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THE MIXED PENALTY METHOD FOR THE SIGNORINI PROBLEM

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Abstract. We investigate two penalty approaches for the Signorini problem in the mixed form. The well-posedness theory has been established for the two mixed penalty problems in both the continuous and discrete sense. For the continuous case, we obtain the error of the penalty for both two approaches. We also study the error estimates of the discrete mixed penalty problems, which depend on the mesh size and the penalty parameter. Based on the two penalty approaches, we design two algorithms and show their convergence. Several numerical experiments are carried out to confirm the theoretical results.

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1. Introduction

The Signorini problem modeling the interaction between a deformable body and a rigid frictionless foundation [34] has many engineering and physics applications that arise in contact problems [13, 18, 35], elastoplastic problems [19, 31], the blood flow problems [33, 37], the transmission problem [17], electropainting and shallow dam problems [1, 2], etc. The Signorini boundary conditions have both equality and inequality constraints (see (1.1c)), which leads to a nonlinear problem. Many mathematical and numerical analyses of this well-known problem have been carried out in recent years, such as the finite element method (FEM) (see [6, 15, 21] for recent analysis result), the hybrid finite element method [7], the discontinuous Petrov–Galerkin (DPG) method [16], the boundary element methods [1] and the mixed FEM [4, 17, 22] and so on (we only list a few references for these methods).

In this study, we develop new approaches combining the mixed formulation and penalty techniques to approximate the simplified Signorini problem and establish the theoretical analysis. These results can be applied to handle the generalized Signorini problem in elastic mechanics without any challenges.

The Signorini problem **(P)** is stated as follows

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{1.1b}$$

$$u \geq 0, \quad \nabla u \cdot \boldsymbol{n} \geq 0, \quad u(\nabla u \cdot \boldsymbol{n}) = 0 \qquad \text{on } \Gamma_S. \tag{1.1c}$$

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Here and hereafter, to avoid the issue of domain perturbation when introducing the triangulation, we assume that Ω is a polygonal/polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$ for simplicity in finite element analysis. The boundary $\partial\Omega = \overline{\Gamma_D \cup \Gamma_S}$ with $\Gamma_D \cap \Gamma_S = \emptyset$, $|\Gamma_D| > 0$ and $|\Gamma_S| > 0$. $f \in L^2(\Omega)$ is a given function. The unit outer normal vector to the boundary $\partial\Omega$ is represented by $\boldsymbol{n}$.

The penalty approach is commonly utilized for approximating the Signorini condition (see [18, 26]), which is to solve the elliptic equation with a nonlinear Robin-type boundary condition, *i.e.*,

$$-\Delta u_\varepsilon = f \qquad \text{in } \Omega, \tag{1.2a}$$

$$u_\varepsilon = 0 \qquad \text{on } \Gamma_D, \tag{1.2b}$$

$$\nabla u_\varepsilon \cdot \boldsymbol{n} = \frac{1}{\varepsilon}[u_\varepsilon]_- \qquad \text{on } \Gamma_S, \tag{1.2c}$$

where ε is the penalty parameter $(0 < \varepsilon \ll 1)$ and $[s]_- := \max(0, -s)$ represents the negative part of s. The variational form of (1.2) is stated as follows.

$\mathbf{(P)}_\varepsilon$ Find $u_\varepsilon \in X$ such that

$$(\nabla u_\varepsilon, \nabla v) - \frac{1}{\varepsilon}\int_{\Gamma_S} [u_\varepsilon]_- v \, \mathrm{d}s = (f, v) \qquad (\forall v \in X),$$

where $X := H^1_{0,\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$. For the elastic problem with the Signorini contact condition, one can refer to [12, 14] for the convergence analysis of the penalty method combining with the finite element approximation, under the $H^{\frac{3}{2}+\nu}$ $(0 < \nu \le \frac{1}{2})$-regularity assumption. The Tresca type friction condition is also considered in [12, 14]. Besides the above penalty approach, the barrier (or interior penalty) functionals are also introduced to approximate the Signorini condition, and a full investigation is given by Oden and Kim [30].

In our study, we utilize the mixed method for the Signorini problem, which results a weaker variational formulation. Let $\boldsymbol{\sigma} := \nabla u$ be an additional unknown. We reformulate (1.1) into the (formally) equivalent mixed system **(MP)**:

$$\boldsymbol{\sigma} - \nabla u = 0 \qquad \text{in } \Omega, \tag{1.3a}$$

$$\nabla \cdot \boldsymbol{\sigma} + f = 0 \qquad \text{in } \Omega, \tag{1.3b}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{1.3c}$$

$$u \ge 0, \quad \boldsymbol{\sigma} \cdot \boldsymbol{n} \ge 0, \quad u\boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_S. \tag{1.3d}$$

The mixed method provides various computational benefits, such as preserving the conservation law locally, preventing the loss of precision in stress, and approximating solutions with low regularity [8]. Additionally, for the Signorini problem, we will show that designing iteration schemes becomes more flexible when considering the mixed formulation. To see that, let us introduce the mixed penalty problems. Set the function spaces:

$$V := H(\mathrm{div}; \Omega), \quad Q := L^2(\Omega), \quad \Lambda_{1/2} := H^{\frac{1}{2}}_{00}(\Gamma_S), \quad \Lambda := L^2(\Gamma_S) = \Lambda^*, \quad \Lambda_{-1/2} := H^{-\frac{1}{2}}(\Gamma_S) = \Lambda^*_{1/2}.$$

For $\boldsymbol{\tau} \in V$, the trace $\boldsymbol{\tau} \cdot \boldsymbol{n}$ on Γ_S is defined by ([8], Lem. 2.1.1)

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{n}, v \rangle_{\Gamma_S} = (\boldsymbol{\tau}, \nabla v) + (\nabla \cdot \boldsymbol{\tau}, v) \qquad (\forall v \in X). \tag{1.4}$$

Since the trace $v|_{\Gamma_S}$ is a surjection from X onto $\Lambda_{1/2}$, it follows from (1.4) that $\boldsymbol{\tau} \cdot \boldsymbol{n}|_{\Gamma_S} \in \Lambda_{-1/2}$. Let $\hat{V}$ be the subspace of V where the trace $\boldsymbol{\tau} \cdot \boldsymbol{n}$ on Γ_S enjoys a better regularity:

$$\hat{V} := \{\boldsymbol{\tau} \in V : \boldsymbol{\tau} \cdot \boldsymbol{n}|_{\Gamma_S} \in \Lambda\}.$$

We introduce two types of mixed penalty problems:

(MP$_\varepsilon$-I) Find $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon) \in V \times Q \times \Lambda_{1/2}$ such that:

$$(\boldsymbol{\sigma}_\varepsilon, \boldsymbol{\tau}) + (u_\varepsilon, \nabla \cdot \boldsymbol{\tau}) - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}, p_\varepsilon \rangle_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in V), \tag{1.5a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_\varepsilon, v) + (f, v) = 0 \qquad (\forall v \in Q), \tag{1.5b}$$

$$\langle \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}, q \rangle_{\Gamma_S} - \frac{1}{\varepsilon}([p_\varepsilon]_-, q)_{\Gamma_S} = 0 \qquad (\forall q \in \Lambda_{1/2}); \tag{1.5c}$$

(MP$_\varepsilon$-II) Find $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon) \in \hat{V} \times Q$ such that:

$$(\boldsymbol{\sigma}_\varepsilon, \boldsymbol{\tau}) + (u_\varepsilon, \nabla \cdot \boldsymbol{\tau}) - \frac{1}{\varepsilon}([\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_-, \boldsymbol{\tau} \cdot \boldsymbol{n})_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in \hat{V}), \tag{1.6a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_\varepsilon, v) + (f, v) = 0 \qquad (\forall v \in Q). \tag{1.6b}$$

The strong forms of **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)** are presented below.

(MP$_\varepsilon$-I$'$)

$$-\nabla \cdot \boldsymbol{\sigma}_\varepsilon = f, \quad \boldsymbol{\sigma}_\varepsilon - \nabla u_\varepsilon = 0 \qquad \text{in } \Omega, \tag{1.7a}$$

$$u_\varepsilon = 0 \qquad \text{on } \Gamma_D, \tag{1.7b}$$

$$\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} = \frac{1}{\varepsilon}[u_\varepsilon]_- \qquad \text{on } \Gamma_S; \tag{1.7c}$$

(MP$_\varepsilon$-II$'$)

$$-\nabla \cdot \boldsymbol{\sigma}_\varepsilon = f, \quad \boldsymbol{\sigma}_\varepsilon - \nabla u_\varepsilon = 0 \qquad \text{in } \Omega, \tag{1.8a}$$

$$u_\varepsilon = 0 \qquad \text{on } \Gamma_D, \tag{1.8b}$$

$$u_\varepsilon = \frac{1}{\varepsilon}[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_- \qquad \text{on } \Gamma_S. \tag{1.8c}$$

Apparently, the above two mixed penalty problems are not equivalent. Nevertheless, we will demonstrate that the solution to both penalty problems will eventually converge to the same solution $(\boldsymbol{\sigma}, u)$ as $\varepsilon \to 0$. When the solution is smooth enough, (1.7c) is equal to (1.2c) because $\boldsymbol{\sigma}_\varepsilon = \nabla u_\varepsilon$ and $p_\varepsilon = u_\varepsilon|_{\Gamma_S}$. However, in the discrete case, we will see that $\boldsymbol{\sigma}_{\varepsilon h} \neq \nabla u_{\varepsilon h}$ and $p_{\varepsilon h} \neq u_{\varepsilon h}|_{\Gamma_S}$.

Note that to make (1.6a) well-defined, we require $\boldsymbol{\sigma}_\varepsilon \in \hat{V}$ for **(MP$_\varepsilon$-II)**, which is a stronger regularity requirement than that of **(MP$_\varepsilon$-I)**. From the point of view of implementation, **(MP$_\varepsilon$-II)** is easier to deal with than **(MP$_\varepsilon$-I)**. For **(MP$_\varepsilon$-I)**, in view of (1.5c), we need to enforce $\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} = \varepsilon^{-1}[p_\varepsilon]_-$ on Γ_S, which can be regarded as the essential boundary condition. We propose an iteration algorithm to solve **(MP$_\varepsilon$-I)** in Section 3 (see Algorithm 1), where we have to modify the value of $\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}$ on Γ_S at each iteration. It is worth mentioning that both two penalty approaches have similar convergence results (see Thms. 2.3 and 2.4). Meanwhile, the same convergence performance for the two penalty formulations is observed in our numerical experiments (see Sect. 5).

