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NUMERICAL SOLUTIONS TO HYPERBOLIC MAXWELL QUASI-VARIATIONAL INEQUALITIES IN BEAN–KIM MODEL FOR TYPE-II SUPERCONDUCTIVITY

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Abstract. This paper is devoted to the finite element analysis for the Bean–Kim model governed by the full 3D Maxwell equations. Describing type-II superconductivity at the macroscopic level, this model leads to a challenging coupled system consisting of the Faraday equation and a hyperbolic quasi-variational inequality (QVI) of the second kind with L^1-type nonlinearity, that arises explicitly from the magnetic field dependency in the critical current. With the involved Maxwell coupling in the 3D $\boldsymbol{H}(\mathbf{curl})$-setting, the hyperbolic QVI character poses the primary challenge in the numerical investigation. Two mixed finite element methods based on implicit Euler and leapfrog time-stepping are proposed. On the one hand, the implicit Euler method results in a nonstandard system of curl-curl elliptic QVI with a first-order curl-type nonlinearity. Though the well-posedness of this system is guaranteed, its numerical realization is not straightforward and requires the use of a two-stage iteration process of high computational complexity. On the other hand, by approximating the electric and magnetic fields at two different time step levels, the leapfrog method turns out to be more suitable as it naturally eliminates the notorious QVI structure. More importantly, utilizing suited subdifferential and optimization techniques, we are able to prove an efficiently computable explicit formula for its exact solution in terms of the electric field, which makes its numerical computation substantially more favorable than the Euler method. As further advantages, the leapfrog method applies to broad scenarios involving low regular data of bounded variation (BV) in time for both the applied current source and the temperature distribution. Through nonstandard technical arguments tailored to the BV data, our analysis proves the conditional stability and, eventually, the uniform convergence of the proposed leapfrog method. This paper is closed by 3D numerical tests showcasing the reasonable and efficient performance of the proposed numerical solution.

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1. Introduction

Superconductivity comprises physical properties of certain materials, causing them to lose their electrical resistance at sufficiently low temperatures. In particular, the discovery of a high-temperature superconductor (HTS) in the late 80's has opened up a new scientific frontier and made a tremendous impact on technological

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advances of the 21st century. HTS falls into the class of type-II superconductors characterized by the presence of the mixed phase and non-abrupt transition between the superconducting and normal states. At the macroscopic level, the Bean–Kim model [5,19] is widely accepted to describe the physical process of type-II superconductors. This model replaces the classical Ohm's law and postulates a nonlinear relation between the electric field $\boldsymbol{E}$ and the current density $\boldsymbol{J}$ as follows:

(B1) there exists a critical current j_c, depending on the magnetic field and the temperature, such that $|\boldsymbol{J}| \leq j_c$;
(B2) the electric field $\boldsymbol{E}$ vanishes if $|\boldsymbol{J}| < j_c$;
(B3) the electric field $\boldsymbol{E}$ is parallel to $\boldsymbol{J}$.

In this regard, the Bean–Kim model specifies the underlying region of the superconducting state by all points where the strict inequality condition in (B2) is satisfied. Due to their inequality and nonsmooth character, incorporating (B1)–(B3) to electromagnetic equations such as Maxwell's equations or eddy current equations gives rise to challenging mathematical problems. In the eddy current case, the Bean–Kim model results in a parabolic quasi-variational inequality (QVI) featuring a first-kind (obstacle-type) character. See [3,6–8,26,27] for numerous contributions in this research direction. The finite element analysis for the parabolic variational inequality (VI) counterpart, resulting from neglecting the temperature and magnetic field dependency in j_c, has been studied in [11]. Furthermore, the numerical analysis for the associated parabolic QVI in a two-dimensional (2D) setting was investigated in [4]. See also [14,15,23] for related contributions towards parabolic QVI. All aforementioned articles are devoted to the eddy current case as a simplification of the Maxwell equations by disregarding the displacement current.

Considering the full Maxwell equations for the electromagnetic fields, the Bean–Kim model (B1)–(B3) leads to a distinctive QVI problem [30]. In contrast to the eddy current counterpart, the full Maxwell case is given by a coupled system involving a hyperbolic QVI of the second kind with L^1-type nonlinearity and the Faraday equation. The nonlinear coupling and the QVI character explicitly arise from the magnetic field dependency in the critical current. The corresponding formulation reads precisely as follows: Find $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)) \cap L^\infty((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{H}(\mathbf{curl}))$ such that

$$\begin{cases} \displaystyle\int_\Omega \epsilon\partial_t\boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \mathbf{curl}\,\boldsymbol{H}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \displaystyle + \int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))(|\boldsymbol{v}|-|\boldsymbol{E}(t)|)\,\mathrm{d}x \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \\ \text{for a.e. } t\in(0,T) \text{ and all } \boldsymbol{v}\in\boldsymbol{L}^2(\Omega) \\ \mu\partial_t\boldsymbol{H}(t) + \mathbf{curl}\,\boldsymbol{E}(t) = 0 \quad \text{for a.e. } t\in(0,T) \\ (\boldsymbol{E},\boldsymbol{H})(0) = (\boldsymbol{E}_0,\boldsymbol{H}_0). \end{cases} \tag{QVI}$$

The well-posedness of (QVI) has been recently studied by the third author in [30] based on the Maxwell theory along with the Rothe method and suitable fixed-point arguments. As readily explored in [30], the pointwise (primal) PDE-formulation of (QVI) is precisely given by the Maxwell equations under the constitutive relations (B1)–(B3) as follows:

$$\begin{cases} \epsilon\partial_t\boldsymbol{E} - \mathbf{curl}\,\boldsymbol{H} + \boldsymbol{J} = \boldsymbol{f} & \text{in } \Omega\times(0,T) \\ \mu\partial_t\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0 & \text{in } \Omega\times(0,T) \\ \boldsymbol{E}\times\boldsymbol{n} = 0 & \text{on } \partial\Omega\times(0,T) \\ \boldsymbol{E}(\cdot,0) = \boldsymbol{E}_0 \quad \boldsymbol{H}(\cdot,0) = \boldsymbol{H}_0 & \text{in } \Omega \\ |\boldsymbol{J}(x,t)| \leq j_c(x,\theta(x,t),\boldsymbol{H}(x,t)) & \text{for a.e. } (x,t)\in\Omega\times(0,T) \\ \boldsymbol{J}(x,t)\cdot\boldsymbol{E}(x,t) = j_c(x,\theta(x,t),\boldsymbol{H}(x,t))|\boldsymbol{E}(x,t)| & \text{for a.e. } (x,t)\in\Omega\times(0,T). \end{cases}$$

In the setting of (QVI), we consider a bounded simply connected Lipschitz polyhedral domain $\Omega\subset\mathbb{R}^3$ containing a type-II superconductor and a finite time horizon $T>0$ where the electromagnetic fields $\boldsymbol{E}\colon\Omega\times$

$(0,T) \to \mathbb{R}^3$ and $\boldsymbol{H}\colon \Omega \times (0,T) \to \mathbb{R}^3$ are acting. Furthermore, $\theta\colon \Omega \times (0,T) \to \mathbb{R}$ and $\boldsymbol{f}\colon \Omega \times (0,T) \to \mathbb{R}^3$ are given data representing the operating temperature distribution and the applied current source. Note that all involved time-dependent functions $\boldsymbol{E}, \boldsymbol{H}, \theta, \boldsymbol{f}$ are considered in (QVI) as abstract functions mapping from the time interval $(0,T)$ into the corresponding Banach space. Furthermore, to model the magnetic field and temperature dependence in the critical current, we consider a specific nonlinear func tion $j_c\colon \Omega \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. The precise assumptions for j_c and all other data involved in (QVI) are specified in Assumption 1.1.

The hyperbolic QVI character along with the Maxwell coupling in the 3D $\boldsymbol{H}(\mathbf{curl})$-setting poses the fundamental challenge in the numerical analysis of (QVI), which is to the best of the authors' knowledge genuinely open. As mentioned earlier, the contribution [4] is devoted to the numerical analysis of the parabolic QVI problem arising from the eddy current case in two dimensions. In particular, the 2D setting leads to an H^1-type problem. Thus, the developed strategies by Barrett and Prigozhin [4] cannot be applied to (QVI).

The present article is therefore concerned with the development of efficient numerical solutions for (QVI) and proposes two fully discrete finite element (FE) methods. The first one ($\mathrm{IE}_{N,h}$) is built upon the implicit Euler time-stepping and a mixed FE spatial discretization comprising the lowest order edge element space $\mathbf{ND}_h$ [25] for the electric field and the piecewise constant FE space $\mathbf{DG}_h$ for the magnetic field. This ansatz results in a nontrivial system (2.3) of elliptic **curl**–**curl**-type QVIs with a first-order **curl**-type nonlinearity. We prove its well-posedness in Theorem 2.1 based on its equivalent reformulation through a system of fixed-point equations

$$F_{Nh}^n(\boldsymbol{E}_h^n) = \boldsymbol{E}_h^n \qquad \forall n \in \{1,\dots,N\}.$$

Here, F_{Nh}^n denotes the solution operator for the VI problem (2.4) resulted by freezing the key component $\mathbf{curl}\,\boldsymbol{E}_h^n$ inside the nonlinearity j_c of (2.3). This fixed-point strategy serves also as a basis for the numerical realization of ($\mathrm{IE}_{N,h}$): At every time step $n \in \{1,\dots,N\}$, we approximate $\boldsymbol{E}_h^n$ by a nonlinear iterative process

$$\boldsymbol{e}_k = F_{Nh}^n(\boldsymbol{e}_{k-1}), \qquad k = 1,2,\dots, \tag{1.1}$$

where, for every iteration $k \in \mathbb{N}$, the solution $\boldsymbol{e}_k$ of the VI problem (2.4) for $\boldsymbol{w} = \boldsymbol{e}_{k-1}$ is numerically computed by the semi-smooth Newton (SSN) method (cf. [17]). Such a nonlinear solver is well-known to be able to address the nonlinearity and inequality constraints inherent in the VI formulation. However, the SSN algorithm is an iterative method and is applied in (1.1) iteratively and for multiple times. This leads to a nonstandard two-stage iteration process which makes the overall numerical procedure of the implicit Euler scheme ($\mathrm{IE}_{N,h}$) rather difficult and computationally intensive, particularly when dealing with large-scale simulations.

Aiming at developing a more efficient numerical solution, we propose another FE method capable of treating the notorious QVI character without invoking any nonlinear iterative process and SSN method. More specifically, the second scheme applies the leapfrog time-stepping [9,29] and reverses the use of the mixed FE discretization, *i.e.*, $\mathbf{DG}_h$ for the electric field and $\mathbf{ND}_h$ for the magnetic field. Compared to the implicit Euler method, the leapfrog time-stepping turns out to be profoundly suitable for (QVI). In fact, by approximating the electric and magnetic fields at different time steps, ($\mathrm{LF}_{N,h}$) naturally simplifies the QVI character to a VI problem. In Theorem 2.3, we derive its well-posedness and, more importantly, prove an explicit formula for its exact solution $\boldsymbol{E}_h^n$ in terms of given data. The proof for the explicit formula (2.15) strongly relies on suited subdifferential and optimization techniques accounting for the proposed $\mathbf{DG}_h$ structure of the electric field. Thanks to (2.15), the exact solution $\boldsymbol{E}_h^n$ of ($\mathrm{LF}_{N,h}$) can be directly and efficiently implemented without the use of any additional solver. This is the primary advantage of ($\mathrm{LF}_{N,h}$) over ($\mathrm{IE}_{N,h}$) that is confirmed by our numerical tests (Sect. 4.1).

The second part of this paper focuses on the stability and convergence analysis of ($\mathrm{LF}_{N,h}$) under a low regularity assumption on the applied current source $\boldsymbol{f} : \Omega \times (0,T) \to \mathbb{R}^3$ and the temperature distribution $\theta : \Omega \times (0,T) \to \mathbb{R}$. We underline that previous mathematical contributions (see [30] and the references therein) mainly rely on the global Lipschitz regularity in time for θ and the global H^1-regularity in time for $\boldsymbol{f}$. Our current analysis allows us to substantially weaken the previous requirement to data of bounded variation in time, *i.e.*,

$$\boldsymbol{f} \in \mathrm{BV}([0,T], \boldsymbol{L}^2(\Omega)) \quad \text{and} \quad \theta \in \mathrm{BV}([0,T], L^2(\Omega)) \cap L^\infty(\Omega \times (0,T)). \tag{1.2}$$

This regularity improvement is pivotal, especially for the temperature distribution θ. As a matter of fact, in real applications, θ is generated by solutions of heat equations. Thus, requiring less regularity in θ makes our approach more applicable to wider scenarios involving nonsmooth data. This is particularly relevant when considering energy (hysteresis) losses in type-II superconductors caused by the local penetration and motion of the magnetic flux occurring in the mixed phase (cf. [13]). In this more complex case, the temperature distribution is specified by a nonsmooth parabolic system and is thus expected to possess poor regularity. The weaker assumption (1.2), however, complicates the stability analysis of ($\mathrm{LF}_{N,h}$) considerably and requires a significant extension of developed techniques. In Theorem 3.2, we prove a conditional stability result through nonstandard technical arguments tailored to the current BV situation (1.2). After proving the stability result, we present our final result in Theorem 3.5 on the uniform convergence of ($\mathrm{LF}_{N,h}$) towards the unique solution to (QVI). The proof is based on the use of the stability result (Thm. 3.2) and the $\boldsymbol{X}_h^{(\mu)}$-regularity of the magnetic solution to ($\mathrm{LF}_{N,h}$), allowing for the application of the discrete compactness property of $\boldsymbol{X}_h^{(\mu)}$ to pass to the limit in the L^1 nonlinearity of the QVI problem. As a byproduct, Theorem 3.5 provides a well-posedness result for (QVI) under the low regularity condition (1.2), exending the recently established result in Corollary 5.2 of [30].

1.1. Preliminaries

For a given Banach space X, we denote its norm by $\|\cdot\|_X$ and the duality pairing with the corresponding dual space X^* by $\langle\cdot,\cdot\rangle$. If X is a Hilbert space, then $(\cdot,\cdot)_X$ stands for its scalar product and $\|\cdot\|_X$ for the induced norm. In the case of $X=\mathbb{R}^n$, we renounce the subscript in the (Euclidean) norm and write $|\cdot|$. The Euclidean scalar product is denoted by a dot. Unless otherwise stated, we identify the dual space X^* with the Hilbert space X itself. The central Hilbert spaces in this paper are $\boldsymbol{H}(\mathbf{curl}) := \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega) : \mathbf{curl}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\}$ and $\boldsymbol{H}(\mathrm{div}) := \{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega) : \mathrm{div}\,\boldsymbol{v}\in\boldsymbol{L}^2(\Omega)\}$ where $\mathbf{curl}$ and div are understood in the distributional sense. Also, note that we use bold letters for vector-valued functions and the respective spaces. As usual, $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions with compact support in Ω. The spaces $\boldsymbol{H}_0(\mathbf{curl})$ and $\boldsymbol{H}_0(\mathrm{div})$ stand for the closure of $\boldsymbol{\mathcal{C}}_0^\infty(\Omega)$ with respect to the $\boldsymbol{H}(\mathbf{curl})$-norm and the $\boldsymbol{H}(\mathrm{div})$-norm, respectively. Material parameters occur on the problem statement, and thus, for a given positive function $\alpha\in L^\infty(\Omega)$, we denote by $\boldsymbol{L}^2_\alpha(\Omega)$ the weighted $\boldsymbol{L}^2(\Omega)$-space with the scalar product $(\alpha\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$. Making use of this notation, we introduce

$$\boldsymbol{X}^{(\mu)}(\Omega) := \boldsymbol{H}(\mathbf{curl})\cap\mu^{-1}\boldsymbol{H}_0(\mathrm{div}{=}0) = \big\{\boldsymbol{v}\in\boldsymbol{H}(\mathbf{curl}) : (\boldsymbol{v},\nabla\phi)_{\boldsymbol{L}^2_\mu(\Omega)} = 0 \quad \forall\phi\in H^1(\Omega)\big\}. \tag{1.3}$$

By $\mathcal{P}$ we denote the set of all partitions of the interval $[0,T]$, *i.e.*,

$$\mathcal{P} := \{P=\{s_0,\dots,s_{n_P}\}\subset[0,T] \ : \ n_P\in\mathbb{N} \text{ and } s_i\le s_{i+1} \text{ for every } 0\le i\le n_{\mathcal{P}}-1\}.$$

Then, for a given Banach space V, we introduce the BV function space

$$\mathrm{BV}([0,T],V) := \{g\colon[0,T]\to V \ : \ \mathrm{TV}(g)<\infty\},$$

where, for a given function $g\colon[0,T]\to V$, the pointwise total variation is defined by

$$\mathrm{TV}(g) := \sup_{P\in\mathcal{P}}\sum_{n=0}^{n_P-1}\|g(s_{n+1})-g(s_n)\|_V. \tag{1.4}$$

By definition, we see that the embedding $\mathrm{BV}([0,T],V)\hookrightarrow L^\infty((0,T),V)$ is valid. Let us close this section by presenting the mathematical assumption for (QVI).

Assumption 1.1 (Regularity assumptions on the material parameters and given data)**.**

(A1) The material parameters $\epsilon, \mu \in L^\infty(\Omega)$ are piecewise constants, *i.e.*, there exists a family of Lipschitz polyhedral domains $\{\Omega_j\}_{j=1}^{j_0}$ in Ω with

$$\Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j \in \{1, \dots, j_0\}, \quad \overline{\Omega} = \bigcup_{j=1}^{j_0} \overline{\Omega}_j,$$

and there exist constants $c_j^\epsilon, c_j^\mu > 0$ such that

$$\epsilon(x) = c_j^\epsilon, \quad \mu(x) = c_j^\mu, \quad \text{for a.e. } x \in \Omega_j \text{ and every } j \in \{1, \dots, j_0\}.$$

Furthermore, we set $\underline{\epsilon} := \min_{j=1,\dots,j_0} c_j^\epsilon$, $\underline{\mu} := \min_{j=1,\dots,j_0} c_j^\mu$ and $\overline{\epsilon} := \max_{j=1,\dots,j_0} c_j^\epsilon$, $\overline{\mu} := \max_{j=1,\dots,j_0} c_j^\mu$.

