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A ROBUST FAMILY OF EXPONENTIAL ATTRACTORS FOR A LINEAR TIME DISCRETIZATION OF THE CAHN-HILLIARD EQUATION WITH A SOURCE TERM

Dieunel Dor[1,2] and Morgan Pierre[1,*]

Abstract. We consider a linear implicit-explicit (IMEX) time discretization of the Cahn-Hilliard equation with a source term, endowed with Dirichlet boundary conditions. For every time step small enough, we build an exponential attractor of the discrete-in-time dynamical system associated to the discretization. We prove that, as the time step tends to 0, this attractor converges for the symmetric Hausdorff distance to an exponential attractor of the continuous-in-time dynamical system associated with the PDE. We also prove that the fractal dimension of the exponential attractor (and consequently, of the global attractor) is bounded by a constant independent of the time step. The results also apply to the classical Cahn-Hilliard equation with Neumann boundary conditions.

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1. Introduction

We consider the Cahn-Hilliard equation with a source term and Dirichlet boundary conditions. It reads

$$u_t + \Delta^2 u - \Delta f(u) + g(u) = 0 \qquad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = \Delta u = 0 \qquad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u|_{t=0} = u_0 \qquad \text{in } \Omega, \tag{1.3}$$

where Ω is a bounded open subset of $\mathbb{R}^d$ ($d = 1$, 2 or 3) with smooth boundary $\partial\Omega$. The unknown function u is the order parameter, f is the nonlinear regular potential and g is the source term.

When $g = 0$, the PDE (1.1) is known as the Cahn-Hilliard equation [6] and it has been thoroughly studied (see [25] and references therein). The generalization with a source term g has drawn a lot of interest in recent years, in particular for biological applications [15, 17, 25, 26].

The PDE (1.1) endowed with Dirichlet boundary conditions (1.2) was analyzed in [14, 21, 22] with various assumptions on f and g (see also [8]). In particular, global-in-time solutions were shown to exist and their

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[1] Laboratoire de Mathématiques et Applications, Université de Poitiers, CNRS, 86073 Poitiers, France.

[2] Universite d'Etat d'Haiti, Ecole Normale Superieure, Laboratoire de Mathematiques et Applications (LAMA-UEH), HT6115, 2 Rue Caseus, Pacot, Port-au-Prince, Haiti.

[*]Corresponding author: Morgan.Pierre@math.univ-poitiers.fr; morgan.pierre@univ-poitiers.fr

asymptotic behaviour was studied. The existence of finite dimensional attractors was established. Numerical simulations were performed, *e.g.*, in [1, 14, 18].

We stress that Neumann boundary conditions are usually prefered for the standard Cahn-Hilliard equation ($g = 0$). The mass is conserved and the analysis is rather similar to the case with Dirichlet boundary conditions. However, if $g \neq 0$, the situation is more delicate. We no longer have the conservation of mass and if g is a proliferation term, some solutions may blow up in finite time [7, 23, 25].

Our purpose in this manuscript is to obtain a global asymptotic result for a linear time discretization of (1.1) and (1.2) with fixed time step $\delta t > 0$. We want a construction of exponential attractors which is robust as δt goes to 0. We use a first order implicit-explicit (IMEX) time discretization where the nonlinearities f and g are treated explicitly and the bilaplacian is treated implicitly. This is a very popular discretization of the classical Cahn-Hilliard equation which allows the use of the fast Fourier transform (FFT) [5, 30]. It has also been successfully used in variants of the Cahn-Hilliard equation including a source term [9, 10].

An exponential attractor is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. In comparison with the global attractor, an exponential attractor is expected to be more robust to perturbations: global attractors are generally upper semicontinuous with respect to perturbations, but the lower semicontinuity can be proved only in some specific cases (see [2, 27, 31, 34] and references therein). In particular, the upper semicontinuity of the global attractor as the mesh step and the time step tend to 0 was proved in [13] for a finite element approximation of the Cahn-Hilliard equation.

Exponential attractors were first introduced by Eden *et al.* [11] with a construction based on a “squeezing property”. In [12], Efendiev, Miranville and Zelik proposed a robust construction of exponential attractors based on a “smoothing property” and an appropriate error estimate. Their construction has been adapted to many situations, including singular perturbations. We refer the reader to the review [27] for details.

In [29], a robust family of exponential attractors was built for a time semidiscretization of a generalized Allen-Cahn equation. An abstract result was first derived, based on the construction in [12], and it was then applied to the backward Euler scheme. The same approach was successfully applied for a time splitting scheme in [4], for a discretized Ginzburg Landau equation in [3] and for a space semidiscretization of the Allen-Cahn equation in [28]. In these papers, the nonlinearity was treated implicitly. Here, we also adapt the approach introduced in [29], but we focus on a case where the nonlinearity is treated explicitly, thus allowing a linear scheme.

Since we use an IMEX scheme, the main condition that we impose on the potential is that f is Lipschitz continous on $\mathbb{R}$ (*cf.* (2.2)). This restriction can be well understood for the classical Cahn-Hilliard equation, which is a gradient flow for the H^{-1} inner product, so that there is a Lyapunov functional (the energy) naturally associated with it. In order for the IMEX scheme to have the same property, it is necessary to assume that f is Lipschitz continuous and that the time step is small enough. This is known as energy stability [5, 30].

For $g \neq 0$, the PDE (1.1) and (1.2) is no longer a gradient flow and there is no Lyapunov functional associated with it. Nonetheless, the PDE is a dissipative system if f satisfies a standard dissipativity assumption (see (2.4)) and if g is subordinated to f (*cf* Rem. 2.3). We prove here that the discrete-in-time dynamical system associated to the IMEX scheme is also dissipative if the time step is small enough (*cf* Sect. 3.2). However, the smallness condition required on the time step to prove dissipativity is much stronger than the one required for energy stability. Typically, f can be the usual cubic nonlinearity which is modified into an affine function outside a compact interval as in (2.8). In turn, a typical choice for g is the symport term

$$g(s) = \frac{ks}{k' + |s|} \quad (s \in \mathbb{R}), \tag{1.4}$$

where $k, k' > 0$ [19, 20, 24]. Our analysis also includes the case $g = 0$ (the classical Cahn-Hilliard equation).

Our manuscript is organized as follows. We first give the a priori estimates for the PDE (1.1)–(1.3) in Section 2. In Section 3, we establish their discrete counterpart for the IMEX scheme. The most technical part is the dissipative H^2 estimate (Prop. 3.6). An error estimate on finite time intervals is proved in Section 4.

The main result is given in the last section. For every time step $\delta t > 0$ small enough, we build an exponential attractor $\mathcal{M}_{\delta t}$ of the discrete-in-time dynamical system associated to the IMEX scheme. We prove that $\mathcal{M}_{\delta t}$ converges to $\mathcal{M}_0$ for the symmetric Hausdorff distance as δt tends to 0, where $\mathcal{M}_0$ is an exponential attractor of the continuous-in-time dynamical system associated with the PDE. We also prove that the fractal dimension of $\mathcal{M}_{\delta t}$ (and consequently, of the global attractor) is bounded by a constant independent of δt. The results also apply to the classical Cahn-Hilliard equation with Neumann boundary conditions, as pointed out in Remark 5.5. As a perspective, we note that it would be very interesting to obtain similar results for a space and time discretization of the PDE (1.1) and (1.2) based on the IMEX scheme.

2. The continuous problem

2.1. Notation and assumptions

We make the following assumptions:

$$f \in C^{1,1}(\mathbb{R}),\ \ f(0) = 0, \tag{2.1}$$

$$f' \text{ is bounded on } \mathbb{R}, \tag{2.2}$$

$$f' \text{ is piecewise } C^1 \text{ and } f'' \text{ is bounded on } \mathbb{R}, \tag{2.3}$$

$$\lim_{|s|\to\infty} \inf f'(s) > 0. \tag{2.4}$$

Assumption (2.4) is the dissipativity condition.

The term of symport g satisfies

$$g \in C^1(\mathbb{R}). \tag{2.5}$$

$$g \text{ is bounded on } \mathbb{R}, \tag{2.6}$$

$$g' \text{ is bounded on } \mathbb{R} \tag{2.7}$$

Example 2.1. The function g defined by (1.4) satisfies (2.5)–(2.7). If $g = 0$, then g also satisfies (2.5)–(2.7) and equation (1.1) is the classical Cahn-Hilliard equation.

Remark 2.2. We note that if $f \in C^2(\mathbb{R})$ with f'' bounded on $\mathbb{R}$, then assumption (2.3) is satisfied. However, the weaker assumption (2.3) is interesting because it allows for the usual C^1 regularization of the cubic nonlinearity $s^3 - s$, defined by

$$f(s) = f_K(s) = \begin{cases} \left(3K^2 - 1\right)s - 2K^3, & s > K, \\ s^3 - s, & s \in [-K, K], \\ \left(3K^2 - 1\right)s + 2K^3, & s < -K, \end{cases} \tag{2.8}$$

where $K \geq 1$. Thus, $f_K \in C^1(\mathbb{R})$ has a linear growth at $\pm\infty$ with

$$\max_{s\in\mathbb{R}} |f_K'(s)| = 3K^2 - 1.$$

This regularization is very popular for the IMEX scheme [5, 30].

Remark 2.3. The growth of g is controlled by the growth of f in order to ensure that the PDE is dissipative (see [21, Rem. 2.1]). If we suppress assumption (2.6), then dissipativity is no longer guaranteed. Indeed, let us choose $f(s) = s$ and $g(s) = -\alpha s$ $(s \in \mathbb{R})$ with $\alpha > \lambda_1^2 + \lambda_1$ and $\lambda_1 > 0$ is the first eigenvalue of the minus Laplacian operator with Dirichlet boundary conditions. We have $-\Delta e_1 = \lambda_1 e_1$ where $e_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ is an eigenfunction associated to λ_1. Then the function $u(t) = e^{\beta t} e_1$ with $\beta = \alpha - \lambda_1^2 - \lambda_1 > 0$ solves (1.1) and (1.2) but $\|u(t)\|_{L^2(\Omega)} \to +\infty$ as $t \to +\infty$.

We set $H := L^2(\Omega)$ and we denote by $(\cdot,\cdot)$ the scalar product both in H and in H^d and by $\|\cdot\|$ the induced norm. The symbol $\|\cdot\|_X$ will indicate the norm in the generic real Banach space X. Next, we set $V := H_0^1(\Omega)$, so that $V' = H^{-1}(\Omega)$ is the topological dual of V. The space V is endowed with the Hilbertian norm $v \mapsto \|\nabla v\|$ which is equivalent to the usual $H^1(\Omega)$-norm, thanks to the Poincaré inequality.

We also denote by $A : D(A) \to H$ the (minus) Laplace operator $A = -\Delta$ with homogeneous Dirichlet boundary condition, with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. By elliptic regularity [16], the norm $v \mapsto \|\Delta v\|$ is equivalent to the usual $H^2(\Omega)$-norm in $D(A)$. Moreover,

$$D\left(A^2\right) = \left\{v \in H^4(\Omega) \ : \ v = \Delta v = 0 \qquad \text{on } \partial\Omega \text{ (in the sense of trace)}\right\},$$

and the norm $v \mapsto \left\|\Delta^2 v\right\|$ is equivalent to the usual $H^4(\Omega)$-norm in $D\left(A^2\right)$.

