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ANALYSIS OF A POSITIVITY-PRESERVING SPLITTING SCHEME FOR SOME SEMILINEAR STOCHASTIC HEAT EQUATIONS

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Abstract. We construct a positivity-preserving Lie–Trotter splitting scheme with finite difference discretization in space for approximating the solutions to a class of semilinear stochastic heat equations with multiplicative space-time white noise. We prove that this explicit numerical scheme converges in the mean-square sense, with rate 1/4 in time and rate 1/2 in space, under appropriate CFL conditions. Numerical experiments illustrate the superiority of the proposed numerical scheme compared with standard numerical methods which do not preserve positivity.

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1. Introduction

Starting with the seminal work [40] on an implicit scheme for stochastic quasi-linear parabolic partial differential equations in 1995, the field of numerical analysis of stochastic partial differential equations (SPDEs) has gained a huge interest during the last decades. We refer the interested readers to [27,28,48,58,81] for references on the theory of SPDEs and to [2,4–6,18,24,29–31,35,38,39,44–47,49,50,52,54–58,63,66–68,70,71,75,80,82–87] for references on the numerical analysis of SPDEs (with a particular focus on works related to strong convergence for parabolic SPDEs).

In this work we propose and study a novel positivity-preserving numerical scheme for a fully discrete approximation of the following semilinear Stochastic Heat Equation (SHE) with multiplicative space-time white noise

$$\begin{cases} \partial_t u(t,x) = \partial_{xx}^2 u(t,x) + g(u(t,x))\dot{W}(t,x), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

for $(t,x) \in [0,T] \times [0,1]$ and where $u_0 \geq 0$ is continuous, $g\colon \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous, of class $\mathcal{C}^1$ and satisfies $g(0) = 0$, and $\dot{W}$ is a space-time white noise, see Section 2 for precise definitions and assumptions. Taking $g(x) = x$ in equation (1) results in the celebrated parabolic Anderson model, see for instance [19]. This

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equation is used to model (particle) branching processes, hydrodynamics with random forcing, and serves as a model for turbulent diffusions.

The positivity-preserving property of the exact solutions to the SPDE (1) is the subject of extensive research: two of the first results in this direction can be found in [65, 76], where this property is proven to be true for noise of the form $u^\gamma \dot{W}$ (where $1 \leq \gamma < 3/2$) and for Lipschitz continuous drift and diffusion coefficients, respectively. The case of a Lipschitz continuous diffusion g is studied in, for example, [28, 64, 72]. For the sake of completeness, we mention the paper [8] on the positivity of the SHE with random initial conditions, the paper [78] on problems with spatially homogeneous Wiener process, the paper [21] on the stochastic fractional heat equation, the paper [20] on problems in $\mathbb{R}^n$, as well as the paper [25] on systems of SHEs with a spatially correlated noise. Note that these references are considering the space domain to be $\mathbb{R}$ or $\mathbb{R}^n$. To the best of our current knowledge, there are no corresponding results for the case of compact domains with homogeneous Dirichlet boundary conditions.

While standard time integrators for SPDEs, such as the Euler–Maruyama scheme [29], the semi-implicit Euler–Maruyama scheme [38], and the stochastic exponential Euler integrator [56] do converge when applied to the problem (1), they do not preserve the positivity property of the exact solution. Note that the semi-implicit Euler scheme and the exponential Euler integrator preserve positivity in the deterministic case ($g \equiv 0$ in Eq. (1)).

In this work, we employ a splitting strategy for the time integration of the SPDE (1). This results in an efficient and positivity-preserving explicit time integrator. In essence, a splitting integrator decomposes the vector field of the original evolution equation in several parts, such that the arising subsystems are exactly integrated (or easily). Splitting schemes have been extensively studied and successfully applied to deterministic differential equations, see for instance [10, 41, 62] and references therein. Splitting schemes are also very popular for an efficient time discretization of stochastic (partial) differential equations. We refer the reader to the following non-exhaustive list of articles: [3, 7, 9, 12–15, 17, 23, 26, 32, 36, 53, 60, 61, 69].

The preservation of positivity by numerical methods have been investigated in several references in both the deterministic and stochastic settings. Without being exhaustive, we mention the following articles on positivity-preserving schemes for stochastic differential equations: [1, 42, 43, 51, 59, 73, 74, 77]. Finally, let us mention the recent reference [88] on a positivity-preserving numerical scheme for the linear stochastic heat equation with finite dimensional noise. We are not aware of works on the numerical analysis of positivity-preserving schemes for SPDEs driven by space-time white noise.

The fully-discrete Lie–Trotter splitting scheme, see equation (15), considered in this article combines a finite difference approximation in space and the explicit recursion

$$u_{m+1}^{\mathrm{LT}} = \exp\big(\tau N^2 D^N\big) \hat{u}_{m+1}^{\mathrm{LT}},$$

where for $n = 1, \dots, N-1$ one has

$$\hat{u}_{m+1,n}^{\mathrm{LT}} = \exp\left(\sqrt{N} f\big(u_{m,n}^{\mathrm{LT}}\big)\Delta_{m,n} W - \frac{N f\big(u_{m,n}^{\mathrm{LT}}\big)^2 \tau}{2}\right) u_{m,n}^{\mathrm{LT}},$$

where $\tau = T/M > 0$ denotes the time-step size, $h = 1/N$ is the mesh size, $\Delta_{m,n} W$ denote space-time Wiener increments, $N^2 D^N$ the $(N-1) \times (N-1)$ matrix of the discrete Laplace operator, and $g(v) = v f(v)$. Observe that the linear diffusion part of (1) is solved exactly, while the noise part is solved exactly in the case of the parabolic Anderson model (where one has $g(v) = v$ and $f(v) = 1$ and thus the second subsystem is a geometric Brownian motion). This shares similarities with the works [33, 79] on stochastic differential equations. For a general mapping g, we freeze the factor f at the previous time point and obtain a geometric Brownian motion in the spirit of the exponential scheme proposed in [11] for finite dimensional problems.

The main results of the paper are the following:

– We obtain a fully discrete explicit approximation of the stochastic heat equation (1) that is positivity-preserving, see Proposition 4.

– We show bounds for the second moment of the numerical approximation under a CFL condition $\tau/h = \mathrm{O}(1)$ in Proposition 5.
– We prove the strong convergence, with rate 1/4, for the temporal discretization under a CFL condition $\tau/h^2 = \mathrm{O}(1)$, see Theorem 6. The strong convergence of the fully discrete scheme is provided in Corollary 7.

We leave the study of weak convergence of the proposed scheme to possible future works. On top of that, we show positivity of the exact solution to the SPDE (1) on compact domains. This follows naturally from the numerical analysis of the proposed approximation, see Proposition 2. Let us mention that the CFL conditions above are not due to the discretization of the Laplace operator, since the linear part is solved exactly. They are due to the discretization of the contribution of the space-time white noise in the temporal evolution. Numerical experiments, see Figure 2, confirm that the CFL condition is necessary when studying the mean-square convergence of the proposed scheme.

In the recent work [16], we have considered a variant of the SHE (1) and of the scheme above in the case of a purely temporal white noise in arbitrary spatial dimension, instead of space-time white noise in spatial dimension 1. It is worth mentioning that for this type of stochastic perturbation, CFL conditions are not needed for the proposed positivity-preserving scheme. Extending the results presented in this paper to more general spatially colored noise and in arbitrary dimension would require new techniques in the construction and in the analysis of the scheme. This is left for future works.

This paper is organized as follows. Section 2 presents the setting, assumptions, and useful results on the considered SHE. We also recall results on the finite difference discretization from [37]. Section 3 contains the definition of the proposed Lie–Trotter splitting as well as the main results of the paper. We postpone their proofs to Section 5. We dedicate Section 4 to numerical experiments illustrating our qualitative and quantitative results on the proposed splitting scheme. The last Section 6 briefly presents an extension to systems of semilinear stochastic heat equations. Appendices A and B contain proofs of auxiliary inequalities used in the proofs of the main results.

2. Setting

This section provides the necessary setting for the description of the considered class of semilinear stochastic heat equations as well as of its solution. We recall the notion of a mild solution and a standard well-posedness result for completeness. In addition, we recall the spatial discretization by finite differences from [37].

For any real-valued continuous function $v\colon [0,1] \to \mathbb{R}$, let $\|v\|_\infty = \max_{x\in[0,1]} |v(x)|$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions. The expectation operator is denoted by $\mathbb{E}[\cdot]$. In the sequel, C denotes a generic constant that may vary from line to line. We sometimes use subscripts on C to indicate dependence on parameters.

2.1. Description of the SPDE

Let us first introduce the main assumptions needed for the numerical analysis of the proposed time integrator for the stochastic heat equation.

Assumption 1. *The initial value* $u_0\colon [0,1] \to \mathbb{R}$ *is a function of class* $\mathcal{C}^3$*, and satisfies the conditions* $u_0(0) = u_0(1) = 0$.

Note that the regularity assumption on the initial value above is for ease of presentation. For weaker conditions, see [37] or [2].

When discussing positivity-preserving properties, a further condition is needed.

Assumption 2. *The initial value* $u_0\colon [0,1] \to \mathbb{R}$ *satisfies* $u_0(x) \geq 0$ *for all* $x \in [0,1]$.

For the nonlinearity in the considered SPDE, we make use of the following.

Assumption 3. *The mapping $g\colon \mathbb{R}\to\mathbb{R}$ is of class $\mathcal{C}^1$, is globally Lipschitz continuous, and satisfies $g(0)=0$.*

We denote by $\mathrm{L_g}$ the Lipschitz constant of g:

$$\mathrm{L_g} = \sup_{v_1,v_2\in\mathbb{R},v_2\neq v_1} \frac{|g(v_2)-g(v_1)|}{|v_2-v_1|}.$$

The moment bounds and the error estimates presented below depend on the value of the Lipschitz constant $\mathrm{L_g}$. This is not indicated in order to simplify the notation.

We then introduce the auxiliary mapping $f\colon \mathbb{R}\to\mathbb{R}$ defined for all $v\in\mathbb{R}\setminus\{0\}$ by

$$f(v) = \frac{g(v)}{v} = \int_0^1 g'(rv)\,\mathrm{d}r \tag{2}$$

and by $f(0)=g'(0)$. Since g' is continuous by Assumption 3, the mapping f is continuous and bounded, and one has the upper bound $\sup_{v\in\mathbb{R}} |f(v)| \le \mathrm{L_g}$.

For a fixed time horizon $T>0$, let $W=\{W(t,x):\ t\in[0,T], x\in[0,1]\}$ be an $\mathcal{F}_t$-adapted Brownian sheet. We recall that a Brownian sheet is a Gaussian random field with mean zero and covariance $\mathbb{E}[W(t,x)W(s,y)] = (t\wedge s)(x\wedge y)$ for all $s,t\in[0,T]$ and $x,y\in[0,1]$, see for instance [81]. We consider the stochastic heat equation in the Itô sense

$$\begin{cases} \mathrm{d}u(t,x) = \partial_{xx}^2 u(t,x)\,\mathrm{d}t + g(u(t,x))\,\mathrm{d}W(t,x),\\ u(t,0)=u(t,1)=0,\\ u(0,x)=u_0(x) \end{cases} \tag{3}$$

for $t\in[0,T]$ and $x\in[0,1]$, where u_0 and g satisfy Assumptions 1 and 3, respectively.

In order to define a mild solution of the stochastic heat equation (3), we introduce the heat kernel

$$G(t,x,y) = 2\sum_{j=1}^{\infty} e^{-j^2\pi^2 t}\sin(j\pi x)\sin(j\pi y),$$

for $t\ge 0, x,y\in[0,1]$, which is the fundamental solution of the (deterministic) heat equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \mathrm{d}v(t,x) = \partial_{xx}^2 v(t,x)\,\mathrm{d}t,\\ v(t,0)=v(t,1)=0,\\ v(0,x)=\delta(x), \end{cases}$$

where the initial value is the Dirac delta function.

A mild solution to the SPDE (3) is a random field $(u(t,x))_{t\in[0,T],x\in[0,1]}$ satisfying the following integral equation almost surely: for all $t\in[0,T]$ and $x\in[0,1]$, one has

$$u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t\int_0^1 G(t-s,x,y)g(u(s,y))\,\mathrm{d}W(s,y). \tag{4}$$

The stochastic integral in (4) is understood in the Itô–Walsh sense, see for instance [28, 48, 81].

We collect some properties of the mild solution $u(t,x)$ to the stochastic heat equation (3) in the following statement, see for instance Proposition 3.7 in [37].

Proposition 1. *Consider the stochastic heat equation* (3) *under Assumptions 1 and 3. There exists a unique mild solution $(u(t,x))_{t\in[0,T],x\in[0,1]}$ to the SPDE* (3)*. In addition, for all $T\in(0,\infty)$, there exists $C_T\in(0,\infty)$ such that*

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]} \mathbb{E}\big[|u(t,x)|^2\big] \le C_T \|u_0\|_\infty^2.$$

Finally, the solution satisfies the following mean-square regularity property: for all $T \in (0,\infty)$*, there exists* $C_T(u_0) \in (0,\infty)$ *such that for all* $x_1, x_2 \in [0,1]$ *and all* $t_1, t_2 \in [0,T]$ *one has*

$$\Big(\mathbb{E}\big[|u(t_2,x_2) - u(t_1,x_1)|^2\big]\Big)^{\frac{1}{2}} \le C_T(u_0)\Big(|t_2 - t_1|^{\frac{1}{4}} + |x_2 - x_1|^{\frac{1}{2}}\Big). \tag{5}$$

In this article, our objective is to propose and analyze consistent numerical schemes which preserve the following property of the exact solution: if the initial value u_0 is nonnegative, then the exact solution to the stochastic heat equation, $u(t,\cdot)$, remains nonnegative for all $t > 0$.

