}), (0, \mathbf{v}_{\le i-1}^\mathrm{t}))]_{i-1}, \qquad i = 1, 2, \ldots, n, \tag{56c}$$

and the leading coefficient of $\mathrm{F}^\mathrm{n}(t, \mathbf{u}, \mathbf{v})$ is

$$[\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}_{\le n}^\mathrm{t}), (0, \mathbf{v}_{\le n}^\mathrm{t}))]_n < 0. \tag{56d}$$

Step (ii). Say $\mathbf{v}_0^{\mathrm{t}+} \ne \mathbf{0}$. Then the subdifferential $\partial\|\mathbf{v}_\mathrm{as}^\mathrm{t}\|$ is a singleton set near $\mathbf{v}_\mathrm{as}^\mathrm{t} = \mathbf{v}_0^{\mathrm{t}+}$, whose single element $\mathbf{v}_\mathrm{as}^\mathrm{t}/\|\mathbf{v}_\mathrm{as}^\mathrm{t}\|$ is analytic at $\mathbf{v}_0^{\mathrm{t}+}$. Thus the Cauchy-Kovalevskaya theorem implies that the initial value problem

$$\mathbf{u}^\mathrm{s}(0) = \mathbf{u}_0, \qquad \mathbf{v}^\mathrm{s}(0) = \mathbf{v}_0^{\mathrm{t}+}$$

$$\dot{\mathbf{u}}^\mathrm{s} = \mathbf{v}^\mathrm{s}, \qquad \dot{\mathbf{v}}^\mathrm{s} = \mathbf{F}^\mathrm{t}(t, (0, \mathbf{u}^\mathrm{s}), (0, \mathbf{v}^\mathrm{s})) + \mu\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}^\mathrm{s}), (0, \mathbf{v}^\mathrm{s}))\frac{\mathbf{v}^\mathrm{s}}{\|\mathbf{v}^\mathrm{s}\|} \qquad \text{on } [0, T_\mathrm{as})$$

has an analytic solution $(\mathbf{u}^\mathrm{s}, \mathbf{v}^\mathrm{s})$ for some $T_\mathrm{as} > 0$. The superscript "s" indicates a particle that is already sliding. Moreover, the first few coefficients of the series solution of this initial value problem are identical to those in equation (56c) and thus we have $\mathrm{lcf}(\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}^\mathrm{s}), (0, \mathbf{v}^\mathrm{s}))) < 0$. Hence, setting

$$(\mathbf{u}_\mathrm{as}, \mathbf{v}_\mathrm{as}, \mathrm{R}_\mathrm{as}^\mathrm{n}) = ((0, \mathbf{u}^\mathrm{s}), (0, \mathbf{v}^\mathrm{s}), -\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}^\mathrm{s}), (0, \mathbf{v}^\mathrm{s})))$$

solves initial value problem (52) for some $T_\mathrm{as} > 0$. Furthermore, this solution has $\mathrm{v}_\mathrm{as}^\mathrm{n} = 0$ on $[0, T_\mathrm{as})$; and by the rules of composition of analytic functions, $\mathrm{R}_\mathrm{as}^\mathrm{n}$ is analytic on $[0, T_\mathrm{as})$; thus $\mathrm{R}_\mathrm{as}^\mathrm{n}$ is differentiable and equation (53) holds for some $T_\mathrm{as} > 0$.

Step (ii$^+$). Say the friction coefficient μ vanishes. Then the subdifferential $\mu\partial\|\mathbf{v}_\mathrm{as}^\mathrm{t}\|$ is $\{\mathbf{0}\}$. Arguing as in Step (ii), we see that the conclusions again follow from the Cauchy-Kovalevskaya theorem and equations (56).

Step (iii). Say $\mathbf{v}_0^{\mathrm{t}+} = \mathbf{0}$ and for some $T_\mathrm{as} > 0$

$$\|\mathbf{F}^\mathrm{t}(t, (0, \mathbf{u}_0^\mathrm{t}), \mathbf{0})\| \le |\mu\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}_0^\mathrm{t}), \mathbf{0})| \qquad \text{on } [0, T_\mathrm{as}).$$

Then using results (56), initial value problem (52) is seen to be satisfied by

$$(\mathbf{u}_\mathrm{as}, \mathbf{v}_\mathrm{as}, \mathrm{R}_\mathrm{as}^\mathrm{n}) = ((0, \mathbf{u}_0^\mathrm{t}), \mathbf{0}, -\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}_0^\mathrm{t}), \mathbf{0})).$$

Furthermore, this solution has $\mathrm{v}_\mathrm{as}^\mathrm{n} = 0$ on $[0, T_\mathrm{as})$; and as F^n is analytic, $\mathrm{R}_\mathrm{as}^\mathrm{n}$ is differentiable on $[0, T_\mathrm{as})$ and conclusion (53) holds for a suitably small $T_\mathrm{as} > 0$.