The term "mixed formulation" is also used for saddle-point problems arising from Lagrange multipliers for the inequality contact boundary conditions [4,5,20], which differs from what we have mentioned in this work. Here we introduce $\boldsymbol{\sigma} = \nabla u$ as a new unknown and reformulate the original problem into the mixed form incorporating the penalty methods. Utilizing mixed method provides additional techniques for solving the Signorini problem. The theories of abstract mixed variational inequalities **(MVI)** (see (2.1)) were first established by Brezzi *et al.* [10]. Slimane *et al.* [35] proposed a mixed formulation with the penalty-type stabilization term $\frac{1}{\varepsilon}(\nabla \cdot \boldsymbol{\sigma}_\varepsilon + f)$ for the unilateral elastic contact problem which enforces the constraint $\nabla \cdot \boldsymbol{\sigma} + f = 0$ in the weak sense with penalty. In [19,31], the elastoplasticity problem was analyzed in the mixed form, where [19] stated a regularized approach for approximation based on a friction-type boundary problem. The DPG method for the Signorini problem is investigated by Führer *et al.* [16], which is adapted to the singularly perturbed case of reaction-dominated diffusion. As far as we know, the penalty for the Signorini problem in mixed form has not been investigated yet.

We provide a summary of our main results here. Firstly, we analyze two penalty approaches, saying (1.7c) and (1.8c), in the continuous sense. We use the monotone operator theory to obtain the well-posedness of **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)** (see Thms. 2.1 and 2.2). Moreover, we obtain the error of penalty (see Thms. 2.3 and 2.4).

Secondly, we study the finite element discretization of two types of mixed penalty approaches. The unique existence is obtained by using the monotone operator theory (see Thms. 3.1 and 3.2). We conduct a convergence analysis of two discrete mixed penalty problems (see Thms. 4.1 and 4.3), with the error estimates dependent on both mesh size h and penalty parameter ε. In addition, we devise two algorithms to solve the discrete mixed penalty problems **(MP$_\varepsilon$-I)**$_h$ and **(MP$_\varepsilon$-II)**$_h$ (see Algorithms 1 and 2, respectively), which mimic the Active/Inactive Set algorithm (or called the primal-dual active set strategy [23, 24, 27]). Such strategy has also been applied to solve the Stokes problem with a frictional condition [25].

The rest of this paper is organized as follows. Section 2 is devoted to the analysis of the two kinds of mixed penalty approaches in the continuous sense. The well-posedness and iteration schemes of the discrete mixed penalty methods are developed in Section 3. The error estimates of the discrete problems are established in Section 4. Section 5 reports the numerical experiments.

Notations. Throughout this paper, we denoted the C by the general positive constant depending on Ω. Note that C is independent of the mesh size h and the penalty parameter ε. The constant C_{E1} of the inverse trace inequality (2.4), the constant C_{T1} of the discrete inverse trace inequality (3.5), the constants C_1 and C_2 of the elliptic regularities (2.8b) and (3.11b), and the constant C_3 and C_4 of the inf-sup conditions (3.13) and (3.9), are all independent of h.

2. The mixed penalty methods

This section is devoted to the well-posedness and error analysis of two penalty approaches in the continuous sense. We set the notations of the inner and dual products:

$$
\begin{aligned}
&(a,b) := \int_\Omega a\cdot b\,\mathrm{d}x \quad (\forall a,b\in L^2(\Omega) \text{ or } L^2(\Omega)^d), \quad (a,b)_{\Gamma_S} := \int_{\Gamma_S} ab\,\mathrm{d}\Gamma \qquad (\forall a,b\in L^2(\Gamma_S)),\\
&\langle a,b\rangle_{\Gamma_S} := \int_{\Gamma_S} ab\,\mathrm{d}\Gamma \qquad (\forall a\in H^{-\frac12}(\Gamma_S), b\in H^{\frac12}_{00}(\Gamma_S)).
\end{aligned}
$$

For $a\in H^{-\frac12}(\Gamma_S)$ and $b\in H^{\frac12}_{00}(\Gamma_S)$, we adopt the same notation in ([28], p. 76), which uses the integral $\int_{\Gamma_S} ab\,\mathrm{d}\Gamma$ to represent a continuous linear functional (associated with a) acting on b. If $a\in L^2(\Gamma_S)$, then $\langle a,b\rangle_{\Gamma_S}$ is same to the L^2 inner product of a and b (noting that $b\in H^{\frac12}_{00}(\Gamma_S)\subset L^2(\Gamma_S)$). We denote by ${}_{B^*}\langle a,b\rangle_B$ the dual product between the Banach space B and its dual B^*.

Set the function spaces:

$$
\begin{aligned}
&\mathring{V} := \{\boldsymbol{\tau}\in V : \nabla\cdot\boldsymbol{\tau} = 0\}, \quad V_{\Gamma_S} := \{\boldsymbol{\tau}\in V : \langle\boldsymbol{\tau}\cdot\boldsymbol{n}, v\rangle_{\Gamma_S} = 0\ (\forall v\in X)\},\\
&\tilde{V} := \{\boldsymbol{\tau}\in V : \langle\boldsymbol{\tau}\cdot\boldsymbol{n}, v\rangle_{\Gamma_S} \geq 0\ (\forall v\in X \text{ with } v|_{\Gamma_S}\geq 0 \text{ a.e. })\},\\
&\tilde{\Lambda} := \{p\in\Lambda : p\geq 0 \text{ a.e.}\}, \quad \tilde{\Lambda}_{1/2} := \{p\in\Lambda_{1/2} : p\geq 0 \text{ a.e.}\},\\
&\tilde{\Lambda}_{-1/2} := \{\mu\in H^{-\frac12}(\Gamma_S) : \langle\mu,p\rangle_{\Gamma_S}\geq 0\ (\forall p\in\tilde{\Lambda}_{1/2})\}.
\end{aligned}
$$

We refer to ([8], (2.1.21)) and [17] for the definition of V_{Γ_S} and $\tilde{V}$, respectively. For briefness, we will write $\boldsymbol{\tau}\cdot\boldsymbol{n}\geq 0$ (resp. $p\geq 0$) for $\boldsymbol{\tau}\in\tilde{V}$ (resp. $p\in\tilde{\Lambda}_{-1/2}$). In fact, for any distribution $\mu\in(C_0^\infty(\Gamma_S))^*$, we can define $\mu\geq 0$ by the technique of ([9], I.1.4).

Let us first introduce the mixed variational inequality, which will be used for convergence analysis.

(MVI) Find $(\boldsymbol{\sigma}, u) \in \tilde{V} \times Q$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) + (u, \nabla \cdot (\boldsymbol{\tau} - \boldsymbol{\sigma})) \geq 0 \qquad (\forall \boldsymbol{\tau} \in \tilde{V}), \tag{2.1a}$$

$$(\nabla \cdot \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad (\forall v \in Q). \tag{2.1b}$$

The well-posedness of the mixed variational inequality **(MVI)** was demonstrated in [10, 17]. Introducing the Lagrange multiplier $p \in \Lambda_{1/2}(p = u|_{\Gamma_S}$ if $u \in H^1(\Omega))$, we present two equivalent mixed variational forms:

(MLM-I) Find $(\boldsymbol{\sigma}, u, p) \in \tilde{V} \times Q \times \Lambda_{1/2}$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}, p \rangle_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in V), \tag{2.2a}$$

$$(\nabla \cdot \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad (\forall v \in Q), \tag{2.2b}$$

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{n} - \boldsymbol{\sigma} \cdot \boldsymbol{n}, p \rangle_{\Gamma_S} \geq 0 \qquad (\forall \boldsymbol{\tau} \in \tilde{V}). \tag{2.2c}$$

(MLM-II) Find $(\boldsymbol{\sigma}, u, p) \in V \times Q \times \tilde{\Lambda}_{1/2}$ such that

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}, p \rangle_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in V), \tag{2.3a}$$

$$(\nabla \cdot \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad (\forall v \in Q), \tag{2.3b}$$

$$\langle \boldsymbol{\sigma} \cdot \boldsymbol{n}, q - p \rangle_{\Gamma_S} \geq 0 \qquad (\forall q \in \tilde{\Lambda}_{1/2}). \tag{2.3c}$$

It is not difficult to check that both **(MLM-I)** and **(MLM-II)** are equivalent to **(MVI)**.

Remark 2.1. By the extension theorem (or called the reverse trace inequality), for any $q \in \Lambda_{1/2}$, there exists a $v_q \in X$ satisfying $v_q|_{\Gamma_S} = q$ and

$$\|v_q\|_X \leq C_{E1} \|q\|_{\Lambda_{1/2}}. \tag{2.4}$$

By using (1.4), (2.4) and $v|_{\Gamma_S} \in \Lambda_{1/2}$ for all $v \in X$, we have

$$\|\boldsymbol{\tau} \cdot \boldsymbol{n}\|_{\Lambda_{-1/2}} := \sup_{q \in \Lambda_{1/2}} \frac{\langle \boldsymbol{\tau} \cdot \boldsymbol{n}, q \rangle_{\Gamma_S}}{\|q\|_{\Lambda_{1/2}}} \leq \sup_{v_q \in X} \frac{(\boldsymbol{\tau}, \nabla v_q) + (\nabla \cdot \boldsymbol{\tau}, v_q)}{C_{E1}^{-1} \|v_q\|_X} \leq C_{E1} \|\boldsymbol{\tau}\|_V. \tag{2.5}$$

Lemma 2.1 ([8])**.** *There exists a constant $C_1 > 0$ such that, for any $f \in Q$ and $\xi \in \Lambda_{-1/2}$, one can find a $\boldsymbol{\sigma} \in V$ satisfying*

$$\nabla \cdot \boldsymbol{\sigma} = f \text{ in } \Omega, \quad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \xi \textit{ on } \Gamma_S, \tag{2.6a}$$

$$\|\boldsymbol{\sigma}\|_V \leq C_1 (\|f\|_Q + \|\xi\|_{\Lambda_{-1/2}}). \tag{2.6b}$$

In the following, we develop the well-posedness theory of **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)** by using the nonlinear monotone operator theory. Then we turn to estimate the error of $\boldsymbol{\sigma} - \boldsymbol{\sigma}_\varepsilon$ and $u - u_\varepsilon$ for both two penalty methods.

