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STABILITY AND SPACE/TIME CONVERGENCE OF STÖRMER-VERLET TIME INTEGRATION OF THE MIXED FORMULATION OF LINEAR WAVE EQUATIONS

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Abstract. This work focuses on the mixed formulation of linear wave equations. It provides a proof of stability and convergence of time discretisation of a semi discrete linear wave equation in mixed form with Störmer-Verlet time integration, that is uniform as the time step reaches its largest allowed value for stability (Courant-Friedrich-Levy condition), contrary to the proofs recalled here from the literature.

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1. Introduction

The mixed formulation of linear wave equations is one possible modeling of wave propagation phenomena. It is chosen over its second order formulation counterpart in situations where the unknowns are more relevant to the physical context, or where these unknowns are more natural for modeling purposes (as for instance coupling with other parts), or even where it is not possible to formulate the equations as a second order equation (as for instance in presence of intricate dissipative or nonlinear phenomena). This encompasses acoustic waves, elastodynamics, electromagnetic waves, etc. On an abstract level, this system reads, for $0 \leq t \leq T$,

$$p(0) = p_0, \tag{1.1a}$$

$$v(0) = v_0, \tag{1.1b}$$

$$\dot{p} + \mathcal{B}^* v = f, \tag{1.1c}$$

$$\dot{v} - \mathcal{B} p = \dot{g}, \tag{1.1d}$$

where $\mathcal{B} : U \subset P \to D$ is an operator and $\mathcal{B}^*$ its adjoint.

Example 1.1. The acoustic wave equation on a bounded domain in one dimension of space corresponds to choosing $\Omega = [0, 1]$, $P = L^2(\Omega)$, $U = H^1(\Omega)$, $D = L^2(\Omega)$ and $\mathcal{B} = -\nabla$.

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The source terms and the initial conditions are supposed regular enough so that the following hypothesis holds, in three Hilbert spaces P, $U = \{p \in P \,|\mathcal{B}p \in D\} \subset P$ and D:

Hypothesis 1.2 (Stability of the continuous system)**.** The source terms and the initial conditions are regular enough such that there exists a constant $C > 0$ such that

$$\|p\|_{\mathcal{C}^3(0,T;P)} + \|p\|_{\mathcal{C}^0(0,T;U)} + \|v\|_{\mathcal{C}^2(0,T;D)} \leq C. \tag{1.2}$$

Numerical methods to solve this system are numerous and can rely on several analysis tools. We want to focus on this work on the time discretisation with an interleaved scheme usually referred to as "centered explicit scheme" and which is formally equivalent to the Störmer-Verlet scheme, which is of primary importance for solving linear wave equations models because of its good mathematical properties, efficiency and ease of implementation [14]. We therefore suppose that the spatial discretisation is done with usual methods such as Finite Differences [18], Finite Elements [3], Finite Volumes [10], or any other method that provides the following semi discrete system, along with some necessary bounds on the space discretisation error. More precisely, we assume that, after following the steps that lead to the semi discrete system, the semi discrete solution (p_h, v_h) is sought in finite dimensional spaces $U_h \subset U \subset P$ and $D_h \subset D$: Find $p_h \in U_h$ and $v_h \in D_h$ such that

$$p_h(0) = p_{h,0}, \tag{1.3a}$$

$$v_h(0) = v_{h,0}, \tag{1.3b}$$

$$\dot{p}_h + B_h^* v_h = f_h, \tag{1.3c}$$

$$\dot{v}_h - B_h p_h = \dot{g}_h, \tag{1.3d}$$

where $B_h : U_h \to D_h$ is a discrete approximation of the operator $\mathcal{B}$, and $B_h^* : D_h \to U_h$ is its adjoint, and $p_{h,0}$, $v_{h,0}$, f_h and g_h are discrete representations of p_0, v_0, f and g in U_h and D_h. We also denote I_h the identity operator of U_h.

Example 1.3. A possible choice of spatial discretisation for the acoustic wave equation given in example 1.1 is to partition the domain Ω into N disjoint intervals $\{I_k\}_{1\leq k\leq N}$ of length at most h and to approximate p and v as p_h and v_h which are piecewise polynomial functions of degree r on each interval I_k, with p_h being continuous at the junction between intervals. This ensures that the approximation is conformal ($U_h \subset U$). In this example, the operator B_h is the restriction of $\mathcal{B}$ to U_h.

For the sequel, we will suppose that the spatial discretisation satisfies the

Hypothesis 1.4 (Stability of the semi discrete system)**.** The spatial discretisation is such that there exists a constant $C > 0$ independent of h such that

$$\|p_h\|_{\mathcal{C}^3(0,T;P)} + \|p_h\|_{\mathcal{C}^0(0,T;U)} + \|v_h\|_{\mathcal{C}^2(0,T;D)} \leq C. \tag{1.4}$$

Hypothesis 1.5 (Convergence of the semi discrete system)**.** The spatial discretisation is such that there exists a function $\delta : \mathbb{R}_*^+ \to \mathbb{R}^+$, such that

$$\|p - p_h\|_{\mathcal{C}^0(0,T;P)} + \|v - v_h\|_{\mathcal{C}^0(0,T;D)} \leq \delta(h), \quad \text{with } \delta(h) \underset{h\to 0}{\longrightarrow} 0. \tag{1.5}$$

Time is discretised using a centered Störmer-Verlet algorithm (see [8], Chap. 5, Eq. (5.12)) on staggered time grids with a time step Δt, which is formally equivalent to using the leap frog scheme on the second order system obtained after elimination of one equation, as developed further later. The unknown p_h is sought for on the grid $\{t^{n+\frac{1}{2}} = (n+\frac{1}{2})\Delta t\}_{1\leq n\leq N}$ while the unknown v_h is sought for on the grid $\{t^n = n\Delta t\}_{0\leq n\leq N}$. Let us introduce the discrete operators δ and μ defined as

$$\delta p_h^n = \frac{p_h^{n+\frac{1}{2}} - p_h^{n-\frac{1}{2}}}{\Delta t}, \quad \mu p_h^n = \frac{p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}}{2}, \quad \delta v_h^{n+\frac{1}{2}} = \frac{v_h^{n+1} - v_h^n}{\Delta t}, \quad \mu v_h^{n+\frac{1}{2}} = \frac{v_h^{n+1} + v_h^n}{2}. \tag{1.6}$$

They satisfy the useful following properties

$$v_h^{n-\frac{1}{2}\pm\frac{1}{2}} = \mu v_h^{n-\frac{1}{2}} \pm \frac{\Delta t}{2}\delta v_h^{n-\frac{1}{2}}, \tag{1.7a}$$

$$\|\delta X\| \leq \frac{2}{\Delta t}\mu\|X\|. \tag{1.7b}$$

The Störmer Verlet scheme reads: Find $\{p_h^{n-\frac{1}{2}}\}_{1\leq n\leq N+1} \in U_h^{N+1}$ and $\{v_h^n\}_{0\leq n\leq N+1} \in D_h^{N+2}$ such that

$$v_h^0 = v_h(0), \tag{1.8a}$$

$$p_h^{\frac{1}{2}} = p_h(0) + \frac{\Delta t}{2}\left[f_h^0 - B_h^* v_h(0)\right], \tag{1.8b}$$

$$\text{For } n \in [1, N],\ \delta p_h^n + B_h^* v_h^n = f_h^n, \tag{1.8c}$$

$$\text{For } n \in [0, N],\ \delta v_h^{n+\frac{1}{2}} - B_h p_h^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}}, \tag{1.8d}$$

where $f_h^n = f_h(t^n)$ and $g_h^n = g_h(t^n)$.

The motivations of the present paper are to provide proof of space/time convergence of the Störmer Verlet scheme in the mixed formulation of wave equations. In simple cases such as the one presented here, eliminating one unknown by applying the ∂_t (resp. δ) operator to (1.3c) (resp. (1.8c)) and using (1.3d) (resp. (1.8d)) to replace v_h yields a second order linear wave equation, if the source terms are regular enough:

$$\ddot{p}_h + B_h^* B_h p_h = \dot{f}_h - B_h^* \dot{g}_h,$$

and is equivalent to applying the leap frog scheme [7, 14]:

$$\delta^2 p_h^{n+\frac{1}{2}} + B_h^* B_h p_h^{n+\frac{1}{2}} = \delta f_h^{n+\frac{1}{2}} - B_h^* \delta g_h^{n+\frac{1}{2}}.$$

So, for a field u_h such that $u_h(0) = u_{h,0}$, $\dot{u}_h = p_h$, and its discrete counterpart $\{u_n^n\}_{0\leq n\leq N}$ such that $u_h^0 = u_{h,0}$, $\delta u_h^{n+\frac{1}{2}} = p_h^{n+\frac{1}{2}}$, the Störmer Verlet scheme on the mixed formulation amounts to a leap frog scheme on the second order wave equation

$$\ddot{u}_h + B_h^* B_h u_h = f_h - B_h^* g_h,$$

such that

$$\delta^2 u_h^{n+\frac{1}{2}} + B_h^* B_h u_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^* g_h^{n+\frac{1}{2}}.$$

Hence we expect similar estimates, at least on this remaining field, as those found in [5, 6] for the leap frog scheme applied to the second order wave equation: If the CFL stability condition is respected, *e.g.*

$$\eta \leq 1, \quad \text{where } \eta = \frac{\Delta t}{2}\sqrt{\rho(B_h^* B_h)}, \tag{CFL-condition}$$

where ρ stands for the spectral radius:

$$\rho(B_h^* B_h) = \sup_{p_h \in U_h} \frac{(B_h^* B_h p_h, p_h)_P}{(p_h, p_h)_P} = \sup_{p \in U_h} \frac{(B_h p_h, B_h p_h)_D}{(p_h, p_h)_P} = \|\!|B_h|\!\|^2, \tag{1.9}$$

then, there exists a constant $C < 0$ independent of h and η such that

$$\|\bar{u}_h^n - u_h^n\| \leq C\Delta t^2 \|u_h\|_{\mathcal{C}^4(0,T;P)}. \tag{1.10}$$

Existing proofs of space/time convergence of the Störmer Verlet scheme on the mixed formulation of wave equations are found in [13], [11], [15], [1], [9]. Some of them are quite generic, while others are performed for a very specific spatial discretisation, but in all those references, the authors either suppose that $\eta \le \frac{1}{2}$ or $\eta < 1$. Indeed, those proofs turn out to not be uniform as Δt approaches its greatest admissible value, in other words the stability and convergence bounding constants depend on η and blow up as $\eta \to 1$. Numerical evidence and equivalence to second order formulation suggest that this limitation is only technical, which should be possible to overcome theoretically.