Step (iv). The only remaining possibility is that

$$\mu > 0, \qquad \mathbf{v}_0^{\mathrm{t}+} = \mathbf{0} \qquad \text{and} \qquad \|\mathbf{F}^\mathrm{t}(t, (0, \mathbf{u}_0^\mathrm{t}), \mathbf{0})\| > |\mu\mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}_0^\mathrm{t}), \mathbf{0})| \qquad \text{on } (0, T_\mathrm{as})$$

for some $T_\mathrm{as} > 0$. In this case, the functions

$$\mathbf{f}^\mathrm{t} : (t, \mathbf{u}^\mathrm{t}, \mathbf{v}^\mathrm{t}) \mapsto \mathbf{F}^\mathrm{t}(t, (0, \mathbf{u}^\mathrm{t}), (0, \mathbf{v}^\mathrm{t})) \qquad \text{and} \qquad \mathrm{f}^\mathrm{n} : (t, \mathbf{u}^\mathrm{t}, \mathbf{v}^\mathrm{t}) \mapsto \mathrm{F}^\mathrm{n}(t, (0, \mathbf{u}^\mathrm{t}), (0, \mathbf{v}^\mathrm{t}))$$

satisfy the hypotheses of Propositions 3.4 and 3.6. Using these propositions, there exists a solution $(\mathbf{u}^\mathrm{t}, \mathbf{v}^\mathrm{t})$ to the bilateral problem (32) on a time interval $[0, T_\mathrm{as})$, and by conclusion (36) this solution satisfies $\mu\mathrm{f}^\mathrm{n}(t, \mathbf{u}^\mathrm{t}, \mathbf{v}^\mathrm{t}) < 0$ on $(0, T_\mathrm{as})$ for some $T_\mathrm{as} > 0$. Hence setting

$$(\mathbf{u}_\mathrm{as}, \mathbf{v}_\mathrm{as}, \mathrm{R}_\mathrm{as}^\mathrm{n}) = ((0, \mathbf{u}^\mathrm{t}), (0, \mathbf{v}^\mathrm{t}), -\mathrm{f}^\mathrm{n}(t, \mathbf{u}^\mathrm{t}, \mathbf{v}^\mathrm{t})) \qquad \text{on } [0, T_\mathrm{as})$$

solves initial value problem (52). This solution satisfies conclusion (53), as $\mathrm{v}_\mathrm{as}^\mathrm{n} \ge 0$, and by conclusion (37) of Proposition 3.6. This completes the proof. ☐

4.2. Uniqueness of the Local Solution

Proposition 4.2. *Let $\mathscr{P}$ be any instance of Problem $\mathscr{P}_{\mathrm{u}}$ in which the force function is an analytic function of its arguments and a Lipschitz function of its position and velocity arguments at each time. Let $\mathbf{u}$ be any solution in* $\mathrm{MMA}([0,T],\mathbb{R}^d)$ *to instance $\mathscr{P}$; and let $\mathbf{u}_{\mathrm{as}}$ be any solution to the unilateral reaction force problem (52) for the same problem data as instance $\mathscr{P}$, that satisfies the conclusions of Proposition 4.1 on a time interval $[0,T_{\mathrm{as}})$ with $T_{\mathrm{as}}>0$. Then solutions $\mathbf{u}$ and $\mathbf{u}_{\mathrm{as}}$ are identical on time interval $[0,\eta)$ for some $\eta>0$.*

Proof. The proof is similar to proofs presented by Ballard and Basseville Theorem 4.2 from [3], and by Charles and Ballard Theorem 4.2 from [6]. The former assume the force function does not depend on the particle's velocity, and the latter extend the proof to remove that assumption, so the following discussion focuses on the latter. Whereas Charles and Ballard assume the solution $\mathbf{u}_{\mathrm{as}}$ is analytic, we can only assume that the solution satisfies the conclusions of Proposition 4.1. Rather than repeating Charles and Ballard's five-page proof, we briefly explain four changes necessary to extend their proof to such non-analytic solutions, and to correct a minor flaw. The discussion below uses our own notation: $\mathbf{v}_{\mathrm{as}}$ and $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ denote the velocity and normal reaction force associated with $\mathbf{u}_{\mathrm{as}}$ respectively, which satisfy Proposition 4.1; $\mathbf{v}^+$ is the right velocity associated with solution $\mathbf{u}$; and we emphasize that Charles and Ballard require the normal position and reaction to be *nonpositive*, whereas we require them to be *nonnegative*.