2.1. The well-posedness of (MP$_\varepsilon$-I)

Theorem 2.1. *There exists a unique solution $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon) \in V \times Q \times \Lambda_{1/2}$ of* **(MP$_\varepsilon$-I)** *satisfying*

$$\|\boldsymbol{\sigma}_\varepsilon\|_V^2 + \|u_\varepsilon\|_Q^2 + \frac{1}{\varepsilon} \|[p_\varepsilon]_-\|_{L^2(\Gamma_S)}^2 + \|p_\varepsilon\|_{\Lambda_{1/2}}^2 \leq C \|f\|_Q^2. \tag{2.7}$$

Proof. (*Unique existence*) According to Lemma 2.1 (in the case of $\xi = 0$), for any $f \in Q$, there exists a $\boldsymbol{\sigma}_f \in V$ satisfying

$$\nabla \cdot \boldsymbol{\sigma}_f = f \text{ in } \Omega, \quad \boldsymbol{\sigma}_f \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_S, \tag{2.8a}$$

$$\|\boldsymbol{\sigma}_f\|_V \le C_1\|f\|_Q, \tag{2.8b}$$

where C_1 is the same as that of (2.6b). Since (2.8) and $V_{\Gamma_S} \subset \hat{V} \subset V$, we have

$$C_1^{-1}\|v\|_Q \le \sup_{\boldsymbol{\tau}\in V_{\Gamma_S}} \frac{(v,\nabla\cdot\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_V} \le \sup_{\boldsymbol{\tau}\in \hat{V}} \frac{(v,\nabla\cdot\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_V} \le \sup_{\boldsymbol{\tau}\in V} \frac{(v,\nabla\cdot\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_V} \quad (\forall v\in Q). \tag{2.9}$$

Given $\boldsymbol{\sigma}\in\hat{V}$, the above inf-sup condition ensures the unique existence of $u\in Q$ satisfying

$$(u,\nabla\cdot\boldsymbol{\xi}) = (-\boldsymbol{\sigma},\boldsymbol{\xi}) \quad \forall\boldsymbol{\xi}\in V_{\Gamma_S}, \tag{2.10a}$$

$$\|u\|_Q \le C_1 \sup_{\boldsymbol{\xi}\in V_{\Gamma_S}} \frac{(u,\nabla\cdot\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_V} = C_1 \sup_{\boldsymbol{\xi}\in V_{\Gamma_S}} \frac{(-\boldsymbol{\sigma},\boldsymbol{\xi})}{\|\boldsymbol{\xi}\|_V} \le C_1\|\boldsymbol{\sigma}\|_Q. \tag{2.10b}$$

Next, we consider the problem:
(MP$_\varepsilon$-I)$^\circ$ Find $(\mathring{\boldsymbol{\sigma}}_\varepsilon, p_\varepsilon)\in\mathring{V}\times\Lambda_{1/2}$ such that:

$$(\mathring{\boldsymbol{\sigma}}_\varepsilon,\mathring{\boldsymbol{\tau}}) - \langle\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n},p_\varepsilon\rangle_{\Gamma_S} = (\boldsymbol{\sigma}_f,\mathring{\boldsymbol{\tau}}) \qquad (\forall\mathring{\boldsymbol{\tau}}\in\mathring{V}), \tag{2.11a}$$

$$\langle\mathring{\boldsymbol{\sigma}}_\varepsilon\cdot\boldsymbol{n},q\rangle_{\Gamma_S} - \frac{1}{\varepsilon}([p_\varepsilon]_-,q)_{\Gamma_S} = 0 \qquad (\forall q\in\Lambda_{1/2}). \tag{2.11b}$$

If (2.11) admits a unique solution $(\mathring{\boldsymbol{\sigma}}_\varepsilon, p_\varepsilon)$, then by taking $\boldsymbol{\sigma}_\varepsilon := \mathring{\boldsymbol{\sigma}}_\varepsilon - \boldsymbol{\sigma}_f$ and using the inf-sup condition (2.9), we assert the unique existence of u_ε satisfying

$$(u_\varepsilon,\nabla\cdot\boldsymbol{\tau}) = -(\boldsymbol{\sigma}_\varepsilon,\boldsymbol{\tau}) + \langle\boldsymbol{\tau}\cdot\boldsymbol{n},p_\varepsilon\rangle_{\Gamma_S} \quad (\forall\boldsymbol{\tau}\in V), \tag{2.12a}$$

$$\|u_\varepsilon\|_Q \le C_1(\|\boldsymbol{\sigma}_\varepsilon\|_Q + \|p_\varepsilon\|_{\Lambda_{1/2}}). \tag{2.12b}$$

To conclude the existence of **(MP$_\varepsilon$-I)**, it remains to show the well-posedness of (2.11). To this end, we set

$$M := \mathring{V}\times\Lambda_{1/2}, \quad M_f := \{(\mathring{\boldsymbol{\sigma}},p)\in M : (\mathring{\boldsymbol{\sigma}},\mathring{\boldsymbol{\tau}}) - \langle\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n},p\rangle_{\Gamma_S} = (\boldsymbol{\sigma}_f,\mathring{\boldsymbol{\tau}})\ (\forall\mathring{\boldsymbol{\tau}}\in\mathring{V})\}.$$

Note that M_f is a closed subspace of M. Now we define the operator $\mathcal{A}^I_\varepsilon : M\to M^*$ by

$${}_{M^*}\langle\mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}},p),(\mathring{\boldsymbol{\tau}},q)\rangle_M := (\mathring{\boldsymbol{\sigma}},\mathring{\boldsymbol{\tau}}) - \langle\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n},p\rangle_{\Gamma_S} + \langle\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n},q\rangle_{\Gamma_S} - \frac{1}{\varepsilon}([p]_-,q)_{\Gamma_S} \quad (\forall(\mathring{\boldsymbol{\sigma}},p),(\mathring{\boldsymbol{\tau}},q)\in M).$$

The following properties hold for $\mathcal{A}^I_\varepsilon$.

(1) $\mathcal{A}^I_\varepsilon$ is coercive on M_f. By using Lemma 2.1 (in the case of $f=0$), for any $\mu\in\Lambda_{-1/2}$, there exist a $\boldsymbol{\tau}_\mu\in\mathring{V}$ such that

$$\boldsymbol{\tau}_\mu\cdot\boldsymbol{n} = \mu \text{ on } \Gamma_S, \quad \|\boldsymbol{\tau}_\mu\|_V \le C_1\|\mu\|_{\Lambda_{-1/2}}. \tag{2.13}$$

Following from (2.13) and the duality of $\Lambda_{1/2}$ and $\Lambda_{-1/2}$, we have the inf-sup condition

$$\begin{aligned}\|p\|_{\Lambda_{1/2}} &= \sup_{\mu\in\Lambda_{-1/2}} \frac{\langle\mu,p\rangle_{\Gamma_S}}{\|\mu\|_{\Lambda_{-1/2}}} \le \sup_{\boldsymbol{\tau}\in V} \frac{\langle\boldsymbol{\tau}\cdot\boldsymbol{n},p\rangle_{\Gamma_S}}{\|\boldsymbol{\tau}\cdot\boldsymbol{n}\|_{\Lambda_{-1/2}}} \\ &\le C_1 \sup_{\boldsymbol{\tau}\in\mathring{V}} \frac{\langle\boldsymbol{\tau}\cdot\boldsymbol{n},p\rangle_{\Gamma_S}}{\|\boldsymbol{\tau}\|_V} \le C_1 \sup_{\boldsymbol{\tau}\in V} \frac{\langle\boldsymbol{\tau}\cdot\boldsymbol{n},p\rangle_{\Gamma_S}}{\|\boldsymbol{\tau}\|_V} \quad (\forall p\in\Lambda_{1/2}).\end{aligned} \tag{2.14}$$

For any pair $(\mathring{\boldsymbol{\sigma}},p)\in M_f$, by using (2.14), we have:

$$\|p\|_{\Lambda_{1/2}} \le C_1 \sup_{\mathring{\boldsymbol{\tau}}\in\mathring{V}} \frac{\langle\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n},p\rangle_{\Gamma_S}}{\|\mathring{\boldsymbol{\tau}}\|_V} = C_1 \sup_{\mathring{\boldsymbol{\tau}}\in\mathring{V}} \frac{(\mathring{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_f,\mathring{\boldsymbol{\tau}})}{\|\mathring{\boldsymbol{\tau}}\|_V} \le C_1(\|\mathring{\boldsymbol{\sigma}}\|_Q + \|\boldsymbol{\sigma}_f\|_Q). \tag{2.15}$$

In addition, for all $(\mathring{\boldsymbol{\sigma}}, p) \in M$,

$$
{}_{M^*}\langle \mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}}, p), (\mathring{\boldsymbol{\sigma}}, p)\rangle_M = \|\mathring{\boldsymbol{\sigma}}\|^2_Q + \frac{1}{\varepsilon}\|[p]_-\|^2_\Lambda. \tag{2.16}
$$

Following from (2.15) and (2.16), we validate the coercivity: for $(\mathring{\boldsymbol{\sigma}}, p) \in M_f$,

$$
\frac{{}_{M^*}\langle \mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}}, p), (\mathring{\boldsymbol{\sigma}}, p)\rangle_M}{\|\mathring{\boldsymbol{\sigma}}\|_Q + \|p\|_{\Lambda_{1/2}}} \geq \frac{\|\mathring{\boldsymbol{\sigma}}\|^2_Q + \varepsilon^{-1}\|[p]_-\|^2_\Lambda}{\|\mathring{\boldsymbol{\sigma}}\|_Q + C_1(\|\mathring{\boldsymbol{\sigma}}\|_Q + \|\boldsymbol{\sigma}_f\|_Q)} \to \infty \quad \text{ as } \|\mathring{\boldsymbol{\sigma}}\|_Q + \|p\|_{\Lambda_{1/2}} \to \infty.
$$

(2) $\mathcal{A}^I_\varepsilon$ is continuous and bounded. By using (2.5) and $|[a]_- - [b]_-| \leq |a-b|$, we have: for any $(\mathring{\boldsymbol{\sigma}}_1, p_1), (\mathring{\boldsymbol{\sigma}}_2, p_2) \in M$,

$$
\begin{aligned}
\|A^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_1, p_1) - A^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_2, p_2)\|_{M^*} &= \sup_{(\mathring{\boldsymbol{\tau}}, q)\in M} \frac{{}_{M^*}\langle \mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_1, p_1) - \mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_2, p_2), (\mathring{\boldsymbol{\tau}}, q)\rangle_M}{\|\mathring{\boldsymbol{\tau}}\|_V + \|q\|_{\Lambda_{1/2}}} \\
&\leq C(\|\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2\|_V + \|p_1 - p_2\|_{\Lambda_{1/2}} + \varepsilon^{-1}\|p_1 - p_2\|_\Lambda).
\end{aligned}
$$

By taking $(\mathring{\boldsymbol{\sigma}}_2, p_2) = (0, 0)$, we get the boundedness of A^I_ε.