Proposition 2. *Consider the stochastic heat equation* (3) *together with Assumptions 1–3. Then, for all* $t \in (0,\infty)$ *and all* $x \in [0,1]$*, almost surely, one has*

$$u(t,x) \ge 0.$$

The proof of Proposition 2 above is postponed to Section 5.5. It is a consequence of the analysis of the fully-discrete numerical scheme and combines two arguments: on the one hand, the numerical scheme satisfies a variant of Proposition 2, see Proposition 4 below, on the other hand, Theorem 6 gives a strong convergence result of the numerical approximation. Note that similar results are known when considering the stochastic heat equation on the real line, see for instance the works [65, 76] and the lecture notes [72]. We are not aware of positivity-preserving results for SPDEs on bounded domains.

2.2. Spatial discretization

Let us recall the spatial discretization based on a finite difference approximation on a uniform grid from [37]. For any integer $N \in \mathbb{N}$, let $h = 1/N$ be the space mesh size, and let $x_n = nh$ for $0 \le n \le N$ be the grid points. Let $\kappa^N : [0,1] \to \{x_0, \ldots, x_N\}$, be the mapping defined by $\kappa^N(x) = x_n$ for $x \in [x_n, x_{n+1})$ if $n \in \{0, \ldots, N-1\}$, and $\kappa^N(1) = \kappa^N(x_N) = x_N = 1$.

Throughout this article, we use the convention that for any vector $v = (v_n)_{1 \le n \le N-1} \in \mathbb{R}^{N-1}$, we append discrete homogeneous Dirichlet boundary conditions $v_0 = 0$ and $v_N = 0$ when needed.

We discretize the initial value u_0 of the stochastic heat equation (3) by $u_{0,n}^N = u_n^N(0) = u(0,x_n)$ for $0 \le n \le N$. Note that discrete homogeneous Dirichlet boundary conditions $u_{0,0}^N = u_{0,N}^N = 0$ are satisfied owing to Assumption 1. Let us then define a piecewise linear extension $u^N(0,\cdot) : [0,1] \to \mathbb{R}$ satisfying $u^N(0,x_n) = u_{0,n}^N$ for all $n = 0, \ldots, N$, meaning that for $x \in (0,1)$ one has

$$u^N(0,x) = N\big(\kappa^N(x) + h - x\big)u\big(0,\kappa^N(x)\big) + N\big(x - \kappa^N(x)\big)u\big(0,\kappa^N(x) + h\big).$$

Let $D^N = (D_{ij}^N)_{1 \le i,j \le N-1}$ denote the matrix coming from a standard finite difference discretization of the Laplace operator at the grid points x_n with homogeneous Dirichlet boundary conditions. The matrix D^N is thus given by

$$D^N = \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \ddots & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{pmatrix}.$$

We then introduce the discrete heat kernel $G^N(t) = (G_{ij}^N(t))_{1 \le i,j \le N-1} = e^{tN^2 D^N}$, for $t \ge 0$. Observe that $G^N(0)$ is the identity matrix of size $(N-1) \times (N-1)$. By convention, set $G_{00}^N(t) = G_{NN}^N(t) = 1$, $G_{0N}^N(t) = G_{N0}^N(t) = 0$

and $G^N_{0j}(t) = G^N_{Nj}(t) = 0$ for all $j \in \{1, \dots, N-1\}$, in order to satisfy homogeneous discrete Dirichlet boundary conditions. Finally, we extend the definition of $G^N(t) = (G^N_{ij}(t))_{1\le i,j\le N-1}$ to $(G^N(t,x,y))_{t\ge 0, x,y\in[0,1]}$ by asking that $G^N(t, x_i, y_j) = NG^N_{ij}(t)$ for $0 \le i, j \le N$ and for $t \ge 0$ and $y \in [0,1]$

$$G^N(t,x,y) = N\big(\kappa^N(x) + h - x\big)G^N\big(t, \kappa^N(x), \kappa^N(y)\big) + N\big(x - \kappa^N(x)\big)G^N\big(t, \kappa^N(x) + h, \kappa^N(y)\big)$$

for $x \in (0,1)$ and $G^N(t,0,y) = G^N(t,1,y) = 0$. As a result, the mapping $(x,y) \mapsto G^N(t,x,y)$ is piecewise linear in x and piecewise constant in y at all times t.

It is worth recalling the following well-known property of the discrete heat kernel: one has $G^N_{ij}(t) \ge 0$ for all $i, j \in \{1, \dots, N-1\}$ and $t \ge 0$. As a consequence, one has $G^N(t,x,y) \ge 0$ for $x, y \in [0,1]$ and $t \ge 0$. This property follows from the fact that tN^2D^N is a Metzler matrix, see for instance [34] for a definition, and the exponential of a Metzler matrix has only non-negative elements.

We are now in position to define the spatial discretization u^N, for $N \in \mathbb{N}$, by the following integral equality

$$u^N(t,x) = \int_0^1 G^N(t,x,y)u^N\big(0, \kappa^N(y)\big)\,\mathrm{d}y + \int_0^t \int_0^1 G^N(t-s,x,y)g\big(u^N\big(s, \kappa^N(y)\big)\big)\,\mathrm{d}W(s,y) \tag{6}$$

for $t \ge 0$ and $x \in [0,1]$. Note that the mapping $x \in [0,1] \mapsto u^N(t,x)$ is linear on $[x_n, x_{n+1}]$, for every $n \in \{0, \dots, N-1\}$, for every $t \ge 0$. In addition, one has $u^N(t,0) = u^N(t,1) = 0$ for every $t \ge 0$. Observe that it is sufficient to compute $u^N_n(t) = u^N(t, x_n)$ for all $1 \le n \le N-1$. This is performed as follows: for all $t \ge 0$ and $1 \le n \le N-1$ one has

$$u^N_n(t) = \sum_{j=1}^{N-1} G^N_{nj}(t)u^N_{0,j} + \sqrt{N}\sum_{j=1}^{N-1}\int_0^t G^N_{nj}(t-s)g\big(u^N_j(s)\big)\,\mathrm{d}W^N_j(s),$$

where

$$W^N_n(t) = \sqrt{N}(W(t, x_{n+1}) - W(t, x_n)).$$

By definition of a Wiener sheet, observe that the processes $(W^N_1(t))_{t\ge 0}, \dots, (W^N_{N-1}(t))_{t\ge 0}$ are independent standard real-valued Wiener processes, for any $N \in \mathbb{N}$.

Introduce the $\mathbb{R}^{N-1}$-valued process u^N defined by $u^N(t) = (u^N_n(t))_{1\le n\le N-1}$ for all $t \ge 0$. This process is the solution of the following stochastic differential equation

$$\mathrm{d}u^N(t) = N^2D^Nu^N(t)\,\mathrm{d}t + \sqrt{N}g\big(u^N(t)\big)\,\mathrm{d}W^N(t) \tag{7}$$

with initial value $u^N(0) = (u^N_0)_{1\le n\le N-1}$, where the notation $(g(u^N(t))\,\mathrm{d}W^N(t))_n = g(u^N_n(t))\,\mathrm{d}W^N_n(t)$ is used.

Let us recall the following convergence result for the spatial discretization, see Theorem 3.1 in [37].

Proposition 3. *Consider the stochastic heat equation* (3) *with a nonlinearity g satisfying Assumption 3. Denote by $(u(t,x))_{t\in[0,T],x\in[0,1]}$ its exact solution and by $(u^N(t,x))_{t\in[0,T],x\in[0,1]}$ the numerical approximation by finite differences with mesh size $h = 1/N$. For all $T \in (0,\infty)$ and any initial value u_0 satisfying Assumption 1, there exists $C_T(u_0) \in (0,\infty)$ such that for all $h = 1/N$ with $N \in \mathbb{N}$ one has*

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}\Big(\mathbb{E}\Big[\big|u^N(t,x) - u(t,x)\big|^2\Big]\Big)^{\frac12} \le C_T(u_0)h^{\frac12}. \tag{8}$$

In the error analysis below, the following auxiliary result from Proposition 2.4 in [2] on the temporal regularity of u^N is used: there exists $C_T(u_0) \in (0,\infty)$ such that for all $t, s \in [0,T]$, one has

$$\sup_{N\in\mathbb{N}}\sup_{x\in[0,1]}\mathbb{E}\Big[\big|u^N(t,x) - u^N(s,x)\big|^2\Big] \le C_T(u_0)|t-s|^{\frac12}. \tag{9a}$$

In order to state one of our results below, a variant of (9a) is required: there exists $C_T(u_0) \in (0,\infty)$ such that for all $t, s \in [0,T]$, one has

$$\sup_{x\in[0,1]} \mathbb{E}\Big[\big|u^N(t,x) - u^N(s,x)\big|^2\Big] \le C_T(u_0)\frac{|t-s|}{h}. \tag{9b}$$

The proof of the inequality (9b) is given in Appendix B. Note that the right-hand side of (9b) depends on $h = 1/N$.

Finally, let us recall moment bounds for the solution u^N of (6), see for instance Proposition 3.5 in [37]: there exists a constant $C_T(u_0) \in (0,\infty)$ such that one has

$$\sup_{N\ge 1}\sup_{t\in[0,T]}\sup_{x\in[0,1]} \mathbb{E}\Big[\big|u^N(t,x)\big|^2\Big] \le C_T(u_0). \tag{10}$$

3. The positivity-preserving splitting scheme

In the core part of this paper, we present and study the strong convergence of an efficient and positivity-preserving time integrator for the stochastic heat equation (3).

Let $T \in (0,\infty)$ and divide the interval $[0,T]$ into $M \in \mathbb{N}$ subintervals $[t_m, t_{m+1}]$ of length $\tau = T/M$, where $t_m = m\tau$ for $m \in \{0,\dots,M\}$. Introduce the mapping $\ell^M \colon [0,T] \to \{t_0,\dots,t_M\}$, defined by $\ell^M(t) = t_m$ for all $t \in [t_m, t_{m+1})$, if $m \in \{0,\dots,M-1\}$, and $\ell^M(T) = \ell^M(t_M) = t_M = T$.

We propose a fully-discrete explicit scheme based on a Lie–Trotter splitting strategy producing approximations $u_m^{\mathrm{LT}} = (u_{m,n}^{\mathrm{LT}})_{1\le n\le N-1}$ of the finite difference approximation $u^N(t_m) = (u_n^N(t_m))_{1\le n\le N-1}$ at the grid times t_m, $m = 0,\dots,M$. We set the initial value to be $u_{0,n}^{\mathrm{LT}} = u_n^N(0) = u_{0,n}^N$ for all $1 \le n \le N-1$. As above, one has $u_{m,0}^{\mathrm{LT}} = 0$ and $u_{m,N}^{\mathrm{LT}} = 0$ for all $m \in \{0,\dots,M\}$. In this way, homogeneous Dirichlet boundary conditions are satisfied by the numerical scheme at all times.

We explain the construction of the scheme in Section 3.1. We then describe the main results of this article: the positivity-preserving property of the splitting scheme (Prop. 4) and the mean-square convergence in time with order 1/4 (Thm. 6 and Cor. 7).

3.1. Description of the time integrator

Let us describe how the splitting scheme is constructed. Given the numerical solution $u_m^{\mathrm{LT}} = (u_{m,n}^{\mathrm{LT}})_{1\le n\le N-1}$ at grid time $t_m = m\tau$ for $0 \le m \le M-1$, the solution u_{m+1}^{LT} at the next grid time $t_{m+1} = t_m + \tau$ is constructed by successively solving two subsystems in $\mathbb{R}^{N-1}$:

– first, the linear Itô SDE system

$$\mathrm{d}v_{m,n}^{M,N,1}(t) = \sqrt{N} v_{m,n}^{M,N,1}(t) f\big(u_{m,n}^{\mathrm{LT}}\big)\,\mathrm{d}W_n^N(t), \tag{11}$$

for $n \in \{1,\dots,N-1\}$ and $t \in [t_m, t_{m+1}]$, with initial value $v_{m,n}^{M,N,1}(t_m) = u_{m,n}^{\mathrm{LT}}$, where we recall (see Eq. (2) in Sect. 2) that the auxiliary function f is such that $g(v) = vf(v)$ for all $v \in \mathbb{R}$;

– second, the linear ODE system

$$\mathrm{d}v_m^{M,N,2}(t) = N^2 D^N v_m^{M,N,2}(t)\,\mathrm{d}t, \tag{12}$$

for $t \in [t_m, t_{m+1}]$, with initial value $v_{m,n}^{M,N,2}(t_m) = v_{m,n}^{M,N,1}(t_{m+1})$.

Observe that the solutions of the two subsystems above are known: the solution of the SDE (11) is given by

$$v_{m,n}^{M,N,1}(t) = \exp\left(\sqrt{N} f\big(u_{m,n}^{\mathrm{LT}}\big)\big(W_n^N(t) - W_n^N(t_m)\big) - \frac{N f\big(u_{m,n}^{\mathrm{LT}}\big)^2 (t - t_m)}{2}\right) u_{m,n}^{\mathrm{LT}}, \tag{13}$$

for all $t \in [t_m, t_{m+1}]$, and the solution of the ODE (12) is given by

$$v_m^{M,N,2}(t) = e^{(t-t_m)N^2 D^N} v_m^{M,N,1}(t_{m+1}), \tag{14}$$

for all $t \in [t_m, t_{m+1}]$.

Gathering the expressions above gives the following expression for the proposed Lie–Trotter splitting scheme

$$u_{m+1}^{\mathrm{LT}} = e^{\tau N^2 D^N} \left(\exp\left(\sqrt{N} f\big(u_{m,n}^{\mathrm{LT}}\big) \Delta_{m,n} W - \frac{N f\big(u_{m,n}^{\mathrm{LT}}\big)^2 \tau}{2} \right) u_{m,n}^{\mathrm{LT}} \right)_{1 \le n \le N-1}, \tag{15}$$

where $\Delta_{m,n} W = W_n^N(t_{m+1}) - W_n^N(t_m)$. Observe that the random variables $(\Delta W_{m,n})_{0 \le m \le M-1, 1 \le n \le N-1}$ are independent standard real-valued Gaussian random variables.

The splitting scheme formula (15) can also be written as

$$u_{m+1,n}^{\mathrm{LT}} = \sum_{k=1}^{N-1} G_{nk}^N(\tau) \exp\left(\sqrt{N} f\big(u_{m,k}^{\mathrm{LT}}\big) \Delta_{m,n} W - \frac{N f\Big(u_{m,k}^{\mathrm{LT}}\Big)^2 \tau}{2} \right) u_{m,k}^{\mathrm{LT}},$$

for all $m \in \{0, \ldots, M-1\}$ and $n \in \{1, \ldots, N-1\}$.