Change 1. In Step 1 of their proof, Charles and Ballard show that the difference in the velocities of the two solutions is bounded by a multiple of the integral of the normal reaction $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$: for some $C>0$ and $\eta>0$,

$$\|\mathbf{v}^+(t)-\mathbf{v}_{\mathrm{as}}(t)\| \le C\int_0^t \mathrm{R}^{\mathrm{n}}_{\mathrm{as}} \qquad \text{for } t\in[0,\eta). \tag{57}$$

Early in the demonstration of this inequality, it is argued that "$\dot{\mathbf{U}}_{\mathrm{a,n}}\le 0$, by taking a smaller time interval $[0,\eta)$ if necessary" from which the conclusion that $\int_{(0,t]}\mathrm{R}^{\mathrm{n}}\mathrm{v}^{\mathrm{n}}_{\mathrm{as}}\ge 0$ is drawn. We adapt this statement as follows. Either the initial condition has $\mathrm{u}^{\mathrm{n}}(0)>0$, in which case continuity implies that $\mathrm{u}^{\mathrm{n}}(t)>0$ on $[0,\eta)$ for some $\eta>0$ and hence $\mathrm{R}^{\mathrm{n}}=0$ on that interval; or $\mathrm{u}^{\mathrm{n}}(0)=0$, in which case Proposition 4.1 gives $\mathrm{v}^{\mathrm{n}}_{\mathrm{as}}\ge 0$ on $[0,\eta)$ for some $\eta>0$.

Change 2. In Step 2, it is argued that either $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ vanishes identically on $[0,\eta)$, in which case the theorem follows immediately from equation (57); or $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ does not vanish identically, in which case $\mathrm{u}^{\mathrm{n}}(t)=0=\mathrm{u}^{\mathrm{n}}_{\mathrm{as}}(t)$ on $[0,\eta)$. The demonstration of the latter fact involves an integration by parts, resulting in an integrand that may be bounded in terms of the modulus of the function

$$h(s,\mathbf{u},\mathbf{v}^+,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}) \coloneqq \frac{\partial \mathrm{F}^{\mathrm{n}}}{\partial s}(s,\mathbf{u},\mathbf{v}^+) + \mathbf{v}^+\cdot\frac{\partial \mathrm{F}^{\mathrm{n}}}{\partial \mathbf{u}}(s,\mathbf{u},\mathbf{v}^+) - \frac{\partial \mathrm{F}^{\mathrm{n}}}{\partial s}(s,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}) - \mathbf{v}_{\mathrm{as}}\cdot\frac{\partial \mathrm{F}^{\mathrm{n}}}{\partial \mathbf{u}_{\mathrm{as}}}(s,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}).$$

Charles and Ballard correctly claim that there exists a real number c such that

$$|h(s,\mathbf{u},\mathbf{v}^+,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}})| \le c\|\mathbf{u}-\mathbf{u}_{\mathrm{as}}\| + c\|\mathbf{v}^+-\mathbf{v}_{\mathrm{as}}\| \qquad \text{on } [0,\eta). \tag{58}$$

Here we provide a clearer[2] justification of that claim. Recall Hadamard's lemma [20]: any smooth real-valued function f defined on an open star-convex neighbourhood of a point $\mathbf{a}$ of $\mathbb{R}^n$ can be expressed in the form

$$f(\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x}-\mathbf{a})\cdot\mathbf{g}(\mathbf{x}),$$

where each component of $\mathbf{g}$ is a smooth function defined on that neighbourhood. The hypothesis that F^{n} is analytic implies that h is smooth. Also, the definition of h gives $h(\mathbf{a})=0$ at the point $\mathbf{a}=(s,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}},\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}})$. Thus Hadamard's lemma gives

$$h(s,\mathbf{u},\mathbf{v}^+,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}) = (\mathbf{u}-\mathbf{u}_{\mathrm{as}})\cdot\mathbf{g}_1(s,\mathbf{u},\mathbf{v}^+,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}) + (\mathbf{v}^+-\mathbf{v}_{\mathrm{as}})\cdot\mathbf{g}_2(s,\mathbf{u},\mathbf{v}^+,\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}})$$