It is well known that A is a positive self-adjoint operator with compact resolvent, so that we can define, for $s \in \mathbb{R}$, its powers $A^s : D\left(A^s\right) \to H$. For each $s \in \mathbb{R}$, the Hilbert space $D\left(A^s\right)$ is equipped with the norm $\|v\|_{2s} = \|A^s v\|$. We have $D\left(A^0\right) = H$ with $\|\cdot\|_0 = \|\cdot\|$ and $D\left(A^{1/2}\right) = V$ with $\|\cdot\|_1 = \|\nabla \cdot\|$. Indeed, an integration by parts shows that

$$\|\nabla v\|^2 = (Av, v) = \left\|A^{1/2} v\right\|^2, \quad \forall v \in D(A). \tag{2.9}$$

We note that for every $s \in \mathbb{R}$, we have

$$\|v\|_s^2 = \left(A^s v, v\right), \quad \forall v \in D\left(A^s\right) \cap H.$$

If $s_1 < s_2$, then the space $D\left(A^{s_2}\right)$ is continuously embedded in $D(A^{s_1})$, *i.e.*

$$\|v\|_{2s_1} \le c_S(s_1, s_2)\|v\|_{2s_2}, \quad \forall v \in D(A^{s_2}), \tag{2.10}$$

where the positive constant c_S depends on s_1 and s_2. In particular for $s_1 = 0$ and $s_2 = 1/2$, we have the Poincaré inequality

$$\|v\| \le c_P \|\nabla v\|, \quad \forall v \in V, \quad c_P = c_S(0, 1/2). \tag{2.11}$$

By (2.1)-(2.2), the map $v \mapsto f(v)$ is Lipschitz continuous from H into H and from V into V and we have

$$\|f(v) - f(w)\| \le L_f \|v - w\|, \quad \forall v, w \in H, \tag{2.12}$$

where $L_f = \sup_{s\in\mathbb{R}} |f'(s)|$. Similarly, by (2.5)–(2.7), $v \mapsto g(v)$ is Lipschitz continuous from H into H and from V into V and we have

$$\|g(v)\| \le c_g, \quad \forall v \in H, \tag{2.13}$$

$$\|g(v) - g(w)\| \le L_g \|v - w\|, \quad \forall v, w \in H. \tag{2.14}$$

We define F by

$$F(s) = \int_0^s f(t)\,\mathrm{d}t \tag{2.15}$$

We deduce from (2.4) that

$$F(s) \ge \gamma_1 s^2 - \gamma_2, \quad \forall s \in \mathbb{R}, \text{ where } \gamma_1 > 0, \ \gamma_2 \ge 0. \tag{2.16}$$

We deduce from (2.2) that

$$F(s) \leq \gamma_3 s^2 + \gamma_4, \quad \forall s \in \mathbb{R}, \text{ where } \gamma_3 > 0,\ \gamma_4 \geq 0. \tag{2.17}$$

By (2.4), we also have

$$f(s)s \geq \gamma_5 s^2 - \gamma_6, \quad \forall s \in \mathbb{R}, \text{ where } \gamma_5 > 0,\ \gamma_6 \geq 0, \tag{2.18}$$

and

$$f'(s) \geq -\gamma_7, \quad \forall s \in \mathbb{R}, \text{ where } \gamma_7 \geq 0. \tag{2.19}$$

Assumption (2.3) implies that f'' has a finite number of discontinuities (the corner points of f'). Here, f'' is the distributional derivative of f' since $C^{1,1}(\mathbb{R}) = W^{2,\infty}(\mathbb{R})$. Moreover, the following chain rule holds [16, Thm. 7.8]: if $v \in H^1(\Omega)$, we have $f'(v) \in H^1(\Omega)$ and

$$\nabla f'(v) = f''(v)\nabla v.$$

Consequently, if $v \in H^2(\Omega)$, then $f(v) \in H^2(\Omega)$ and

$$\Delta f(v) = f'(v)\Delta v + f''(v)|\nabla v|^2. \tag{2.20}$$

We use here that $H^1(\Omega) \subset L^4(\Omega)$ since $d \leq 3$, so that $|\nabla v|^2 \in L^2(\Omega)$.

The abstract version of (1.1) and (1.2) reads

$$\frac{\mathrm{d}u}{\mathrm{d}t} + A^2 u + Af(u) + g(u) = 0 \text{ in } D\left(A^{-1}\right), \qquad \text{for a.e. } t > 0. \tag{2.21}$$

It is associated to the variational formulation

$$\frac{d}{dt}(u, v) + (Au, Av) + (f(u), Av) + (g(u), v) = 0 \quad \text{ in } \mathcal{D}'(0,\infty),\ \forall v \in D(A).$$

2.2. The continuous semigroup

We first state the well-posedness result of our model.

Theorem 2.4. *For every $u_0 \in V$, there exists a unique solution u of (1.1)–(1.3) which satisfies*

$$u \in C^0([0,T], V) \cap L^2(0,T; D(A)) \quad \text{ and } \quad u_t \in L^2(0,T; V'), \quad \forall T > 0.$$

Moreover, if $u_0 \in D(A)$, then u satisfies

$$u \in C^0([0,T], D(A)) \cap L^2(0,T; D(A^2)), \quad \forall T > 0.$$

Proof. For $u_0 \in V$, the proof of existence is based on Proposition 2.5, Proposition 2.6 and a standard Galerkin scheme. In order to prove that u is continuous from $[0,T]$ into V, we use a standard argument [25, 32]. We first show that u is weakly continuous into V, thanks to the Strauss lemma, and we note that $t \mapsto \|\nabla u(t)\|^2$ is absolutely continuous since $\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2$ belongs to $L^1(0,T)$ by (2.32). For $u_0 \in D(A)$, the proof of existence is based on Proposition 2.7.

The proof of uniqueness and of the continuous dependence with respect to the initial data in the L^2-norm follow from Lemma 2.10. □

As a consequence, we have the continuous (with respect to the L^2-norm) semigroup $S_0(t)$ defined as

$$S_0(t) : D(A) \to D(A),\ u_0 \mapsto u(t), \quad t \geq 0. \tag{2.22}$$

2.3. Dissipative estimates

In this subsection, we first establish some priori estimates for the solution u to the system (1.1)–(1.3). These formal estimates could be rigorously justified by a Galerkin approximation. These dissipative estimates are useful in the proof of Theorem 2.4. They also show that the semigroup is dissipative. It is an important step in the construction of an exponential attractor. As a by-product, we obtain the existence of a global attractor.

Proposition 2.5 (Dissipative estimate in $L^2(\Omega)$). *We have*

$$\|u(t)\|^2 + e^{-\varepsilon_0 t}\int_0^t \|\Delta u(s)\|^2 \,\mathrm{d}s \le C_0 \|u(0)\|^2 e^{-\varepsilon_0 t} + M_0, \quad t \ge 0, \tag{2.23}$$

where the positive constants ε_0, C_0 and M_0 are independent of $u(0)$.

Proof. By multiplying (1.1) by $(-\Delta)^{-1}u$ and integrating over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{-1}^2 + 2\|\nabla u\|^2 + 2(f(u),u) + 2(g(u),(-\Delta)^{-1}u) = 0 \tag{2.24}$$

Using (2.18), we find

$$(f(u),u) \ge \gamma_5\|u\|^2 - \gamma_6|\Omega|,$$

where $|\Omega| = \int_\Omega 1\,\mathrm{d}x$. Using (2.13), the continuous injection $H \subset D(A^{-1})$ and Young's inequality, we find

$$\begin{aligned}\left|(g(u),(-\Delta)^{-1}u)\right| &\le \|g(u)\| \;\left\|(-\Delta)^{-1}u\right\| \\ &\le c_g c_S \|u\| \\ &\le \frac{\gamma_5}{2}\|u\|^2 + \frac{c_g^2 c_S^2}{2\gamma_5},\end{aligned}$$

where $c_S = c_S(-1,0)$ (*cf.* (2.10)). Combining the above estimates in (2.24), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{-1}^2 + c_1\left(\left\|\nabla u(t)\right\|^2 + \left\|u(t)\right\|^2\right) \le c_2, \tag{2.25}$$

where $c_1 = \min(2,\gamma_5) > 0$ and $c_2 = 2\gamma_6|\Omega| + c_g^2c_S^2/\gamma_5 \ge 0$.

Next, we multiply (1.1) by u and we integrate over Ω. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\|\Delta u\|^2 + 2(\nabla f(u),\nabla u) + 2(g(u),u) = 0. \tag{2.26}$$

Thanks to (2.19), we have

$$(f'(u)\nabla u,\nabla u) \ge -\gamma_7\|\nabla u\|^2, \quad \gamma_7 > 0.$$

Using the Cauchy-Schwarz inequality, (2.13) and Young's inequality, we deduce that

$$\begin{aligned}|(g(u),u)| &\le \|g(u)\| \;\|u\| \\ &\le c_g\|u\| \\ &\le \gamma_7\|u\|^2 + \frac{c_g^2}{4\gamma_7}\cdot\end{aligned}$$

Combining the above estimates in (2.26), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + 2\left\|\Delta u(t)\right\|^2 \le 2\gamma_7\left(\left\|\nabla u(t)\right\|^2 + \left\|u(t)\right\|^2\right) + \frac{c_g^2}{2\gamma_7}\cdot \tag{2.27}$$

Summing (2.25) and α times (2.27) where $\alpha > 0$ is such that $2\alpha\gamma_7 \leq c_1/2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + \frac{c_1}{2}\left(\|\nabla u(t)\|^2 + \|u(t)\|^2\right) + 2\alpha\,\|\Delta u(t)\|^2 \leq c_2 + \frac{\alpha c_g^2}{2\gamma_7},$$

where

$$E_1(t) = \alpha\|u(t)\|^2 + \|u(t)\|_{-1}^2.$$

Since $H^{-1}(\Omega)$ is continuously embedded in H (see (2.10)), we have

$$E_1(t) \leq \left(\alpha + \tilde{c}_S^2\right)\|u(t)\|^2, \quad \tilde{c}_S = c_S(-1/2, 0) > 0, \tag{2.28}$$

and this yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) + c_3\left(E_1(t) + \|\Delta u(t)\|^2\right) \leq c_4, \tag{2.29}$$

where $c_3 = \min\left\{c_1/\left(2\alpha + 2\tilde{c}_S^2\right), 2\alpha\right\}$ and $c_4 = c_2 + \alpha c_g^2/(2\gamma_7)$. Applying Gronwall's lemma to (2.29) (see, *e.g.*, [32]), we find

$$E_1(t) + c_3 e^{-c_3 t}\int_0^t \|\Delta u(s)\|^2\,\mathrm{d}s \leq E_1(0)e^{-c_3 t} + \frac{c_4}{c_3}, \quad t \geq 0. \tag{2.30}$$

Using (2.28) for $t = 0$ and $E_1(t) \geq \alpha\|u(t)\|^2$ for all $t \geq 0$, we obtain Proposition 2.5. □

Proposition 2.6 (Dissipative estimate in $H^1(\Omega)$)**.** *We have*

$$\|\nabla u(t)\|^2 + e^{-\varepsilon_1 t}\int_0^t \|u_t(s)\|_{-1}^2 ds \leq C_1\|\nabla u(0)\|^2 e^{-\varepsilon_1 t} + M_1, \quad t \geq 0, \tag{2.31}$$

where the positive constants ε_1, C_1 and M_1 are independent of $u(0)$.

Proof. Testing (1.1) by $(-\Delta)^{-1}u_t$ and integrating over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + 2\|u_t\|_{-1}^2 + 2(f(u), u_t) + 2(g(u), (-\Delta)^{-1}u_t) = 0. \tag{2.32}$$

Using (2.13), the injection $H^{-1}(\Omega) \subset D(A^{-1})$ and Young's inequality, we get

$$\begin{aligned}\left|(g(u), (-\Delta)^{-1}u_t)\right| &\leq \|g(u)\|\ \left\|(-\Delta)^{-1}u_t\right\| \\ &\leq c_g\|u_t\|_{-1} \\ &\leq \frac{1}{2}\|u_t\|_{-1}^2 + \frac{c_g^2}{2}.\end{aligned}$$

By (2.15), we get

$$(f(u), u_t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u)\,\mathrm{d}x.$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u(t)\|^2 + 2(F(u), 1)\right) + \|u_t(t)\|_{-1}^2 \leq c_g^2, \quad t \geq 0. \tag{2.33}$$

Adding (2.29) and β times (2.33) where $\beta > 0$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) + c_5\left(E_1(t) + \|\Delta u(t)\|^2 + \|u_t(t)\|_{-1}^2\right) \leq c_4 + \beta c_g^2, \quad t \geq 0, \tag{2.34}$$

where $c_5 = \min(c_3, \beta) > 0$ and

$$E_2(t) = E_1(t) + \beta\|\nabla u(t)\|^2 + 2\beta(F(u), 1).$$

By (2.17), (2.28) and the Poincaré inequality (2.11), we have

$$\begin{aligned} E_2(t) &\leq \left(\alpha + \tilde{c}_S^2\right)\|u\|^2 + \beta\|\nabla u\|^2 + 2\beta\left(\gamma_3\|u\|^2 + \gamma_4|\Omega|\right) \\ &\leq c_6\|\nabla u(t)\|^2 + 2\beta\gamma_4|\Omega|, \end{aligned} \tag{2.35}$$

where $c_6 = \left(\alpha + \tilde{c}_S^2 + 2\beta\gamma_3\right)c_P^2 + \beta > 0$. Since $D(A) \subset H_0^1(\Omega)$ by (2.10), we deduce from (2.34) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2(t) + c_7\left(E_2(t) + \|u_t(t)\|_{-1}^2\right) \leq c_8, \quad t \geq 0, \tag{2.36}$$

where $c_7 = \min\{c_5, c_5/\left(c_6 c_S(1/2,1)^2\right)\}$ and

$$c_8 = c_4 + \beta c_g^2 + \frac{2c_5\beta\gamma_4|\Omega|}{c_6 c_S(1/2,1)^2}.$$

Gronwall's lemma yields

$$E_2(t) + c_7 e^{-c_7 t}\int_0^t \|u_t(s)\|_{-1}^2\,\mathrm{d}s \leq E_2(0)e^{-c_7 t} + \frac{c_8}{c_7}, \quad t \geq 0. \tag{2.37}$$

By (2.16), we have

$$E_2(t) \geq c_9\left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) - 2\beta\gamma_2|\Omega|, \quad t \geq 0, \tag{2.38}$$

where $c_9 = \min\{\alpha + 2\beta\gamma_1, \beta\} > 0$. This estimate, (2.37) and (2.35) give the result of Proposition 2.6. $\square$

Proposition 2.7 (Dissipative estimate in $H^2(\Omega)$)**.** *We have*

$$\|\Delta u(t)\|^2 \leq Q_2(\|\Delta u(0)\|)e^{-\varepsilon_2 t} + M_2, \quad t \geq 0, \tag{2.39}$$

and

$$\int_0^t \left\|\Delta^2 u(s)\right\|^2 ds \leq Q_2(\|\Delta u(0)\|) + C_2 t, \quad t \geq 0, \tag{2.40}$$

where Q_2 is a monotonic function and the positive constants ε_2, M_2 and C_2 are independent of $u(0)$.

