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AN EXACTLY DIVERGENCE-FREE HYBRIDIZED DISCONTINUOUS GALERKIN METHOD FOR THE GENERALIZED BOUSSINESQ EQUATIONS WITH SINGULAR HEAT SOURCE

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Abstract. The purpose of this work is to propose and analyze a hybridized discontinuous Galerkin (HDG) method for the generalized Boussinesq equations with singular heat source. We use polynomials of order $k,\ k-1$ and k to approximate the velocity, the pressure and the temperature. By introducing Lagrange multipliers for the pressure, the approximate velocity field obtained by the HDG method is shown to be exactly divergence-free and $H(\mathrm{div})$-conforming. Under a smallness assumption on the problem data and solutions, we prove by Brouwer's fixed point theorem that the discrete system has a solution in two dimensions. If the viscosity and thermal conductivity are further assumed to be positive constants, *a priori* error estimates with convergence rate $O(h)$ and efficient and reliable *a posteriori* error estimates are derived. Finally numerical examples illustrate the theoretical analysis and show the performance of the obtained *a posteriori* error estimator.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d,\ d \in \{2,3\}$ be an open, bounded polygonal or polyhedral domain with Lipschitz boundary $\partial\Omega$. In this work, we consider the following stationary generalized Boussinesq equations consisting of incompressible Navier–Stokes equations and convection diffusion equations with singular heat source:

$$-\mathrm{div}(\nu(T)\nabla\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{g}T = \mathbf{F}, \qquad \text{in } \Omega, \tag{1a}$$

$$\mathrm{div}\mathbf{u} = 0, \qquad \text{in } \Omega, \tag{1b}$$

$$-\mathrm{div}(\kappa(T)\nabla T) + \mathbf{u}\cdot\nabla T = \mathcal{H}\delta_z, \quad \text{in } \Omega, \tag{1c}$$

$$\mathbf{u} = \mathbf{0}, \quad T = 0, \qquad \text{on } \partial\Omega, \tag{1d}$$

where the velocity $\mathbf{u}$, the pressure p, and the temperature T are unknowns. $\mathbf{g} \in (L^\infty(\Omega))^d$ is the external force per unit mass, usually acting in the opposite direction to gravity. δ_z is the Dirac delta supported at the interior point $z \in \Omega$. Functions ν and κ that depend on the temperature denote the fluid viscosity and

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thermal conductivity. $\mathbf{F} \in (L^2(\Omega))^d$ and $\mathcal{H} \in \mathbb{R}$. Throughout this paper, we suppose that ν and κ are Lipschitz continuous and bounded from above and below, *i.e.*, there are positive constants $\nu_L,\ \kappa_L,\ \nu_-,\ \nu_+,\ \kappa_-,\ \kappa_+$ such that

$$|\nu(x_1) - \nu(x_2)| \le \nu_L |x_1 - x_2|, \quad |\kappa(x_1) - \kappa(x_2)| \le \kappa_L |x_1 - x_2|, \qquad \forall x_1, x_2 \in \mathbb{R}, \tag{2a}$$

$$0 < \nu_- \le \nu(t) \le \nu_+, \qquad 0 < \kappa_- \le \kappa(t) \le \kappa_+, \qquad \forall t \in \mathbb{R}. \tag{2b}$$

If the right hand side of (1c) is replaced by a datum in L^2 space, the coupled problem (1) describing an incompressible nonisothermal fluid flow has been extensively investigated both theoretically and numerically [2,5,6,13,15,16,24,38,42,44–46], and relevant industrial applications include heat exchangers, cooling of electronic equipments, cooling of nuclear reactors, climate predictions, oceanic flows, and many others.

To the best of our knowledge, one of the first work concerning the investigation of solution techniques for the classical Boussinesq problem where ν and κ are positive constants is [15]. For the same problem in two dimensions, Farhloul *et al.* [24] studied the regularity of solutions and proposed and analyzed a mixed formulation where the gradients of velocity and temperature are introduced as new unknows. In [42], Oyarzúa and Serón considered a divergence-conforming finite element method for the primal formulation of the fluid flow and the dual-mixed formulation of the heat equation. The resulting discrete velocity field was exactly divergence-free. Thus, it exactly preserves the essential constraint (1b) and was probably energy-stable without the need of modifying the underlying differential equations.

For the generalized Boussinesq problem where the fluid viscosity ν and thermal conductivity κ are temperature-dependent, Lorca and Boldrini [37] investigated the existence and regularity of solutions by spectral Galerkin methods and fixed point arguments under certain assumptions on ν and κ. In addition, the uniqueness of small solutions was established if the domain was sufficiently smooth. For more complicated boundary conditions, a similar result corresponding to the existence, uniqueness and regularity of solutions was obtained in [45] for two dimensional case. Based on the result in [45], Pérez *et al.* [46] analyzed a conforming finite element method, proved its well-posedness, and derived *a priori* error estimates under a smallness assumption on the problem data. In [43], Oyarzúa and Zúñiga studied a conforming finite element method for the primal formulation of the fluid flow and the primal-mixed formulation of the heat equation, proved the existence of discrete solutions, and derived *a priori* error estimates under a smallness assumption on the problem data. In [5,6], Allendes *et al.* investigated the lowest-order stabilized finite element method, proved the existence of discrete solutions by Brouwer's fixed point theorem, and derived *a priori* and *a posteriori* error estimates under a smallness assumption on the problem data and solutions. Note that the discrete velocity field obtained in [5,6] could be divergence-free by adding the jump of the pressure to the conforming velocity field.

Since solutions of multi-scale nonlinear equations are sensitive to small errors and high-order accurate schemes reach a smaller error tolerance with a few degrees of freedom than low-order accurate schemes, it is worth to develop a high-order and physically conservative scheme to Boussinesq problems. In [44], Oyarzúa *et al.* studied the generalized Boussinesq problem and employed divergence-conforming Brezzi–Douglas–Marini (BDM) elements of order $k \ge 1$ for the velocity, discontinuous elements of order $k-1$ for the pressure, and continuous elements of order k for the temperature to derive a discrete system with exactly divergence-free velocity approximations. Under a smallness assumption on the problem data and solutions, the existence of discrete solutions and *a priori* error estimates were proved. In [51], Ueckermann and Lermusiaux designed a numerical scheme for the time-dependent classical Boussinesq problem by using hybridizable discontinuous Galerkin methods, projection methods, and implicit-explicit Runge–Kutta time-integration schemes. It is worth mentioning that Gatica *et al.* [11–13] published a series of papers to investigate *a priori* and *a posteriori* error estimates of high-order and fully-mixed finite element methods for an augmented fully-mixed formulation of Boussinesq problems with temperature-dependent viscosity and temperature-dependent or -independent thermal conductivity.

As for the Boussinesq problem (1), we only found one relevant literature [7]. There Allendes *et al.* proved by Leray–Schauder fixed point theorem the existence of continuous solutions in Muckenhoupt weighted space under a smallness assumption on the problem data and solutions. Then they utilized inf-sup stable finite element methods to approximate the model and under a similar assumption with the continuous case the existence of discrete solutions was obtained by Brouwer's fixed point theorem. Finally, reliable and efficient *a posteriori* error estimators for the lowest-order Taylor-Hood and mini elements were derived under additional assumption that ν and κ were two positive constants. An instance where singular sources appear is in the modelling of the motion of active thin structures [27], there the right hand side is a linear combination of Dirac deltas supported at interior points of Ω. Another instance emerges in Darcy [10] and Darcy-Forchheimer [9] problems coupled with heat equation and PDE-constrained optimal control problems [8, 18, 26, 41].

In this paper, our purpose is to propose and analyze a divergence-free and $H(\text{div})$-conforming HDG method for the Boussinesq problem (1). The main source of difficulty and interest is that the external heat source is singular, say a Dirac measure concentrated at an interior point so that the problem can not be understood with the usual energy setting. In Section 2, a weak formulation of problem (1) is provided and the existence of continuous solutions is proved in $H_0^1(\Omega) \times L_0^2(\Omega) \times W_0^{1,s}(\Omega)$ space by Leray–Schauder fixed point theorem under a smallness assumption on the problem data and solutions. Based on idea of [48,54], we propose an HDG method with divergence-free and $H(\text{div})$-conforming velocity field. Under a similar assumption with the continuous case, we prove by Brouwer's fixed point theorem that the discrete system has a solution in two dimensions. Notice that HDG methods were first proposed by Cockburn *et al.* [20] for second order elliptic problems. Later, it was extended to convection diffusion problem [25, 39, 54], incompressible Navier–Stokes problem [19, 35, 40, 47], and so on. The results so far show that HDG methods are competitive with both continuous Galerkin and DG methods. Compared with the standard finite element methods, HDG methods are easier to design an exactly mass-conservative scheme for incompressible flow. Compared with DG methods, HDG methods can yield a discrete system with significantly reduced degrees of freedom by introducing Lagrange multipliers.

In order to obtain *a priori* and *a posteriori* error estimates, from Lemma 2.4 we assume that the domain $\Omega \subset \mathbb{R}^2$ and from Section 3 we assume that ν and κ are two positive constants. By duality argument and BDM interpolation, *a priori* error estimates with convergence rate $O(h)$ for L^2-norms of the velocity gradient, the pressure and the temperature are obtained in Section 3 by imposing a smallness assumption on the problem data and solutions (see Thm. 3.2 and Cor. 3.1). Note that due to the singular nature of δ_z *a priori* error estimates for the duality problem of (1c) (see Thm. 3.1) are necessary for error estimates of the temperature. Moreover error estimates of the velocity are independent of the pressure (see Lem. 3.5) because of the divergence-free and $H(\text{div})$-conforming properties of the discrete velocity field.

In Section 4, we develop a computable *a posteriori* error estimator and prove that it is globally reliable and locally efficient with respect to an energy norm by using Oswald interpolation and BDM lifting operator (see Thms. 4.1 and 4.2). Since the approximate velocity field is divergence-free and $H(\text{div})$-conforming, the indicator $\|\nabla \cdot \mathbf{u}_h\|_{0,\mathcal{T}_h}$ is not included by our *a posteriori* error estimator. On the other hand, the indicators $\|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^2(\partial K)}$ and $\|T_h - \widehat{T}_h\|_{L^s(\partial K)}$, that are special for HDG methods, appear in our *a posteriori* error estimator.

In Section 5, a fixed-point algorithm is designed to solve the discrete system and a series of numerical examples are provided to validate the theoretical analysis and show the performance of the obtained *a posteriori* error estimator. We conclude this paper by some conclusions in Section 6.

2. Notation, weak formulation and HDG discretization

Here, we adopt the standard notation [1] for Sobolev spaces $W^{m,s}(\mathfrak{D})$ (m nonegative integer and $1 \le s \le \infty$) equipped with the norms $\|\cdot\|_{m,s,\mathfrak{D}}$ and seminorms $|\cdot|_{m,s,\mathfrak{D}}$, where $\mathfrak{D} \subset \mathbb{R}^d$ or $\mathfrak{D} \subset \mathbb{R}^{d-1}$ denotes an open and bounded set. If $s = 2$, we write $H^m(\mathfrak{D})$ in place of $W^{m,2}(\mathfrak{D})$ endowed with the norms $\|\cdot\|_{m,\mathfrak{D}}$ and seminorms $|\cdot|_{m,\mathfrak{D}}$. For $s \in (1,\infty)$, let s' be the conjugate number of s such that $\frac{1}{s} + \frac{1}{s'} = 1$, then the dual space of $W_0^{m,s}(\Omega)$ is denoted by $W^{-m,s'}(\Omega)$, where $W_0^{m,s}(\Omega)$ is the completion of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,s}(\Omega)$ and $\mathcal{C}_0^\infty(\Omega)$ is the space of functions with continuous derivatives of arbitrary order and compact support in Ω. The space $L_0^2(\Omega)$ is a space of functions in $L^2(\Omega)$ with vanishing mean value over Ω. Furthermore, we define

$$\mathcal{V} = \left\{ \mathbf{v} \in \left(H_0^1(\Omega)\right)^d : \mathrm{div}\mathbf{v} = 0 \text{ in } \Omega \right\}.$$

In the subsequent analysis, we denote by $C_{s,q} > 0$ the embedding constant such that (see [1], Thm. 5.4) for any $s \le d$

$$W_0^{1,s}(\Omega) \hookrightarrow L^q(\Omega) \quad s \le q \le \frac{ds}{d-s}, \quad \text{and} \quad \|\varphi\|_{L^q(\Omega)} \le C_{s,q}\|\nabla\varphi\|_{L^s(\Omega)}. \tag{3}$$

Moreover, we shall frequently use notations C and c with or without subscripts to denote generic positive constants independent of discretization parameters. Note that the values of C and c might change at each occurrence.

The variational formulation of problem (1) amounts to finding $(\mathbf{u}, p, T) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times W_0^{1,s}(\Omega)$, $s \in (\frac{2d}{d+2}, \frac{d}{d-1})$, such that

$$A_F(T; \mathbf{u}, \mathbf{v}) + O_F(\mathbf{u}; \mathbf{u}, \mathbf{v}) - B(\mathbf{v}, p) - D(T, \mathbf{v}) = (\mathbf{F}, \mathbf{v}), \tag{4a}$$

$$B(\mathbf{u}, q) = 0, \tag{4b}$$

$$A_T(T; T, w) - O_T(\mathbf{u}; T, w) = \langle \mathcal{H}\delta_z, w\rangle, \tag{4c}$$

for any $(\mathbf{v}, q, w) \in (H_0^1(\Omega))^d \times L_0^2(\Omega) \times W_0^{1,s'}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality paring between spaces $W^{-1,s}(\Omega)$ and $W_0^{1,s'}(\Omega)$,

$$\begin{aligned} &A_F(\theta; \mathbf{u}, \mathbf{v}) = (\nu(\theta)\nabla\mathbf{u}, \nabla\mathbf{v}), && O_F(\mathbf{w}; \mathbf{u}, \mathbf{v}) = ((\mathbf{w}\cdot\nabla)\mathbf{u}, \mathbf{v}),\\ &A_T(\theta; T, w) = (\kappa(\theta)\nabla T, \nabla w), && O_T(\mathbf{w}; T, \varphi) = (\mathbf{w}T, \nabla\varphi),\\ &B(\mathbf{v}, q) = (\mathrm{div}\mathbf{v}, q), && D(T, \mathbf{v}) = (\mathbf{g}T, \mathbf{v}), \end{aligned}$$

and $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\Omega)$. Utilizing the embedding result (3), we have the following bound for the convective term:

$$\begin{aligned} O_T(\mathbf{u}; T, w) &\le \|\mathbf{u}T\|_{L^s(\Omega)}\|\nabla w\|_{L^{s'}(\Omega)} \le \|\mathbf{u}\|_{L^d(\Omega)}\|T\|_{L^{\frac{ds}{d-s}}(\Omega)}\|\nabla w\|_{L^{s'}(\Omega)}\\ &\le C_{2,d}C_{s,\frac{ds}{d-s}}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla T\|_{L^s(\Omega)}\|\nabla w\|_{L^{s'}(\Omega)}, \quad \frac{2d}{d+2} < s < \frac{d}{d-1}. \end{aligned} \tag{5}$$

Thus according to assumption (2), embedding theory, and the above bound for the convective term, it is easy to observe that the terms appeared in (4) are all well-defined.

Remark 2.1. If $\mathbf{g} \in (L^2(\Omega))^d$ and $d = 3$, Hölder inequality and embedding result (3) yield

$$D(T, \mathbf{v}) \le \|\mathbf{g}\|_{L^2(\Omega)}\|T\|_{L^{\frac{3s}{3-s}}}\|\mathbf{v}\|_{L^{\frac{6s}{5s-6}}(\Omega)}, \quad \frac{6}{5} < s < \frac{3}{2}.$$

By a careful calculation,

$$\frac{6s}{5s-6} > 6,$$

which indicates that the space $L^{\frac{6s}{5s-6}}(\Omega)$ can not be embedded in $H^1(\Omega)$, hence the bilinear form D is not well defined. This is why we restrict the data $\mathbf{g}$ in the space $(L^\infty(\Omega))^d$ in this paper.

2.1. Existence of weak solutions

This subsection is devoted to explore the existence of solutions of weak formulation (4) based on the approach introduced in Section 3 of [7].

First of all, for any given $\theta \in L^2(\Omega)$, we consider the Navier–Stokes equations (4a) and (4b) with T replaced by θ: Find $(\mathbf{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ such that

$$A_F(\theta; \mathbf{u}, \mathbf{v}) + O_F(\mathbf{u}; \mathbf{u}, \mathbf{v}) - B(\mathbf{v}, p) - D(\theta, \mathbf{v}) = (\mathbf{F}, \mathbf{v}) \tag{6a}$$

$$B(\mathbf{u}, q) = 0 \tag{6b}$$

for any $(\mathbf{v}, q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$.

Theorem 2.1. *For any given $\theta \in L^2(\Omega)$ problem (6) has at least one solution $(\boldsymbol{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$ satisfying*

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \le \nu_-^{-1} C_{2,2}\big(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^\infty(\Omega)}\|\theta\|_{L^2(\Omega)}\big). \tag{7}$$

In addition, if

$$C_{2,4}^2 C_{2,2}\big(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|\boldsymbol{g}\|_{L^\infty(\Omega)}\|\theta\|_{L^2(\Omega)}\big) < \nu_-^2, \tag{8}$$

the solution is unique. Here $C_{2,2}$ and $C_{2,4}$ are the constants in (3).

Proof. For any given $\theta \in L^2(\Omega)$, it is easy to observe that $\mathbf{F} + \theta\mathbf{g} \in (H^{-1}(\Omega))^d$. Thus from Theorem 2.1 of [28] we know that problem (6) has at least one solution $(\mathbf{u}, p) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$. Let $\mathbf{v} = \mathbf{u}$ in (6a), we yield

$$\nu_-\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 \le C_{2,2}\big(\|\mathbf{F}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^\infty(\Omega)}\|\theta\|_{L^2(\Omega)}\big)\|\nabla \mathbf{u}\|_{L^2(\Omega)},$$

which derives the bound (7) by dividing $\nu_-\|\nabla \mathbf{u}\|_{L^2(\Omega)}$ from both sides.

Next we assume that $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{V}$ satisfying the bound (7) are the velocity part of two solutions of problem (6). Then

$$\begin{aligned} A_F(\theta; \mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) &= -(O_F(\mathbf{u}_1; \mathbf{u}_1, \mathbf{v}) - O_F(\mathbf{u}_2; \mathbf{u}_2, \mathbf{v})) \\ &= -O_F(\mathbf{u}_1; \mathbf{u}_1 - \mathbf{u}_2, \mathbf{v}) - O_F(\mathbf{u}_1 - \mathbf{u}_2; \mathbf{u}_2, \mathbf{v}), \end{aligned}$$

for any $\mathbf{v} \in \mathcal{V}$. Let $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ in the above equality, we have

$$\begin{aligned} \nu_-\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^2(\Omega)}^2 &\le C_{2,4}^2\|\nabla \mathbf{u}_2\|_{L^2(\Omega)}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^2(\Omega)}^2 \\ &\le C_{2,4}^2\nu_-^{-1}C_{2,2}\big(\|\mathbf{F}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^\infty(\Omega)}\|\theta\|_{L^2(\Omega)}\big)\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{L^2(\Omega)}^2, \end{aligned}$$

which, together with (8), leads to $\mathbf{u}_1 = \mathbf{u}_2$. This shows that if the condition (8) holds the solution of problem (6) is unique. □

Next, we study the existence and uniqueness of solutions of problem (4c) for any given $\mathbf{w} \in \mathcal{V}$ and $\theta \in L^2(\Omega)$: Find $T \in W_0^{1,s}(\Omega)$, $\frac{2d}{d+2} < s < \frac{d}{d-1}$, such that

$$A_T(\theta; T, \varphi) - O_T(\mathbf{w}; T, \varphi) = \langle \mathcal{H}\delta_z, \varphi\rangle, \qquad \forall \varphi \in W_0^{1,s'}(\Omega). \tag{9}$$

If $\mathbf{w} = 0$, problem (9) has a unique solution $T \in W_0^{1,s}(\Omega)$ (see [50], Thm. C), $\frac{6-d+\epsilon}{5-d+\epsilon} < s < \frac{d}{d-1}$, where $\epsilon > 0$ is a constant depending on the domain Ω. Then according to Theorem 2.6 of [23] the following inf-sup condition holds, *i.e.*, there exists a positive constant β depending on $\kappa(\theta)$, s and Ω such that

$$\beta\|\nabla w\|_{L^s(\Omega)} \le \sup_{\varphi \in W_0^{1,s'}(\Omega)} \frac{(\kappa(\theta)\nabla w, \nabla \varphi)}{\|\nabla \varphi\|_{L^{s'}(\Omega)}}, \quad \forall w \in W_0^{1,s}(\Omega),\ \frac{6-d+\epsilon}{5-d+\epsilon} < s < \frac{d}{d-1}\cdot \tag{10}$$

Theorem 2.2. *Let $\frac{6-d+\epsilon}{5-d+\epsilon} < s < \frac{d}{d-1}$, then for any given $\boldsymbol{w} \in \mathcal{V}$ and $\theta \in L^2(\Omega)$ problem (9) has a unique solution $T \in W_0^{1,s}(\Omega)$ satisfying*

$$\|\nabla T\|_{L^s(\Omega)} \le \frac{|\mathcal{H}|}{\beta(1-\alpha)} \|\delta_z\|_{-1,s,\Omega}, \tag{11}$$

if

$$\frac{1}{\beta} C_{2,d} C_{s,\frac{ds}{d-s}} \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \le \alpha < 1, \tag{12}$$

where $C_{2,d}$ and $C_{s,\frac{ds}{d-s}}$ are the constants in (3) and β is the constant in the inf-sup condition (10).