Moreover, we aim at providing a uniform proof that does not eliminate one of the two unknowns, because this procedure could prevent to generalize the result to the addition of terms as coupling [1] or dissipation [2]. The paper is organized as follows. In Section 2, several identified proofs of stability are put in the same framework in order to exhibit their blow up as the time step approaches its greatest allowed value. Then, uniform proofs of stability (Props. 3.12 and 3.15 of Sect. 3) and convergence (Thm. 4.1 of Sect. 4) are proposed.

2. Stability far from the CFL

In this section, several proofs of the literature are put under the same form on a very simple discrete system, in order to show their underlying mechanisms and exhibit the fact that the bounding stability constants depend on η, and even blow up as $\eta \to 1$.
Let us manipulate the discrete equations in the following way. Let $(\alpha,\beta,\gamma) \in (\mathbb{R}^+)^3$. We take the scalar product of (1.8c)centered at time t^n with μp_h^n in U_h. We then apply the operator μ to (1.8d) at times $t^{n+\frac12}$ and $t^{n-\frac12}$, and take the scalar product with βv_h^n in D_h. We take the scalar product of (1.8d) centered at $t^{n-\frac12}$ with $\alpha\mu v_h^{n-\frac12}$ in D_h. Finally we take the scalar product of (1.8d) centered at $t^{n+\frac12}$ with $(\alpha+\gamma)\mu v_h^{n+\frac12}$ in D_h. We add up and get

$$\frac{\alpha}{2\Delta t}\left(\|v_h^n\|_D^2 - \|v_h^{n-1}\|_D^2\right) + \frac{\alpha+\gamma}{2\Delta t}\left(\|v_h^{n+1}\|_D^2 - \|v_h^n\|_D^2\right) + \frac{1}{2\Delta t}\left(\|p_h^{n+\frac12}\|_P^2 - \|p_h^{n-\frac12}\|_P^2\right) + \frac{\beta}{2\Delta t}\left((v_h^{n+1},v_h^n)_D - (v_h^n,v_h^{n-1})_D\right)$$
$$= \alpha(B_h p_h^{n-\frac12}, \mu v_h^{n-\frac12})_D + (\alpha+\gamma)(B_h p_h^{n+\frac12}, \mu v_h^{n+\frac12})_D + \beta(B_h\mu p_h^n, v_h^n)_D - (v_h^n, B_h\mu p_h^n)_D$$
$$+ \alpha(\delta g_h^{n-\frac12}, \mu v_h^{n-\frac12})_D + (\alpha+\gamma)(\delta g_h^{n+\frac12}, \mu v_h^{n+\frac12})_D + \beta(\mu\delta g_h^n, v_h^n)_D + (f_h^n, \mu p_h^n)_P. \quad (2.1)$$

This is a discrete energy variation identity

$$\delta\mathcal{E}^n_{\alpha,\beta,\gamma} = \alpha\left[(B_h p_h^{n-\frac12}, \mu v_h^{n-\frac12})_D + (B_h p_h^{n+\frac12}, \mu v_h^{n+\frac12})_D\right] + \gamma(B_h p_h^{n+\frac12}, \mu v_h^{n+\frac12})_D + (\beta-1)(B_h\mu p_h^n, v_h^n)_D$$
$$+ \alpha\left[(\delta g_h^{n-\frac12}, \mu v_h^{n-\frac12})_D + (\delta g_h^{n+\frac12}, \mu v_h^{n+\frac12})_D\right] + \gamma(\delta g_h^{n+\frac12}, \mu v_h^{n+\frac12})_D + \beta(\mu\delta g_h^n, v_h^n)_D + (f_h^n, \mu p_h^n)_P, \quad (2.2)$$

with

$$\mathcal{E}^{n-\frac12}_{\alpha,\beta,\gamma} = \frac12\|p_h^{n-\frac12}\|_P^2 + \alpha\left[\frac12\|v_h^{n-1}\|_D^2 + \frac12\|v_h^n\|_D^2\right] + \frac{\gamma}{2}\|v_h^n\|_D^2 + \frac{\beta}{2}(v_h^n, v_h^{n-1})_D. \quad (2.3)$$

Note that this identity is centered around $t^{n-\frac12}$ if and only if $\gamma \equiv 0$, which leads to interpret γ as an off-centering parameter.

2.1. Choice $\beta = 1$ and $\alpha = \gamma = 0$

The choice $\beta = 1$ and $\alpha = \gamma = 0$ is done in [14] and proof of Lemma 5.3 of [9]. It leads to a cross term in the energy

$$\mathcal{E}^{n-\frac12}_{0,1,0} = \frac12\|p_h^{n-\frac12}\|_P^2 + \frac12(v_h^n, v_h^{n-1})_D \equiv \frac12\|p_h^{n-\frac12}\|_P^2 + \frac12\|\mu v_h^{n-\frac12}\|_D^2 - \frac{\Delta t^2}{8}\|\delta v_h^{n-\frac12}\|_D^2, \quad (2.4)$$

by using (1.7a).

2.1.1. Special case

If $\delta g_h \equiv 0$, we have that $\delta v_h^{n-\frac12} = B_h p_h^{n-\frac12}$, therefore

$$\mathcal{E}_{0,1,0}^{n-\frac12} = \frac12 (\widetilde{I}_h p_h^{n-\frac12}, p_h^{n-\frac12})_P + \frac12 \|\mu v_h^{n-\frac12}\|_D^2, \tag{2.5}$$

with $\widetilde{I}_h = I_h - \frac{\Delta t^2}{4} B_h^* B_h$ a modified identity operator. It is positive under the usual (CFL-condition). Then if $\eta \le 1$,

$$\mathcal{E}_{0,1,0}^{n-\frac12} \ge \frac{1-\eta^2}{2} \|p_h^{n-\frac12}\|_P^2 \quad \text{ and } \mathcal{E}_{0,1,0}^{n-\frac12} \ge \frac12 \|\mu v_h^{n-\frac12}\|_D^2. \tag{2.6}$$

If moreover $\eta < 1$ (strict CFL condition), we have

$$\delta \mathcal{E}_{0,1,0}^n = (f_h^n, \mu p_h^n)_P \le \|f_h^n\|_P \, \mu \|p_h\|_P^n \le \|f_h^n\|_P \frac{\sqrt2}{\sqrt{1-\eta^2}} \mu (\sqrt{\mathcal{E}_{0,1,0}})^n. \tag{2.7}$$

A discrete summation from $n = 1$ to $n = N$ yields, after telescopic elimination,

$$\sqrt{\mathcal{E}_{0,1,0}^{N+\frac12}} \le \sqrt{\mathcal{E}_{0,1,0}^{\frac12}} + \frac{\sqrt2}{\sqrt{1-\eta^2}} \Delta t \sum_{n=1}^N \|f_h^n\|_P, \tag{2.8}$$

which grants the stability of the energy, under the strict CFL condition. It then provides the stability of $\|p_h^{n-\frac12}\|$ and $\|\mu v_h^{n-\frac12}\|$ by using (2.6). The stability constants blow up as $\eta \to 1$.

Remark 2.1. By introducing auxiliary functions on top of more usual elliptic projections, [9] write a system of equations on the error terms which are of the form of (1.8) with $\delta g_h = 0$ (see Eq. (5.5)–(5.6)). They suppose $\eta \le \frac12$ in the second part of their hypothesis (A4).

2.1.2. General case

If $\delta g_h \ne 0$, a Young's inequality with any $\varepsilon > 0$ leads to

$$\begin{aligned} \|\delta v_h^{n-\frac12}\|_D^2 = \|B_h p_h^{n-\frac12} + \delta g_h^{n-\frac12}\|_D^2 &= \|B_h p_h^{n-\frac12}\|_D^2 + 2(B_h p_h^{n-\frac12}, \delta g_h^{n-\frac12})_D + \|\delta g_h^{n-\frac12}\|_D^2 \\ &\underset{\text{Young}}{\le} \|B_h p_h^{n-\frac12}\|_D^2 + \|\delta g_h^{n-\frac12}\|_D^2 + \varepsilon \|B_h p_h^{n-\frac12}\|_D^2 + \frac1\varepsilon \|\delta g_h^{n-\frac12}\|_D^2 \\ &\le (1+\varepsilon) \|B_h p_h^{n-\frac12}\|_D^2 + (1+\frac1\varepsilon) \|\delta g_h^{n-\frac12}\|_D^2. \end{aligned} \tag{2.9}$$

Hence

$$\begin{aligned} \mathcal{E}_{0,1,0}^{n-\frac12} &\ge \frac12 \|p_h^{n-\frac12}\|_P^2 + \frac12 \|\mu v_h^{n-\frac12}\|_D^2 - \frac{\Delta t^2}{8} \left[(1+\varepsilon) \|B_h p_h^{n-\frac12}\|_D^2 + (1+\frac1\varepsilon) \|\delta g_h^{n-\frac12}\|_D^2 \right] \\ &\ge \frac12 (\widetilde{I}_{h,\varepsilon} \, p_h^{n-\frac12}, p_h^{n-\frac12})_P + \frac12 \|\mu v_h^{n-\frac12}\|_D^2 - \frac{\Delta t^2}{8} (1+\frac1\varepsilon) \|\delta g_h^{n-\frac12}\|_D^2, \end{aligned} \tag{2.10}$$

with $\widetilde{I}_{h,\varepsilon} = I_h - \frac{\Delta t^2}{4}(1+\varepsilon) B_h^* B_h$ a modified identity operator. It is positive under the ε-CFL condition

$$\Delta t^2 \rho(B_h^* B_h) \le \frac{4}{1+\varepsilon} < 4. \tag{2.11}$$

Recall the definition of η from (CFL-condition). The ε-CFL condition is equivalent to supposing that $\eta < 1$ (strict CFL condition), since it suffices to pose $\varepsilon = \frac{1-\eta^2}{1+\eta^2} > 0$ to satisfy (2.11). Therefore

$$\mathcal{E}_{0,1,0}^{n-\frac12} + \frac{\Delta t^2}{4} \frac{1}{1-\eta^2} \|\delta g_h^{n-\frac12}\|_D^2 \ge \frac{1-\eta^2}{1+\eta^2} \|p_h^{n-\frac12}\|_P^2 + \frac12 \|\mu v_h^{n-\frac12}\|_D^2 \ge 0. \tag{2.12}$$

We then have

$$\delta\mathcal{E}^n_{0,1,0} = (\mu\delta g^n_h, v^n_h)_D + (f^n_h, \mu p^n_h)_P. \tag{2.13}$$

The first RHS term can not be bounded by the square root of the energy since $\|v^n_h\|_D \not\lesssim \|\mu v_h^{n-\frac{1}{2}}\|_D$. We cannot conclude.