One of the key properties of the proposed splitting scheme is the following: if the initial value $u_0^{\mathrm{LT}} = (u_{0,n}^{\mathrm{LT}})_{1 \le n \le N-1}$ only has nonnegative elements, then for all $m \in \{1, \ldots, M\}$ the numerical solution $u_m^{\mathrm{LT}} = (u_{m,n}^{\mathrm{LT}})_{1 \le n \le N-1}$ at time $t_m = m\tau$ also only has nonnegative elements almost surely. In other words, the proposed scheme is positivity-preserving. This is stated in the next proposition.

Proposition 4. *Let $M \in \mathbb{N}$ and $N \in \mathbb{N}$ be arbitrary integers and let $T \in (0, \infty)$. Let Assumptions 1–3 be satisfied. Let the sequence $u_0^{\mathrm{LT}}, \ldots, u_M^{\mathrm{LT}}$ be given by the splitting scheme (15), with $h = 1/N$ and $\tau = T/M$, with initial value $u_{0,n}^{\mathrm{LT}} = u_0(x_n) \ge 0$ for all $n \in \{1, \ldots, N\}$. Then, almost surely, one has*

$$u_{m,n}^{\mathrm{LT}} \ge 0,$$

for all $m \in \{1, \ldots, M\}$ and $n \in \{1, \ldots, N-1\}$.

Proof of Proposition 4. The proof proceeds by recursion on the time index m.

- Note that $u_{0,n}^{\mathrm{LT}} = u(0, x_n) \ge 0$ for all $n \in \{0, \ldots, N\}$.
- Assume that the property $u_{m,n}^{\mathrm{LT}} \ge 0$, for all $n \in \{1, \ldots, N-1\}$, holds at time $t_m = m\tau$. We prove that under this assumption, it also holds at time $t_{m+1} = (m+1)\tau$.
  The argument is straightforward: the solutions of the subsystems (11) and (12) are nonnegative at all times when they have nonnegative initial values. More precisely, first one has

  $$v_{m,n}^{M,N,1}(t_{m+1}) = e^{\sqrt{N} f\left(u_{m,n}^{\mathrm{LT}}\right) \Delta_{m,n} W - N \frac{f\left(u_{m,n}^{\mathrm{LT}}\right)^2 \tau}{2}} u_{m,n}^{\mathrm{LT}} \ge 0$$

  for all $n \in \{1, \ldots, N-1\}$. Second, using the inequality $G_{nk}^N(\tau) \ge 0$ (see Sect. 2.2), one has

  $$u_{m+1,n}^{\mathrm{LT}} = v_{m,n}^{M,N,2}(t_{m+1}) = \sum_{k=1}^{N-1} G_{nk}^N(\tau) v_{m,k}^{M,N,1}(t_{m+1}) \ge 0.$$

  Thus the positivity property of the numerical solution holds at time $t_{m+1} = (m+1)\tau$.

As a consequence, the property $u_{m,n}^{\mathrm{LT}} \ge 0$, for all $n \in \{1, \ldots, N-1\}$, holds for any $m \in \{0, \ldots, M\}$. The proof of Proposition 4 is completed. □

3.2. Convergence results

Let us now prove that the proposed numerical scheme provides accurate approximation of the exact solution. In this article, we show mean-square error estimates and give orders of convergence with respect to $\tau = T/M$ and $h = 1/N$.

We impose a CFL stability condition in the sequel to ensure stability and convergence of the Lie–Trotter splitting scheme (15) when applied to the stochastic heat equation (3); more precisely, we introduce conditions of the type $\tau \le \gamma h$ or $\tau \le \gamma h^2$ in the statements below, for some (nonrandom) arbitrary parameter $\gamma \in (0,\infty)$. The conditions on τ and h above are equivalent to the conditions $\gamma M \ge TN$ and $\gamma M \ge TN^2$ on M and N respectively.

Owing to Proposition 3, it is sufficient to focus on the error $u_{m,n}^{\rm LT} - u_n^N(t_m)$ to obtain estimates for the total error $u_{m,n}^{\rm LT} - u(t_m, x_n)$. Proposition 5 shows moment bounds of the numerical solution and is used to prove our main result in Theorem 6. As a corollary we obtain convergence of $u_{m,n}^{\rm LT}$ to the exact solution $u(t_m, x_n)$ at the grid points using results from [37].

Note that Assumption 2 on the positivity of the initial value is not needed in the statements on the moment bounds and on the convergence of the scheme below.

Proposition 5. *Assume that Assumptions 1 and 3 are satisfied. Let the sequence $u_0^{\rm LT}, \ldots, u_M^{\rm LT}$ be given by the Lie–Trotter splitting scheme (15).*

For all $\gamma \in (0,\infty)$ and all $T \in (0,\infty)$, there exists $C_{\gamma,T} \in (0,\infty)$ such that for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau \le \gamma h$, one has

$$\sup_{0\le m\le M} \sup_{1\le n\le N-1} \mathbb{E}\Big[\big|u_{m,n}^{\rm LT}\big|^2\Big] \le C_{\gamma,T} \|u_0\|_\infty^2. \tag{16}$$

The proof of this proposition also provides moment bounds for a space-time continuous version $u^{\rm LT}(t,x)$, defined by equation (22) below, of the Lie–Trotter splitting scheme (15):

$$\sup_{t\in[0,T]} \sup_{x\in[0,1]} \mathbb{E}\Big[\big|u^{\rm LT}(t,x)\big|^2\Big] \le C_{\gamma,T} \|u_0\|_\infty^2.$$

We are now in position to state the main convergence result of this article. For ease of presentation, we only consider errors at space-time grid points.

Theorem 6. *Assume that Assumptions 1 and 3 are satisfied. Let the sequence $u_0^{\rm LT}, \ldots, u_M^{\rm LT}$ be given by the Lie–Trotter scheme (15), and let $(u^N(t))_{t\ge 0, 0\le n\le N}$ be given by the spatial semi-discretization scheme (7).*

For all $\gamma \in (0,\infty)$ and $T \in (0,\infty)$, there exists $C_{\gamma,T}(u_0) \in (0,\infty)$ such that for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau \le \gamma h$, one has

$$\sup_{0\le m\le M} \sup_{0\le n\le N} \Big(\mathbb{E}\Big[\big|u_{m,n}^{\rm LT} - u_n^N(t_m)\big|^2\Big]\Big)^{\frac12} \le C_{\gamma,T}(u_0)\Big(\tau^{\frac14} + \Big(\frac{\tau}{h}\Big)^{\frac12}\Big). \tag{17a}$$

Moreover, under the same assumptions as above, one has

$$\sup_{0\le m\le M} \sup_{0\le n\le N} \Big(\mathbb{E}\Big[\big|u_{m,n}^{\rm LT} - u_n^N(t_m)\big|^2\Big]\Big)^{\frac12} \le C_{\gamma,T}(u_0)\Big(\frac{\tau}{h}\Big)^{\frac12}. \tag{17b}$$

Finally, for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau \le \gamma h^2$, one has

$$\sup_{0\le m\le M} \sup_{0\le n\le N} \Big(\mathbb{E}\Big[\big|u_{m,n}^{\rm LT} - u_n^N(t_m)\big|^2\Big]\Big)^{\frac12} \le C_{\gamma,T}(u_0)\tau^{\frac14}. \tag{18}$$

Observe that the second error estimate (17b) is a variant of (17a) where the error term $\tau^{\frac{1}{4}}$ is discarded. The error term $(\frac{\tau}{h})^{\frac{1}{2}}$ converges faster to 0 when τ goes to 0, for arbitrary fixed h, than the error term $\tau^{\frac{1}{4}}$, however the latter error term is independent of h. The order of convergence $1/4$ with respect to the time-step size is natural and due to the temporal regularity properties of the exact solution, see (9a). It would be possible to prove versions of the two error estimates (17a) and (17b) for standard schemes like the semi-implicit Euler–Maruyama scheme and the stochastic exponential Euler scheme described in Section 4. Moreover, for those schemes it is also possible to obtain an upper bound of the error by $\tau^{\frac{1}{4}}$, uniformly with respect to h, in other words the error term $(\frac{\tau}{h})^{\frac{1}{2}}$ which depends on h can be discarded for those schemes. This is not the case for the proposed positivity-preserving splitting scheme (15): to retrieve the error term $\tau^{\frac{1}{4}}$ in (18) the condition $\tau \leq \gamma h^2$ needs to be imposed. Based on the numerical experiments reported in Section 4, the error estimates (17a) and (17b) seem to be optimal.

The proofs of (17a) and (17b) follow essentially from the same arguments, except that the inequalities (9a) and (25a) are used for the proof of (17a), and the inequalities (9b) and (25b) are used for the proof of (17b).

Proving (18) from the error estimate (17a) under the stronger condition $\tau \leq \gamma h^2$ is straightforward.

Combining Theorem 6 and Proposition 3, one directly obtains error estimates for the fully-discrete scheme.

Corollary 7. *Consider the setting and assumptions of Theorem 6. For all $\gamma \in (0,\infty)$ and $T \in (0,\infty)$, there exists $C_{\gamma,T}(u_0) \in (0,\infty)$ such that for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau \leq \gamma h^2$, one has*

$$\sup_{0\leq m\leq M} \sup_{0\leq n\leq N} \left(\mathbb{E}\left[\left|u^{\mathrm{LT}}_{m,n} - u(t_m, x_n)\right|^2\right]\right)^{\frac{1}{2}} \leq C_{\gamma,T}(u_0) h^{\frac{1}{2}}. \tag{19}$$

We postpone the proofs of the above results to Section 5.

4. Numerical experiments

In this section we provide numerical experiments to support and verify the above theoretical results[1]. Recall that $\tau = T/M > 0$ is the time-step size and $h = 1/N > 0$ is the space mesh size. Let us define $\Delta_m W = (\Delta_{m,1}W, \ldots, \Delta_{m,N-1}W) \in \mathbb{R}^{N-1}$ for all $m = 0, \ldots, M-1$. Here, we recall the notation $\Delta_{m,n}W = W_n^N(t_{m+1}) - W_n^N(t_m)$ from above. We compare the proposed Lie–Trotter splitting scheme (15), denoted LT below, to the following classical time integrators when applied to the spatially discretized system (7):

- the Euler–Maruyama scheme (denoted EM below), see for instance [29]

$$u^{\mathrm{EM}}_{m+1} = u^{\mathrm{EM}}_m + \tau N^2 D^N u^{\mathrm{EM}}_m + \sqrt{N} g\big(u^{\mathrm{EM}}_m\big)\Delta_m W,$$

- the semi-implicit Euler–Maruyama scheme (denoted SEM below), see for instance [38]

$$u^{\mathrm{SEM}}_{m+1} = u^{\mathrm{SEM}}_m + \tau N^2 D^N u^{\mathrm{SEM}}_{m+1} + \sqrt{N} g\big(u^{\mathrm{SEM}}_m\big)\Delta_m W,$$

- the stochastic exponential Euler integrator (denoted SEXP below), see for instance [56]

$$u^{\mathrm{SEXP}}_{m+1} = e^{\tau N^2 D^N}\Big(u^{\mathrm{SEXP}}_m + \sqrt{N} g\big(u^{\mathrm{SEXP}}_m\big)\Delta_m W\Big).$$

4.1. Preservation of the positivity

We start by illustrating the positivity-preserving property of the Lie–Trotter scheme (LT) and show the lack of positivity-preserving behavior for the Euler–Maruyama scheme (EM), the semi-implicit Euler–Maruyama scheme (SEM), and the stochastic exponential scheme (SEXP). To do this, we use the same noise samples for all time integrators when applied to the space-discretization of the SPDE (3) as described in Section 2.2 with

[1]The codes are available under `https://doi.org/10.5281/zenodo.10300733`

TABLE 1. Proportion of samples containing only positive values out of 50 simulated sample paths for the Lie–Trotter splitting scheme (LT), the stochastic exponential Euler integrator (SEXP), the semi-implicit Euler–Maruyama scheme (SEM), and Euler–Maruyama scheme (EM) for the diffusion coefficient $g(v) = 2.5v$ and several choices of discretization parameters τ and h.

(τ, h)	LT	SEXP	SEM	EM
$(10^{-3}, 10^{-2})$	50/50	0/50	0/50	0/50
$(10^{-4}, 10^{-3})$	50/50	50/50	50/50	0/50
$(10^{-5}, 10^{-3})$	50/50	50/50	50/50	0/50

the initial condition $u_0(x) = \sin(\pi x)$ and final time $T = 20$. We consider this problem with the three choices of multiplicative term given by $g(v) = \lambda v$, $g(v) = \lambda \ln(1 + v)$, and $g(v) = \lambda(v + \sin(v))$. The real-valued parameter $\lambda > 0$ is introduced to avoid the need to run numerical experiments with very long time horizons T in order to obtain negative values for the numerical schemes SEXP and SEM. Note that the logarithmic nonlinearity $g(v) = \lambda \ln(1 + v)$ does not satisfy Assumption 3. However, this function is of class $\mathcal{C}^1$ and is globally Lipschitz continuous on the interval $[0, \infty)$ and it satisfies $g(0) = 0$. Considering this nonlinearity in the proposed LT scheme is not an issue since it preserves positivity. However, the other integrators SEXP, SEM or EM do not preserve positivity and may be problematic when this nonlinearity is considered. The numerical results are presented in Tables 1 and 2, where the notation $k/50$ indicates that k out of 50 samples remain positive.

In Table 1, we let $g(v) = 2.5v$ and we consider 50 sample paths for each of the time integrators for several choices of the discretization parameters τ and h. Table 1 confirms that the LT scheme preserves positivity. This is not the case for SEXP, SEM and EM. We observe that fewer samples of SEXP and SEM contain negative values for small time-steps τ. This is expected as each of the time integrators SEXP, SEM, and even EM, converges (for every fixed h) to the exact, everywhere positive, solution of the space-discretized system of SDEs in equation (7).