Proof. We multiply (1.1) by $\Delta^2 u$ and integrate over Ω, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + 2\left\|\Delta^2 u\right\|^2 + 2\left(g(u), \Delta^2 u\right) = 2\left(\Delta f(u), \Delta^2 u\right). \tag{2.41}$$

Using (2.13) and Young's inequality, we find

$$\begin{aligned} \left|(g(u), \Delta^2 u)\right| &\leq \|g(u)\| \ \left\|\Delta^2 u\right\| \\ &\leq c_g\left\|\Delta^2 u\right\| \\ &\leq \frac{1}{4}\left\|\Delta^2 u\right\|^2 + c_g^2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \left|(\Delta f(u), \Delta^2 u)\right| &\leq \|\Delta f(u)\| \ \left\|\Delta^2 u\right\| \\ &\leq \|\Delta f(u)\|^2 + \frac{1}{4}\left\|\Delta^2 u\right\|^2. \end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2+\|\Delta^2 u\|^2\leq 2\|\Delta f(u)\|^2+2c_g^2. \tag{2.42}$$

Using the chain rule (2.20), assumptions (2.2) and (2.3) and interpolation inequalities, we obtain (see [25,32])

$$\begin{aligned}\|\Delta f(u)\| &\leq \|f'(u)\|_{L^\infty(\Omega)}\ \|\Delta u\|+\|f''(u)\|_{L^\infty(\Omega)}\ \|\nabla u\|^2_{L^4(\Omega)}\\ &\leq L_f\|\Delta u\|+c_{f''}\|\nabla u\|^2_{L^4(\Omega)}\\ &\leq c'_{f'}\|u\|^{\frac{2}{3}}_{H^1(\Omega)}\|u\|^{\frac{1}{3}}_{H^4(\Omega)}+c'_{f''}\|u\|^2_{H^{\frac{7}{4}}},\quad H^{\frac{3}{4}}\subset L^4(\Omega)\\ &\leq c'_{f'}\|u\|^{\frac{2}{3}}_{H^1(\Omega)}\|u\|^{\frac{1}{3}}_{H^4(\Omega)}+c''_{f''}\|u\|^{\frac{3}{4}}_{H^1(\Omega)}\|u\|^{\frac{1}{4}}_{H^4(\Omega)}\\ &\leq (\text{Thanks to estimate (2.31)})\\ &\leq c''_{f'}(R_1)\|u\|^{\frac{1}{3}}_{H^4(\Omega)}+c'''_{f''}(R_1)\|u\|^{\frac{1}{4}}_{H^4(\Omega)},\quad t\geq 0,\end{aligned}$$

where $R_1=\|\nabla u(0)\|$. Since the norm $v\mapsto\left\|\Delta^2 v\right\|$ is equivalent to the usual $H^4(\Omega)$-norm in $D\left(A^2\right)$, we have

$$\|\Delta f(u)\,\|^2\leq c_{10}\|\,\Delta^2 u\|^{\frac{2}{3}}+c_{11}\left\|\Delta^2 u\right\|^{\frac{1}{2}}, \tag{2.43}$$

where $c_{10}=c_{10}\left(f',R_1\right)$ and $c_{11}=c_{11}\left(f'',R_1\right)$. Hence, by Young's inequality,

$$\begin{aligned}\|\Delta f(u)\|^2&\leq\frac{1}{8}\left\|\Delta^2 u\right\|^2+\frac{2}{3}\left(\frac{8}{3}\right)^{\frac{1}{2}}c_{10}^{\frac{3}{2}}+\frac{1}{8}\left\|\Delta^2 u\right\|^2+\frac{3}{4}\times 2^{\frac{1}{3}}c_{11}^{\frac{4}{3}}\\ &\leq\frac{1}{4}\left\|\Delta^2 u\right\|^2+c_{12},\quad c_{12}=c_{12}(c_{10},c_{11})\geq 0.\end{aligned} \tag{2.44}$$

This estimate, combined with (2.42), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(t)\|^2+\frac{1}{2}\left\|\Delta^2 u(t)\right\|^2 ds\leq c_{13},\quad c_{13}=2c_g^2+c_{12}(R_1)\geq 0,\quad t\geq 0. \tag{2.45}$$

Using the dissipative estimate (2.31), we choose a time

$$t_1=t_1(R_1)=\frac{\ln\left(R_1^2\right)}{\varepsilon_1} \tag{2.46}$$

such that $\|\nabla u(t)\|^2\leq C_1+M_1$, for all $t\geq t_1$, where the constants C_1 and M_1 are independent of R_1. Integrating (2.45) on the interval $[0,t]$ for $t\leq t_1$, we obtain

$$\|\Delta u(t)\|^2+\frac{1}{2}\int_0^t\left\|\Delta^2 u(s)\right\|^2\,\mathrm{d}s\leq\|\Delta u(0)\|^2+c_{13}t_1(R_1),\quad t\in[0,t_1]. \tag{2.47}$$

For $t\geq t_1(R_1)$, the constants c_{10} and c_{11} in (2.43) do not depend on R_1 and consequently, we obtain as in (2.45),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(t)\|^2+\frac{1}{2}\left\|\Delta^2 u(t)\right\|^2 ds\leq c_{13},\quad t\geq t_1(R_1),$$

where c_{13} does not depend on R_1. Since $D\left(A^2\right)\subset D(A)$ by (2.10), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(t)\|^2+c_{14}\left\|\Delta u(t)\right\|^2+\frac{1}{4}\|\Delta^2 u(t)\|^2\leq c_{13},\quad t\geq t_1(R_1), \tag{2.48}$$

where $c_{14} = 1/c_S(1,2)^2 > 0$ does not depend on R_1. Applying Gronwall's lemma, we obtain

$$\|\Delta u(t)\|^2 \le \|\Delta u(t_1)\|^2 e^{-c_{14}(t-t_1)} + \frac{c_{13}}{c_{14}}, \quad t \ge t_1(R_1). \tag{2.49}$$

By integration on $[t_1, t]$, we also deduce from (2.48) that

$$\frac{1}{4}\int_{t_1}^{t} \left\|\Delta^2 u(s)\right\|^2 \mathrm{d}s \le \|\Delta u(t_1)\|^2 + c_{13}(t-t_1), \quad t \ge t_1(R_1). \tag{2.50}$$

Estimate (2.40) follows from (2.47) and (2.50). From (2.47), we deduce that for $t \in [0, t_1(R_1)]$,

$$\begin{aligned} \|\Delta u(t)\|^2 &\le \|\Delta u(0)\|^2 + c_{13}(R_1)t_1(R_1) \\ &\le \left[\|\Delta u(0)\|^2 + c_{13}(R_1)t_1(R_1)\right] e^{c_{14}(t_1(R_1)-t)} \\ &\le Q(\|\Delta u(0)\|)e^{-c_{14}t}, \end{aligned} \tag{2.51}$$

where Q is a monotonic function of $\|\Delta u(0)\|$. In the last line above, we used that $R_1 = \|\nabla u(0)\|$, the continuous injection $V \subset D(A)$ (*cf.* (2.10)) and the fact that c_{13} and t_1 are increasing functions of R_1. The dissipative estimate (2.39) follows from (2.51) and (2.49). □

We deduce from Proposition 2.7 the existence of bounded absorbing set in $D(A)$ and consequently, of a global attractor associated with our semigroup $S_0(t)$ [32].

Theorem 2.8. *The semigroup $S_0(t)$ has a global attractor $\mathcal{A} \subset D(A)$ which is invariant ($S_0(t)\mathcal{A} = \mathcal{A}$), bounded in $H^2(\Omega)$, compact in $L^2(\Omega)$, and which attracts the bounded sets of $D(A)$ for the $L^2(\Omega)$-norm.*

The following estimate shows that the semigroup is Hölder continuous in time.

Lemma 2.9. *Let $T > 0$. If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then*

$$\|u(t_1) - u(t_2)\|^2 \le Q(T, \|u_0\|_2)|t_1 - t_2|, \quad \forall t_1, t_2 \in [0, T]. \tag{2.52}$$

Proof. We multiply (1.1) by u_t and integrate over Ω. We find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(t)\|^2 + 2\|u_t(t)\|^2 - 2(\Delta f(u), u_t) + 2(g(u), u_t) = 0. \tag{2.53}$$

Using the Cauchy-Schwarz inequality, (2.13) and Young's inequality, we obtain

$$\begin{aligned} |(g(u), u_t)| &\le \|g(u)\| \; \|u_t\| \\ &\le c_g \|u_t\| \\ &\le \frac{1}{4}\|u_t\|^2 + c_g^2. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |(\Delta f(u), u_t)| &\le \|\Delta f(u)\| \; \|u_t\| \\ &\le \frac{1}{4}\|u_t\|^2 + \|\Delta f(u)\|^2 \\ &\le (\text{Estimate (2.44)}) \\ &\le \frac{1}{4}\|u_t\|^2 + \frac{1}{2}\left\|\Delta^2 u\right\|^2 + c_{12}(R_1), \end{aligned}$$

where $R_1 = \|\nabla u(0)\|$. Combining the above estimates in (2.53), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(t)\|^2 + \|u_t(t)\|^2 \leq \left\|\Delta^2 u(t)\right\|^2 + 2c_{12}(R_1), \quad t \geq 0.$$

Integrating on $[0,T]$, we obtain

$$\|\Delta u(T)\|^2 + \int_0^T \|u_t(t)\|^2 dt \leq \int_0^T \left\|\Delta^2 u(t)\right\|^2 \,\mathrm{d}t + 2c_{12}(R_1)T.$$

Thus, by Proposition 2.7,

$$\int_0^T \|u_t(t)\|^2 \,\mathrm{d}t \leq Q(\|u(0)\|_2, T), \tag{2.54}$$

where Q is a continuous and monotonic function of its arguments.

Let $t_1, t_2 \in [0,T]$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|u(t_1) - u(t_2)\| &= \left\| \int_{t_1}^{t_2} u_t(s)\,\mathrm{d}s \right\| \\ &\leq \left| \int_{t_1}^{t_2} \|u_t(s)\| \,\mathrm{d}s \right| \\ &\leq |t_1 - t_2|^{\frac{1}{2}} \left| \int_{t_1}^{t_2} \|u_t(s)\|^2 \,\mathrm{d}s \right|^{\frac{1}{2}}. \end{aligned} \tag{2.55}$$

Lemma 2.9 follows from (2.54) and (2.55). ☐

2.4. Estimates for the difference of two solutions

Let now u_1 and u_2 be two solutions of system (1.1)–(1.3) with initial data $u_{0,1}$ and $u_{0,2}$, respectively. We set $u = u_1 - u_2$ and $u_0 = u_{0,1} - u_{0,2}$ and we have, for all $T > 0$,

$$u_t + A^2 u + A\left(f(u_1) - f(u_2)\right) + \left(g(u_1) - g(u_2)\right) = 0 \quad \text{in} \quad L^2\left(0,T; D(A^{-1})\right), \tag{2.56}$$

$$u|_{t=0} = u_0 \,(= u_{0,1} - u_{0,2}) \quad \text{in} \quad V. \tag{2.57}$$

We first prove:

Lemma 2.10 (Uniqueness)**.** *For all $t \geq 0$, we have*

$$\|u_1(t) - u_2(t)\|^2 + \int_0^t \|\Delta u(s)\|^2 \,\mathrm{d}s \leq e^{c_{f,g} t}\|u_{0,1} - u_{0,2}\|^2, \tag{2.58}$$

where the positive constant $c_{f,g}$ depends only on L_f and L_g.