2.2. Choice $\alpha = \frac{1}{2}$, $\beta = \gamma = 0$

The choice $\alpha = \frac{1}{2}$, $\beta = 0$ and $\gamma = 0$ is done in [1]. This leads to

$$\mathcal{E}^{n-\frac{1}{2}}_{\frac{1}{2},0,0} = \frac{1}{2}\|p_h^{n-\frac{1}{2}}\|^2_P + \frac{1}{4}\|v_h^{n-1}\|^2_D + \frac{1}{4}\|v^n_h\|^2_D, \tag{2.14}$$

which is positive by definition, and satisfies

$$\begin{aligned}
\delta\mathcal{E}^n_{\frac{1}{2},0,0} &= \frac{1}{2}(B_h p_h^{n-\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D + \frac{1}{2}(B_h p_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D - (B_h \mu p^n_h, v^n_h)_D \\
&\quad + \frac{1}{2}\left[(\delta g_h^{n-\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D + (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D\right] + (f^n_h, \mu p^n_h)_P \qquad (2.15)\\
&= \frac{1}{2}(B_h p_h^{n-\frac{1}{2}}, \frac{v^n_h + v^{n-1}_h}{2})_D + \frac{1}{2}(B_h p_h^{n+\frac{1}{2}}, \frac{v^n_h + v^{n+1}_h}{2})_D - (B_h \frac{p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}}{2}, v^n_h)_D + \dots \qquad (2.16)\\
&= \frac{1}{4}(B_h p_h^{n-\frac{1}{2}}, v_h^{n-1})_D + \frac{1}{4}(B_h p_h^{n+\frac{1}{2}}, v_h^{n+1})_D - \frac{1}{4}(B_h p_h^{n+\frac{1}{2}}, v^n_h)_D - \frac{1}{4}(B_h p_h^{n-\frac{1}{2}}, v^n_h)_D + \dots \qquad (2.17)\\
&= \frac{\Delta t}{4}(B_h p_h^{n+\frac{1}{2}}, \delta v_h^{n+\frac{1}{2}})_D - \frac{\Delta t}{4}(B_h p_h^{n-\frac{1}{2}}, \delta v_h^{n-\frac{1}{2}})_D + \frac{1}{2}\left[(\delta g_h^{n-\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D + (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D\right] \\
&\quad + (f^n_h, \mu p^n_h)_P. \qquad (2.18)
\end{aligned}$$

The two first terms of the right-hand side of the previous equation have no defined sign but appear to behave as perturbations of the positive energy (2.14). This prompts to denoting

$$\widetilde{\mathcal{E}}^{n-\frac{1}{2}}_{\frac{1}{2},0,0} = \mathcal{E}^{n-\frac{1}{2}}_{\frac{1}{2},0,0} - \frac{\Delta t^2}{4}(B_h p_h^{n-\frac{1}{2}}, \delta v_h^{n-\frac{1}{2}})_D = \frac{1}{2}\|p_h^{n-\frac{1}{2}}\|^2_P + \frac{1}{4}\|v_h^{n-1}\|^2_D + \frac{1}{4}\|v^n_h\|^2_D - \frac{\Delta t^2}{4}(B_h p_h^{n-\frac{1}{2}}, \delta v_h^{n-\frac{1}{2}})_D, \tag{2.19}$$

which has no defined sign. If $\delta g_h^{n-\frac{1}{2}} = 0$, it is immediately positive under the usual (CFL-condition) since $\delta v_h^{n-\frac{1}{2}} = B_h p_h^{n-\frac{1}{2}}$. Otherwise, the analysis of the positivity (or more precisely the boundedness-by-below) of $\widetilde{\mathcal{E}}^n_{\frac{1}{2},0,0}$ relies on bounding $(B_h p_h^{n-\frac{1}{2}}, \delta v_h^{n-\frac{1}{2}})_D$ with the same techniques as above (Cauchy-Schwarz and Young's inequalities):

$$(B_h p_h^{n-\frac{1}{2}}, \delta v_h^{n-\frac{1}{2}})_D \leq (1+\varepsilon)\|B_h p_h^{n-\frac{1}{2}}\|^2_D + \frac{1}{4\varepsilon}\|\delta g_h^{n-\frac{1}{2}}\|^2_D. \tag{2.20}$$

We get

$$\widetilde{\mathcal{E}}^{n-\frac{1}{2}}_{\frac{1}{2},0,0} \geq \frac{1}{2}(\widetilde{I}_{h,\varepsilon}\, p_h^{n-\frac{1}{2}}, p_h^{n-\frac{1}{2}})_P + \frac{1}{4}\|v_h^{n-1}\|^2_D + \frac{1}{4}\|v^n_h\|^2_D - \frac{\Delta t^2}{16\varepsilon}\|\delta g_h^{n-\frac{1}{2}}\|^2_D, \tag{2.21}$$

with $\widetilde{I}_{h,\varepsilon} = I_h - \frac{\Delta t^2}{4}(1+\varepsilon)B^*_h B_h$ a modified identity operator which is positive under the ε-CFL condition (2.11), *i.e.* under the strict CFL condition $\eta < 1$. We get that

$$\widetilde{\mathcal{E}}^{n-\frac{1}{2}}_{\frac{1}{2},0,0} + \frac{\Delta t^2}{16}\frac{1+\eta^2}{1-\eta^2}\|\delta g_h^{n-\frac{1}{2}}\|^2_D \geq \frac{1-\eta^2}{1+\eta^2}\|p_h^{n-\frac{1}{2}}\|^2_P + \frac{1}{4}\|v_h^{n-1}\|^2_D + \frac{1}{4}\|v^n_h\|^2_D \geq \frac{1-\eta^2}{1+\eta^2}\|p_h^{n-\frac{1}{2}}\|^2_P + \frac{1}{2}\|\mu v_h^{n-\frac{1}{2}}\|^2_D \geq 0. \tag{2.22}$$

The modified energy satisfies

$$\delta\widetilde{\mathcal{E}}^n_{\frac{1}{2},0,0} = \frac{1}{2}\left[(\delta g_h^{n-\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D + (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D\right] + (f^n_h, \mu p^n_h)_P. \tag{2.23}$$

A discrete summation from 0 to N yields the stability of the energy. Under the strict CFL condition, it yields the stability of $\|p_h^{n-\frac{1}{2}}\|_P$ and $\|\mu v_h^{n-\frac{1}{2}}\|_D$ by using (2.22). This method improves the previous one since it is valid with source terms on each equation. However, the stability constants still blow up as $\eta \to 1$.

2.3. Choice $\alpha = \beta = 0$ and $\gamma = 1$

The choice $\beta = 0$, $\alpha = 0$ and $\gamma = 1$ is done in the convergence proof of [13] and is remarkable since it breaks the symmetry of the manipulation. This leads to

$$\mathcal{E}_{0,0,1}^{n-\frac{1}{2}} = \frac{1}{2}\|p_h^{n-\frac{1}{2}}\|_P^2 + \frac{1}{2}\|v_h^n\|_D^2, \tag{2.24}$$

with

$$\begin{aligned}\delta\mathcal{E}_{0,0,1}^n &= (B_h p_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D - (B_h \mu p_h^n, v_h^n)_D + (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D + (f_h^n, \mu p_h^n)_P \\ &= (B_h p_h^{n+\frac{1}{2}}, \frac{v_h^{n+1} + v_h^n}{2})_D - (B_h \frac{p_h^{n+\frac{1}{2}} + p_h^{n-\frac{1}{2}}}{2}, v_h^n)_D + (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D + (f_h^n, \mu p_h^n)_P \\ &= \frac{1}{2}(B_h p_h^{n+\frac{1}{2}}, v_h^{n+1})_D - \frac{1}{2}(B_h p_h^{n-\frac{1}{2}}, v_h^n)_D + (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D + (f_h^n, \mu p_h^n)_P.\end{aligned} \tag{2.25}$$

We denote

$$\widetilde{\mathcal{E}}_{0,0,1}^{n-\frac{1}{2}} = \mathcal{E}_{0,0,1}^{n-\frac{1}{2}} - \frac{\Delta t}{2}(B_h p_h^{n-\frac{1}{2}}, v_h^n)_D = \frac{1}{2}\|p_h^{n-\frac{1}{2}}\|_P^2 + \frac{1}{2}\|v_h^n\|_D^2 - \frac{\Delta t}{2}(B_h p_h^{n-\frac{1}{2}}, v_h^n)_D, \tag{2.26}$$

which has no definite sign. However, a Young's inequality with $\varepsilon > 0$ yields

$$(B_h p_h^{n-\frac{1}{2}}, v_h^n)_D \le \|B_h p_h^{n-\frac{1}{2}}\|_D \; \|v_h^n\|_D \le |||B_h||| \; \|p_h^{n-\frac{1}{2}}\|_P \; \|v_h^n\|_D \le |||B_h||| \left[\frac{\varepsilon}{2}\|p_h^{n-\frac{1}{2}}\|_P^2 + \frac{1}{2\varepsilon}\|v_h^n\|_D^2\right]. \tag{2.27}$$

Hence

$$\widetilde{\mathcal{E}}_{0,0,1}^{n-\frac{1}{2}} \ge \frac{1}{2}\left(1 - \frac{\Delta t |||B_h||| \varepsilon}{2}\right)\|p_h^{n-\frac{1}{2}}\|_P^2 + \frac{1}{2}\left(1 - \frac{\Delta t |||B_h|||}{2\varepsilon}\right)\|v_h^n\|_D^2. \tag{2.28}$$

The largest positivity condition is obtained with $\varepsilon = 1$ and reads $\Delta t \, |||B_h||| \le 2$. Under the usual (CFL-condition) $\eta \le 1$, we have

$$\widetilde{\mathcal{E}}_{0,0,1}^{n-\frac{1}{2}} \ge \frac{1}{2}(1-\eta)\|p_h^{n-\frac{1}{2}}\|_P^2 + \frac{1}{2}(1-\eta)\|v_h^n\|_D^2 \ge 0. \tag{2.29}$$

Note that the modified energy is non-negative even in the equality case. It satisfies

$$\delta\widetilde{\mathcal{E}}_{0,0,1}^n = (\delta g_h^{n+\frac{1}{2}}, \mu v_h^{n+\frac{1}{2}})_D + (f_h^n, \mu p_h^n)_P, \tag{2.30}$$

which is bounded by the square root of a the modified energy with constants that blow up as $\eta \to 1$. This methods improves the previous ones since a discrete summation yields the stability of a sharper modified energy under the strict CFL condition for source terms on each equation. Moreover, it provides bounds on $\|p_h^{n-\frac{1}{2}}\|_P$ and $\|v_h^n\|_D$ directly. However, the stability constants still blow up as $\eta \to 1$.