In Table 2 we instead fix the discretization parameters $\tau = 10^{-5}$ and $h = 10^{-3}$ and consider different types of multiplicative terms $g(v)$. We again use 50 samples in each of the entries of Table 2. From the results of Table 2, one can observe the poor performance of the EM scheme in all cases. This table also illustrates the fact that increasing the size of the multiplicative term prevents SEM and SEXP to remain positive. It should be clear that increasing the value of λ even more, or the length of the time interval, would hinder the numerical solutions to stay positive for all time integrators except for the proposed Lie–Trotter splitting scheme.

4.2. Mean-square errors

For the next numerical experiment, we discretize the stochastic heat equation (3) with initial value $u_0(x) = \sin(\pi x)$ by a finite-difference scheme in space with mesh size $h = 2^{-8}$. The resulting system of stochastic differential equation (7) is then discretized by the time integrators LT, SEXP, and SEM. The classical EM scheme is not appropriate in this setting and numerical results are thus not presented. The following choices for the function g are considered: $g(v) = v$ and $g(v) = \frac{v}{1+v^2}$ and $g(v) = \ln(1 + v)$, for $v \geq 0$, and $g(v) = v\exp(-v^2)$. Figure 1 displays, in a loglog plot, the mean-square errors

$$\sup_{0 \leq m \leq M} \sup_{0 \leq n \leq N} \left(\mathbb{E}\left[\left| u^{\text{num}}_{m,n} - u^{\text{ref}}(t_m, x_n) \right|^2 \right] \right)^{\frac{1}{2}}$$

measured at the space-time grid points (t_m, x_n) for the time interval $[0, 0.5]$. The reference solution u^{ref} is computed using the LT splitting scheme with time-step size $\tau_{\text{ref}} = 2^{-16}$. Here, 200 samples have been used to approximate the expectations. We have checked that the Monte Carlo error is negligible to observe mean-square convergence. In this figure, one can observe a rate of convergence $1/2$ instead of $1/4$ in the mean-square error

TABLE 2. Proportion of samples containing only positive values out of 50 simulated sample paths for the Lie–Trotter splitting scheme (LT), the stochastic exponential Euler integrator (SEXP), the semi-implicit Euler–Maruyama scheme (SEM), and the Euler–Maruyama scheme (EM) for several choices of diffusion terms $g(v)$. The discretization parameters are $\tau = 10^{-5}$ and $h = 10^{-3}$.

$g(v)$	LT	SEXP	SEM	EM
$2.5\ln(1+v)$	50/50	50/50	50/50	0/50
$3.5\ln(1+v)$	50/50	50/50	50/50	0/50
$5\ln(1+v)$	50/50	47/50	26/50	0/50
$2.5v$	50/50	50/50	50/50	0/50
$3.5v$	50/50	50/50	50/50	0/50
$5v$	50/50	4/50	50/50	0/50
$2.5(v+\sin(v))$	50/50	44/50	50/50	0/50
$3.5(v+\sin(v))$	50/50	0/50	0/50	0/50
$5(v+\sin(v))$	50/50	0/50	0/50	0/50

estimates (18) for the splitting scheme in Theorem 6. This is related to the mean-square error estimates (17a) and the role of the CFL condition $\tau \leq \gamma h^2$ to obtain (18).

To illustrate this, we compute the mean-square errors of the Lie–Trotter splitting scheme when applied to the finite difference discretization of the stochastic heat equation on the time interval $[0, 0.5]$ with different values of the mesh size, namely $h = 2^{-4}, 2^{-6}, 2^{-8}, 2^{-10}$. The time-step sizes are from 2^{-4} to 2^{-16}. The reference solution u^{ref} is computed using the LT splitting scheme with time-step size $\tau_{\text{ref}} = 2^{-16}$ for each choice of the mesh size. This is presented only for the two nonlinearities $g(v) = 1.5v$ and $g(v) = 1.5\frac{v}{1+v^2}$. We have used 200 samples to approximate the expectations. The results are presented in Figure 2. In these experiments we observe upper bounds which are not uniform with respect to h, in fact we observe the contribution of the error term $\tau^{\frac{1}{2}}/h^{\frac{1}{2}}$ in the mean-square error estimates (17a) and (17b).

In order to illustrate Corollary 7 and the convergence of the Lie–Trotter splitting scheme when τ and h go to 0, we compute its mean-square errors with the time-step sizes $\tau = h^2$ with $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$. The reference solution u^{ref} is obtained using the LT splitting scheme with time-step size $\tau_{\text{ref}} = h_{\text{ref}}^2$, where $h_{\text{ref}} = 2^{-10}$. The expectation is approximated using 100 samples. The stochastic heat equation (3) is considered in the time interval $[0, 0.5]$ with $g(v) = v$. The results are presented in Figure 3. Under this coupling between the time-step sizes and the mesh sizes, one clearly observes that the error behaves likes $\tau^{\frac{1}{4}} = h^{\frac{1}{2}}$ as stated in Corollary 7.

In the final numerical experiment, we consider the same parameters as above and the function $g(v) = v^{1.25}$. Observe that this nonlinearity is not globally Lipschitz continuous and is thus not covered by the the results from Section 3.2. A convergence plot for the splitting scheme (15) is provided in Figure 4. As above, we observe a mean-square order of convergence 1/2, but which should not be uniform with respect to h, similarly to what is observed in Figure 2. To prove such rate of convergence is beyond the scope of this paper and will be the subject of a future work.

5. Proofs of the main results

The objective of this section is to provide the proofs of the results stated in Section 3.2, namely the moment bounds in Proposition 5 and the mean-square error estimates in Theorem 6 and in Corollary 7. We also prove Proposition 2, which ensures positivity of the exact solution. Preliminary auxiliary tools are given in Sections 5.1 and 5.2, before proceeding with the detailed proofs.

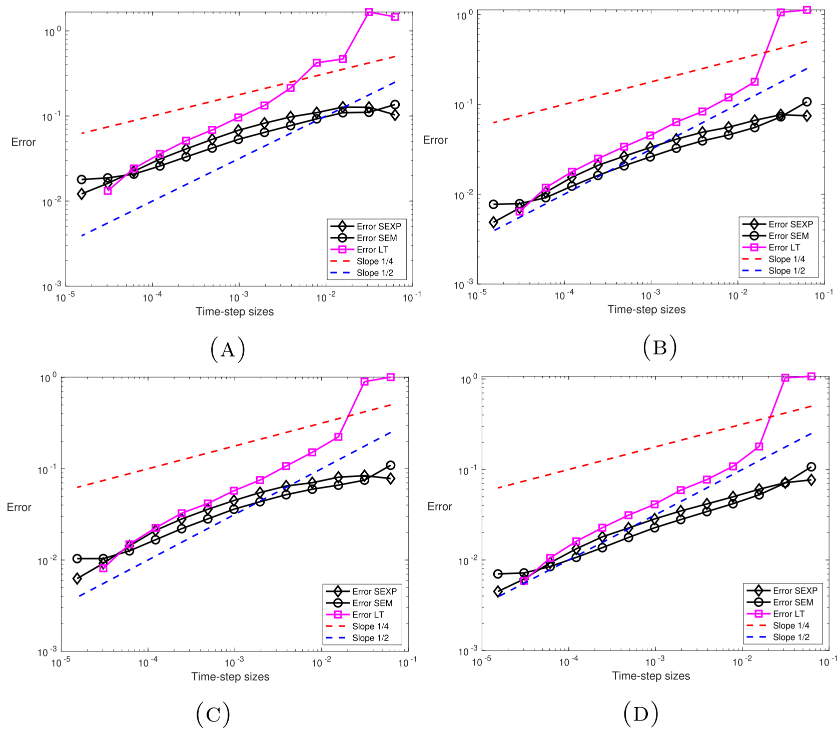

FIGURE 1. Mean-square errors on the time interval $[0, 0.5]$ of the splitting scheme (LT), the stochastic exponential Euler integrator (SEXP), and the semi-implicit Euler–Maruyama scheme (SEM). Mesh size $h = 2^{-8}$ and average over 200 samples. (a) $g(v) = v$. (b) $g(v) = \frac{v}{(1+v^2)}$. (c) $g(v) = \ln(1 + v)$. (d) $g(v) = v \exp(-v^2)$.

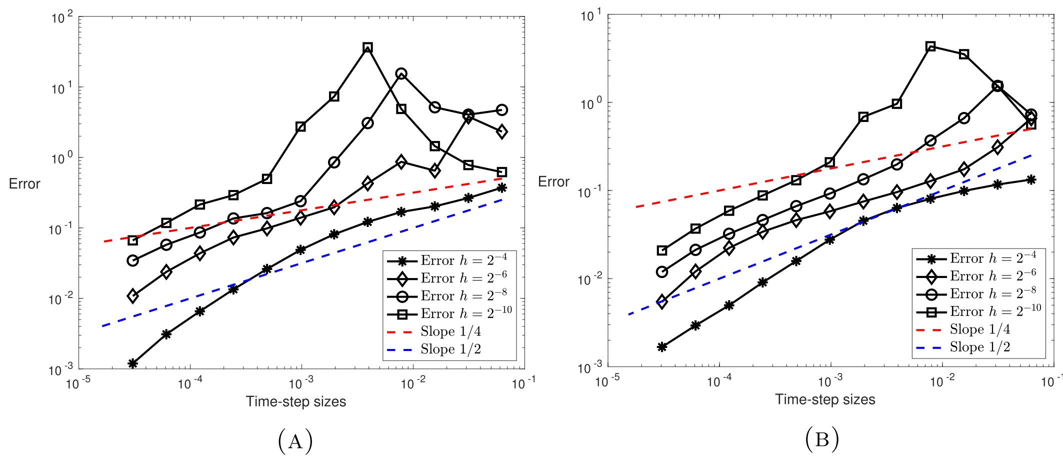

FIGURE 2. Mean-square errors on the time interval $[0, 0.5]$ of the splitting scheme for several values of the spatial mesh h. Average over 200 samples. (a) $g(v) = v$. (b) $g(v) = \frac{v}{(1+v^2)}$.

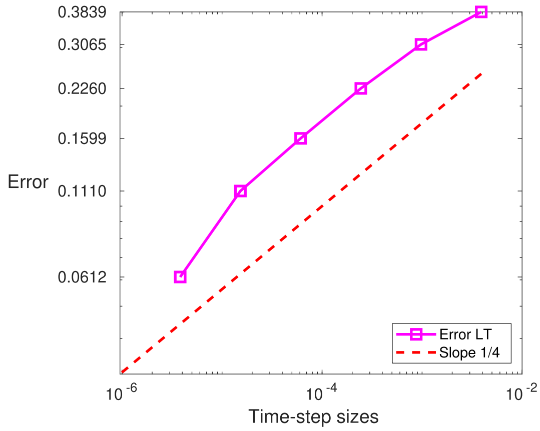

FIGURE 3. Mean-square errors on the time interval $[0, 0.5]$ of the splitting scheme (LT) when applied to the stochastic heat equation (3) with $g(v) = v$. The time-step sizes are coupled to the mesh sizes $\tau = h^2$. The average is over 100 samples.

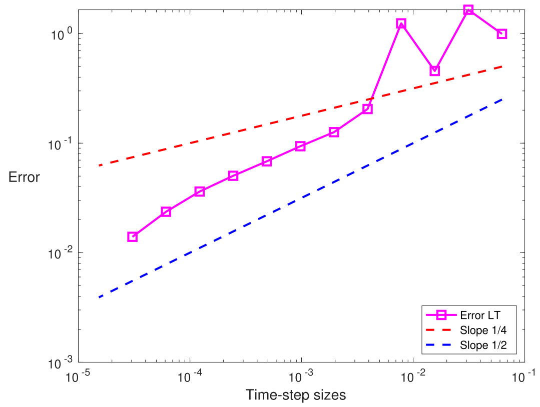

FIGURE 4. Mean-square errors on the time interval $[0, 0.5]$ of the splitting scheme (LT) when applied to the stochastic heat equation (3) with $g(v) = v^{1.25}$. Mesh size $h = 2^{-8}$ and average over 200 samples.

5.1. Auxiliary process

In this section, for any $M \in \mathbb{N}$ and $N \in \mathbb{N}$, we define an auxiliary stochastic process $(u^{\mathrm{LT}}(t,x))_{t\in[0,T],x\in[0,1]}$ satisfying $u^{\mathrm{LT}}(t_m, x_n) = u^{\mathrm{LT}}_{m,n}$ for all $m \in \{0,\dots,M\}$ and $n \in \{1,\dots,N-1\}$. The auxiliary process u^{LT} is piecewise continuous with respect to the spatial variable x, while its temporal evolution on each interval (t_m, t_{m+1}) follows a stochastic differential equation similar to (11).

Recall that the auxiliary mappings $\kappa^N : [0,1] \to \{x_0,\dots,x_N\}$ and $\ell^M : [0,T] \to \{t_0,\dots,t_M\}$ are defined in Sections 2.2 and 3 respectively.

Let $n \in \{1,\dots,N-1\}$ and $m \in \{0,\dots,M-1\}$, then for all $t \in [t_m, t_{m+1}]$ set

$$u^{\mathrm{LT}}_{m,n}(t) = \sum_{k=1}^{N-1} G^N_{nk}\big(\ell^M(t) - t_m\big) v^{M,N,1}_{m,k}(t), \tag{20}$$

where $v_{m,n}^{M,N,1}(t)=\exp\Big(\sqrt{N}f(u_{m,n}^{\rm LT})(W_n^N(t)-W_n^N(t_m))-\frac{Nf(u_{m,n}^{\rm LT})^2(t-t_m)}{2}\Big)u_{m,n}^{\rm LT}$ is the explicit expression (13) of the solution at time $t\in[t_m,t_{m+1}]$ of the auxiliary stochastic subsystem (11) used in the construction of the splitting integrator. Observe that $u_{m,n}^{\rm LT}(t)=v_{m,n}^{M,N,1}(t)$ for all $t\in[t_m,t_{m+1})$, and, in particular, that $u_{m,n}^{\rm LT}(t_m)=u_{m,n}^{\rm LT}$. Moreover, by the construction of the splitting scheme, see (15), it holds that $u_{m,n}^{\rm LT}(t_{m+1})=u_{m+1,n}^{\rm LT}$.