(A2) For every $(y, \boldsymbol{z}) \in \mathbb{R} \times \mathbb{R}^3$, $j_c(\cdot, y, \boldsymbol{z}) \colon \Omega \to \mathbb{R}$ is nonnegative and Lebesgue-measurable.

(A3) For every $M > 0$, there exists a constant $C(M) > 0$ such that

$$0 \leq j_c(x, y, \boldsymbol{z}) \leq C(M)$$

for a.e. $x \in \Omega$, every $y \in [-M, M]$, and every $\boldsymbol{z} \in \mathbb{R}^3$.

(A4) For every $M > 0$, there exists a constant $L(M) > 0$ such that

$$|j_c(x, y_1, \boldsymbol{z}_1) - j_c(x, y_2, \boldsymbol{z}_2)| \leq L(M)(|y_1 - y_2| + |\boldsymbol{z}_1 - \boldsymbol{z}_2|)$$

for a.e. $x \in \Omega$, every $y_1, y_2 \in [-M, M]$, and $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^3$.

(A5) It holds that $\boldsymbol{f} \in \mathrm{BV}([0,T], \boldsymbol{L}^2(\Omega))$ and $\theta \in \mathrm{BV}([0,T], L^2(\Omega)) \cap L^\infty(\Omega_T)$ with $\Omega_T := \Omega \times (0,T)$.

(A6) The initial data fulfills $(\boldsymbol{E}_0, \boldsymbol{H}_0) \in \big(\boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl})\big) \times \boldsymbol{X}^{(\mu)}(\Omega)$.

The assumptions (A2)–(A4) are physically reasonable (see, *e.g.*, [2, 10]). An example for j_c satisfying (A2)–(A4) is given by $j_c(x, y, \boldsymbol{z}) = \frac{c \max\{0, 1 - \frac{|y|}{y_c}\}^2}{1 + |\boldsymbol{z}|^\beta} \chi_\omega(x)$ for some constants $c > 0, y_c > 0, \beta > 1$ and $\omega \subset \Omega$. This model is related to certain type-II superconductors as experimentally measured in [2, 10].

2. Fully discrete schemes

In the following, we propose two fully discrete schemes, both based on a mixed finite element method with suitable time-stepping. To begin with, we choose a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω, *i.e.*,

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \qquad \forall h > 0$$

such that $\epsilon_{|T}$ and $\mu_{|T}$ are constant for all $T \in \mathcal{T}_h$. Here, $h > 0$ stands for the largest diameter of $T \in \mathcal{T}_h$. The family $\{\mathcal{T}_h\}_{h>0}$ is assumed to be quasi-uniform: There exist constants $\rho > 0$ and $\nu > 0$ such that

$$\frac{h_T}{\rho_T} \leq \rho \quad \text{and} \quad \frac{h}{h_T} \leq \nu \qquad \forall T \in \mathcal{T}_h \quad \forall h > 0,$$

where h_T and ρ_T denote, respectively, the diameter of T and the diameter of the largest ball contained in T. All finite element spaces considered are summarized as follows:

$$\begin{aligned}
\mathbf{ND}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathbf{curl}) \ : \ \boldsymbol{v}_h|_T = \boldsymbol{a}_T + \mathbf{b}_T \times \boldsymbol{x} \text{ with } \boldsymbol{a}_T, \boldsymbol{b}_T \in \mathbb{R}^3 \quad \forall T \in \mathcal{T}_h\} \\
\mathbf{ND}_h^0 &:= \{\boldsymbol{v}_h \in \boldsymbol{H}_0(\mathbf{curl}) \ : \ \boldsymbol{v}_h|_T = \boldsymbol{a}_T + \mathbf{b}_T \times \boldsymbol{x} \text{ with } \boldsymbol{a}_T, \boldsymbol{b}_T \in \mathbb{R}^3 \quad \forall T \in \mathcal{T}_h\} \\
\mathbf{DG}_h &:= \{\boldsymbol{w}_h \in \boldsymbol{L}^2(\Omega) \ : \ \boldsymbol{w}_h|_T = \boldsymbol{a}_T \text{ with } \boldsymbol{a}_T \in \mathbb{R}^3 \quad \forall T \in \mathcal{T}_h\} \\
\Theta_h &:= \{\psi_h \in H^1(\Omega) \ : \ \psi_h|_T = \boldsymbol{a}_T \cdot \boldsymbol{x} + b_T \text{ with } \boldsymbol{a}_T \in \mathbb{R}^3, b_T \in \mathbb{R} \quad \forall T \in \mathcal{T}_h\}
\end{aligned}$$

$$\boldsymbol{X}_h^{(\mu)} := \{\boldsymbol{w}_h \in \mathbf{ND}_h \ : \ (\boldsymbol{w}_h, \nabla\psi_h)_{\boldsymbol{L}^2_\mu(\Omega)} = 0 \quad \forall \psi_h \in \Theta_h\}.$$

Note that the space $\boldsymbol{X}_h^{(\mu)}$ is the discrete counterpart to (1.3) and contains all discrete μ-divergence-free edge elements. Towards discretizing (QVI) in time, for every $N \in \mathbb{N}$, we set

$$\tau := \frac{T}{N}, \quad 0 = t_0 < t_1 < \cdots < t_N = T \quad \text{with} \quad t_n := n\tau. \tag{2.1}$$

2.1. Mixed FEM and implicit Euler time-stepping

Our first method is based on the implicit Euler time-stepping and a mixed FEM consisting of

- the Nédélec finite element space $\mathbf{ND}_h^0$ for the electric field $\boldsymbol{E}$
- the piecewise constant finite element space $\mathbf{DG}_h$ for the magnetic field $\boldsymbol{H}$.

Given a discrete approximation $(\boldsymbol{E}_h^0, \boldsymbol{H}_h^0) \in \mathbf{ND}_h^0 \times \mathbf{DG}_h$ for $(\boldsymbol{E}_0, \boldsymbol{H}_0)$, the proposed Euler ansatz leads to the following fully discrete formulation:

$$\begin{cases} \text{For every } n \in \{1, \dots, N\} \text{ find } (\boldsymbol{E}_h^n, \boldsymbol{H}_h^n) \in \mathbf{ND}_h^0 \times \mathbf{DG}_h \text{ such that} \\ \displaystyle\int_\Omega \epsilon\delta\boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^n) - \boldsymbol{H}_h^n \cdot \mathbf{curl}(\boldsymbol{v}_h - \boldsymbol{E}_h^n)\,\mathrm{d}x + \int_\Omega j_c(\cdot, \theta^n, \boldsymbol{H}_h^n)(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^n|)\,\mathrm{d}x \\ \qquad \displaystyle\geq \int_\Omega \boldsymbol{f}^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^n)\,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{ND}_h^0 \\ \mu\delta\boldsymbol{H}_h^n + \mathbf{curl}\,\boldsymbol{E}_h^n = 0 \end{cases} \tag{IE$_{N,h}$}$$

with and the backward in time difference quotients

$$\delta\boldsymbol{E}_h^n := \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau} \quad \text{and} \quad \delta\boldsymbol{H}_h^n := \frac{\boldsymbol{H}_h^n - \boldsymbol{H}_h^{n-1}}{\tau} \qquad \forall n \in \{1, \dots, N\} \tag{2.2}$$

and

$$\theta^n := \theta(t_n) \quad \text{and} \quad \boldsymbol{f}^n := \boldsymbol{f}(t_n) \qquad \forall n \in \{1, \dots, N\}.$$

By (2.2), the discrete Faraday equation $\mu\delta\boldsymbol{H}_h^n + \mathbf{curl}\,\boldsymbol{E}_h^n = 0$ is nothing but $\boldsymbol{H}_h^n = -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_h^n + \boldsymbol{H}_h^{n-1}$. Applying this formula leads therefore to a reformulation of (IE$_{N,h}$) to a system of elliptic **curl**–**curl** type QVIs:

$$\begin{cases} \text{For every } n \in \{1, \dots, N\} \text{ find } (\boldsymbol{E}_h^n, \boldsymbol{H}_h^n) \in \mathbf{ND}_h^0 \times \mathbf{DG}_h \text{ such that} \\ \displaystyle\int_\Omega \epsilon\tau^{-1}\boldsymbol{E}_h^n \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^n) + \tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_h^n \cdot \mathbf{curl}(\boldsymbol{v}_h - \boldsymbol{E}_h^n)\,\mathrm{d}x \\ \qquad \displaystyle+ \int_\Omega j_c\big(\cdot, \theta^n, -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_h^n + \boldsymbol{H}_h^{n-1}\big)(|\boldsymbol{v}_h| - |\boldsymbol{E}_h^n|)\,\mathrm{d}x \\ \qquad \displaystyle\geq \int_\Omega \big(\boldsymbol{f}^n + \epsilon\tau^{-1}\boldsymbol{E}_h^{n-1}\big) \cdot (\boldsymbol{v}_h - \boldsymbol{E}_h^n) + \boldsymbol{H}_h^{n-1} \cdot \mathbf{curl}(\boldsymbol{v}_h - \boldsymbol{E}_h^n)\,\mathrm{d}x \qquad \forall \boldsymbol{v}_h \in \mathbf{ND}_h^0 \\ \boldsymbol{H}_h^n = -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_h^n + \boldsymbol{H}_h^{n-1}. \end{cases} \tag{2.3}$$

Theorem 2.1. *Let Assumption 1.1 be satisfied. Then, for every $h > 0$ and $N \in \mathbb{N}$, the fully discrete problem (IE$_{N,h}$) admits a solution $\{(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n)\}_{n=1}^N \subset \mathbf{ND}_h^0 \times \mathbf{DG}_h$. If $N \in \mathbb{N}$ satisfies $N > TL(\|\theta\|_{L^\infty(\Omega_T)})(2\underline{\epsilon}\underline{\mu})^{-1}$, then the solution is unique.*

Proof. Let $h > 0$, $N \in \mathbb{N}$ and $n \in \{1, \dots, N\}$. Suppose that $(\boldsymbol{E}_h^{n-1}, \boldsymbol{H}_h^{n-1}) \subset \mathbf{ND}_h^0 \times \mathbf{DG}_h$ is already computed in agreement with (2.3). For a given $\boldsymbol{w} \in \mathbf{ND}_h^0$, we consider the following auxiliary variational inequality:

$$\begin{cases} \text{Find } \boldsymbol{e} \in \mathbf{ND}_h^0 \text{ such that} \\ \displaystyle\int_\Omega \epsilon\tau^{-1}\boldsymbol{e}\cdot(\boldsymbol{v}_h - \boldsymbol{e}) + \tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}(\boldsymbol{v}_h - \boldsymbol{e})\,\mathrm{d}x \\ \quad \displaystyle+ \int_\Omega j_c\big(\cdot, \theta^n, -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{w} + \boldsymbol{H}_h^{n-1}\big)(|\boldsymbol{v}_h| - |\boldsymbol{e}|)\,\mathrm{d}x \\ \quad \displaystyle\geq \int_\Omega \big(\boldsymbol{f}^n + \epsilon\tau^{-1}\boldsymbol{E}_h^{n-1}\big)\cdot(\boldsymbol{v}_h - \boldsymbol{e}) + \boldsymbol{H}_h^{n-1}\cdot\mathbf{curl}(\boldsymbol{v}_h - \boldsymbol{e})\,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{ND}_h^0\,. \end{cases} \tag{2.4}$$

Note that (2.4) arises from (2.3) by freezing the term $\mathbf{curl}\,\boldsymbol{E}_h^n$ in the nonlinearity j_c using $\mathbf{curl}\,\boldsymbol{w}$. From the classical theory for variational inequalities (see [21], Thms. 2.1' and 2.2), it follows that (2.4) admits a unique solution $\boldsymbol{e} \in \mathbf{ND}_h^0$. Therefore, the mapping

$$F_{Nh}^n\colon \mathbf{ND}_h^0 \to \mathbf{ND}_h^0, \quad \boldsymbol{w} \mapsto \boldsymbol{e},$$

that assigns to every $\boldsymbol{w} \in \mathbf{ND}_h^0$ the unique solution $\boldsymbol{e} \in \mathbf{ND}_h^0$ of (2.4), is well-defined. By construction, we have

$$(\boldsymbol{E}_h^n, \boldsymbol{H}_h^n) \text{ fulfills } (\mathrm{IE}_{N,h}) \text{ at the } n\text{-th time step } \Leftrightarrow\ F_{Nh}^n(\boldsymbol{E}_h^n) = \boldsymbol{E}_h^n \text{ and } \boldsymbol{H}_h^n = -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}_h^n + \boldsymbol{H}_h^{n-1}. \tag{2.5}$$

To verify the latter condition, we invoke the Brouwer fixed-point theorem, *i.e.*, we show that F_{Nh}^n is continuous and maps a closed ball into itself. To this aim, we test (2.4) with $\boldsymbol{v}_h = 0$, which results in

$$\begin{aligned} \min\big\{\underline{\epsilon}\tau^{-1}, \tau\overline{\mu}^{-1}\big\}\|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl})}^2 &\leq \int_\Omega \epsilon\tau^{-1}\boldsymbol{e}\cdot\boldsymbol{e} + \tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\boldsymbol{e}\,\mathrm{d}x \\ &\leq \int_\Omega \big(\boldsymbol{f}^n + \epsilon\tau^{-1}\boldsymbol{E}_h^{n-1}\big)\cdot\boldsymbol{e} + \boldsymbol{H}_h^{n-1}\cdot\mathbf{curl}\,\boldsymbol{e}\,\mathrm{d}x + \int_\Omega j_c\big(\cdot, \theta^n, -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{w} + \boldsymbol{H}_h^{n-1}\big)|\boldsymbol{e}|\,\mathrm{d}x \\ &\overbrace{\leq}^{(\mathrm{A3})} \big\|\boldsymbol{f}^n + \epsilon\tau^{-1}\boldsymbol{E}_h^{n-1}\big\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{e}\|_{\boldsymbol{L}^2(\Omega)} + \big\|\boldsymbol{H}_h^{n-1}\big\|_{\boldsymbol{L}^2(\Omega)}\|\mathbf{curl}\,\boldsymbol{e}\|_{\boldsymbol{L}^2(\Omega)} + \sqrt{|\Omega|}C\Big(\|\theta\|_{L^\infty(\Omega_T)}\Big)\|\boldsymbol{e}\|_{\boldsymbol{L}^2(\Omega)}, \end{aligned}$$

which implies for any $\boldsymbol{w} \in \mathbf{ND}_h^0$ that

$$\|F_{Nh}^n(\boldsymbol{w})\|_{\boldsymbol{H}(\mathbf{curl})} = \|\boldsymbol{e}\|_{\boldsymbol{H}(\mathbf{curl})} \leq C^\star := \frac{\big\|\boldsymbol{f}^n + \epsilon\tau^{-1}\boldsymbol{E}_h^{n-1}\big\|_{\boldsymbol{L}^2(\Omega)} + \big\|\boldsymbol{H}_h^{n-1}\big\|_{\boldsymbol{L}^2(\Omega)} + \sqrt{|\Omega|}C\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\min\big\{\underline{\epsilon}\tau^{-1}, \tau\overline{\mu}^{-1}\big\}}. \tag{2.6}$$

Next, let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbf{ND}_h^0$ be arbitrarily fixed. Testing (2.4) for $\boldsymbol{w} = \boldsymbol{w}_1$ (resp. $\boldsymbol{w} = \boldsymbol{w}_2$) with the test function $\boldsymbol{v}_h = F_{Nh}^n(\boldsymbol{w}_2)$ (resp. $\boldsymbol{v}_h = F_{Nh}^n(\boldsymbol{w}_1)$), we obtain after adding the resulting inequalities that

$$\begin{aligned} &\frac{1}{\tau}\|F_{Nh}^n(\boldsymbol{w}_1) - F_{Nh}^n(\boldsymbol{w}_2)\|_{\boldsymbol{L}_\epsilon^2(\Omega)}^2 + \tau\|\mathbf{curl}(F_{Nh}^n(\boldsymbol{w}_1) - F_{Nh}^n(\boldsymbol{w}_2))\|_{\boldsymbol{L}_{\mu^{-1}}^2(\Omega)}^2 \\ &\quad\leq \int_\Omega \big|j_c\big(\cdot, \theta^n, -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{w}_1 + \boldsymbol{H}_h^{n-1}\big) - j_c\big(\cdot, \theta^n, -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{w}_2 + \boldsymbol{H}_h^{n-1}\big)\big|\big||F_{Nh}^n(\boldsymbol{w}_1)| - |F_{Nh}^n(\boldsymbol{w}_2)|\big|\,\mathrm{d}x \\ &\quad\overbrace{\leq}^{(A4)} \tau\underline{\mu}^{-1/2}\underline{\epsilon}^{-1/2}L\Big(\|\theta\|_{L^\infty(\Omega_T)}\Big)\|\mathbf{curl}(\boldsymbol{w}_1 - \boldsymbol{w}_2)\|_{\boldsymbol{L}_{\mu^{-1}}^2(\Omega)}\|F_{Nh}^n(\boldsymbol{w}_1) - F_{Nh}^n(\boldsymbol{w}_2)\|_{\boldsymbol{L}_\epsilon^2(\Omega)} \\ &\quad\leq \frac{\tau^3 L\Big(\|\theta\|_{L^\infty(\Omega_T)}\Big)^2}{2\underline{\epsilon}\underline{\mu}}\|\mathbf{curl}(\boldsymbol{w}_1 - \boldsymbol{w}_2)\|_{\boldsymbol{L}_{\mu^{-1}}^2(\Omega)}^2 + \frac{1}{2\tau}\|F_{Nh}^n(\boldsymbol{w}_1) - F_{Nh}^n(\boldsymbol{w}_2)\|_{\boldsymbol{L}_\epsilon^2(\Omega)}^2. \end{aligned} \tag{2.7}$$

Consequently, the mapping $F^n_{Nh}\colon \mathbf{ND}^0_h \to \mathbf{ND}^0_h$ is Lipschitz-continuous and, denoting by $B(0,C^\star) \subset \mathbf{ND}^0_h$ the closed ball with zero at its center and radius $C^\star$, (2.6) implies that $F^n_{Nh|B(0,C^\star)}\colon B(0,C^\star) \to B(0,C^\star)$ is continuous. Thus, the Brouwer fixed-point theorem implies the existence of $\boldsymbol{E}^n_h \in \mathbf{ND}^0_h$ with $F^n_{Nh}(\boldsymbol{E}^n_h) = \boldsymbol{E}^n_h$. In conclusion, setting $\boldsymbol{H}^n_h := -\tau\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}^n_h + \boldsymbol{H}^{n-1}_h$, (2.5) implies that $(\boldsymbol{E}^n_h, \boldsymbol{H}^n_h) \in \mathbf{ND}^0_h \times \mathbf{DG}_h$ satisfies ($\mathrm{IE}_{N,h}$) at the n-th time step.