3. Stability at the CFL

We provide here two strategies to obtain stability even as $\eta \to 1$. This will not be possible directly on the unknowns of the system, but on post averaged or post processed quantities.

3.1. Case $\delta g_h \equiv 0$

We follow the ideas developed for the second order equation in [14]. The leading principle is to use the second equation of (2.6) on two consecutive time steps to compensate the crossing term appearing in a new energy obtained on the system (1.8) averaged between two time steps. More precisely,

$$\mathcal{H}^n_{0,1,0} = \frac{1}{2}\|\mu p_h^n\|_P^2 + \frac{1}{2}(\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D \tag{3.1}$$

satisfies

$$\delta \mathcal{H}^{n-\frac{1}{2}}_{0,1,0} = (\mu f_h^{n-\frac{1}{2}}, \mu^2 p_h^{n-\frac{1}{2}})_P. \tag{3.2}$$

Let us pose

$$\begin{aligned}\mathcal{F}^n_{0,1,0} &= \mathcal{H}^n_{0,1,0} + \mathcal{E}^{n+\frac{1}{2}}_{0,1,0} + \mathcal{E}^{n-\frac{1}{2}}_{0,1,0} = \frac{1}{2}\|\mu p_h^n\|_P^2 + \frac{1}{2}(\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D + \mathcal{E}^{n+\frac{1}{2}}_{0,1,0} + \mathcal{E}^{n-\frac{1}{2}}_{0,1,0} \\ &\geq_{(2.6)} \frac{1}{2}\|\mu p_h^n\|_P^2 + \frac{1}{2}(\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D + \frac{1}{2}\|\mu v_h^{n+\frac{1}{2}}\|_D^2 + \frac{1}{2}\|\mu v_h^{n-\frac{1}{2}}\|_D^2 \quad \text{if } \eta \leq 1 \\ &\geq \frac{1}{2}\|\mu p_h^n\|_P^2,\end{aligned} \tag{3.3}$$

because

$$2(\mu v_h^{n+\frac{1}{2}}, \mu v_h^{n-\frac{1}{2}})_D = \|\mu v_h^{n+\frac{1}{2}} + \mu v_h^{n-\frac{1}{2}}\|_D^2 - \|\mu v_h^{n+\frac{1}{2}}\|_D^2 - \|\mu v_h^{n-\frac{1}{2}}\|_D^2. \tag{3.4}$$

If satisfies

$$\begin{aligned}\delta \mathcal{F}^{n-\frac{1}{2}}_{0,1,0} &= (\mu f_h^{n-\frac{1}{2}}, \mu^2 p_h^{n-\frac{1}{2}})_P + (f_h^n, \mu p_h^n)_P + (f_h^{n-1}, \mu p_h^{n-1})_P \\ &\leq \|\mu f_h^{n-\frac{1}{2}}\|_P \, \|\mu^2 p_h^{n-\frac{1}{2}}\|_P + \|f_h^n\|_P \, \|\mu p_h^n\|_P + \|f_h^{n-1}\|_P \, \|\mu p_h^{n-1}\|_P \\ &\leq \left[\frac{1}{2}\|\mu f_h^{n-\frac{1}{2}}\|_P + \|f_h^n\|_P\right] \|\mu p_h^n\|_P + \left[\frac{1}{2}\|\mu f_h^{n-\frac{1}{2}}\|_P + \|f_h^{n-1}\|_P\right] \|\mu p_h^{n-1}\|_P \\ &\leq \left[\frac{1}{2}\|\mu f_h^{n-\frac{1}{2}}\|_P + \|f_h^n\|_P\right] \sqrt{2\mathcal{F}^n_{0,1,0}} + \left[\frac{1}{2}\|\mu f_h^{n-\frac{1}{2}}\|_P + \|f_h^{n-1}\|_P\right] \sqrt{2\mathcal{F}^{n-1}_{0,1,0}} \\ &\leq \left[\|\mu f_h^{n-\frac{1}{2}}\|_P + \|f_h^n\|_P + \|f_h^{n-1}\|_P\right] \sqrt{2} \left[\sqrt{\mathcal{F}^n_{0,1,0}} + \sqrt{\mathcal{F}^{n-1}_{0,1,0}}\right].\end{aligned} \tag{3.5}$$

Adding the telescopic sum from $n = 1$ to $n = N$ we get

$$\sqrt{\mathcal{F}^N_{0,1,0}} \leq \sqrt{\mathcal{F}^{\frac{1}{2}}_{0,1,0}} + \Delta t \sum_{n=1}^{N} \|\mu f_h^{n-\frac{1}{2}}\|_P + \|f_h^n\|_P + \|f_h^{n-1}\|_P, \tag{3.6}$$

which yields uniform stability estimates on the averaged fields:

$$\|\mu p_h^n\|_P \leq \sqrt{2}\sqrt{\mathcal{F}^n_{0,1,0}} \quad \text{and} \quad \|\mu v_h^{n+\frac{1}{2}}\|_D^2 + \|\mu v_h^{n-\frac{1}{2}}\|_D^2 \leq 2\mathcal{F}^n_{0,1,0}. \tag{3.7}$$

However, trying to bound directly p_h only gets that

$$\|p_h^{n-\frac{1}{2}}\|_P^2 + \|p_h^{n+\frac{1}{2}}\|_P^2 \leq \frac{2}{1-\eta^2}\mathcal{F}^n_{0,1,0}, \tag{3.8}$$

which blows up as $\eta \to 1$.

3.2. General case

In this section, we apply the methodology developed in [5, 6] and adapt it to the mixed formulation and the Störmer-Verlet scheme. It is based on the manipulation described above, with $\beta = 1$ and $\alpha = \gamma = 0$ as in [14] and proof of Lemma 5.3 [9]. A naive manipulation would be to take the scalar product of (1.8c) with μp_h^n and of μ(1.8d) with $v_h^{n+\frac{1}{2}}$. One would get that

$$\delta\mathcal{E}_{pv,h}^n = (f_h^n, \mu p_h^n)_P + (\mu\delta g_h^n, v_h^n)_D \quad \text{with} \quad \mathcal{E}_{pv,h}^{n+\frac{1}{2}} = \frac{1}{2}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2}(v_h^{n+1}, v_h^n)_D. \tag{3.9}$$

Transferring the usual concepts for energy analysis directly here would imply to use (1.7a) and (1.8d) to replace the term $\Delta t^2(\delta v_h^{n+\frac{1}{2}}, \delta v_h^{n+\frac{1}{2}})_D/4$. However it induces a difficulty due to the presence of the source term δg_h, as mentionned in Section 2.1.2. To mitigate this issue, let us proceed to a change of variables and suppose that g_h is regular enough uniformly with h, which occurs for the classical discretization described in Example 1.3 as long as the source term is spatially distributed in the domain with a regularity related to the polynomial order r:

Hypothesis 3.1. In the sequel we will suppose that there exists a constant $C_g > 0$ such that for all $h > 0$,

$$\|B_h^* g_h\|_{\mathcal{C}^0(0,T,P)} \le C_g. \tag{3.10}$$

The couple $(\{p_h^{n+\frac{1}{2}}\}_n, \{w_h^n\}_n)$ defined as $w_h^n = v_h^n - g_h^n \in D_h$ is solution to

$$w_h^0 = v_h^0 - g_h^0, \tag{3.11a}$$

$$p_h^{\frac{1}{2}} = p_h(0) + \frac{\Delta t}{2}\left[f_h^0 - B_h^* v_h(0)\right], \tag{3.11b}$$

$$\delta p_h^n + B_h^* w_h^n = f_h^n - B_h^* g_h^n, \tag{3.11c}$$

$$\delta w_h^{n+\frac{1}{2}} - B_h p_h^{n+\frac{1}{2}} = 0. \tag{3.11d}$$

Proposition 3.2 (Discrete power balance)**.** *Any solution to* (1.8) *satisfies, for* $n \in [1, N]$,

$$\delta\mathcal{E}_{pw,h}^n = (f_h^n, \mu p_h^n)_P - (B_h^* g_h^n, \mu p_h^n)_P \quad \textit{with} \quad \mathcal{E}_{pw,h}^{n+\frac{1}{2}} = \frac{1}{2}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2}(w_h^{n+1}, w_h^n)_D, \tag{3.12}$$

where $w_h^n = v_h^n - g_h^n$.