(3) $\mathcal{A}^I_\varepsilon$ is strongly monotone on M_f. For any $(\mathring{\boldsymbol{\sigma}}_1, p_1), (\mathring{\boldsymbol{\sigma}}_2, p_2) \in M$,

$$
{}_{M^*}\langle A^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_1, p_1) - \mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_2, p_2), (\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2, p_1 - p_2)\rangle_M \geq \|\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2\|^2_Q + \frac{1}{\varepsilon}\|[p_1]_- - [p_2]_-\|^2_\Lambda, \tag{2.17}
$$

where we have used

$$
\begin{aligned}
-([p_1]_- - [p_2]_-)(p_1 - p_2) &= |[p_1]_- - [p_2]_-|^2 + [p_1]_-[p_2]_+ + [p_2]_-[p_1]_+ \\
&\geq |[p_1]_- - [p_2]_-|^2. \quad ([s]_+ := \max(0, s)).
\end{aligned} \tag{2.18}
$$

In addition, for any $(\mathring{\boldsymbol{\sigma}}_1, p_1), (\mathring{\boldsymbol{\sigma}}_2, p_2) \in M_f$, we have

$$
(\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2, \mathring{\boldsymbol{\tau}}) - \langle \mathring{\boldsymbol{\tau}} \cdot \boldsymbol{n}, p_1 - p_2\rangle_{\Gamma_S} = 0 \quad (\forall \mathring{\boldsymbol{\tau}} \in \mathring{V}).
$$

Applying the inf-sup condition (2.14) to the above equation, similar to (2.15), we get

$$
\|p_1 - p_2\|_{\Lambda_{1/2}} \leq C_1\|\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2\|_Q. \tag{2.19}
$$

A combination of (2.17) and (2.19) implies the strong monotonicity of A^I_ε on M_f, that is

$$
{}_{M^*}\langle A^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_1, p_1) - \mathcal{A}^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_2, p_2), (\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2, p_1 - p_2)\rangle_M \geq C(\|\mathring{\boldsymbol{\sigma}}_1 - \mathring{\boldsymbol{\sigma}}_2\|^2_Q + \|p_1 - p_2\|^2_{\Lambda_{1/2}}) + \frac{1}{\varepsilon}\|[p_1]_- - [p_2]_-\|^2_\Lambda.
$$

According to the monotone operator theory (*cf.* [36]), the equation $A^I_\varepsilon(\mathring{\boldsymbol{\sigma}}_\varepsilon, p_\varepsilon) = (\boldsymbol{\sigma}_f, 0)$, or equivalently (2.11), admits a unique solution $(\mathring{\boldsymbol{\sigma}}_\varepsilon, p_\varepsilon) \in M$.

(*A-prior estimate*) Substituting $(\boldsymbol{\tau}, v, q) = (\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon)$ into (1.5), we get

$$
\|\boldsymbol{\sigma}_\varepsilon\|^2_Q + \frac{1}{\varepsilon}\|[p_\varepsilon]_-\|^2_\Lambda = \|\boldsymbol{\sigma}_\varepsilon\|^2_Q + \varepsilon\|\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}\|^2_\Lambda = (f, u_\varepsilon) \leq \|f\|_Q\|u_\varepsilon\|_Q.
$$

Together with (2.12b), (2.15) and (2.8b), we conclude (2.7). □

2.2. The well-posedness of (MP$_\varepsilon$-II)

Theorem 2.2. *There exists a unique solution $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon) \in \hat{V} \times Q$ of* **(MP$_\varepsilon$-II)** *with the estimate*

$$
\|\boldsymbol{\sigma}_\varepsilon\|^2_V + \|u_\varepsilon\|^2_Q + \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_-\|^2_\Lambda \leq C\|f\|^2_Q. \tag{2.20}
$$

Proof. (*A-prior estimate*) Substituting $(\boldsymbol{\tau}, v) = (\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ into (1.6), we obtain

$$\|\boldsymbol{\sigma}_\varepsilon\|_Q^2 + \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_-\|_\Lambda^2 = (f, u_\varepsilon) \leq \|f\|_Q \|u_\varepsilon\|_Q. \tag{2.21}$$

By the inf-sup condition (2.10b),

$$\|u_\varepsilon\|_Q \leq C_1 \sup_{\boldsymbol{\tau}\in V_{\Gamma_S}} \frac{(u_\varepsilon, \nabla\cdot\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_V} \overset{(1.6a)}{=} C_1 \sup_{\boldsymbol{\tau}\in V_{\Gamma_S}} \frac{-(\boldsymbol{\sigma}_\varepsilon, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_V} \leq C_1 \|\boldsymbol{\sigma}_\varepsilon\|_Q. \tag{2.22}$$

Together with (2.21) and $\nabla\cdot\boldsymbol{\sigma}_\varepsilon = -f$, we conclude (2.20).
(*Unique existence*) Recall the definition of $\boldsymbol{\sigma}_f$ (see (2.8)) and consider the problem:
(MP$_\varepsilon$-II)$^\circ$ Find $\mathring{\boldsymbol{\sigma}}_\varepsilon \in \hat{\mathring{V}} := \mathring{V} \cap \hat{V}$ such that

$$(\mathring{\boldsymbol{\sigma}}_\varepsilon, \mathring{\boldsymbol{\tau}}) - \frac{1}{\varepsilon}([\mathring{\boldsymbol{\sigma}}_\varepsilon\cdot\boldsymbol{n}]_-, \mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}_f, \mathring{\boldsymbol{\tau}}) \quad (\forall \mathring{\boldsymbol{\tau}} \in \hat{\mathring{V}}). \tag{2.23}$$

Obviously, if $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ solves **(MP$_\varepsilon$-II)**, then $\mathring{\boldsymbol{\sigma}}_\varepsilon := \boldsymbol{\sigma}_\varepsilon + \boldsymbol{\sigma}_f$ satisfies (2.23). Conversely, if (2.23) admits a solution $\mathring{\boldsymbol{\sigma}}_\varepsilon$, then by taking $\boldsymbol{\sigma}_\varepsilon := \mathring{\boldsymbol{\sigma}}_\varepsilon - \boldsymbol{\sigma}_f$ and using the inf-sup condition (2.10b), there exists a unique $u_\varepsilon \in Q$ such that $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ solves **(MP$_\varepsilon$-II)**. The well-posedness of **(MP$_\varepsilon$-II)$^\circ$** is demonstrated below. Define the operator $\mathcal{A}_\varepsilon^{\mathrm{II}} : \hat{\mathring{V}} \to \hat{\mathring{V}}^*$,

$$_{\hat{\mathring{V}}^*}\langle \mathcal{A}_\varepsilon^{\mathrm{II}}\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\tau}}\rangle_{\hat{\mathring{V}}} := (\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\tau}}) - \frac{1}{\varepsilon}([\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n}]_-, \mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n})_{\Gamma_S} \quad (\forall \mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\tau}} \in \hat{\mathring{V}}).$$

The following properties hold for $\mathcal{A}_\varepsilon^{\mathrm{II}}$.

(1) $\mathcal{A}_\varepsilon^{\mathrm{II}}$ is coercive:

$$_{\hat{\mathring{V}}^*}\langle \mathcal{A}_\varepsilon^{\mathrm{II}}\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\sigma}}\rangle_{\hat{\mathring{V}}} = \|\mathring{\boldsymbol{\sigma}}\|_V^2 + \frac{1}{\varepsilon}\|[\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n}]_-\|_\Lambda^2 \quad (\forall \mathring{\boldsymbol{\sigma}} \in \hat{\mathring{V}}).$$

(2) $\mathcal{A}_\varepsilon^{\mathrm{II}}$ is continuous and bounded. In fact, we have: for any $\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\tau}} \in \hat{\mathring{V}}$,

$$\|\mathcal{A}_\varepsilon^{\mathrm{II}}\mathring{\boldsymbol{\sigma}} - \mathcal{A}_\varepsilon^{\mathrm{II}}\mathring{\boldsymbol{\tau}}\|_{\hat{\mathring{V}}^*} \leq \|\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{\tau}}\|_V + \varepsilon^{-1}\|[\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n}]_- - [\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n}]_-\|_\Lambda.$$

(3) $\mathcal{A}_\varepsilon^{\mathrm{II}}$ is strictly monotone: for any $\mathring{\boldsymbol{\sigma}}, \mathring{\boldsymbol{\tau}} \in \hat{\mathring{V}}$,

$$\begin{aligned} _{\hat{\mathring{V}}^*}\langle A_\varepsilon^{\mathrm{II}}\mathring{\boldsymbol{\sigma}} - A_\varepsilon^{\mathrm{II}}\mathring{\boldsymbol{\tau}}, \mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{\tau}}\rangle_{\hat{\mathring{V}}} &= \|\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{\tau}}\|_Q^2 - \frac{1}{\varepsilon}(([\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n}]_- - [\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n}]_-), (\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n} - \mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n}))_{\Gamma_S} \\ &\geq \|\mathring{\boldsymbol{\sigma}} - \mathring{\boldsymbol{\tau}}\|_V^2 + \frac{1}{\varepsilon}\|[\mathring{\boldsymbol{\sigma}}\cdot\boldsymbol{n}]_- - [\mathring{\boldsymbol{\tau}}\cdot\boldsymbol{n}]_-\|_\Lambda^2. \end{aligned}$$

It follows from properties (1)–(3) that (2.23) admits a unique solution $\mathring{\boldsymbol{\sigma}}_\varepsilon$. □

2.3. The penalty error of (MP$_\varepsilon$-I)

To simplify the notation, we set

$$\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_\varepsilon, \quad e_{p,\varepsilon} := p - p_\varepsilon, \quad e_{u,\varepsilon} := u - u_\varepsilon.$$

Theorem 2.3. *Let $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon)$ and $(\boldsymbol{\sigma}, u, p)$ be the unique solution of* **(MP$_\varepsilon$-I)** *and* **(MLM-I)** *respectively. Assume that $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in \Lambda$. Then we have the error estimate*

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \|e_{p,\varepsilon}\|_{\Lambda_{1/2}} + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\cdot\boldsymbol{n}\|_\Lambda \leq C\sqrt{\varepsilon}\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_\Lambda. \tag{2.24}$$

Additionally, if $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in \Lambda_{1/2}$, then we have

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \|e_{p,\varepsilon}\|_{\Lambda_{1/2}} + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\cdot\boldsymbol{n}\|_\Lambda \leq C\varepsilon\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{\Lambda_{1/2}}. \tag{2.25}$$

Remark 2.2. Before starting the proof, we make a comment on the regularity assumptions, because the error estimates of Theorems 2.3, 2.4, 4.1 and 4.3 all rely strongly on the regularity of $(\boldsymbol{\sigma}, u, p)$. Note that $(\boldsymbol{\sigma}, p) = (\nabla u, u|_{\Gamma_S})$ if the solution u of **(P)** is sufficiently smooth. It is known that u has the H^2-regularity if the domain is C^2-smooth or convex polygonal [9]. However, contact problems are restricted in terms of regularity on Γ_S due to the Signorini condition (1.3d). Singularities arise at contact $(u = 0)$-noncontact $(u > 0)$ transitions [13]. Recently, Apel and Nicaise [3] has studied the regularity of the Signorini problem in a general polygonal domain and has investigated the leading singularity at the transition points.