As a result, for any $n\in\{1,\dots,N-1\}$, the mapping $u_n^{\rm LT}\colon t\in[0,T]\mapsto u_n^{\rm LT}(t)$ defined such that $u_n^{\rm LT}(t)=u_{m,n}^{\rm LT}(t)$ for $t\in[t_m,t_{m+1}]$ is well-defined. It is continuous on each interval $[t_m,t_{m+1})$, and one has $u_n^{\rm LT}(t_m)=u_{m,n}^{\rm LT}$ for all $m\in\{0,\dots,M\}$.

We claim that the following identity holds: for all $M\in\mathbb{N}$ and $N\in\mathbb{N}$, for all $n\in\{1,\dots,N-1\}$ and $t\in[0,T]$, one has

$$u_n^{\rm LT}(t)=\sum_{k=1}^{N-1}G_{nk}^N\big(\ell^M(t)\big)u_{0,k}^{\rm LT}+\sqrt{N}\int_0^t\sum_{k=1}^{N-1}G_{nk}^N\big(\ell^M(t)-\ell^M(s)\big)u_k^{\rm LT}(s)f\big(u_k^{\rm LT}\big(\ell^M(s)\big)\big)\,{\rm d}W_k^N(s). \tag{21}$$

The proof is based on a straightforward recursion argument on the time index m, for $t\in[t_m,t_{m+1})$.

Recall from Section 2.2 that one has the identities $NG_{nk}^N(t)=G^N(t,x_n,x_k)$ and $\sqrt{N}{\rm W}_n^N(t)=N(W(t,x_{n+1})-W(t,x_n))$. We are now in position to provide the definition of the auxiliary process $u^{\rm LT}$: for $t\in[0,T]$ and $x\in[0,1]$, define

$$\begin{aligned}u^{\rm LT}(t,x)&=\int_0^1G^N(t,x,y)u_0\big(\kappa^N(y)\big)\,{\rm d}y\\&\quad+\int_0^t\int_0^1G^N\big(\ell^M(t)-\ell^M(s),x,y\big)u^{\rm LT}\big(s,\kappa^N(y)\big)f\big(u^{\rm LT}\big(\ell^M(s),\kappa^N(y)\big)\big)\,{\rm d}W(s,y).\end{aligned} \tag{22}$$

In the identity (22) above, it is worth recalling that $x\in[0,1]\mapsto G^N(t,x,y)$ is a piecewise linear mapping, whereas $y\in[0,1]\mapsto G^N(t,x,y)$ is a piecewise constant mapping, with $G^N(t,x_n,x_k)=NG_{nk}^N(t)$ for all $1\le n,k\le N-1$ and $t\in[0,T]$.

Combining (21) and (22), one obtains the identity $u^{\rm LT}(t,x_n)=u_n^{\rm LT}(t)$ for all $t\in[0,T]$ and $n\in\{1,\dots,N-1\}$, and therefore one obtains the required property $u^{\rm LT}(t_m,x_n)=u_n^{\rm LT}(t_m)=u_{m,n}^{\rm LT}$. Note that, for any $t\in[0,T]$, the mapping $x\in[0,1]\mapsto u^{\rm LT}(t,x)$ is piecewise linear, more precisely it is linear on each subinterval $[x_n,x_{n+1}]$.

5.2. Auxiliary inequalities

In this subsection we state several inequalities used in the convergence analysis of the splitting scheme.

- For any continuous function $v\colon[0,1]\to\mathbb{R}$ with $v(0)=v(1)=0$, one has (see for instance [37], Eq. (3.5))

$$\sup_{N\in\mathbb{N}}\sup_{t\ge0}\sup_{x\in[0,1]}\left|\int_0^1G^N(t,x,y)v\big(\kappa^N(y)\big)\,{\rm d}y\right|\le\sup_{x\in[0,1]}|v(x)|. \tag{23}$$

- For all $T\in(0,\infty)$, there exists $C_T\in(0,\infty)$ such that for all $t\in(0,T]$ one has (see for instance [2], Lem. 2.3)

$$\sup_{N\in\mathbb{N}}\sup_{x\in[0,1]}\int_0^1\big|G^N(t,x,y)\big|^2\,{\rm d}y\le\frac{C_T}{\sqrt{t}}\cdot \tag{24}$$

- For all $T\in(0,\infty)$, there exists $C_T\in(0,\infty)$ such that for all $t\in(0,T]$ and all $M\in\mathbb{N}$ one has

$$\sup_{N\in\mathbb{N}}\sup_{x\in[0,1]}\int_0^t\int_0^1\big|G^N(t-s,x,y)-G^N\big(t-\ell^M(s),x,y\big)\big|^2\,{\rm d}y\,{\rm d}s\le C_T\sqrt{\tau}, \tag{25a}$$

  and for all $N\ge1$ one has

$$\sup_{x\in[0,1]}\int_0^t\int_0^1\big|G^N(t-s,x,y)-G^N\big(t-\ell^M(s),x,y\big)\big|^2\,{\rm d}y\,{\rm d}s\le C_T\frac{\tau}{h}\cdot \tag{25b}$$

Since we are not aware of a detailed proof of the inequalities (25a) and(25b) in the literature, we provide a proof in Appendix A. Note that the proof is similar to the proof of Lemma 2.3 in [2].

Let us also recall the following discrete Grönwall inequality, see for instance Lemma A.4 in [50]: assume that a sequence $(a_m)_{0\le m\le M}$ of nonnegative numbers satisfies the inequality

$$a_m \le A + C\tau \sum_{k=0}^{m-1} \frac{a_k}{\sqrt{t_m - t_k}},$$

where we recall that $t_k = k\tau = \frac{kT}{M}$, for some $A, C \in (0,\infty)$. Then, there exists $C_T \in (0,\infty)$, depending only on C and on T, such that one has

$$\sup_{0\le m\le M} a_m \le C_T A. \tag{26}$$

5.3. Moment bounds

The objective of this section is to prove Proposition 5. Recall that this requires to impose the condition $\tau \le \gamma h$ where we recall that $\tau = T/M$, $h = 1/N$ and where $\gamma \in (0,\infty)$ is an arbitrary parameter.

Proof of Proposition 5. Using the definition (22) of the auxiliary process u^{LT}, for all $m \in \{1,\dots,M\}$ and $n \in \{1,\dots,N-1\}$, one has

$$\begin{aligned} u^{\mathrm{LT}}_{m,n} &= u^{\mathrm{LT}}(t_m, x_n) \\ &= \int_0^1 G^N(t_m, x_n, y) u_0\big(\kappa^N(y)\big)\,\mathrm{d}y \\ &\quad + \int_0^t \int_0^1 G^N\big(t_m - \ell^M(s), x_n, y\big) u^{\mathrm{LT}}\big(s, \kappa^N(y)\big) f\big(u^{\mathrm{LT}}\big(\ell^M(s), \kappa^N(y)\big)\big)\,\mathrm{d}W(s,y). \end{aligned}$$

Using Itô's isometry formula, one obtains

$$\begin{aligned} \mathbb{E}\Big[\big|u^{\mathrm{LT}}_{m,n}\big|^2\Big] &= \mathbb{E}\left[\left|\int_0^1 G^N(t_m, x_n, y) u_0\big(\kappa^N(y)\big)\,\mathrm{d}y\right|^2\right] \\ &\quad + \int_0^t \int_0^1 \big|G^N\big(t_m - \ell^M(s), x_n, y\big)\big|^2 \mathbb{E}\Big[\big|u^{\mathrm{LT}}\big(s, \kappa^N(y)\big)\big|^2 \big|f\big(u^{\mathrm{LT}}\big(\ell^M(s), \kappa^N(y)\big)\big)\big|^2\Big]\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

On the one hand, using the auxiliary inequality (23) and Assumption 1, one obtains

$$\mathbb{E}\left[\left|\int_0^1 G^N(t_m, x_n, y) u_0\big(\kappa^N(y)\big)\,\mathrm{d}y\right|^2\right] \le \|u_0\|_\infty^2.$$

On the other hand, recall that Assumption 3 implies that f is bounded by $\mathrm{L_g}$. In addition, for all $k \in \{0,\dots,m-1\}$ and all $s \in [t_k, t_{k+1})$, one has

$$\mathbb{E}\Big[\big|u^{\mathrm{LT}}\big(s, \kappa^N(y)\big)\big|^2\Big] = \mathbb{E}\left[\left|v^{M,N,1}_{k,n}(s)\right|^2\right]$$

where $n \in \{1,\dots,N-1\}$ is such that $\kappa^N(y) = x_n$ and $(v^{M,N,1}_{k,n}(s))_{s\in[t_k,t_{k+1}]}$ is the solution of the auxiliary stochastic subsystem (11). Using the expression (13) for the solution of (11), and the identity

$$\mathbb{E}\big[e^{\alpha Z}\big] = e^{\frac{\alpha^2\sigma^2}{2}},$$

if $Z \sim \mathcal{N}(0,\sigma^2)$ is a centered real-valued Gaussian random variable with variance σ^2, one obtains

$$\mathbb{E}\Big[\big|v_{k,n}^{M,N,1}(s)\big|^2\big|\mathcal{F}_{t_k}\Big] = e^{Nf\left(u_{k,n}^{\mathrm{LT}}\right)^2(s-t_k)}\big|u_{k,n}^{\mathrm{LT}}\big|^2.$$

Then applying the tower property of conditional expectation, and using the boundedness of f and the condition $N\tau \leq \gamma$ one obtains

$$\mathbb{E}\Big[\big|v_{k,n}^{M,N,1}(s)\big|^2\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\big|v_{k,n}^{M,N,1}(s)\big|^2\big|\mathcal{F}_{t_k}\Big]\Big] = \mathbb{E}\Big[e^{Nf\left(u_{k,n}^{\mathrm{LT}}\right)^2(s-t_k)}\big|u_{k,n}^{\mathrm{LT}}\big|^2\Big] \leq e^{N\tau \mathrm{L}_{\mathrm{g}}^2}\mathbb{E}\Big[\big|u_{k,n}^{\mathrm{LT}}\big|^2\Big] \leq e^{\mathrm{L}_{\mathrm{g}}^2\gamma}\mathbb{E}\Big[\big|u_{k,n}^{\mathrm{LT}}\big|^2\Big].$$

Using the auxiliary inequality (24), gathering the upper bounds above yields the following inequality: for all $m \in \{1,\ldots,M\}$ one has

$$\sup_{1\leq n\leq N-1}\mathbb{E}\Big[\big|u_{m,n}^{\mathrm{LT}}\big|^2\Big] \leq \|u_0\|_\infty^2 + C_{\gamma,T}\tau\sum_{k=0}^{m-1}\frac{1}{\sqrt{t_m-t_k}}\sup_{1\leq n\leq N-1}\mathbb{E}\Big[\big|u_{k,n}^{\mathrm{LT}}\big|^2\Big].$$

Using the discrete Grönwall inequality (26) then gives

$$\sup_{0\leq m\leq M}\sup_{1\leq n\leq N-1}\mathbb{E}\Big[\big|u_{m,n}^{\mathrm{LT}}\big|^2\Big] \leq C_{\gamma,T}\|u_0\|_\infty^2, \tag{27}$$

where $C_{\gamma,T} \in (0,\infty)$ is independent of M, N and $\|u_0\|_\infty^2$. This shows moment bounds of the numerical solution at the grid. It remains to extend this moment bound for $u^{\mathrm{LT}}(t,x_n)$ when t is no longer assumed to be a grid point t_m.

For all $t \in [0,T)$ and $n \in \{0,\ldots,N-1\}$, let $m \in \{0,\ldots,M-1\}$ be such that $t_m = \ell^M(t)$, using the same arguments as above one has

$$\mathbb{E}\Big[\big|u^{\mathrm{LT}}(t,x_n)\big|^2\Big] = \mathbb{E}\Big[\big|v_{m,n}^{M,N,1}(t)\big|^2\Big] \leq e^{\mathrm{L}_{\mathrm{g}}^2\gamma}\mathbb{E}\Big[\big|u_{m,n}^{\mathrm{LT}}\big|^2\Big] \leq C_{\gamma,T}\|u_0\|_\infty^2,$$

where the inequality (27) is used in the last step. As a consequence, one has

$$\sup_{t\in[0,T]}\sup_{1\leq n\leq N-1}\mathbb{E}\Big[\big|u^{\mathrm{LT}}(t,x_n)\big|^2\Big] \leq C_{\gamma,T}\|u_0\|_\infty^2. \tag{28}$$

Finally, since $x \mapsto u^{\mathrm{LT}}(t,x)$ is linear on each subinterval $[x_n,x_{n+1}]$, one obtains

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}\mathbb{E}\Big[\big|u^{\mathrm{LT}}(t,x)\big|^2\Big] \leq \sup_{t\in[0,T]}\sup_{1\leq n\leq N-1}\mathbb{E}\Big[\big|u^{\mathrm{LT}}(t,x_n)\big|^2\Big] \leq C_{\gamma,T}\|u_0\|_\infty^2. \tag{29}$$

The proof of Proposition 5 is thus completed. □

A straightforward consequence of Proposition 5 is the following result.

Lemma 8. *Let Assumptions 1 and 3 be satisfied. Let* $(u^{\mathrm{LT}}(t,x))_{t\in[0,T],x\in[0,1]}$ *be given by the mild formula* (22).