Proof. Take the scalar product of (3.11c) with $\mu p_h^n \in U_h$ and of $\mu(3.11d)^n$ with $w_h^n \in D_h$. □

Proposition 3.3 (Reformulation of the energy)**.** *The discrete energy* $\mathcal{E}_{pw,h}^{n+\frac{1}{2}}$ *also writes*

$$\mathcal{E}_{pw,h}^{n+\frac{1}{2}} = \frac{1}{2}\tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2}(\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D, \tag{3.13}$$

where $\tilde{m}$ *is the bilinear form defined on* U_h *such that for all* $(p, q) \in U_h \times U_h$,

$$\tilde{m}(p, q) = (p, q)_P - \frac{\Delta t^2}{4}(B_h p, B_h q)_D = (\widetilde{\mathcal{I}}_h p, q)_P \quad \textit{where} \;\; \widetilde{\mathcal{I}}_h = \mathcal{I}_h - \frac{\Delta t^2}{4}B_h^* B_h. \tag{3.14}$$

Proof. Using (1.7a) we get

$$(w_h^{n+1}, w_h^n)_D = (\mu w_h^{n+\frac{1}{2}} + \frac{\Delta t}{2}\delta w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}} - \frac{\Delta t}{2}\delta w_h^{n+\frac{1}{2}})_D \tag{3.15}$$

$$= (\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D - \frac{\Delta t^2}{4}(\delta w_h^{n+\frac{1}{2}}, \delta w_h^{n+\frac{1}{2}})_D \tag{3.16}$$

$$\underset{(3.11d)}{=} (\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D - \frac{\Delta t^2}{4}(B_h p_h^{n+\frac{1}{2}}, B_h p_h^{n+\frac{1}{2}})_D. \tag{3.17}$$

Hence

$$\mathcal{E}_{pw,h}^{n+\frac{1}{2}} = \frac{1}{2}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})_P + \frac{1}{2}(\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D - \frac{\Delta t^2}{4}(B_h p_h^{n+\frac{1}{2}}, B_h p_h^{n+\frac{1}{2}})_D \tag{3.18}$$

$$= \frac{1}{2}\tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \frac{1}{2}(\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D. \tag{3.19}$$

□

Proposition 3.4 (CFL condition)**.** *The energy* (3.13) *is positive if*

$$\eta \leq 1, \tag{3.20}$$

where η is defined in (CFL-condition)*.*

Corollary 3.5. *Suppose* (3.20) *is satisfied. Then,*

$$\tilde{m}(p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) \leq 2\mathcal{E}_{pw,h}^{n+\frac{1}{2}}, \quad (\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D \leq 2\mathcal{E}_{pw,h}^{n+\frac{1}{2}}. \tag{3.21}$$

To show the stability uniformly as Δt approaches its greatest allowed value, let us exploit the spectral properties of the operator $B_h^* B_h$ as in [5].

Proposition 3.6 (Spectral decomposition of $B_h^* B_h$)**.** *The operator $B_h^* B_h : U_h \to U_h$ is diagonalizable in $\mathbb{R}$. We call $(\lambda_{h,i}, e_{h,i})$ its eigenpairs which are chosen orthonormal in P.*

Proof. In finite dimensional spaces, any symmetric real operator is diagonalizable in an orthonormal basis. □

Following [5], we introduce the polynomial

$$P_k(x) = 1 - \frac{1}{4}x, \tag{3.22}$$

which is non-negative on the interval $[0, 4]$. Then $\widetilde{\mathcal{I}}_h$ can be expressed as a polynomial of the operator $\Delta t^2 B_h^* B_h$:

$$\widetilde{\mathcal{I}}_h = \mathcal{I}_h - \frac{\Delta t^2}{4} B_h^* B_h = P_k(\Delta t^2 B_h^* B_h). \tag{3.23}$$

Note that this polynomial appears naturally in the definition of the bilinear form $\tilde{m}$ which refined the kinetic energy part of the discrete energy, hence the subscript "k" in the polynomial notation. For more intricate integration schemes, another polynomial P_p is introduced to treat the potential part of the discrete energy, see [5]. Also note that the polynomial is used with $\Delta t^2 B_h^* B_h$ as argument to define $\widetilde{\mathcal{I}}_h$, which relates the upper bound of the interval $[0, 4]$ to the (CFL-condition). If Δt reaches its greatest admissible value, the operator $\widetilde{\mathcal{I}}_h$ has a kernel, which prevents from using the kinetic part of the energy as a norm on the solution. This is why a uniform bound is sought for through a partitioning of the interval $[0, 4]$, that will allow a uniform control on the different spectral components of the discrete solution.

Lemma 3.7 (Partitioning)**.** *The interval $[0, 4]$ can be partitioned as $J_k \cup J_p$ with $J_k \cap J_p = \emptyset$ such that there exist $C_k > 0$ and $C_p > 0$ such that for all $x \in J_k$, $P_k(x) \geq C_k$ and for all $x \in J_p$, $x \geq C_p$.*

Proof. See appendix of [5] for the scheme called "TS" with $\theta = 0$. We get $J_k = [0, 2]$, $J_p = [2, 4]$, $C_k = \frac{1}{2}$ and $C_p = 2$. □

Proposition 3.8. *Let us define the two projectors Π_k and Π_p such that for all $q_h \in U_h$*

$$\Pi_k q_h = \sum_{\substack{\Delta t^2\lambda_{h,i}\in J_k\\ \lambda_{h,i}\in\rho(\mathcal{B}_h^* B_h)}} (q_h, e_{h,i})_P\, e_{h,i}, \qquad \Pi_p q_h = \sum_{\substack{\Delta t^2\lambda_{h,i}\in J_p\\ \lambda_{h,i}\in\rho(\mathcal{B}_h^* B_h)}} (q_h, e_{h,i})_P\, e_{h,i}. \tag{3.24}$$

where ρ stands for the spectral radius of the discrete operator (see Eq. (1.9)). Then, for all $q_h \in U_h$

$$\|\Pi_k q_h\|_P^2 \le C_k^{-1}(\widetilde{\mathcal{I}}_h q_h, q_h)_P, \tag{3.25a}$$
$$\|\Pi_p q_h\|_P^2 \le \Delta t^2 C_p^{-1}(B_h q_h, B_h q_h)_D. \tag{3.25b}$$

Proof. See [5]. □

Proposition 3.9. *Any $(p_h, w_h) \in U_h \times D_h$ solution to (3.11) satisfies*

$$\|\Pi_k \mu p_h^n\|_P \le \frac{C_k^{-1/2}}{2}\left(\sqrt{2\mathcal{E}_{pw,h}^{n+\frac12}} + \sqrt{2\mathcal{E}_{pw,h}^{n-\frac12}}\right) \tag{3.26a}$$

$$\|\Pi_p \mu p_h^n\|_P \le C_p^{-1/2}\left(\sqrt{2\mathcal{E}_{pw,h}^{n+\frac12}} + \sqrt{2\mathcal{E}_{pw,h}^{n-\frac12}}\right) \tag{3.26b}$$

Proof. Take $q_h = \mu p_h^n$ in (3.25a). Using the triangular inequality and (3.21), we directly have

$$(\widetilde{\mathcal{I}}_h \mu p_h^n, \mu p_h^n)_P^{1/2} \le \frac12(\widetilde{\mathcal{I}}_h p_h^{n+\frac12}, p_h^{n+\frac12})_P^{1/2} + \frac12(\widetilde{\mathcal{I}}_h p_h^{n-\frac12}, p_h^{n-\frac12})_P^{1/2} \le \frac12\sqrt{2\mathcal{E}_{pw,h}^{n+\frac12}} + \frac12\sqrt{2\mathcal{E}_{pw,h}^{n-\frac12}}. \tag{3.27}$$

Take $q_h = \mu p_h^n$ in (3.25b). Since $\delta\mu w_h^n = B_h \mu p_h^n$ from μ(3.11d),

$$\Delta t(B_h\mu p_h^n, B_h\mu p_h^n)_D^{1/2} = \Delta t(\delta\mu w_h^n, \delta\mu w_h^n)_D^{1/2} \underset{(1.7b)}{\le} \left((\mu w_h^{n+\frac12}, \mu w_h^{n+\frac12})_D^{1/2} + (\mu w_h^{n-\frac12}, \mu w_h^{n-\frac12})_D^{1/2}\right). \tag{3.28}$$

From (3.21) we then have

$$\Delta t(B_h\mu p_h^n, B_h\mu p_h^n)_D^{1/2} \le \sqrt{2\mathcal{E}_{pw,h}^{n+\frac12}} + \sqrt{2\mathcal{E}_{pw,h}^{n-\frac12}}. \tag{3.29}$$

□

Proposition 3.10 (Majoration of the averaged unknowns)**.**

$$\|\mu w_h^{n+\frac12}\|_D \le \sqrt{2\mathcal{E}_{pw,h}^{n+\frac12}}, \quad \|\mu p_h^n\|_P \le \gamma\left(\sqrt{2\mathcal{E}_{pw,h}^{n+\frac12}} + \sqrt{2\mathcal{E}_{pw,h}^{n-\frac12}}\right), \tag{3.30}$$

where

$$\gamma = \frac{C_k^{-1/2}}{2} + C_p^{-1/2}. \tag{3.31}$$

Proof. For $\mu w_h^{n+\frac12}$ the result is immediate using (3.21). For μp_h^n we decompose it as $\Pi_k\mu p_h^n + \Pi_p\mu p_h^n$, and use the orthogonality between the projection spaces. We then use (3.26) to get the expected result. □

Proposition 3.11 (Stability of the energy)**.** *The energy satisfies*

$$\sqrt{\mathcal{E}_{pw,h}^{n+\frac12}} \le \sqrt{\mathcal{E}_{pw,h}^{\frac12}} + \sqrt{2}\gamma\Delta t\sum_{k=1}^{n}\left[\|f_h^k\|_P + \|B_h^* g_h^k\|_P\right], \tag{3.32}$$

where

$$\sqrt{\mathcal{E}_{pw,h}^{\frac12}} \le \frac{1+\sqrt2}{2}\|p_h(0)\|_P + \frac{1+\sqrt2}{2}\frac{\Delta t}{2}\|f_h^0\|_P + \frac{1+2\sqrt2}{2}\|v_h(0)\|_D + \frac{1}{\sqrt2}\|g_h^0\|_D. \tag{3.33}$$

Proof. Apply the Cauchy-Schwarz inequality on (3.12). Since g_h satisfies Hypothesis 3.1, we have

$$\frac{\mathcal{E}_{pw,h}^{n+\frac{1}{2}} - \mathcal{E}_{pw,h}^{n-\frac{1}{2}}}{\Delta t} \leq \left(\|f_h^n\|_P + \|B_h^* g_h^n\|_P\right) \|\mu p_h^n\|_P. \tag{3.34}$$