Towards uniqueness, let $N \in \mathbb{N}$ satisfy

$$N^2 > \frac{T^2 L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{2\underline{\epsilon}\underline{\mu}} \quad\Rightarrow\quad \tau^2 < \frac{2\underline{\epsilon}\underline{\mu}}{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2} \quad\Rightarrow\quad K := \frac{\tau^2 L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{2\underline{\epsilon}\underline{\mu}} \in (0,1).$$

As a result of (2.7), it holds that

$$\|\mathbf{curl}(F^n_{Nh}(\boldsymbol{w}_1) - F^n_{Nh}(\boldsymbol{w}_2))\|^2_{\boldsymbol{L}^2_{\mu^{-1}}(\Omega)} \le K\|\mathbf{curl}(\boldsymbol{w}_1 - \boldsymbol{w}_2)\|^2_{\boldsymbol{L}^2_{\mu^{-1}}(\Omega)} \qquad \forall \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbf{ND}^0_h\,. \tag{2.8}$$

Now, assume that $(\boldsymbol{E}^n_h, \boldsymbol{H}^n_h), (\hat{\boldsymbol{E}}^n_h, \hat{\boldsymbol{H}}^n_h) \in \mathbf{ND}^0_h \times \mathbf{DG}_h$ satisfy ($\mathrm{IE}_{N,h}$) at the n-th time step. In view of (2.5), it holds that $F^n_{Nh}(\boldsymbol{E}^n_h) = \boldsymbol{E}^n_h$ and $F^n_{Nh}(\hat{\boldsymbol{E}}^n_h) = \hat{\boldsymbol{E}}^n_h$. Thus, applying (2.8) with $\boldsymbol{w}_1 = \boldsymbol{E}^n_h$ and $\boldsymbol{w}_2 = \hat{\boldsymbol{E}}^n_h$, we obtain due to $K \in (0,1)$ that $\mathbf{curl}\,\boldsymbol{E}^n_h = \mathbf{curl}\,\hat{\boldsymbol{E}}^n_h$. For this reason, $\hat{\boldsymbol{E}}^n_h$ solves the auxiliary variational inequality (2.4) for $\boldsymbol{w} = \boldsymbol{E}^n_h$. In conclusion, $\hat{\boldsymbol{E}}^n_h = F^n_{Nh}(\boldsymbol{E}^n_h) = \boldsymbol{E}^n_h$ and consequently $\hat{\boldsymbol{H}}^n_h = \boldsymbol{H}^n_h$. This completes the proof. □

Remark 2.2 (Numerical realization of ($\mathrm{IE}_{N,h}$))**.** In view of the fixed-point reformulation (2.5), we numerically execute the scheme ($\mathrm{IE}_{N,h}$) by approximating $\boldsymbol{E}^n_h$ at every time step $n \in \{1,\dots,N\}$ by the nonlinear iterative process

$$\boldsymbol{e}_k = F^n_{Nh}(\boldsymbol{e}_{k-1}), \quad k = 1,2,\dots, \tag{2.9}$$

with a warm start $\boldsymbol{e}_0 := \boldsymbol{E}^{n-1}_h$. At every iteration k, the solution $\boldsymbol{e}_k$ of the corresponding VI problem (2.4) for $\boldsymbol{w} = \boldsymbol{e}_{k-1}$ is numerically computed by the SSN method [17], which is again an iterative method. To improve its convergence, the SSN iteration is initialized using a warm start given by the solution of the previous outer iteration $\boldsymbol{e}_{k-1}$.

Altogether, the numerical execution of ($\mathrm{IE}_{N,h}$) results in a two-stage iteration process comprising the outer one (2.9) and the inner SSN iteration. Even when considering a warm guess for (2.9) and SSN iterations, its numerical realization turns out to be extremely expensive in terms of computational complexity (see Sect. 4.1). Furthermore, we note that the problem ($\mathrm{IE}_{N,h}$) (resp. its reformulation (2.3)) features the first-order term $\mathbf{curl}\,\boldsymbol{E}^n_h$ within the nonlinearity j_c. Due to the lack of discrete compactness property for the space $\mathbf{curl}\,\mathbf{ND}^0_h$, the convergence of ($\mathrm{IE}_{N,h}$) is highly unclear. For these reasons, the remainder of this paper will focus on another fully discrete scheme capable of addressing these issues.

2.2. Mixed FEM and Leapfrog time-stepping

Let us begin by introducing the classical projection operator $\boldsymbol{P}_h\colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$, defined by

$$\forall \boldsymbol{u} \in \boldsymbol{L}^2(\Omega) \quad \boldsymbol{P}_h\boldsymbol{u} := \operatorname*{arg\,min}_{\boldsymbol{v}_h \in \mathbf{DG}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \quad\Leftrightarrow\quad \int_\Omega (\boldsymbol{P}_h\boldsymbol{u} - \boldsymbol{u}) \cdot \boldsymbol{v}_h \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,. \tag{2.10}$$

We also make use of the scalar counterpart of $\boldsymbol{P}_h$, that is, the operator $P_h\colon L^2(\Omega) \to \mathrm{DG}_h$ with DG_h containing all piecewise constant scalar functions. Moreover, we denote by $\boldsymbol{\Phi}_h\colon \boldsymbol{H}(\mathbf{curl}) \to \mathbf{ND}_h$ the operator assigning to every $\boldsymbol{y} \in \boldsymbol{H}(\mathbf{curl})$ the unique solution $\boldsymbol{y}_h \in \mathbf{ND}_h$ to the variational mixed problem

$$\begin{cases} (\mathbf{curl}\,\boldsymbol{y}_h, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} = (\mathbf{curl}\,\boldsymbol{y}, \mathbf{curl}\,\boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} & \forall \boldsymbol{v}_h \in \mathbf{ND}_h \\ (\boldsymbol{y}_h, \nabla\psi_h)_{\boldsymbol{L}^2_\mu(\Omega)} = (\boldsymbol{y}, \nabla\psi_h)_{\boldsymbol{L}^2_\mu(\Omega)} & \forall \psi_h \in \Theta_h. \end{cases} \tag{2.11}$$

The well-posedness for (2.11) follows from the classical theory of mixed problems (see, *e.g.*, [24], Thm. 2.45) in combination with the discrete Poincaré–Friedrichs-type inequality ([16], Thm. 4.7). Introducing the intermediate time steps

$$t_{n-\frac{1}{2}} := \frac{t_n + t_{n-1}}{2} = t_n - \frac{\tau}{2} \qquad \forall n \in \{1, \dots, N\},$$

we apply the leapfrog time-stepping [29] to (QVI) by approximating

- the quasi-variational inequality in (QVI) at the intermediate time steps $t_{n-\frac{1}{2}}$
- the Faraday equation in (QVI) at the time steps t_n

and making use of the following central difference and mean value approximations:

$$\partial_t \boldsymbol{E}\Big(t_{n-\frac{1}{2}}\Big) \approx \frac{\boldsymbol{E}(t_n) - \boldsymbol{E}(t_{n-1})}{\tau}, \quad \partial_t \boldsymbol{H}(t_n) \approx \frac{\boldsymbol{H}\Big(t_{n+\frac{1}{2}}\Big) - \boldsymbol{H}\Big(t_{n-\frac{1}{2}}\Big)}{\tau}, \quad \boldsymbol{E}\Big(t_{n-\frac{1}{2}}\Big) \approx \frac{\boldsymbol{E}(t_n) + \boldsymbol{E}(t_{n-1})}{2}.$$

Then, switching the role of the finite element spaces as follows:

- the piecewise constant finite element space $\mathbf{DG}_h$ for the electric field $\boldsymbol{E}$
- the Nédélec finite element space $\mathbf{ND}_h$ for the magnetic field $\boldsymbol{H}$,

we propose the following fully discrete scheme for (QVI):

$$\begin{cases} \text{For every } n \in \{1, \dots, N\} \text{ find } \boldsymbol{E}_h^n \in \mathbf{DG}_h \text{ such that} \\ \displaystyle\int_\Omega \epsilon \delta \boldsymbol{E}_h^n \cdot \Big(\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\Big) - \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \Big(\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\Big)\, \mathrm{d}x + \int_\Omega P_h j_c\Big(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}\Big)\Big(|\boldsymbol{v}_h| - \Big|\widehat{\boldsymbol{E}}_h^n\Big|\Big)\, \mathrm{d}x \\ \qquad \displaystyle\geq \int_\Omega \boldsymbol{f}_h^{n-\frac{1}{2}} \cdot \Big(\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\Big)\, \mathrm{d}x \quad \forall \boldsymbol{v}_h \in \mathbf{DG}_h \\ \text{and for every } n \in \{1, \dots, N-1\} \text{ find } \boldsymbol{H}_h^{n+\frac{1}{2}} \in \mathbf{ND}_h \text{ such that} \\ \displaystyle\int_\Omega \mu \delta \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h\, \mathrm{d}x + \int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\, \boldsymbol{w}_h\, \mathrm{d}x = 0 \quad \forall \boldsymbol{w}_h \in \mathbf{ND}_h, \\ \boldsymbol{E}_h^0 := \boldsymbol{P}_h \boldsymbol{E}_0 \in \mathbf{DG}_h, \quad \boldsymbol{H}_h^{\frac{1}{2}} := \boldsymbol{\Phi}_h \boldsymbol{H}_0 \in \boldsymbol{X}_h^{(\mu)} \end{cases} \tag{$\mathrm{LF}_{N,h}$}$$

with

$$\widehat{\boldsymbol{E}}_h^n := \frac{\boldsymbol{E}_h^n + \boldsymbol{E}_h^{n-1}}{2} \quad \forall n \in \{1, \dots, N\} \quad \text{and} \quad \delta \boldsymbol{H}_h^{n+\frac{1}{2}} := \frac{\boldsymbol{H}_h^{n+\frac{1}{2}} - \boldsymbol{H}_h^{n-\frac{1}{2}}}{\tau} \quad \forall n \in \{1, \dots, N-1\} \tag{2.12}$$

and

$$\boldsymbol{f}_h^{n-\frac{1}{2}} := \boldsymbol{P}_h \boldsymbol{f}^{n-\frac{1}{2}}, \quad \boldsymbol{f}^{n-\frac{1}{2}} := \boldsymbol{f}\Big(t_{n-\frac{1}{2}}\Big), \quad \text{and} \quad \theta^{n-\frac{1}{2}} := \theta\Big(t_{n-\frac{1}{2}}\Big) \qquad \forall n \in \{1, \dots, N\}. \tag{2.13}$$

Note that the regularity of the initial discrete magnetic field $\boldsymbol{H}_h^{\frac{1}{2}} = \boldsymbol{\Phi}_h \boldsymbol{H}_0 \in \boldsymbol{X}_h^{(\mu)}$ follows directly from the definition of $\boldsymbol{\Phi}_h$ in (2.11) and the regularity $\boldsymbol{H}_0 \in \boldsymbol{X}^{(\mu)}(\Omega)$ (see (1.3) for the definition of $\boldsymbol{X}^{(\mu)}(\Omega)$).

In every n-th time step, the nonlinearity j_c in ($\mathrm{LF}_{N,h}$) is evaluated at the discrete magnetic field $\boldsymbol{H}_h^{n-\frac{1}{2}}$ for the previous time step such that ($\mathrm{LF}_{N,h}$) is not of QVI type anymore and features a more simple VI character. More importantly, thanks to the reverse choice of the mixed FEM ($\mathbf{DG}_h$ for $\boldsymbol{E}$ and $\mathbf{ND}_h$ for $\boldsymbol{H}$), an explicit formula (see below) is available to compute the electrical field at every time step t_n, making the numerical realization of ($\mathrm{LF}_{N,h}$) significantly more efficient than ($\mathrm{IE}_{N,h}$). In the sequel, for the sake of a simpler notation, we introduce

$$\varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}) := \int_\Omega j_{c,h}^{n-\frac{1}{2}} |\boldsymbol{v}|\, \mathrm{d}x \quad \text{with} \quad j_{c,h}^{n-\frac{1}{2}} := P_h j_c\Big(\cdot, \theta^{n-\frac{1}{2}}, \boldsymbol{H}_h^{n-\frac{1}{2}}\Big) \qquad \forall n \in \{1, \dots, N\}. \tag{2.14}$$

Theorem 2.3. *Let Assumption 1.1 be satisfied. Then, for every $h > 0$ and $N \in \mathbb{N}$, the fully discrete scheme* (LF$_{N,h}$) *admits a unique solution $\{\boldsymbol{E}_h^n\}_{n=1}^N \subset \mathbf{DG}_h$ and $\{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \boldsymbol{X}_h^{(\mu)}$. Furthermore, for every $n = 1, \dots, N$, the solution $\boldsymbol{E}_h^n$ is explicitly given by the following formula*

$$\boldsymbol{E}_h^n = 2\widehat{\boldsymbol{E}}_h^n - \boldsymbol{E}_h^{n-1}, \quad \widehat{\boldsymbol{E}}_h^n = \frac{\tau\epsilon^{-1}}{2}(\boldsymbol{\omega}_h^n - \Pi_h^n \boldsymbol{\omega}_h^n), \quad \boldsymbol{\omega}_h^n := \boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}, \tag{2.15}$$

with

$$\Pi_h^n \boldsymbol{v}_h = \frac{j_{c,h}^{n-\frac{1}{2}} \boldsymbol{v}_h}{\max\Big(|\boldsymbol{v}_h|, j_{c,h}^{n-\frac{1}{2}}\Big)} \qquad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,. \tag{2.16}$$

Proof. Let $N \in \mathbb{N}$, $h > 0$, and $n \in \{1, \dots, N\}$. Supposing that the unique solution $(\boldsymbol{E}_h^{n-1}, \boldsymbol{H}_h^{n-\frac{1}{2}}) \in \mathbf{DG}_h \times \mathbf{ND}_h$ at the previous time step is already computed, we show that the solution $(\boldsymbol{E}_h^n, \boldsymbol{H}_h^{n+\frac{1}{2}})$ of (LF$_{N,h}$) uniquely exists. To this aim, let us first note that

$$\delta \boldsymbol{E}_h^n = \frac{\boldsymbol{E}_h^n - \boldsymbol{E}_h^{n-1}}{\tau} \overbrace{=}^{(2.12)} \frac{2}{\tau}\Big(\widehat{\boldsymbol{E}}_h^n - \boldsymbol{E}_h^{n-1}\Big). \tag{2.17}$$

Thus, applying (2.17) to (LF$_{N,h}$) results in the following variational inequality in terms of $\widehat{\boldsymbol{E}}_h^n$:

$$\begin{aligned}\int_\Omega \frac{2\epsilon}{\tau}\widehat{\boldsymbol{E}}_h^n \cdot \Big(\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\Big)\,\mathrm{d}x + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}\Big(\widehat{\boldsymbol{E}}_h^n\Big) \\ \geq \int_\Omega \Big(\boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\, \boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}\Big) \cdot \Big(\boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\Big)\,\mathrm{d}x \qquad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,.\end{aligned} \tag{2.18}$$

Thanks to its $\boldsymbol{L}^2$-structure, Theorems 2.1' and 2.2 of [21] implies that (2.18) admits a unique solution $\widehat{\boldsymbol{E}}_h^n \in \mathbf{DG}_h$. In view of (2.17), it follows that $\boldsymbol{E}_h^n = 2\widehat{\boldsymbol{E}}_h^n - \boldsymbol{E}_h^{n-1} \in \mathbf{DG}_h$ is the unique solution to the variational inequality in (LF$_{N,h}$). With $\boldsymbol{E}_h^n$ at hand, and if $n < N$, we directly obtain $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \mathbf{ND}_h$ as the unique solution to the discrete linear equation

$$\int_\Omega \mu \boldsymbol{H}_h^{n+\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x = \int_\Omega \mu \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \boldsymbol{w}_h\,\mathrm{d}x - \tau \int_\Omega \boldsymbol{E}_h^n \cdot \mathbf{curl}\, \boldsymbol{w}_h\,\mathrm{d}x \qquad \forall \boldsymbol{w}_h \in \mathbf{ND}_h\,. \tag{2.19}$$