Proof. If T is the solution of problem (9), we have by (5)

$$A_T(\theta; T, \varphi) \le \Big(|\mathcal{H}| \|\delta_z\|_{-1,s,\Omega} + C_{2,d} C_{s,\frac{ds}{d-s}} \|\nabla \mathbf{w}\|_{L^2(\Omega)} \|\nabla T\|_{L^s(\Omega)} \Big) \|\nabla \varphi\|_{L^{s'}(\Omega)}, \tag{13}$$

which, together with the inf-sup condition (10) and (12), results in

$$\|\nabla T\|_{L^s(\Omega)} \le \frac{|\mathcal{H}|}{\beta(1-\alpha)} \|\delta_z\|_{-1,s,\Omega}.$$

Now we consider a mapping $\mathcal{F}$: for any given $w \in W_0^{1,s}(\Omega)$, $\mathcal{F}(w) = \psi \in W_0^{1,s}(\Omega)$ being the solution of

$$A_T(\theta; \psi, \varphi) - O_T(\mathbf{w}; w, \varphi) = \langle \mathcal{H}\delta_z, \varphi \rangle, \qquad \forall \varphi \in W_0^{1,s'}(\Omega).$$

Due to the inf-sup condition (10), we know that the above equation has a unique solution. Moreover, by using (13) and (12), it is easy to observe that the mapping $\mathcal{F}$ maps the space $\mathcal{W}$ to itself, where $\mathcal{W} := \{\varphi \in W_0^{1,s}(\Omega) : \|\nabla \varphi\|_{L^s(\Omega)} \le \frac{|\mathcal{H}|}{\beta(1-\alpha)} \|\delta_z\|_{-1,s,\Omega}\}$.

Let $w_1, w_2 \in \mathcal{W}$, then the inf-sup condition (10), (13), and (12) yield

$$\beta \|\nabla(\mathcal{F}(w_1) - \mathcal{F}(w_2))\|_{L^s(\Omega)} \le \alpha\beta \|\nabla(w_1 - w_2)\|_{L^s(\Omega)}.$$

This shows that the mapping $\mathcal{F}$ is contractive on the space $\mathcal{W}$. Thus according to Banach's fixed point theorem, we know that problem (9) has a unique solution satisfying (11) under condition (12). □

With Theorems 2.1 and 2.2 at hand, now we are ready to study the existence of solutions of problem (4). To achieve the goal, we consider a mapping $\mathcal{F}^* : \mathcal{V} \times W_0^{1,s}(\Omega) \to \mathcal{V} \times W_0^{1,s}(\Omega)$ ($\frac{6-d+\epsilon}{5-d+\epsilon} < s < \frac{d}{d-1}$): For any given $(\mathbf{w}, \theta) \in \mathcal{V} \times W_0^{1,s}(\Omega)$, $\mathcal{F}^*(\mathbf{w}, \theta) = (\mathbf{u}, T) \in \mathcal{V} \times W_0^{1,s}(\Omega)$ being the solution of

$$A_F(\theta; \mathbf{u}, \mathbf{v}) + O_F(\mathbf{u}; \mathbf{u}, \mathbf{v}) - D(\theta, \mathbf{v}) = (\mathbf{F}, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathcal{V}, \tag{14a}$$

$$A_T(\theta; T, w) - O_T(\mathbf{u}; T, w) = \langle \mathcal{H}\delta_z, w \rangle, \qquad \forall w \in W_0^{1,s'}(\Omega). \tag{14b}$$

Lemma 2.1. *For $\frac{6-d+\epsilon}{5-d+\epsilon} < s < \frac{d}{d-1}$, if the singular heat source satisfies*

$$|\mathcal{H}| \|\delta_z\|_{-1,s,\Omega} \le \beta(1-\alpha) S, \tag{15}$$

the mapping $\mathcal{F}^*$ is well-defined on the space $\mathcal{W}_u \times \mathcal{W}_T$ and maps $\mathcal{W}_u \times \mathcal{W}_T$ to itself. Here

$$S = \frac{\min\{G\nu_-, \nu_-^2 / C_{2,4}^2\} - C_{2,2} \|\boldsymbol{F}\|_{L^2(\Omega)}}{C_{s,2} C_{2,2} \|\boldsymbol{g}\|_{L^\infty(\Omega)}}, \quad G = \frac{\alpha\beta}{C_{2,d} C_{s,\frac{ds}{d-s}}},$$

$$\mathcal{W}_u = \{\boldsymbol{w} \in \mathcal{V} : \|\nabla \boldsymbol{w}\|_{L^2(\Omega)} \le G\}, \quad \mathcal{W}_T = \Big\{\theta \in W_0^{1,s}(\Omega) : \|\nabla \theta\|_{L^s(\Omega)} \le S\Big\},$$

$C_{2,2}, C_{2,4}, C_{s,2}, C_{2,d}, C_{s,\frac{ds}{d-s}}$ are the constants in (3), β is the constant in the inf-sup condition (10), and α is given in Theorem 2.2.

Proof. For any $\theta \in \mathcal{W}_T$, Theorem 2.1 shows that equation (14a) has a unique solution $\mathbf{u} \in \mathcal{V}$ satisfying

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} \leq \nu_-^{-1} C_{2,2}\big(\|\mathbf{F}\|_{L^2(\Omega)} + C_{s,2}\|\mathbf{g}\|_{L^\infty(\Omega)}\|\nabla\theta\|_{L^s(\Omega)}\big) \leq G.$$

Hence $\mathbf{u} \in \mathcal{W}_u$ satisfying the condition (12), then according to Theorem 2.2 we know that problem (14b) has a unique solution $T \in W_0^{1,s}(\Omega)$. With the help of (11) and (15), we have

$$\|\nabla T\|_{L^s(\Omega)} \leq \frac{|\mathcal{H}|}{\beta(1-\alpha)}\|\delta_z\|_{-1,s,\Omega} \leq S.$$

Thus $T \in \mathcal{W}_T$. $\square$

Theorem 2.3 (Existence)**.** *If the singular heat source satisfies the condition* (15)*, problem* (4) *has a solution* $(\boldsymbol{u}, p, T) \in \mathcal{W}_u \times L_0^2(\Omega) \times \mathcal{W}_T$*, where the spaces* $\mathcal{W}_u$ *and* $\mathcal{W}_T$ *are defined in Lemma 2.1.*

Proof. If the mapping $\mathcal{F}^*$ has a fixed point $(\mathbf{u}, T) \in \mathcal{W}_u \times \mathcal{W}_T$, by equation (4a) we can derive a unique pressure $p \in L_0^2(\Omega)$ corresponding to the velocity $\mathbf{u}$ with the inf-sup condition, and the triplet $(\mathbf{u}, p, T) \in \mathcal{W}_u \times L_0^2(\Omega) \times \mathcal{W}_T$ is a solution of problem (4).

Hence we only need to prove that the mapping $\mathcal{F}^*$ has a fixed point on the space $\mathcal{W}_u \times \mathcal{W}_T$ in this theorem. On the one hand, it is easy to observe that the space $\mathcal{W}_u \times \mathcal{W}_T$ is nonempty, closed, bounded, and convex, and $\mathcal{F}^*$ maps the space $\mathcal{W}_u \times \mathcal{W}_T$ to itself (see Lem. 2.1). On the other hand, according to Leray–Schauder fixed point theorem, we know that we still need to prove that the mapping $\mathcal{F}^*$ is compact. To this end, let $\{\mathbf{w}_n, \theta_n\}_{n\geq 0} \subset \mathcal{W}_u \times \mathcal{W}_T$ be such that $(\mathbf{w}_n, \theta_n) \rightharpoonup (\mathbf{w}, \theta)$ in $(H_0^1(\Omega))^d \times W_0^{1,s}(\Omega)$, then we prove $(\mathbf{u}_n, T_n) \to (\mathbf{u}, T)$ in $(H_0^1(\Omega))^d \times W_0^{1,s}(\Omega)$, where $(\mathbf{u}_n, T_n) = \mathcal{F}^*(\mathbf{w}_n, \theta_n)$ and $(\mathbf{u}, T) = \mathcal{F}^*(\mathbf{w}, \theta)$.

Since $(\mathbf{u}_n, T_n) \subset \mathcal{W}_u \times \mathcal{W}_T$ is bounded, there exists a subsequence (still labeled by n) such that $(\mathbf{u}_n, T_n) \rightharpoonup (\widetilde{\mathbf{u}}, \widetilde{T}) \subset \mathcal{W}_u \times \mathcal{W}_T$ in $(H_0^1(\Omega))^d \times W_0^{1,s}(\Omega)$, and which, together with $\theta_n \to \theta$ in $L^2(\Omega)$ and $\mathbf{u}_n \to \widetilde{\mathbf{u}}$ in $(L^m(\Omega))^d$ for $m = d, 4$, yields

$$D(\theta_n, \mathbf{v}) \to D(\theta, \mathbf{v}), \quad O_T(\mathbf{u}_n; T_n, w) \to O_T\Big(\widetilde{\mathbf{u}}; \widetilde{T}, w\Big), \quad O_F(\mathbf{u}_n; \mathbf{u}_n, \mathbf{v}) \to O_F(\widetilde{\mathbf{u}}; \widetilde{\mathbf{u}}, \mathbf{v}) \tag{16}$$

$$(\nu(\theta)\nabla(\mathbf{u}_n - \widetilde{\mathbf{u}}), \nabla \mathbf{v}) \to 0, \quad \Big(\kappa(\theta)\nabla\Big(T_n - \widetilde{T}\Big), \nabla w\Big) \to 0, \tag{17}$$

as $n \to \infty$, for any test functions $\mathbf{v} \in (H_0^1(\Omega))^d$ and $w \in W_0^{1,s'}(\Omega)$. Furthermore, with the Lipschitz continuity of ν and κ and the Lebesgue dominated convergence theorem, we have

$$((\nu(\theta_n) - \nu(\theta))\nabla \mathbf{u}_n, \nabla \mathbf{v}) \to 0, \quad ((\kappa(\theta_n) - \kappa(\theta))\nabla T_n, \nabla w) \to 0, \quad \text{as } n \to \infty, \tag{18}$$

which, together with (17), leads to

$$A_F(\theta_n; \mathbf{u}_n, \mathbf{v}) \to A_F(\theta; \widetilde{\mathbf{u}}, \mathbf{v}) \quad \text{and} \quad A_T(\theta_n; T_n, w) \to A_T\Big(\theta; \widetilde{T}, w\Big).$$

Hence $(\widetilde{\mathbf{u}}, \widetilde{T}) = (\mathbf{u}, T)$ and $(\mathbf{u}_n, T_n) \rightharpoonup (\mathbf{u}, T)$ in $(H_0^1(\Omega))^d \times W_0^{1,s}(\Omega)$.

Next, we are going to prove strong convergence. By a simple calculation and inf-sup condition (10), we infer that

$$\nu_-\|\nabla(\mathbf{u}_n - \mathbf{u})\|_{L^2(\Omega)}^2 \leq ((\nu(\theta_n) - \nu(\theta))\nabla \mathbf{u}, \nabla(\mathbf{u}_n - \mathbf{u})) + O_F(\mathbf{u} - \mathbf{u}_n; \mathbf{u}, \mathbf{u}_n - \mathbf{u}) + D(\theta - \theta_n, \mathbf{u} - \mathbf{u}_n), \tag{19}$$

and

$$\beta\|\nabla(T_n - T)\|_{L^s(\Omega)} \leq \sup_{\forall w \in W_0^{1,s'}(\Omega)} \frac{(\kappa(\theta_n)\nabla(T_n - T), \nabla w)}{\|\nabla w\|_{L^{s'}(\Omega)}}$$

$$= \sup_{\forall w \in W_0^{1,s'}(\Omega)} \frac{((\kappa(\theta) - \kappa(\theta_n))\nabla T, \nabla w) + O_T(\mathbf{u}_n; T_n, w) - O_T(\mathbf{u}; T, w)}{\|\nabla w\|_{L^{s'}(\Omega)}}. \tag{20}$$

Then the strong convergence of $\mathbf{u}$ in $(L^m(\Omega))^d$ for $m = 2, 4$, (16), and (18) derive that the right hand sides of (19) and (20) tend to zero as $n \to \infty$. Thus we conclude the proof. □

Remark 2.2. In order to guarantee the existence of solutions for problem (4), we know from Lemma 2.1 and Theorem 2.3 that the heat source $|\mathcal{H}|\|\delta\|_{-1,s,\Omega}$ and the data $\|\mathbf{F}\|_{L^2(\Omega)}$ need to be sufficiently small.

Remark 2.3. Due to the low regularity of T induced by the Dirac delta, the approach introduced in Theorem 2.3 of [44] to prove the uniqueness of solutions by assuming that the maximum norm of temperature gradient is sufficiently small is unavailable in this paper.

2.2. HDG discretization

Let $\mathcal{T}_h$ be a conforming and shape-regular triangulation of the domain Ω. Denote by $\mathcal{E}_h^o$ the set of all interior edges/faces of $\mathcal{T}_h$ and $\mathcal{E}_h^\partial$ the set of all boundary edges/faces. We define $\mathcal{E}_h = \mathcal{E}_h^o \bigcup \mathcal{E}_h^\partial$ and $\partial\mathcal{T}_h := \{\partial K : K \in \mathcal{T}_h\}$, where ∂K is the boundary of the element K. For any $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$, let h_K and h_F be the diameters of K and F. For the interior point $z \in \Omega$, let $\mathcal{T}_z := \{K \in \mathcal{T}_h : z \in \overline{K}\}$ and $\sharp\mathcal{T}_z$ be the number of elements in $\mathcal{T}_z$. In addition, we set

$$h = \max_{K \in \mathcal{T}_h} h_K, \quad \underline{h} = \min_{K \in \mathcal{T}_h} h_K,$$

and define

$$\begin{aligned}
(\cdot,\cdot)_{\mathfrak{S}} &= \sum_{K \in \mathfrak{S}} (\cdot,\cdot)_K, & \langle\cdot,\cdot\rangle_{\partial\mathfrak{S}} &= \sum_{K \in \mathfrak{S}} \langle\cdot,\cdot\rangle_{\partial K},\\
\|\cdot\|_{L^p(\mathfrak{S})}^p &= \sum_{K \in \mathfrak{S}} \|\cdot\|_{L^p(K)}^p \quad 1 \le p < \infty, & \|\cdot\|_{L^\infty(\mathfrak{S})} &= \max_{K \in \mathfrak{S}} \|\cdot\|_{L^\infty(K)},\\
\|\cdot\|_{L^p(\partial\mathfrak{S})}^p &= \sum_{K \in \mathcal{T}_h} \|\cdot\|_{L^p(\partial K)}^p \quad 1 \le p < \infty, & \|\cdot\|_{L^\infty(\partial\mathfrak{S})} &= \max_{K \in \mathfrak{S}} \|\cdot\|_{L^\infty(\partial K)},
\end{aligned}$$

for any $\mathfrak{S} \subset \mathcal{T}_h$, where $(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_{\partial K}$ denote the inner products in $L^2(K)$ and $L^2(\partial K)$.

Next, we introduce discontinuous finite element spaces corresponding to the partition $\mathcal{T}_h$:

$$\begin{aligned}
V_h &= \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^d : \mathbf{v}|_K \in \left(\mathcal{P}^k(K)\right)^d, \ \forall K \in \mathcal{T}_h\right\},\\
Q_h &= \left\{q \in L_0^2(\Omega) : q|_K \in \mathcal{P}^{k-1}(K), \ \forall K \in \mathcal{T}_h\right\},\\
W_h &= \left\{w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K), \ \forall K \in \mathcal{T}_h\right\},\\
\widehat{V}_h &= \left\{\widehat{\mathbf{v}} \in \left(L^2(\mathcal{E}_h)\right)^d : \widehat{\mathbf{v}}|_F \in \left(\mathcal{P}^k(F)\right)^d \ \forall F \in \mathcal{E}_h, \ \widehat{\mathbf{v}} = 0 \text{ on } \mathcal{E}_h^\partial\right\},\\
\widehat{Q}_h &= \left\{\widehat{q} \in L^2(\mathcal{E}_h) : \widehat{q}|_F \in \mathcal{P}^k(F), \ \forall F \in \mathcal{E}_h\right\},\\
\widehat{W}_h &= \left\{\widehat{w} \in L^2(\mathcal{E}_h) : \widehat{w}|_F \in \mathcal{P}^k(F) \ \forall F \in \mathcal{E}_h, \ \widehat{w} = 0 \text{ on } \mathcal{E}_h^\partial\right\},
\end{aligned}$$

where $\mathcal{P}^m(\mathfrak{D})$ is the set of polynomials of degree no larger than m on the domain $\mathfrak{D}$ and $k \ge 1$.

Based on idea of [35, 47, 54], we take the following discretizations for the bilinear forms A_F, A_T, B and the convective terms O_F, O_T:

$$A_F^h(\theta_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) = (\nu(\theta_h)\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} \frac{(\nu_- + \nu_+)\alpha_F}{2h_K} \langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial K}$$

$$
\begin{aligned}
&- \langle (\nu(\theta_h)\nabla \mathbf{u}_h)\mathbf{n}, \mathbf{v}_h - \widehat{\mathbf{v}}_h \rangle_{\partial\mathcal{T}_h} - \langle (\nu(\theta_h)\nabla \mathbf{v}_h)\mathbf{n}, \mathbf{u}_h - \widehat{\mathbf{u}}_h \rangle_{\partial\mathcal{T}_h},\\
O_F^h(\mathbf{w}_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) &= -(\mathbf{u}_h \otimes \mathbf{w}_h, \nabla \mathbf{v}_h)_{\mathcal{T}_h} + \frac{1}{2}\langle (\mathbf{w}_h\cdot\mathbf{n})(\mathbf{u}_h + \widehat{\mathbf{u}}_h), \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad + \frac{1}{2}\langle |\mathbf{w}_h\cdot\mathbf{n}|(\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h},\\
B^h((\mathbf{v}_h, \widehat{\mathbf{v}}_h), (q_h, \widehat{q}_h)) &= (q_h, \nabla\cdot\mathbf{v}_h)_{\mathcal{T}_h} - \langle (\mathbf{v}_h - \widehat{\mathbf{v}}_h)\cdot\mathbf{n}, \widehat{q}_h\rangle_{\partial\mathcal{T}_h},
\end{aligned}
$$

and

$$
\begin{aligned}
A_T^h\Big(\theta_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) &= (\kappa(\theta_h)\nabla T_h, \nabla w_h)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h} \frac{(\kappa_- + \kappa_+)\alpha_T}{2h_K}\Big\langle T_h - \widehat{T}_h, w_h - \widehat{w}_h\Big\rangle_{\partial K}\\
&\quad - \langle \kappa(\theta_h)\nabla T_h\cdot\mathbf{n}, w_h - \widehat{w}_h\rangle_{\partial\mathcal{T}_h} - \Big\langle \kappa(\theta_h)\nabla w_h\cdot\mathbf{n}, T_h - \widehat{T}_h\Big\rangle_{\partial\mathcal{T}_h},\\
O_T^h\Big(\mathbf{w}_h; \Big(T_h, \widehat{T}_h\Big), (\varphi_h, \widehat{\varphi}_h)\Big) &= (\mathbf{w}_h T_h, \nabla\varphi_h)_{\mathcal{T}_h} - \frac{1}{2}\Big\langle (\mathbf{w}_h\cdot\mathbf{n})\Big(T_h + \widehat{T}_h\Big), \varphi_h - \widehat{\varphi}_h\Big\rangle_{\partial\mathcal{T}_h}\\
&\quad - \frac{1}{2}\Big\langle |\mathbf{w}_h\cdot\mathbf{n}|\Big(T_h - \widehat{T}_h\Big), \varphi_h - \widehat{\varphi}_h\Big\rangle_{\partial\mathcal{T}_h},
\end{aligned}
$$

where $\mathbf{n}$ denotes the unit outward normal vector to ∂K, α_F and α_T are the stabilization parameters, and $\otimes$ denotes the tensor product.

Then the HDG discretization of problem (4) reads as follows: Find $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h \times W_h \times \widehat{W}_h$ such that

$$
\begin{aligned}
&A_F^h(T_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + O_F^h(\mathbf{u}_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h))\\
&\qquad - B^h((\mathbf{v}_h, \widehat{\mathbf{v}}_h), (p_h, \widehat{p}_h)) - D(T_h, \mathbf{v}_h) = (\mathbf{F}, \mathbf{v}_h), \qquad (21a)
\end{aligned}
$$

$$
B^h((\mathbf{u}_h, \widehat{\mathbf{u}}_h), (q_h, \widehat{q}_h)) = 0, \qquad (21b)
$$

$$
A_T^h\Big(T_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) - O_T^h\Big(\mathbf{u}_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) = \frac{\mathcal{H}}{\sharp\mathcal{T}_z}\sum_{K\in\mathcal{T}_z} w_h|_K(z), \qquad (21c)
$$

for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h, q_h, \widehat{q}_h, w_h, \widehat{w}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h \times W_h \times \widehat{W}_h$.

In equation (21b), let $q_h = 0$, we have

$$
\langle (\mathbf{u}_h - \widehat{\mathbf{u}}_h)\cdot\mathbf{n}, \widehat{q}_h\rangle_{\partial\mathcal{T}_h} = 0, \qquad \forall \widehat{q}_h \in \widehat{Q}_h,
$$

Since $\widehat{\mathbf{u}}_h$ and $\widehat{q}_h$ are single-valued on each edge/face, the jump $[\mathbf{u}_h\cdot\mathbf{n}]$ is zero on each edge/face. This shows that the approximate velocity field obtained by (21) is $H(\mathrm{div})$-conforming. For each interior edge/face F, let K^+ and K^- be two elements sharing the common edge/face F, then the jump is defined as $[\mathbf{u}_h\cdot\mathbf{n}] = \mathbf{u}_h^+\cdot\mathbf{n}^+ + \mathbf{u}_h^-\cdot\mathbf{n}^-$, where $\mathbf{u}_h^+$ and $\mathbf{u}_h^-$ are restrictions of $\mathbf{u}_h$ on $F\cap\partial K^+$ and $F\cap\partial K^-$, and $\mathbf{n}^+$ and $\mathbf{n}^-$ denote unit outward normal vectors to $F\cap\partial K^+$ and $F\cap\partial K^-$. As for the boundary edge/face, we set $[\mathbf{u}_h\cdot\mathbf{n}] = \mathbf{u}_h\cdot\mathbf{n}$.