Proof. On multiplying (2.56) by u in H, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + 2\|Au\|^2 + 2(f(u_1) - f(u_2), Au) + 2(g(u_1) - g(u_2), u) = 0. \tag{2.59}$$

From (2.14), we deduce that

$$\begin{aligned} |(g(u_1) - g(u_2), u)| &\leq \|g(u_1) - g(u_2)\| \; \|u\| \\ &\leq L_g \|u\|^2. \end{aligned}$$

Using (2.12) and Young's inequality, we find

$$\begin{aligned}|(f(u_1)-f(u_2),Au)| &\le \|f(u_1)-f(u_2)\|\ \|Au\|\\ &\le L_f\|u\|\ \|Au\|\\ &\le \frac{L_f^2}{2}\|u\|^2+\frac{1}{2}\|Au\|^2.\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\|Au(t)\|^2\le c_{f,g}\|u(t)\|^2,\quad c_{f,g}=\left(2L_g+L_f^2\right)>0,\quad t\ge 0. \tag{2.60}$$

We finally conclude (2.58) from (2.60) and Gronwall's lemma. □

Next, we show a L^2-H^1 smoothing property which is essential in the construction of an exponential attractor.

Lemma 2.11. *If $\|u_i(0)\|_2\le R_2$ (i=1,2), then for all $t>0$, we have*

$$\|u(t)\|_1^2\le \frac{c_S}{t}\exp(c(R_2)t)\|u_0\|^2. \tag{2.61}$$

Proof. We multiply (2.56) by $2tA^{-1}u_t$ in H. We deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}(t\|\nabla u\|^2)+2t\|u_t\|_{-1}^2+2t\left(g(u_1)-g(u_2),(-\Delta)^{-1}u_t\right)+2t\left(f(u_1)-f(u_2),u_t\right)=\|\nabla u\|^2. \tag{2.62}$$

Using (2.14), (2.10), Poincaré's inequality (2.11) and Young's inequality, we have

$$\begin{aligned}\left|(g(u_1)-g(u_2),(-\Delta)^{-1}u_t)\right| &\le \|g(u_1)-g(u_2)\|\ \left\|(-\Delta)^{-1}u_t\right\|\\ &\le L_g\|u\|\ c_S\|u_t\|_{-1}\\ &\le L_gc_Sc_P\|\nabla u\|\ \|u_t\|_{-1}\\ &\le L_g^2c_S^2c_P^2\|\nabla u\|^2+\frac{1}{4}\|u_t\|_{-1}^2.\end{aligned}$$

Next, we use the Cauchy-Schwarz inequality and (2.9):

$$\begin{aligned}|(f(u_1)-f(u_2),u_t)| &= \left|(A^{\frac{1}{2}}(f(u_1)-f(u_2)),A^{-\frac{1}{2}}u_t)\right|\\ &\le \left\|\nabla\left(\int_0^1 f'(u_1+s(u_2-u_1))\,\mathrm{d}su\right)\right\|\ \|u_t\|_{-1}\\ &\le \left\|\int_0^1 f'(u_1+s(u_2-u_1))\,\mathrm{d}s\nabla u\right\|\ \|u_t\|_{-1}\\ &\quad+\left\|\int_0^1 f''(u_1+s(u_2-u_1))(\nabla u_1+s\nabla(u_2-u_1))\,\mathrm{d}su\right\|\ \|u_t\|_{-1}\end{aligned}$$

Consequently, using (2.2) and (2.3), we have

$$\begin{aligned}|(f(u_1)-f(u_2),u_t)| &\le c_f\left(\|\nabla u\|+\||u|\,|\nabla u_1|\|+\||u|\,|\nabla u_2|\|\right)\|u_t\|_{-1}\\ &\le c_f(\|\nabla u\|+\|u\|_{L^4(\Omega)}\left(\|\nabla u_1\|_{L^4(\Omega)}+\|\nabla u_2\|_{L^4(\Omega)}\right)\|u_t\|_{-1}\\ &\le \text{(thanks to the continuous embedding } H^1(\Omega)\subset L^4(\Omega))\\ &\le c(\|\nabla u\|+\|\nabla u\|\left(\|u_1\|_{H^2(\Omega)}+\|u_2\|_{H^2(\Omega)}\right)\|u_t\|_{-1}\\ &\le \text{(since } u_1,\ u_2 \text{ are bounded in } H^2(\Omega) \text{ by (2.39))}\\ &\le c(R_2)\|\nabla u\|^2+\frac{1}{4}\|u_t\|_{-1}^2.\end{aligned} \tag{2.63}$$

Combining the above estimates in (2.62), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|\nabla u(t)\|^2\right)+t\|u_t(t)\|_{-1}^2 \leq c't\|\nabla u(t)\|^2+\|\nabla u(t)\|^2, \quad t\geq 0,$$

where $c'=2L_g^2c_S^2c_P^2+2c(R_2)$. By Gronwall's lemma,

$$t\|\nabla u(t)\|^2 \leq e^{c't}\int_0^t \|\nabla u(s)\|^2\,\mathrm{d}s, \quad t\geq 0.$$

We conclude from (2.58) and (2.10) that (2.61) holds with $c(R_2)=c'+c_{f,g}$. □

3. The time semidiscrete problem

3.1. The discrete semigroup

For the time semidiscretization, we apply the semi-implicit Euler scheme to (1.1). In the remainder of the manuscript, $\delta t>0$ denotes the time step. The scheme reads : let $u^0\in D(A)=H^2(\Omega)\cap H_0^1(\Omega)$ and for $n=0,1,2,\cdots$, let $u^{n+1}\in D(A)$ solve

$$\frac{u^{n+1}-u^n}{\delta t}+A^2u^{n+1}+Af\left(u^n\right)+g\left(u^n\right)=0. \tag{3.1}$$

This is a linear scheme known as the IMEX (Implicit-Explicit) scheme: at each time step, u^{n+1} is computed by solving a linear system whose right-hand side involves u^n. By elliptic regularity, for $u^n\in D(A)$, u^{n+1} is unique and belongs to $D(A)$. The following result shows that the discrete semigroup $S_{\delta t}^n u^0=u^n$ is well-defined on $D(A)$.

Theorem 3.1. *Assume that $\delta t\leq 1/(2L_g)$, where L_g is the constant in (2.14). Then for every $u^n\in D(A)$, there exists a unique $u^{n+1}\in D(A)$ which solves (3.1). Moreover, the mapping $S_{\delta t}:u^n\mapsto u^{n+1}$ is Lipschitz continuous for the $L^2(\Omega)$-norm from $D(A)$ into $D(A)$.*

The Lipschitz continuity in $L^2(\Omega)$ follows from Lemma 3.7.

The following regularity result will prove useful for the dissipative estimate in $H^2(\Omega)$.

Lemma 3.2. *If $u^n\in D(A)$, then $u^{n+1}=S_{\delta t}u^n$ belongs to $D\left(A^2\right)$ and*

$$\delta t\left\|A^2u^{n+1}\right\|^2\leq 2\left\|\Delta u^n\right\|^2+C\delta t\left(\left\|\Delta u^n\right\|^4+1\right), \tag{3.2}$$

where the positive constant C is independent of δt and u^n. Moreover,

$$\left\|\Delta u^{n+1}\right\|^2\leq\left\|\Delta u^n\right\|^2+C\delta t\left(\left\|\Delta u^n\right\|^4+1\right). \tag{3.3}$$

Proof. By (3.1), u^{n+1} solves

$$u^{n+1}-u^n+\delta tA^2u^{n+1}=\delta t\,h \tag{3.4}$$

where $h=\Delta f\left(u^n\right)-g\left(u^n\right)$. By the chain rule (2.20), $h\in L^2(\Omega)$ with

$$\begin{aligned}\|h\| &\leq \|f'(u^n)\|_{L^\infty(\Omega)}\left\|\Delta u^n\right\|+\|f''(u^n)\|_{L^\infty(\Omega)}\left\|\nabla u^n\right\|_{L^4(\Omega)}^2+\|g(u^n)\|\\ &\leq \left(H^1(\Omega)\subset L^4(\Omega)\right),\ (2.2),\ (2.3)\text{ and } (2.13))\\ &\leq L_f\left\|\Delta u^n\right\|+c_{f''}c_S^2\left\|\Delta u^n\right\|^2+c_g.\end{aligned} \tag{3.5}$$

Since u^n, u^{n+1} and h belong to $L^2(\Omega)$, we deduce from (3.4) that $u^{n+1} \in D\left(A^2\right)$. Next, we take the L^2-scalar product of (3.4) with $u^{n+1} - u^n$. This yields

$$\begin{aligned}\left\|u^{n+1}-u^n\right\|^2+\delta t\left\|\Delta u^{n+1}\right\|^2 &= \delta t\left(\Delta u^{n+1}, \Delta u^n\right)+\delta t\left(h, u^{n+1}-u^n\right)\\ &\leq \delta t\left\|\Delta u^{n+1}\right\|\left\|\Delta u^n\right\|+\delta t\|h\|\left\|u^{n+1}-u^n\right\|.\end{aligned}$$

From Young's inequality, we deduce that

$$\left\|u^{n+1}-u^n\right\|^2+\delta t\left\|\Delta u^{n+1}\right\|^2 \leq \delta t\left\|\Delta u^n\right\|^2+\delta t^2\|h\|^2. \tag{3.6}$$

Now, we take the L^2-scalar product of (3.4) with $A^2 u^{n+1}$. We find

$$\begin{aligned}\delta t\left\|A^2 u^{n+1}\right\|^2 &= -\left(u^{n+1}-u^n, A^2 u^{n+1}\right)+\delta t\left(h, A^2 u^{n+1}\right)\\ &\leq \frac{1}{\delta t}\left\|u^{n+1}-u^n\right\|^2+\frac{\delta t}{4}\left\|A^2 u^{n+1}\right\|^2+\delta t\|h\|^2+\frac{\delta t}{4}\left\|A^2 u^{n+1}\right\|^2.\end{aligned}$$

Thus, by (3.6),

$$\delta t\left\|A^2 u^{n+1}\right\|^2 \leq 2\|\Delta u^n\|^2+4\delta t\|h\|^2.$$

Using (3.5), we find (3.2). From (3.6) and (3.5), we also deduce that

$$\begin{aligned}\left\|\Delta u^{n+1}\right\|^2 &\leq \left\|\Delta u^n\right\|^2+\delta t\|h\|^2\\ &\leq \left\|\Delta u^n\right\|^2+C\delta t\left(\left\|\Delta u^n\right\|^4+1\right).\end{aligned}$$

This is (3.3). □

3.2. Dissipative estimates, uniform with respect to the time step

In this subsection, we first establish some priori estimates for the scheme (3.1). The following well-known identity will be frequently used:

$$(a-b, a)=\frac{1}{2}\left(\|a\|^2-\|b\|^2+\|a-b\|^2\right), \quad a, b \in L^2(\Omega) \tag{3.7}$$

In (3.7), we may also replace the inner product and the norm in $L^2(\Omega)$ by another inner product and the norm associated to it. We recall a discrete Gronwall lemma.

Lemma 3.3. *Let $C, \gamma > 0$ and (a_n), (b_n) be two sequences of nonnegative real numbers such that*

$$a_{n+1}+\delta t b_{n+1} \leq (1-\gamma\delta t) a_n+\delta t C, \quad \forall n \geq 0, \tag{3.8}$$

where $\delta t \in (0, 1/(2\gamma)]$. Then for all $n \geq 0$, we have

$$a_n \leq e^{-n\gamma\delta t} a_0+\frac{C}{\gamma} \tag{3.9}$$

and

$$\delta t \sum_{k=0}^{n-1} b_{k+1} \leq a_0+n\delta t C. \tag{3.10}$$

By convention, for $n = 0$, the sum in the left-hand side of (3.10) is zero.

Proof. Since

$$a_{n+1} \leq (1-\gamma\delta t)a_n + \delta t C, \quad \forall n \geq 0,$$

we find by induction that for all $n \geq 0$,

$$\begin{aligned} a_n &\leq (1-\gamma\delta t)^n a_0 + \delta t C \sum_{k=0}^{n-1} (1-\gamma\delta t)^k \\ &\leq (1-\gamma\delta t)^n a_0 + \frac{C}{\gamma}\cdot \end{aligned}$$

By convexity, we have $1-s \leq e^{-s}$ for all $s \geq 0$. Thus, for all $n \geq 0$,

$$a_n \leq e^{-n\gamma\delta t} a_0 + \frac{C}{\gamma}\cdot$$

This is (3.9). By (3.8), we also have

$$a_{n+1} + \delta t b_{n+1} \leq a_n + \delta t C, \quad \forall n \geq 0.$$

By summing from $n = 0$ to $n = N-1$, we find

$$a_N + \delta t \sum_{n=0}^{N-1} b_{n+1} \leq a_0 + N\delta t C.$$

This yields (3.10). □

Proposition 3.4 (Dissipative estimate in $L^2(\Omega)$)**.** *If δt is small enough, then*

$$\|u_n\|^2 \leq C_0\|u_0\|^2 e^{-\varepsilon_0 n\delta t} + M_0, \quad \forall n \geq 0, \tag{3.11}$$

and

$$\delta t \sum_{k=0}^{n-1} \|\Delta u_{k+1}\|^2 \leq C_0'\|u_0\|^2 + n\delta t M_0', \quad \forall n \geq 0, \tag{3.12}$$

where the positive constants C_0, ε_0, M_0, C_0' and M_0' are independent of u_0 and δt.