Use (3.30) to get

$$\frac{\left(\sqrt{\mathcal{E}_{pw,h}^{n+\frac{1}{2}}} + \sqrt{\mathcal{E}_{pw,h}^{n-\frac{1}{2}}}\right)\left(\sqrt{\mathcal{E}_{pw,h}^{n+\frac{1}{2}}} - \sqrt{\mathcal{E}_{pw,h}^{n-\frac{1}{2}}}\right)}{\Delta t} \leq \left(\|f_h^n\|_P + \|B_h^* g_h^n\|_P\right) \gamma\sqrt{2}\left(\sqrt{\mathcal{E}_{pw,h}^{n+\frac{1}{2}}} + \sqrt{\mathcal{E}_{pw,h}^{n-\frac{1}{2}}}\right). \tag{3.35}$$

Cancel out to get

$$\sqrt{\mathcal{E}_{pw,h}^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_{pw,h}^{n-\frac{1}{2}}} + \Delta t \left(\|f_h^n\|_P + \|B_h^* g_h^n\|_P\right) \|\gamma\sqrt{2}. \tag{3.36}$$

Add this telescopic sum from $k = 1$ to n to get the first expected result:

$$\sqrt{\mathcal{E}_{pw,h}^{n+\frac{1}{2}}} \leq \sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^{n} \left[\|f_h^k\|_P + \|B_h^* g_h^k\|_P\right]. \tag{3.37}$$

Finally, let us express $\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}}$ by using the expression (3.13):

$$\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} = \left[\frac{1}{2}\tilde{m}(p_h^{\frac{1}{2}}, p_h^{\frac{1}{2}}) + \frac{1}{2}(\mu w_h^{\frac{1}{2}}, \mu w_h^{\frac{1}{2}})_D\right]^{1/2} \leq \frac{1}{\sqrt{2}}\tilde{m}(p_h^{\frac{1}{2}}, p_h^{\frac{1}{2}})^{1/2} + \frac{1}{\sqrt{2}}\|\mu w_h^{\frac{1}{2}}\|_D. \tag{3.38}$$

First,

$$\tilde{m}(p_h^{\frac{1}{2}}, p_h^{\frac{1}{2}}) = \frac{1}{2}\|p_h^{\frac{1}{2}}\|_P^2 - \frac{\Delta t^2}{4}\|B_h p_h^{\frac{1}{2}}\|_D^2 \leq \frac{1}{2}\|p_h^{\frac{1}{2}}\|_P^2. \tag{3.39}$$

Moreover,

$$\|\mu w_h^{\frac{1}{2}}\|_D \leq \frac{1}{2}\|w_h^0\|_D + \frac{1}{2}\|w_h^1\|_D, \tag{3.40}$$

where

$$w_h^0 = v_h(0) - g_h^0 \quad \Rightarrow \|w_h^0\|_D = \|v_h(0)\|_D + \|g_h^0\|_D \quad \text{and} \quad w_h^1 = w_h^0 + \Delta t B_h p_h^{\frac{1}{2}}. \tag{3.41}$$

Hence

$$\|\mu w_h^{\frac{1}{2}}\|_D \leq \frac{1}{2}\|w_h^0\|_D + \frac{1}{2}\left[\|w_h^0\|_D + \Delta t\|B_h p_h^{\frac{1}{2}}\|_D\right]. \tag{3.42}$$

Since (CFL-condition) is satisfied, we have that

$$\eta = \frac{\Delta t}{2}|||B_h||| = \frac{\Delta t}{2}\sup_{p_h \in U_h}\frac{\|B_h p_h\|_D}{\|p_h\|_P} \leq 1 \quad \Rightarrow \frac{\Delta t}{2}\|B_h p_h\|_D \leq \eta\|p_h\|_P \leq \|p_h\|_P, \forall p_h \in U_h. \tag{3.43}$$

So

$$\|\mu w_h^{\frac{1}{2}}\|_D \leq \|w_h^0\|_D + \|p_h^{\frac{1}{2}}\|_P. \tag{3.44}$$

Summing up,

$$\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} \leq \frac{1}{2}\|p_h^{\frac{1}{2}}\|_P + \frac{1}{\sqrt{2}}\|w_h^0\|_D + \frac{1}{\sqrt{2}}\|p_h^{\frac{1}{2}}\|_P \leq \frac{1+\sqrt{2}}{2}\|p_h^{\frac{1}{2}}\|_P + \frac{1}{\sqrt{2}}\|v_h(0)\|_D + \frac{1}{\sqrt{2}}\|g_h^0\|_D. \tag{3.45}$$

We write that

$$p_h^{\frac{1}{2}} = p_h(0) + \frac{\Delta t}{2}\left[f_h^0 - B_h^* v_h(0)\right] \quad \Rightarrow \quad \|p_h^{\frac{1}{2}}\|_P \leq \|p_h(0)\|_P + \frac{\Delta t}{2}\|f_h^0\|_P + \frac{\Delta t}{2}\|B_h^* v_h(0)\|_P. \tag{3.46}$$

Since (CFL-condition) is satisfied, we also have that

$$\eta = \frac{\Delta t}{2}|||B_h||| = \frac{\Delta t}{2}|||B_h^*||| = \frac{\Delta t}{2}\sup_{v_h \in D_h}\frac{\|B_h^* v_h\|_P}{\|v_h\|_D} \leq 1 \quad \Rightarrow \frac{\Delta t}{2}\|B_h^* v_h\|_P \leq \eta\|v_h\|_D \leq \|v_h\|_D \quad \forall v_h \in D_h, \tag{3.47}$$

which yields

$$\|p_h^{\frac{1}{2}}\|_P \leq \|p_h(0)\|_P + \frac{\Delta t}{2}\|f_h^0\|_P + \|v_h(0)\|_D. \tag{3.48}$$

Hence

$$\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} \leq \frac{1+\sqrt{2}}{2}\left[\|p_h(0)\|_P + \frac{\Delta t}{2}\|f_h^0\|_P + \|v_h(0)\|_D\right] + \frac{1}{\sqrt{2}}\|v_h(0)\|_D + \frac{1}{\sqrt{2}}\|g_h^0\|_D \tag{3.49}$$

$$\leq \frac{1+\sqrt{2}}{2}\|p_h(0)\|_P + \frac{1+\sqrt{2}}{2}\frac{\Delta t}{2}\|f_h^0\|_P + \frac{1+2\sqrt{2}}{2}\|v_h(0)\|_D + \frac{1}{\sqrt{2}}\|g_h^0\|_D. \tag{3.50}$$

□

Proposition 3.12 (Stability of the averaged unknowns)**.**

$$\|\mu w_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2}\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t\sum_{k=1}^{n}\left[\|f_h^k\|_P + \|B_h^* g_h^k\|_P\right], \tag{3.51}$$

$$\|\mu p_h^n\| \leq 2\sqrt{2}\gamma\left(\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t\sum_{k=1}^{n}\left[\|f_h^k\|_P + \|B_h^* g_h^k\|_P\right]\right), \tag{3.52}$$

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \sqrt{2}\sqrt{\mathcal{E}_{pw,h}^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t\sum_{k=1}^{n}\left[\|f_h^k\|_P + \|B_h^* g_h^k\|_P\right] + \frac{\|g_h^{n+1}\|_D + \|g_h^n\|_D}{2}. \tag{3.53}$$

Proof. Use jointly Prop. 3.10 and 3.11 to get the two first equations. Moreover, the initial system was posed on the unknown $v_h^n = w_h^n + g_h^n$. Using the previous equations, one can only hope a direct control on $\mu v_h^{n+\frac{1}{2}}$ as

$$\|\mu v_h^{n+\frac{1}{2}}\|_D \leq \|\mu w_h^{n+\frac{1}{2}}\|_D + \|\mu g_h^{n+\frac{1}{2}}\|_D. \tag{3.54}$$

□

Remark 3.13. This result generalizes the result recalled in Section 3.1 to the case of two source terms. It confirms that the uniform stability is only obtained on the averaged unknowns.

Remark 3.14. One can not hope for more control on the unknowns $p_h^{n+\frac{1}{2}}$ and w_h^n, unfortunately, at least without supposing more restrictive bounds on Δt. Only the Π_k-projection of $p_h^{n+\frac{1}{2}}$ is controlled directly. However, a global control is achieved on a post-processed field.

Proposition 3.15 (Stability of the underlying field)**.** *Let* $\{u_h^n\}_{0\leq n\leq N+1}$ *be the series of elements of* U_h *defined as*

$$u_h^{n+1} = u_h^n + \Delta t\, p_h^{n+\frac{1}{2}}, \quad u_h^0 \text{ such that } B_h u_h^0 = w_h^0. \tag{3.55}$$

Then,

$$\|u_h^{n+1}\| \leq \sqrt{2}\|u_h^0\| + t^{n+1}2\gamma\sqrt{2\mathcal{E}_{pw,h}^{\frac{1}{2}}} + 4\gamma^2\Delta t^2\sum_{j=1}^{n+1}\sum_{k=1}^{j}\left[\|f^k\|_P + \|B_h^* g_h^k\|_P\right]. \tag{3.56}$$

Proof. From the orthogonality between the projection spaces, we get that

$$\|u_h^{n+1}\|_P^2 = \|\Pi_k u_h^{n+1}\|_P^2 + \|\Pi_p u_h^{n+1}\|_P^2. \tag{3.57}$$

Here we treat the two terms differently. For the first one we use (3.25a) with $q_h = p_h^{n+\frac{1}{2}}$ to get

$$\|\Pi_k u_h^{n+1}\|_P \leq \|\Pi_k u_h^n\|_P + \Delta t \|\Pi_k \delta u_h^{n+\frac{1}{2}}\|_P \leq \|\Pi_k u_h^n\|_P + \Delta t \|\Pi_k p_h^{n+\frac{1}{2}}\|_P \leq \|\Pi_k u_h^n\|_P + \Delta t C_k^{-1/2} \sqrt{2\mathcal{E}_{pw,h}^{n+\frac{1}{2}}}. \tag{3.58}$$