For all $\gamma \in (0,\infty)$ *and all* $T \in (0,\infty)$*, there exists* $C_{\gamma,T} \in (0,\infty)$ *such that for all* $\tau = T/M$ *and* $h = 1/N$ *satisfying the condition* $\tau \leq \gamma h$*, for all* $m \in \{0,\ldots,M-1\}$ *and all* $t \in [t_m,t_{m+1})$*, one has*

$$\sup_{1\leq n\leq N-1}\Big(\mathbb{E}\Big[\big|u^{\mathrm{LT}}(t,x_n) - u^{\mathrm{LT}}(t_m,x_n)\big|^2\Big]\Big)^{\frac{1}{2}} \leq C_{\gamma,T}\|u_0\|_\infty\Big(\frac{\tau}{h}\Big)^{\frac{1}{2}}. \tag{30}$$

Proof of Lemma 8. Let $n \in \{1, \dots, N-1\}$ and $m \in \{0, \dots, M-1\}$, then for all $t \in [t_m, t_{m+1})$ one has

$$\begin{aligned} u^{\mathrm{LT}}(t, x_n) - u^{\mathrm{LT}}(t_m, x_n) &= v_{m,n}^{M,N,1}(t) - v_{m,n}^{M,N,1}(t_m) \\ &= \sqrt{N} \int_{t_m}^{t} v_{m,n}^{M,N,1}(s) f\big(u_{m,n}^{\mathrm{LT}}\big)\, \mathrm{d}W_n^N(s) \\ &= \sqrt{N} \int_{t_m}^{t} u^{\mathrm{LT}}(s, x_n) f\big(u_{m,n}^{\mathrm{LT}}\big)\, \mathrm{d}W_n^N(s), \end{aligned}$$

where we recall that the auxiliary process $(v_{m,n}^{M,N,1}(t))_{t_m \le t \le t_{m+1}}$ is defined by the auxiliary subsystem (11) which gives the first step of the splitting procedure, see Section 3.1.

Since the mapping f is bounded, using Itô's isometry formula, the condition $\tau N \le \gamma$ and the moment bounds (16) from Proposition 5, one obtains

$$\mathbb{E}\Big[\big|u^{\mathrm{LT}}(t, x_n) - u^{\mathrm{LT}}(t_m, x_n)\big|^2\Big] \le \mathrm{L}_{\mathrm{g}}^2 N \tau \mathbb{E}\Big[\big|u_{m,n}^{\mathrm{LT}}\big|^2\Big] \le \mathrm{L}_{\mathrm{g}}{}^2 C_{\gamma,T} \|u_0\|_\infty^2 \frac{\tau}{h}.$$

The proof of Lemma 8 is thus completed. □

5.4. Convergence analysis

This section is devoted to the proof of the mean-square convergence of the splitting scheme given in Theorem 6.

Proof of Theorem 6. Recall that $u_{m,n}^{\mathrm{LT}} = u^{\mathrm{LT}}(t_m, x_n)$ for all $n \in \{1, \dots, N-1\}$ and $m \in \{0, \dots, M\}$, where $(u^{\mathrm{LT}}(t,x))_{t \in [0,1], x \in [0,1]}$ is the process defined by (20).

For all $n \in \{1, \dots, N-1\}$ and $m \in \{1, \dots, M\}$, let us define

$$E_{m,n} = u^N(t_m, x_n) - u_{m,n}^{\mathrm{LT}} \quad \text{and} \quad E_m = \sup_{1 \le n \le N-1} \mathbb{E}\Big[|E_{m,n}|^2\Big].$$

Using the expression (6) for $u^N(t,x)$ and the expression (22) for $u^{\mathrm{LT}}(t,x)$, one obtains the following decomposition of the error: for all $n \in \{1, \dots, N-1\}$ and $m \in \{1, \dots, M\}$, one has

$$\begin{aligned} E_{m,n} &= u^N(t_m, x_n) - u^{\mathrm{LT}}(t_m, x_n) \\ &= \int_0^{t_m} \int_0^1 G^N(t_m - s, x, y) g\big(u^N\big(s, \kappa^N(y)\big)\big)\, \mathrm{d}W(s,y) \\ &\quad - \int_0^{t_m} \int_0^1 G^N\big(t_m - \ell^M(s), x, y\big) u^{\mathrm{LT}}\big(s, \kappa^N(y)\big) f\big(u^{\mathrm{LT}}\big(\ell^M(s), \kappa^N(y)\big)\big)\, \mathrm{d}W(s,y) \\ &= E_{m,n}^{(1)} + E_{m,n}^{(2)}, \end{aligned}$$

where we set

$$\begin{aligned} E_{m,n}^{(1)} &= \int_0^{t_m} \int_0^1 G^N(t_m - s, x, y)\big[g\big(u^N\big(s, \kappa^N(y)\big)\big) - u^{\mathrm{LT}}\big(s, \kappa^N(y)\big) f\big(u^{\mathrm{LT}}\big(\ell^M(s), \kappa^N(y)\big)\big)\big]\, \mathrm{d}W(s,y), \\ E_{m,n}^{(2)} &= \int_0^{t_m} \int_0^1 \big[G^N(t_m - s, x, y) - G^N\big(t_m - \ell^M(s), x, y\big)\big] u^{\mathrm{LT}}\big(s, \kappa^N(y)\big) f\big(u^{\mathrm{LT}}\big(\ell^M(s), \kappa^N(y)\big)\big)\, \mathrm{d}W(s,y). \end{aligned}$$

Let us first deal with the error term $E_{m,n}^{(1)}$. Recall that $g(u) = uf(u)$, therefore one has the decomposition $E_{m,n}^{(1)} = E_{m,n}^{(1,1)} + E_{m,n}^{(1,2)} + E_{m,n}^{(1,3)}$, where

$$E_{m,n}^{(1,1)} = \int_0^{t_m} \int_0^1 G^N(t_m - s, x, y)\big[g\big(u^N\big(s, \kappa^N(y)\big)\big) - g\big(u^N\big(\ell^M(s), \kappa^N(y)\big)\big)\big]\, \mathrm{d}W(s,y)$$

$$E_{m,n}^{(1,2)} = \int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)\big[g\big(u^N\big(\ell^M(s),\kappa^N(y)\big)\big) - g\big(u^{\rm LT}\big(\ell^M(s),\kappa^N(y)\big)\big)\big]\,{\rm d}W(s,y)$$
$$E_{m,n}^{(1,3)} = \int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)\big[u^{\rm LT}\big(\ell^M(s),\kappa^N(y)\big) - u^{\rm LT}\big(s,\kappa^N(y)\big)\big]f\big(u^{\rm LT}\big(\ell^M(s),\kappa^N(y)\big)\big)\,{\rm d}W(s,y).$$

Using Itô's isometry formula, the global Lipschitz continuity assumption on g, one obtains

$$\begin{aligned}\mathbb{E}\Big[\Big|E_{m,n}^{(1,1)}\Big|^2\Big] &\le {\rm L}_{\rm g}^2\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\mathbb{E}\Big[\big|u^N\big(s,\kappa^N(y)\big) - u^N\big(\ell^M(s),\kappa^N(y)\big)\big|^2\Big]\,{\rm d}y\,{\rm d}s\\ &\le C_T(u_0){\rm L}_{\rm g}^2\sqrt{\tau}\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\,{\rm d}y\,{\rm d}s\\ &\le C_T(u_0)\sqrt{\tau},\end{aligned}$$

where we have used the temporal regularity estimate (9a) for u^N and the auxiliary inequality (24).

Similarly, using Itô's isometry formula, the global Lipschitz continuity assumption on g, one obtains

$$\begin{aligned}\mathbb{E}\Big[\Big|E_{m,n}^{(1,2)}\Big|^2\Big] &\le {\rm L}_{\rm g}^2\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\mathbb{E}\Big[\big|u^N\big(\ell^M(s),\kappa^N(y)\big) - u^{\rm LT}\big(\ell^M(s),\kappa^N(y)\big)\big|^2\Big]\,{\rm d}y\,{\rm d}s\\ &\le C\sum_{k=0}^{m-1}E_k\int_{t_k}^{t_{k+1}}\int_0^1 G^N(t_m-s,x,y)^2\,{\rm d}y\,{\rm d}s.\end{aligned}$$

Using the inequality (24), for all $k\in\{0,\ldots,m-1\}$, one has

$$\begin{aligned}\int_{t_k}^{t_{k+1}}\int_0^1 G^N(t_m-s,x,y)^2\,{\rm d}y\,{\rm d}s &\le \int_{t_k}^{t_{k+1}}\frac{C_T}{\sqrt{t_m-s}}\,{\rm d}s\\ &= 2C_T\Big(\sqrt{t_m-t_k}-\sqrt{t_m-t_{k+1}}\Big)\\ &= 2C_T\sqrt{t_m-t_k}\left(1-\sqrt{1-\frac{\tau}{t_m-t_k}}\right)\\ &\le \frac{2C_T\tau}{\sqrt{t_m-t_k}},\end{aligned}$$

where we have used the inequality $1-\sqrt{1-z}\le z$ for all $z\in[0,1]$ in the last step. Therefore one has

$$\mathbb{E}\Big[\Big|E_{m,n}^{(1,2)}\Big|^2\Big] \le C_T\tau\sum_{k=0}^{m-1}\frac{E_k}{\sqrt{t_m-t_k}}.$$

Finally, for the third term, using Itô's isometry formula and the boundedness of f, one obtains

$$\begin{aligned}\mathbb{E}\Big[\Big|E_{m,n}^{(1,3)}\Big|^2\Big] &\le {\rm L}_{\rm g}^2\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\mathbb{E}\Big[\big|u^{\rm LT}\big(\ell^M(s),\kappa^N(y)\big) - u^{\rm LT}\big(s,\kappa^N(y)\big)\big|^2\Big]\,{\rm d}y\,{\rm d}s\\ &\le C_{\gamma,T}(u_0)\frac{\tau}{h}\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\,{\rm d}y\,{\rm d}s\\ &\le C_{\gamma,T}(u_0)\frac{\tau}{h}\end{aligned}$$

using the temporal regularity estimate (30) from Lemma 8 for $u^{\rm LT}$ and the auxiliary inequality (24).

Let us now deal with the error term $E_{m,n}^{(2)}$. Using Itô's formula, the boundedness of f and the moment bounds (16) from Proposition 5, one obtains

$$\begin{aligned}\mathbb{E}\left[\left|E_{m,n}^{(2)}\right|^2\right] &\leq \mathrm{L}_{\mathrm{g}}^2 \int_0^{t_m}\int_0^1 \left|G^N\left(t-\ell^M(s),x,y\right)-G^N\left(t_m-\ell^M(s),x,y\right)\right|^2 \mathbb{E}\left[\left|u^{\mathrm{LT}}\left(s,\kappa^N(y)\right)\right|^2\right]\mathrm{d}y\,\mathrm{d}s\\ &\leq C_{\gamma,T}(u_0)\int_0^{t_m}\int_0^1 \left|G^N\left(t-\ell^M(s),x,y\right)-G^N\left(t_m-\ell^M(s),x,y\right)\right|^2\mathrm{d}y\,\mathrm{d}s\\ &\leq C_{\gamma,T}(u_0)\sqrt{\tau},\end{aligned}$$

owing to the auxiliary inequality (25a) in the last step.

Gathering the estimates, for all $m\in\{1,\dots,M\}$, one has

$$E_m \leq C_{\gamma,T}(u_0)\left(\sqrt{\tau}+\frac{\tau}{h}\right)+C_T\tau\sum_{k=0}^{m-1}\frac{E_k}{\sqrt{t_m-t_k}}\cdot$$

Applying the discrete Grönwall inequality (26) (see Sect. 5.2) then yields

$$\sup_{0\leq m\leq M} E_m \leq C_{\gamma,T}(u_0)\left(\sqrt{\tau}+\frac{\tau}{h}\right)\cdot$$

This gives the error estimate (17a).

Proving the error estimate (17b) requires minor changes in the arguments above: making use of the inequalities (9b) and (25b) instead of (9a) and (25a) to prove upper bounds for the terms $\mathbb{E}[|E_{m,n}^{(1,1)}|^2]$ and $\mathbb{E}[|E_{m,n}^{(2)}|^2]$. One obtains

$$\begin{aligned}\mathbb{E}\left[\left|E_{m,n}^{(1,1)}\right|^2\right] &\leq \mathrm{L}_{\mathrm{g}}^2\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\mathbb{E}\left[\left|u^N\left(s,\kappa^N(y)\right)-u^N\left(\ell^M(s),\kappa^N(y)\right)\right|^2\right]\mathrm{d}y\,\mathrm{d}s\\ &\leq C_T(u_0)\mathrm{L}_{\mathrm{g}}^2\frac{\tau}{h}\int_0^{t_m}\int_0^1 G^N(t_m-s,x,y)^2\,\mathrm{d}y\,\mathrm{d}s \leq C_T(u_0)\frac{\tau}{h}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left|E_{m,n}^{(2)}\right|^2\right] &\leq C_{\gamma,T}(u_0)\int_0^{t_m}\int_0^1 \left|G^N\left(t-\ell^M(s),x,y\right)-G^N\left(t_m-\ell^M(s),x,y\right)\right|^2\mathrm{d}y\,\mathrm{d}s\\ &\leq C_{\gamma,T}(u_0)\frac{\tau}{h}\cdot\end{aligned}$$

The rest of the proof is as above. Applying the discrete Grönwall inequality (26), one obtains the estimate

$$\sup_{0\leq m\leq M} E_m \leq C_{\gamma,T}(u_0)\frac{\tau}{h}$$

which is the error bound (17b).

When the condition $\tau\leq\gamma h^2$ is satisfied, one has $\tau/h\leq\sqrt{\gamma}\tau^{\frac{1}{2}}$ and one obtains the error estimate (18) from either (17a) or (17b). This concludes the proof of Theorem 6. □

Let us also provide the proof of Corollary 7.

Proof of Corollary 7. It suffices to combine the error estimate (8) from Proposition 3 for the spatial discretization error, and the error estimate (17a) from Theorem 6 for the temporal discretization error. One then obtains the error estimate for the splitting scheme

$$\left(\mathbb{E}\left[\left|u_{m,n}^{\mathrm{LT}}-u(t_m,x_n)\right|^2\right]\right)^{\frac{1}{2}} \leq \left(\mathbb{E}\left[\left|u_{m,n}^{\mathrm{LT}}-u^N(t_m,x_n)\right|^2\right]\right)^{\frac{1}{2}}+\left(\mathbb{E}\left[\left|u^N(t_m,x_n)-u(t_m,x_n)\right|^2\right]\right)^{\frac{1}{2}}$$

$$\leq C_{\gamma,T}(u_0)\tau^{\frac{1}{4}} + C_T(u_0)h^{\frac{1}{2}}$$
$$\leq C_{\gamma,T}(u_0)\gamma^{\frac{1}{4}}h^{\frac{1}{2}} + C_T(u_0)h^{\frac{1}{2}},$$

under the condition $\tau \leq \gamma h^2$. This gives the error estimate (19) and concludes the proof of Corollary 7. □

5.5. Proof of Proposition 2

We conclude this section with the proof of the positivity property of the exact solution to the stochastic heat equation (3) on a bounded domain.