Proof. We multiply (3.1) by $A^{-1}u^{n+1}$ in H. Using (3.7), we have

$$\begin{aligned} \frac{1}{2\delta t}\left(\left\|u^{n+1}\right\|_{-1}^2 - \|u^n\|_{-1}^2 + \left\|u^{n+1}-u^n\right\|_{-1}^2\right) + \left\|\nabla u^{n+1}\right\|^2 + \left(f\left(u^{n+1}\right), u^{n+1}\right) \\ + \left(f(u^n) - f\left(u^{n+1}\right), u^{n+1}\right) + \left(g\left(u^n\right), (-\Delta)^{-1} u^{n+1}\right) = 0. \end{aligned} \tag{3.13}$$

Thanks to (2.18), we have

$$\left(f\left(u^{n+1}\right), u^{n+1}\right) \geq \gamma_5 \left\|u^{n+1}\right\|^2 - \gamma_6|\Omega|.$$

Using the Cauchy-Schwarz inequality, (2.12) and Young's inequality, we find

$$\begin{aligned} \left|\left(f(u^n) - f\left(u^{n+1}\right), u^{n+1}\right)\right| &\leq \left\|f\left(u^n\right) - f\left(u^{n+1}\right)\right\| \left\|u^{n+1}\right\| \\ &\leq L_f \left\|u^n - u^{n+1}\right\| \left\|u^{n+1}\right\| \\ &\leq \frac{\gamma_5}{4}\left\|u^{n+1}\right\|^2 + \frac{L_f^2}{\gamma_5}\left\|u^n - u^{n+1}\right\|^2. \end{aligned}$$

Moreover, by (2.10) and (2.13),

$$\begin{aligned}\left|\left(g\left(u^{n}\right),(-\Delta)^{-1}u^{n+1}\right)\right| &\leq \|g\left(u^{n}\right)\| \;\; \|(-\Delta)^{-1}u^{n+1}\| \\ &\leq c_g c_S \left\|u^{n+1}\right\| \\ &\leq \frac{\gamma_5}{4}\left\|u^{n+1}\right\|^2 + \frac{c_g^2 c_S^2}{\gamma_5}.\end{aligned}$$

Let us combine the above estimates in (3.13). We find

$$\begin{aligned}&\frac{1}{2\delta t}\left\|u^{n+1}\right\|_{-1}^2 + \frac{\gamma_5}{2}\left\|u^{n+1}\right\|^2 + \left\|\nabla u^{n+1}\right\|^2 + \frac{1}{2\delta t}\left\|u^{n+1}-u^n\right\|_{-1}^2 \\ &\leq \frac{1}{2\delta t}\left\|u^n\right\|_{-1}^2 + \frac{L_f^2}{\gamma_5}\left\|u^{n+1}-u^n\right\|^2 + \frac{c_g^2 c_S^2}{\gamma_5} + \gamma_6|\Omega|. \qquad (3.14)\end{aligned}$$

Now, we multiply (3.1) by u^{n+1} in H. We obtain

$$\begin{aligned}&\frac{1}{2\delta t}\left(\left\|u^{n+1}\right\|^2 - \left\|u^n\right\|^2 + \left\|u^{n+1}-u^n\right\|^2\right) + \left\|\Delta u^{n+1}\right\|^2 \\ &-\left(f\left(u^n\right),\Delta u^{n+1}\right) + \left(g\left(u^n\right),u^{n+1}\right) = 0. \qquad (3.15)\end{aligned}$$

By (2.12), (2.1) and Young's inequality, we have

$$\begin{aligned}\left|\left(f\left(u^n\right),\Delta u^{n+1}\right)\right| &\leq \|f(u^n)\| \;\; \left\|\Delta u^{n+1}\right\| \\ &\leq L_f\left\|u^n\right\| \;\; \left\|\Delta u^{n+1}\right\| \\ &\leq \frac{L_f^2}{2}\left\|u^n\right\|^2 + \frac{1}{2}\left\|\Delta u^{n+1}\right\|^2.\end{aligned}$$

Owing to (2.13) and Young's inequality, we have

$$\begin{aligned}\left|\left(g\left(u^n\right),u^{n+1}\right)\right| &\leq \|g\left(u^n\right)\| \;\; \left\|u^{n+1}\right\| \\ &\leq c_g\left\|u^{n+1}\right\| \\ &\leq \frac{1}{4}\left\|u^{n+1}\right\|^2 + c_g^2.\end{aligned}$$

Combining the above estimates in (3.15), we obtain

$$\frac{1}{2\delta t}\left\|u^{n+1}\right\|^2 + \frac{1}{2\delta t}\left\|u^{n+1}-u^n\right\|^2 + \frac{1}{2}\left\|\Delta u^{n+1}\right\|^2 \leq \frac{1}{2\delta t}\left\|u^n\right\|^2 + \frac{L_f^2}{2}\left\|u^n\right\|^2 + \frac{1}{4}\left\|u^{n+1}\right\|^2 + c_g^2. \qquad (3.16)$$

On summing (3.14) and α (3.16) with $\alpha > 0$, we conclude that

$$\begin{aligned}&\frac{1}{2\delta t}\left\|u^{n+1}\right\|_{-1}^2 + \frac{\alpha}{2\delta t}\left\|u^{n+1}\right\|^2 + \left(\frac{\gamma_5}{2} - \frac{\alpha}{4}\right)\left\|u^{n+1}\right\|^2 + \left\|\nabla u^{n+1}\right\|^2 \\ &+ \frac{1}{2\delta t}\left\|u^{n+1}-u^n\right\|_{-1}^2 + \frac{\alpha}{2\delta t}\left\|u^{n+1}-u^n\right\|^2 + \frac{\alpha}{2}\left\|\Delta u^{n+1}\right\|^2 \\ &\quad\leq \frac{1}{2\delta t}\left\|u^n\right\|_{-1}^2 + \frac{\alpha}{2\delta t}\left\|u^n\right\|^2 + \frac{L_f^2}{\gamma_5}\left\|u^{n+1}-u^n\right\|^2 + \alpha\frac{L_f^2}{2}\left\|u^n\right\|^2 + c', \qquad (3.17)\end{aligned}$$

where $c' = \dfrac{c_g^2 c_S^2}{\gamma_5} + \gamma_6|\Omega| + \alpha c_g^2$. Now, we use the Poincaré inequality (2.11) for the term $\|\nabla u^{n+1}\|^2$, we choose $\alpha > 0$ small enough so that $\alpha \leq \gamma_5$ and $\alpha L_f^2/2 \leq 1/c_P^2$, and we set

$$a_n = \|u^n\|_{-1}^2 + \alpha\|u^n\|^2 + \frac{2\delta t}{c_P^2}\|u^n\|^2. \qquad (3.18)$$

Then (3.17) yields

$$\frac{1}{2\delta t}a_{n+1}+\frac{\gamma_5}{4}\left\|u^{n+1}\right\|^2+\frac{\alpha}{2\delta t}\left\|u^{n+1}-u^n\right\|^2+\frac{\alpha}{2}\left\|\Delta u^{n+1}\right\|^2\le\frac{1}{2\delta t}a_n+\frac{L_f^2}{\gamma_5}\left\|u^{n+1}-u^n\right\|^2+c'.$$

Then, for δt small enough so that $\alpha/(2\delta t)\ge L_f^2/\gamma_5$, we obtain

$$\frac{1}{2\delta t}a_{n+1}+\frac{\gamma_5}{4}\left\|u^{n+1}\right\|^2+\frac{\alpha}{2}\left\|\Delta u^{n+1}\right\|^2\le\frac{1}{2\delta t}a_n+c'.$$

By (2.10),

$$a_n\le\left(c_S^2+\alpha+\frac{2\delta t^\star}{c_P^2}\right)\left\|u^n\right\|^2, \tag{3.19}$$

where $\delta t^\star$ is the maximum value of the time step. This yields

$$\frac{1}{2\delta t}a_{n+1}+\frac{c_1'}{2}a_{n+1}+\frac{\alpha}{2}\left\|\Delta u^{n+1}\right\|^2\le\frac{1}{2\delta t}a_n+c',\quad c_1'=\gamma_5/\left(2c_S^2+2\alpha+\frac{4\delta t^\star}{c_P^2}\right)>0.$$

Thus,

$$a_{n+1}+\frac{\alpha\delta t}{1+c_1'\delta t}\left\|\Delta u^{n+1}\right\|^2\le\frac{1}{1+c_1'\delta t}a_n+\frac{2\delta t}{1+c_1'\delta t}c'.$$

Next, we use that

$$\frac{1}{2}\le\frac{1}{1+s}\le 1-\frac{s}{2},\quad\forall s\in[0,1]. \tag{3.20}$$

By choosing $s=c_1'\delta t$ we obtain that for δt small enough ($\delta t\le 1/c_1'$),

$$a_{n+1}+\frac{\alpha}{2}\delta t\left\|\Delta u^{n+1}\right\|^2\le\left(1-\frac{c_1'}{2}\delta t\right)a_n+2c'\delta t. \tag{3.21}$$

We deduce from Lemma 3.3 that for all $n\ge 0$,

$$a_n\le e^{-c_1'n\delta t/2}a_0+\frac{4c'}{c_1'}$$

and

$$\frac{\alpha}{2}\delta t\sum_{k=0}^{n-1}\left\|\Delta u^{k+1}\right\|^2\le a_0+2c'n\delta t.$$

Finally, we note that $a_n\ge\alpha\left\|u^n\right\|^2$. These estimates, together with (3.19), conclude the proof of Proposition 3.4. ☐

Proposition 3.5 (Dissipative estimate in $H^1(\Omega)$)**.** *If δt is small enough, then*

$$\left\|\nabla u^n\right\|^2\le C_1e^{-\varepsilon_1 n\delta t}\left\|\nabla u^0\right\|^2+M_1,\quad\forall n\ge 0, \tag{3.22}$$

and

$$\frac{1}{\delta t}\sum_{k=0}^{n-1}\left\|u^{k+1}-u^k\right\|_{-1}^2\le C_1'\left\|\nabla u^0\right\|^2+M_1'(1+n\delta t),\quad\forall n\ge 0,$$

where the positive constants C_1, ε_1, M_1, C_1' and M_1' are independent of u_0 and δt.

Proof. We multiply (3.1) by $A^{-1}\dfrac{u^{n+1}-u^n}{\delta t}$ in H. We obtain

$$\begin{aligned} &\frac{1}{\delta t^2}\left\|u^{n+1}-u^n\right\|_{-1}^2+\frac{1}{2\delta t}\left(\left\|\nabla u^{n+1}\right\|^2-\left\|\nabla u^n\right\|^2+\left\|\nabla\left(u^{n+1}-u^n\right)\right\|^2\right)\\ &+\frac{1}{\delta t}\left(F\left(u^{n+1}\right)-F\left(u^n\right),1\right)+\left(g\left(u^n\right),(-\Delta)^{-1}\frac{u^{n+1}-u^n}{\delta t}\right)\\ &\quad=\frac{1}{2\delta t}\int_\Omega f'\left(\zeta_{u^{n+1},u^n}\right)\left(u^{n+1}-u^n\right)^2\,\mathrm{d}x. \end{aligned} \tag{3.23}$$

Here, we used that for all $r,s\in\mathbb{R}$,

$$F(s)=F(r)+f(r)(s-r)+f'(\xi_{s,r})\frac{(s-r)^2}{2},\quad \text{for some } \xi_{s,r}\in[r,s],$$

and $(r,s)\mapsto f'(\xi_{s,r})$ is a continuous function on $\mathbb{R}^2$, since $F\in C^2(\mathbb{R})$.