On the other hand, it is easy to show by the property of $H(\mathrm{div})$-conformity that $(\nabla\cdot\mathbf{u}_h, 1)_{\mathcal{T}_h} = 0$. Thus we can set $q_h = \nabla\cdot\mathbf{u}_h$ and $\widehat{q}_h = 0$ in equation (21b). This leads to

$$
\|\nabla\cdot\mathbf{u}_h\|_{L^2(\mathcal{T}_h)}^2 = 0,
$$

which means that the approximate velocity field derived by (21) is pointwise divergence-free within each element.

We conclude this subsection by recalling some useful properties for A_F^h, O_F^h, A_T^h and O_T^h. Before this, we introduce mesh-dependent seminorms

$$
|||(v,\mu)|||_h^2 = \|\nabla v\|_{L^2(\mathcal{T}_h)}^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1}\|v-\mu\|_{L^2(\partial K)}^2,
$$

$$|||(v,\mu)|||_{h'}^2 = |||(v,\mu)\,|||_h^2 + \sum_{K\in\mathcal{T}_h} h_K \|\nabla v\cdot\mathbf{n}\|_{L^2(\partial K)}^2,$$

and set

$$\|v\|_{1,h} = |||(v,\{v\})|||_h,$$

where the average of v, $\{v\}$, is defined as follows. On the interior edge/face, we have $\{v\} = \frac{1}{2}(v^+ + v^-)$. As for the boundary edge/face, we set $\{v\} = 0$. Note that $\|v\|_{1,h}$ is nothing but the standard discrete H^1-norm of v.

Lemma 2.2. *If $1 \le q < \infty$ for $d = 2$ and $1 \le q \le 6$ for $d = 3$, we have*

$$\|v\|_{L^q(\Omega)} \le C_1 \|v\|_{1,h}, \tag{22}$$

$$\|v\|_{L^q(\Omega)} \le C_1\, |||\,(v,\mu)|||_h, \tag{23}$$

for any $v \in W_h + H_0^1(\Omega)$ and $\mu \in \widehat{W}_h + H_0^{1/2}(\mathcal{E}_h)$, where $H_0^{1/2}(\mathcal{E}_h) = \{\mu \in H^{1/2}(\mathcal{E}_h) : \mu = 0$ on $\mathcal{E}_h^\partial\}$. In addition, the following estimate holds

$$\left(\sum_{K\in\mathcal{T}_h} h_K \|v\|_{L^4(\partial K)}^4\right)^{1/4} \le C_2 \|v\|_{1,h} \le C_2\, |||\,(v,\mu)\,|||_h\,. \tag{24}$$

Proof. The inequalities (22) and (23) have been proved in Proposition A.2 of [19], Theorem 2.1 of [21], and Theorem 5.3 of [22]. As for the inequality (24), it is a direct result of (22) and (7.7) of [33]. □

In Lemmas 4.2 and 4.3 of [48] and Lemmas 5.2 and 5.3 of [54], it was shown that for sufficiently large α_F and α_T the bilinear forms A_F^h and A_T^h are coercive and bounded, *i.e.*, there are positive constants $C_F^s, C_F^b, C_T^s, C_T^b$ independent of mesh parameters such that for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in V_h \times \widehat{V}_h$, $(\mathbf{w}, \widehat{\mathbf{w}}) \in V(h) \times \widehat{V}(h)$, $(v_h, \widehat{v}_h) \in W_h \times \widehat{W}_h$, $(w, \widehat{w}) \in W(h) \times \widehat{W}(h)$

$$\nu_- C_F^s\, |||\,(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|||_h^2 \le A_F^h(\cdot; (\mathbf{v}_h, \widehat{\mathbf{v}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)), \tag{25}$$

$$A_F^h(\cdot; (\mathbf{w}, \widehat{\mathbf{w}}), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) \le \nu_+ C_F^b\, |||\,(\mathbf{w}, \widehat{\mathbf{w}})\,|||_{h'}\, |||(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|||_h, \tag{26}$$

and

$$\kappa_- C_T^s\, |||\,(v_h, \widehat{v}_h)|||_h^2 \le A_T^h(\cdot; (v_h, \widehat{v}_h), (v_h, \widehat{v}_h)), \tag{27}$$

$$A_T^h(\cdot; (w, \widehat{w}), (v_h, \widehat{v}_h)) \le \kappa_+ C_T^b\, |||\,(w, \widehat{w})\,|||_{h'}\, |||(v_h, \widehat{v}_h)|||_h, \tag{28}$$

where $V(h) = V_h + (H_0^1(\Omega))^d \cap (H^2(\Omega))^d$, $\widehat{V}(h) = \widehat{V}_h + (H_0^{3/2}(\mathcal{E}_h))^d$, $W(h) = W_h + H_0^1(\Omega) \cap H^2(\Omega)$, $\widehat{W}(h) = \widehat{W}_h + H_0^{3/2}(\mathcal{E}_h)$, and $H_0^{3/2}(\mathcal{E}_h) = \{\mu \in H^{3/2}(\mathcal{E}_h) : \mu = 0 \text{ on } \mathcal{E}_h^\partial\}$.

In Propositions 3.4 and 3.6 of [19], it was shown that the convective terms O_F^h and $-O_T^h$ satisfy the following continuity and positivity, *i.e.*, there are positive constants C_{OF} and C_{OT} independent of mesh parameters such that for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in V_h \times \widehat{V}_h$, $(\mathbf{w}, \widehat{\mathbf{w}}) \in V(h) \times \widehat{V}(h)$, $\mathbf{w}_1, \mathbf{w}_2 \in V(h)$, $\boldsymbol{\xi} \in \mathcal{V}_h$, $(v_h, \widehat{v}_h) \in W_h \times \widehat{W}_h$, $(w, \widehat{w}) \in W(h) \times \widehat{W}(h)$

$$O_F^h(\boldsymbol{\xi}; (\mathbf{v}_h, \widehat{\mathbf{v}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) = \frac{1}{2}\langle|\boldsymbol{\xi}\cdot\mathbf{n}|(\mathbf{v}_h - \widehat{\mathbf{v}}_h), (\mathbf{v}_h - \widehat{\mathbf{v}}_h)\rangle_{\partial\mathcal{T}_h} \ge 0, \tag{29}$$

$$\left|O_F^h(\mathbf{w}_1; (\mathbf{w}, \widehat{\mathbf{w}}), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - O_F^h(\mathbf{w}_2; (\mathbf{w}, \widehat{\mathbf{w}}), (\mathbf{v}_h, \widehat{\mathbf{v}}_h))\right| \le C_{OF} \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h}\, |||\,(\mathbf{w}, \widehat{\mathbf{w}})\,|||_h\, |||(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|||_h, \tag{30}$$

and

$$O_T^h(\boldsymbol{\xi}; (v_h, \widehat{v}_h), (v_h, \widehat{v}_h)) = -\frac{1}{2}\langle|\boldsymbol{\xi}\cdot\mathbf{n}|(v_h - \widehat{v}_h), (v_h - \widehat{v}_h)\rangle_{\partial\mathcal{T}_h} \le 0, \tag{31}$$

$$\left|O_T^h(\mathbf{w}_1; (w, \widehat{w}), (v_h, \widehat{v}_h)) - O_T^h(\mathbf{w}_2; (w, \widehat{w}), (v_h, \widehat{v}_h))\right| \le C_{OT} \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h}\, |||\,(w, \widehat{w})\,|||_h\, |||(v_h, \widehat{v}_h)|||_h, \tag{32}$$

where $\mathcal{V}_h = \{\mathbf{v}_h \in V_h : B^h((\mathbf{v}_h, \mathbf{0}), (q_h, \widehat{q}_h)) = 0,\ \forall (q_h, \widehat{q}_h) \in Q_h \times \widehat{Q}_h\}$.

2.3. Existence of discrete solutions

In this subsection, we devote to prove the existence of solutions for problem (21) under a similar assumption to Theorem 2.3. Before this, we introduce the following well-known trace and inverse estimates.

Lemma 2.3 ([17], Lem. 4.5.3 and [29], (2.3)). *For any $K \in \mathcal{T}_h$, $v \in \mathcal{P}^k(K)$ and $w \in W^{1,s}(K)$,*

$$\|v\|_{s,l,K} \le C_{inv} h_K^{m-s+\frac{d}{l}-\frac{d}{q}} \|v\|_{m,q,K}, \quad 1 \le q \le l \le \infty, \quad 0 \le m \le s \le k+1, \tag{33}$$

and

$$\|w\|_{L^s(\partial K)} \le C_{tra}\Big(h_K^{-\frac{1}{s}} \|w\|_{L^s(K)} + h_K^{1-\frac{1}{s}} \|\nabla w\|_{L^s(K)}\Big), \quad 1 \le s \le \infty. \tag{34}$$

Similar with the continuous case, we first consider the discrete problems (21a) and (21b) with T_h replaced by a given $\theta_h \in W_h$: Find $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$ such that

$$A_F^h(\theta_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + O_F^h(\mathbf{u}_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - B^h((\mathbf{v}_h, \widehat{\mathbf{v}}_h), (p_h, \widehat{p}_h)) - D(\theta_h, \mathbf{v}_h) = (\mathbf{F}, \mathbf{v}_h), \tag{35a}$$

$$B^h((\mathbf{u}_h, \widehat{\mathbf{u}}_h), (q_h, \widehat{q}_h)) = 0, \tag{35b}$$

for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h, q_h, \widehat{q}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$.

Theorem 2.4. *Assuming*

$$\|\boldsymbol{F}\|_{L^2(\Omega)} + \|\boldsymbol{g}\theta_h\|_{L^2(\Omega)} < \frac{(\nu_- C_F^s)^2}{C_1 C_{OF}}, \tag{36}$$

where C_1 is the constant in Lemma 2.2, C_F^s and C_{OF} are the constants in (25) and (30), then problem (35) has a unique solution $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$ satisfying

$$|||(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)|||_h \le \frac{C_1}{\nu_- C_F^s}\Big(\|\boldsymbol{F}\|_{L^2(\Omega)} + \|\boldsymbol{g}\theta_h\|_{L^2(\Omega)}\Big). \tag{37}$$

Proof. Let $\mathbf{f} = \mathbf{F} + \mathbf{g}\theta_h \in (L^2(\Omega))^d$, then the proof of the theorem is the same as Lemma 1 of [35], thus we omit it. $\square$

Next for any given $\theta_h \in W_h$ and $\mathbf{w}_h \in \mathcal{V}_h$, we study the discrete problem (21c): Find $(T_h, \widehat{T}_h) \in W_h \times \widehat{W}_h$ such that

$$A_T^h\Big(\theta_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) - O_T^h\Big(\mathbf{w}_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) = \frac{\mathcal{H}}{\sharp \mathcal{T}_z} \sum_{K \in \mathcal{T}_z} w_h|_K(z), \tag{38}$$

for any $(w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h$.

Theorem 2.5. *For any given $\theta_h \in W_h$ and $\boldsymbol{w}_h \in \mathcal{V}_h$, problem (38) has a unique solution $(T_h, \widehat{T}_h) \in W_h \times \widehat{W}_h$. If $d = 2$,*

$$\left|\left|\left|\Big(T_h, \widehat{T}_h\Big)\right|\right|\right|_h \le \frac{2 C_{inv} C_1}{\kappa_- C_T^s} |\mathcal{H}|, \tag{39}$$

where C_{inv} is the constant in Lemma 2.3, C_1 is the constant in Lemma 2.2, and C_T^s is the constant in (27).

Proof. Since the bilinear form A_T^h is coercive and the convective term $-O_T^h$ possesses positivity (31), problem (38) has a unique solution. If $d = 2$, by (18) of [36] we have

$$\kappa_- C_T^s \left\| \left(T_h, \widehat{T}_h \right) \right\|_h^2 \leq A_T^h \left(\theta_h; \left(T_h, \widehat{T}_h \right), \left(T_h, \widehat{T}_h \right) \right) - O_T^h \left(\mathbf{w}_h; \left(T_h, \widehat{T}_h \right), \left(T_h, \widehat{T}_h \right) \right)$$
$$= \frac{\mathcal{H}}{\sharp \mathcal{T}_z} \sum_{K \in \mathcal{T}_z} T_h|_K(z) \leq 2C_{\text{inv}} C_1 |\mathcal{H}| \left\| \left(T_h, \widehat{T}_h \right) \right\|_h,$$

which derives the bound (39) by dividing $\kappa_- C_T^s \; ||| \; (T_h, \widehat{T}_h) |||_h$ from both sides. □

Remark 2.4. Compared with Proposition 14 of [7], the discrete scheme (38) does not need the smallness assumption on $\mathbf{w}_h$ to guarantee the existence and uniqueness of solutions. The reason is that the convective term O_T^h in this paper includes the upwind schemes.

Now we are ready to study the existence of solutions for problem (21). To achieve the goal, we define a mapping $\mathcal{F}_h^*$: For any $(\mathbf{w}_h, \widehat{\mathbf{w}}_h, \theta_h, \widehat{\theta}_h) \in \mathcal{V}_h \times \widehat{V}_h \times W_h \times \widehat{W}_h$, let $\mathcal{F}_h^*(\mathbf{w}_h, \widehat{\mathbf{w}}_h, \theta_h, \widehat{\theta}_h) = (\mathbf{u}_h, \widehat{\mathbf{u}}_h, T_h, \widehat{T}_h) \in \mathcal{V}_h \times \widehat{V}_h \times W_h \times \widehat{W}_h$ be the solution of

$$A_F^h(\theta_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + O_F^h(\mathbf{u}_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - D(\theta_h, \mathbf{v}_h) = (\mathbf{F}, \mathbf{v}_h), \tag{40a}$$
$$A_T^h \left(\theta_h; \left(T_h, \widehat{T}_h \right), (w_h, \widehat{w}_h) \right) - O_T^h \left(\mathbf{u}_h; \left(T_h, \widehat{T}_h \right), (w_h, \widehat{w}_h) \right) = \frac{\mathcal{H}}{\sharp \mathcal{T}_z} \sum_{K \in \mathcal{T}_z} w_h|_K(z), \tag{40b}$$

for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h, w_h, \widehat{w}_h) \in \mathcal{V}_h \times \widehat{V}_h \times W_h \times \widehat{W}_h$.

Lemma 2.4. *If $d = 2$ and $\mathcal{H}$ satisfies*

$$|\mathcal{H}| \leq \frac{S^* \kappa_- C_T^s}{2C_{inv} C_1}, \tag{41}$$

the mapping $\mathcal{F}_h^*$ is well defined on the space $\mathcal{W}^h = \mathcal{W}_u^h \times \mathcal{W}_T^h$ and maps $\mathcal{W}^h$ to itself. Here

$$S^* = \frac{G^* \nu_- C_F^s - C_1 \|\boldsymbol{F}\|_{L^2(\Omega)}}{C_1^2 \|\boldsymbol{g}\|_{L^\infty(\Omega)}}, \quad G^* = \frac{\nu_- C_F^s}{2C_{OF}},$$
$$\mathcal{W}_u^h = \left\{ (\boldsymbol{w}_h, \widehat{\boldsymbol{w}}_h) \in \mathcal{V}_h \times \widehat{V}_h : |||(\boldsymbol{w}_h, \widehat{\boldsymbol{w}}_h)|||_h \leq G^* \right\},$$
$$\mathcal{W}_T^h = \left\{ \left(\theta_h, \widehat{\theta}_h \right) \in W_h \times \widehat{W}_h : \left\| \left(\theta_h, \widehat{\theta}_h \right) \right\|_h \leq S^* \right\},$$

C_{inv}, C_1, C_T^s, C_F^s, and C_{OF} are the constants in Lemmas 2.3, 2.2, (27), (25), and (30).

Proof. A simple calculation yields

$$\|\mathbf{F}\|_{L^2(\Omega)} + \|\mathbf{g}\theta_h\|_{L^2(\Omega)} \leq \|\mathbf{F}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{L^\infty(\Omega)} \|\theta_h\|_{L^2(\Omega)} \leq \frac{1}{2} \frac{(\nu_- C_F^s)^2}{C_1 C_{OF}}.$$

According to Theorem 2.4, we know that equation (40a) has a unique solution $(\mathbf{u}_h, \widehat{\mathbf{u}}_h) \in \mathcal{V}_h \times \widehat{V}_h$ satisfying

$$|||(\mathbf{u}_h, \widehat{\mathbf{u}}_h)|||_h \leq \frac{C_1}{\nu_- C_F^s} \frac{(\nu_- C_F^s)^2}{2C_1 C_{OF}} = G^*.$$

On the other hand, Theorem 2.5 shows that equation (40b) has a unique solution $(T_h, \widehat{T}_h) \in W_h \times \widehat{W}_h$. Moreover if $d = 2$, (39) and (41) yield

$$\left\| \left(T_h, \widehat{T}_h \right) \right\|_h \leq \frac{2C_{\text{inv}} C_1}{\kappa_- C_T^s} |\mathcal{H}| \leq S^*.$$

Thus $\mathcal{F}_h^*$ is well defined and maps the space $\mathcal{W}^h$ to itself. □

Theorem 2.6 (Existence)**.** *If $d = 2$ and $\mathcal{H}$ satisfies the condition (41), discrete problem (21) has a solution $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h) \in \mathcal{W}_u^h \times Q_h \times \widehat{Q}_h \times \mathcal{W}_T^h$, where the spaces $\mathcal{W}_u^h$ and $\mathcal{W}_T^h$ are defined in Lemma 2.4.*

Proof. Similar with Theorem 2.3, we only need to prove that the mapping $\mathcal{F}_h^*$ has a fixed point in the space $\mathcal{W}^h$. Since the problem (40) is finite dimensional, by Brouwer's fixed point theorem we know that the remaining task is to prove the continuity of $\mathcal{F}_h^*$.

Let $(\mathbf{w}_{h,m}, \widehat{\mathbf{w}}_{h,m}, \theta_{h,m}, \widehat{\theta}_{h,m}) \in \mathcal{W}^h$ be a sequence converging to $(\mathbf{w}_h, \widehat{\mathbf{w}}_h, \theta_h, \widehat{\theta}_h) \in \mathcal{W}^h$ in $|||(\cdot,\cdot)|||_h$. In addition, we let $\mathcal{F}_h^*(\mathbf{w}_{h,m}, \widehat{\mathbf{w}}_{h,m}, \theta_{h,m}, \widehat{\theta}_{h,m}) = (\mathbf{u}_{h,m}, \widehat{\mathbf{u}}_{h,m}, T_{h,m}, \widehat{T}_{h,m}) \in \mathcal{W}^h$ and $\mathcal{F}_h^*(\mathbf{w}_h, \widehat{\mathbf{w}}_h, \theta_h, \widehat{\theta}_h) = (\mathbf{u}_h, \widehat{\mathbf{u}}_h, T_h, \widehat{T}_h) \in \mathcal{W}^h$. By a simple calculation, we have for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h, w_h, \widehat{w}_h) \in \mathcal{V}_h \times \widehat{V}_h \times W_h \times \widehat{W}_h$

$$\begin{aligned}
&A_F^h(\theta_{h,m}; (\mathbf{u}_{h,m} - \mathbf{u}_h, \widehat{\mathbf{u}}_{h,m} - \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + O_F^h(\mathbf{u}_{h,m}; (\mathbf{u}_{h,m} - \mathbf{u}_h, \widehat{\mathbf{u}}_{h,m} - \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) \\
&\quad = A_F^h(\theta_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - A_F^h(\theta_{h,m}; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + D(\theta_{h,m} - \theta_h, \mathbf{v}_h) \\
&\qquad + O_F^h(\mathbf{u}_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - O_F^h(\mathbf{u}_{h,m}; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)).
\end{aligned}$$

According to the definition of A_F^h, we get by Lemma 2.3

$$\begin{aligned}
&A_F^h(\theta_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - A_F^h(\theta_{h,m}; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) \\
&\quad = (\kappa(\theta_h) - \kappa(\theta_{h,m})\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\mathcal{T}_h} - \langle \kappa(\theta_h) - \kappa(\theta_{h,m})(\nabla \mathbf{u}_h)\mathbf{n}, \mathbf{v}_h - \widehat{\mathbf{v}}_h \rangle_{\partial\mathcal{T}_h} \\
&\qquad - \langle \kappa(\theta_h) - \kappa(\theta_{h,m})(\nabla \mathbf{v}_h)\mathbf{n}, \mathbf{u}_h - \widehat{\mathbf{u}}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad \le C\|\theta_h - \theta_{h,m}\|_{L^2(\Omega)} \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_h \Big(\|\nabla \mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} + \underline{h}^{-1} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^\infty(\partial\mathcal{T}_h)} \Big).
\end{aligned} \tag{42}$$

Set $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) = (\mathbf{u}_{h,m} - \mathbf{u}_h, \widehat{\mathbf{u}}_{h,m} - \widehat{\mathbf{u}}_h)$ in the above, then (25), (30), and Lemma 2.2 yield

$$\begin{aligned}
&\nu_- C_F^s \|(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})\|_h^2 \\
&\quad \le C\|\theta_h - \theta_{h,m}\|_{L^2(\Omega)} \|(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})\|_h \Big(\|\nabla \mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} + \underline{h}^{-1} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^\infty(\partial\mathcal{T}_h)} \Big) \\
&\qquad + C_1 \|\theta_h - \theta_{h,m}\|_{L^2(\Omega)} \|\mathbf{g}\|_{L^\infty(\Omega)} \|(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})\|_h \\
&\qquad + C_{OF} \|(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})\|_h \; ||| \; (\mathbf{u}_h, \widehat{\mathbf{u}}_h) \; |||_h \; \|(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})\|_h.
\end{aligned}$$

Due to $(\mathbf{u}_h, \widehat{\mathbf{u}}_h) \in \mathcal{W}_u^h$,

$$\begin{aligned}
\frac{1}{2}\nu_- C_F^s \|(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})\|_h &\le C\|\theta_h - \theta_{h,m}\|_{L^2(\Omega)} \Big(\|\nabla \mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} + \underline{h}^{-1} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^\infty(\partial\mathcal{T}_h)} \Big) \\
&\quad + C_1 \|\theta_h - \theta_{h,m}\|_{L^2(\Omega)} \|\mathbf{g}\|_{L^\infty(\Omega)},
\end{aligned} \tag{43}$$

which leads to $|||(\mathbf{u}_h - \mathbf{u}_{h,m}, \widehat{\mathbf{u}}_h - \widehat{\mathbf{u}}_{h,m})|||_h \to 0$ as $m \to \infty$.