For the second one, we use (3.25b) with $q_h = \mu u_h^{n+\frac{1}{2}}$ to get

$$\|\Pi_p u_h^{n+1}\|_P \leq \|\Pi_p u_h^n\|_P + 2\|\Pi_p \mu u_h^{n+\frac{1}{2}}\|_P \leq \|\Pi_p u_h^n\|_P + 2\Delta t C_p^{-1/2} (B_h \mu u_h^{n+\frac{1}{2}}, B_h \mu u_h^{n+\frac{1}{2}})_D. \tag{3.59}$$

From the definition of u_h in (3.55) and using (3.11d), we get that $B_h \mu u_h^{n+\frac{1}{2}} = \mu w_h^{n+\frac{1}{2}}$. Hence

$$\|\Pi_p u_h^{n+1}\|_P \leq \|\Pi_p u_h^n\|_P + 2\Delta t C_p^{-1/2} (\mu w_h^{n+\frac{1}{2}}, \mu w_h^{n+\frac{1}{2}})_D \leq \|\Pi_p u_h^n\|_P + 2\Delta t C_p^{-1/2} \sqrt{2\mathcal{E}_{pw,h}^{n+\frac{1}{2}}}. \tag{3.60}$$

Both telescopic sums are canceled up from 0 to n, and then added up.

$$\|u_h^{n+1}\|_P \leq \sqrt{2}\|u_h^0\|_P + 2\gamma \Delta t \sum_{k=0}^{n} \sqrt{2\mathcal{E}_{pw,h}^{k+\frac{1}{2}}}. \tag{3.61}$$

Use (3.37) to conclude. □

Remark 3.16. In practice, $\gamma = \sqrt{2}$, see appendix of [5] for the scheme called "TS" with $\theta = 0$.

4. Convergence at the CFL

Subtracting directly equations (1.3) to1 (1.8) yields truncation errors on each appearance of the operator δ and needs additional manipulation to use the results of the previous section, necessitating non optimal assumptions on the semi discrete field. Instead, we propose to introduce the underlying field at both the fully and semi discrete level, and to manipulate the resulting equations before analysis. Let $u_h \in U_h$ be defined as

$$\dot{u}_h = p_h, \tag{4.1a}$$
$$u_h(0) = 0. \tag{4.1b}$$

Inserting this field in equation (1.3) yields

$$\ddot{u}_h + B_h^* v_h = f_h, \tag{4.2a}$$
$$\dot{v}_h - B_h \dot{u}_h = \dot{g}_h. \tag{4.2b}$$

The second line can be integrated in time to get

$$v_h - v_{h,0} - B_h u_h + B_h u_{h,0} = g_h - g_h(0). \tag{4.3}$$

Along with the other equations, we get

$$\ddot{u}_h + B_h^* v_h = f_h, \tag{4.4a}$$
$$v_h - B_h u_h = g_h - g_h^0 + v_{h,0}, \tag{4.4b}$$
$$\dot{u}_h = p_h. \tag{4.4c}$$

Note that Hypothesis 1.4 amounts to supposing that

$$\|u_h\|_{\mathcal{C}^4(0,T;P)} \leq C. \tag{4.5}$$

On the fully discrete level, as in the previous section, let us define $u_h \in U_h$ as

$$\delta u_h^{n+\frac{1}{2}} = p_h^{n+\frac{1}{2}}, \tag{4.6a}$$

$$u_h^0 = 0. \tag{4.6b}$$

Inserting this field in equation (1.8) yields

$$\delta^2 u_h^n + B_h^* v_h^n = f_h^n, \tag{4.7a}$$

$$\delta v_h^{n+\frac{1}{2}} - B_h \delta u_h^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}}. \tag{4.7b}$$

The second line can be summed up in time to get

$$\delta^2 u_h^n + B_h^* v_h^n = f_h^n, \tag{4.8a}$$

$$v_h^n - B_h^* u_h^n = g_h^n - g_h^0 + v_h^0, \tag{4.8b}$$

$$\delta u_h^{n+\frac{1}{2}} = p_h^{n+\frac{1}{2}}. \tag{4.8c}$$

Let $\bar{u}_h^n = u_h(t^n)$ and $\bar{v}_h^n = v_h(t^n)$, where v_h is solution to the semi-discrete equation (1.3), u_h is the solution to the semi-discrete equation (4.1), therefore solutions to the semi-discrete system (4.4).

Theorem 4.1. *Suppose that* (CFL-condition) *is satisfied. Then there exists a constant $C > 0$ independent of h and η, such that*

$$\|\bar{p}_h^n - \mu p_h^n\|_P \leq C\Delta t^2 \|p_h\|_{\mathcal{C}^3(0,T,P)}, \tag{4.9}$$

$$\|\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D \leq C\Delta t^2 \left(\|v_h\|_{\mathcal{C}^2(0,T;D)} + \|p_h\|_{\mathcal{C}^3(0,T,P)}\right), \tag{4.10}$$

$$\|\bar{u}_h^{n+1} - u_h^{n+1}\|_P \leq C\Delta t^2 \|p_h\|_{\mathcal{C}^3(0,T;P)}. \tag{4.11}$$

where C depends on the final time T as a quadratic polynomial.

Proof. Let us subtract the two first lines of systems (4.4) and (4.8).

$$\ddot{\bar{u}}_h^n - \delta^2 u_h^n + B_h^* \bar{v}_h^n - B_h^* v_h^n = f_h^n - f_h^n, \tag{4.12a}$$

$$\bar{v}_h^n - v_h^n - B_h \bar{u}_h^n + B_h u_h^n = g_h^n - g_h^n - g_h^0 + g_h^0 + v_{h,0} - v_h^0. \tag{4.12b}$$

Let us define the error terms $e_u^n = \bar{u}_h^n - u_h^n$ and $e_v^n = \bar{v}_h^n - v_h^n$, which satisfy

$$\delta^2 e_u^n + B_h^* e_v^n = \varepsilon_u^n, \tag{4.13a}$$

$$e_v^n - B_h e_u^n = 0, \tag{4.13b}$$

where the right hand side of the second line vanishes because of the initial condition v_h^0 is defined as (1.8a), and a Taylor expansion shows that there exists $t_1^n \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}]$ such that

$$\varepsilon_u^n = \frac{\Delta t^2}{12} \bar{u}_h^{(4)}(t_1^n). \tag{4.14}$$

Eliminating e_v in this equation leads to a classical Leap-Frog scheme on e_u, for which the results of [5] can be applied. To get more precise results on the errors made for the unknowns v_h and p_h, let us proceed differently. Introduce the sequence $\eta_p^{n+\frac{1}{2}}$ such that

$$\delta e_u^{n+\frac{1}{2}} = \eta_p^{n+\frac{1}{2}}. \tag{4.15}$$

Notice that $\eta_p^{n+\frac{1}{2}}$ is not equal to $e_p^{n+\frac{1}{2}} = \bar{p}_h^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}}$ since this quantity satisfies

$$\delta e_u^{n+\frac{1}{2}} = e_p^{n+\frac{1}{2}} + \dot{\varepsilon}_u^{n+\frac{1}{2}}, \tag{4.16}$$

where there exists $t_2^{n+\frac{1}{2}} \in [t^n, t^{n+1}]$ such that

$$\dot{\varepsilon}_u^{n+\frac{1}{2}} = \frac{\Delta t^2}{12} \bar{u}_h^{(3)}(t_2^{n+\frac{1}{2}}). \tag{4.17}$$

Hence, in U_h,

$$\eta_p^{n+\frac{1}{2}} = e_p^{n+\frac{1}{2}} + \dot{\varepsilon}_u^{n+\frac{1}{2}}. \tag{4.18}$$

Finally, the error terms e_u, e_v and η_p are solution to

$$\delta \eta_p^n + B_h^* e_v^n = \varepsilon_u^n, \tag{4.19a}$$

$$\delta e_v^{n+\frac{1}{2}} - B_h \eta_p^{n+\frac{1}{2}} = 0, \tag{4.19b}$$

$$\delta e_u^{n+\frac{1}{2}} = \eta_p^{n+\frac{1}{2}}. \tag{4.19c}$$

The result follows from the stability analysis performed in Section 3.2. Indeed, the two first lines of system (4.19) are of the form (1.8–3.55), with the substitutions

$$\begin{aligned} p_h^{n+\frac{1}{2}} &\leftarrow \eta_p^{n+\frac{1}{2}}, \\ v_h^n \equiv w_h^n &\leftarrow e_v^n, \\ u_h^n &\leftarrow e_u^n, \\ f^n &\leftarrow \varepsilon_u^n, \\ g_h^n &\leftarrow 0. \end{aligned}$$

Hence,

$$\|\mu e_v^{n+\frac{1}{2}}\|_D \leq \sqrt{2}\sqrt{\mathcal{E}_e^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^{n} \left[\|\varepsilon_u^k\|_P\right], \tag{4.20}$$

$$\|\mu\eta_p^n\| \leq 2\sqrt{2}\gamma \left(\sqrt{\mathcal{E}_e^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^{n} \left[\|\varepsilon_u^k\|_P\right]\right), \tag{4.21}$$

$$\|e_u^{n+1}\| \leq \sqrt{2}\|e_u^0\| + t^{n+1} 2\gamma \sqrt{2\mathcal{E}_e^{\frac{1}{2}}} + 4\gamma^2\Delta t^2 \sum_{j=1}^{n+1}\sum_{k=1}^{j} \left[\|\varepsilon_u^k\|_P\right], \tag{4.22}$$

where $\mathcal{E}_e^{\frac{1}{2}}$ is defined as

$$\mathcal{E}_e^{\frac{1}{2}} = \frac{1}{2}\tilde{m}(\eta_p^{\frac{1}{2}}, \eta_p^{\frac{1}{2}}) + \frac{1}{2}(\mu e_v^{\frac{1}{2}}, \mu e_v^{\frac{1}{2}}). \tag{4.23}$$

Therefore, using (4.18),

$$\|\mu e_p^n\|_P \leq 2\sqrt{2}\gamma \left(\sqrt{\mathcal{E}_e^{\frac{1}{2}}} + \sqrt{2}\gamma\Delta t \sum_{k=1}^{n} \left[\|\varepsilon_u^k\|_P\right]\right) + \frac{\|\dot{\varepsilon}_u^{n+\frac{1}{2}}\|_P + \|\dot{\varepsilon}_u^{n-\frac{1}{2}}\|_P}{2}. \tag{4.24}$$