Using (2.2) and Young's inequality, we obtain

$$\begin{aligned} \left|\int_\Omega f'\left(\zeta_{u^{n+1},u^n}\right)\left(u^{n+1}-u^n\right)^2\,\mathrm{d}x\right| &\le \|f'\|_{L^\infty(\Omega)}\left\|u^{n+1}-u^n\right\|^2\\ &\le L_f\left\|u^{n+1}-u^n\right\|^2\\ &\le L_f\left\|\nabla\left(u^{n+1}-u^n\right)\right\|\left\|u^{n+1}-u^n\right\|_{-1}\\ &\le \frac{1}{2}\left\|\nabla\left(u^{n+1}-u^n\right)\right\|^2+\frac{L_f^2}{2}\left\|u^{n+1}-u^n\right\|_{-1}^2. \end{aligned}$$

In the third inequality above, we used the interpolation inequality

$$\|v\|^2=\left(A^{1/2}v,A^{-1/2}v\right)\le\|v\|_1\|v\|_{-1},\quad \forall v\in V.$$

By (2.13) and (2.10),

$$\begin{aligned} \left|\left(g\left(u^n\right),(-\Delta)^{-1}\frac{u^{n+1}-u^n}{\delta t}\right)\right| &\le \left\|g\left(u^n\right)\right\|\left\|(-\Delta)^{-1}\frac{u^{n+1}-u^n}{\delta t}\right\|\\ &\le c_gc_S\left\|\frac{u^{n+1}-u^n}{\delta t}\right\|_{-1}\\ &\le \frac{1}{2\delta t^2}\left\|u^{n+1}-u^n\right\|_{-1}^2+\frac{c_g^2c_S^2}{2}. \end{aligned}$$

Combining the above estimates in (3.23), we get

$$\begin{aligned} &\frac{1}{2\delta t}\left\|\nabla u^{n+1}\right\|^2+\frac{1}{\delta t}\left(F\left(u^{n+1}\right),1\right)+\frac{1}{2\delta t^2}\left\|u^{n+1}-u^n\right\|_{-1}^2+\frac{1}{4\delta t}\left\|\nabla(u^{n+1}-u^n)\right\|^2\\ &\quad\le\frac{1}{2\delta t}\left\|\nabla u^n\right\|^2+\frac{1}{\delta t}\left(F\left(u^n\right),1\right)+\frac{L_f^2}{4\delta t}\left\|u^{n+1}-u^n\right\|_{-1}^2+\frac{c_g^2c_S^2}{2}. \end{aligned} \tag{3.24}$$

Let δt be small enough so that (3.21) holds and $\delta t\le 1/L_f^2$. Adding (3.21) and $2\delta t\beta$ times (3.24) where $\beta>0$, we find that

$$E_2^{n+1}+\frac{\alpha}{2}\delta t\left\|\Delta u^{n+1}\right\|^2+\frac{\beta}{2\delta t}\left\|u^{n+1}-u^n\right\|_{-1}^2\le E_2^n+\left(2c'+\beta c_g^2c_S^2\right)\delta t, \tag{3.25}$$

where

$$E_2^n=a_n+\beta\left\|\nabla u^n\right\|^2+2\beta\left(F\left(u^n\right),1\right)$$

and a_n is defined by (3.18). By (2.16), $F(s)+\gamma_2 \geq 0$ for all $s \in \mathbb{R}$, so that

$$\tilde{E}_2^n = E_2^n + 2\beta(\gamma_2, 1) \geq 0, \quad \forall n \geq 0.$$

Moreover, by (2.17), (3.19) and the Poincaré inequality, we have

$$\tilde{E}_2^n \leq c_1 \left\|\nabla u^n\right\|^2 + c_2, \quad c_1,\ c_2 > 0. \tag{3.26}$$

By (2.10),

$$\left\|\Delta u^n\right\|^2 \geq \frac{1}{c_1 c_S}\tilde{E}_2^n - \frac{c_2}{c_1 c_S},$$

so that (3.25) yields

$$\tilde{E}_2^{n+1} + c_3\delta t \tilde{E}_2^{n+1} + \frac{\beta}{2\delta t}\left\|u^{n+1}-u^n\right\|_{-1}^2 \leq \tilde{E}_2^n + c_4\delta t,$$

where $c_3 = \alpha/(2c_1c_S)$ and $c_4 = 2c' + \beta c_g^2 c_S^2 + \alpha c_2/(2c_1c_S)$. Thus,

$$\tilde{E}_2^{n+1} + \frac{1}{1+c_3\delta t}\frac{\beta}{2\delta t}\left\|u^{n+1}-u^n\right\|_{-1}^2 \leq \frac{1}{1+c_3\delta t}\tilde{E}_2^n + \frac{c_4}{1+c_3\delta t}\delta t.$$

By (3.20), for δt small enough ($\delta t \leq 1/c_3$), we have

$$\tilde{E}_2^{n+1} + \frac{\beta\delta t}{4}\left\|\frac{u^{n+1}-u^n}{\delta t}\right\|_{-1}^2 \leq \left(1 - \frac{c_3}{2}\delta t\right)\tilde{E}_2^n + c_4\delta t.$$

We may apply Lemma 3.3, which yields

$$\tilde{E}_2^n \leq e^{-nc_3\delta t/2}\tilde{E}_2^0 + \frac{2c_4}{c_3} \tag{3.27}$$

and

$$\frac{\beta}{4\delta t}\sum_{k=0}^{n-1}\left\|u^{k+1}-u^k\right\|_{-1}^2 \leq \tilde{E}_2^0 + n\delta t c_4, \tag{3.28}$$

for all $n \geq 0$. Finally, we note that

$$\tilde{E}_2^n \geq \beta\left\|\nabla u^n\right\|^2,$$

and this estimate, together with (3.26)–(3.28), concludes the proof. □

Proposition 3.6 (Dissipative estimate in $H^2(\Omega)$)**.** *For δt small enough, we have*

$$\left\|\Delta u^n\right\|^2 \leq Q_2\left(\left\|\Delta u^0\right\|\right)e^{-\varepsilon_2 n\delta t} + M_2, \quad \forall n \geq 0, \tag{3.29}$$

and

$$\sum_{k=0}^{n-1}\left\|\Delta\left(u^{k+1}-u^k\right)\right\|^2 \leq Q_2\left(\left\|\Delta u^0\right\|\right) + M_2' n\delta t, \quad n \geq 0, \tag{3.30}$$

where Q_2 is a monotonic function and the positive constants ε_2, M_2 and M_2' are independent of u^0 and δt.

Proof. By Lemma 3.2, we know that for all $n \geq 1$, $u^n \in D\left(A^2\right)$. By multiplying (3.1) by A^2u^{n+1} in H and using (3.7), we obtain

$$\begin{aligned}&\frac{1}{2\delta t}\left(\left\|\Delta u^{n+1}\right\|^2 - \left\|\Delta u^n\right\|^2 + \left\|\Delta\left(u^{n+1}-u^n\right)\right\|^2\right) + \left\|\Delta^2 u^{n+1}\right\|^2\\ &= -\left(g\left(u^n\right), \Delta^2 u^{n+1}\right) + \left(\Delta f\left(u^n\right), \Delta^2 u^{n+1}\right).\end{aligned} \tag{3.31}$$

Using (2.13) and Young's inequality yields

$$\begin{aligned}\left|\left(g\left(u^n\right), \Delta^2 u^{n+1}\right)\right| &\leq \|g(u^n)\| \ \left\|\Delta^2 u^{n+1}\right\| \\ &\leq c_g \left\|\Delta^2 u^{n+1}\right\| \\ &\leq \frac{1}{4}\left\|\Delta^2 u^{n+1}\right\|^2 + c_g^2.\end{aligned}$$

Similarly, we have

$$\left|\left(\Delta f\left(u^n\right), \Delta^2 u^{n+1}\right)\right| \leq \|\Delta f\left(u^n\right)\| \ \left\|\Delta^2 u^{n+1}\right\| \leq \frac{1}{4}\left\|\Delta^2 u^{n+1}\right\|^2 + \|\Delta f\left(u^n\right)\|^2.$$

Therefore, we have

$$\frac{1}{2\delta t}\left(\left\|\Delta u^{n+1}\right\|^2 - \|\Delta u^n\|^2 + \left\|\Delta\left(u^{n+1}-u^n\right)\right\|^2\right) + \frac{1}{2}\left\|\Delta^2 u^{n+1}\right\|^2 \leq \|\Delta f\left(u^n\right)\|^2 + c_g^2. \tag{3.32}$$

Arguing as in the continuous case (*cf.* (2.44)) and using the H^1-estimate (3.22), we find that for all $n \geq 1$, we have

$$\|\Delta f\left(u^n\right)\| \leq \frac{1}{4}\left\|\Delta^2 u^n\right\|^2 + c_{12}(R_1), \tag{3.33}$$

where c_{12} depends on $R_1 = \|\nabla u(0)\|$. The estimate (3.32) becomes

$$\frac{1}{2\delta t}\left\|\Delta u^{n+1}\right\|^2 + \frac{1}{2}\left\|\Delta^2 u^{n+1}\right\|^2 + \frac{1}{2\delta t}\left\|\Delta\left(u^{n+1}-u^n\right)\right\|^2 \leq \frac{1}{2\delta t}\|\Delta u^n\|^2 + \frac{1}{4}\left\|\Delta^2 u^n\right\|^2 + c_5,$$

where $c_5(R_1) = c_g^2 + c_{12}(R_1)$. We multiply this estimate by $2\delta t$ and we obtain

$$\tilde{a}_{n+1} + \frac{\delta t}{2}\left\|\Delta^2 u^{n+1}\right\|^2 + \left\|\Delta\left(u^{n+1}-u^n\right)\right\|^2 \leq \tilde{a}_n + 2c_5\delta t, \tag{3.34}$$

where

$$\tilde{a}_n = \|\Delta u^n\|^2 + \frac{\delta t}{2}\left\|\Delta^2 u^n\right\|^2.$$

Using that δt is bounded from above and that (2.10) holds for $s_1 = 1$ and $s_2 = 2$, we find

$$\tilde{a}_n \leq c_6\left\|\Delta^2 u^n\right\|^2, \quad \forall n \geq 1$$

for some constant $c_6 > 0$ independent of δt. Thus, (3.34) implies

$$\tilde{a}_{n+1} + \frac{\delta t}{2c_6}\tilde{a}_{n+1} + \|\Delta(u^{n+1}-u^n)\|^2 \leq \tilde{a}_n + 2c_5\delta t, \quad \forall n \geq 1.$$

Therefore, by (3.20), for $\delta t \leq 2c_6$, we have

$$\tilde{a}_{n+1} + \frac{1}{2}\left\|\Delta\left(u^{n+1}-u^n\right)\right\|^2 \leq (1 - c_7\delta t)\tilde{a}_n + 2c_5\delta t \quad \forall n \geq 1, \tag{3.35}$$

where $c_7 = 1/(4c_6)$. By Lemma 3.3,

$$\tilde{a}_n \leq e^{-(n-1)c_7\delta t}\tilde{a}_1 + \frac{2c_5}{c_7}, \quad \forall n \geq 1, \tag{3.36}$$

and

$$\frac{1}{2}\sum_{k=1}^{n-1}\left\|\Delta\left(u^{k+1}-u^k\right)\right\|^2 \leq \tilde{a}_1 + (n-1)\delta t\frac{2c_5}{c_7}, \quad \forall n \geq 1. \tag{3.37}$$

Thanks to the dissipative estimate (3.22), there exists a time $t_1 = t_1(R_1)$ such that

$$\|\nabla u^n\|^2 \leq C_1 + M_1, \quad \forall n \geq t_1/\delta t,$$

where C_1 and M_1 are independent of R_1 and δt. Let $n_1 = \lceil t_1/\delta t \rceil$, where $\lceil \cdot \rceil$ denotes the integer ceiling function. Then for $n \geq n_1$, the constant c_{12} in (3.33) no longer depends on R_1. Consequently, (3.35) holds for all $n \geq n_1$ with c_5 and c_7 independent of R_1. By induction (as in Lem. 3.3), we have

$$\tilde{a}_n \leq e^{-(n-n_1)c_7\delta t}\tilde{a}_{n_1} + \frac{2c_5}{c_7}, \quad \forall n \geq n_1, \tag{3.38}$$

and

$$\frac{1}{2}\sum_{k=n_1}^{n-1} \left\|\Delta\left(u^{k+1} - u^k\right)\right\|^2 \leq \tilde{a}_{n_1} + (n - n_1)\delta t\frac{2c_5}{c_7}, \quad \forall n \geq n_1. \tag{3.39}$$

The dissipative estimate (3.29) follows from (3.38) (for $n \geq n_1$), (3.36) (for $1 \leq n \leq n_1$) and Lem. 3.2 (for $n = 0$). Estimate (3.30) follows from (3.39) (for $n \geq n_1$), (3.37) (for $1 \leq n \leq n_1$), (3.36) (for $\tilde{a}_{n_1}$) and Lem. 3.2 (for $n = 0$). □

3.3. Estimates for the difference of solutions, uniform with respect to the time step

Let v^n and w^n be two sequences generated by the scheme (3.1) and corresponding to the initial data v^0 and w^0 respectively. We denote $u^n = v^n - w^n$ their difference, which satisfies

$$\frac{u^{n+1} - u^n}{\delta t} + A^2u^{n+1} + A\left(f\left(v^n\right) - f\left(w^n\right)\right) + \left(g\left(v^n\right) - g\left(w^n\right)\right) = 0, \quad \forall n \geq 0. \tag{3.40}$$