Similar with the velocity, a simple calculation can derive that for any $(w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h$

$$\begin{aligned}
&A_T^h\Big(\theta_{h,m}; \Big(T_{h,m} - T_h, \widehat{T}_{h,m} - \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) - O_T^h\Big(\mathbf{u}_{h,m}; \Big(T_{h,m} - T_h, \widehat{T}_{h,m} - \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) \\
&\quad = A_T^h\Big(\theta_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) - A_T^h\Big(\theta_{h,m}; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) \\
&\qquad + O_T^h\Big(\mathbf{u}_{h,m}; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big) - O_T^h\Big(\mathbf{u}_h; \Big(T_h, \widehat{T}_h\Big), (w_h, \widehat{w}_h)\Big),
\end{aligned} \tag{44}$$

which, together with (27), (31), (32), and Lemma 2.3, results in

$$\kappa_- C_T^s \Big\| \Big(T_{h,m} - T_h, \widehat{T}_{h,m} - \widehat{T}_h\Big) \Big\|_h^2$$

$$\leq C\|\theta_h-\theta_{h,m}\|_{L^2(\Omega)}\left\|\left(T_h-T_{h,m},\widehat{T}_h-\widehat{T}_{h,m}\right)\right\|_h\left(\|\nabla T_h\|_{L^\infty(\mathcal{T}_h)}+\underline{h}^{-1}\left\|T_h-\widehat{T}_h\right\|_{L^\infty(\partial\mathcal{T}_h)}\right)$$
$$+C_{OT}\|(\mathbf{u}_h-\mathbf{u}_{h,m},\widehat{\mathbf{u}}_h-\widehat{\mathbf{u}}_{h,m})\|_h\left\|\left(T_h,\widehat{T}_h\right)\right\|_h\left\|\left(T_h-T_{h,m},\widehat{T}_h-\widehat{T}_{h,m}\right)\right\|_h, \tag{45}$$

by setting $(w_h,\widehat{w}_h)=(T_{h,m}-T_h,\widehat{T}_{h,m}-\widehat{T}_h)$. This shows that $\|(T_{h,m}-T_h,\widehat{T}_{h,m}-\widehat{T}_h)\|_h\to 0$ as $m\to\infty$. Thus $\mathcal{F}_h^*$ is a continuous mapping from $\mathcal{W}^h$ to itself. □

Remark 2.5. In three dimensions, since (23) is only correct for $1\leq q\leq 6$, a similar result with (39) can not be established. However, in order to prove the existence of solutions for the discrete problem (21) by Brouwer's fixed point theorem, a suitable bound for T_h is necessary, hence we only consider the case of $d=2$ in Theorem 2.6.

3. *A priori* ERROR ESTIMATES

Since the existence of solutions of problem (21) can only be guaranteed theoretically for two dimensional case, we assume from now on that $d=2$. In addition, we shall require that the viscosity ν and thermal conductivity κ are two positive constants.

For ease of exposition, in the subsequent analysis we use $a\lesssim b$ to denote $a\leq Cb$, where C with or without subscript denotes a generic positive constant independent of discretization parameters. The value of C might change at each occurrence.

Based on analysis in Section 2.2 we know that the approximate velocity field obtained by (21) is divergence-free and $H(\mathrm{div})$-conforming. Thus the velocity error should be independent of the pressure error. This is the so-called pressure-robustness. To achieve the attractive property, we introduce the BDM interpolation operator in the following lemma.

Lemma 3.1 ([30], Lem. 7)**.** *If the mesh consists of triangles ($d=2$) or tetrahedron ($d=3$) there is an interpolation $\Pi_{\mathrm{BDM}}:(H_0^1(\Omega))^d\to V_h$ satisfying the following properties for $\boldsymbol{v}\in(H^{k+1}(K))^d$ and $K\in\mathcal{T}_h$:*

(1) $[\Pi_{\mathrm{BDM}}\boldsymbol{v}\cdot\boldsymbol{n}]=0$;

(2) $\|\boldsymbol{v}-\Pi_{\mathrm{BDM}}\boldsymbol{v}\|_{m,K}\leq C^1_{\mathrm{BDM}}h_K^{l-m}\|\boldsymbol{v}\|_{l,K}$, $\quad m=0,1,2$ *and* $\max\{1,m\}\leq l\leq k+1$;

(3) $\|\nabla\cdot(\boldsymbol{v}-\Pi_{\mathrm{BDM}}\boldsymbol{v})\|_{m,K}\leq C^2_{\mathrm{BDM}}h^{l-m}\|\nabla\cdot\boldsymbol{v}\|_{l,K}$, $\quad m=0,1$ *and* $m\leq l\leq k$;

(4) $\int_K q(\nabla\cdot\boldsymbol{v}-\nabla\cdot\Pi_{\mathrm{BDM}}\boldsymbol{v})\,\mathrm{d}x=0$, $\quad\forall q\in\mathcal{P}^{k-1}(K)$;

(5) $\int_F\widehat{q}(\boldsymbol{v}-\Pi_{\mathrm{BDM}}\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}s=0$, $\quad\forall\widehat{q}\in\mathcal{P}^k(F)$, *where F is an edge/face of ∂K.*

First of all, we consider error estimates to the temperature. Before this, for any $f\in L^2(\Omega)$ we introduce a duality problem: Find $\phi\in H_0^1(\Omega)$ such that

$$(\kappa\nabla\phi,\nabla w)-(\mathbf{u}_h\cdot\nabla\phi,w)=(f,w),\qquad\forall w\in H_0^1(\Omega), \tag{46}$$

where $\mathbf{u}_h\in\mathcal{V}_h$ is the velocity part of solutions of problem (21).

Lemma 3.2 (Regularity for duality problem)**.** *For any $f\in L^2(\Omega)$, problem (46) has a unique solution $\phi\in H_0^1(\Omega)$ satisfying*

$$\kappa\|\nabla\phi\|_{L^2(\Omega)}\leq C_{2,2}\|f\|_{L^2(\Omega)}. \tag{47}$$

If domain Ω is convex and

$$\frac{4C_{inv}C_1C_3}{\kappa}\,\|(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\|_h\leq 1, \tag{48}$$

the solution $\phi\in H^2(\Omega)$ and

$$\kappa\|\phi\|_{2,\Omega}\leq 2C_3\|f\|_{L^2(\Omega)}, \tag{49}$$

where C_3 is a positive constant depending on the domain Ω, $C_{2,2}$, C_1, and C_{inv} are the constants in (3), *Lemmas 2.2, and 2.3, and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ is the velocity part of solutions of problem* (21).

Proof. Since $\mathbf{u}_h$ is divergence-free and $H(\mathrm{div})$-conforming, it is easy to observe that $(\mathbf{u}_h \cdot \nabla w, w) = 0$ for any $w \in H_0^1(\Omega)$. Thus the existence of a unique solution for problem (46) can be guaranteed by the Lax–Milgram theorem, and the bound (47) holds.

On the other hand, Lemma 2.2 and (18) of [36] yield

$$\|\mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} \leq 2C_{\mathrm{inv}} C_1 \, \|\!| \, (\mathbf{u}_h, \widehat{\mathbf{u}}_h) \, |\!\|_h \, . \tag{50}$$

Hence $\mathbf{u}_h \cdot \nabla\phi \in L^2(\Omega)$, and under assumption that the domain Ω is convex we have $\phi \in H^2(\Omega)$ satisfying

$$\begin{aligned}\kappa\|\phi\|_{2,\Omega} &\leq C_3\Big(\|f\|_{L^2(\Omega)} + \|\mathbf{u}_h \cdot \nabla\phi\|_{L^2(\Omega)}\Big)\\ &\leq C_3\|f\|_{L^2(\Omega)} + \frac{\kappa}{2}\|\nabla\phi\|_{L^2(\Omega)},\end{aligned}$$

which concludes the proof. $\square$

Corresponding to problem (46), we consider an HDG discretization: Find $(\phi_h, \widehat{\phi}_h) \in W_h \times \widehat{W}_h$ such that

$$A_T^h\Big(\cdot; \Big(\phi_h, \widehat{\phi}_h\Big), (w_h, \widehat{w}_h)\Big) + O_T^h\Big(\mathbf{u}_h; \Big(\phi_h, \widehat{\phi}_h\Big), (w_h, \widehat{w}_h)\Big) = (f, w_h), \qquad \forall (w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h, \tag{51}$$

where A_T^h and O_T^h are defined in Subsection 2.2.

Lemma 3.3. *If*

$$\frac{2C_{inv}C_1}{\kappa C_T^s} \, \|\!| \, (\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h) |\!\|_h \leq 1, \tag{52}$$

discrete problem (51) *has a unique solution $(\phi_h, \widehat{\phi}_h) \in W_h \times \widehat{W}_h$ satisfying*

$$\Big\|\!\Big|\Big(\phi_h, \widehat{\phi}_h\Big)\Big|\!\Big\|_h \leq \frac{2C_1}{\kappa C_T^s}\|f\|_{L^2(\Omega)}, \tag{53}$$

where C_1, C_T^s, and C_{inv} are the constants in Lemma 2.2, (27)*, and Lemma 2.3, and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)$ is the velocity part of solutions of problem* (21).

Proof. Using (50), positivity (31), and condition (52), we have for any $(w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h$

$$\begin{aligned}-O_T^h(\mathbf{u}_h; (w_h, \widehat{w}_h), (w_h, \widehat{w}_h)) &\leq \frac{1}{2}h\|\mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} \, \|\!| \, (w_h, \widehat{w}_h) |\!\|_h^2\\ &\leq \frac{1}{2}\kappa C_T^s \, \|\!| \, (w_h, \widehat{w}_h) |\!\|_h^2,\end{aligned}$$

which along with coercivity (27) leads to

$$A_T^h(\cdot; (w_h, \widehat{w}_h), (w_h, \widehat{w}_h)) + O_T^h(\mathbf{u}_h; (w_h, \widehat{w}_h), (w_h, \widehat{w}_h)) \geq \frac{1}{2}\kappa C_T^s \, \|\!| \, (w_h, \widehat{w}_h) \, |\!\|_h^2 \, . \tag{54}$$

Thus discrete problem (51) has a unique solution under condition (52) and the bound (53) holds. $\square$

With Lemmas 3.2 and 3.3 at hand, we state *a priori* error estimates in the following theorem for the duality problem (46).

Theorem 3.1. *For any $f \in L^2(\Omega)$, let ϕ and $(\phi_h, \widehat{\phi}_h)$ be the solutions of problems (46) and (51). If domain Ω is convex, partition $\mathcal{T}_h$ is quasi-uniform, and conditions (48) and (52) hold, we have*

$$\left\| \left(\phi - \phi_h, \phi - \widehat{\phi}_h \right) \right\|_h \lesssim h \|\phi\|_{2,\Omega}. \tag{55}$$

Proof. For any $(w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h$, coercivity (54) and consistency

$$A_T^h(\cdot; (\phi, \phi), (w, \widehat{w})) + O_T^h(\mathbf{u}_h; (\phi, \phi), (w, \widehat{w})) = (f, w), \qquad \forall (w, \widehat{w}) \in H^1(\mathcal{T}_h) \times H_0^{1/2}(\mathcal{E}_h), \tag{56}$$

yield

$$\begin{aligned} \frac{1}{2}\kappa C_T^s \left\| \left(w_h - \phi_h, \widehat{w}_h - \widehat{\phi}_h \right) \right\|_h^2 \leq\ & A_T^h\Big(\cdot; (w_h - \phi, \widehat{w}_h - \phi), \Big(w_h - \phi_h, \widehat{w}_h - \widehat{\phi}_h\Big)\Big) \\ & + O_T^h\Big(\mathbf{u}_h; (w_h - \phi, \widehat{w}_h - \phi), \Big(w_h - \phi_h, \widehat{w}_h - \widehat{\phi}_h\Big)\Big), \end{aligned}$$

which, together with triangle inequality, bounds (28) and (32), and condition (52), leads to

$$\left\| \left(\phi - \phi_h, \phi - \widehat{\phi}_h \right) \right\|_h \leq \left(1 + \frac{2C_T^b}{C_T^s} + \frac{C_{OT}}{C_{\text{inv}} C_1} \right) \inf_{(w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h} \left\| (\phi - w_h, \phi - \widehat{w}_h) \right\|_{h'} . \tag{57}$$

Thus we conclude the proof by using interpolation error estimates. $\square$

Now, we are ready to prove error estimates of the temperature.

Lemma 3.4. *Let $(\boldsymbol{u}, T)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, T_h, \widehat{T}_h)$ be the velocity and temperature parts of solutions of problems (4) and (21). If domain Ω is convex, partition $\mathcal{T}_h$ is quasi-uniform, and conditions (48) and (52) hold, we have*

$$\|T - T_h\|_{L^2(\Omega)} \leq C_4 \kappa^{-1} \big(h|\mathcal{H}| + \|T\|_{L^2(\Omega)} \left\| (\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \widehat{\boldsymbol{u}}_h) \right\|_h \big). \tag{58}$$

Proof. In order to estimate $\|T - T_h\|_{L^2(\Omega)}$, let $T^h \in H_0^1(\Omega)$ be the solution of

$$A_T\big(\cdot; T^h, w\big) - O_T\big(\mathbf{u}_h; T^h, w\big) = (\mathcal{H}\delta_h, w), \qquad \forall w \in H_0^1(\Omega), \tag{59}$$

where $\delta_h = 0$ on $\mathcal{T}_h \backslash \mathcal{T}_z$ and

$$(\delta_h, w_h)_K = \frac{1}{\sharp \mathcal{T}_z} w_h(z), \qquad \forall w_h \in \mathcal{P}^k(K),\ K \in \mathcal{T}_z.$$

Under condition (48), we know from the proof of Lemma 3.2 that problem (59) has a unique solution $T^h \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$\kappa \|T^h\|_{2,\Omega} \leq 2C_3 |\mathcal{H}| \|\delta_h\|_{L^2(\Omega)} \lesssim |\mathcal{H}| h^{-1}. \tag{60}$$

Using integration by parts, consistency (56), and bounds (28) and (32), we infer that

$$\begin{aligned} \big(T^h - T_h, f\big) &= A_T^h\Big(\cdot; (\phi, \phi), \Big(T^h - T_h, T^h - \widehat{T}_h\Big)\Big) + O_T^h\Big(\mathbf{u}_h; (\phi, \phi), \Big(T^h - T_h, T^h - \widehat{T}_h\Big)\Big) \\ &= A_T^h\Big(\cdot; \Big(T^h - T_h, T^h - \widehat{T}_h\Big), (\phi, \phi)\Big) - O_T^h\Big(\mathbf{u}_h; \Big(T^h - T_h, T^h - \widehat{T}_h\Big), (\phi, \phi)\Big) \\ &= A_T^h\Big(\cdot; \Big(T^h - T_h, T^h - \widehat{T}_h\Big), \Big(\phi - \phi_h, \phi - \widehat{\phi}_h\Big)\Big) \\ &\quad - O_T^h\Big(\mathbf{u}_h; \Big(T^h - T_h, T^h - \widehat{T}_h\Big), \Big(\phi - \phi_h, \phi - \widehat{\phi}_h\Big)\Big) \end{aligned}$$

$$\leq \left(\kappa C_T^b + C_{OT}\|\!|(\mathbf{u}_h, \widehat{\mathbf{u}}_h)\|\!|_h\right) \left\|\!\left|\left(T^h - T_h, T^h - \widehat{T}_h\right)\right|\!\right\|_{h'} \left\|\!\left|\left(\phi - \phi_h, \phi - \widehat{\phi}_h\right)\right|\!\right\|_{h'},$$

which, together with Lemma 3.2, Theorem 3.1, (52), (60), Lemma 5.5 of [54], and the duality argument, results in

$$\left\|T^h - T_h\right\|_{L^2(\Omega)} \lesssim \kappa^{-1}|\mathcal{H}|h. \tag{61}$$

Obviously, the remaining task is to prove *a priori* error estimates for $\|T - T^h\|_{L^2(\Omega)}$. According to (4c), (59), and the definition of δ_h, we have

$$\begin{aligned}\left(T - T^h, f\right) &= \left(\kappa\nabla\left(T - T^h\right), \nabla\phi\right) - \left(\mathbf{u}_h\left(T - T^h\right), \nabla\phi\right)\\ &= ((\mathbf{u} - \mathbf{u}_h)T, \nabla\phi) + \mathcal{H}\phi(z) - (\mathcal{H}\delta_h, \phi)\\ &= ((\mathbf{u} - \mathbf{u}_h)T, \nabla\phi) + \mathcal{H}(\phi(z) - \Pi_L\phi(z)) - (\mathcal{H}\delta_h, \phi - \Pi_L\phi),\end{aligned}$$

where Π_L denotes the Lagrange interpolation operator onto continuous linear finite element spaces, which along with interpolation error estimates of Π_L, Lemma 3.2, and the duality argument derives that

$$\left\|T - T^h\right\|_{L^2(\Omega)} \lesssim \kappa^{-1}\left(\|T\|_{L^2(\Omega)} \,|\!|\!|\, (\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \widehat{\mathbf{u}}_h) \,|\!|\!|_h + |\mathcal{H}|h\right). \tag{62}$$

Combining (61) and (62) we conclude the proof. ◻

Next we are going to prove error estimates of the velocity.

Lemma 3.5. *Let $(\boldsymbol{u}, p, T)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ be the solutions of problems (4) and (21). If domain Ω is convex, partition $\mathcal{T}_h$ is quasi-uniform, and*

$$\max\left\{|\!|\!|(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h)|\!|\!|_h, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)}\right\} \leq G^*, \tag{63}$$

we have

$$|\!|\!|(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \widehat{\boldsymbol{u}}_h)|\!|\!|_h \leq 2C_5 h|\boldsymbol{u}|_{2,\Omega}\left(\frac{3}{2} + \frac{C_F^b}{C_F^s}\right) + \frac{2C_1}{\nu C_F^s}\|\boldsymbol{g}\|_{L^\infty(\Omega)}\|T - T_h\|_{L^2(\Omega)}, \tag{64}$$

where G^* is defined in Lemma 2.4, C_1 is the constant in Lemma 2.2, C_F^s, C_F^b, and C_{OF} are the constants in (25), (26), and (30), and C_5 is the constant in (67).

Proof. By integration by parts, we have the following consistency:

$$\begin{aligned}&A_F^h(\cdot; (\mathbf{u}, \mathbf{u}), (\mathbf{w}_h, \widehat{\mathbf{w}}_h)) + O_F^h(\mathbf{u}; (\mathbf{u}, \mathbf{u}), (\mathbf{w}_h, \widehat{\mathbf{w}}_h)) - B^h((\mathbf{w}_h, \widehat{\mathbf{w}}_h), (p, p)) - D(T, \mathbf{w}_h)\\ &\quad = (\nu\nabla\mathbf{u}, \nabla\mathbf{w}_h)_{\mathcal{T}_h} - \langle(\nu\nabla\mathbf{u})\mathbf{n}, \mathbf{w}_h - \widehat{\mathbf{w}}_h\rangle_{\partial\mathcal{T}_h} - (\mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{w}_h)_{\mathcal{T}_h}\\ &\qquad + \langle(\mathbf{u}\cdot\mathbf{n})\mathbf{u}, \mathbf{w}_h - \widehat{\mathbf{w}}_h\rangle_{\partial\mathcal{T}_h} - (p, \nabla\cdot\mathbf{w}_h)_{\mathcal{T}_h} + \langle(\mathbf{w}_h - \widehat{\mathbf{w}}_h)\cdot\mathbf{n}, p\rangle_{\partial\mathcal{T}_h}\\ &\qquad - (T\mathbf{g}, \mathbf{w}_h) = (\mathbf{F}, \mathbf{w}_h), \quad \forall(\mathbf{w}_h, \widehat{\mathbf{w}}_h) \in V_h \times \widehat{V}_h.\end{aligned} \tag{65}$$

In the rest of the proof, we set $\mathbf{v}_h = \Pi_{\mathrm{BDM}}\mathbf{u}$ and $\widehat{\mathbf{v}}_h = \Pi_{\widehat{V}}\mathbf{u}$, where $\Pi_{\widehat{V}}$ is the standard L^2-projection operator onto $\widehat{V}_h$. Then coercivity (25) and consistency (65) yield

$$\begin{aligned}\nu C_F^s \,|\!|\!|\, (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)|\!|\!|_h^2 &\leq A_F^h(\cdot; (\mathbf{v}_h - \mathbf{u}, \widehat{\mathbf{v}}_h - \mathbf{u}), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h))\\ &\quad + A_F^h(\cdot; (\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \widehat{\mathbf{u}}_h), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h))\\ &= A_F^h(\cdot; (\mathbf{v}_h - \mathbf{u}, \widehat{\mathbf{v}}_h - \mathbf{u}), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h))\\ &\quad + \left[O_F^h(\mathbf{u}_h; (\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)) - O_F^h(\mathbf{u}; (\mathbf{u}, \mathbf{u}), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h))\right]\end{aligned}$$

$$+ D(T - T_h, \mathbf{v}_h - \mathbf{u}_h) = I_1 + I_2 + I_3. \tag{66}$$

Using interpolation error estimates of Π_{BDM} and $\Pi_{\widehat{V}}$ and Lemma 2.3, it is easy to obtain that

$$|||(\mathbf{v}_h - \mathbf{u}, \widehat{\mathbf{v}}_h - \mathbf{u})|||_{h'} \le C_5 h |\mathbf{u}|_{2,\Omega}. \tag{67}$$

Thus bound (26) and Lemma 2.2 infer that

$$I_1 \le \nu C_F^b C_5 h |\mathbf{u}|_{2,\Omega} \,|||\, (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)|||_h, \tag{68}$$

and

$$I_3 \le C_1 \|\mathbf{g}\|_{L^\infty(\Omega)} \|T - T_h\|_{L^2(\Omega)} \,|||\, (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h) \,|||_h \,. \tag{69}$$

Now, we turn to estimate I_2. By using positivity (29), (30), and (67), we have

$$\begin{aligned}
I_2 \le\; & O_F^h(\mathbf{u}_h; (\mathbf{v}_h - \mathbf{u}, \widehat{\mathbf{v}}_h - \mathbf{u}), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)) \\
& + O_F^h(\mathbf{u}_h; (\mathbf{u}, \mathbf{u}), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)) - O_F^h(\mathbf{u}; (\mathbf{u}, \mathbf{u}), (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)) \\
\le & C_{OF} C_5 h |\mathbf{u}|_{2,\Omega} \,|||\, (\mathbf{u}_h, \widehat{\mathbf{u}}_h) \,|||_h \,|||(\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)|||_h \\
& + C_{OF} \|\nabla \mathbf{u}\|_{L^2(\Omega)} \,|||\, (\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \widehat{\mathbf{u}}_h) \,|||_h \,|||(\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h) \,|||_h \,.
\end{aligned} \tag{70}$$

Combining (66) and (68)–(70),

$$\begin{aligned}
\nu C_F^s \,|||\, & (\mathbf{v}_h - \mathbf{u}_h, \widehat{\mathbf{v}}_h - \widehat{\mathbf{u}}_h)|||_h \\
& \le C_5 h |\mathbf{u}|_{2,\Omega} \big(\nu C_F^b + C_{OF} \,|||\, (\mathbf{u}_h, \widehat{\mathbf{u}}_h)|||_h\big) + C_1 \|\mathbf{g}\|_{L^\infty(\Omega)} \|T - T_h\|_{L^2(\Omega)} \\
& \quad + C_{OF} \|\nabla \mathbf{u}\|_{L^2(\Omega)} \,|||\, (\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \widehat{\mathbf{u}}_h)|||_h,
\end{aligned}$$

which, together with (67), triangle inequality, and assumption (63), derives the desired result (64). □

Remark 3.1. Under assumption that the domain Ω is convex, Farhloul *et al.* has proved in Theorem 3.1 of [24] that the velocity $\mathbf{u}$ and pressure p for problem (4) belong to the spaces $(H^2(\Omega))^d$ and $H^1(\Omega)$.