[2] Charles and Ballard page 19 from [6] argue that $h(s,\mathbf{u},\mathbf{v},\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}) = (\mathbf{u}-\mathbf{u}_{\mathrm{as}})\cdot\mathbf{G}_1(s,\mathbf{u}-\mathbf{u}_{\mathrm{as}},\mathbf{v}-\mathbf{v}_{\mathrm{as}}) + (\mathbf{v}-\mathbf{v}_{\mathrm{as}})\cdot\mathbf{G}_2(s,\mathbf{u}-\mathbf{u}_{\mathrm{as}},\mathbf{v}-\mathbf{v}_{\mathrm{as}})$ for some functions $\mathbf{G}_1$ and $\mathbf{G}_2$. But this is false: for instance, if $\mathrm{F}^{\mathrm{n}}(s,\mathbf{u},\mathbf{v}) = s\|\mathbf{v}\|^2$ then $h(s,\mathbf{u},\mathbf{v},\mathbf{u}_{\mathrm{as}},\mathbf{v}_{\mathrm{as}}) = (\mathbf{v}-\mathbf{v}_{\mathrm{as}})\cdot(\mathbf{v}+\mathbf{v}_{\mathrm{as}})$.

for some smooth functions $\mathbf{g}_1$ and $\mathbf{g}_2$. As the solutions are in MMA, they have bounded variation, so there exists a real number B such that

$$\|\mathbf{u}-\mathbf{u}_0\| \le B, \qquad \|\mathbf{v}^+ - \mathbf{v}_0^+\| \le B, \qquad \|\mathbf{u}_{\mathrm{as}}-\mathbf{u}_0\| \le B, \qquad \|\mathbf{v}_{\mathrm{as}} - \mathbf{v}_0^+\| \le B, \qquad \text{on } [0,\eta).$$

As functions $\mathbf{g}_1$ and $\mathbf{g}_2$ are continuous, they are bounded for arguments ranging over these compact sets. Therefore, inequality (58) holds.

Change 3. Also in Step 2, it is claimed that if the solution satisfies $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}} > 0$ on $(0, T_{\mathrm{as}})$ and C_5 is a nonnegative real number, then there exists an $\eta > 0$ such that

$$\mathrm{R}^{\mathrm{n}}_{\mathrm{as}} - C_5 \int_0^t \mathrm{R}^{\mathrm{n}}_{\mathrm{as}} > 0 \qquad \text{on } (0,\eta). \tag{59}$$

This claim also holds if function $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ is not necessarily analytic, but is only known to satisfy the conclusions of Proposition 4.1. Indeed by those conclusions, $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ is differentiable, thus it is continuous on $[0, T_{\mathrm{as}})$. As $\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}$ is positive and continuous on $(0, T_{\mathrm{as}})$, the function $g(t) \coloneqq \int_0^t \mathrm{R}^{\mathrm{n}}_{\mathrm{as}}/\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}(t)$ is also positive and continuous on $(0, T_{\mathrm{as}})$; and it follows that

$$0 \le \lim_{t\to 0^+} g(t) \le \lim_{t\to 0^+} t \sup_{0\le s<t} \frac{\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}(s)}{\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}(t)} = 0.$$

But for any continuous function g with $\lim_{t\to 0^+} g(t) = 0$ and $g(t) > 0$ on $(0, T_{\mathrm{as}})$, there is an $\eta > 0$ such that $g(t) < 1/C_5$ on $(0,\eta)$: this is equivalent to inequality (59).

Change 4. Also in Step 2, the claim that there exist real numbers m and α such that $t\,|\dot{\mathrm{R}}^{\mathrm{n}}_{\mathrm{as}}(t)| \le (m+\alpha t)\,|\mathrm{R}^{\mathrm{n}}_{\mathrm{as}}(t)|$ for $t \in [0,\eta)$ holds by Proposition 4.1.

Conclusion. The remainder of Step 2 makes no further appeal to the properties of the solutions; and Steps 3 and 4 are only relevant in situations with multiple particles. This completes the proof. □

5. Future work

It would be interesting to extend our main result to anisotropic friction, by replacing the subdifferential $\partial\|x\|$ with the subdifferential of the support function of a convex set [4,18]. A natural open question is the the extension to multiple interacting particles [6]. Although is straightforward to find formal series solutions for multi-particle problems, it is not at all clear why an inequality of form (2) would hold for the normal reaction densities, even in the case of two particles.

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