Proof of Proposition 2. Owing to Corollary 7 and to the temporal regularity estimate (5) satisfied by the solution u of the SPDE in equation (3), one obtains the following result (recall that $\tau = T/M$ and $h = 1/N$): there exists $C_{\gamma,T}(u_0) \in (0,\infty)$ such that for all $N \in \mathbb{N}$ and $M \in \mathbb{N}$, such that $M \geq \frac{TN^2}{\gamma}$, for all $t \in [0,T]$ and $x \in [0,1]$, one has

$$\left(\mathbb{E}\left[\left|u(t,x) - u^{\mathrm{LT}}\left(\ell^M(t), \kappa^N(x)\right)\right|^2\right]\right)^{\frac{1}{2}} \leq C_{\gamma,T}(u_0)N^{-\frac{1}{2}}. \tag{31}$$

Let $t \in [0,T]$ and $x \in [0,1]$ be fixed, then there exists a sequence $(N_k)_{k\in\mathbb{N}}$ such that $N_k \to \infty$ and $u^{\mathrm{LT}}(\ell^{M_k}(t), \kappa^{N_k}(x))$, where $(M_k)_{k\in\mathbb{N}}$ is any sequence satisfying $M_k \geq \frac{TN_k^2}{\gamma}$ for every $k \in \mathbb{N}$, converges to $u(t,x)$ almost surely. Since $u^{\mathrm{LT}}(\ell^{M_k}(t), \kappa^{N_k}(x)) \geq 0$ almost surely owing to Proposition 4, one obtains $u(t,x) \geq 0$ almost surely. □

6. Generalization to systems

In this section, we briefly describe how to generalize the construction of the splitting scheme (15) and the analysis above to stochastic systems of the type

$$\begin{cases} \mathrm{d}u_1(t,x) = \partial_{xx}^2 u_1(t,x)\,\mathrm{d}t + g_1(u_1(t,x), u_2(t,x))\,\mathrm{d}W_1(t,x), \\ \mathrm{d}u_2(t,x) = \partial_{xx}^2 u_2(t,x)\,\mathrm{d}t + g_2(u_1(t,x), u_2(t,x))\,\mathrm{d}W_2(t,x), \\ u_1(t,0) = u_1(t,1) = 0, \quad u_2(t,0) = u_2(t,1) = 0, \\ u_1(0,x) = u_{1,0}(x), \qquad u_2(0,x) = u_{2,0}(x), \end{cases} \tag{32}$$

for $(t,x) \in [0,T] \times [0,1]$, where $g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are globally Lipschitz continuous mappings, with initial values $u_{1,0}, u_{2,0}$ satisfying Assumptions 1 and 2. The two evolution equations are driven by space-time white noise. The Wiener sheets W_1 and W_2 can either be equal or independent. For ease of presentation we only deal with systems of two equations, while considering systems of arbitrary size would also be possible.

In this setting, to obtain solutions which only have nonnegative values, it is necessary to replace Assumption 3 by the following.

Assumption 4. *The mappings $g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are of class $\mathcal{C}^1$ and globally Lipschitz continuous. In addition, they satisfy $g_1(0,v_2) = 0$ and $g_2(v_1,0) = 0$ for all $(v_1,v_2) \in \mathbb{R}^2$.*

One then has the following generalization of Proposition 2.

Proposition 9. *Consider the SPDE system (32). Let Assumption 4 be satisfied and assume that the initial values $u_{1,0}, u_{2,0}$ satisfy Assumptions 1 and 2. Then, for all $t \in (0,\infty)$ and all $x \in [0,1]$, almost surely, one has*

$$u_1(t,x) \geq 0, \quad u_2(t,x) \geq 0.$$

As in Sections 2 and 3, the mesh size and the time-step sizes are denoted by $h = 1/N$ and $\tau = T/M$ respectively, and the space and time grid points are denoted by $x_n = nh$ and $t_m = m\tau$, with $0 \le n \le N$ and $0 \le m \le M$. In addition, introduce the mappings $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f_1(v_1,v_2) = \frac{g_1(v_1,v_2)}{v_1} = \int_0^1 \partial_{v_1} g_1(rv_1,v_2)\,\mathrm{d}r, \quad f_2(v_1,v_2) = \frac{g_2(v_1,v_2)}{v_2} = \int_0^1 \partial_{v_2} g_2(v_1,rv_2)\,\mathrm{d}r.$$

Owing to Assumption 4, the mappings f_1 and f_2 are bounded and continuous mappings. Finally, for all $t \ge 0$ and $n \in \{1,\ldots,N-1\}$ define

$$W_{1,n}^N(t) = \sqrt{N}(W_1(t,x_{n+1}) - W_1(t,x_n)), \quad W_{2,n}^N(t) = \sqrt{N}(W_2(t,x_{n+1}) - W_2(t,x_n))$$

and define the noise increments

$$\Delta_{m,n}W_1 = W_{1,n}^N(t_{m+1}) - W_{1,n}^N(t_m), \quad \Delta_{m,n}W_2 = W_{2,n}^N(t_{m+1}) - W_{2,n}^N(t_m)$$

for all $n \in \{1,\ldots,N-1\}$ and $m \in \{0,\ldots,M-1\}$.

Using the finite difference method and the same notation as in Section 2.2, one obtains the spatial semi-discretization scheme for the SPDE system (32) with mesh size h as follows:

$$\begin{cases} \mathrm{d}u_1^N(t) = N^2 D^N u_1^N(t)\,\mathrm{d}t + \sqrt{N} g_1\big(u_1^N(t), u_2^N(t)\big)\,\mathrm{d}W_1^N(t) \\ \mathrm{d}u_2^N(t) = N^2 D^N u_2^N(t)\,\mathrm{d}t + \sqrt{N} g_2\big(u_1^N(t), u_2^N(t)\big)\,\mathrm{d}W_2^N(t). \end{cases} \tag{33}$$

We are now in position to state the definition of the fully-discrete scheme based on a Lie–Trotter splitting strategy and inspired by (15) for the approximation of solutions of (32): for all $m \in \{0,\ldots,M-1\}$, set

$$\begin{cases} u_{1,m+1}^{\mathrm{LT}} = e^{\tau N^2 D^N}\left(\exp\left(\sqrt{N} f_1\big(u_{1,m,n}^{\mathrm{LT}}, u_{2,m,n}^{\mathrm{LT}}\big)\Delta_{m,n}W_1 - \frac{N f_1\big(u_{1,m,n}^{\mathrm{LT}}, u_{2,m,n}^{\mathrm{LT}}\big)^2\tau}{2}\right) u_{1,m,n}^{\mathrm{LT}}\right)_{1\le n\le N-1} \\ u_{2,m+1}^{\mathrm{LT}} = e^{\tau N^2 D^N}\left(\exp\left(\sqrt{N} f_2\big(u_{1,m,n}^{\mathrm{LT}}, u_{2,m,n}^{\mathrm{LT}}\big)\Delta_{m,n}W_2 - \frac{N f_2\big(u_{1,m,n}^{\mathrm{LT}}, u_{2,m,n}^{\mathrm{LT}}\big)^2\tau}{2}\right) u_{2,m,n}^{\mathrm{LT}}\right)_{1\le n\le N-1}, \end{cases} \tag{34}$$

with initial values $u_{1,0}^{\mathrm{LT}} = (u_{1,0}(x_n))_{1\le n\le N-1}$ and $u_{1,0}^{\mathrm{LT}} = (u_{2,0}(x_n))_{1\le n\le N-1}$.

The scheme (34) is positivity-preserving in the following sense.

Proposition 10. *Let $M \in \mathbb{N}$ and $N \in \mathbb{N}$ be arbitrary integers and let $T \in (0,\infty)$. Let Assumption 4 be satisfied, and assume that the initial values $u_{1,0}, u_{2,0}$ satisfy Assumptions 1 and 2. Let the sequence $u_{1,0}^{\mathrm{LT}},\ldots,u_{1,M}^{\mathrm{LT}}$ and $u_{2,0}^{\mathrm{LT}},\ldots,u_{2,M}^{\mathrm{LT}}$ be given by the splitting scheme (34), with $h = 1/N$ and $\tau = T/M$, with initial values $u_{1,0,n}^{\mathrm{LT}} = u_{1,0}(x_n) \ge 0$ and $u_{1,0,n}^{\mathrm{LT}} = u_{2,0}(x_n) \ge 0$ for all $n \in \{1,\ldots,N\}$. Then, almost surely, one has*

$$u_{1,m,n}^{\mathrm{LT}} \ge 0, \quad u_{2,m,n}^{\mathrm{LT}} \ge 0,$$

for all $m \in \{1,\ldots,M\}$ and $n \in \{1,\ldots,N-1\}$.

The proof of Proposition 10 is a straightforward modification of the proof of Proposition 4. Moreover, one has the following variant of Proposition 5.

Proposition 11. *Let Assumption 4 be satisfied and assume that the initial values $u_{1,0}, u_{2,0}$ satisfy Assumptions 1 and 2. Let the sequences $u_{1,0}^{\mathrm{LT}},\ldots,u_{1,M}^{\mathrm{LT}}$ and $u_{2,0}^{\mathrm{LT}},\ldots,u_{2,M}^{\mathrm{LT}}$ be given by the Lie–Trotter splitting scheme (34).*

For all $\gamma \in (0,\infty)$ and all $T \in (0,\infty)$, there exists $C_{\gamma,T} \in (0,\infty)$ such that for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau \le \gamma h$, one has

$$\sup_{0\le m\le M}\sup_{1\le n\le N-1}\mathbb{E}\Big[\big|u_{1,m,n}^{\mathrm{LT}}\big|^2\Big] + \sup_{0\le m\le M}\sup_{1\le n\le N-1}\mathbb{E}\Big[\big|u_{2,m,n}^{\mathrm{LT}}\big|^2\Big] \le C_{\gamma,T}\big(\|u_{1,0}\|_\infty^2 + \|u_{2,0}\|_\infty^2\big). \tag{35}$$

Finally, one has the following generalization of Theorem 6.

Theorem 12. *Let Assumption 4 be satisfied and assume that the initial values $u_{1,0}, u_{2,0}$ satisfy Assumptions 1 and 2. Let the sequences $u_{1,0}^{\rm LT}, \ldots, u_{1,M}^{\rm LT}$ and $u_{2,0}^{\rm LT}, \ldots, u_{2,M}^{\rm LT}$ be given by the Lie–Trotter splitting scheme (34), and let $(u_1^N(t))_{t\geq 0, 0\leq n\leq N}$ and $(u_2^N(t))_{t\geq 0,0\leq n\leq N}$ be given by the spatial semi-discretization scheme (33).*

For all $\gamma \in (0,\infty)$ and $T\in(0,\infty)$, there exists $C_{\gamma,T}(u_{1,0},u_{2,0}) \in (0,\infty)$ such that for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau \leq \gamma h$, one has

$$
\begin{aligned}
\sup_{0\leq m\leq M}\sup_{0\leq n\leq N}\Big(\mathbb{E}\Big[\big|u_{1,m,n}^{\rm LT}-u_{1,n}^N(t_m)\big|^2\Big]\Big)^{\frac12} &\leq C_{\gamma,T}(u_{1,0},u_{2,0})\Big(\tau^{\frac14}+\Big(\frac{\tau}{h}\Big)^{\frac12}\Big),\\
\sup_{0\leq m\leq M}\sup_{0\leq n\leq N}\Big(\mathbb{E}\Big[\big|u_{2,m,n}^{\rm LT}-u_{2,n}^N(t_m)\big|^2\Big]\Big)^{\frac12} &\leq C_{\gamma,T}(u_{1,0},u_{2,0})\Big(\tau^{\frac14}+\Big(\frac{\tau}{h}\Big)^{\frac12}\Big).
\end{aligned}
\tag{36}
$$

In addition, for all $\tau = T/M$ and $h = 1/N$ satisfying the condition $\tau\leq\gamma h^2$, one has

$$
\begin{aligned}
\sup_{0\leq m\leq M}\sup_{0\leq n\leq N}\Big(\mathbb{E}\Big[\big|u_{1,m,n}^{\rm LT}-u_{1,n}^N(t_m)\big|^2\Big]\Big)^{\frac12} &\leq C_{\gamma,T}(u_{1,0},u_{2,0})\tau^{\frac14},\\
\sup_{0\leq m\leq M}\sup_{0\leq n\leq N}\Big(\mathbb{E}\Big[\big|u_{2,m,n}^{\rm LT}-u_{2,n}^N(t_m)\big|^2\Big]\Big)^{\frac12} &\leq C_{\gamma,T}(u_{1,0},u_{2,0})\tau^{\frac14}.
\end{aligned}
\tag{37}
$$

Note that the error term $\tau^{\frac14}$ could be removed from the equation (36) like in Theorem 6.

The proofs of Proposition 11 and of Theorem 12 are omitted since they follow from the same arguments as those of Proposition 5 and of Theorem 6. Finally, one obtains the following variant of Corollary 7.

Corollary 13. *Consider the setting and assumptions of Theorem 12. For all $\gamma\in(0,\infty)$ and $T\in(0,\infty)$, there exists $C_{\gamma,T}(u_{1,0},u_{2,0})\in(0,\infty)$ such that for all $\tau = T/M$ and $h=1/N$ satisfying the condition $\tau\leq\gamma h^2$, one has*

$$
\begin{aligned}
\sup_{0\leq m\leq M}\sup_{0\leq n\leq N}\Big(\mathbb{E}\Big[\big|u_{1,m,n}^{\rm LT}-u_1(t_m,x_n)\big|^2\Big]\Big)^{\frac12} &\leq C_{\gamma,T}(u_{1,0},u_{2,0})h^{\frac12},\\
\sup_{0\leq m\leq M}\sup_{0\leq n\leq N}\Big(\mathbb{E}\Big[\big|u_{2,m,n}^{\rm LT}-u_2(t_m,x_n)\big|^2\Big]\Big)^{\frac12} &\leq C_{\gamma,T}(u_{1,0},u_{2,0})h^{\frac12}.
\end{aligned}
\tag{38}
$$

To conclude this presentation of the positivity-preserving Lie–Trotter splitting scheme (34) for the approximation of solutions of the SPDE system (32), we report some numerical experiments.