To fulfill the assumption $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda$ of Theorems 2.3 and 2.4, we need $u \in H^s(\Omega)$ $(s \geq 3/2)$ such that $\nabla u \cdot \boldsymbol{n} \in H^{s-\frac{3}{2}}(\Gamma_S) \subset \Lambda$ $(s \geq 3/2)$. To guarantee that $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda_{1/2}$ (the assumption of Thm. 2.3 to get $O(\varepsilon)$-convergence), we require that $u \in H^s(\Omega)$ $(s \geq 2)$ so that $\nabla u \cdot \boldsymbol{n} \in H^{s-\frac{3}{2}}(\Gamma_S) \subset \Lambda_{1/2}$ $(s \geq 2)$. In Theorems 4.1 and 4.3, we require more regularity assumptions (see Rem. 4.1 for details).

Proof of Theorem 2.3. Under the assumption that $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda$, it follows from (1.5) and (2.2) that

$$(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}, \boldsymbol{\tau}) + (e_{u,\varepsilon}, \nabla \cdot \boldsymbol{\tau}) - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}, e_{p,\varepsilon} \rangle_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau} \in V), \tag{2.26a}$$

$$\nabla \cdot \boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} = 0 \qquad \text{in } \Omega. \tag{2.26b}$$

Substituting $\boldsymbol{\tau} = \boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}$ into (2.26a) results

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q^2 - \langle \boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}, e_{p,\varepsilon} \rangle_{\Gamma_S} = \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q^2 - (\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}, e_{p,\varepsilon})_{\Gamma_S} = 0. \tag{2.27}$$

Note that $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda$, $\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} = \frac{1}{\varepsilon}[p_\varepsilon]_- \in \Lambda$, and $p, p_\varepsilon \in \Lambda_{1/2} \subset \Lambda$, we can replace the dual-product $\langle \cdot, \cdot \rangle_{\Gamma_S}$ of (2.27) with the inner-product $(\cdot, \cdot)_{\Gamma_S}$.

By using $(\boldsymbol{\sigma} \cdot \boldsymbol{n}, p)_{\Gamma_S} = 0$ and $\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} = \frac{1}{\varepsilon}[p_\varepsilon]_-$, we calculate $-(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}, e_{p,\varepsilon})_{\Gamma_S}$ as follows.

$$\begin{aligned}
-(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}, e_{p,\varepsilon})_{\Gamma_S} &= -(\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}, p - p_\varepsilon)_{\Gamma_S} \\
&= -(\boldsymbol{\sigma} \cdot \boldsymbol{n}, p)_{\Gamma_S} + (\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}, p)_{\Gamma_S} + (\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}, [p_\varepsilon]_+ - [p_\varepsilon]_-)_{\Gamma_S} \\
&= 0 + \frac{1}{\varepsilon}([p_\varepsilon]_-, p)_{\Gamma_s} + (\boldsymbol{\sigma} \cdot \boldsymbol{n}, [p_\varepsilon]_+)_{\Gamma_s} - \left(\frac{1}{\varepsilon}[p_\varepsilon]_-, [p_\varepsilon]_+\right)_{\Gamma_s} + (\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}, -\varepsilon \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n})_{\Gamma_S} \\
&= \left(\frac{1}{\varepsilon}[p_\varepsilon]_-, p\right)_{\Gamma_S} + (\boldsymbol{\sigma} \cdot \boldsymbol{n}, [p_\varepsilon]_+)_{\Gamma_S} - 0 + \varepsilon \|\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}\|_\Lambda^2 \\
&\quad - \varepsilon(\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}, \boldsymbol{\sigma} \cdot \boldsymbol{n})_{\Gamma_S}.
\end{aligned} \tag{2.28}$$

In view of $\boldsymbol{\sigma} \cdot \boldsymbol{n} \geq 0$ and $p \geq 0$, we obtain form (2.27) and (2.28) that

$$\begin{aligned}
\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q^2 + \varepsilon \|\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}\|_\Lambda^2 &\leq \varepsilon(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}, \boldsymbol{\sigma} \cdot \boldsymbol{n})_{\Gamma_S} \\
&\leq \begin{cases} \varepsilon \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}\|_{\Lambda_{-1/2}} \|\boldsymbol{\sigma} \cdot \boldsymbol{n}\|_{\Lambda_{1/2}} & \text{if } \boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda_{1/2}. \\ \frac{\varepsilon}{2}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}\|_\Lambda^2 + \frac{\varepsilon}{2}\|\boldsymbol{\sigma} \cdot \boldsymbol{n}\|_\Lambda^2. & \end{cases}
\end{aligned}$$

Since $\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}\|_{\Lambda_{-1/2}} \leq C_{E1} \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V = C_{E1} \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q$ (by (2.5) and $\nabla \cdot \boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} = 0$), we get

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V^2 + \varepsilon \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \cdot \boldsymbol{n}\|_\Lambda^2 \leq \begin{cases} C\varepsilon^2 \|\boldsymbol{\sigma} \cdot \boldsymbol{n}\|_{\Lambda_{1/2}}^2 & \text{if } \boldsymbol{\sigma} \cdot \boldsymbol{n} \in \Lambda_{1/2}, \\ C\varepsilon \|\boldsymbol{\sigma} \cdot \boldsymbol{n}\|_\Lambda^2. & \end{cases} \tag{2.29}$$

Applying the inf-sup condition (2.9) to (2.26a) with V being replaced by V_{Γ_S}, we have

$$\|e_{u,\varepsilon}\|_Q \leq C_1 \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q. \tag{2.30}$$

Furthermore, applying the inf-sup condition (2.14) to (2.26a) yields

$$\|e_{p,\varepsilon}\|_{\Lambda_{1/2}} \leq C_1(\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q + \|e_{u,\varepsilon}\|_Q). \tag{2.31}$$

A combination of (2.29)–(2.31) results in (2.24) and (2.25). ∎

2.4. The penalty error of (MP$_\varepsilon$-II)

We set $p_\varepsilon := \frac{1}{\varepsilon}[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_- \in \tilde{\Lambda}$. Notice that $e_{p,\varepsilon} = p - p_\varepsilon \in \Lambda$.

Theorem 2.4. *Assume that $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in \Lambda$. Let $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ and $(\boldsymbol{\sigma}, u)$ be the unique solution of* **(MP$_\varepsilon$-II)** *and* **(MVI)**. *We have the error estimate*

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \sqrt{\varepsilon}\|e_{p,\varepsilon}\|_\Lambda \le C\sqrt{\varepsilon}\|p\|_\Lambda. \tag{2.32}$$

In addition, if $[\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}]_- \in \Lambda_{1/2}$ (or equivalently, $p_\varepsilon \in \Lambda_{1/2}$), then we have

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|e_{u,\varepsilon}\|_Q + \sqrt{\varepsilon}\|e_{p,\varepsilon}\|_\Lambda + \|e_{p,\varepsilon}\|_{\Lambda_{1/2}} \le C\varepsilon\|p\|_{\Lambda_{-1/2}}. \tag{2.33}$$

Proof. Subtracting (1.6a) from (2.3a), we get

$$(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}, \boldsymbol{\tau}) + (e_{u,\varepsilon}, \nabla\cdot\boldsymbol{\tau}) - (\boldsymbol{\tau}\cdot\boldsymbol{n}, e_{p,\varepsilon})_{\Gamma_S} = 0 \quad (\forall \boldsymbol{\tau} \in \hat{V}). \tag{2.34}$$

It follows from (2.3b) and (1.6b) that $\nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} = 0$. Under the assumption $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in \Lambda$), substituting $\boldsymbol{\tau} = \boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon} \in \hat{V}$ into (2.34) yields

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q^2 + 0 - (\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\cdot\boldsymbol{n}, e_{p,\varepsilon})_{\Gamma_S} = 0. \tag{2.35}$$

The second term of the above equation is calculated as follows.

$$\begin{aligned}
-(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\cdot\boldsymbol{n}, e_{p,\varepsilon})_{\Gamma_S} &= -(\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}, p - p_\varepsilon)_{\Gamma_S} \\
&= -(\boldsymbol{\sigma}\cdot\boldsymbol{n}, p)_{\Gamma_S} + (\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}, p - p_\varepsilon)_{\Gamma_S} + (\boldsymbol{\sigma}\cdot\boldsymbol{n}, p_\varepsilon)_{\Gamma_S} \\
&\ge ([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_+ - [\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_-, p - p_\varepsilon)_{\Gamma_S} \quad (\text{by } (\boldsymbol{\sigma}\cdot\boldsymbol{n}, p)_{\Gamma_S} = 0,\ \boldsymbol{\sigma}\cdot\boldsymbol{n} \ge 0,\ p_\varepsilon \ge 0) \\
&= ([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_+, p)_{\Gamma_S} - \frac{1}{\varepsilon}([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_+, [\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_-) + (-\varepsilon p_\varepsilon, p - p_\varepsilon)_{\Gamma_S} \quad (\text{by } \varepsilon p_\varepsilon = [\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_-) \\
&\ge \varepsilon\|p - p_\varepsilon\|_\Lambda^2 - \varepsilon(p, p - p_\varepsilon)_{\Gamma_S} \quad (\text{by } p \ge 0,\ [s]_+[s]_- = 0\ (\forall s \in \mathbb{R})).
\end{aligned} \tag{2.36}$$

In view of $p \in \Lambda_{1/2}$, we insert (2.36) into (2.35) to obtain

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_Q^2 + \varepsilon\|p - p_\varepsilon\|_\Lambda^2 \le \varepsilon(p, p - p_\varepsilon)_{\Gamma_S} \le \begin{cases} \varepsilon\|p\|_\Lambda\|p - p_\varepsilon\|_\Lambda, & \\ \varepsilon\|p\|_{\Lambda_{-1/2}}\|p - p_\varepsilon\|_{\Lambda_{1/2}} & \text{if } p_\varepsilon \in \Lambda_{1/2}. \end{cases} \tag{2.37}$$

Replacing the test function space $\hat{V}$ of (2.34) with the subspace V_{Γ_S} and applying the inf-sup condition (2.22), we get (2.30).