Altogether, by inductive reasoning, we conclude that (LF$_{N,h}$) admits a unique solution $\{\boldsymbol{E}_h^n\}_{n=1}^N \subset \mathbf{DG}_h$ and $\{\boldsymbol{H}_h^{n+\frac{1}{2}}\}_{n=1}^{N-1} \subset \mathbf{ND}_h$. Moreover, since $\mathbf{curl}\, \nabla \equiv 0$ and $\boldsymbol{H}_h^{\frac{1}{2}} \in \boldsymbol{X}_h^{(\mu)}$, inserting $\boldsymbol{w}_h = \nabla \psi_h \in \mathbf{ND}_h$ for all $\psi_h \in \Theta_h$ in (2.19) yields, by inductive reasoning, that $\boldsymbol{H}_h^{n+\frac{1}{2}} \in \boldsymbol{X}_h^{(\mu)}$ holds for all $n \in \{1, \dots, N-1\}$. Let us now prove the explicit representation in (2.15). First of all, according to the definition of the subdifferential, it holds that

$$\begin{aligned}\partial \varphi_h^{n-\frac{1}{2}}(0) &= \Big\{\boldsymbol{v}_h \in \mathbf{DG}_h \;:\; (\boldsymbol{v}_h, \boldsymbol{p}_h)_{\boldsymbol{L}^2(\Omega)} \leq \varphi_h^{n-\frac{1}{2}}(\boldsymbol{p}_h) - \overbrace{\varphi_h^{n-\frac{1}{2}}(0)}^{=0} \quad \forall \boldsymbol{p}_h \in \mathbf{DG}_h\Big\} \\ &\overbrace{=}^{(2.14)} \Big\{\boldsymbol{v}_h \in \mathbf{DG}_h \;:\; (\boldsymbol{v}_h, \boldsymbol{p}_h)_{\boldsymbol{L}^2(\Omega)} \leq \Big(j_{c,h}^{n-\frac{1}{2}}, |\boldsymbol{p}_h|\Big)_{L^2(\Omega)} \quad \forall \boldsymbol{p}_h \in \mathbf{DG}_h\Big\}.\end{aligned} \tag{2.20}$$

By the piecewise constant structure of $\mathbf{DG}_h$, setting

$$\boldsymbol{p}_h = \begin{cases} \boldsymbol{v}_h, & \text{on } T \\ 0, & \text{on } \Omega \setminus T \end{cases}$$

for $T \in \mathcal{T}_h$ in (2.20) yields

$$\partial\varphi_h^{n-\frac{1}{2}}(0) = \Big\{ \boldsymbol{v}_h \in \mathbf{DG}_h \ : \ \big|\boldsymbol{v}_{h|T}\big| \le j_{c,h}^{n-\frac{1}{2}}{}_{|T} \quad \text{for all } T \in \mathcal{T}_h \Big\}. \tag{2.21}$$

By $\Pi_h^n : \mathbf{DG}_h \to \mathbf{DG}_h$, we now denote the Hilbert projector on $\partial\varphi_h^{n-\frac{1}{2}}(0)$, that assigns every $\boldsymbol{v}_h \in \mathbf{DG}_h$ the unique minimizer to

$$\begin{aligned} \min_{\boldsymbol{w}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \|\boldsymbol{v}_h - \boldsymbol{w}_h\|^2_{\boldsymbol{L}^2(\Omega)} &= \min_{\boldsymbol{w}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \sum_{T\in\mathcal{T}_h} \int_T |\boldsymbol{v}_h - \boldsymbol{w}_h|^2 \,\mathrm{d}x \\ &= \min_{\boldsymbol{w}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \sum_{T\in\mathcal{T}_h} |T| \big|\boldsymbol{v}_{h|T} - \boldsymbol{w}_{h|T}\big|^2. \end{aligned} \tag{2.22}$$

Let us verify that Π_h^n admits the explicit form (2.16). Indeed, in view of (2.21), it follows that, for every $\boldsymbol{v}_h \in \mathbf{DG}_h$ and $T \in \mathcal{T}_h$, $(\Pi_h^n \boldsymbol{v}_h)_{|T}$ minimizes the problem

$$\min_{x\in\mathbb{R}^3} \big|\boldsymbol{v}_{h|T} - x\big|^2 \quad \text{s.t.} \quad |x| \le j_{c,h}^{n-\frac{1}{2}}{}_{|T}. \tag{2.23}$$

But, the solution of the three-dimensional minimization problem (2.23) is exactly given by the projection of the vector $\boldsymbol{v}_{h|T} \in \mathbb{R}^3$ into the euclidean ball with radius $j_{c,h}^{n-\frac{1}{2}}{}_{|T}$, *i.e.*, $(\Pi_h^n \boldsymbol{v}_h)_{|T} = \frac{j_{c,h}^{n-\frac{1}{2}}{}_{|T} \boldsymbol{v}_{h|T}}{\max\left(|\boldsymbol{v}_{h|T}|, j_{c,h}^{n-\frac{1}{2}}{}_{|T}\right)}$. In conclusion, (2.16) is valid for the Hilbert projector $\Pi_h^n : \mathbf{DG}_h \to \mathbf{DG}_h$.

Now, let us verify that $\widehat{\boldsymbol{E}}_h^n$ given by (2.15) solves (2.18). To this aim, we recall from the classical Hilbert projection theorem that the projection $\Pi_h^n \boldsymbol{\omega}_h^n$, with $\omega_h^n \in \mathbf{DG}_h$ as in (2.15), is characterized by the solution to the variational inequality

$$(\boldsymbol{\omega}_h^n - \Pi_h^n \boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \Pi_h^n \boldsymbol{\omega}_h^n)_{\boldsymbol{L}^2(\Omega)} \le 0 \qquad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0). \tag{2.24}$$

Using once again the piecewise constant structure, we set for a given test function $\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)$

$$\widetilde{\boldsymbol{v}_h} = \left\{ \begin{array}{cc} \boldsymbol{v}_h & \text{in } T \\ \Pi_h^n \boldsymbol{\omega}_h^n & \text{in } \Omega \setminus T \end{array} \right\} \quad \in \partial\varphi_h^{n-\frac{1}{2}}(0)$$

in (2.24) to obtain its equivalence to

$$(\boldsymbol{\omega}_h^n - \Pi_h^n \boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \Pi_h^n \boldsymbol{\omega}_h^n)_{\boldsymbol{L}^2(T)} \le 0 \qquad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0) \quad \forall T \in \mathcal{T}_h. \tag{2.25}$$

Using that ϵ is positive and piecewise constant, we multiply the inequality in (2.25) with $\tau\epsilon^{-1}/2$, which yields

$$\left(\frac{\tau\epsilon^{-1}}{2} (\boldsymbol{\omega}_h^n - \Pi_h^n \boldsymbol{\omega}_h^n), \boldsymbol{v}_h - \Pi_h^n \boldsymbol{\omega}_h^n \right)_{\boldsymbol{L}^2(T)} \le 0 \qquad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0) \quad \forall T \in \mathcal{T}_h. \tag{2.26}$$

Consequently,

$$\left(\widehat{\boldsymbol{E}}_h^n, \boldsymbol{v}_h - \Pi_h^n \boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \overbrace{=}^{(2.15)} \left(\frac{\tau\epsilon^{-1}}{2} (\boldsymbol{\omega}_h^n - \Pi_h^n \boldsymbol{\omega}_h^n), \boldsymbol{v}_h - \Pi_h^n \boldsymbol{\omega}_h^n \right)_{\boldsymbol{L}^2(\Omega)} \overbrace{\le}^{(2.26)} 0 \qquad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0),$$

which implies that

$$\left(\widehat{\boldsymbol{E}}_h^n, \boldsymbol{v}_h\right)_{\boldsymbol{L}^2(\Omega)} \le \left(\widehat{\boldsymbol{E}}_h^n, \Pi_h^n \boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)$$

$$\Rightarrow \quad \max_{\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \left(\widehat{\boldsymbol{E}}_h^n, \boldsymbol{v}_h\right)_{\boldsymbol{L}^2(\Omega)} = \left(\widehat{\boldsymbol{E}}_h^n, \Pi_h^n \boldsymbol{\omega}_h^n\right)_{\boldsymbol{L}^2(\Omega)}, \tag{2.27}$$

since $\Pi_h^n \boldsymbol{\omega}_h^n \in \partial\varphi_h^{n-\frac{1}{2}}(0)$. On the other hand, by setting $\boldsymbol{p}_h = \widehat{\boldsymbol{E}}_h^n$ in (2.20), we obtain

$$\max_{\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \left(\widehat{\boldsymbol{E}}_h^n, \boldsymbol{v}_h\right)_{\boldsymbol{L}^2(\Omega)} \leq \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right) \underbrace{=}_{(2.14)} \int_\Omega j_{c,h}^{n-\frac{1}{2}} \left|\widehat{\boldsymbol{E}}_h^n\right| \mathrm{d}x. \tag{2.28}$$

Introducing

$$\boldsymbol{q}_h^n(x) = \begin{cases} \dfrac{j_{c,h}^{n-\frac{1}{2}}(x)\widehat{\boldsymbol{E}}_h^n(x)}{\left|\widehat{\boldsymbol{E}}_h^n(x)\right|} & \text{if } \widehat{\boldsymbol{E}}_h^n(x) \neq 0 \\ 0 & \text{elsewhere,} \end{cases}$$

we have

$$\int_\Omega j_{c,h}^{n-\frac{1}{2}} \left|\widehat{\boldsymbol{E}}_h^n\right| \mathrm{d}x = \left(\boldsymbol{q}_h^n, \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \leq \max_{\boldsymbol{v}_h \in \partial\varphi_h^{n-\frac{1}{2}}(0)} \left(\widehat{\boldsymbol{E}}_h^n, \boldsymbol{v}_h\right)_{\boldsymbol{L}^2(\Omega)}, \tag{2.29}$$

since $\boldsymbol{q}_h^n \in \partial\varphi_h^{n-\frac{1}{2}}(0)$ according to (2.20). Hence, combining (2.27) to (2.29) yields that

$$\varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right) = \left(\Pi_h^n \boldsymbol{\omega}_h^n, \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)}, \tag{2.30}$$

from which it follows that

$$\begin{aligned}
&\left(\frac{2\epsilon}{\tau}\widehat{\boldsymbol{E}}_h^n, \boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right) \\
&\quad \overbrace{=}^{(2.15)} \left(\boldsymbol{\omega}_h^n - \Pi_h^n\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right) \\
&\quad = \left(\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} + \left(\Pi_h^n\boldsymbol{\omega}_h^n, \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} - (\Pi_h^n\boldsymbol{\omega}_h^n, \boldsymbol{v}_h)_{\boldsymbol{L}^2(\Omega)} + \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}_h) - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right) \\
&\quad \overbrace{\geq}^{(2.20)} \left(\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} + \left(\Pi_h^n\boldsymbol{\omega}_h^n, \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right) \\
&\quad \overbrace{=}^{(2.30)} \left(\boldsymbol{\omega}_h^n, \boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \overbrace{=}^{(2.15)} \left(\boldsymbol{f}_h^{n-\frac{1}{2}} + \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \frac{2\epsilon}{\tau}\boldsymbol{E}_h^{n-1}, \boldsymbol{v}_h - \widehat{\boldsymbol{E}}_h^n\right)_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{v}_h \in \mathbf{DG}_h\,.
\end{aligned}$$

This concludes that $\widehat{\boldsymbol{E}}_h^n$ given by (2.15) is the unique solution to (2.18). □

Remark 2.4 (Numerical realization of ($\mathrm{LF}_{N,h}$))**.** The explicit formula (2.15) serves as a fundament for the efficient numerical realization of ($\mathrm{LF}_{N,h}$). By means of this, the numerical execution of ($\mathrm{LF}_{N,h}$) at every time step $n \in \{1, \ldots, N\}$ is reduced to simply setting $\boldsymbol{E}_h^n$ according to (2.15) and solving the linear problem (2.19).

3. Conditional stability and convergence analysis of ($\mathrm{LF}_{N,h}$)

The goal of this section is to establish a conditional stability result for ($\mathrm{LF}_{N,h}$). To begin with, let us recall the following well-known properties for the projection operator $\boldsymbol{P}_h \colon \boldsymbol{L}^2(\Omega) \to \mathbf{DG}_h$ (see (2.10) for its definition):

$$\|\boldsymbol{P}_h \boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \quad \forall h > 0 \tag{3.1}$$

as well as

$$\begin{aligned} &\|\boldsymbol{P}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \to 0 \text{ as } h\to 0 && \forall \boldsymbol{v}\in \boldsymbol{L}^2(\Omega) \\ &\|\boldsymbol{P}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \le Ch && \forall \boldsymbol{v}\in \boldsymbol{H}^1(\Omega) \quad \forall h>0 \end{aligned} \tag{3.2}$$

with a constant $C>0$ independent of $\boldsymbol{v}$ and h. The scalar operator P_h satisfies also analogous properties to (3.1) and (3.2). Moreover, we recall that the operator $\boldsymbol{\Phi}_h\colon \boldsymbol{H}(\mathbf{curl})\to \mathbf{ND}_h$ satisfies

$$\|\boldsymbol{\Phi}_h\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} \le C\|\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} \qquad \forall h>0 \quad \forall \boldsymbol{y}\in \boldsymbol{H}(\mathbf{curl}) \tag{3.3}$$

$$\lim_{h\to 0}\|\boldsymbol{\Phi}_h\boldsymbol{y}-\boldsymbol{y}\|_{\boldsymbol{H}(\mathbf{curl})} = 0 \qquad \forall \boldsymbol{y}\in \boldsymbol{H}(\mathbf{curl}), \tag{3.4}$$

where the constant $C>0$ is independent of h and $\boldsymbol{y}$. Note that the properties (3.3) and (3.4) follows again from the classical theory of mixed problems (see again [24], Thm. 2.45). Let us also recall the well-known inverse estimate (cf. [1], Thm. 1.3): There exists a constant $C_{\mathrm{inv}}>0$, independent of h, such that

$$\|\mathbf{curl}\,\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \le \frac{C_{\mathrm{inv}}}{h}\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \qquad \forall \boldsymbol{v}_h\in \mathbf{ND}_h\,. \tag{3.5}$$

By the explicit nature of the scheme ($\mathrm{LF}_{N,h}$), we need to employ a growth restriction on τ resulting from the equidistant partition (2.1):

$$\exists \alpha\in(0,1): \quad \frac{\tau}{h} \le \sqrt{\frac{\underline{\epsilon}\underline{\mu}}{2C_{\mathrm{inv}}}(1-\alpha)} \quad \Leftrightarrow \quad \alpha \le 1-\frac{2C_{\mathrm{inv}}\tau^2}{\underline{\epsilon}\underline{\mu}h^2}. \tag{3.6}$$

For the upcoming estimates, we will also use the following identities

$$\widehat{\boldsymbol{E}}_h^n - \widehat{\boldsymbol{E}}_h^{n-1} \overbrace{=}^{(2.12)} \frac{\boldsymbol{E}_h^n+\boldsymbol{E}_h^{n-1}}{2} - \frac{\boldsymbol{E}_h^{n-1}+\boldsymbol{E}_h^{n-2}}{2} \overbrace{=}^{(2.2)} \frac{\tau}{2}\big(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1}\big) \tag{3.7}$$

$$\sum_{n=1}^{i_0}\boldsymbol{a}_n\cdot(\boldsymbol{b}_n+\boldsymbol{b}_{n-1}) = \sum_{n=1}^{i_0-1}(\boldsymbol{a}_{n+1}+\boldsymbol{a}_n)\cdot\boldsymbol{b}_n + \boldsymbol{a}_{i_0}\cdot\boldsymbol{b}_{i_0} + \boldsymbol{a}_1\cdot\boldsymbol{b}_0 \tag{3.8}$$

for any $\{\boldsymbol{a}_n\}_{n=1}^{i_0}\subset\mathbb{R}^d$ and $\{\boldsymbol{b}_n\}_{n=0}^{i_0}\subset\mathbb{R}^d$ with $d, i_0\in\mathbb{N}$.

Lemma 3.1. *Under Assumption 1.1 and the CFL condition (3.6), there exists a constant $C>0$, independent of $N\in\mathbb{N}$ and $h>0$, such that the unique solution to ($\mathrm{LF}_{N,h}$) satisfies*

$$\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \Big\|\delta\boldsymbol{H}_h^{\frac{3}{2}}\Big\|_{\boldsymbol{L}^2_\mu(\Omega)} \le C \qquad \forall N\in\mathbb{N} \quad \forall h>0.$$

Proof. Let $N\in\mathbb{N}$ and $h>0$. Testing ($\mathrm{LF}_{N,h}$) for $n=1$ with $\boldsymbol{v}_h = \widehat{\boldsymbol{E}}_h^1-\delta\boldsymbol{E}_h^1\in\mathbf{DG}_h$ implies

$$\|\delta\boldsymbol{E}_h^1\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \int_\Omega\Big(\boldsymbol{f}_h^{\frac12}+\mathbf{curl}\,\boldsymbol{H}_h^{\frac12}\Big)\cdot\delta\boldsymbol{E}_h^1\,\mathrm{d}x + \int_\Omega P_h j_c\Big(\cdot,\theta^{\frac12},\boldsymbol{H}_h^{\frac12}\Big)\Big(\Big|\widehat{\boldsymbol{E}}_h^1-\delta\boldsymbol{E}_h^1\Big|-\Big|\widehat{\boldsymbol{E}}_h^1\Big|\Big)\,\mathrm{d}x.$$

Thanks to the uniform boundedness of $\boldsymbol{H}_h^{\frac12}=\boldsymbol{\Phi}_h(\boldsymbol{H}_0)$ with respect to $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl})}$ (see (3.3)), (A3), (A6), and (3.1), we obtain

$$\|\delta\boldsymbol{E}_h^1\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \underline{\epsilon}^{-\frac12}\Big(\Big\|\boldsymbol{f}_h^{\frac12}\Big\|_{\boldsymbol{L}^2(\Omega)} + \Big\|\mathbf{curl}\,\boldsymbol{H}_h^{\frac12}\Big\|_{\boldsymbol{L}^2(\Omega)} + \Big\|P_h j_c\Big(\cdot,\theta^{\frac12},\boldsymbol{H}_h^{\frac12}\Big)\Big\|_{\boldsymbol{L}^2(\Omega)}\Big) \le C. \tag{3.9}$$