The first numerical experiment illustrates the positivity-preserving property of the Lie–Trotter splitting scheme (LT) when applied to the system of SPDEs (32) driven by two independent noise. The initial values are taken to be $u_{1,0}(x) = u_{2,0}(x) = \sin(\pi x)$, the final time is $T = 5$ and the multiplicative terms are $g_1(v_1, v_2) = 7\sin(v_1)\cos(v_2)$ and $g_2(v_1,v_2) = 7\cos(v_1)\sin(v_2)$. The discretization parameters are $\tau = 2^{-2}$ and $h = 2^{-8}$. The proportion of samples containing only positive values out of 500 simulated samples for all considered time integrators are presented in Table 3.

The second numerical experiment illustrates the mean-square convergence of the Lie–Trotter splitting scheme when applied to systems of semilinear SHEs. Figure 5 presents, in a loglog plot, the mean-square errors measured at the space-time grid for the time interval $[0, 0.5]$. The discretization parameters are $h = 2^{-8}$ and $\tau = 2^{-4}, 2^{-5}, \ldots, 2^{-16}$ (the last one being used for the reference solution). We have used 200 samples to approximate the expected values. The expected mean-square orders of convergence is observed in this figure. Here, the mesh size h is fixed, hence the convergence rate in the estimate (36) is observed to be 1/2 like in Figure 1. But the upper bounds are not uniform in h as observed in Figure 2 (we do not repeat this numerical experiment for this system of SPDEs).

TABLE 3. Proportion of samples containing only positive values out of 500 simulated sample paths for the Lie–Trotter splitting scheme (LT), the stochastic exponential Euler integrator (SEXP), the semi-implicit Euler–Maruyama scheme (SEM), and the Euler–Maruyama scheme (EM). First and second component. The multiplicative terms are $g_1(v_1,v_2) = 7\sin(v_1)\cos(v_2)$ and $g_2(v_1,v_2) = 7\cos(v_1)\sin(v_2)$. The discretization parameters are $\tau = 2^{-2}$ and $h = 2^{-8}$.

LT (first, second)	SEXP (first, second)	SEM (first, second)	EM (first, second)
500/500, 500/500	500/500, 499/500	498/500, 496/500	0/500, 0/500

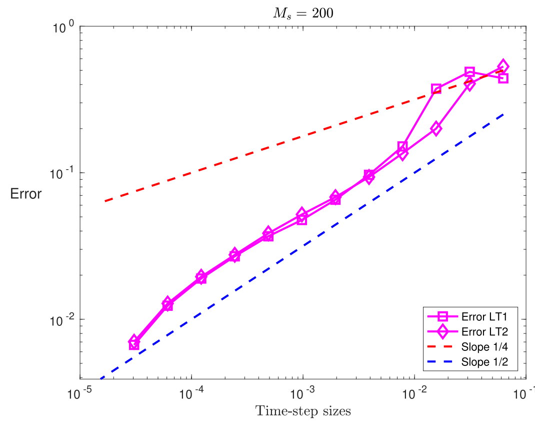

FIGURE 5. Mean-square errors of the Lie–Trotter splitting scheme (first component denoted by LT1, second by LT2) when applied to the system of stochastic heat equations with multiplicative terms $g_1(v_1,v_2) = \sin(v_1)\cos(v_2)$ and $g_2(v_1,v_2) = \cos(v_1)\sin(v_2)$. Mesh size $h = 2^{-8}$ and average over 200 samples.

APPENDIX A. PROOF OF AUXILIARY INEQUALITIES

Proof of the auxiliary inequalities (25a) *and* (25b). Let us recall some notation. For all $N \in \mathbb{N}$, all $t \geq 0$ and $x, y \in [0,1]$, one has

$$G^N(t,x,y) = \sum_{j=1}^{N-1} e^{-\lambda_j^N t} \varphi_j^N(x)\varphi_j\big(\kappa^N(y)\big),$$

where $\lambda_j^N = 4N^2 \sin(\frac{j\pi}{2N})^2$, $\varphi_j(\cdot) = \sqrt{2}\sin(j\pi\cdot)$ and φ_j^N is the linear interpolation of φ_j at the space grid points $x_n = nh$ for $n = 1, \ldots, N-1$.

Using the orthogonality property

$$\int_0^1 \varphi_j\big(\kappa^N(y)\big)\varphi_k\big(\kappa^N(y)\big)\,\mathrm{d}y = \delta_{jk},$$

one obtains

$$\begin{aligned}
&\int_0^t \int_0^1 \big|G^N(t-s,x,y) - G^N\big(t-\ell^M(s),x,y\big)\big|^2 \,\mathrm{d}y\,\mathrm{d}s \\
&= \int_0^t \int_0^1 \left|\sum_{j=1}^{N-1} \Big(e^{-\lambda_j^N(t-s)} - e^{-\lambda_j^N\left(t-\ell^M(s)\right)}\Big)\varphi_j^N(x)\varphi_j\big(\kappa^N(y)\big)\right|^2 \mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \sum_{j=1}^{N-1} \Big(e^{-\lambda_j^N (t-s)} - e^{-\lambda_j^N \left(t-\ell^M(s)\right)} \Big)^2 \varphi_j^N(x)^2 \,\mathrm{d}s \\
&\le 2 \int_0^t \sum_{j=1}^{N-1} \Big(e^{-\lambda_j^N (t-s)} - e^{-\lambda_j^N \left(t-\ell^M(s)\right)} \Big)^2 \,\mathrm{d}s \\
&\le 2 \sum_{j=1}^{N-1} \int_0^t e^{-2\lambda_j^N (t-s)} \Big(1 - e^{-\lambda_j^N \left(s-\ell^M(s)\right)} \Big)^2 \,\mathrm{d}s \\
&\le C \sum_{j=1}^{N-1} \frac{\min\big(1, \lambda_j^N \tau\big)^2}{\lambda_j^N}.
\end{aligned}
$$

Let us first prove the inequality (25a). One checks that there exists $c \in (1, \infty)$ such that for all $N \ge 1$ and $j \in \{1, \dots, N-1\}$ one has

$$
c^{-1} \le \frac{\lambda_j^N}{j^2} \le c.
$$

Let $L \in \mathbb{N}$ be an arbitrary positive integer. Owing to the inequalities above, one obtains

$$
\begin{aligned}
\sum_{j=1}^{N-1} \frac{\min\big(1, \lambda_j^N \tau\big)^2}{\lambda_j^N} &\le C \sum_{j=1}^{\infty} \frac{\min\big(1, j^2 \tau\big)^2}{j^2} \\
&\le C \sum_{j=1}^{L} j^2 \tau^2 + C \sum_{j=L+1}^{\infty} j^{-2} \\
&\le C \tau^2 L^3 + C L^{-1},
\end{aligned}
$$

using standard comparison of series and integrals arguments. Choosing $L = \lfloor \tau^{-\frac{1}{2}} \rfloor \ge 1$ (where $\lfloor \cdot \rfloor$ denotes the integer part), and recalling that $\tau \in (0,1)$, one obtains

$$
\int_0^t \int_0^1 \big| G^N(t-s, x, y) - G^N\big(t - \ell^M(s), x, y\big) \big|^2 \,\mathrm{d}y \,\mathrm{d}s \le C \tau^{\frac{1}{2}}.
$$

The value of C is independent of $N \in \mathbb{N}$, $t \in (0, T]$ and $x \in [0,1]$. The proof of the auxiliary inequality (25a) is thus completed.

Let us now prove the inequality (25b). It suffices to observe that $\min(1, \lambda_j^N \tau)^2 \le \lambda_j^N \tau$, thus one has

$$
\sum_{j=1}^{N-1} \frac{\min\big(1, \lambda_j^N \tau\big)^2}{\lambda_j^N} \le \sum_{j=1}^{N-1} \tau \le N\tau = \frac{\tau}{h}.
$$

As a result one obtains

$$
\int_0^t \int_0^1 \big| G^N(t-s, x, y) - G^N\big(t - \ell^M(s), x, y\big) \big|^2 \,\mathrm{d}y \,\mathrm{d}s \le C \frac{\tau}{h}.
$$

The value of C is independent of $N \in \mathbb{N}$, $t \in (0, T]$ and $x \in [0,1]$. The proof of the auxiliary inequality (25b) is thus completed. □

Appendix B. Proof of the inequality 9b

Proof of the auxiliary inequality (9b). One can decompose

$$u^N(t,x) = \int_0^1 G^N(t,x,y)u_0\big(\kappa^N(y)\big) + w^N(t,x),$$

where the auxiliary random field w^N is defined as follows: for all $t \in [0,T]$ and $x \in [0,1]$ set

$$w^N(t,x) = u^N(t,x) - \int_0^1 G^N(t,x,y)u_0\big(\kappa^N(y)\big)\,\mathrm{d}y = \int_0^t \int_0^1 G^N(t-r,x,y)g\big(u^N\big(r,\kappa^N(y)\big)\big)\,\mathrm{d}W(r,y).$$

First, using Proposition 2.4 in [2] with $\beta = 1$, one obtains the following result: there exists $C_T(u_0) \in (0,\infty)$ such that for all $0 \le s \le t \le T$ and all $x \in [0,1]$ one has

$$\begin{aligned}\left|\int_0^1 G^N(t,x,y)u_0\big(\kappa^N(y)\big)\,\mathrm{d}y - \int_0^1 G^N(s,x,y)u_0\big(\kappa^N(y)\big)\,\mathrm{d}y\right|^2 &\le C_T(u_0)\sum_{j=1}^{N-1}\frac{\min\big(1,\lambda_j^N(t-s)\big)^2}{\lambda_j^N}\\ &\le C_T(u_0)\frac{|t-s|}{h},\end{aligned}$$

using the same strategy as in the proof of the inequality (25b) in Appendix A above.

It remains to deal with the time increments of w^N: for all $0 \le s \le t \le T$ and all $x \in [0,1]$ one has

$$\begin{aligned} w^N(t,x) - w^N(s,x) &= \int_0^s \int_0^1 \big(G^N(t-r,x,y) - G^N(s-r,x,y)\big)g\big(u^N\big(r,\kappa^N(y)\big)\big)\,\mathrm{d}W(r,y)\\ &\quad + \int_s^t \int_0^1 G^N(t-r,x,y)g\big(u^N\big(r,\kappa^N(y)\big)\big)\,\mathrm{d}W(r,y).\end{aligned}$$

Applying Ito's isometry, one has

$$\begin{aligned}\mathbb{E}\Big[\big|w^N(t,x) - w^N(s,x)\big|^2\Big] &\le 2\int_0^s \int_0^1 \big|G^N(t-r,x,y) - G^N(s-r,x,y)\big|^2 \mathbb{E}\Big[\big|g\big(u^N\big(r,\kappa^N(y)\big)\big)\big|^2\Big]\,\mathrm{d}y\,\mathrm{d}r\\ &\quad + 2\int_s^t \int_0^1 \big|G^N(t-r,x,y)\big|^2 \mathbb{E}\Big[\big|g\big(u^N\big(r,\kappa^N(y)\big)\big)\big|^2\Big]\,\mathrm{d}y\,\mathrm{d}r\\ &\le C_T(u_0)\bigg(\int_0^s \int_0^1 \big|G^N(t-r,x,y) - G^N(s-r,x,y)\big|^2\,\mathrm{d}y\,\mathrm{d}r\\ &\quad + \int_s^t \int_0^1 \big|G^N(t-r,x,y)\big|^2\,\mathrm{d}y\,\mathrm{d}r\bigg),\end{aligned}$$

using the fact that g is a globally Lipschitz function and the moment bounds (10) for u^N.

To bound the two integrals above, the technique used in the proof of the inequality (25b) in Appendix A is used again. For the first integral, using the inequality $\min(1,\lambda_j^N(t-s))^2 \le \lambda_j^N|t-s|$, one has

$$\int_0^s \int_0^1 \big|G^N(t-r,x,y) - G^N(s-r,x,y)\big|^2\,\mathrm{d}y\,\mathrm{d}r \le C\sum_{j=1}^{N-1}\frac{\min\big(1,\lambda_j^N(t-s)\big)^2}{\lambda_j^N} \le C\frac{|t-s|}{h}\cdot$$

For the second integral, integrating and using the inequality $1-\exp(-x) \le x$ for $x \ge 0$, one has

$$\begin{aligned}\int_s^t \int_0^1 \big|G^N(t-r,x,y)\big|^2\,\mathrm{d}y\,\mathrm{d}r &\le C\sum_{j=1}^{N-1}\frac{1-e^{-2\lambda_j^N(t-s)}}{2\lambda_j^N} \le C\sum_{j=1}^{N-1}\frac{\lambda_j^N|t-s|}{\lambda_j^N}\\ &\le CN|t-s| \le C\frac{|t-s|}{h}\cdot\end{aligned}$$

As a result, one obtains for all $0 \leq s \leq t \leq T$ and all $x \in [0,1]$

$$\mathbb{E}\Big[\big|w^N(t,x) - w^N(s,x)\big|^2\Big] \leq C_T(u_0)\frac{|t-s|}{h}\cdot$$

Gathering the estimates, one obtains for all $N \geq 1$ and all $0 \leq s \leq t \leq T$

$$\sup_{x\in[0,1]} \mathbb{E}\Big[\big|u^N(t,x) - u^N(s,x)\big|^2\Big] \leq C_T(u_0)\frac{|t-s|}{h}$$

and the proof of the inequality (9b) is completed. □

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Data availability statement

The research data associated with this article are available in Zenodo, under the reference [22] https://doi.org/10.5281/zenodo.10300733.

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