Lemma 3.7. *Assume that $\delta t < 1/(2L_g)$. Then for all $n \geq 1$, we have*

$$\|u^n\|^2 + \sum_{k=0}^{n-1}\left\|u^{k+1} - u^k\right\|^2 + \delta t\sum_{k=0}^{n-1}\left\|\Delta u^{k+1}\right\|^2 \leq \exp\left(c_{f,g}n\delta t\right)\left\|u^0\right\|^2. \tag{3.41}$$

Proof. We multiply (3.40) by u^{n+1} in H. We obtain

$$\begin{aligned}\frac{1}{2\delta t}\left(\left\|u^{n+1}\right\|^2 - \|u^n\|^2 + \left\|u^{n+1} - u^n\right\|^2\right) + \left\|\Delta u^{n+1}\right\|^2 \\ = \left(f\left(v^n\right) - f\left(w^n\right), \Delta u^{n+1}\right) - \left(g\left(v^n\right) - g\left(w^n\right), u^{n+1}\right).\end{aligned} \tag{3.42}$$

Owing to (2.14) and Young's inequality, we have

$$\begin{aligned}\left|\left(g\left(v^n\right) - g\left(w^n\right), u^{n+1}\right)\right| &\leq \|g(v^n) - g\left(w^n\right)\| \; \left\|u^{n+1}\right\| \\ &\leq L_g\|u^n\| \; \left\|u^{n+1}\right\| \\ &\leq \frac{L_g}{2}\left\|u^{n+1}\right\|^2 + \frac{L_g}{2}\|u^n\|^2.\end{aligned}$$

By (2.12) and Young's inequality,

$$\begin{aligned}\left|\left(f\left(v^n\right) - f\left(w^n\right), \Delta u^{n+1}\right)\right| &\leq \|f\left(v^n\right) - f\left(w^n\right)\| \; \left\|\Delta u^{n+1}\right\| \\ &\leq L_f\|u^n\| \; \left\|\Delta u^{n+1}\right\| \\ &\leq \frac{L_f^2}{2}\|u^n\|^2 + \frac{1}{2}\left\|\Delta u^{n+1}\right\|^2.\end{aligned}$$

Plugging this in (3.42) times $2\delta t$, we find

$$(1-L_g\delta t)\left\|u^{n+1}\right\|^2+\left\|u^{n+1}-u^n\right\|^2+\delta t\left\|\Delta u^{n+1}\right\|^2\leq(1+c\delta t)\left\|u^n\right\|^2, \tag{3.43}$$

where $c=L_f^2+L_g$. We divide this estimate by $(1-L_g\delta t)$ and we obtain that for $\delta t\leq 1/(2L_g)$,

$$\left\|u^{n+1}\right\|^2+\left\|u^{n+1}-u^n\right\|^2+\delta t\left\|\Delta u^{n+1}\right\|^2\leq(1+c'\delta t)\left\|u^n\right\|^2,\quad\forall n\geq 0, \tag{3.44}$$

where $c'=c'(c,L_g)$. We apply the estimate

$$1+s\leq\exp(s),\quad\forall s\in\mathbb{R}, \tag{3.45}$$

to $s=c'\delta t$ and we obtain (3.41) by induction, with $c_{f,g}=c'$. □

Next, we show a L^2-H^1 smoothing property.

Lemma 3.8. *Let $R_2>0$ and $\delta t<1/(2L_g)$. If $\left\|v^0\right\|_2\leq R_2$ and $\left\|w^0\right\|_2\leq R_2$, then for all $n\geq 1$, we have*

$$n\delta t\left\|u^n\right\|_1^2\leq c_S\exp\left(c\left(R_2\right)n\delta t\right)\left\|u^0\right\|^2. \tag{3.46}$$

Proof. We multiply (3.40) by $A^{-1}\left(u^{n+1}-u^n\right)/\delta t$ in H and we find

$$\begin{aligned}&\frac{1}{\delta t^2}\left\|u^{n+1}-u^n\right\|_{-1}^2+\frac{1}{2\delta t}\left(\left\|\nabla u^{n+1}\right\|^2-\left\|\nabla u^n\right\|^2+\left\|\nabla(u^{n+1}-u^n)\right\|^2\right)\\&+\left(f\left(v^n\right)-f\left(w^n\right),\frac{u^{n+1}-u^n}{\delta t}\right)\\&+\left(g\left(v^n\right)-g\left(w^n\right),A^{-1}\frac{u^{n+1}-u^n}{\delta t}\right)=0.\end{aligned} \tag{3.47}$$

Using (2.14), (2.10), the Poincaré inequality (2.11) and Young's inequality, we get

$$\begin{aligned}\left|\left(g\left(v^n\right)-g\left(w^n\right),(-\Delta)^{-1}\frac{u^{n+1}-u^n}{\delta t}\right)\right|&\leq\left\|g\left(v^n\right)-g\left(w^n\right)\right\|\left\|(-\Delta)^{-1}\frac{u^{n+1}-u^n}{\delta t}\right\|\\&\leq L_g\left\|u^n\right\|c_S\left\|\frac{u^{n+1}-u^n}{\delta t}\right\|_{-1}\\&\leq L_g^2c_S^2c_P^2\left\|\nabla u^n\right\|^2+\frac{1}{4\delta t^2}\left\|u^{n+1}-u^n\right\|_{-1}^2.\end{aligned}$$

Thanks to (3.29), we know that (v^n) and (w^n) are bounded in $H^2(\Omega)$. Arguing as in the continuous case (see (2.63)), we obtain

$$\begin{aligned}\left|\left(f\left(v^n\right)-f\left(w^n\right),\frac{u^{n+1}-u^n}{\delta t}\right)\right|&\leq\left\|A^{\frac12}\left(f\left(v^n\right)-f\left(w^n\right)\right)\right\|\left\|\frac{u^{n+1}-u^n}{\delta t}\right\|_{-1}\\&\leq c(R_2)\left\|\nabla u^n\right\|^2+\frac{1}{4\delta t^2}\left\|u^{n+1}-u^n\right\|_{-1}^2.\end{aligned}$$

We combine the above estimates in (3.47) and we deduce that

$$\frac{1}{2\delta t}\left\|\nabla u^{n+1}\right\|^2\leq\frac{1}{2\delta t}\left\|\nabla u^n\right\|^2+c'\left\|\nabla u^n\right\|^2,\quad\forall n\geq 0,$$

where $c'=c'(R_2)=L_g^2c_S^2c_P^2+c(R_2)$. We multiply this by $2n\delta t$ and we add $\left\|\nabla u^{n+1}\right\|^2$ on both sides. This yields

$$(n+1)\left\|\nabla u^{n+1}\right\|^2\leq(1+2c'\delta t)\,n\left\|\nabla u^n\right\|^2+\left\|\nabla u^{n+1}\right\|^2,\quad\forall n\geq 0.$$

Let $d_n = n\left\|\nabla u^n\right\|^2$. By (2.10), we have

$$d_{n+1} \leq (1 + 2c'\delta t)\, d_n + c_S \left\|\Delta u^{n+1}\right\|^2, \quad \forall n \geq 0.$$

Using $d_0 = 0$, we deduce by induction that

$$d_n \leq (1 + 2c'\delta t)^n \left(c_S \sum_{k=0}^{n-1} \left\|\Delta u^{k+1}\right\|^2 \right), \quad \forall n \geq 1.$$

The conclusion (3.46) follows from (3.41) and from (3.45) with $s = 2c'\delta t$. □

4. Finite time uniform error estimate

The error estimate between the continuous semigroup and the discrete semigroup is the last essential step in order to build a robust family of exponential attractors.

For the error estimate on a finite time interval, we follow the methodology in [29,33]. We consider a sequence (u^n) in $D(A)$ generated by (3.1). To the sequence (u^n), we associate three functions $u_{\delta t}$, $\overline{u}_{\delta t}$, $\underline{u}_{\delta t} : \mathbb{R}_+ \to D(A)$, namely

$$\begin{aligned} &u_{\delta t} = u^n + \frac{t - n\delta t}{\delta t}\left(u^{n+1} - u^n\right), \quad t \in [n\delta t, (n+1)\delta t), \\ &\overline{u}_{\delta t} = u^{n+1}, \quad t \in [n\delta t, (n+1)\delta t), \\ &\underline{u}_{\delta t} = u^n, \quad t \in [n\delta t, (n+1)\delta t). \end{aligned}$$

We note that $u_{\delta t} \in C^0([0,T], D(A))$ is piecewise linear, $\overline{u}_{\delta t} \in L^\infty(0,T; D(A))$ and $\underline{u}_{\delta t} \in L^\infty(0,T; D(A))$, for all $T > 0$. The scheme (3.1) can be rewritten

$$\frac{\mathrm{d}u_{\delta t}}{\mathrm{d}t} + A^2 \overline{u}_{\delta t} + Af(\underline{u}_{\delta t}) + g(\underline{u}_{\delta t}) = 0 \quad \text{in } D(A^{-1}), \text{ for a.e. } t > 0. \tag{4.1}$$

Equivalently, we have

$$\frac{du_{\delta t}}{dt} + A^2 u_{\delta t} + Af(u_{\delta t}) + g(u_{\delta t}) = A^2(u_{\delta t} - \overline{u}_{\delta t}) + A(f(u_{\delta t}) - f(\underline{u}_{\delta t})) + (g(u_{\delta t}) - g(\underline{u}_{\delta t})) \tag{4.2}$$

in $D(A^{-1})$, for a.e. $t > 0$. We denote by u the solution to (2.21) with initial condition $u_0 \in D(A)$ and we set

$$e_{\delta t}(t) = u_{\delta t}(t) - u(t).$$

The error estimate reads:

Theorem 4.1. *For all $T > 0$ and for all $R_2 > 0$, there is a constant $C(T, R_2)$ independent of δt such that $u^0 = u_0$ and $\left\|u^0\right\|_2 \leq R_2$ imply*

$$\sup_{t \in [0, N\delta t]} \|e_{\delta t}(t)\| \leq C(T, R_2)\ (\delta t)^{\frac{1}{2}}, \tag{4.3}$$

where $N = \lfloor T/\delta t \rfloor$ and $\lfloor\cdot\rfloor$ denotes the integer floor function.

Proof. On subtracting (2.21) from (4.2), we find

$$\frac{\mathrm{d}e_{\delta t}}{\mathrm{d}t} + \Delta^2 e_{\delta t} + A(f(u_{\delta t}) - f(u)) + (g(u_{\delta t}) - g(u)) = A^2(u_{\delta t} - \overline{u}_{\delta t}) + A(f(u_{\delta t}) - f(\underline{u}_{\delta t})) + (g(u_{\delta t}) - g(\underline{u}_{\delta t})). \tag{4.4}$$

We multiply (4.4) by $e_{\delta t}$ in H. We obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_{\delta t}(t)\right\|^2+\left\|\Delta e_{\delta t}(t)\right\|^2-\left(f\left(u_{\delta t}\right)-f(u),\Delta e_{\delta t}\right)+\left(g(u_{\delta t})-g(u),e_{\delta t}\right)\\&=\left(\Delta(u_{\delta t}-\overline{u}_{\delta t}),\Delta e_{\delta t}\right)-\left(f(u_{\delta t})-f(\underline{u}_{\delta t}),\Delta e_{\delta t}\right)+\left(g(u_{\delta t})-g(\underline{u}_{\delta t}),e_{\delta t}\right).\end{aligned}\tag{4.5}$$

Estimate (2.14) and Young's inequality yield

$$\begin{aligned}|(g(u_{\delta t})-g(u),e_{\delta t})|&\le\|g(u_{\delta t})-g(u)\|\ \|e_{\delta t}\|\\&\le L_g\|e_{\delta t}\|^2\end{aligned}$$

and

$$\begin{aligned}|(g(u_{\delta t})-g(\underline{u}_{\delta t}),e_{\delta t})|&\le\|g(u_{\delta t})-g(\underline{u}_{\delta t})\|\ \|e_{\delta t}\|\\&\le L_g\|u_{\delta t}-\underline{u}_{\delta t}\|\ \|e_{\delta t}\|\\&\le\frac{L_g^2}{4}\|u_{\delta t}-\underline{u}_{\delta t}\|^2+\|e_{\delta t}\|^2.\end{aligned}$$

Moreover, by (2.12),

$$\begin{aligned}|(f(u_{\delta t})-f(u),\Delta e_{\delta t})|&\le\|f(u_{\delta t})-f(u)\|\ \|\Delta e_{\delta t}\|\\&\le L_f\|e_{\delta t}\|\ \|\Delta e_{\delta t}\|\\&\le L_f^2\|e_{\delta t}\|^2+\frac{1}{4}\|\Delta e_{\delta t}\|^2\end{aligned}$$

and

$$\begin{aligned}|(f(u_{\delta t})-f(\underline{u}_{\delta t}),\Delta e_{\delta t})|&\le\|f(u_{\delta t})-f(\underline{u}_{\delta t})\|\ \|\Delta e_{\delta t}\|\\&\le L_f\|u_{\delta t}-\underline{u}_{\delta t}\|\ \|\Delta e_{\delta t}\|\\&\le L_f^2\|u_{\delta t}-\underline{u}_{\delta t}\|^2+\frac{1}{4}\|\Delta e_{\delta t}\|^2.\end{aligned}$$

We also have

$$\begin{aligned}|(\Delta(u_{\delta t}-\overline{u}_{\delta t}),\Delta e_{\delta t})|&\le\|\Delta(u_{\delta t}-\overline{u}_{\delta t})\|\ \|\Delta e_{\delta t}\|\\&\le\|\Delta(u_{\delta t}-\overline{u}_{\delta t})\|^2+\frac{1}{4}\|\Delta e_{\delta t}\|^2.\end{aligned}$$

Inserting the estimates above into (4.5), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_{\delta t}(t)\right\|^2+\frac{1}{2}\left\|\Delta e_{\delta t}(t)\right\|^2\le c_1\left\|e_{\delta t}(t)\right\|^2+c_2\left\|u_{\delta t}-\underline{u}_{\delta t}\right\|^2+2\left\|\Delta(u_{\delta t}-\overline{u}_{\delta t})\right\|^2,\tag{4.6}$$

where $c_1=2L_g+2+2L_f^2$ and $c_2=L_g^2/2+2L_f^2$.