Additionally, equation ($\mathrm{LF}_{N,h}$) for $n=1$ tested with $\boldsymbol{w}_h = \delta \boldsymbol{H}_h^{\frac{3}{2}}$ implies

$$\begin{aligned}
\left\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)} &= \int_\Omega \mu \delta \boldsymbol{H}_h^{\frac{3}{2}} \cdot \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x \overbrace{=}^{(\mathrm{LF}_{N,h})} - \int_\Omega \boldsymbol{E}_h^1 \cdot \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x \\
&= -\int_\Omega \left(\tau \delta \boldsymbol{E}_h^1 + \boldsymbol{E}_h^0 - \boldsymbol{E}_0\right) \cdot \mathbf{curl}\, \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x - \int_\Omega \mathbf{curl}\, \boldsymbol{E}_0 \cdot \delta \boldsymbol{H}_h^{\frac{3}{2}} \,\mathrm{d}x \\
&\overbrace{\leq}^{(3.5)} \frac{C_{\mathrm{inv}}}{\sqrt{\underline{\epsilon}\underline{\mu}}} \left(\frac{\tau}{h} \left\|\delta \boldsymbol{E}_h^1\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{\left\|\boldsymbol{E}_h^0 - \boldsymbol{E}_0\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)}}{h} + \left\|\mathbf{curl}\, \boldsymbol{E}_0\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \right) \left\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\right\|_{\boldsymbol{L}^2_\mu(\Omega)}.
\end{aligned}$$

Finally, the assertion follows from (3.6), (3.9), and (3.2). □

Theorem 3.2. *Under Assumption 1.1 and the CFL condition (3.6), there exist a number $N_0 \in \mathbb{N}$ and a constant $C > 0$, independent of $N \in \mathbb{N}$ and $h > 0$, such that the unique solution to ($\mathrm{LF}_{N,h}$) satisfies*

$$\max_{1\leq n\leq N} \left\|\delta \boldsymbol{E}_h^n\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \max_{1\leq n\leq N-1} \left\|\delta \boldsymbol{H}_h^{n+\frac{1}{2}}\right\|_{\boldsymbol{L}^2_\mu(\Omega)} \leq C \qquad \forall N \geq N_0 \quad \forall h > 0.$$

Proof. Let $N \in \mathbb{N}$ and $h > 0$. First, we fix $n \in \{2, \dots, N\}$ and test the n-th inequality in ($\mathrm{LF}_{N,h}$) with $\boldsymbol{v}_h = \widehat{\boldsymbol{E}}_h^{n-1}$ and the $(n-1)$-th inequality with $\boldsymbol{v}_h = \widehat{\boldsymbol{E}}_h^{n}$. Then, by (3.7), adding the resulting inequalities implies

$$\begin{aligned}
&\int_\Omega \epsilon \left(\delta \boldsymbol{E}_h^n - \delta \boldsymbol{E}_h^{n-1}\right) \cdot \left(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\right) - \tau\, \mathbf{curl}\, \delta \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \left(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\right) \mathrm{d}x \\
&\quad - \frac{2}{\tau} \left(\varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^{n-1}\right) - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^{n}\right) + \varphi_h^{n-\frac{3}{2}}\left(\widehat{\boldsymbol{E}}_h^{n}\right) - \varphi_h^{n-\frac{3}{2}}\left(\widehat{\boldsymbol{E}}_h^{n-1}\right) \right) \\
&\quad \leq \int_\Omega \left(\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}} \right) \cdot \left(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\right) \mathrm{d}x.
\end{aligned} \tag{3.10}$$

After summing (3.10) up over $\{2, \dots, i_0\}$ for a fixed $i_0 \in \{2, \dots, N\}$, we obtain

$$\begin{aligned}
&\left\|\delta \boldsymbol{E}_h^{i_0}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \tau \sum_{n=2}^{i_0} \int_\Omega \mathbf{curl}\, \delta \boldsymbol{H}_h^{n-\frac{1}{2}} \cdot \left(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\right) \mathrm{d}x \\
&\qquad \leq \left\|\delta \boldsymbol{E}_h^1\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \sum_{n=2}^{i_0} \int_\Omega \left(\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}} \right) \cdot \left(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\right) \mathrm{d}x \\
&\qquad\quad + \frac{2}{\tau} \sum_{n=2}^{i_0} \left(\varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^{n-1}\right) - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^{n}\right) + \varphi_h^{n-\frac{3}{2}}\left(\widehat{\boldsymbol{E}}_h^{n}\right) - \varphi_h^{n-\frac{3}{2}}\left(\widehat{\boldsymbol{E}}_h^{n-1}\right) \right).
\end{aligned} \tag{3.11}$$

Let us now estimate the terms in (3.11) separately. For the first sum in the right-hand side of (3.11), we have

$$\begin{aligned}
&\sum_{n=2}^{i_0} \int_\Omega \left(\boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}} \right) \cdot \left(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1}\right) \mathrm{d}x \\
&\quad \leq \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n=2}^{i_0} \left\| \boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}} \right\|_{\boldsymbol{L}^2(\Omega)} \left\|\delta \boldsymbol{E}_h^n\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n=2}^{i_0} \left\| \boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}} \right\|_{\boldsymbol{L}^2(\Omega)} \left\|\delta \boldsymbol{E}_h^{n-1}\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\quad \leq \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n=2}^{i_0-1} \left\| \boldsymbol{f}_h^{n-\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{3}{2}} \right\|_{\boldsymbol{L}^2(\Omega)} \left\|\delta \boldsymbol{E}_h^n\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n=1}^{i_0-1} \left\| \boldsymbol{f}_h^{n+\frac{1}{2}} - \boldsymbol{f}_h^{n-\frac{1}{2}} \right\|_{\boldsymbol{L}^2(\Omega)} \left\|\delta \boldsymbol{E}_h^n\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\qquad + \frac{2}{\underline{\epsilon}} \left\| \boldsymbol{f}_h^{i_0-\frac{1}{2}} - \boldsymbol{f}_h^{i_0-\frac{3}{2}} \right\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{8} \left\|\delta \boldsymbol{E}_h^{i_0}\right\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{3.12}$$

Furthermore, introducing the sets

$$\mathcal{I}_{i_0} := \Big\{ n \in \{1,\dots,i_0-1\} : \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le 1 \Big\} \quad \text{and} \quad \mathcal{I}_{i_0}^{\mathrm{c}} := \Big\{ n \in \{1,\dots,i_0-1\} : \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} > 1 \Big\}$$

and recalling the definition (1.4) with $V = \boldsymbol{L}^2(\Omega)$, it holds that

$$\begin{aligned}
&\sum_{n=2}^{i_0-1} \Big\| \boldsymbol{f}_h^{n-\frac12} - \boldsymbol{f}_h^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \sum_{n\in\mathcal{I}_{i_0}\setminus\{1\}} \Big\| \boldsymbol{f}_h^{n-\frac12} - \boldsymbol{f}_h^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} + \sum_{n\in\mathcal{I}_{i_0}^{\mathrm{c}}\setminus\{1\}} \Big\| \boldsymbol{f}_h^{n-\frac12} - \boldsymbol{f}_h^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\overbrace{\le}^{(1.4),(2.13),(3.1)} \mathrm{TV}(\boldsymbol{f}) + \sum_{n\in\mathcal{I}_{i_0}^{\mathrm{c}}\setminus\{1\}} \Big\| \boldsymbol{f}^{n-\frac12} - \boldsymbol{f}^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{3.13}$$

In an analogous way, we have

$$\sum_{n=1}^{i_0-1} \Big\| \boldsymbol{f}_h^{n+\frac12} - \boldsymbol{f}_h^{n-\frac12} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \le \mathrm{TV}(\boldsymbol{f}) + \sum_{n\in\mathcal{I}_{i_0}^{\mathrm{c}}} \Big\| \boldsymbol{f}^{n+\frac12} - \boldsymbol{f}^{n-\frac12} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \tag{3.14}$$

Then, applying (3.13) and (3.14) to (3.12) leads to

$$\begin{aligned}
&\sum_{n=2}^{i_0} \int_\Omega \Big(\boldsymbol{f}_h^{n-\frac12} - \boldsymbol{f}_h^{n-\frac32} \Big) \cdot \big(\delta \boldsymbol{E}_h^n + \delta \boldsymbol{E}_h^{n-1} \big) \,\mathrm{d}x \\
&\quad \le \frac{2}{\underline{\epsilon}} \mathrm{TV}(\boldsymbol{f})^2 + \frac18 \|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{2}{\sqrt{\underline{\epsilon}}} \mathrm{TV}(\boldsymbol{f}) + \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n\in\mathcal{I}_{i_0}^{\mathrm{c}}\setminus\{1\}} \Big\| \boldsymbol{f}^{n-\frac12} - \boldsymbol{f}^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\qquad + \frac{1}{\sqrt{\underline{\epsilon}}} \sum_{n\in\mathcal{I}_{i_0}^{\mathrm{c}}} \Big\| \boldsymbol{f}^{n+\frac12} - \boldsymbol{f}^{n-\frac12} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned} \tag{3.15}$$

Moreover, thanks to (A6), we make use of the local Lipschitz-properties of j_c (cf. (A4) and (A5)) in order to estimate the second sum on the right-hand side of (3.11) as follows

$$\begin{aligned}
&\frac{2}{\tau} \sum_{n=2}^{i_0} \Big(\varphi_h^{n-\frac12}\Big(\widehat{\boldsymbol{E}}_h^{n-1}\Big) - \varphi_h^{n-\frac12}\Big(\widehat{\boldsymbol{E}}_h^{n}\Big) + \varphi_h^{n-\frac32}\Big(\widehat{\boldsymbol{E}}_h^{n}\Big) - \varphi_h^{n-\frac32}\Big(\widehat{\boldsymbol{E}}_h^{n-1}\Big) \Big) \\
&\overbrace{=}^{(2.14)} \sum_{n=2}^{i_0} \int_\Omega \frac{2}{\tau} P_h\Big(j_c\Big(\cdot, \theta^{n-\frac12}, \boldsymbol{H}_h^{n-\frac12}\Big) - j_c\Big(\cdot, \theta^{n-\frac32}, \boldsymbol{H}_h^{n-\frac32}\Big)\Big) \Big(\Big|\widehat{\boldsymbol{E}}_h^{n-1}\Big| - \Big|\widehat{\boldsymbol{E}}_h^{n}\Big| \Big) \,\mathrm{d}x \\
&\overbrace{\le}^{(3.1),(3.7)} \sum_{n=2}^{i_0} \Big\| j_c\Big(\cdot, \theta^{n-\frac12}, \boldsymbol{H}_h^{n-\frac12}\Big) - j_c\Big(\cdot, \theta^{n-\frac32}, \boldsymbol{H}_h^{n-\frac32}\Big) \Big\|_{L^2(\Omega)} \Big(\|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \|\delta \boldsymbol{E}_h^{n-1}\|_{\boldsymbol{L}^2(\Omega)} \Big) \,\mathrm{d}x \\
&\overbrace{\le}^{(A4),(A5)} \frac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}} \sum_{n=2}^{i_0} \Big\| \theta^{n-\frac12} - \theta^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{\tau L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}\underline{\mu}}} \sum_{n=2}^{i_0} \Big\| \delta \boldsymbol{H}_h^{n-\frac12} \Big\|_{\boldsymbol{L}^2_\mu(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\quad + \frac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}} \sum_{n=1}^{i_0-1} \Big\| \theta^{n+\frac12} - \theta^{n-\frac12} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \frac{\tau L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}\underline{\mu}}} \sum_{n=1}^{i_0-1} \Big\| \delta \boldsymbol{H}_h^{n+\frac12} \Big\|_{\boldsymbol{L}^2_\mu(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \\
&\le \frac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}} \left(\sum_{n=2}^{i_0-1} \Big\| \theta^{n-\frac12} - \theta^{n-\frac32} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \sum_{n=1}^{i_0-1} \Big\| \theta^{n+\frac12} - \theta^{n-\frac12} \Big\|_{\boldsymbol{L}^2(\Omega)} \|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \right)
\end{aligned}$$

$$+\frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}}\Big\|\theta^{i_0-\frac12}-\theta^{i_0-\frac32}\Big\|^2_{\boldsymbol{L}^2(\Omega)}+\frac18\big\|\delta\boldsymbol{E}_h^{i_0}\big\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}$$
$$+\frac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}\sum_{n=2}^{i_0}\Big\|\delta\boldsymbol{H}_h^{n-\frac12}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{\tau}{8T}\sum_{n=1}^{i_0}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \tag{3.16}$$

Now, using again the definition (1.4) this time with $V=L^2(\Omega)$ and following the same argumentation as in (3.13), we obtain

$$\sum_{n=2}^{i_0-1}\Big\|\theta^{n-\frac12}-\theta^{n-\frac32}\Big\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)}+\sum_{n=1}^{i_0-1}\Big\|\theta^{n+\frac12}-\theta^{n-\frac12}\Big\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)}$$
$$\le 2\mathrm{TV}(\theta)+\sum_{n\in\mathcal{I}_{i_0}^c\setminus\{1\}}\Big\|\theta^{n-\frac12}-\theta^{n-\frac32}\Big\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\sum_{n\in\mathcal{I}_{i_0}^c}\Big\|\theta^{n+\frac12}-\theta^{n-\frac12}\Big\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}, \tag{3.17}$$

and hence applying (3.17) to (3.16) results in

$$\frac{2}{\tau}\sum_{n=2}^{i_0}\Big(\varphi_h^{n-\frac12}\Big(\widehat{\boldsymbol{E}}_h^{n-1}\Big)-\varphi_h^{n-\frac12}\Big(\widehat{\boldsymbol{E}}_h^{n}\Big)+\varphi_h^{n-\frac32}\Big(\widehat{\boldsymbol{E}}_h^{n}\Big)-\varphi_h^{n-\frac32}\Big(\widehat{\boldsymbol{E}}_h^{n-1}\Big)\Big)\le\frac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}$$
$$\times\left(2\mathrm{TV}(\theta)+\sum_{n\in\mathcal{I}_{i_0}^c\setminus\{1\}}\Big\|\theta^{n-\frac12}-\theta^{n-\frac32}\Big\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\sum_{n\in\mathcal{I}_{i_0}^c}\Big\|\theta^{n+\frac12}-\theta^{n-\frac12}\Big\|_{\boldsymbol{L}^2(\Omega)}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right)$$
$$+\frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}}\mathrm{TV}(\theta)^2+\frac14\big\|\delta\boldsymbol{E}_h^{i_0}\big\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\frac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}\sum_{n=2}^{i_0}\Big\|\delta\boldsymbol{H}_h^{n-\frac12}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}$$
$$+\frac{\tau}{8T}\sum_{n=1}^{i_0-1}\|\delta\boldsymbol{E}_h^n\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}. \tag{3.18}$$

Now, the estimation of the second term on the left-hand side of (3.11) requires the following formula:

$$-\tau\int_\Omega\mathbf{curl}\Big(\delta\boldsymbol{H}_h^{n+\frac12}+\delta\boldsymbol{H}_h^{n-\frac12}\Big)\cdot\delta\boldsymbol{E}_h^n\,\mathrm{d}x=\Big\|\delta\boldsymbol{H}_h^{n+\frac12}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}-\Big\|\delta\boldsymbol{H}_h^{n-\frac12}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\qquad n\in\{2,\dots,N-1\}. \tag{3.19}$$

In fact, substracting the $(n-1)$-th from the n-th equations of ($\mathrm{LF}_{N,h}$) results in

$$-\int_\Omega\big(\boldsymbol{E}_h^n-\boldsymbol{E}_h^{n-1}\big)\cdot\mathbf{curl}\,\boldsymbol{w}_h\,\mathrm{d}x=\int_\Omega\mu\Big(\delta\boldsymbol{H}_h^{n+\frac12}-\delta\boldsymbol{H}_h^{n-\frac12}\Big)\cdot\boldsymbol{w}_h\,\mathrm{d}x\qquad\forall\boldsymbol{w}_h\in\mathbf{ND}_h\,.$$

Hence, choosing $\boldsymbol{w}_h=\delta\boldsymbol{H}_h^{n+\frac12}+\delta\boldsymbol{H}_h^{n-\frac12}\in\mathbf{ND}_h$ implies (3.19). Now, utilizing (3.8) and (3.5), we obtain an estimate for the second term on the left-hand side of (3.11) as follows:

$$\tau\sum_{n=2}^{i_0}\int_\Omega\mathbf{curl}\,\delta\boldsymbol{H}_h^{n-\frac12}\cdot\big(\delta\boldsymbol{E}_h^n+\delta\boldsymbol{E}_h^{n-1}\big)\,\mathrm{d}x$$
$$\overbrace{=}^{(3.8)}\tau\sum_{n=2}^{i_0-1}\int_\Omega\mathbf{curl}\Big(\delta\boldsymbol{H}_h^{n+\frac12}+\delta\boldsymbol{H}_h^{n-\frac12}\Big)\cdot\delta\boldsymbol{E}_h^n+\mathbf{curl}\,\delta\boldsymbol{H}_h^{i_0-\frac12}\cdot\delta\boldsymbol{E}_h^{i_0}+\mathbf{curl}\,\delta\boldsymbol{H}_h^{\frac32}\cdot\delta\boldsymbol{E}_h^1\,\mathrm{d}x$$
$$\overbrace{\le}^{(3.19),(3.5)}-\Big\|\delta\boldsymbol{H}_h^{i_0-\frac12}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\Big\|\delta\boldsymbol{H}_h^{\frac32}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\Big\|\delta\boldsymbol{H}_h^{i_0-\frac12}\Big\|^2_{\boldsymbol{L}^2_\mu(\Omega)}$$

$$+\frac{1}{8}\|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\left\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{1}{8}\|\delta \boldsymbol{E}_h^{1}\|_{\boldsymbol{L}^2_\epsilon(\Omega)}. \tag{3.20}$$