Moreover, if $p_\varepsilon \in \Lambda_{1/2}$, then $e_{p,\varepsilon} \in \Lambda_{1/2}$ and we have (instead of (2.34)):

$$(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}, \boldsymbol{\tau}) + (e_{u,\varepsilon}, \nabla\cdot\boldsymbol{\tau}) - \langle\boldsymbol{\tau}\cdot\boldsymbol{n}, e_{p,\varepsilon}\rangle_{\Gamma_S} = 0 \quad (\forall \boldsymbol{\tau} \in V),$$

which, by using the inf-sup condition (2.14), implies (2.31). Combining (2.37), (2.30) and (2.31), we conclude (2.32) and (2.33). □

3. The discrete mixed penalty problems

In this section, we utilize the RT_0/P_0 element to discretize the mixed penalty problems **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)**. We will focus on examining the well-posedness of the discrete mixed penalty problems. Furthermore, we introduce two iteration algorithms based on two mixed penalty problems respectively, and examine their convergence.

To avoid the issue of geometric domain approximation, we assume that Ω is a polygon($d = 2$) or polyhedron($d = 3$) and introduce a quasi-uniform regular triangulation $\mathcal{T}_h$ to Ω, which is a sufficient condition to derive the discrete inverse trace inequality (see (3.5)). The mesh $\mathcal{T}_h$ is quasi-uniform, *i.e.*, there exists

$C > 0$ independent of h and E such that $Ch \geq h_E$ ($\forall E \in \mathcal{T}_h$), where $h = \max_{E\in\mathcal{T}_h} h_E$ is the mesh size, and $h_E = \mathrm{diam}(E)$ is the diameter of E. We denote by $\mathcal{E}_h$ the set of element faces, by $\mathcal{E}_{h,\Gamma_S}$ the mesh on Γ_S inherited from $\mathcal{T}_h$. We introduce the finite element spaces:

$$\begin{aligned}
&V_h := \{\boldsymbol{\tau}_h \in V : \boldsymbol{\tau}_h|_E \in RT_0(E)\ (\forall E \in \mathcal{T}_h)\}, \quad Q_h := \{v_h \in Q : v_h|_E \in P_0(E)\ (\forall E \in \mathcal{T}_h)\},\\
&\Lambda_h := \{p_h \in \Lambda : p_h|_e \in P_0(e)\,(\forall e \in \mathcal{E}_{h,\Gamma_S})\} = \{\boldsymbol{\tau}_h \cdot \boldsymbol{n}|_{\Gamma_S} : \boldsymbol{\tau}_h \in V_h\},\\
&V_{h,\Gamma_S} := \{\boldsymbol{\tau}_h \in V_h : \boldsymbol{\tau}_h \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_S\}, \quad \mathring{V}_h := \{\boldsymbol{\tau}_h \in V_h : \nabla \cdot \boldsymbol{\tau}_h|_E = 0\ (\forall E \in \mathcal{T}_h)\},
\end{aligned}$$

where $RT_0(E) := \{(a,b)^\top + c(x,y)^\top : a,b,c \in \mathbb{R},\ (x,y)^\top \in E\}$, $P_0(E) := \{a : a \in \mathbb{R},\ (x,y)^\top \in E\}$, and $P_0(e) := \{a : a \in \mathbb{R},\ (x,y)^\top \in e\}$.

We define the L^2-projection $P_h : Q \to Q_h$

$$(f - P_h f, v_h) = 0 \qquad \forall v_h \in Q_h. \tag{3.1}$$

The following estimates hold for P_h:

$$\|P_h f\|_Q \leq \|f\|_Q \quad (\forall f \in Q), \qquad \|P_h f - f\|_Q \leq Ch\|f\|_{H^1(\Omega)} \qquad (\forall f \in H^1(\Omega)).$$

We set $f_h := P_h f$.

Set $M(E) := \{\boldsymbol{\tau} \in L^s(E)^d : \nabla \cdot \boldsymbol{\tau} \in L^2(E)\}$ for some $s \in (2, \infty]$. We introduce the interpolation operator (*cf.* [8]):

$$\Phi_h : M(E) \to RT_0(E), \quad \int_e (\boldsymbol{\tau} - \Phi_h \boldsymbol{\tau}) \cdot \boldsymbol{n} \, \mathrm{d}s = 0 \qquad (\forall e \in \partial E). \tag{3.2}$$

Note that when $s = 2$, we have $\boldsymbol{\tau} \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\partial E)$. For any $\varphi_h \in R_k(\partial E) := \{\varphi_h \in L^2(\partial E) : \varphi_h|_e \in P_k(e)\ (\forall e \in \partial E)\}$, the integral $\int_{\partial E} \boldsymbol{\tau} \cdot \boldsymbol{n}\varphi_h \, \mathrm{d}s$ is not well-defined (see [8], Sect. 2.5.1). As it stated in Section 2.5.1 of [8], we need $\boldsymbol{\tau} \in L^s(E)^d$ ($s > 2$) and $\nabla \cdot \boldsymbol{\tau} \in L^2(E)$ to ensure that the integral $\int_{\partial E} \boldsymbol{\tau} \cdot \boldsymbol{n}\varphi_h \, \mathrm{d}s$ is well-defined, as well as the projection $\int_{\partial E}(\boldsymbol{\tau} - \Phi_h\boldsymbol{\tau}) \cdot \boldsymbol{n}\varphi_h \, \mathrm{d}s = 0$ (see [8], Example 2.5.3). Here we consider the case $\varphi_h \in R_0(\partial E)$.

We set $M := V \cap L^s(\Omega)^d$ and define the interpolation operator:

$$\Pi_h : M \to V_h, \qquad \Pi_h \boldsymbol{\tau}|_E = \Phi_h(\boldsymbol{\tau}|_E) \quad (\forall E \in \mathcal{T}_h).$$

For Φ_h and Π_h, the following properties hold (*cf.* [8], Props. 2.5.1–2.5.4):

$$\begin{aligned}
&\mathrm{div}(\Phi_h \boldsymbol{\tau}) = P_{h,E}(\mathrm{div}\boldsymbol{\tau}) && (\forall \boldsymbol{\tau} \in M(E)), && (3.3a)\\
&\mathrm{div}(\Pi_h \boldsymbol{\tau}) = P_h(\mathrm{div}\boldsymbol{\tau}) && (\forall \boldsymbol{\tau} \in M(E)), && (3.3b)\\
&\|\boldsymbol{\tau} - \Phi_h\boldsymbol{\tau}\|_{L^2(E)} \leq Ch\|\boldsymbol{\tau}\|_{H^1(E)} && (\forall \boldsymbol{\tau} \in H^1(E)), && (3.3c)\\
&\|\nabla \cdot (\boldsymbol{\tau} - \Phi_h\boldsymbol{\tau})\|_{L^2(E)} \leq Ch\|\nabla \cdot \boldsymbol{\tau}\|_{H^1(E)} && (\forall \boldsymbol{\tau} \in H^1(E) \text{ with } \nabla \cdot \boldsymbol{\tau} \in H^1(E)), && (3.3d)\\
&\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_Q \leq Ch\|\boldsymbol{\tau}\|_{H^1} && (\forall \boldsymbol{\tau} \in H^1(\Omega)), && (3.3e)\\
&\|\nabla \cdot (\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\|_Q \leq Ch\|\nabla \cdot \boldsymbol{\tau}\|_{H^1} && (\forall \boldsymbol{\tau} \in H^1(\Omega) \text{ with } \nabla \cdot \boldsymbol{\tau} \in H^1(\Omega)), && (3.3f)
\end{aligned}$$

where $P_{h,E}$ represents the L^2-projection from $L^2(E)$ to $P_0(E)$.

Remark 3.1. The properties (3.3a) and (3.3b) can also be expressed as commutativity of the diagrams

Furthermore, we define the L^2-projection operator $P_{\Lambda_h} : \Lambda \to \Lambda_h$ by

$$(P_{\Lambda_h} p - p, q_h)_{\Gamma_S} = 0 \quad (\forall q_h \in \Lambda_h),$$

which satisfies

$$\begin{aligned}
&\|P_{\Lambda_h} p\|_\Lambda \leq \|p\|_\Lambda && (\forall p \in \Lambda),\\
&\|P_{\Lambda_h} p - p\|_\Lambda \leq Ch\|p\|_{H^1(\Gamma_S)} && (\forall p \in H^1(\Gamma_S)).
\end{aligned}$$

3.1. The discretization of $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})$

The discretization of $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})$ is stated as follows. $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h$ Find $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h}) \in V_h \times Q_h \times \Lambda_h$ such that

$$(\boldsymbol{\sigma}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}, \nabla \cdot \boldsymbol{\tau}_h) - (\boldsymbol{\tau}_h \cdot \boldsymbol{n}, p_{\varepsilon h})_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), \tag{3.4a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h), \tag{3.4b}$$

$$\left(\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n} - \frac{1}{\varepsilon}[p_{\varepsilon h}]_-, q_h\right)_{\Gamma_S} = 0 \qquad (\forall q_h \in \Lambda_h). \tag{3.4c}$$

Remark 3.2. By using the discrete inverse trace inequality, there is a constant C_{T1} independent of h such that,

$$\|\boldsymbol{\tau}_h \cdot \boldsymbol{n}\|_\Lambda \le C_{T1} h^{-\frac{1}{2}} \|\boldsymbol{\tau}_h\|_Q \le C_{T1} h^{-\frac{1}{2}} \|\boldsymbol{\tau}_h\|_V \qquad (\forall \boldsymbol{\tau}_h \in V_h). \tag{3.5}$$

Theorem 3.1 (The well-posedness of $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h$). *There exists a unique solution $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h}) \in V_h \times Q_h \times \Lambda_h$ to $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h$ with the estimate*

$$\|\boldsymbol{\sigma}_{\varepsilon h}\|_V + \|u_{\varepsilon h}\|_Q + \frac{1}{\sqrt{\varepsilon}}\|[p_{\varepsilon h}]_-\|_\Lambda + \|p_{\varepsilon h}\|_\Lambda \le C\|f_h\|_Q. \tag{3.6}$$

Proof. The proof is a discrete analogy to the continuous case (see the proof of Thm. 2.1). Given $f \in Q$, let w be the unique solution to the following elliptic problem:

$$\begin{aligned} \Delta w &= f_h && \text{in } \Omega, \\ w &= 0 && \text{on } \Gamma_D, \\ \nabla w \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_S. \end{aligned}$$

We have $w \in W^{1,s}(\Omega)$ with $\|w\|_{W^{1,s}(\Omega)} \le C\|f_h\|_Q$ (for some $s \in (2, \infty)$). Set

$$\boldsymbol{\sigma}_{h,f} := \Pi_h(\nabla w) \in V_{h,\Gamma_S}, \tag{3.7}$$

which satisfies

$$\begin{aligned} &\nabla \cdot \boldsymbol{\sigma}_{h,f} = f_h \quad (i.e.\ (\nabla \cdot \boldsymbol{\sigma}_{h,f}, v_h) = (f, v_h) = (f_h, v_h) \quad (\forall v_h \in Q_h)), \\ &\|\boldsymbol{\sigma}_{h,f}\|_V = C\|f_h\|_Q \le C\|f\|_Q. \end{aligned}$$