Let $T>0$ and $N=\lfloor T/\delta t\rfloor$. Thanks to $e_{\delta t}(0)=0$ and the classical Gronwall lemma applied to (4.6), we obtain

$$\begin{aligned}\|e_{\delta t}(t)\|^2\le\exp\left(c_1T\right)&\int_0^{N\delta t}c_2\left\|u_{\delta t}(s)-\underline{u}_{\delta t}(s)\right\|^2\mathrm{d}s\\&+\exp\left(c_1T\right)\int_0^{N\delta t}2\left\|\Delta(u_{\delta t}(s)-\overline{u}_{\delta t}(s))\right\|^2\mathrm{d}s,\quad\forall t\in[0,N\delta t].\end{aligned}\tag{4.7}$$

On the interval $[n\delta t, (n+1)\delta t)$, we have

$$\|u_{\delta t}(s) - \underline{u}_{\delta t}(s)\| \leq \|u^{n+1} - u^n\| \qquad \text{and} \qquad \|\Delta(u_{\delta t}(s) - \overline{u}_{\delta t}(s))\| \leq \left\|\Delta(u^{n+1} - u^n)\right\|.$$

Thus,

$$\int_0^{N\delta t} c_2 \|u_{\delta t}(s) - \underline{u}_{\delta t}(s)\|^2 \, \mathrm{d}s \leq c_2 \delta t \sum_{k=0}^{N-1} \left\|u^{k+1} - u^k\right\|^2$$

and

$$\int_0^{N\delta t} \|\Delta(u_{\delta t}(s) - \overline{u}_{\delta t}(s))\|^2 \, \mathrm{d}s \leq \delta t \sum_{k=0}^{N-1} \left\|\Delta(u^{k+1} - u^k)\right\|^2.$$

Plugging these estimates into (4.7), we obtain

$$\begin{aligned} \|e_{\delta t}(t)\|^2 &\leq \exp(c_1 T) \left(c_2 \sum_{k=0}^{N-1} \left\|u^{k+1} - u^k\right\|^2 + 2 \sum_{k=0}^{N-1} \left\|\Delta(u^{k+1} - u^k)\right\|^2 \right) \delta t \\ &\leq c_3 \exp(c_1 T) \sum_{k=0}^{N-1} \left\|\Delta\left(u^{k+1} - u^k\right)\right\|^2 \delta t, \end{aligned} \tag{4.8}$$

where $c_3 = c_2 c_S^2 + 2$. By (3.30),

$$\|e_{\delta t}(t)\|^2 \leq c_3 \exp(c_1 T) \left(Q_2\left(\left\|\Delta u^0\right\|\right) + M_2' T\right) \delta t, \quad \forall t \in [0, N\delta t].$$

This concludes the proof. □

5. Convergence of exponential attractors

5.1. Some definitions

Before stating our main result, we recall some definitions (see *e.g.* [12, 32]). We recall that $H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. A continuous-in-time semigroup $\{S(t),\ t \in \mathbb{R}_+\}$ on $D(A)$ is a family of (nonlinear) operators such that $S(t)$ is a continuous operator (for the $L^2(\Omega)$-norm) from $D(A)$ into itself, for all $t \in \mathbb{R}_+$, with $S(0) = Id$ (identity) and

$$S(t+s) = S(t) \circ S(s), \quad \forall s, t \in \mathbb{R}_+.$$

A discrete-in-time semigroup $\{S(t),\ t \in \mathbb{N}\}$ on $D(A)$ is a family of (nonlinear) operators which satisfy these properties with $\mathbb{R}_+$ replaced by $\mathbb{N}$. A discrete-in-time semigroup is usually denoted $\{S^n,\ n \in \mathbb{N}\}$, where $S(= S(1))$ is a continuous (nonlinear) operator from $D(A)$ into itself.

A (continuous or discrete) semigroup $\{S(t),\ t \geq 0\}$ defines a (continuous or discrete) dynamical system: if u_0 is the state of the dynamical system at time 0, then $u(t) = S(t)u_0$ is the state at time $t \geq 0$. The term "dynamical system" will sometimes be used instead of "semigroup".

Definition 5.1 (Global attractor)**.** Let $\{S(t),\ t \geq 0\}$ be a continuous or discrete semigroup on $D(A)$. A bounded set $\mathcal{A} \subset D(A)$ is called the global attractor of the dynamical system if the following three conditions are satisfied:

(1) $\mathcal{A}$ is compact in H;
(2) $\mathcal{A}$ is invariant, *i.e.* $S(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;
(3) $\mathcal{A}$ attracts all bounded sets in $D(A)$, *i.e.*, for every bounded set B in $D(A)$,

$$\lim_{t \to +\infty} dist_H(S(t)B, \mathcal{A}) = 0.$$

Here, $dist_H$ denotes the non-symmetric Hausdorff semidistance in H between two subsets, which is defined as

$$dist_H(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \|b_1 - b_2\|_H.$$

It is easy to see, thanks to the invariance and the attracting property, that the global attractor, when it exists, is unique [32].

Let $X \subset H$ be a (relatively compact) subset of H. For $\varepsilon > 0$, we denote $N_\varepsilon(X, H)$ the minimum number of balls of H of radius $\varepsilon > 0$ which are necessary to cover X. The *fractal dimension of* X (see *e.g.* [11,32]) is the number

$$dim_F(X) = \limsup_{\varepsilon \to 0} \frac{\log(N_\varepsilon(X, H))}{\log(1/\varepsilon)} \in [0, +\infty].$$

Definition 5.2 (Exponential attractor)**.** Let $\{S(t),\ t \geq 0\}$ be a continuous or discrete semigroup on $D(A)$. A bounded set $\mathcal{M} \subset D(A)$ is an exponential attractor of the dynamical system if the following three conditions are satisfied:

(1) $\mathcal{M}$ is compact in H and has finite fractal dimension;
(2) $\mathcal{M}$ is positively invariant, *i.e.* $S(t)\mathcal{M} \subset \mathcal{M}$, for all $t \geq 0$;
(3) $\mathcal{M}$ attracts exponentially the bounded subsets of $D(A)$ in the following sense:

$$\forall B \subset D(A) \text{ bounded}, \quad dist_H(S(t)B, \mathcal{M}) \leq \mathcal{Q}(\|B\|_H)e^{-\alpha t}, \quad t \geq 0,$$

where the positive constant α and the monotonic function $\mathcal{Q}$ are independent of B. Here, $\|B\|_H = \sup_{b \in B} \|b\|_H$.

It is easy to see that the exponential attractor, if it exists, contains the global attractor.

5.2. The main result

We have seen that $\{S_0(t),\ t \in \mathbb{R}_+\}$ defined by (2.22) is a continuous-in-time dynamical system on $D(A)$, and that for every $\delta t > 0$ small enough, $\{S_{\delta t}^n,\ n \in \mathbb{N}\}$ defines a discrete-in-time dynamical system on $D(A)$ (Thm. 3.1). We have:

Theorem 5.3. *Let $\delta t^\star > 0$ be small enough. For every $\delta t \in (0, \delta t^\star]$, the discrete dynamical system $\{S_{\delta t}^n,\ n \in \mathbb{N}\}$ possesses an exponential attractor $\mathcal{M}_{\delta t}$ in $D(A)$, and the continuous dynamical system $\{S_0(t),\ t \in \mathbb{R}_+\}$ possesses an exponential attractor $\mathcal{M}_0$ in $D(A)$ such that:*

(1) the fractal dimension of $\mathcal{M}_{\delta t}$ is bounded, uniformly with respect to $\delta t \in [0, \delta t^\star]$,

$$dim_F \mathcal{M}_{\delta t} \leq c_1,$$

where c_1 is independent of δt;

(2) $\mathcal{M}_{\delta t}$ attracts the bounded sets of $D(A)$, uniformly with respect to $\delta t \in (0, \delta t^\star]$, i.e. for all $\delta t \in (0, \delta t^\star]$,

$$\forall B \subset D(A) \text{ bounded}, \ dist_H\,(S_{\delta t}^n B, \mathcal{M}_{\delta t}) \leq \mathcal{Q}(\|B\|_H)e^{-c_2 n \delta t}, \quad n \in \mathbb{N},$$

where the positive constant c_2 and the monotonic function $\mathcal{Q}$ are independent of δt;

(3) the family $\{\mathcal{M}_{\delta t},\ \delta t \in [0, \delta t^\star]\}$ is continuous at 0,

$$dist_{sym}(\mathcal{M}_{\delta t}, \mathcal{M}_0) \leq c_3 (\delta t)^{c_4},$$

where c_3 and $c_4 \in (0, 1)$ are independent of δt and $dist_{sym}$ denotes the symmetric Hausdorff distance between sets, defined by

$$dist_{sym}(B_1, B_2) := \max\{dist_H(B_1, B_2), dist_H(B_2, B_1)\}.$$

Proof. We apply Theorem 2.5 in [29] with the spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ and the set

$$\mathcal{B} = \{v \in D(A) \ : \ \|v\|_2 \leq M_2 + 1\},$$

where M_2 is the constant in (2.39) and (3.29). We note that V is compactly imbedded in H and that an H-V smoothing property holds, uniformly with respect to δt (Lems. 2.11 and 3.8). Moreover, $\mathcal{B}$ is absorbing in $D(A)$, uniformly with respect to $\delta t \in [0, \delta t_0]$, where $\delta t_0 > 0$ is chosen small enough. The estimates of Sections 2–4 show that assumptions (H1)–(H9) of Theorem 2.5 in [29] are satisfied. Thus, the conclusions of Theorem 5.3 hold for $\delta t \in [0, \delta t^\star]$, for some $\delta t^\star \in (0, \delta t_0]$ small enough. We note that Theorem 2.5 in [29] is stated for a family of semigroups which act on the whole space H, but with a minor modification of the proof, it can be applied to our situation where the semigroup acts on $D(A)$ and is continuous for the H-norm. The main tool is the construction of exponential attractors based on a uniform smoothing property proposed by Efendiev, Miranville and Zelik in [12, Thm. 4.4]. □

As in [29, Cor. 6.2], we have:

Corollary 5.4. *For every $\delta t \in [0, \delta t^\star]$, the semigroup $\{S_{\delta t}(t),\ t \geq 0\}$ possesses a global attractor $\mathcal{A}_{\delta t}$ in $D(A)$ which is bounded in $D(A)$ and compact in H. Moreover, $dist_H(\mathcal{A}_{\delta t}, \mathcal{A}_0) \to 0$ as $\delta t \to 0^+$, and the fractal dimension of $\mathcal{A}_{\delta t}$ is bounded by a constant independent of δt.*

Remark 5.5. Let us replace the Dirichlet boundary conditions (1.2) with Neumann boundary conditions, which read

$$\partial_n u = \partial_n \Delta u = 0 \text{ on } \partial\Omega \times (0, +\infty), \tag{5.1}$$

where n is the unit outer normal to $\partial\Omega$.

If $g = 0$ in (1.1), we deal with the classical Cahn-Hilliard equation and a result similar to Theorem 5.3 can be obtained. In this case, we have the conservation of mass and it is convenient to introduce the function spaces

$$H_\beta = \left\{ v \in H \ : \ \int_\Omega v = \beta \right\} \text{ and } \mathcal{H}_\alpha = \bigcup_{|\beta| \leq \alpha} H_\beta,$$

as in [4,32], where $\beta \in \mathbb{R}$ and $\alpha > 0$.

If $g \neq 0$, the situation is more delicate because we no longer have the conservation of mass [21, Rem. 5.7]. If g is a proliferation term, the mass may even blow up in finite time [7,23,25].

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