In conclusion, applying (3.15), (3.18) and (3.20) to (3.11) yields

$$\begin{aligned}
&\frac{1}{2}\|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\left(1-\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}-\frac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}\right)\left\|\delta \boldsymbol{H}_h^{i_0-\frac{1}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\\
&\quad\le\frac{9}{8}\|\delta \boldsymbol{E}_h^{1}\|_{\boldsymbol{L}^2_\epsilon(\Omega)}+\left(1+\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\right)\left\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{2}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\boldsymbol{f})+\frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\theta)\\
&\qquad+\frac{2}{\underline{\epsilon}}\mathrm{TV}(\boldsymbol{f})^2+\frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}}\mathrm{TV}(\theta)^2+\frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n\in\mathcal{I}^{\mathrm{c}}_{i_0}\setminus\{1\}}\left\|\boldsymbol{f}^{n-\frac{1}{2}}-\boldsymbol{f}^{n-\frac{3}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\\
&\qquad+\frac{1}{\sqrt{\underline{\epsilon}}}\sum_{n\in\mathcal{I}^{\mathrm{c}}_{i_0}}\left\|\boldsymbol{f}^{n+\frac{1}{2}}-\boldsymbol{f}^{n-\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\frac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}\\
&\qquad\times\left(\sum_{n\in\mathcal{I}^{\mathrm{c}}_{i_0}\setminus\{1\}}\left\|\theta^{n-\frac{1}{2}}-\theta^{n-\frac{3}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\sum_{n\in\mathcal{I}^{\mathrm{c}}_{i_0}}\left\|\theta^{n+\frac{1}{2}}-\theta^{n-\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right)\\
&\qquad+\frac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}\sum_{n=2}^{i_0-1}\left\|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}+\frac{\tau}{8T}\sum_{n=1}^{i_0-1}\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}.
\end{aligned}\tag{3.21}$$

Thanks to the CFL condition (3.6), there exists a fixed $N_0\in\mathbb{N}$ such that

$$1-\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}-\frac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}\ge\alpha-\frac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}=\alpha-\frac{8T^2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{N\underline{\epsilon}\underline{\mu}}\ge\frac{\alpha}{2}\quad\forall N\ge N_0. \tag{3.22}$$

Now we suppose that $N\ge N_0$. By means of (3.6) and Lemma 3.1, the first two terms on the right-hand side of (3.21) are bounded, so that we can define a nonnegative constant

$$\begin{aligned}
\beta:=\sup_{(N,h)\in\mathbb{N}\times(0,\infty)}&\left(\frac{9}{8}\|\delta \boldsymbol{E}_h^{1}\|_{\boldsymbol{L}^2_\epsilon(\Omega)}+\left(1+\frac{2C_{\mathrm{inv}}^2\tau^2}{\underline{\epsilon}\underline{\mu}h^2}\right)\left\|\delta \boldsymbol{H}_h^{\frac{3}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right)+\frac{2}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\boldsymbol{f})\\
&+\frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\theta)+\frac{2}{\underline{\epsilon}}\mathrm{TV}(\boldsymbol{f})^2+\frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}}\mathrm{TV}(\theta)^2.
\end{aligned}\tag{3.23}$$

To simplify the estimate (3.21), for $n\in\{1,\dots,i_0-1\}$, let us introduce the coefficient

$$k_n:=\begin{cases}
\dfrac{1}{\sqrt{\underline{\epsilon}}}\left\|\boldsymbol{f}^{\frac{3}{2}}-\boldsymbol{f}^{\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}+\dfrac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}\left\|\theta^{\frac{3}{2}}-\theta^{\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}+\dfrac{\tau}{8T}, & n=1\\
\dfrac{1}{\sqrt{\underline{\epsilon}}}\left\|\boldsymbol{f}^{n-\frac{1}{2}}-\boldsymbol{f}^{n-\frac{3}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}+\dfrac{1}{\sqrt{\underline{\epsilon}}}\left\|\boldsymbol{f}^{n+\frac{1}{2}}-\boldsymbol{f}^{n-\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)} & \\
\quad+\dfrac{L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}\left(\left\|\theta^{n-\frac{1}{2}}-\theta^{n-\frac{3}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}+\left\|\theta^{n+\frac{1}{2}}-\theta^{n-\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}\right) & \\
\quad+\dfrac{\tau}{8T}+\dfrac{8\tau TL\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}}, & n\in\{2,\dots,i_0-1\},
\end{cases}$$

by the use of which, in combination with (3.22) and (3.23), it follows that

$$\frac{\alpha}{2}\left(\|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\left\|\delta \boldsymbol{H}_h^{i_0-\frac{1}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right)\le\beta+\sum_{n=1}^{i_0-1}k_n\left(\|\delta \boldsymbol{E}_h^{n}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}+\left\|\delta \boldsymbol{H}_h^{n-\frac{1}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)}\right).$$

Invoking the discrete version of Gronwall's inequality, this implies

$$\|\delta \boldsymbol{E}_h^{i_0}\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} + \left\|\delta \boldsymbol{H}_h^{i_0-\frac{1}{2}}\right\|^2_{\boldsymbol{L}^2_\mu(\Omega)} \leq \beta \exp\left(\sum_{n=1}^{i_0-1} k_n\right). \tag{3.24}$$

Finally, since

$$\sum_{n=1}^{i_0-1} k_n \leq \frac{2}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\boldsymbol{f}) + \frac{2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)}{\sqrt{\underline{\epsilon}}}\mathrm{TV}(\theta) + \frac{1}{8} + \frac{8T^2L\big(\|\theta\|_{L^\infty(\Omega_T)}\big)^2}{\underline{\epsilon}\underline{\mu}},$$

and since $i_0 \in \{2,\dots,N\}, N \geq N_0$, and $h > 0$ were chosen arbitrarily, equation (3.24) and Lemma 3.1 conclude that

$$\max_{1\leq n\leq N}\|\delta \boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \max_{1\leq n\leq N-1}\left\|\delta \boldsymbol{H}_h^{n+\frac{1}{2}}\right\|_{\boldsymbol{L}^2_\mu(\Omega)} \leq C \qquad \forall N \geq N_0 \quad \forall h > 0. \tag{3.25}$$

This completes the proof. □

Corollary 3.3. *Under Assumption 1.1 and the CFL condition (3.6), there exist a constant $C > 0$, independent of $N \in \mathbb{N}$ and $h > 0$, such that the unique solution to ($\mathrm{LF}_{N,h}$) satisfies*

$$\max_{1\leq n\leq N}\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \max_{1\leq n\leq N-1}\left\|\boldsymbol{H}_h^{n+\frac{1}{2}}\right\|_{\boldsymbol{L}^2_\mu(\Omega)} + \max_{1\leq n\leq N-1}\left\|\mathbf{curl}\,\boldsymbol{H}_h^{n+\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)} \leq C \qquad \forall N \geq N_0 \quad \forall h > 0$$

with $N_0 \in \mathbb{N}$ as in Theorem 3.2.

Proof. Let $N \geq \mathbb{N}_0$ and $h > 0$. Using the reversed triangle inequality, it follows by definition of the difference quotients together with Theorem 3.2 that

$$\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq \tau C + \left\|\boldsymbol{E}_h^{n-1}\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq \cdots \leq n\tau C + \left\|\boldsymbol{E}_h^0\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq TC + \left\|\boldsymbol{E}_h^0\right\|_{\boldsymbol{L}^2_\epsilon(\Omega)} \leq C \qquad \forall n \in \{1,\dots,N\}.$$

Using the same argumentation for the discrete magnetic fields, it follows that

$$\max_{1\leq n\leq N}\|\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2_\epsilon(\Omega)} + \max_{1\leq n\leq N-1}\left\|\boldsymbol{H}_h^{n+\frac{1}{2}}\right\|_{\boldsymbol{L}^2_\mu(\Omega)} \leq C \qquad \forall N \in \mathbb{N} \quad \forall h > 0. \tag{3.26}$$

Now, we insert $\boldsymbol{v}_h = \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \widehat{\boldsymbol{E}}_h^n \in \mathbf{DG}_h$ for $n \in \{1,\dots,N\}$ into ($\mathrm{LF}_{N,h}$) to deduce that

$$\begin{aligned}\left\|\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\right\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \int_\Omega \left(\epsilon\delta\boldsymbol{E}_h^n - \boldsymbol{f}_h^{n-\frac{1}{2}}\right)\cdot \mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\,\mathrm{d}x + \varphi_h^{n-\frac{1}{2}}\left(\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}} + \widehat{\boldsymbol{E}}_h^n\right) - \varphi_h^{n-\frac{1}{2}}\left(\widehat{\boldsymbol{E}}_h^n\right)\\ &\leq \left(\|\epsilon\delta\boldsymbol{E}_h^n\|_{\boldsymbol{L}^2(\Omega)} + \left\|\boldsymbol{f}_h^{n-\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)} + \left\|P_h j_c\left(\cdot,\theta^{n-\frac{1}{2}},\boldsymbol{H}_h^{n-\frac{1}{2}}\right)\right\|_{\boldsymbol{L}^2(\Omega)}\right)\left\|\mathbf{curl}\,\boldsymbol{H}_h^{n-\frac{1}{2}}\right\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Ultimately, using (3.25), (3.26), and (A3), the assertion follows. □

3.1. Uniform convergence

We start by introducing the following piecewise linear and piecewise constant interpolations:

$$\begin{aligned}&\boldsymbol{E}_{N,h}(t) \coloneqq \boldsymbol{E}_h^{n-1} + (t - t_{n-1})\delta\boldsymbol{E}_h^n, \quad \overline{\boldsymbol{E}}_{N,h}(t) \coloneqq \boldsymbol{E}_h^n,\\ &\overline{\boldsymbol{E}}^a_{N,h}(t) \coloneqq \widehat{\boldsymbol{E}}_h^n \quad \text{for } t \in (t_{n-1},t_n),\ n \in \{1,\dots,N\}.\end{aligned} \tag{3.27}$$

A similar notation is used for the discrete magnetic fields

$$\begin{cases} \boldsymbol{H}_{N,h}(t) \coloneqq \boldsymbol{H}_h^{n-\frac{1}{2}} + (t-t_{n-1})\delta \boldsymbol{H}_h^{n+\frac{1}{2}} & \text{for } t \in (t_{n-1},t_n),\ n \in \{1,\dots,N-1\} \\ \boldsymbol{H}_{N,h}(t) \coloneqq \boldsymbol{H}_h^{N-\frac{1}{2}} & \text{for } t \in (t_{N-1},t_N) \\ \overline{\boldsymbol{H}}_{N,h}(t) \coloneqq \boldsymbol{H}_h^{n-\frac{1}{2}} & \text{for } t \in (t_{n-1},t_n),\ n \in \{1,\dots,N\}. \end{cases} \tag{3.28}$$

Note that the piecewise linear interpolations extend continuously to the interval $[0,T]$, *i.e.*, $(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) \in \mathcal{C}([0,T],\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega))$. Furthermore, our construction (3.28) is based on the fact that $\delta\boldsymbol{H}_h^{n+\frac{1}{2}}$ in ($\mathrm{LF}_{N,h}$) is only computed for $n \in \{1,\dots,N-1\}$. In particular, this construction implies

$$\left\|\boldsymbol{H}_{N,h}(t) - \overline{\boldsymbol{H}}_{N,h}(t)\right\|_{\boldsymbol{L}^2_\mu(\Omega)} = \|\partial_t \boldsymbol{H}_{N,h}(t)\|_{\boldsymbol{L}^2_\mu(\Omega)} = 0 \qquad \forall t \in (T-\tau,T]. \tag{3.29}$$

As a consequence of (3.27)–(3.29), we obtain that

$$\left\|\overline{\boldsymbol{E}}_{N,h} - \boldsymbol{E}_{N,h}\right\|_{L^\infty((0,T),\boldsymbol{L}^2_\epsilon(\Omega))} + \left\|\overline{\boldsymbol{E}}^a_{N,h} - \overline{\boldsymbol{E}}_{N,h}\right\|_{L^\infty((0,T),\boldsymbol{L}^2_\epsilon(\Omega))} + \left\|\overline{\boldsymbol{H}}_{N,h} - \boldsymbol{H}_{N,h}\right\|_{L^\infty((0,T),\boldsymbol{L}^2_\mu(\Omega))} \le C\tau. \tag{3.30}$$

Moreover, $\overline{\theta}_N$, $\overline{\boldsymbol{f}}_{N,h}$ are defined similarly as in (3.28), namely

$$\overline{\theta}_N(t) \coloneqq \theta^{n-\frac{1}{2}} \quad \text{and} \quad \overline{\boldsymbol{f}}_{N,h}(t) \coloneqq \boldsymbol{f}_h^{n-\frac{1}{2}} \quad \text{for } t \in (t_{n-1},t_n),\ n \in \{1,\dots,N\}. \tag{3.31}$$

Further, we introduce the function $\varphi_{N,h} \coloneqq [0,T]\times\boldsymbol{L}^2(\Omega) \to \mathbb{R}$ by

$$\varphi_{N,h}(t,\boldsymbol{v}) \coloneqq \varphi_h^{n-\frac{1}{2}}(\boldsymbol{v}) = \int_\Omega P_h j_c(\cdot,\overline{\theta}_N(t),\overline{\boldsymbol{H}}_{N,h}(t))|\boldsymbol{v}|\,\mathrm{d}x \quad \text{for } t \in (t_{n-1},t_n),\ n \in \{1,\dots,N\}. \tag{3.32}$$

By virtue of (3.27)–(3.32), we may now equivalently rewrite ($\mathrm{LF}_{N,h}$) to

$$\begin{cases} \displaystyle\int_\Omega \epsilon\partial_t \boldsymbol{E}_{N,h}(t)\cdot\left(\boldsymbol{v}_h - \overline{\boldsymbol{E}}^a_{N,h}(t)\right) - \mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(t)\cdot(\boldsymbol{v}_h - \overline{\boldsymbol{E}}^a_{N,h}(t))\,\mathrm{d}x \\ \displaystyle\quad + \varphi_{N,h}(t,\boldsymbol{v}_h) - \varphi_{N,h}(t,\overline{\boldsymbol{E}}^a_{N,h}(t)) \ge \int_\Omega \overline{\boldsymbol{f}}_{N,h}(t)\cdot\left(\boldsymbol{v}_h - \overline{\boldsymbol{E}}^a_{N,h}(t)\right)\mathrm{d}x \\ \text{for a.e. } t \in (0,T) \text{ and all } \boldsymbol{v}_h \in \mathbf{DG}_h \\ \displaystyle\int_\Omega \mu\partial_t \boldsymbol{H}_{N,h}(t)\cdot\boldsymbol{w}_h\,\mathrm{d}x + \int_\Omega \overline{\boldsymbol{E}}_{N,h}(t)\cdot\mathbf{curl}\,\boldsymbol{w}_h = 0 \\ \text{for a.e. } t \in (0,T-\tau) \text{ and all } \boldsymbol{w}_h \in \mathbf{ND}_h\,. \end{cases} \tag{3.33}$$

Let us now state the key compactness property of the space $\boldsymbol{X}_h^{(\mu)}$ which is crucial for the convergence analysis of (3.33) towards (QVI) (cf. [18] for the original result with $\mu \equiv 1$).

Lemma 3.4. *Let $\{\boldsymbol{z}_h\}_{h>0} \subset \boldsymbol{H}(\mathbf{curl})$ be bounded and satisfy $\boldsymbol{z}_h \in \boldsymbol{X}_h^{(\mu)}$ for every $h>0$. Then, there exist a $\boldsymbol{z} \in \boldsymbol{X}^{(\mu)}(\Omega)$ and a subsequence $\{\boldsymbol{z}_{h_n}\}_{n=1}^\infty \subset \{\boldsymbol{z}_h\}_{h>0}$ with $h_n \to 0$ as $n \to \infty$ such that*

$$\boldsymbol{z}_{h_n} \to \boldsymbol{z} \quad \text{strongly in } \boldsymbol{L}^2_\mu(\Omega) \quad \text{and} \quad \mathbf{curl}\ \boldsymbol{z}_{h_n} \rightharpoonup \mathbf{curl}\,\boldsymbol{z} \quad \text{weakly in } \boldsymbol{L}^2(\Omega).$$

Theorem 3.5. *Let Assumption 1.1 be satisfied. Then, (QVI) admits a unique solution*

$$(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}\big((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega)\big) \cap L^\infty\Big((0,T),\boldsymbol{H}_0(\mathbf{curl})\times\boldsymbol{X}^{(\mu)}(\Omega)\Big).$$

For $N = N(h) \in \mathbb{N}$ with $N_0 \le N(h) \to \infty$ as $h \to 0$, satisfying the CFL-condition (3.6) and $N_0 \in \mathbb{N}$ as in Theorem 3.2, it holds that

$$\lim_{h\to 0}\|(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) - (\boldsymbol{E},\boldsymbol{H})\|_{\mathcal{C}\left([0,T],\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega)\right)} = \lim_{h\to 0}\left\|\left(\overline{\boldsymbol{E}}_{N,h},\overline{\boldsymbol{H}}_{N,h}\right) - (\boldsymbol{E},\boldsymbol{H})\right\|_{L^\infty\left((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega)\right)} = 0.$$

Proof. We divide the proof into two steps.