Now we consider the following problem:

$(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h^\circ$ Find $(\mathring{\boldsymbol{\sigma}}_{\varepsilon h}, p_{\varepsilon h}) \in \mathring{V}_h \times \Lambda_h$ such that

$$(\mathring{\boldsymbol{\sigma}}_{\varepsilon h}, \mathring{\boldsymbol{\tau}}_h) - (\mathring{\boldsymbol{\tau}}_h \cdot \boldsymbol{n}, p_{\varepsilon h})_{\Gamma_S} = (\boldsymbol{\sigma}_{h,f}, \mathring{\boldsymbol{\tau}}_h) \qquad (\forall \mathring{\boldsymbol{\tau}}_h \in \mathring{V}_h), \tag{3.8a}$$

$$\left(\mathring{\boldsymbol{\sigma}}_{\varepsilon h} \cdot \boldsymbol{n} - \frac{1}{\varepsilon}[p_{\varepsilon h}]_-, q_h\right)_{\Gamma_S} = 0 \qquad (\forall q_h \in \Lambda_h). \tag{3.8b}$$

We only need to establish the well-posedness of $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h^\circ$. In fact, if $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h$ admits a solution $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$, then it is clear that $(\mathring{\boldsymbol{\sigma}}_{\varepsilon h} := \boldsymbol{\sigma}_{\varepsilon h} + \boldsymbol{\sigma}_{h,f}, p_{\varepsilon h})$ solve $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h^\circ$. Conversely, let $(\mathring{\boldsymbol{\sigma}}_{\varepsilon h}, p_{\varepsilon h})$ solve $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h^\circ$. Taking $\boldsymbol{\sigma}_{\varepsilon h} := \mathring{\boldsymbol{\sigma}}_{\varepsilon h} - \boldsymbol{\sigma}_{h,f}$ and applying the inf-sup condition (the discrete version of (2.9))

$$\|u_h\|_Q \le C_4 \sup_{\boldsymbol{\tau}_h \in V_{h,\Gamma_S}} \frac{(u_h, \nabla \cdot \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_V} \le C_4 \sup_{\boldsymbol{\tau}_h \in V_h} \frac{(u_h, \nabla \cdot \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_V} \qquad (\forall u_h \in Q_h) \tag{3.9}$$

to (3.8a), we assert the unique existence of $u_{\varepsilon h} \in Q$ such that $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$ satisfies $(\mathbf{MP}_\varepsilon\text{-}\mathbf{I})_h$. We set

$$M_h := \mathring{V}_h \times \Lambda_h, \quad M_{h,f} := \{(\mathring{\boldsymbol{\sigma}}_h, p_h) \in M_h : (\mathring{\boldsymbol{\sigma}}_h, \mathring{\boldsymbol{\tau}}_h) - (\mathring{\boldsymbol{\tau}}_h \cdot \boldsymbol{n}, p_h)_{\Gamma_S} = (\boldsymbol{\sigma}_{h,f}, \mathring{\boldsymbol{\tau}}_h)\ (\forall \mathring{\boldsymbol{\tau}}_h \in \mathring{V}_h)\},$$

and define the operator $\mathcal{A}^I_{\varepsilon h} : M_h \to M_h^*$ by

$$
\begin{aligned}
{}_{M_h^*}\langle \mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\sigma}}_h, p_h), (\mathring{\boldsymbol{\tau}}_h, q_h)\rangle_{M_h} &:= (\mathring{\boldsymbol{\sigma}}_h, \mathring{\boldsymbol{\tau}}_h) \\
&\quad - (\mathring{\boldsymbol{\tau}}_h \cdot \boldsymbol{n}, p_h)_{\Gamma_S} + (\mathring{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, q_h)_{\Gamma_S} - \frac{1}{\varepsilon}([p_h]_-, q_h)_{\Gamma_S} \qquad (\forall (\mathring{\boldsymbol{\sigma}}_h, p_h), (\mathring{\boldsymbol{\tau}}_h, q_h) \in M_h).
\end{aligned} \tag{3.10}
$$

Given $p_h \in \Lambda_h \subset L^2(\Gamma_S)$, there exists a $w \in W^{1,s}(\Omega)$ (for some $s \in (2, \infty)$) satisfying

$$
\Delta w = 0 \text{ on } \Omega, \quad w = 0 \text{ on } \Gamma_D, \quad \nabla w \cdot \boldsymbol{n} = p_h \text{ on } \Gamma_S, \tag{3.11a}
$$

$$
\|w\|_{W^{1,s}(\Omega)} \le C_2 \|p_h\|_\Lambda. \tag{3.11b}
$$

In view of $\nabla \cdot \boldsymbol{\tau} = 0$, we set $\boldsymbol{\tau} := \nabla w \in M$ and $\boldsymbol{\tau}_h := \Pi_h \boldsymbol{\tau} \in V_h$, which satisfies

$$
\boldsymbol{\tau}_h \cdot \boldsymbol{n} = \boldsymbol{\tau} \cdot \boldsymbol{n} = p_h, \quad \nabla \cdot \boldsymbol{\tau}_h = \nabla \cdot \boldsymbol{\tau} = 0, \tag{3.12a}
$$

$$
\|\boldsymbol{\tau}_h\|_V \le C\|\boldsymbol{\tau}\|_M = C\|\nabla w\|_{L^s(\Omega)} \le C_3 \|p_h\|_\Lambda \quad (C_3 := CC_2). \tag{3.12b}
$$

From (3.12), we have the following inf-sup condition

$$
\|p_h\|_\Lambda = \sup_{q_h \in \Lambda_h} \frac{(p_h, q_h)_{\Gamma_S}}{\|q_h\|_\Lambda} \le C_3 \sup_{\boldsymbol{\tau}_h \in \mathring{V}_h} \frac{(p_h, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S}}{\|\boldsymbol{\tau}_h\|_V} \le C_3 \sup_{\boldsymbol{\tau}_h \in V_h} \frac{(p_h, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S}}{\|\boldsymbol{\tau}_h\|_V} \qquad (\forall p_h \in \Lambda_h), \tag{3.13}
$$

which implies

$$
\begin{aligned}
\|p_h\|_\Lambda \le C_3 \sup_{\mathring{\boldsymbol{\tau}}_h \in \mathring{V}_h} \frac{(p_h, \mathring{\boldsymbol{\tau}}_h \cdot \boldsymbol{n})_{\Gamma_S}}{\|\mathring{\boldsymbol{\tau}}_h\|_V} &= C_3 \sup_{\mathring{\boldsymbol{\tau}}_h \in \mathring{V}_h} \frac{(\mathring{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_{h,f}, \mathring{\boldsymbol{\tau}}_h)}{\|\mathring{\boldsymbol{\tau}}_h\|_V} \\
&\le C_3(\|\mathring{\boldsymbol{\sigma}}_h\|_Q + \|\boldsymbol{\sigma}_{h,f}\|_Q) \qquad (\forall (\mathring{\boldsymbol{\sigma}}_h, p_h) \in M_{h,f}).
\end{aligned} \tag{3.14}
$$

The following properties hold for $\mathcal{A}^I_{\varepsilon h}$.

(1) $\mathcal{A}^I_{\varepsilon h}$ is coercive on $M_{h,f}$: for all $(\mathring{\boldsymbol{\sigma}}_h, p_h) \in M_{h,f}$,

$$
\begin{aligned}
\frac{{}_{M_h^*}\langle \mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\sigma}}_h, p_h), (\mathring{\boldsymbol{\sigma}}_h, p_h)\rangle_{M_h}}{\|\mathring{\boldsymbol{\sigma}}_h\|_V + \|p_h\|_\Lambda} &= \frac{\|\mathring{\boldsymbol{\sigma}}_h\|_Q^2 + \frac{1}{\varepsilon}\|[p_h]_-\|_\Lambda^2}{\|\mathring{\boldsymbol{\sigma}}_h\|_V + \|p_h\|_\Lambda} \\
&\ge \frac{\|\mathring{\boldsymbol{\sigma}}_h\|_Q^2 + \frac{1}{\varepsilon}\|[p_h]_-\|_\Lambda^2}{\|\mathring{\boldsymbol{\sigma}}_h\|_V + C_3(\|\mathring{\boldsymbol{\sigma}}_h\|_Q + \|\boldsymbol{\sigma}_{h,f}\|_Q)} \to \infty \quad \text{as } \|\mathring{\boldsymbol{\sigma}}_h\|_V + \|p_h\|_\Lambda \to \infty \quad (\text{by } (3.14)).
\end{aligned}
$$

(2) $\mathcal{A}^I_{\varepsilon h}$ is continuous and bounded: for any $(\mathring{\boldsymbol{\sigma}}_{h1}, p_{h1}), (\mathring{\boldsymbol{\sigma}}_{h2}, p_{h2}) \in M_h$,

$$
\begin{aligned}
&\|\mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\sigma}}_{h1}, p_{h1}) - \mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\sigma}}_{h2}, p_{h1})\|_{M_h^*} \\
&\quad \le \|\mathring{\boldsymbol{\sigma}}_{h1} - \mathring{\boldsymbol{\sigma}}_{h2}\|_V + C_{T1} h^{-\frac{1}{2}} \|p_{h1} - p_{h2}\|_\Lambda + C_{T1} h^{-\frac{1}{2}} \|\mathring{\boldsymbol{\sigma}}_{h1} - \mathring{\boldsymbol{\sigma}}_{h2}\|_V + \frac{1}{\varepsilon}\|p_{h1} - p_{h2}\|_\Lambda,
\end{aligned}
$$

where we have used $|[a]_- - [b]_-| \le |a - b|$ and (3.5).