So,

$$\|\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D \leq \|\bar{v}_h^{n+\frac{1}{2}} - \mu\bar{v}_h^{n+\frac{1}{2}}\|_D + \|\mu\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D \tag{4.25}$$

$$\leq \|\varepsilon_v^{n+\frac{1}{2}}\|_D + \|\mu\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D, \tag{4.26}$$

where there exists $t_3^{n+\frac{1}{2}} \in [t^n, t^{n+1}]$ such that

$$\varepsilon_v^{n+\frac{1}{2}} = \mu \bar{v}_h^{n+\frac{1}{2}} - \bar{v}_h^{n+\frac{1}{2}} = \frac{\Delta t^2}{3} \bar{v}_h^{(2)}(t_3^{n+\frac{1}{2}}). \tag{4.27}$$

Similarly,

$$\|\bar{p}_h^n - \mu p_h^n\|_P \le \|\bar{p}_h^n - \mu \bar{p}_h^n\|_P + \|\mu \bar{p}_h^n - \mu p_h^n\|_P \tag{4.28}$$
$$\le \|\varepsilon_p^n\|_P + \|\mu \bar{p}_h^n - \mu p_h^n\|_P, \tag{4.29}$$

where there exists $t_4^n \in [t^{n-\frac{1}{2}}, t^{n+\frac{1}{2}}]$ such that

$$\varepsilon_p^n = \mu \bar{p}_h^n - \bar{p}_h^n = \frac{\Delta t^2}{3} \bar{p}_h^{(2)}(t_4^n). \tag{4.30}$$

Hence,

$$\|\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D \le \|\varepsilon_v^{n+\frac{1}{2}}\|_D + \sqrt{2}\sqrt{\mathcal{E}_e^{\frac{1}{2}}} + \sqrt{2}\gamma \Delta t \sum_{k=1}^{n} \left[\|\varepsilon_u^k\|_P\right], \tag{4.31}$$

$$\|\bar{p}_h^n - \mu p_h^n\|_P \le \|\varepsilon_p^n\|_P + 2\sqrt{2}\gamma \left(\sqrt{\mathcal{E}_e^{\frac{1}{2}}} + \sqrt{2}\gamma \Delta t \sum_{k=1}^{n} \left[\|\varepsilon_u^k\|_P\right] \right) + \frac{\|\dot{\varepsilon}_u^{n+\frac{1}{2}}\| + \|\dot{\varepsilon}_u^{n-\frac{1}{2}}\|}{2}, \tag{4.32}$$

$$\|\bar{u}_h^{n+1} - u_h^{n+1}\|_P \le \sqrt{2}\|e_u^0\|_P + t^{n+1} 2\gamma \sqrt{2\mathcal{E}_e^{\frac{1}{2}}} + 4\gamma^2 \Delta t^2 \sum_{j=1}^{n+1} \sum_{k=1}^{j} \left[\|\varepsilon_u^k\|_P\right]. \tag{4.33}$$

Next, let us show that $\mathcal{E}_e^{\frac{1}{2}}$ is small. Using the naive formulation of the energy (3.12), we write that

$$\mathcal{E}_e^{\frac{1}{2}} = \frac{1}{2}(\eta_p^{\frac{1}{2}}, \eta_p^{\frac{1}{2}})_P + \frac{1}{2}(e_v^1, e_v^0)_D \le \frac{1}{2}\|\eta_p^{\frac{1}{2}}\|_P^2 + \frac{1}{2}\|e_v^1\|_D \; \|e_v^0\|_D. \tag{4.34}$$

Since the initial conditions v_h^0 and $p_h^{\frac{1}{2}}$ are defined as (1.8a) and (1.8b), we directly have that $e_v^0 = 0$. It remains to estimate $\eta_p^{\frac{1}{2}} = e_p^{\frac{1}{2}} + \dot{\varepsilon}_u^{\frac{1}{2}}$. The value of $e_p^{\frac{1}{2}}$ can be found by subtracting both sides of (1.8b) to $\bar{p}_h^{\frac{1}{2}} - \frac{\Delta t}{2} B_h^* v_h^0$ in order to make also appear e_v^0.

$$\underbrace{\bar{p}_h^{\frac{1}{2}} - p_h^{\frac{1}{2}}}_{e_p^{\frac{1}{2}}} + \frac{\Delta t}{2} B_h^* \underbrace{(\bar{v}_h^0 - v_h^0)}_{e_v^0 \equiv 0} = \frac{\Delta t}{2} \underbrace{\left[B_h^* \bar{v}_h^0 - f_h^0\right]}_{\overset{=}{(1.3c)} -\bar{p}_h^0} + \bar{p}_h^{\frac{1}{2}} - p_h(0). \tag{4.35}$$

A Taylor expansion of $\bar{p}_h$ yields that there exists $t_5^0 \in [0, t^{\frac{1}{2}}]$ such that

$$\bar{p}_h(t^{\frac{1}{2}}) = \bar{p}_h(0) + \frac{\Delta t}{2} \bar{p}_h^{(1)}(0) + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 \bar{p}_h^{(2)}(t_5^0). \tag{4.36}$$

Hence we get that

$$e_p^{\frac{1}{2}} = \frac{\Delta t^2}{8} \bar{p}_h^{(2)}(t_5^0). \tag{4.37}$$

Take $\tilde{p}_h = e_p^{\frac{1}{2}}$ and apply Cauchy Schwarz inequality to get that

$$\|e_p^{\frac{1}{2}}\|_P \le \frac{\Delta t^2}{8} \|p_h\|_{\mathcal{C}^2(0,T;P)}. \tag{4.38}$$

So

$$\sqrt{2\mathcal{E}_e^{\frac{1}{2}}} \leq \|e_p^{\frac{1}{2}}\|_P + \|\dot{\varepsilon}_u^{\frac{1}{2}}\|_P, \tag{4.39}$$

$$\leq \frac{5\Delta t^2}{24}\|p_h\|_{\mathcal{C}^2(0,T;P)}. \tag{4.40}$$

Finally, we have that

$$\varepsilon_u^n = \frac{\Delta t^2}{12}\bar{p}_h^{(3)}(t_1^n) \quad \text{and} \quad \dot{\varepsilon}_u^{n+\frac{1}{2}} = \frac{\Delta t^2}{12}\bar{p}_h^{(2)}(t_2^{n+\frac{1}{2}}) \quad \text{and} \quad e_u^0 = 0, \tag{4.41}$$

where the last equation comes from (4.6b). Altogether, we get

$$\|\bar{p}_h^n - \mu p_h^n\|_P \leq \frac{\Delta t^2}{12}\left(5 + 5\gamma + 4\gamma^2 T\right)\|p_h\|_{\mathcal{C}^3(0,T,P)}, \tag{4.42}$$

$$\|\bar{v}_h^{n+\frac{1}{2}} - \mu v_h^{n+\frac{1}{2}}\|_D \leq \frac{\Delta t^2}{3}\|v_h\|_{\mathcal{C}^2(0,T;D)} + \left(5 + 2\sqrt{2}\gamma T\right)\frac{\Delta t^2}{24}\|p_h\|_{\mathcal{C}^3(0,T,P)}, \tag{4.43}$$

$$\|\bar{u}_h^{n+1} - u_h^{n+1}\|_P \leq \left(5\,T\gamma + 4\gamma^2 T^2\right)\frac{\Delta t^2}{12}\|p_h\|_{\mathcal{C}^3(0,T;P)}. \tag{4.44}$$

□

Along with Hypothesis 1.5, this proves the uniform space/time convergence of the Störmer-Verlet integration scheme of the average unknowns and a post processed field.

Remark 4.2. Note that the post processed semi discrete field is actually solution to the equation

$$\ddot{u}_h + B_h^* B_h u_h = f_h - B_h^*\left[g_h - g_h^0 + v_{h,0}\right], \tag{4.45}$$

while the post processed discrete field is solution to the leap-frog scheme

$$\delta^2 u_h^n + B_h^* B_h u_h^{n+\frac{1}{2}} = f_h^n - B_h^*\left[g_h^{n+\frac{1}{2}} - g_h^0 + v_h^0\right]. \tag{4.46}$$

Performing the same eliminations for the variable p_h yields a similar second order equation and leap-frog scheme but with differentiated sources in time. This distinction is of importance when using the mixed formulation model coupled with other models, in the context of multi-physics phenomena for instance.

5. Conclusion and prospects

This work proposes a proof of stability and convergence of the Störmer-Verlet integration scheme, towards the semi-discrete solution of wave equations in mixed formulation, uniform as the time step tends towards its largest admissible value. The error constants obtained depend neither on the spatial discretization parameters, nor on the distance of the time step from its largest admissible value, thus providing space/time convergence, if certain stability and convergence assumptions are satisfied by the spatial discretization. This makes it possible to generalize results from the literature in which, for simplicity's sake, the stability condition is supposed to be strictly satisfied. The assumptions of the theorem are consistent with existing uniform results for the leap-frop scheme applied to the second-order formulation of wave equations. It appears that the natural variables that converge uniformly in space/time are not the unknowns directly calculated by the scheme, but their consecutive average between two time steps. However, it is possible to reconstruct a field that converges at each discretization instant to a semi-discrete field. Preliminary numerical results on simple cases in 1D suggest that pointwise fields also converge at second order in space-time for $\eta = 1$ providing that the source terms are regular enough, but also exhibit a change of convergence behavior as $\eta \to 1$ if the source terms are not regular enough. This suggests that with an adequate change of variable or manipulation of the source terms assumed regular enough, the unknowns may be controlled directly. Note that this trick is only expected to work for linear wave equations with no external coupling, while the present result on averaged variables could be extended to the case of dissipative equations, couplings between two domains, addition of nonlinear terms, and so on. It could also be generalized in order to analyze other integration schemes, such as implicit or dissipative schemes. A preliminary work on implicit generalizations of Störmer-Verlet and Crank-Nicolson integration schemes applied to the mixed formulation of linear wave equations can be found in [4].

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