Step 1: Existence of a unique solution**.** Thanks to Theorem 3.2, Corollary 3.3 as well as (3.27) and (3.28), the families $\{\boldsymbol{E}_{N,h}\}_{h>0}$, $\{\overline{\boldsymbol{E}}_{N,h}\}_{h>0}$, $\{\overline{\boldsymbol{E}}^a_{N,h}\}_{h>0}$, $\{\boldsymbol{H}_{N,h}\}_{h>0}$, $\{\overline{\boldsymbol{H}}_{N,h}\}_{h>0}$, $\{\partial_t\boldsymbol{E}_{N,h}\}_{h>0}$, and $\{\partial_t\boldsymbol{H}_{N,h}\}_{h>0}$ are uniformly bounded in their respective spaces. Therefore, by standard arguments there exists $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega)) \cap L^\infty((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{H}(\mathbf{curl}))$ such that

$$\begin{aligned}
(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) &\rightharpoonup^* (\boldsymbol{E},\boldsymbol{H}) && \text{weakly-* in } L^\infty\big((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{H}(\mathbf{curl})\big)\\
\Big(\overline{\boldsymbol{E}}_{N,h},\overline{\boldsymbol{H}}_{N,h},\overline{\boldsymbol{E}}^a_{N,h}\Big) &\rightharpoonup^* (\boldsymbol{E},\boldsymbol{H},\boldsymbol{E}) && \text{weakly-* in } L^\infty\big((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{H}(\mathbf{curl})\times\boldsymbol{L}^2_\epsilon(\Omega)\big)\\
\partial_t(\boldsymbol{E}_{N,h},\boldsymbol{H}_{N,h}) &\rightharpoonup^* \partial_t(\boldsymbol{E},\boldsymbol{H}) && \text{weakly-* in } L^\infty\big((0,T),\boldsymbol{L}^2_\epsilon(\Omega)\times\boldsymbol{L}^2_\mu(\Omega)\big)
\end{aligned} \tag{3.34}$$

as well as

$$\boldsymbol{E}_{N,h}(t) \rightharpoonup \boldsymbol{E}(t) \text{ weakly in } \boldsymbol{L}^2_\epsilon(\Omega) \quad\text{and}\quad \boldsymbol{H}_{N,h}(t) \rightharpoonup \boldsymbol{H}(t) \text{ weakly in } \boldsymbol{L}^2_\mu(\Omega) \qquad \forall t\in[0,T]. \tag{3.35}$$

The above convergence at $t=0$ particularly yields $\boldsymbol{E}_0=\boldsymbol{E}(0)$ and $\boldsymbol{H}_0=\boldsymbol{H}(0)$. From Theorems 2.3, 3.2, and Corollary 3.3, we know that $\{\boldsymbol{H}_{N,h}(t)\}_{h>0}\subset \boldsymbol{X}^{(\mu)}_h$ for every $t\in[0,T]$ is uniformly bounded in $\boldsymbol{H}(\mathbf{curl})$. For this reason, by Lemma 3.4 and (3.35), we obtain after choosing a subsequence that

$$\lim_{h\to0}\|\boldsymbol{H}_{N,h}(t)-\boldsymbol{H}(t)\|_{\boldsymbol{L}^2_\mu(\Omega)}=0 \qquad \forall t\in[0,T]. \tag{3.36}$$

Moreover, equation (3.34) implies the uniform boundedness and equicontinuity of the family $\{\boldsymbol{H}_{N,h}\}_{h>0}$ in $\mathcal{C}([0,T],\boldsymbol{L}^2_\mu(\Omega))$. Therefore, the Arzelá–Ascoli theorem for Banach space-valued functions (cf. [20], Thm. 3.1) implies

$$\lim_{h\to0}\|\boldsymbol{H}_{N,h}-\boldsymbol{H}\|_{\mathcal{C}([0,T],\boldsymbol{L}^2_\mu(\Omega))}=0 \overbrace{\Rightarrow}^{(3.30)} \lim_{h\to0}\big\|\overline{\boldsymbol{H}}_{N,h}-\boldsymbol{H}\big\|_{L^\infty((0,T),\boldsymbol{L}^2_\mu(\Omega))}=0. \tag{3.37}$$

Let us now prove that $(\boldsymbol{E},\boldsymbol{H})$ solves (QVI). We recall the classical identity:

$$\int_0^t(\partial_t\boldsymbol{E}_{N,h}(s),\boldsymbol{E}_{N,h}(s))_{\boldsymbol{L}^2_\epsilon(\Omega)}\,\mathrm{d}s=\frac12\|\boldsymbol{E}_{N,h}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\frac12\|\boldsymbol{E}^0_h\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} \qquad \forall t\in[0,T]. \tag{3.38}$$

Combining (3.38) with (3.35) yields

$$\begin{aligned}
\liminf_{h\to0}\int_0^t(\partial_t\boldsymbol{E}_{N,h}(s),\overline{\boldsymbol{E}}^a_{N,h}(s))_{\boldsymbol{L}^2_\epsilon(\Omega)}\,\mathrm{d}s &\overbrace{=}^{(3.30)} \liminf_{h\to0}\int_0^t(\partial_t\boldsymbol{E}_{N,h}(s),\boldsymbol{E}_{N,h}(s))_{\boldsymbol{L}^2_\epsilon(\Omega)}\,\mathrm{d}s\\
&\overbrace{=}^{(3.38)} \liminf_{h\to0}\left(\frac12\|\boldsymbol{E}_{N,h}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\frac12\Bigg\|\overbrace{\boldsymbol{E}^0_h}^{=\boldsymbol{P}_h\boldsymbol{E}_0}\Bigg\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}\right) \overbrace{\geq}^{(3.2),(3.35)} \frac12\|\boldsymbol{E}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}-\frac12\Bigg\|\overbrace{\boldsymbol{E}_0}^{=\boldsymbol{E}(0)}\Bigg\|^2_{\boldsymbol{L}^2_\epsilon}\\
&=\int_0^t(\partial_t\boldsymbol{E}(s),\boldsymbol{E}(s))_{\boldsymbol{L}^2_\epsilon(\Omega)}\,\mathrm{d}s,
\end{aligned} \tag{3.39}$$

where the above inequality holds since the squared norm is weakly lower semicontinuous. Analogously,

$$\liminf_{h\to0}\int_0^t\big(\partial_t\boldsymbol{H}_{N,h}(s),\overline{\boldsymbol{H}}_{N,h}(s)\big)_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}s\geq\int_0^t(\partial_t\boldsymbol{H}(s),\boldsymbol{H}(s))_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}s. \tag{3.40}$$

Let now $t\in(0,T]$ and $(\boldsymbol{v},\boldsymbol{w})\in\boldsymbol{L}^2(\Omega)\times\boldsymbol{H}(\mathbf{curl})$ be fixed. Using the density properties of $\mathbf{DG}_h\subset\boldsymbol{L}^2(\Omega)$ and $\mathbf{ND}_h\subset\boldsymbol{H}(\mathbf{curl})$, we deduce the existence of a family $\{(\boldsymbol{v}_h,\boldsymbol{w}_h)\}_{h>0}\subset\boldsymbol{L}^2(\Omega)\times\boldsymbol{H}(\mathbf{curl})$ satisfying $(\boldsymbol{v}_h,\boldsymbol{w}_h)\in\mathbf{DG}_h\times\mathbf{ND}_h$ for all $h>0$ and

$$\lim_{h\to0}\|\boldsymbol{v}_h-\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}=\lim_{h\to0}\|\boldsymbol{w}_h-\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl})}=0. \tag{3.41}$$

Testing the equation in (3.33) with $\boldsymbol{w}_h \in \mathbf{ND}_h$ and integrating over $[0,t]$, we obtain that

$$\begin{aligned}\int_0^t (\partial_t \boldsymbol{H}(s), \boldsymbol{w})_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}s &\overbrace{=}^{(3.34)} \lim_{h\to 0}\int_0^t (\partial_t \boldsymbol{H}_{N,h}(s), \boldsymbol{w}_h)_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}s \\ &\overbrace{=}^{(3.29)} \lim_{h\to 0}\int_0^{T-\tau} (\partial_t \boldsymbol{H}_{N,h}(s), \boldsymbol{w}_h)_{\boldsymbol{L}^2_\mu(\Omega)}\chi_{[0,t]}(s)\,\mathrm{d}s \\ &\overbrace{=}^{(3.33)} \lim_{h\to 0} -\int_0^{T-\tau} \big(\overline{\boldsymbol{E}}_{N,h}(s), \mathbf{curl}\,\boldsymbol{w}_h\big)_{\boldsymbol{L}^2(\Omega)}\chi_{[0,t]}(s)\,\mathrm{d}s \overbrace{=}^{(3.34)} -\int_0^t (\boldsymbol{E}(s), \mathbf{curl}\,\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s. \end{aligned} \tag{3.42}$$

Now, taking the derivative of (3.42) with respect to t yields

$$\begin{aligned} &\int_\Omega \mu\partial_t \boldsymbol{H}(t)\cdot \boldsymbol{w}\,\mathrm{d}x = -\int_\Omega \boldsymbol{E}(t)\cdot \mathbf{curl}\,\boldsymbol{w}\,\mathrm{d}x \quad \forall \boldsymbol{w}\in \boldsymbol{H}(\mathbf{curl}) \\ \Rightarrow\ &\boldsymbol{E}(t)\in \boldsymbol{H}_0(\mathbf{curl}) \text{ with } \mathbf{curl}\,\boldsymbol{E}(t) = -\mu\partial_t \boldsymbol{H}(t) \text{ for a.e. } t\in(0,T). \end{aligned} \tag{3.43}$$

Since $(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}((0,T), \boldsymbol{L}^2_\epsilon(\Omega)\times \boldsymbol{L}^2_\mu(\Omega)) \cap L^\infty((0,T), \boldsymbol{L}^2_\epsilon(\Omega)\times \boldsymbol{H}(\mathbf{curl}))$, equation (3.43) implies the regularity

$$(\boldsymbol{E},\boldsymbol{H}) \in W^{1,\infty}\big((0,T), \boldsymbol{L}^2_\epsilon(\Omega)\times \boldsymbol{L}^2_\mu(\Omega)\big) \cap L^\infty\Big((0,T), \boldsymbol{H}_0(\mathbf{curl})\times \boldsymbol{X}^{(\mu)}(\Omega)\Big).$$

Altogether, it remains to prove that $(\boldsymbol{E},\boldsymbol{H})$ fulfills

$$\begin{aligned}\int_\Omega \epsilon\partial_t \boldsymbol{E}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) - \mathbf{curl}\,\boldsymbol{H}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t)) + j_c(\cdot,\theta(t),\boldsymbol{H}(t))(|\boldsymbol{v}|-|\boldsymbol{E}(t)|)\,\mathrm{d}x \\ \geq \int_\Omega \boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{E}(t))\,\mathrm{d}x \quad \text{for a.e. } t\in(0,T) \text{ and all } \boldsymbol{v}\in \boldsymbol{L}^2(\Omega). \end{aligned} \tag{3.44}$$

To prove (3.44), we first deduce that

$$\begin{aligned}\limsup_{h\to 0}\int_0^t \Big(\mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(s), \overline{\boldsymbol{E}}^a_{N,h}(s)\Big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s &\overbrace{=}^{(3.30)} \limsup_{h\to 0}\int_0^t \big(\mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(s), \overline{\boldsymbol{E}}_{N,h}(s)\big)_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s \\ &\overbrace{=}^{(3.33)} \limsup_{h\to 0}\Bigg(-\int_0^{T-\tau} \big(\partial_t \boldsymbol{H}_{N,h}(s), \overline{\boldsymbol{H}}_{N,h}(s)\big)_{\boldsymbol{L}^2_\mu(\Omega)}\chi_{[0,t]}(s)\,\mathrm{d}s \\ &\qquad + \int_{T-\tau}^T \big(\mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(s), \overline{\boldsymbol{E}}_{N,h}(s)\big)_{\boldsymbol{L}^2(\Omega)}\chi_{[0,t]}(s)\,\mathrm{d}s\Bigg) \\ &\overbrace{\leq}^{(3.40)} -\int_0^t (\partial_t \boldsymbol{H}(s), \boldsymbol{H}(s))_{\boldsymbol{L}^2_\mu(\Omega)}\,\mathrm{d}s \overbrace{=}^{(3.43)} \int_0^t (\mathbf{curl}\,\boldsymbol{H}(s), \boldsymbol{E}(s))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s. \end{aligned} \tag{3.45}$$

Furthermore, since $\boldsymbol{f}$ is of bounded variation, it is continuous almost everywhere in $[0,T]$ (cf. [12], Sect. 2.5.16). Let $t\in[0,T]$ be such that $\boldsymbol{f}$ is continuous at t. For every $h>0$, we define the index

$$n_h := \operatorname*{arg\,min}_{\substack{n\in\{1,\dots,N(h)\}\\ t_n\geq t}} |t_n - t|.$$

Then, by construction of the index n_h and the discretization of $[0,T]$, it holds that

$$\left|t_{n_h-\frac{1}{2}} - t\right| \leq t_{n_h} - t_{n_h-1} \to 0 \quad \text{as } h\to 0.$$

By the continuity of $\boldsymbol{f}$ at t, it then follows

$$\overline{\boldsymbol{f}}_{N,h}(t) = \boldsymbol{P}_h \boldsymbol{f}(t_{n_h-\frac{1}{2}}) \to \boldsymbol{f}(t) \quad \text{in } \boldsymbol{L}^2(\Omega) \quad \text{as } h \to 0.$$

We conclude that $\overline{\boldsymbol{f}}_{N,h}(t) \to \boldsymbol{f}(t)$ for almost every $t \in (0,T)$. Since $\boldsymbol{f}$ is also of class $L^\infty((0,T),\boldsymbol{L}^2(\Omega))$, an application of the Lebesgue dominated convergence theorem implies

$$\overline{\boldsymbol{f}}_{N,h} \to \boldsymbol{f} \quad \text{strongly in } L^2((0,T),\boldsymbol{L}^2(\Omega)) \quad \text{as } h \to 0. \tag{3.46}$$

Analogously, the regularity condition (A5) implies that

$$\overline{\theta}_N \to \theta \quad \text{strongly in } L^2((0,T),L^2(\Omega)) \quad \text{as } h \to 0. \tag{3.47}$$

Moreover,

$$\begin{aligned}
\varphi_{N,h}\Big(t,\overline{\boldsymbol{E}}^a_{N,h}(t)\Big) &\overset{(3.32)}{=} \int_\Omega P_h j_c\big(\cdot,\overline{\theta}_N(t),\overline{\boldsymbol{H}}_{N,h}(t)\big)\Big|\overline{\boldsymbol{E}}^a_{N,h}(t)\Big|\,\mathrm{d}x \\
&= \int_\Omega P_h j_c\big(\cdot,\overline{\theta}_N(t),\overline{\boldsymbol{H}}_{N,h}(t)\big) - j_c(\cdot,\theta(t),\boldsymbol{H}(t))\Big(\Big|\overline{\boldsymbol{E}}^a_{N,h}(t)\Big| - |\boldsymbol{E}(t)|\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))\Big(\Big|\overline{\boldsymbol{E}}^a_{N,h}(t)\Big| - |\boldsymbol{E}(t)|\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega P_h j_c\big(\cdot,\overline{\theta}_N(t),\overline{\boldsymbol{H}}_{N,h}(t)\big)|\boldsymbol{E}(t)|\,\mathrm{d}x.
\end{aligned} \tag{3.48}$$

Due to (3.47), (3.37), (3.30), (A4), and (3.2), it follows that the first term on the right-hand side of (3.48) vanishes as $h \to 0$. Furthermore, as the mapping $\boldsymbol{v} \mapsto \int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))|\boldsymbol{v}|\,\mathrm{d}x$ is convex and continuous, it is weakly lower semi-continuous. Therefore, along with (3.30) and (3.35), the second term in the right-hand side of (3.48) satisfies

$$\liminf_{h\to 0} \int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))\Big(\Big|\overline{\boldsymbol{E}}^a_{N,h}(t)\Big| - |\boldsymbol{E}(t)|\Big)\,\mathrm{d}x \geq 0.$$

In conclusion, it holds that

$$\liminf_{h\to 0} \varphi_{N,h}\Big(t,\overline{\boldsymbol{E}}^a_{N,h}(t)\Big) \geq \int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))|\boldsymbol{E}(t)|\,\mathrm{d}x. \tag{3.49}$$

Finally, we test the variational inequality in (3.33) with $\boldsymbol{v}_h \in \mathbf{DG}_h$ and integrate the resulting inequality over $[0,t]$. Then, by applying the limit superior, we obtain

$$\begin{aligned}
\int_0^t (\boldsymbol{f}(s),\boldsymbol{v}-\boldsymbol{E}(s))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s &\overset{(3.46)}{=} \lim_{h\to 0}\int_0^t (\overline{\boldsymbol{f}}_{N,h}(s),\boldsymbol{v}_h-\overline{\boldsymbol{E}}_{N,h}(s))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s \\
&\overset{(3.33)}{\leq} \limsup_{h\to 0}\Bigg[\int_0^t (\partial_t \boldsymbol{E}_{N,h}(s),\boldsymbol{v}_h-\overline{\boldsymbol{E}}^a_{N,h}(s))_{\boldsymbol{L}^2_\epsilon(\Omega)}\,\mathrm{d}s \\
&\qquad - \int_0^t (\mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(s),\boldsymbol{v}_h-\overline{\boldsymbol{E}}^a_{N,h}(s))_{\boldsymbol{L}^2(\Omega)} + \varphi_{N,h}(s,\boldsymbol{v}_h) - \varphi_{N,h}(s,\overline{\boldsymbol{E}}^a_{N,h}(s))\,\mathrm{d}s\Bigg] \\
&\overset{(3.37),(3.39),(3.41),(3.45)-(3.49)}{\leq} \int_0^t (\partial_t \boldsymbol{E}(s),\boldsymbol{v}-\boldsymbol{E}(s))_{\boldsymbol{L}^2_\epsilon(\Omega)}\,\mathrm{d}s - \int_0^t (\mathbf{curl}\,\boldsymbol{H}(s),\boldsymbol{v}-\boldsymbol{E}(s))_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}s \\
&\qquad + \int_0^t\int_\Omega j_c(\cdot,\theta(s),\boldsymbol{H}(s))(|\boldsymbol{v}|-|\boldsymbol{E}(t)|)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.50}$$

Taking the derivative of (3.50) with respect to t, it follows that (3.44) is satisfied. In conclusion, $(\boldsymbol{E},\boldsymbol{H})$ is a solution to (QVI). The uniqueness follows from Corollary 5.2 of [30].