With Lemmas 3.4 and 3.5 at hand, now we are ready to prove the main result of this section in the following theorem.

Theorem 3.2. *Let $(\boldsymbol{u}, p, T)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ be the solutions of problems (4) and (21). If domain Ω is convex, partition $\mathcal{T}_h$ is quasi-uniform, conditions (48), (52) and (63) hold, and*

$$\max\big\{|\mathcal{H}|, \|T\|_{L^2(\Omega)}\big\} \le \frac{\nu\kappa C_F^s}{4C_1 C_4 \|\boldsymbol{g}\|_{L^\infty(\Omega)}}, \tag{71}$$

we have

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \widehat{\boldsymbol{u}}_h)|||_h \lesssim h, \tag{72}$$

and

$$\|T - T_h\|_{L^2(\Omega)} \lesssim \nu h, \tag{73}$$

where C_1 is the constant in Lemma 2.2, C_4 is the constant in Lemma 3.4, and C_F^s is the constant in (25).

Proof. According to Lemmas 3.4, 3.5, and condition (71), we have

$$|||(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h)|||_h \le 2C_5 h|\mathbf{u}|_{2,\Omega}\left(\frac{3}{2}+\frac{C_F^b}{C_F^s}\right)+\frac{1}{2}h+\frac{1}{2}\,|||\,(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h)|||_h,$$

which results in (72).

As for (73), it is a direct result of Lemma 3.4 and (72). $\square$

Finally, we conclude this section by proving error estimates of the pressure.

Corollary 3.1. *Let $(\boldsymbol{u},p,T)$ and $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h,p_h,\widehat{p}_h,T_h,\widehat{T}_h)$ be the solutions of problems (4) and (21). If domain Ω is convex, partition $\mathcal{T}_h$ is quasi-uniform, and conditions (48), (52), (63) and (71) hold, then we have*

$$\|p-p_h\|_{L^2(\Omega)} \lesssim h. \tag{74}$$

Proof. From (76) of [48], we know that there exists a positive constant γ independent of mesh parameters such that

$$\gamma\|w_h\|_{L^2(\Omega)} \le \sup_{(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\in V_h\times\widehat{V}_h}\frac{B^h((\mathbf{v}_h,\widehat{\mathbf{v}}_h),(w_h,\widehat{w}_h))}{|||(\mathbf{v}_h,\widehat{\mathbf{v}}_h)|||_h},\qquad \forall(w_h,\widehat{w}_h)\in Q_h\times\widehat{Q}_h. \tag{75}$$

On the other hand, for any $(w_h,\widehat{w}_h)\in Q_h\times\widehat{Q}_h$ and $(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\in V_h\times\widehat{V}_h$, consistency (65) yields

$$\begin{aligned}
B^h((\mathbf{v}_h,\widehat{\mathbf{v}}_h),(w_h-p_h,\widehat{w}_h-\widehat{p}_h)) &= B^h((\mathbf{v}_h,\widehat{\mathbf{v}}_h),(w_h-p,\widehat{w}_h-p))+B^h((\mathbf{v}_h,\widehat{\mathbf{v}}_h),(p-p_h,p-\widehat{p}_h))\\
&= B^h((\mathbf{v}_h,\widehat{\mathbf{v}}_h),(w_h-p,\widehat{w}_h-p))+D(T_h-T,\mathbf{v}_h)\\
&\quad + A_F^h(\cdot;(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h),(\mathbf{v}_h,\widehat{\mathbf{v}}_h))+\big[O_F^h(\mathbf{u};(\mathbf{u},\mathbf{u}),(\mathbf{v}_h,\widehat{\mathbf{v}}_h))\\
&\quad - O_F^h(\mathbf{u}_h;(\mathbf{u}_h,\widehat{\mathbf{u}}_h),(\mathbf{v}_h,\widehat{\mathbf{v}}_h))\big] = J_1+J_2+J_3+J_4.
\end{aligned}$$

In the rest of the proof, we set $w_h=\Pi_Q p$ and $\widehat{w}_h=\Pi_{\widehat{Q}}p$, where Π_Q and $\Pi_{\widehat{Q}}$ denote the standard L^2-projection onto $\mathcal{P}^{k-1}(K)$ and $\mathcal{P}^k(F)$ for $K\in\mathcal{T}_h$ and $F\in\mathcal{E}_h$. Thus interpolation error estimates of Π_Q and $\Pi_{\widehat{Q}}$ and the definition of B^h infer that

$$J_1 \lesssim h\|\nabla p\|_{L^2(\Omega)}\,|||\,(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\,|||_h\,. \tag{76}$$

Moreover, bound (26) and Theorem 3.2 can derive that

$$J_2+J_3 \lesssim \nu h\,|||\,(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\,|||_h\,. \tag{77}$$

Hence combining (75)–(77) and (70), we get

$$\|w_h-p_h\|_{L^2(\Omega)} \lesssim h,$$

which along with triangle inequality and interpolation error estimates of Π_Q concludes the proof. $\square$

4. *A posteriori* ERROR ESTIMATES

This section is devoted to develop a computable *a posteriori* error estimator and show its global reliability and local efficiency for errors in the velocity, the pressure and the temperature. Due to the singularity induced by the Dirac function δ_z and the coupling characteristic of system (4), it is obvious that this is not an easy task.

As a start, we introduce the well-known Oswald interpolation, which shows that for any $w_h\in W_h$ we can find a continuous function $\widetilde{w}_h\in H_0^1(\Omega)\cap W_h$ such that (see [36], Lem. 3.3 and [34], Thm. 2.2)

$$\sum_{K\in\mathcal{T}_h} h_K^s\|w_h-\widetilde{w}_h\|_{L^q(K)}^q \lesssim \sum_{F\in\mathcal{E}_h} h_F^{1+s}\|[w_h]\|_{L^q(F)}^q, \tag{78}$$

and

$$\sum_{K\in\mathcal{T}_h} h_K^s \|\nabla(w_h-\widetilde{w}_h)\|_{L^q(K)}^q \lesssim \sum_{F\in\mathcal{E}_h} h_F^{1+s-q}\|[w_h]\|_{L^q(F)}^q, \tag{79}$$

for $1\le q<\infty$, where s is a real number. Here for each interior edge $F\in\mathcal{E}_h^o$, let K^+ and K^- be two elements sharing the common edge F, then we define $[w_h]|_F = w_h^+ - w_h^-$, where w_h^+ and w_h^- are restrictions of w_h on $F\cap\partial K^+$ and $F\cap\partial K^-$. As for the boundary edge $F\in\mathcal{E}_h^\partial$, we set $[w_h]|_F = w_h$.

4.1. Reliability

Since the temperature T belongs to $W_0^{1,s}(\Omega)$ ($\frac{4+\epsilon}{3+\epsilon}<s<2$), we use the duality argument to prove *a posteriori* error estimates for the temperature. Before this , we introduce some properties corresponding to the temperature.

Lemma 4.1. *Let $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h,p_h,\widehat{p}_h,T_h,\widehat{T}_h)$ be the solution of problem (21), then we have*

$$\begin{aligned}0 = &-\sum_{K\in\mathcal{T}_h}\frac{\alpha_T\kappa}{h_K}\left\langle T_h-\widehat{T}_h,\widehat{w}_h\right\rangle_{\partial K} + \langle\kappa\nabla T_h\cdot\boldsymbol{n},\widehat{w}_h\rangle_{\partial\mathcal{T}_h}\\ &-\frac{1}{2}\left\langle(\boldsymbol{u}_h\cdot\boldsymbol{n})\left(T_h+\widehat{T}_h\right),\widehat{w}_h\right\rangle_{\partial\mathcal{T}_h} - \frac{1}{2}\left\langle|\boldsymbol{u}_h\cdot\boldsymbol{n}|\left(T_h-\widehat{T}_h\right),\widehat{w}_h\right\rangle_{\partial\mathcal{T}_h}, \qquad \forall\widehat{w}_h\in\widehat{W}_h,\end{aligned} \tag{80}$$

and

$$\begin{aligned}(\kappa\Delta T_h-\boldsymbol{u}_h\cdot\nabla T_h,w_h)_{\mathcal{T}_h} = &\langle\kappa\nabla T_h\cdot\boldsymbol{n},w_h\rangle_{\partial\mathcal{T}_h} - \left\langle\kappa\nabla w_h\cdot\boldsymbol{n},T_h-\widehat{T}_h\right\rangle_{\partial\mathcal{T}_h}\\ &-\langle(\boldsymbol{u}_h\cdot\boldsymbol{n})T_h,w_h\rangle_{\partial\mathcal{T}_h} - \frac{\mathcal{H}}{\sharp\mathcal{T}_z}\sum_{K\in\mathcal{T}_h} w_h|_K(z)\end{aligned} \tag{81}$$

for any continuous $w_h\in H_0^1(\Omega)\cap W_h$.

Proof. The equality (80) can be derived by setting $w_h=0$ in (21c), and (81) is a result of using integration by parts and (80). □

Lemma 4.2 (Upper bound for the temperature)**.** *Let $(\boldsymbol{u},T)$ and $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h,T_h,\widehat{T}_h)$ be the velocity and temperature parts of solutions of problems (4) and (21). If*

$$2C_1C_6C_{inv}C_{s,s}\kappa^{-1}\,|||\,(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)|||_h \le \frac{1}{2}, \tag{82}$$

we have

$$\|\nabla(T-T_h)\|_{L^s(\mathcal{T}_h)} \le C\eta_s + 2C_1C_6C_{s,\frac{2s}{2-s}}\kappa^{-1}\|\nabla T\|_{L^s(\Omega)}\,|||\,(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\widehat{\boldsymbol{u}}_h)|||_h, \tag{83}$$

for $\frac{4+\epsilon}{3+\epsilon}<s<2$, where C_1, C_6, and C_{inv} are the constants in Lemma 2.2, (86), and Lemma 2.3, $C_{s,s}$ and $C_{s,\frac{2s}{2-s}}$ are the constants in (3),

$$\begin{aligned}\eta_s^s &= \sum_{K\in\mathcal{T}_h}\left(\eta_{s,K}^s+\eta_{s,\partial K}^s\right) + \sum_{F\in\mathcal{E}_h^o}\eta_{s,F}^s,\\ \eta_{s,\partial K}^s &= h_K^{1-s}\left\|T_h-\widehat{T}_h\right\|_{L^s(\partial K)}^s, \quad \eta_{s,F}^s = h_F\|[\nabla T_h\cdot\boldsymbol{n}]\|_{L^s(F)}^s,\end{aligned}$$

and

$$\eta_{s,K}^s = \begin{cases} h_K^s \big\|\Delta T_h - \kappa^{-1}\boldsymbol{u}_h\cdot\nabla T_h\big\|_{L^s(K)}^s, & z \text{ is a node of the partition } \mathcal{T}_h, \\ \kappa^{-s}|\mathcal{H}|^s h_K^{2-s}\chi(K\cap\mathcal{T}_z) + h_K^s\big\|\Delta T_h - \kappa^{-1}\boldsymbol{u}_h\cdot\nabla T_h\big\|_{L^s(K)}^s, & \text{otherwise.} \end{cases}$$

Here $\chi(K\cap\mathcal{T}_z)=1$ *if* $K\cap\mathcal{T}_z$ *is nonempty, otherwise,* $\chi(K\cap\mathcal{T}_z)=0$*.*

Proof. Since the space $W_0^{1,s}(\Omega)$ is reflexive,

$$\|\nabla w\|_{L^s(\Omega)} = \sup_{f\in W^{-1,s'}(\Omega)/0} \frac{\langle w,f\rangle}{\|f\|_{-1,s',\Omega}}, \qquad \forall w\in W_0^{1,s}(\Omega), \tag{84}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring between the spaces $W_0^{1,s}(\Omega)$ and $W^{-1,s'}(\Omega)$. For any $f\in W^{-1,s'}(\Omega)$ with $s'\in(2,4+\epsilon)$, let $\phi\in W_0^{1,s'}(\Omega)$ be the solution of

$$(\kappa\nabla\phi,\nabla v) = \langle f,v\rangle, \qquad \forall v\in W_0^{1,s}(\Omega). \tag{85}$$

According to Theorem 0.5 of [32], we know that problem (85) has a unique solution satisfying

$$\kappa\|\phi\|_{1,s',\Omega} \le C_6\|f\|_{-1,s',\Omega}. \tag{86}$$

Throughout this proof, let $\widetilde{T}_h$ be the Oswald interpolation of T_h, then we have

$$\begin{aligned} \left\langle T-\widetilde{T}_h, f\right\rangle &= \left(\kappa\nabla\left(T-\widetilde{T}_h\right),\nabla\phi\right) = \mathcal{H}\phi(z) + (\mathbf{u}T,\nabla\phi) \\ &\quad + \left(\kappa\nabla\left(T_h-\widetilde{T}_h\right),\nabla\phi\right)_{\mathcal{T}_h} + (\kappa\Delta T_h,\phi)_{\mathcal{T}_h} - \langle\kappa\nabla T_h\cdot\mathbf{n},\phi\rangle_{\partial\mathcal{T}_h} \\ &= \mathcal{H}\phi(z) + \left(\kappa\nabla\left(T_h-\widetilde{T}_h\right),\nabla\phi\right)_{\mathcal{T}_h} + (\mathbf{u}T,\nabla\phi) + (\mathbf{u}_h\cdot\nabla T_h,\phi)_{\mathcal{T}_h} \\ &\quad + (\kappa\Delta T_h - \mathbf{u}_h\cdot\nabla T_h,\phi)_{\mathcal{T}_h} - \langle\kappa\nabla T_h\cdot\mathbf{n},\phi\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Set $w_h=\Pi_L\phi$ in (81) and inset (81) to the above equality, we yield

$$\left\langle T-\widetilde{T}_h, f\right\rangle = E_1+E_2+E_3+E_4, \tag{87}$$

where Π_L denotes the Lagrange interpolation operator onto continuous linear finite element spaces,

$$\begin{aligned} E_1 &= (\kappa\Delta T_h - \mathbf{u}_h\cdot\nabla T_h, \phi-\Pi_L\phi)_{\mathcal{T}_h} - \left\langle\kappa\nabla\Pi_L\phi\cdot\mathbf{n}, T_h-\widehat{T}_h\right\rangle_{\partial\mathcal{T}_h}, \\ E_2 &= \mathcal{H}\phi(z) - \frac{\mathcal{H}}{\sharp\mathcal{T}_z}\sum_{K\in\mathcal{T}_h}\Pi_L\phi|_K(z), \\ E_3 &= \left(\kappa\nabla\left(T_h-\widetilde{T}_h\right),\nabla\phi\right)_{\mathcal{T}_h} - \langle\kappa\nabla T_h\cdot\mathbf{n},\phi-\Pi_L\phi\rangle_{\partial\mathcal{T}_h}, \\ E_4 &= (\mathbf{u}T,\nabla\phi) + (\mathbf{u}_h\cdot\nabla T_h,\phi)_{\mathcal{T}_h} - \langle(\mathbf{u}_h\cdot\mathbf{n})T_h,\Pi_L\phi\rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

By interpolation error estimates of Π_L, Lemma 2.3, and (79), it is easy to infer that

$$E_1 \lesssim \left(\sum_{K\in\mathcal{T}_h} h_K^s\|\Delta T_h - \kappa^{-1}\mathbf{u}_h\cdot\nabla T_h\|_{L^s(K)}^s + h_K^{1-s}\left\|T_h-\widehat{T}_h\right\|_{L^s(\partial K)}^s\right)^{\frac{1}{s}} \kappa\|\nabla\phi\|_{L^{s'}(\Omega)}, \tag{88}$$

and

$$E_3 \lesssim \left(\sum_{K\in\mathcal{T}_h} h_K^{1-s} \left\| T_h - \widehat{T}_h \right\|_{L^s(\partial K)}^s + \sum_{F\in\mathcal{E}_h^o} h_F \|[\nabla T_h \cdot \mathbf{n}]\|_{L^s(F)}^s \right)^{\frac{1}{s}} \kappa \|\nabla\phi\|_{L^{s'}(\Omega)}. \tag{89}$$

If the point z is a node of the partition $\mathcal{T}_h$, it is obvious that $E_2 = 0$, otherwise, we have by interpolation error estimates of Π_L

$$E_2 \lesssim \frac{|\mathcal{H}|}{\sharp\mathcal{T}_z} \sum_{K\in\mathcal{T}_z} h_K^{\frac{2}{s}-1} \|\nabla\phi\|_{L^{s'}(K)}. \tag{90}$$

Finally, we are going to deal with E_4. A simple application of integration by parts, Lemma 2.2, (18) of [36], (78), (79), and (86) yield

$$\begin{aligned}
E_4 &= (\mathbf{u}T - \mathbf{u}_h T_h, \nabla\phi) + \left\langle (\mathbf{u}_h\cdot\mathbf{n})\left(T_h - \widehat{T}_h\right), \phi - \Pi_L\phi \right\rangle_{\partial\mathcal{T}_h} \\
&= ((\mathbf{u}-\mathbf{u}_h)T, \nabla\phi) + (\mathbf{u}_h(T-T_h), \nabla\phi) + \left\langle (\mathbf{u}_h\cdot\mathbf{n})\left(T_h - \widehat{T}_h\right), \phi - \Pi_L\phi \right\rangle_{\partial\mathcal{T}_h} \\
&\le \left(\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \|T\|_{L^{\frac{2s}{2-s}}(\Omega)} + \|\mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} \|T - T_h\|_{L^s(\Omega)} \right) \|\nabla\phi\|_{0,s',\Omega} \\
&\quad + Ch\|\mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)} \left(\sum_{K\in\mathcal{T}_h} h_K^{1-s} \left\| T_h - \widehat{T}_h \right\|_{L^s(K)}^s \right)^{\frac{1}{s}} \|\nabla\phi\|_{0,s',\Omega} \\
&\le \left(C_1 \, \||\, (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h) \,\||_h \, \|T\|_{L^{\frac{2s}{2-s}}(\Omega)} + 2C_1 C_{\mathrm{inv}} C_{s,s} \|\nabla(T-T_h)\|_{L^s(\mathcal{T}_h)} \, \||\, (\mathbf{u}_h, \widehat{\mathbf{u}}_h) \||_h \right) \\
&\quad \times C_6 \kappa^{-1} \|f\|_{-1,s',\Omega} + C \left(\sum_{K\in\mathcal{T}_h} h_K^{1-s} \|T_h - \widehat{T}_h\|_{L^s(K)}^s \right)^{\frac{1}{s}} \|f\|_{-1,s',\Omega}.
\end{aligned} \tag{91}$$

Thus combining (84) and (87)–(91), we have

$$\begin{aligned}
\left\| \nabla\left(T - \widetilde{T}_h\right) \right\|_{L^s(\Omega)} &\le C\eta_s + C_6\kappa^{-1} \Big(C_1 \, \||\, (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h) \,\||_h \, \|T\|_{L^{\frac{2s}{2-s}}(\Omega)} \\
&\quad + 2C_1 C_{\mathrm{inv}} C_{s,s} \|\nabla(T-T_h)\|_{L^s(\mathcal{T}_h)} \, \||\, (\mathbf{u}_h, \widehat{\mathbf{u}}_h) \||_h \Big),
\end{aligned} \tag{92}$$

which, together with triangle inequality, Oswald interpolation error estimate (79), condition (82), and (3), concludes the proof. ◻

Obviously, we further need to prove *a posteriori* error estimates for $\||(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h)\||_h$. Similar with the temperature, we first introduce some properties corresponding to the velocity in the following lemma.