Step 2: Uniform convergence**.** In view of (3.37), it only remains to prove

$$\lim_{h\to 0}\|\boldsymbol{E}_{N,h}-\boldsymbol{E}\|_{\mathcal{C}([0,T],\boldsymbol{L}^2_\epsilon(\Omega))}=0. \tag{3.51}$$

In order to achieve this, we test (QVI) with $\boldsymbol{v}=\overline{\boldsymbol{E}}^a_{N,h}(t)$ to obtain

$$\begin{aligned}\int_\Omega \epsilon\partial_t\boldsymbol{E}(t)\cdot\Big(\overline{\boldsymbol{E}}^a_{N,h}(t)-\boldsymbol{E}(t)\Big)\,\mathrm{d}x-\int_\Omega \mathbf{curl}\,\boldsymbol{H}(t)\cdot\Big(\overline{\boldsymbol{E}}^a_{N,h}(t)-\boldsymbol{E}(t)\Big)\,\mathrm{d}x\\ +\int_\Omega j_c(\cdot,\theta(t),\boldsymbol{H}(t))\Big(\Big|\overline{\boldsymbol{E}}^a_{N,h}(t)\Big|-|\boldsymbol{E}(t)|\Big)\,\mathrm{d}x\\ \geq\int_\Omega \boldsymbol{f}(t)\cdot\Big(\overline{\boldsymbol{E}}^a_{N,h}(t)-\boldsymbol{E}(t)\Big)\,\mathrm{d}x\quad\text{for a.e. } t\in(0,T).\end{aligned} \tag{3.52}$$

Next, we insert $\boldsymbol{v}_h=\boldsymbol{P}_h\boldsymbol{E}(t)\in\mathbf{DG}_h$ into (3.33) and deduce that

$$\begin{aligned}\int_\Omega \epsilon\partial_t\boldsymbol{E}_{N,h}(t)\cdot\Big(\boldsymbol{E}(t)-\overline{\boldsymbol{E}}^a_{N,h}(t)\Big)\,\mathrm{d}x-\int_\Omega \mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(t)\cdot\Big(\boldsymbol{P}_h\boldsymbol{E}(t)-\overline{\boldsymbol{E}}^a_{N,h}(t)\Big)\,\mathrm{d}x\\ +\int_\Omega \epsilon\partial_t\boldsymbol{E}_{N,h}(t)\cdot(\boldsymbol{P}_h\boldsymbol{E}(t)-\boldsymbol{E}(t))\,\mathrm{d}x+\varphi_{N,h}(t,\boldsymbol{P}_h\boldsymbol{E}(t))-\varphi_{N,h}\Big(t,\overline{\boldsymbol{E}}^a_{N,h}(t)\Big)\\ \geq\int_\Omega \overline{\boldsymbol{f}}_{N,h}(t)\cdot\Big(\boldsymbol{E}(t)-\overline{\boldsymbol{E}}^a_{N,h}(t)\Big)\,\mathrm{d}x+\int_\Omega \overline{\boldsymbol{f}}_{N,h}(t)\cdot(\boldsymbol{P}_h\boldsymbol{E}(t)-\boldsymbol{E}(t))\,\mathrm{d}x.\end{aligned} \tag{3.53}$$

On the other hand, due to (2.10) in combination with $\mathbf{curl}\,\mathbf{ND}_h\subset\mathbf{DG}_h$, it holds that

$$\begin{aligned}\int_\Omega \mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(t)\cdot\Big(\boldsymbol{P}_h\boldsymbol{E}(t)-\overline{\boldsymbol{E}}^a_{N,h}(t)\Big)\,\mathrm{d}x\\ =\int_\Omega \mathbf{curl}\,\overline{\boldsymbol{H}}_{N,h}(t)\cdot\Big(\boldsymbol{E}(t)-\overline{\boldsymbol{E}}^a_{N,h}(t)\Big)\,\mathrm{d}x\quad\text{for a.e. } t\in(0,T).\end{aligned} \tag{3.54}$$

Now, adding (3.52) and (3.53), using (3.54), and integrating in time over $(0,t)$ yield

$$\begin{aligned}\int_0^t\int_\Omega \epsilon\partial_t(\boldsymbol{E}_{N,h}(s)-\boldsymbol{E}(s))\cdot(\boldsymbol{E}_{N,h}(s)-\boldsymbol{E}(s))\,\mathrm{d}x\,\mathrm{d}s\\ \leq\int_0^t\Bigg[\int_\Omega(\boldsymbol{f}(s)-\overline{\boldsymbol{f}}_{N,h}(s))\cdot\Big(\boldsymbol{E}(s)-\overline{\boldsymbol{E}}^a_{N,h}(s)\Big)-\overline{\boldsymbol{f}}_{N,h}(s)\cdot(\boldsymbol{P}_h\boldsymbol{E}(s)-\boldsymbol{E}(s))\,\mathrm{d}x\\ +\int_\Omega \epsilon\partial_t\boldsymbol{E}_{N,h}(s)\cdot(\boldsymbol{P}_h\boldsymbol{E}(s)-\boldsymbol{E}(s))\,\mathrm{d}x+\int_\Omega \epsilon\partial_t(\boldsymbol{E}_{N,h}(s)-\boldsymbol{E}(s))\cdot\Big(\boldsymbol{E}_{N,h}(s)-\overline{\boldsymbol{E}}^a_{N,h}(s)\Big)\,\mathrm{d}x\\ +\Big(\varphi_{N,h}(s,\boldsymbol{P}_h\boldsymbol{E}(s))-\int_\Omega j_c(\cdot,\theta(s),\boldsymbol{H}(s))|\boldsymbol{E}(s)|\,\mathrm{d}x\Big)\\ +\Big(\int_\Omega j_c(\cdot,\theta(s),\boldsymbol{H}(s))\Big|\overline{\boldsymbol{E}}^a_{N,h}(s)\Big|\,\mathrm{d}x-\varphi_{N,h}\Big(s,\overline{\boldsymbol{E}}^a_{N,h}(s)\Big)\Big)\\ +\int_\Omega \mathbf{curl}(\overline{\boldsymbol{H}}_{N,h}(s)-\boldsymbol{H}(s))\cdot\Big(\overline{\boldsymbol{E}}^a_{N,h}(s)-\boldsymbol{E}(s)\Big)\,\mathrm{d}x\Bigg]\,\mathrm{d}s=:\sum_{i=1}^7 C_h^i.\end{aligned} \tag{3.55}$$

Let us now prove the convergence $C_h^i\to 0$ for $h\to 0$, $i\in\{1,\dots,7\}$. First of all, as a result of (3.46) in combination with (3.34) and (3.2), it follows that $C_h^i\to 0$ as $h\to 0$ for $i=1,2,3$. The convergence $|C_h^4|\to 0$ as $h\to 0$ follows immediately from (3.30). Now, similar to (3.48), combining (3.1) with (3.2) and (A3), (A4) implies $C_h^i\to 0$ as $h\to 0$ for $i=5,6$. Finally, the last term C_h^7 is treated as follows:

$$\lim_{h\to 0}\int_0^t\int_\Omega \mathbf{curl}(\overline{\boldsymbol{H}}_{N,h}(s)-\boldsymbol{H}(s))\cdot\Big(\overline{\boldsymbol{E}}^a_{N,h}(s)-\boldsymbol{E}(s)\Big)\,\mathrm{d}x\,\mathrm{d}s$$

$$= \lim_{h\to 0} \int_0^t \int_\Omega \mathbf{curl}\, \overline{\boldsymbol{H}}_{N,h}(s) \cdot \big(\overline{\boldsymbol{E}}_{N,h}(s) - \boldsymbol{E}(s)\big)\, \mathrm{d}x\, \mathrm{d}s$$

$$\overbrace{=}^{(3.33)\&(3.43)} \lim_{h\to 0} \int_0^t \int_\Omega \mu \partial_t (\boldsymbol{H}(s) - \boldsymbol{H}_{N,h}(s)) \cdot \overline{\boldsymbol{H}}_{N,h}(s)\, \mathrm{d}x\, \mathrm{d}s \overbrace{=}^{(3.34)\&(3.37)} 0.$$

Now, combining the convergence of $|C_h^i| \to 0$ as $h \to 0$ for $i \in \{1, \dots, 7\}$ with the estimate (3.55), we conclude

$$\frac{1}{2}\|\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)} - \frac{1}{2}\|\boldsymbol{E}_{N,h}(0) - \boldsymbol{E}(0)\|^2_{\boldsymbol{L}^2_\epsilon(\Omega)}$$

$$\overbrace{=}^{(3.38)} \int_0^t \int_\Omega \epsilon \partial_t (\boldsymbol{E}_{N,h}(s) - \boldsymbol{E}(s)) \cdot (\boldsymbol{E}_{N,h}(s) - \boldsymbol{E}(s))\, \mathrm{d}x\, \mathrm{d}s \to 0 \quad \text{as } h \to 0.$$

Ultimately, since

$$\lim_{h\to 0} \|\boldsymbol{E}_{N,h}(0) - \boldsymbol{E}(0)\|_{\boldsymbol{L}^2_\epsilon(\Omega)} = \lim_{h\to 0} \|\boldsymbol{P}_h \boldsymbol{E}_0 - \boldsymbol{E}_0\|_{\boldsymbol{L}^2_\epsilon(\Omega)} = 0,$$

the above convergence yields the pointwise convergence

$$\lim_{h\to 0} \|\boldsymbol{E}_{N,h}(t) - \boldsymbol{E}(t)\|_{\boldsymbol{L}^2_\epsilon(\Omega)} = 0 \qquad \forall t \in [0, T],$$

which implies the desired uniform convergence (3.51) by the use of the Arzelá–Ascoli theorem.

□

4. Numerical test

Evidently, experimental data in the physics literature report the magnetic field and temperature dependency in the critical current density. With our numerical test, we strive to show the reasonable performance of ($\mathrm{LF}_{N,h}$) as well as the impact of the magnetic field dependence in j_c on the numerical simulation. We specify $\Omega = (-1, 1)^3$, $T = 1$, $\epsilon, \mu \equiv 1$, and $(\boldsymbol{E}_0, \boldsymbol{H}_0) = (\mathbf{0}, \mathbf{0})$. All computations were implemented on the open-source platform FEniCS [22]. Based on experimental data, the authors in [10, 19, 28] report that the critical current for type-II superconductors vanishes for sufficiently large $|\boldsymbol{H}|$. Following this observation, the authors in [10] suggest the choice

$$j_c(\cdot, \theta, \boldsymbol{H}) := \frac{c(1-\theta)^2}{1 + |\boldsymbol{H}|^\beta} \chi_{\Omega_{\mathrm{sc}}}(\cdot) \tag{4.1}$$

to model the magnetic field and temperature dependence in j_c for materials covered in specifically designed silver-sheated tapes. In the setting of (4.1), $c = 4 \cdot 10^3$, $\theta(x, t) = t$, and

$$\Omega_{\mathrm{sc}} := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2 + x_3^2} \le 0.2 \right\} \subset \Omega := (-1, 1)^3$$

denotes a type-II superconductor. Furthermore, our test considers two different configurations: $\beta = 0$ and $\beta = 3$. Note that $\beta = 0$ corresponds to the VI case where the magnetic field dependency is neglected in j_c. For the applied current source, we choose

$$\boldsymbol{f} \colon [0, 1] \times \Omega \to \mathbb{R}^3, \quad \boldsymbol{f}(t, x_1, x_2, x_3) := \begin{cases} 10\big(1 + 4t^2\big)\left(0, \dfrac{-x_3}{\sqrt{x_2^2 + x_3^2}}, \dfrac{x_2}{\sqrt{x_2^2 + x_3^2}}\right) & \text{if } (x_1, x_2, x_3) \in P \\ 0 & \text{if } (x_1, x_2, x_3) \notin P, \end{cases}$$

where $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_1| \le 0.5, 0.3 \le \sqrt{x_2^2 + x_3^2} \le 0.5\}$ models a cylindrical pipe coil (Fig. 1).

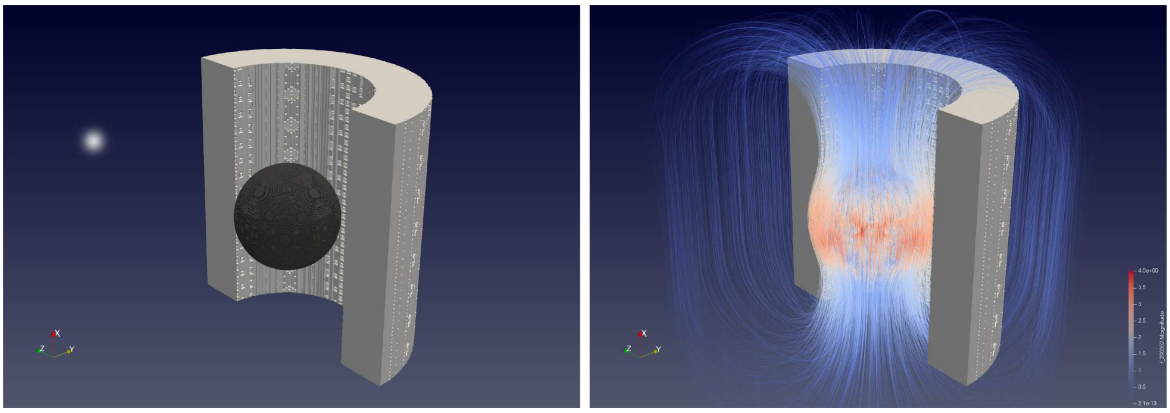

FIGURE 1. Clipped pipe coil with Ω_{sc} in its center (*left*) and the computed solution by ($\mathrm{LF}_{N,h}$) (*right*).

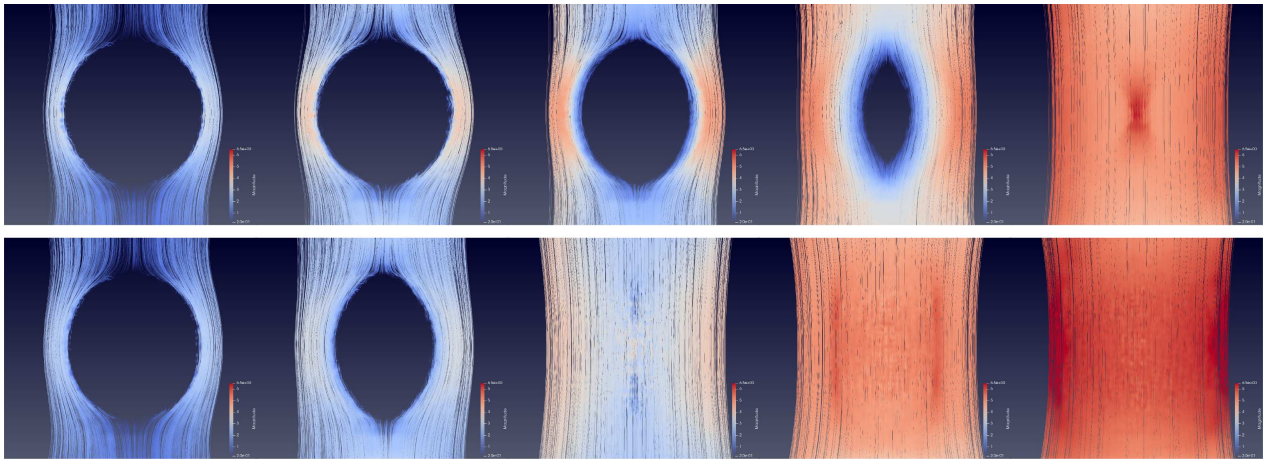

FIGURE 2. Computed magnetic field for $\beta = 0$ (*upper*) and $\beta = 3$ (*lower*) at t_n with $n = 200, 300, 440, 540, 600$.

We implemented ($\mathrm{LF}_{N,h}$) according to Remark 2.4 with $N = 600$ and roughly 3 million DoFs in the mixed finite element space. As observable in Figure 2, for sufficiently small time values, the Meissner effect is fully present for both configurations of β. This is in line with the choice of the zero initial value. With the evolution of time, the strength of the magnetic field enhances, resulting in the mixed phase of the type-II superconductor, where Ω_{sc} gradually leaves its superconducting state. Here, we monitor a distinctive transition between the superconducting and normal states for $\beta = 3$ and $\beta = 0$. In particular, the breakdown of the superconductor state is hugely accelerated for $\beta = 3$. This confirms that the magnetic field dependency in j_c (the QVI character) in general cannot be disregarded when considering a not-so-weak magnetic field. This is in agreement with the experimental data reported in the original work by Kim *et al.* [19].

In conclusion, our 3D test exhibits the reasonable numerical performance of ($\mathrm{LF}_{N,h}$) and its capability to deal with (QVI) and to simulate the Meissner effect as well as the mixed phase in type-II superconductivity at the macroscopic level.

TABLE 1. Comparison for DoF($\mathbf{DG}_h \times \mathbf{ND}_h$) $\approx$ 100.000.

	($\mathrm{LF}_{N,h}$)	($\mathrm{IE}_{N,h}$)							
Error level		1e−3				1e−5			
Time step N	32	4	8	16	32	4	8	16	32
# Linear systems	32	25	36	57	117	37	52	83	159

4.1. Computational Complexity of ($\mathrm{LF}_{N,h}$) *vs.* ($\mathrm{IE}_{N,h}$)

We close this paper by carrying out a brief numerical comparison between ($\mathrm{LF}_{N,h}$) and ($\mathrm{IE}_{N,h}$). The test considers the same numerical setup as before with the choice $\beta = 3$ in the nonlinearity (4.1). With roughly 100.000 DoFs in the mixed finite element space $\mathbf{DG}_h \times \mathbf{ND}_h$, we compare the computational complexity of ($\mathrm{IE}_{N,h}$) and ($\mathrm{LF}_{N,h}$) based on their respective numerical realization as described in Remarks 2.2 and 2.4. Table 1 depicts the number of linear systems to be solved in ($\mathrm{LF}_{N,h}$) and ($\mathrm{IE}_{N,h}$). Here, for the scheme ($\mathrm{IE}_{N,h}$), we consider two different error levels as stopping criteria for the outer and inner (SSN) iteration process (see Rem. 2.2). Moreover, for both configurations of error levels, we conduct our test with $N = 4, 8, 16, 32$. The numerical test for ($\mathrm{LF}_{N,h}$) is performed for $N = 32$. Our numerical tests show that the computed solutions generated by ($\mathrm{LF}_{N,h}$) and ($\mathrm{IE}_{N,h}$) are close to each other. Both solutions exhibit similar physical behavior in terms of Meissner's effect and the mixed transition between the superconducting and normal states. However, we monitor largely deviating numbers of linear systems to be solved in the implicit Euler method ($\mathrm{IE}_{N,h}$). At $N = 32$ and the error level 1e−3, ($\mathrm{IE}_{N,h}$) requires to solve around four times more linear systems than ($\mathrm{LF}_{N,h}$). At the finer error level 1e−5, around five times more linear systems have to be solved in ($\mathrm{IE}_{N,h}$). This certainly showcases the superiority of ($\mathrm{LF}_{N,h}$) over ($\mathrm{IE}_{N,h}$) in terms of computational complexity.

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