Lemma 4.3. *Let $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h)$ be the velocity and pressure parts of solutions of problem (21), then*

$$\begin{aligned}
0 = &- \sum_{K\in\mathcal{T}_h} \frac{\alpha_F \nu}{h_K} \langle \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h, \widehat{\boldsymbol{v}}_h \rangle_{\partial K} + \langle (\nu\nabla\boldsymbol{u}_h)\boldsymbol{n}, \widehat{\boldsymbol{v}}_h \rangle_{\partial\mathcal{T}_h} \\
&- \frac{1}{2} \langle (\boldsymbol{u}_h\cdot\boldsymbol{n})(\boldsymbol{u}_h + \widehat{\boldsymbol{u}}_h), \widehat{\boldsymbol{v}}_h \rangle_{\partial\mathcal{T}_h} - \frac{1}{2} \langle |\boldsymbol{u}_h\cdot\boldsymbol{n}|(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \widehat{\boldsymbol{v}}_h \rangle_{\partial\mathcal{T}_h}, \qquad \forall \widehat{\boldsymbol{v}}_h \in \widehat{V}_h,
\end{aligned} \tag{93}$$

and

$$(\nu\Delta\boldsymbol{u}_h - \nabla\cdot(\boldsymbol{u}_h\otimes\boldsymbol{u}_h) - \nabla p_h + \boldsymbol{g}T_h, \boldsymbol{v}_h)_{\mathcal{T}_h} = \langle (\nu\nabla\boldsymbol{u}_h)\boldsymbol{n}, \boldsymbol{v}_h \rangle_{\partial\mathcal{T}_h} - \langle (\nu\nabla\boldsymbol{v}_h)\boldsymbol{n}, \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h \rangle_{\partial\mathcal{T}_h}$$

$$- \langle (\boldsymbol{u}_h\cdot\boldsymbol{n})\boldsymbol{u}_h, \boldsymbol{v}_h\rangle_{\partial\mathcal{T}_h} - \langle (\boldsymbol{v}_h\cdot\boldsymbol{n}), p_h-\widehat{p}_h\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{F},\boldsymbol{v}_h), \quad (94)$$

for any continuous $\boldsymbol{v}_h \in (H_0^1(\Omega))^2\cap V_h$.

Proof. The equality (93) is a direct result by setting $\mathbf{v}_h = 0$ in (21a). Let $\widehat{\mathbf{v}}_h = 0$ in (21a), then (94) can be derived by applying integration by parts and (93). ☐

Lemma 4.4 (Upper bound for the velocity)**.** *Let* $(\boldsymbol{u}, p, T)$ *and* $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h.\widehat{p}_h, T_h, \widehat{T}_h)$ *be the solutions of problems* (4) *and* (21)*. If*

$$2\sqrt{2}\nu^{-1}C_1\big(C_1 \,|||\, (\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\,|||_h + \|\boldsymbol{u}\|_{L^4(\Omega)}\big) \le \frac{1}{2}, \quad (95)$$

we have

$$\nu^{\frac12}\,|||\,(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}-\widehat{\boldsymbol{u}}_h)|||_h \le C\xi + 4\sqrt{\delta}\nu^{-\frac12}\|p-p_h\|_{L^2(\Omega)} + 4\sqrt{2}\nu^{-\frac12}C_{2,2}\|\boldsymbol{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\Omega)}, \quad (96)$$

for arbitrary $\delta > 0$*, where* C_1 *and* $C_{2,2}$ *are the constants in Lemma 2.2 and* (3)*.*

$$\xi^2 = \sum_{K\in\mathcal{T}_h}\big(\xi_K^2+\xi_{\partial K}^2\big) + \sum_{F\in\mathcal{E}_h^o}\xi_F^2,$$

and

$$\begin{aligned}
\xi_K^2 &= \nu^{-1}h_K^2\|\boldsymbol{F}+\nu\Delta\boldsymbol{u}_h - \nabla\cdot(\boldsymbol{u}_h\otimes\boldsymbol{u}_h) - \nabla p_h + \boldsymbol{g}T_h\|_{L^2(K)}^2,\\
\xi_{\partial K}^2 &= \nu h_K^{-1}\|\boldsymbol{u}_h-\widehat{\boldsymbol{u}}_h\|_{L^2(\partial K)}^2 + \nu^{-1}h_K\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2, \quad \xi_F^2 = \nu h_F\|[(\nabla\boldsymbol{u}_h)\boldsymbol{n}]\|_{L^2(F)}^2.
\end{aligned}$$

Proof. Throughout this proof, let $\widetilde{\mathbf{u}}_h$ be the Oswald interpolation of $\mathbf{u}_h$. Then property (94) and integration by parts yield

$$\begin{aligned}
\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}^2 &= (\nu\nabla(\mathbf{u}-\mathbf{u}_h),\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h} + (\nu\nabla(\mathbf{u}_h-\widetilde{\mathbf{u}}_h),\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}\\
&= -(\nu\Delta(\mathbf{u}-\mathbf{u}_h),\mathbf{u}-\widetilde{\mathbf{u}}_h)_{\mathcal{T}_h} + \langle(\nu\nabla(\mathbf{u}-\mathbf{u}_h))\mathbf{n},\mathbf{u}-\widetilde{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad + (\nu\nabla(\mathbf{u}_h-\widetilde{\mathbf{u}}_h),\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}\\
&= (\mathbf{F}+\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h + \mathbf{g}T_h, \mathbf{u}-\widetilde{\mathbf{u}}_h-\mathbf{v}_h)\\
&\quad - (\mathbf{u}_h\otimes\mathbf{u}_h - \mathbf{u}\otimes\mathbf{u},\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)) - (p_h-p,\nabla\cdot(\mathbf{u}_h-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}\\
&\quad + (\mathbf{g}(T-T_h),\mathbf{u}-\widetilde{\mathbf{u}}_h) + (\nu\nabla(\mathbf{u}_h-\widetilde{\mathbf{u}}_h),\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}\\
&\quad + \langle(\nu\nabla(\mathbf{u}-\mathbf{u}_h))\mathbf{n},\mathbf{u}-\widetilde{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h} + \langle(\mathbf{u}_h\otimes\mathbf{u}_h-\mathbf{u}\otimes\mathbf{u})\mathbf{n},\mathbf{u}-\widetilde{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad + \langle(\mathbf{u}-\widetilde{\mathbf{u}}_h)\cdot\mathbf{n},p_h-p\rangle_{\partial\mathcal{T}_h} - \langle(\nu\nabla\mathbf{v}_h)\mathbf{n},\mathbf{u}_h-\widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad - \langle(\mathbf{u}_h\cdot\mathbf{n})\mathbf{u}_h,\mathbf{v}_h\rangle_{\partial\mathcal{T}_h} - \langle(\mathbf{v}_h\cdot\mathbf{n}),p_h-\widehat{p}_h\rangle_{\partial\mathcal{T}_h} + \langle(\nu\nabla\mathbf{u}_h)\mathbf{n},\mathbf{v}_h\rangle_{\partial\mathcal{T}_h},
\end{aligned}$$

for any continuous $\mathbf{v}_h \in (H_0^1(\Omega))^2\cap V_h$. Since $\nu\nabla\mathbf{u} - \mathbf{u}\otimes\mathbf{u} - p\mathbf{I}$ is normal continuous on each interior edge $F\in\mathcal{E}_h^o$, the above equality can be rewritten as

$$\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}^2 = F_1+F_2+F_3, \quad (97)$$

where

$$\begin{aligned}
F_1 &= (\mathbf{F}+\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h+\mathbf{g}T_h,\mathbf{u}-\widetilde{\mathbf{u}}_h-\mathbf{v}_h),\\
F_2 &= -(\mathbf{u}_h\otimes\mathbf{u}_h-\mathbf{u}\otimes\mathbf{u},\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)) - (p_h-p,\nabla\cdot(\mathbf{u}_h-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}\\
&\quad + (\mathbf{g}(T-T_h),\mathbf{u}-\widetilde{\mathbf{u}}_h) + (\nu\nabla(\mathbf{u}_h-\widetilde{\mathbf{u}}_h),\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h},
\end{aligned}$$

$$
\begin{aligned}
F_3 = & -\langle(\nu\nabla\mathbf{u}_h)\mathbf{n},\mathbf{u}-\widetilde{\mathbf{u}}_h-\mathbf{v}_h\rangle_{\partial\mathcal{T}_h}+\langle((\mathbf{u}_h-\widehat{\mathbf{u}}_h)\otimes\mathbf{u}_h)\mathbf{n},\mathbf{u}-\widetilde{\mathbf{u}}_h-\mathbf{v}_h\rangle_{\partial\mathcal{T}_h}\\
&+\langle(\mathbf{u}-\widetilde{\mathbf{u}}_h-\mathbf{v}_h)\cdot\mathbf{n},p_h-\widehat{p}_h\rangle_{\partial\mathcal{T}_h}-\langle(\nu\nabla\mathbf{v}_h)\mathbf{n},\mathbf{u}_h-\widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

In the rest of the proof we set $\mathbf{v}_h=\Pi_{\mathrm{SZ}}(\mathbf{u}-\widetilde{\mathbf{u}}_h)$, where Π_{SZ} denotes the Scott–Zhang interpolation operator onto continuous linear finite element spaces (see [17], (4.8.10)). Using interpolation error estimates of Π_{SZ}, (3), (79), Lemmas 2.2 and 2.3, we have

$$
\begin{aligned}
F_1 &\lesssim \left(\sum_{K\in\mathcal{T}_h}h_K^2\|\mathbf{F}+\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h+\mathbf{g}T_h\|_{L^2(K)}^2\right)^{\frac12}\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)},\\
F_2 &\le C_1\big(C_1 \,|||\,(\mathbf{u}_h,\widehat{\mathbf{u}}_h)\,|||_h+\|\mathbf{u}\|_{L^4(\Omega)}\big)\,|||\,(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h)\,|||_h\,\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}\\
&\quad+C\Big(\|p-p_h\|_{L^2(\Omega)}+\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}\Big)\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{L^2(\partial K)}^2\right)^{\frac12}\\
&\quad+C_{2,2}\|\mathbf{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\Omega)}\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)},
\end{aligned}
$$

and

$$
\begin{aligned}
F_3 &\lesssim \left(\sum_{F\in\mathcal{E}_h^o}h_F\|[(\nu\nabla\mathbf{u}_h)\mathbf{n}]\|_{L^2(F)}^2+\sum_{K\in\mathcal{T}_h}h_K\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2\right)^{\frac12}\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}\\
&\quad+(|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)\,|||_h+\nu)\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{L^2(\partial K)}^2\right)^{\frac12}.
\end{aligned}
$$

Thus combining (97), Hölder inequality, and the above estimates for $F_1,\ F_2,\ F_3$, we derive

$$
\begin{aligned}
\frac14\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}^2 &\le C\xi^2+\delta\nu^{-1}\|p-p_h\|_{L^2(\Omega)}^2+2\nu^{-1}C_{2,2}^2\|\mathbf{g}\|_{L^\infty(\Omega)}^2\|T-T_h\|_{L^2(\Omega)}^2\\
&\quad+2\nu^{-1}C_1^2\Big(C_1\,|||\,(\mathbf{u}_h,\widehat{\mathbf{u}}_h)\,|||_h+\|\mathbf{u}\|_{L^4(\Omega)}\Big)^2\,|||\,(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h)|||_h^2, \qquad (98)
\end{aligned}
$$

which along with triangle inequality, (79), and assumption (95) leads to

$$
\begin{aligned}
\nu^{\frac12}\,|||\,(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h)|||_h &\le \nu^{\frac12}\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{L^2(\Omega)}+C\left(\sum_{K\in\mathcal{T}_h}\xi_{\partial K}^2\right)^{\frac12}\\
&\le C\xi+2\sqrt{\delta}\nu^{-\frac12}\|p-p_h\|_{L^2(\Omega)}+2\sqrt{2}\nu^{-\frac12}C_{2,2}\|\mathbf{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\Omega)}\\
&\quad+\frac12\nu^{\frac12}\,|||\,(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\widehat{\mathbf{u}}_h)\,|||_h\,.
\end{aligned}
$$

This concludes the proof. □

Remark 4.1. Obviously, the error estimate result in Lemma 4.4 for the velocity is not like Lemma 3.5 that does not depend on the pressure error. This is because the Oswald interpolation of the velocity used commonly in *a posteriori* error analysis of DG methods is not divergence-free.

Next we provide *a posteriori* error analysis for the pressure.

Lemma 4.5 (Upper bound for the pressure)**.** *Let* $(\boldsymbol{u},p,T)$ *and* $(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h,p_h,\widehat{p}_h,T_h,\widehat{T}_h)$ *be the solutions of problems* (4) *and* (21), *we have*

$$
\varrho\nu^{-\frac12}\|p-p_h\|_{L^2(\Omega)}\le C\xi+\big(1+\nu^{-1}C_1\big(C_1\,|||\,(\boldsymbol{u}_h,\widehat{\boldsymbol{u}}_h)\,|||_h+\|\boldsymbol{u}\|_{L^4(\Omega)}\big)\big)
$$

$$\times \nu^{\frac{1}{2}} \ |||\ (\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\widehat{\boldsymbol{u}}_h)\ |||_h + \nu^{-\frac{1}{2}} C_{2,2} \|\boldsymbol{g}\|_{L^\infty(\Omega)} \|T-T_h\|_{L^2(\Omega)}, \tag{99}$$

where C_1 *and* $C_{2,2}$ *are the constants in Lemma 2.2 and* (3), ξ *is defined in Lemma 4.4, and* ϱ *is the constant in the inf-sup condition* (100).

Proof. According to Theorem 5.1 of [28], there exists a constant $\varrho > 0$ such that the following inf-sup condition holds:

$$\varrho \|w\|_{L^2(\Omega)} \leq \sup_{\mathbf{v}\in \left(H_0^1(\Omega)\right)^2 \setminus \{\mathbf{0}\}} \frac{(w, \nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\|_{L^2(\Omega)}}, \qquad \forall w \in L_0^2(\Omega). \tag{100}$$

Since $\mathcal{C}_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, for any $\mathbf{v} \in (\mathcal{C}_0^\infty(\Omega))^2$ integration by parts and (4a) yield

$$\begin{aligned}(p-p_h, \nabla\cdot\mathbf{v}) &= (\nu\nabla\mathbf{u}, \nabla\mathbf{v}) - (\mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{v}) - (\mathbf{g}T, \mathbf{v}) - (\mathbf{F}, \mathbf{v}) + (\nabla p_h, \mathbf{v})_{\mathcal{T}_h} - \langle \mathbf{v}\cdot\mathbf{n}, p_h\rangle_{\partial\mathcal{T}_h} \\ &= -\left(\mathbf{F} + \nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h + \mathbf{g}T_h, \mathbf{v}\right)_{\mathcal{T}_h} \\ &\quad + (\nu\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v})_{\mathcal{T}_h} + (\mathbf{u}_h\otimes\mathbf{u}_h - \mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{v}) + (\mathbf{g}(T_h - T), \mathbf{v}) \\ &\quad + \langle(\nu\nabla\mathbf{u}_h)\mathbf{n}, \mathbf{v}\rangle_{\partial\mathcal{T}_h} - \langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n}, \mathbf{v}\rangle_{\partial\mathcal{T}_h} - \langle\mathbf{v}\cdot\mathbf{n}, p_h\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Set $\mathbf{v}_h = \Pi_L\mathbf{v}$ in (94) and insert (94) to the above inequality, we have

$$(p-p_h, \nabla\cdot\mathbf{v}) = G_1 + G_2 + G_3, \tag{101}$$

where

$$\begin{aligned}G_1 &= -\left(\mathbf{F} + \nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h + \mathbf{g}T_h, \mathbf{v} - \Pi_L\mathbf{v}\right)_{\mathcal{T}_h}, \\ G_2 &= (\nu\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v})_{\mathcal{T}_h} + (\mathbf{u}_h\otimes\mathbf{u}_h - \mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{v}) + (\mathbf{g}(T_h-T), \mathbf{v}), \\ G_3 &= \langle(\nu\nabla\mathbf{u}_h)\mathbf{n}, \mathbf{v}-\Pi_L\mathbf{v}\rangle_{\partial\mathcal{T}_h} - \langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n}, \mathbf{v}-\Pi_L\mathbf{v}\rangle_{\partial\mathcal{T}_h} \\ &\quad - \langle(\mathbf{v}-\Pi_L\mathbf{v})\cdot\mathbf{n}, p_h - \widehat{p}_h\rangle_{\partial\mathcal{T}_h} + \langle(\nu\nabla\Pi_L\mathbf{v})\mathbf{n}, \mathbf{u}_h - \widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h},\end{aligned}$$

Π_L is the Lagrange interpolation operator onto continuous linear finite element spaces. Note that G_1, G_2 and G_3 are similar with F_1, F_2 and F_3 that are defined in Lemma 4.4. Thus we have

$$\begin{aligned}(p-p_h, \nabla\cdot\mathbf{v}) \leq &\Big(C\xi + \nu^{\frac{1}{2}}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\mathcal{T}_h)} + \nu^{-\frac{1}{2}} C_1\big(C_1 \ |||\ (\mathbf{u}_h, \widehat{\mathbf{u}}_h)\ |||_h + \|\mathbf{u}\|_{L^4(\Omega)}\big) \\ &\times |||(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h)\ |||_h + \nu^{-\frac{1}{2}} C_{2,2}\|\mathbf{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\Omega)}\Big)\nu^{\frac{1}{2}}\|\nabla\mathbf{v}\|_{L^2(\Omega)},\end{aligned}$$

which together with (100) concludes the proof. $\square$

With Lemmas 4.2, 4.4, and 4.5 at hand, now we prove the main result of this subsection in the following theorem.

Theorem 4.1 (Globle reliability)**.** *Let* $(\boldsymbol{u}, p, T)$ *and* $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ *be the solutions of problems* (4) *and* (21). *If conditions* (82) *and* (95) *hold and*

$$32 C_1 C_6 C_{s,2} C_{2,2} C_{s,\frac{2s}{2-s}} \kappa^{-1}\nu^{-1}\|\boldsymbol{g}\|_{L^\infty(\Omega)}\|\nabla T\|_{L^s(\Omega)} \leq \frac{1}{2}, \tag{102}$$

we have

$$\begin{aligned}\nu^{\frac{1}{2}}\Big(|||(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\widehat{\boldsymbol{u}}_h)\ |||_h + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)}\Big) + \nu^{-\frac{1}{2}}\|p-p_h\|_{L^2(\Omega)} + \|T-T_h\|_{L^2(\Omega)} \\ + \ |||\Big(T-T_h, T-\widehat{T}_h\Big)|||_{s,h} \lesssim \eta_s + \xi,\end{aligned} \tag{103}$$

for $\frac{4+\epsilon}{3+\epsilon} < s < 2$, where η_s and ξ are defined in Lemmas 4.2 and 4.4, C_1 is the constant in Lemma 2.2, C_6 is the constant in (86), $C_{s,2}$, $C_{2,2}$ and $C_{s,\frac{2s}{2-s}}$ are the constants in (3), and

$$\left\|\left|\left(T-T_h, T-\widehat{T}_h\right)\right|\right\|_{s,h}^s = \|\nabla(T-T_h)\|_{L^s(\mathcal{T}_h)}^s + \sum_{K\in\mathcal{T}_h} h_K^{1-s}\left\|T_h-\widehat{T}_h\right\|_{L^s(\partial K)}^s.$$

Proof. Insert the result (99) to (96), we infer by condition (95) that

$$\nu^{\frac{1}{2}} \; \|\!|\; (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h)\|\!|_h \le C\xi + 4\sqrt{\delta}\varrho^{-1}\left(1+\frac{1}{4\sqrt{2}}\right)\nu^{\frac{1}{2}} \; \|\!|\; (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h)\|\!|_h$$
$$+ \left(\sqrt{2}+\sqrt{\delta}\varrho^{-1}\right)4C_{2,2}\nu^{-\frac{1}{2}}\|\mathbf{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\Omega)}.$$

By choosing $\sqrt{\delta} = \frac{\varrho}{8+\sqrt{2}}$, a simple calculation can derive that

$$4\sqrt{\delta}\varrho^{-1}\left(1+\frac{1}{4\sqrt{2}}\right) = \frac{1}{2} \quad \text{and} \quad 4\left(\sqrt{2}+\sqrt{\delta}\varrho^{-1}\right) < 8.$$

Hence

$$\nu^{\frac{1}{2}} \; \|\!|\; (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h)\|\!|_h \le C\xi + 16C_{2,2}\nu^{-\frac{1}{2}}\|\mathbf{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\Omega)}. \tag{104}$$

On the other hand, a simple application of (78) and Lemma 2.3 deduces that

$$\left\|\widetilde{T}_h - T_h\right\|_{L^2(\mathcal{T}_h)} \lesssim \sum_{K\in\mathcal{T}_h} h_K^{\frac{1}{2}}\left\|T_h-\widehat{T}_h\right\|_{L^2(\partial K)} \lesssim \sum_{K\in\mathcal{T}_h} h_K^{\frac{2}{s'}} h_K^{\frac{1}{s}-1}\left\|T_h-\widehat{T}_h\right\|_{L^s(\partial K)}$$
$$\le \left(\sum_{K\in\mathcal{T}_h} h_K^2\right)^{\frac{1}{s'}}\left(\sum_{K\in\mathcal{T}_h}\eta_{s,\partial K}^s\right)^{\frac{1}{s}} \lesssim \left(\sum_{K\in\mathcal{T}_h}\eta_{s,\partial K}^s\right)^{\frac{1}{s}},$$

where $\widetilde{T}_h$ is the Oswald interpolation of T_h, which, together with triangle inequality, equations (3), (79), and (83), leads to

$$\|T-T_h\|_{L^2(\Omega)} \le C_{s,2}\left\|\nabla\left(T-\widetilde{T}_h\right)\right\|_{L^s(\Omega)} + \left\|\widetilde{T}_h - T_h\right\|_{L^2(\Omega)}$$
$$\le C\eta_s + 2C_1C_6C_{s,2}C_{s,\frac{2s}{2-s}}\kappa^{-1}\|\nabla T\|_{L^s(\Omega)} \; \|\!|\; (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h) \;\|\!|_h\,. \tag{105}$$

Insert (105) to (104) we yield by assumption (102) that

$$\nu^{\frac{1}{2}} \; \|\!|\; (\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\widehat{\mathbf{u}}_h)\|\!|_h \lesssim \eta_s + \xi,$$

which along with Lemmas 2.2, 4.2, 4.5, (105), and the definitions of $|\!|\!|(\cdot,\cdot)|\!|\!|_h$ and $|\!|\!|(\cdot,\cdot)|\!|\!|_{s,h}$ concludes the proof. □

4.2. Efficiency

In this subsection, we analyze the efficiency of *a posteriori* error estimators obtained in Subsection 4.1. To achieve the goal, we introduce the following standard element and edge bubble functions (see [3, 53]). For each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h^o$, we define

$$B_K = 27\prod_{v\in\mathcal{N}(K)}\lambda_v, \qquad B_F = 4\prod_{v'\in\mathcal{N}(F)}\lambda_{v'},$$

where $\mathcal{N}(D)$ denotes the set of vertices on D, and λ_v is the linear Lagrange basis function associated to the vertex v.

Since the heat equation involves singular heat source δ_z, we need to introduce the following bubble functions (see [4, 14]). For each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h^o$, we define the element bubble function

$$\psi_K(x) = \begin{cases} B_K(x)\frac{|x-z|^2}{h_K^2}, & \text{if } z \in K, \\ B_K(x), & \text{otherwise,} \end{cases}$$

and the edge bubble function

$$\psi_F(x) = \begin{cases} B_F(x)\frac{|x-z|^2}{h_F^2}, & \text{if } z \in \mathcal{S}_F, \\ B_F(x), & \text{otherwise,} \end{cases}$$

where $\mathcal{S}_F := interior(\bigcup\{\overline{K} : F \in \partial K, K \in \mathcal{T}_h\})$

Lemma 4.6 (Bubble function properties)**.** *For each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h^o$, let B_K, B_F, ψ_K and ψ_F be defined as above. Then we have*

$$\|w_h\|_{L^2(K)} \lesssim \left\|w_h B_K^{\frac{1}{2}}\right\|_{L^2(K)}, \quad \|w_h\|_{L^2(K)} \lesssim \left\|w_h \psi_K^{\frac{1}{2}}\right\|_{L^2(K)}, \qquad \forall w_h \in \mathcal{P}^j(K), \tag{106}$$

and

$$\|v_h\|_{L^2(F)} \lesssim \|v_h B_F^{\frac{1}{2}}\|_{L^2(F)}, \quad \|v_h\|_{L^2(F)} \lesssim \|v_h \psi_F^{\frac{1}{2}}\|_{L^2(F)}, \qquad \forall v_h \in \mathcal{P}^j(F), \tag{107}$$

where j is a nonnegative integer.

Now, we are ready to investigate the local efficiency of *a posteriori* error estimators obtained in Section 4.1. First of all, we consider the local error indicators $\eta_{s,K}$ and ξ_K defined in Lemmas 4.2 and 4.4.

Lemma 4.7 (Local estimates for $\eta_{s,K}$ and ξ_K)**.** *Let $(\boldsymbol{u}, p, T)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ be the solutions of problems (4) and (21). If the point z is a node of the partition $\mathcal{T}_h$, we have*

$$\begin{aligned} \xi_K \lesssim\, & osc(\boldsymbol{F}, K) + \nu^{\frac{1}{2}}\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|_{L^2(K)} + \nu^{-\frac{1}{2}}\|p - p_h\|_{L^2(K)} \\ & + \nu^{-\frac{1}{2}}\left(\|\boldsymbol{u}\|_{L^4(K)} + \|\boldsymbol{u}_h\|_{L^4(K)}\right)\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^4(K)} \\ & + \nu^{-\frac{1}{2}} h_K \|\boldsymbol{g}\|_{L^\infty(\Omega)}\|T - T_h\|_{L^2(K)} + osc(\boldsymbol{g}, K), \end{aligned} \tag{108}$$

and

$$\eta_{s,K} \lesssim \|\nabla(T - T_h)\|_{L^s(K)} + \kappa^{-1}\|\boldsymbol{u}_h\|_{L^\infty(\mathcal{T}_h)}\|T - T_h\|_{L^s(K)} + \kappa^{-1}\|T\|_{L^{\frac{2s}{2-s}}(K)}\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(K)}, \tag{109}$$

for $\frac{4+\epsilon}{3+\epsilon} < s < 2$ and $K \in \mathcal{T}_h$, where

$$osc(\boldsymbol{F}, K) = \nu^{-\frac{1}{2}} h_K \|\Pi_V \boldsymbol{F} - \boldsymbol{F}\|_{L^2(K)}, \quad osc(\boldsymbol{g}, K) = \nu^{-\frac{1}{2}} h_K \|T_h\|_{L^\infty(K)}\|\boldsymbol{g} - \Pi_V \boldsymbol{g}\|_{L^2(K)},$$

Π_V *denotes the standard L^2-projection operator onto V_h.*

Proof. Let $\mathbf{v}_h = \Pi_V \mathbf{F} + \nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h + T_h\Pi_V\mathbf{g}$, then Lemmas 2.3, 4.6, and integration by parts yield

$$\begin{aligned}\|\mathbf{v}_h\|^2_{L^2(K)} \lesssim (\mathbf{v}_h, \mathbf{v}_h B_K)_K &= (\Pi_V\mathbf{F}-\mathbf{F}, \mathbf{v}_h B_K)_K + \nu(\nabla(\mathbf{u}-\mathbf{u}_h), \nabla(\mathbf{v}_h B_K))_K \\ &- (\mathbf{u}\otimes\mathbf{u} - \mathbf{u}_h\otimes\mathbf{u}_h + (p-p_h)\mathbf{I}, \nabla(\mathbf{v}_h B_K))_K \\ &- (\mathbf{g}(T-T_h) + (\mathbf{g}-\Pi_V\mathbf{g})T_h, \mathbf{v}_h B_K)_K \\ \lesssim &\Big(\|\Pi_V\mathbf{F}-\mathbf{F}\|_{L^2(K)} + \nu h_K^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(K)} \\ &+ h_K^{-1}\|p-p_h\|_{L^2(K)} + h_K^{-1}\Big(\|\mathbf{u}\|_{L^4(K)} + \|\mathbf{u}_h\|_{L^4(K)}\Big)\|\mathbf{u}-\mathbf{u}_h\|_{L^4(K)} \\ &+ \|\mathbf{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(K)} + \|T_h\|_{L^\infty(K)}\|\mathbf{g}-\Pi_V\mathbf{g}\|_{L^2(K)}\Big)\|\mathbf{v}_h\|_{L^2(K)},\end{aligned}$$

which along with triangle inequality derives (108).

In order to prove (109), we let $w_h = \Delta T_h - \kappa^{-1}\mathbf{u}_h\cdot\nabla T_h$. Thus Lemmas 2.3, 4.6, and integration by parts infer that

$$\begin{aligned}\kappa\|w_h\|^2_{L^2(K)} &\lesssim (\kappa\Delta T_h - \mathbf{u}_h\cdot\nabla T_h, w_h\psi_K)_K \\ &= (\kappa\nabla(T-T_h), \nabla(w_h\psi_K))_K + (\mathbf{u}\cdot\nabla T - \mathbf{u}_h\cdot\nabla T_h, w_h\psi_K)_K \\ &= (\kappa\nabla(T-T_h), \nabla(w_h\psi_K))_K - (\mathbf{u}_h(T-T_h), \nabla(w_h\psi_K))_K - ((\mathbf{u}-\mathbf{u}_h)T, \nabla(w_h\psi_K))_K \\ &\lesssim h^{-\frac{2}{s}}\Big(\kappa\|\nabla(T-T_h)\|_{L^s(K)} + \|\mathbf{u}_h\|_{L^\infty(\mathcal{T}_h)}\|T-T_h\|_{L^s(\Omega)}\Big)\|w_h\|_{L^2(K)} \\ &\quad + h_K^{-\frac{2}{s}}\|T\|_{L^{\frac{2s}{2-s}}(K)}\|\mathbf{u}-\mathbf{u}_h\|_{L^2(K)}\|w_h\|_{L^2(K)},\end{aligned}$$

which, together with inverse estimates, leads to (109). ☐

Remark 4.2. For $d=2$, we know from the proof of Theorem 2.5 that $\|T_h\|_{L^\infty(K)}$ is bounded, hence oscillation $osc(\mathbf{g}, K)$ is well-defined.

If the point z is not a node of the partition $\mathcal{T}_h$, we know from the definition of $\eta_{s,K}$ that the efficiency of the term $\kappa^{-1}|\mathcal{H}|h_K^{\frac{2}{s}-1}$ needs to be further proved for each $K\in\mathcal{T}_z$. To accomplish this task, we introduce a smooth bubble function $B_{K,z}$ (see [4], Sect. 3) such that

$$\operatorname{supp}(B_{K,z}) \subset \mathcal{S}_K, \quad B_{K,z}(z) = 1, \quad \|B_{K,z}\|_{L^\infty(\mathcal{S}_K)} = 1, \quad \|\nabla B_{K,z}\|_{L^\infty(\mathcal{S}_K)} \lesssim h_K^{-1}, \tag{110}$$

where $\mathcal{S}_K := interior(\bigcup\{\overline{K}_i : \overline{K}_i\cap\overline{K}\neq\emptyset, K_i\in\mathcal{T}_h\})$. In addition, function $B_{K,z}$ satisfies the following estimates

$$|B_{K,z}|_{l,s',\mathcal{S}_K} \lesssim h_K^{\frac{2}{s'}-l}, \qquad l = 0,1,\ 1\le s'\le\infty. \tag{111}$$

Lemma 4.8. *Let $(\boldsymbol{u}, p, T)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ be the solutions of problems (4) and (21), then we have*

$$\begin{aligned}\kappa^{-1}|\mathcal{H}|h_K^{\frac{2}{s}-1} \lesssim \|\nabla(T-T_h)\|_{L^s(\mathcal{S}_K)} + \kappa^{-1}\|\boldsymbol{u}_h\|_{L^\infty(\mathcal{S}_K)}\|T-T_h\|_{L^s(\mathcal{S}_K)} + \Bigg(\sum_{F\in\mathcal{E}_h^o(\mathcal{S}_K)}\eta_{s,F}^s\Bigg)^{\frac{1}{s}} \\ + \kappa^{-1}h_K\|\boldsymbol{u}_h\|_{L^\infty(\mathcal{S}_K)}\Bigg(\sum_{K'\in\mathcal{S}_K}\eta_{s,\partial K}^s\Bigg)^{1/s} + \kappa^{-1}\|T\|_{L^{\frac{2s}{2-s}}(\mathcal{S}_K)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\mathcal{S}_K)},\end{aligned} \tag{112}$$

for $\frac{4+\epsilon}{3+\epsilon} < s < 2$ and $K\in\mathcal{T}_z$, where $\mathcal{E}_h^o(\mathcal{S}_K)$ denotes the set of all interior edges in $\mathcal{S}_K$.

Proof. We let $w_h = |\mathcal{H}|^{s-1}\text{sign}(\mathcal{H})B_{K,z}$, then integration by parts, equations (110), and (111) result in

$$\begin{aligned}
|\mathcal{H}|^s &= \langle \mathcal{H}\delta_z, w_h\rangle = -(\kappa\Delta(T-T_h), w_h)_{\mathcal{S}_K} + (\mathbf{u}\cdot\nabla T - \mathbf{u}_h\cdot\nabla T_h, w_h)_{\mathcal{S}_K} \\
&\quad - (\kappa\Delta T_h - \mathbf{u}_h\cdot\nabla T_h, w_h)_{\mathcal{S}_K} \\
&= (\kappa\nabla(T-T_h), \nabla w_h)_{\mathcal{S}_K} + \langle \kappa\nabla T_h\cdot\mathbf{n}, w_h\rangle_{\partial\mathcal{S}_K} - (\mathbf{u}T - \mathbf{u}_hT_h, \nabla w_h)_{\mathcal{S}_K} \\
&\quad - \langle \mathbf{u}_h\cdot\mathbf{n}, T_hw_h\rangle_{\partial\mathcal{S}_K} - (\kappa\Delta T_h - \mathbf{u}_h\cdot\nabla T_h, w_h)_{\mathcal{S}_K} \\
&\lesssim \kappa|\mathcal{H}|^{s-1}h_K^{\frac{2}{s'}-1}\left(\|\nabla(T-T_h)\|_{L^s(\mathcal{S}_K)} + \sum_{F\in\mathcal{E}_h^o(\mathcal{S}_K)} h_K^{\frac{1}{s}}\|[\nabla T_h\cdot\mathbf{n}]\|_{L^s(F)}\right) \\
&\quad + |\mathcal{H}|^{s-1}h_K^{\frac{2}{s'}-1}\left(\|\mathbf{u}_h\|_{L^\infty(\mathcal{S}_K)}\|T-T_h\|_{L^s(\mathcal{S}_K)} + \|T\|_{L^{\frac{2s}{2-s}}(\mathcal{S}_K)}\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\mathcal{S}_K)}\right) \\
&\quad + |\mathcal{H}|^{s-1}h_K^{\frac{2}{s'}}\|\mathbf{u}_h\|_{L^\infty(\mathcal{S}_K)}\left(\sum_{K'\in\mathcal{S}_K} h_{K'}^{\frac{1}{s}-1}\left\|T_h - \widehat{T}_h\right\|_{L^s(\partial K')}\right) \\
&\quad + \kappa|\mathcal{H}|^{s-1}h_K^{\frac{2}{s'}-1}\sum_{K'\in\mathcal{S}_K} h_K\left\|\Delta T_h - \kappa^{-1}\mathbf{u}_h\cdot\nabla T_h\right\|_{L^s(K')},
\end{aligned}$$

which, together with Lemma 4.7, concludes the proof. □

Next we are going to investigate local estimates for the indicators $\eta_{s,F}$ and ξ_F. Before this, we recall the continuation operator $\mathcal{P}_F : L^\infty(F) \to L^\infty(\mathcal{S}_F)$ satisfying (see [52], Sect. 3)

$$\|\mathcal{P}_F w_h\|_{L^2(\mathcal{S}_F)} \lesssim h_F^{\frac{1}{2}}\|w_h\|_{L^2(F)}, \qquad \forall w_h \in \mathcal{P}^j(F),\ F\in\mathcal{E}_h^o, \tag{113}$$

where j is a nonnegative integer and $\mathcal{S}_F$ is defined in the above of Lemma 4.6.

Lemma 4.9 (Local estimates for $\eta_{s,F}$ and ξ_F)**.** *Let $(\boldsymbol{u}, p, T)$ and $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ be the solutions of problems (4) and (21), we have*

$$\begin{aligned}
\xi_F &\lesssim \nu^{\frac{1}{2}}\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_{L^2(\mathcal{S}_F)} + \nu^{-\frac{1}{2}}\|p-p_h\|_{L^2(\mathcal{S}_F)} + \nu^{-\frac{1}{2}}h_F^{\frac{1}{2}}\|p_h-\widehat{p}_h\|_{L^2(\partial\mathcal{S}_F)} \\
&\quad + \nu^{-\frac{1}{2}}\left(\|\boldsymbol{u}\|_{L^4(\mathcal{S}_F)} + \|\boldsymbol{u}_h\|_{L^4(\mathcal{S}_F)}\right)\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^4(\mathcal{S}_F)} + \nu^{-\frac{1}{2}}h_F\|\boldsymbol{g}\|_{L^\infty(\Omega)}\|T-T_h\|_{L^2(\mathcal{S}_F)} \\
&\quad + \nu^{-\frac{1}{2}}\|\boldsymbol{u}_h\|_{L^\infty(\mathcal{S}_F)}h_F^{\frac{1}{2}}\|\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\|_{L^2(\partial\mathcal{S}_F)} + osc(\boldsymbol{F},\mathcal{S}_F) + osc(\boldsymbol{g},\mathcal{S}_F),
\end{aligned} \tag{114}$$

and

$$\begin{aligned}
\eta_{s,F} &\lesssim \|\nabla(T-T_h)\|_{L^s(\mathcal{S}_F)} + \kappa^{-1}\|\boldsymbol{u}_h\|_{L^\infty(\mathcal{S}_F)}\|T-T_h\|_{L^s(\mathcal{S}_F)} \\
&\quad + \kappa^{-1}\|T\|_{L^{\frac{2s}{2-s}}(\mathcal{S}_F)}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\mathcal{S}_F)} + \kappa^{-1}h_F\|\boldsymbol{u}_h\|_{L^\infty(\mathcal{S}_F)}h_F^{\frac{1}{s}-1}\left\|T_h-\widehat{T}_h\right\|_{L^s(\partial\mathcal{S}_F)},
\end{aligned} \tag{115}$$

for $\frac{4+\epsilon}{3+\epsilon} < s < 2$ and $F\in\mathcal{E}_h^o$, where

$$osc^2(\boldsymbol{F},\mathfrak{S}) = \sum_{K\in\mathfrak{S}} osc^2(\boldsymbol{F},K), \quad osc^2(\boldsymbol{g},\mathfrak{S}) = \sum_{K\in\mathfrak{S}} osc^2(\boldsymbol{g},K).$$

for any $\mathfrak{S}\subset\mathcal{T}_h$.

Proof. Let $\mathbf{v}_h = B_F\mathcal{P}_F([(\nabla\mathbf{u}_h)\mathbf{n}])$, then integration by parts, Lemmas 2.3 and 4.6 yield

$$\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{L^2(F)}^2 \lesssim \langle[(\nabla\mathbf{u}_h)\mathbf{n}], \mathbf{v}_h\rangle_F = \int_{\mathcal{S}_F}\nabla\cdot\left((\nabla\mathbf{u}_h)^T\mathbf{v}_h\right)$$

$$
\begin{aligned}
&= (\Delta \mathbf{u}_h, \mathbf{v}_h)_{\mathcal{S}_F} + (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\mathcal{S}_F} \\
&= \big(\mathbf{F} + \nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h + \mathbf{g} T_h, \nu^{-1} \mathbf{v}_h\big)_{\mathcal{S}_F} \\
&\quad - (\nabla(\mathbf{u} - \mathbf{u}_h), \nabla \mathbf{v}_h)_{\mathcal{S}_F} + \big(\mathbf{u} \otimes \mathbf{u} - \mathbf{u}_h \otimes \mathbf{u}_h + (p - p_h)\mathbf{I}, \nu^{-1} \nabla \mathbf{v}_h\big)_{\mathcal{S}_F} \\
&\quad + \big(\mathbf{g}(T - T_h), \nu^{-1} \mathbf{v}_h\big)_{\mathcal{S}_F} + \big\langle (\mathbf{u}_h \otimes \mathbf{u}_h + p_h \mathbf{I})\mathbf{n}, \nu^{-1} \mathbf{v}_h \big\rangle_{\partial \mathcal{S}_F} \\
&\lesssim \nu^{-1} \Big(\|\mathbf{F} + \nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h + \mathbf{g} T_h\|_{L^2(\mathcal{S}_F)} \\
&\quad + \nu h_F^{-1} \|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{L^2(\mathcal{S}_F)} + h_F^{-1} \|p - p_h\|_{L^2(\mathcal{S}_F)} \\
&\quad + h_F^{-1} \Big(\|\mathbf{u}\|_{L^4(\mathcal{S}_F)} + \|\mathbf{u}_h\|_{L^4(\mathcal{S}_F)} \Big) \|\mathbf{u} - \mathbf{u}_h\|_{L^4(\mathcal{S}_F)} \\
&\quad + \|\mathbf{g}\|_{L^\infty(\Omega)} \|T - T_h\|_{L^2(\mathcal{S}_F)} + h_F^{-\frac{1}{2}} \|p_h - \widehat{p}_h\|_{L^2(\partial \mathcal{S}_F)} \\
&\quad + \|\mathbf{u}_h\|_{L^\infty} h_F^{-\frac{1}{2}} \|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{L^2(\partial \mathcal{S}_F)} \Big) \|\mathbf{v}_h\|_{L^2(\mathcal{S}_F)}
\end{aligned}
$$

which along with (113) and Lemma 4.7 derives (114).

In order to prove (115), we set $w_h = \psi_F \mathcal{P}_F([\nabla T_h \cdot \mathbf{n}])$. Thus integration by parts, Lemmas 2.3 and 4.6 deduce that

$$
\begin{aligned}
\|[\nabla T_h \cdot \mathbf{n}]\|_{L^2(F)}^2 &\lesssim \langle [\nabla T_h \cdot \mathbf{n}], w_h \rangle_F = \int_{\mathcal{S}_F} \nabla \cdot (w_h \nabla T_h) \\
&= (\Delta T_h, w_h)_{\mathcal{S}_F} + (\nabla T_h, \nabla w_h)_{\mathcal{S}_F} \\
&= \big(\Delta T_h - \kappa^{-1} \mathbf{u}_h \cdot \nabla T_h, w_h\big)_{\mathcal{S}_F} - (\nabla(T - T_h), \nabla w_h)_{\mathcal{S}_F} + \big(\mathbf{u}_h \cdot \nabla T_h - \mathbf{u} \cdot \nabla T, \kappa^{-1} w_h\big)_{\mathcal{S}_F} \\
&= \big(\Delta T_h - \kappa^{-1} \mathbf{u}_h \cdot \nabla T_h, w_h\big)_{\mathcal{S}_F} - (\nabla(T - T_h), \nabla w_h)_{\mathcal{S}_F} \\
&\quad + \big(\mathbf{u} T - \mathbf{u}_h T_h, \kappa^{-1} \nabla w_h\big)_{\mathcal{S}_F} + \Big\langle (\mathbf{u}_h \cdot \mathbf{n}) \Big(T_h - \widehat{T}_h\Big), \kappa^{-1} w_h \Big\rangle_{\partial \mathcal{S}_F} \\
&\lesssim \Big(\big\|\Delta T_h - \kappa^{-1} \mathbf{u}_h \cdot \nabla T_h\big\|_{L^s(\mathcal{S}_F)} + h_F^{-1} \|\nabla(T - T_h)\|_{L^s(\mathcal{S}_F)} \Big) \|w_h\|_{L^{s'}(\mathcal{S}_F)} \\
&\quad \times \left(\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathcal{S}_F)} \|T\|_{L^{\frac{2s}{2-s}}(\mathcal{S}_F)} + \|\mathbf{u}_h\|_{L^\infty(\mathcal{S}_F)} \|T - T_h\|_{L^s(\mathcal{S}_F)} \right) \kappa^{-1} h_F^{-1} \|w_h\|_{L^{s'}(\mathcal{S}_F)} \\
&\quad + \kappa^{-1} \|\mathbf{u}_h\|_{L^\infty(\mathcal{S}_F)} h_F^{\frac{1}{s}-1} \Big\| T_h - \widehat{T}_h \Big\|_{L^s(\partial \mathcal{S}_F)} \|w_h\|_{L^{s'}(\mathcal{S}_F)}.
\end{aligned}
$$

It concludes the proof by using inverse estimates, Lemma 4.7, and (113). □

In the following lemma, we show that the indicator $\nu^{-1} h_K \|p_h - \widehat{p}_h\|_{L^2(\partial K)}^2$ can be controlled by ξ_K and $\nu h_K^{-1} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^2(\partial K)}^2$.

Lemma 4.10. *Let $(\boldsymbol{u}_h, \widehat{\boldsymbol{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ be the solution of problem (21), then we have*

$$
\nu^{-\frac{1}{2}} h_K^{\frac{1}{2}} \|p_h - \widehat{p}_h\|_{L^2(\partial K)} \lesssim \xi_K + \Big(1 + \nu^{-1} h_K \|\boldsymbol{u}_h\|_{L^\infty(K)}\Big) \nu^{\frac{1}{2}} h_K^{-\frac{1}{2}} \|\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\|_{L^2(\partial K)}. \tag{116}
$$

Proof. For $K \in \mathcal{T}_h$, let $\mathbf{v}_h = \mathbf{0}$ on $\mathcal{T}_h \backslash K$. Then by a similar manipulation with the proof of (94) we have

$$
\begin{aligned}
\langle \mathbf{v}_h \cdot \mathbf{n}, p_h - \widehat{p}_h \rangle_{\partial K} = &- (\mathbf{F} + \nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h + \mathbf{g} T_h, \mathbf{v}_h)_K \\
&+ \alpha_F \nu h_K^{-1} \langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \mathbf{v}_h \rangle_{\partial K} - \langle (\nu \nabla \mathbf{v}_h)\mathbf{n}, \mathbf{u}_h - \widehat{\mathbf{u}}_h \rangle_{\partial K} \\
&+ \frac{1}{2} \langle (\mathbf{u}_h \cdot \mathbf{n})(\widehat{\mathbf{u}}_h - \mathbf{u}_h), \mathbf{v}_h \rangle_{\partial K} + \frac{1}{2} \langle |\mathbf{u}_h \cdot \mathbf{n}| (\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{v}_h \rangle_{\partial K} \\
\lesssim &\Big(\xi_K + \nu^{\frac{1}{2}} h_K^{-\frac{1}{2}} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^2(\partial K)}
\end{aligned}
$$

$$+\nu^{-\frac{1}{2}} h_K^{\frac{1}{2}} \|\mathbf{u}_h\|_{L^\infty(K)} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^2(\partial K)} \Big) \nu^{\frac{1}{2}} h_K^{-1} \|\mathbf{v}_h\|_{L^2(K)}. \tag{117}$$

In the rest of the proof, we set $\mathbf{v}_h = L^{\mathrm{BDM}}(p_h - \widehat{p}_h)$. Here $L^{\mathrm{BDM}} : \mathcal{P}^k(\partial K) \to (\mathcal{P}^k(K))^2$ denotes a BDM lifting operator (see [49], Def. 1), which has the property

$$\left(L^{\mathrm{BDM}} \widehat{w}_h\right) \cdot \mathbf{n} = \widehat{w}_h, \quad \left\|L^{\mathrm{BDM}} \widehat{w}_h\right\|_{L^2(K)} \lesssim h_K^{\frac{1}{2}} \|\widehat{w}_h\|_{L^2(\partial K)}, \qquad \forall \widehat{w}_h \in \mathcal{P}^k(\partial K). \tag{118}$$

Thus combine (117) and (118) we conclude the proof. □

With Lemmas 4.7, 4.8, 4.9, and 4.10 at hand, now we state the main result of this subsection in the following theorem.

Theorem 4.2 (Local efficiency)**.** *Let* $(\boldsymbol{u}, p, T)$ *and* $(\boldsymbol{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h)$ *be the solutions of problems* (4) *and* (21)*. If conditions* (82)*,* (95) *and* (102) *hold, we have*

$$\begin{aligned}\xi(K) \lesssim \nu^{\frac{1}{2}} \Big(\||(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\widehat{\mathbf{u}}_h)\||_{h,\mathcal{S}_K^*} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^4(\mathcal{S}_K^*)} + \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\mathcal{S}_K^*)} \Big) \\ + \nu^{-\frac{1}{2}} \|p-p_h\|_{L^2(\mathcal{S}_K^*)} + \nu^{-\frac{1}{2}} h_K \|\boldsymbol{g}\|_{L^\infty(\Omega)} \|T-T_h\|_{L^2(\mathcal{S}_K^*)} + osc(\boldsymbol{F}, \mathcal{S}_K^*) + osc(\boldsymbol{g}, \mathcal{S}_K^*),\end{aligned} \tag{119}$$

and

$$\eta_s(K) \lesssim \left\|\left|\left(T-T_h, T-\widehat{T}_h\right)\right|\right\|_{s,h,\mathcal{S}_K} + \|T-T_h\|_{L^s(\mathcal{S}_K)} + \nu \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\mathcal{S}_K)}, \tag{120}$$

for $\frac{4+\epsilon}{3+\epsilon} < s < 2$*, where* $\mathcal{S}_K^* = \{K' \in \mathcal{T}_h : \partial K' \cap \partial K \neq \emptyset\}$*,* $\mathcal{S}_K$ *is defined in the above of Lemma 4.8,* $osc(\boldsymbol{F}, \mathcal{S}_K^*)$ *and* $osc(\boldsymbol{g}, \mathcal{S}_K^*)$ *are defined in Lemma 4.9,* $\||(\cdot,\cdot)\||_{h,\mathcal{S}_F^*}$ *and* $\||(\cdot,\cdot)\||_{s,h,\mathcal{S}_K}$ *denote restrictions of the discrete seminorms* $\||(\cdot,\cdot)\||_h$ *and* $\||(\cdot,\cdot)\||_{s,h}$ *on* $\mathcal{S}_K^*$ *and* $\mathcal{S}_K$ *respectively, and*

$$\xi(K) = \left(\xi_K^2 + \xi_{\partial K}^2 + \sum_{F \in \partial K \backslash \Omega} \xi_F^2 \right)^{\frac{1}{2}}, \quad \eta_s(K) = \left(\eta_{s,K}^s + \eta_{s,\partial K}^s + \sum_{F \in \partial K \backslash \Omega} \eta_{s,F}^s \right)^{\frac{1}{s}}.$$

Proof. Using condition (95), we infer that for each $K \in \mathcal{T}_h$

$$\nu^{-\frac{1}{2}} \left(\|\mathbf{u}\|_{L^4(K)} + \|\mathbf{u}_h\|_{L^4(K)} \right) \lesssim \nu^{-\frac{1}{2}} \left(\|\mathbf{u}\|_{L^4(K)} + C_1 \||(\mathbf{u}_h, \widehat{\mathbf{u}}_h)\||_h \right) \lesssim \nu^{\frac{1}{2}},$$

and

$$1 + \nu^{-1} \|\mathbf{u}_h\|_{L^\infty(K)} \leq 1 + C\nu^{-1} \||(\mathbf{u}_h, \widehat{\mathbf{u}}_h)\||_h \leq C.$$

Thus combining (108), (114), and (116) we can derive the desired result (119).

On the other hand, according to the conditions (82) and (102), we have for each $K \in \mathcal{T}_h$

$$1 + \kappa^{-1} \|\mathbf{u}_h\|_{L^\infty(K)} \lesssim C, \quad \kappa^{-1} \|T\|_{L^{\frac{2s}{2-s}}(K)} \lesssim \nu.$$

Hence by (109), Lemma 4.8, and (115), we obtain the result (120). □

5. Numerical experiments

In this section, we provide a series of numerical examples to illustrate the convergence rates in Theorem 3.2 and Corollary 3.1 and show the performance of the obtained *a posteriori* error estimator.

TABLE 1. Example 5.1: The convergence histories for the velocity gradient, the pressure, the temperature and the velocity divergence in L^2-norm for $k = 1$.

h	$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{L^2(\mathcal{T}_h)}$	Order	$\|p-p_h\|_{L^2(\Omega)}$	Order	$\|T-T_h\|_{L^2(\Omega)}$	Order	$\|\nabla\cdot\mathbf{u}_h\|_{L^2(\mathcal{T}_h)}$
$\frac{1}{8}$	1.1574e−03		4.8412e−03		4.8809e−02		6.2608e−18
$\frac{1}{16}$	5.2366e−04	1.14	2.7962e−03	0.80	1.9514e−02	1.32	1.1545e−17
$\frac{1}{32}$	2.2871e−04	1.20	1.4883e−03	0.91	7.3253e−03	1.41	2.2061e−17
$\frac{1}{64}$	1.0633e−04	1.10	7.6067e−04	0.97	2.4522e−03	1.58	1.6995e−17

For a given partition $\mathcal{T}_h$, we solve the discrete problem (21) by a fixed-point algorithm presented in Algorithm 5.1. Once the discrete solution is obtained, we compute the *a posteriori* error indicator $\eta_s^s(K) + \xi^2(K)$ for each element $K \in \mathcal{T}_h$, and select a marking set $\mathcal{M}_h$ such that

$$\vartheta\big(\eta_s^s + \xi^2\big) \le \sum_{K\in\mathcal{M}_h} \big(\eta_s^s(K) + \xi^2(K)\big),$$

where $0 < \vartheta < 1$ is the marking parameter. Then the longest edge bisection algorithm is applied for the set $\mathcal{M}_h$ to generate a new mesh.

Throughout this section, the convergence history is plotted in log-log coordinates, the stabilization parameters are set as $\alpha_F = \alpha_T = 16k^2$, $\mathbf{F} = \mathbf{0}$ and $\mathbf{g} = [1,0]^T$, where k is the polynomial order.

Algorithm 5.1 (Fixed-point algorithm)**.**
Input: Initial guess $(\mathbf{u}_h^0, \widehat{\mathbf{u}}_h^0, p_h^0, \widehat{p}_h^0, T_h^0, \widehat{T}_h^0) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h \times W_h \times \widehat{W}_h$ and tol $= 10^{-8}$.

1: For $i \ge 0$, find $(\mathbf{u}_h^{i+1}, \widehat{\mathbf{u}}_h^{i+1}, p_h^{i+1}, \widehat{p}_h^{i+1}) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$ such that

$$\begin{aligned} A_T^h\Big(T_h^i; \Big(\mathbf{u}_h^{i+1}, \widehat{\mathbf{u}}_h^{i+1}\Big), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)\Big) + O_F^h\Big(\mathbf{u}_h^i; \Big(\mathbf{u}_h^{i+1}, \widehat{\mathbf{u}}_h^{i+1}\Big), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)\Big)& \\ -B^h\big((\mathbf{v}_h, \widehat{\mathbf{v}}_h), (p_h^{i+1}, \widehat{p}_h^{i+1})\big) - D\big(T_h^i, \mathbf{v}_h\big) &= (\mathbf{F}, \mathbf{v}_h), \\ B^h\Big(\Big(\mathbf{u}_h^{i+1}, \widehat{\mathbf{u}}_h^{i+1}\Big), (q_h, \widehat{q}_h)\Big) &= 0, \end{aligned}$$

for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h, q_h, \widehat{q}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$. Then, we find $(T_h^{i+1}, \widehat{T}_h^{i+1}) \in W_h \times \widehat{W}_h$ such that

$$A_T^h\Big(T_h^i; \Big(T_h^{i+1}, \widehat{T}_h^{i+1}\Big), (w_h, \widehat{w}_h)\Big) - O_T^h\Big(\mathbf{u}_h^{i+1}; \Big(T_h^{i+1}, \widehat{T}_h^{i+1}\Big), (w_h, \widehat{w}_h)\Big) = \frac{\mathcal{H}}{\sharp\mathcal{T}_z} \sum_{K\in\mathcal{T}_z} w_h|_K(z),$$

for any $(w_h, \widehat{w}_h) \in W_h \times \widehat{W}_h$.

2: If $\|(\mathbf{u}_h^{i+1}, p_h^{i+1}, T_h^{i+1}) - (\mathbf{u}_h^i, p_h^i, T_h^i)\|_2 \ge$ tol, set $i = i+1$, and go to step **1**. Otherwise, **return** $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h, T_h, \widehat{T}_h) = (\mathbf{u}_h^{i+1}, \widehat{\mathbf{u}}_h^{i+1}, p_h^{i+1}, \widehat{p}_h^{i+1}, T_h^{i+1}, \widehat{T}_h^{i+1})$

Example 5.1. On the convex domain $\Omega = (0,1)^2$, we set $\nu = \kappa = 1$, $z = (0.5, 0.5)$ and $\mathcal{H} = 1$.

In Table 1, the computed L^2-errors for the velocity gradient, the pressure, the temperature, and the velocity divergence are provided for $k = 1$. Since there have no exact solutions, we take the discrete solutions on the uniform mesh with 65 536 triangles as the reference solutions. Obviously, these errors can all achieve the convergence order $O(h)$ and the velocity divergence is almost zero.

In Figure 1, the adaptive mesh after 22 adaptive iterations and corresponding plots of the discrete velocity field and temperature are shown. Notice that the initial mesh consists of 64 triangles and we set $\vartheta = 0.4$. It is easy to observe that the mesh nodes are concentrated around the interior point $z = (0.5, 0.5)$. In Figure 2, we present the convergence history of *a posteriori* error estimator $\eta_s + \xi$ $(s = 1.2, 1.4, 1.6)$ for $k = 1$ and $k = 2$ on adaptive meshes, and convergence rate $O(N^{-k/2})$ can be obtained.

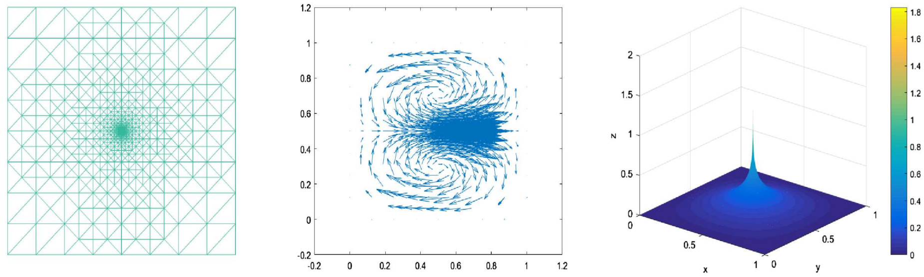

FIGURE 1. Example 5.1: The adaptive mesh after 22 adaptive iterations (*left*) and the corresponding plots of the discrete velocity field (*middle*) and the temperature (*right*) for $k = 2$ and $s = 1.4$.

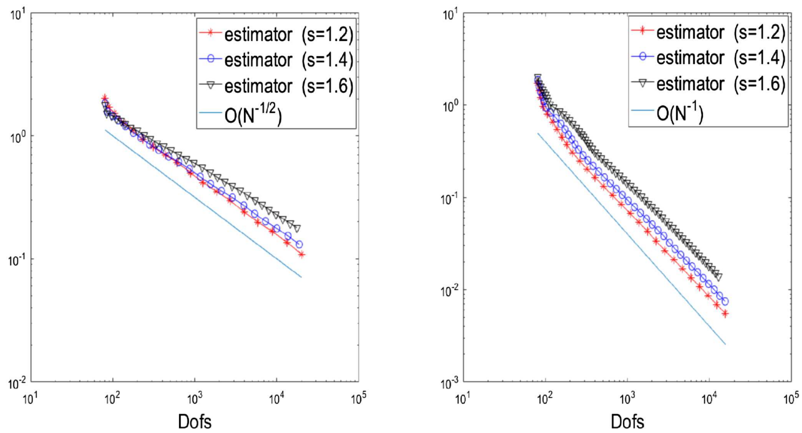

FIGURE 2. Example 5.1: The convergence histories of the *a posteriori* error estimators $\eta_s+\xi$ ($s = 1.2, 1.4, 1.6$) for $k = 1$ (*left*) and $k = 2$ (*right*).

Example 5.2. In this example, we consider the problem (1) on a L-shaped domain $\Omega = (-1, 1)^2\backslash[0, 1)\times(-1, 0]$. Here the right hand side of the heat equation is a linear combination of Dirac functions. Specifically, we will replace (1c) with

$$-\kappa\Delta T + \mathbf{u}\cdot\nabla T = \sum_{v\in\mathcal{V}} \mathcal{H}_v\delta_v,$$

and set

$$\mathcal{V} = \{(0.5, 0.5), (-0.5, 0.5), (-0.5, -0.5)\}, \qquad \mathcal{H}_v = \{0.2, -0.3, 0.4\}.$$

Throughout this example, we set $\vartheta = 0.3$. The initial mesh of the domain Ω is presented in Figure 3. In Figure 4, the adaptive meshes generated by $\eta_{1.4} + \xi$ are shown for $k = 1$ and $\nu = \kappa = 1$. It is easy to observe that the mesh nodes are concentrated around the interior points in $\mathcal{V}$. In Figure 5, plots of the discrete velocity field and temperature are provided on an adaptive mesh generated by $\eta_{1.4} + \xi$ for $k = 1$ and $\nu = \kappa = 1$. In Figure 6, the convergence history of *a posteriori* error estimator $\eta_s + \xi$ ($s = 1.4, 1.6$) for different k, ν and κ are given. Obviously, these error estimators all can derive optimal convergence order $O(N^{-k/2})$.

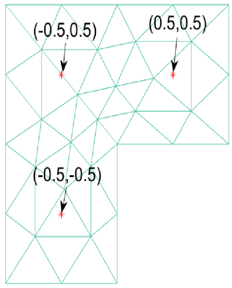

FIGURE 3. Example 5.2: The initial mesh of the domain Ω.

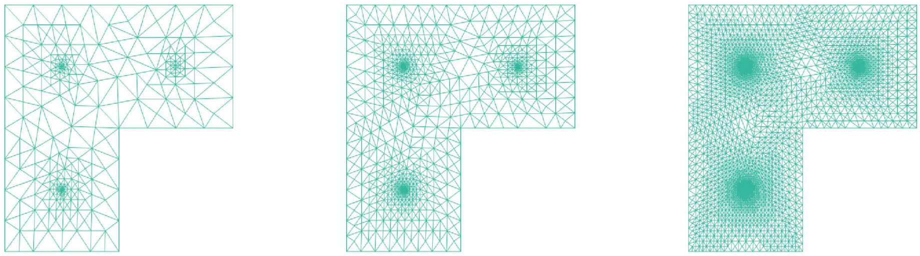

FIGURE 4. Example 5.2: The adaptive meshes after 14, 20 and 26 adaptive iterations for $k = 1$, $s = 1.4$ and $\nu = \kappa = 1$.

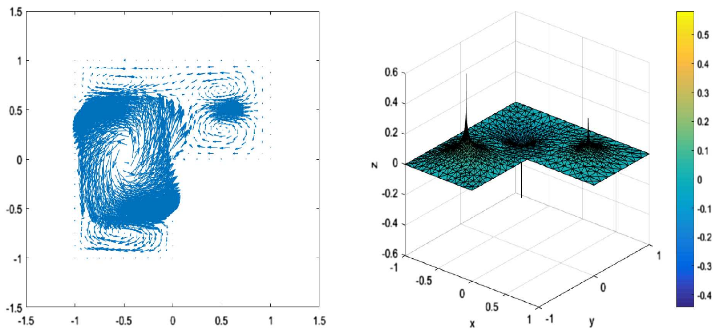

FIGURE 5. Example 5.2: The plots of the discrete velocity field $\mathbf{u}_h$ (*left*) and temperature T_h (*right*) on the adaptive meshes after 22 adaptive iterations for $k = 1$, $s = 1.4$ and $\nu = \kappa = 1$.

6. Conclusions

In this paper, we consider a hybridized discontinuous Galekin method, which is exactly divergence-free and $H(\text{div})$-conforming, for the generalized Boussinesq problem (1) with singular heat source. Specifically, we use polynomials of order k, $k-1$ and k to approximate the velocity, the pressure and the temperature. By Brouwer's fixed point theorem, the resulting HDG discrete system is proved to has a solution in two dimensions under a smallness assumption on the problem data and solutions. If we further assume that the viscosity and thermal conductivity are two positive constants, *a priori* error estimates with convergence rate $O(h)$ and efficient and reliable *a posteriori* error estimates are derived for errors in the velocity, the pressure and the temperature.

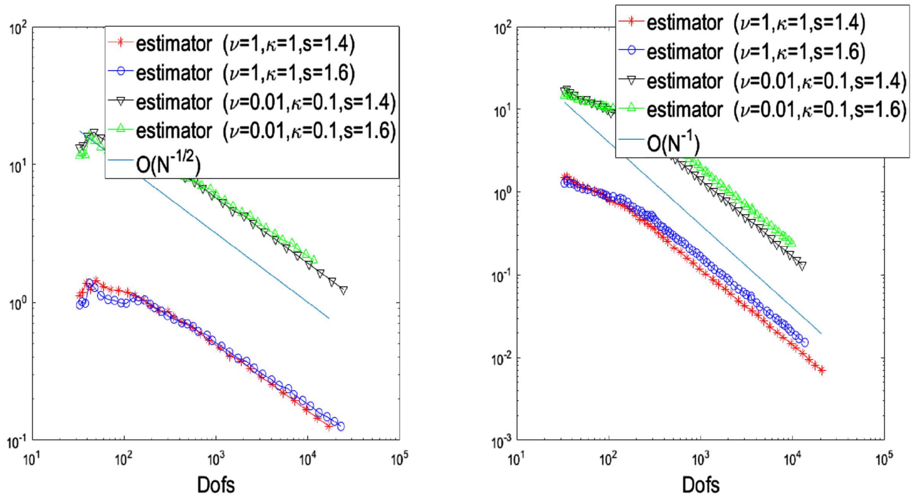

FIGURE 6. Example 5.2: The convergence histories of the *a posteriori* error estimators $\eta_s+\xi$ ($s = 1.4, 1.6$) for $k = 1$ (*left*) and $k = 2$ (*right*).

Finally, two numerical examples illustrate the theoretical analysis and show the performance of the obtained *a posteriori* error estimator.

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