(3) $\mathcal{A}^I_{\varepsilon h}$ is strongly monotone on $M_{h,f}$. In fact, for any $(\mathring{\boldsymbol{\sigma}}_h, p_h), (\mathring{\boldsymbol{\tau}}_h, q_h) \in M_{h,f}$, as with (3.14), we have

$$
\|p_h - q_h\|_\Lambda \le C_3 \sup_{\mathring{\boldsymbol{\xi}} \in \mathring{V}_h} \frac{\big((\mathring{\boldsymbol{\sigma}}_h - \boldsymbol{\sigma}_{h,f}) - (\mathring{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_{h,f}), \mathring{\boldsymbol{\xi}}\big)}{\|\mathring{\boldsymbol{\xi}}\|_V} \le C_3 \|\mathring{\boldsymbol{\sigma}}_h - \mathring{\boldsymbol{\tau}}_h\|_Q. \tag{3.15}
$$

Furthermore, by using (2.18) and (3.15), we obtain

$$
\begin{aligned}
{}_{M_h^*}\langle \mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\sigma}}_h, p_h) - \mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\tau}}_h, qt_h), (\mathring{\boldsymbol{\sigma}}_h - \mathring{\boldsymbol{\tau}}_h, p_h - q_h)\rangle_{M_h} &\ge \|\mathring{\boldsymbol{\sigma}}_h - \mathring{\boldsymbol{\tau}}_h\|_Q^2 + \frac{1}{\varepsilon}\|[p_h]_- - [q_h]_-\|_\Lambda^2 \\
&\ge C(\|\mathring{\boldsymbol{\sigma}}_h - \mathring{\boldsymbol{\tau}}_h\|_Q^2 + \|p_h - q_h\|_\Lambda^2).
\end{aligned}
$$

It follows from (1) to (3) that $\mathcal{A}^I_{\varepsilon h}(\mathring{\boldsymbol{\sigma}}_{\varepsilon h}, p_{\varepsilon h}) = (\boldsymbol{\sigma}_{h,f}, 0)$, equivalently $\mathbf{(MP_\varepsilon\text{-}I)}^\circ_h$, admits a unique solution.

We have shown the unique existence of $\mathbf{(MP_\varepsilon\text{-}I)}_h$. It remains to prove (3.6). Substituting $(\boldsymbol{\tau}_h, v_h, q_h) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$ into (3.4), we obtain

$$\|\boldsymbol{\sigma}_{\varepsilon h}\|^2_Q + \frac{1}{\varepsilon}\|[p_{\varepsilon h}]_-\|^2_\Lambda = (f_h, u_{\varepsilon h}). \tag{3.16}$$

Applying (3.9) and (3.13) to (3.4a) results

$$\|u_{\varepsilon h}\|_Q \le C_4\|\boldsymbol{\sigma}_{\varepsilon h}\|_Q, \tag{3.17a}$$
$$\|p_{\varepsilon h}\|_\Lambda \le C_3^{-1}(\|\boldsymbol{\sigma}_{\varepsilon h}\|_Q + \|u_{\varepsilon h}\|_Q). \tag{3.17b}$$

The *a-priori* estimate (3.6) follows from (3.16) and (3.17). □

We devise the following Active/Inactive Set method to solve the nonlinear mixed penalty problem $\mathbf{(MP_\varepsilon\text{-}I)}_h$.

Algorithm 1. An iteration algorithm for $\mathbf{(MP_\varepsilon\text{-}I)}_h$.

(Step 1) Find $(\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, u^{(0)}_{\varepsilon h}) \in V_h \times Q_h$ such that

$$\boldsymbol{\sigma}^{(0)}_{\varepsilon h} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_S, \tag{3.18a}$$
$$(\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u^{(0)}_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_{h,\Gamma_S}), \tag{3.18b}$$
$$(\nabla\cdot\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{3.18c}$$

Find $p^{(0)}_{\varepsilon h} \in \Lambda_h$ such that

$$(p^{(0)}_{\varepsilon h}, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u^{(0)}_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) \qquad (\forall \boldsymbol{\tau}_h \in V_h). \tag{3.19}$$

Set $n = 1$.

(Step 2) Divide Γ_S into two parts:

$$\Gamma^{1,(n)}_S := \{e \in \Gamma_S : p^{(n-1)}_{\varepsilon h} > 0\} \text{ (The active set)}, \quad \Gamma^{2,(n)}_S := \Gamma_S \backslash \Gamma^{1,(n)}_S \text{ (The inactive set)}.$$

Find $(\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, u^{(n)}_{\varepsilon h}) \in V_h \times Q_h$ such that

$$\boldsymbol{\sigma}^{(n)}_{\varepsilon h} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma^{1,(n)}_S, \tag{3.20a}$$
$$(\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u^{(n)}_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) + \varepsilon(\boldsymbol{\tau}_h\cdot\boldsymbol{n}, \boldsymbol{\sigma}^{(n)}_{\varepsilon h}\cdot\boldsymbol{n})_{\Gamma^{2,(n)}_S} = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_{h,\Gamma^{1,(n)}_S}), \tag{3.20b}$$
$$(\nabla\cdot\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{3.20c}$$

(Step 3) Modify the value of $\boldsymbol{\sigma}^{(n)}_{\varepsilon h}$ such that

$$\boldsymbol{\sigma}^{(n)}_{\varepsilon h}\cdot\boldsymbol{n} \ge 0 \quad \text{on } \Gamma^{2,(n)}_S \text{ (see Rem. 3.3 for details)}. \tag{3.21}$$

Find $p^{(n)}_{\varepsilon h} \in \Lambda_h$ such that

$$(p^{(n)}_{\varepsilon h}, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u^{(n)}_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) \quad (\forall \boldsymbol{\tau}_h \in V_h). \tag{3.22}$$

(Step 4) If $\Gamma^{1,(n)}_S = \Gamma^{1,(n-1)}_S$ $(n > 1)$, then stop iterating. Otherwise, $n \to n+1$ and go to (Step2) and (Step3).

Remark 3.3. For any face $e \in \Gamma^{2,(n)}_S$, we denote by E_e the element with face e, by $\{x_i\}_{i=1}^{d+1}$ the vertices of E_e, and by $\{e_i\}_{i=1}^{d+1}$ the faces of E_e, where the vertex x_i is opposite to the face e_i. Note that $\{\boldsymbol{\phi}_i(x) = h^{-1}_{e_i}(x - x_i)\}_{i=1}^{d+1}$

is a basis of $RT_0(E_e)$, where h_{e_i} is the length of the altitude from x_i to e_i. We can express $\boldsymbol{\sigma}_{\varepsilon h}^{(n)}$ in E_e by

$$\boldsymbol{\sigma}_{\varepsilon h}^{(n)} = \sum_{i=1}^{d+1} \sigma_i^{(n)} \boldsymbol{\phi}_i(x),$$

where $\{\sigma_i^{(n)}\}_{i=1}^{d+1}$ are $(d+1)$ constants. Assume that $e_1 = e \subset \Gamma_S^{2,(n)}$. So we have $\boldsymbol{\phi}_1(x) \cdot \boldsymbol{n} = 1$ and $\boldsymbol{\phi}_i(x) \cdot \boldsymbol{n} = 0$ $(i = 2, 3, \cdots, d+1)$ on E_e. Essentially, (Step 3) is to reset $\sigma_1^{(n)} = 0$ if $\sigma_1^{(n)} < 0$.

Proposition 3.1. *Assume that after n_0 iterations, $\Gamma_S^{1,(n_0+1)} = \Gamma_S^{1,(n_0)}$. Then we have: for all $n \geq n_0$*

$$\left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)}, u_{\varepsilon h}^{(n+1)}\right) = \left(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}\right) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}).$$

Proof. Under the assumption $\Gamma_S^{1,(n+1)} = \Gamma_S^{1,(n)}$, we obtain from (3.20) that: for all $\boldsymbol{\tau}_h \in V_{h,\Gamma_S^{1,(n)}}$,

$$\left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}, \boldsymbol{\tau}_h\right) + \left(u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}, \nabla \cdot \boldsymbol{\tau}_h\right) + \varepsilon\left(\left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\right) \cdot \boldsymbol{n}, \boldsymbol{\tau}_h \cdot \boldsymbol{n}\right)_{\Gamma_S^{2,(n)}} = 0 \tag{3.23a}$$

$$\nabla \cdot \left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\right) = 0. \tag{3.23b}$$

Testing (3.23a) by $\boldsymbol{\tau}_h = \boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)} \in V_{h,\Gamma_S^{1,(n)}}$, together with (3.23b), we obtain

$$\left\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\right\|_Q^2 + \varepsilon\left\|\left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\right) \cdot \boldsymbol{n}\right\|_{L^2(\Gamma_S^{2,(n)})} = 0.$$

Hence, $\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} = \boldsymbol{\sigma}_{\varepsilon h}^{(n)}$. Applying inf-sup condition (3.9) to (3.23a) yields $u_{\varepsilon h}^{(n+1)} = u_{\varepsilon h}^{(n)}$. It remains to check that $(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$. It follows from (3.20a), (3.21), (3.20b), (3.22) and the definition of $\Gamma_S^{1,(n+1)}$ and $\Gamma_S^{2,(n+1)}$ that

$$\begin{aligned} 0 < p_{\varepsilon h}^{(n)},\ \boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n} = 0 \quad &\text{on } \Gamma_S^{1,(n+1)}, \\ 0 \geq p_{\varepsilon h}^{(n)} = -\varepsilon \boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n} \quad &\text{on } \Gamma_S^{2,(n+1)}. \end{aligned}$$

Hence, we see that $[p_{\varepsilon h}^{(n)}]_- = \varepsilon \boldsymbol{\sigma}_{\varepsilon h}^{(n)} \cdot \boldsymbol{n}$ on Γ_S, which says $(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$. $\square$

Remark 3.4. The Algorithm 1 (also Algorithm 2 for $\textbf{(MP}_\varepsilon\textbf{-II)}_h$ presented in Sect. 3.2) mimic the Active/Inactive Set method in [23–25,27]. Similar to these works, the analysis is carried out under the assumptions that the active/inactive sets are invariant after n_0 steps of iteration. However, it is uncertain whether this condition will be satisfied or not. Different from the results obtained in Theorem 4.1(ii) from [25], we discover that $(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)})$ actually equals to $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ if the active/inactive sets are invariant after the n-th step of iteration.

3.2. The discretization of $(\mathbf{MP}_\varepsilon\textbf{-II})$

Now we turn our attention to investigating the discretization of $\textbf{(MP}_\varepsilon\textbf{-II)}$.

$\textbf{(MP}_\varepsilon\textbf{-II)}_h$ Find $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}) \in V_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}, \nabla \cdot \boldsymbol{\tau}_h) - \frac{1}{\varepsilon}([\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}]_-, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), \tag{3.24a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{3.24b}$$

Introducing the Lagrange multiplier $p_{\varepsilon h} := \frac{1}{\varepsilon}[\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}]_- \in \Lambda_h$, we can write (3.24a) into the equivalent form:

$$\begin{aligned} (\boldsymbol{\sigma}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u_{\varepsilon h}, \nabla \cdot \boldsymbol{\tau}_h) - (p_{\varepsilon h}, \boldsymbol{\tau}_h \cdot \boldsymbol{n})_{\Gamma_S} = 0 \qquad &(\forall \boldsymbol{\tau}_h \in V_h), \\ \left(p_{\varepsilon h} - \frac{1}{\varepsilon}[\boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n}]_-, q_h\right)_{\Gamma_S} = 0 \qquad &(\forall q_h \in \Lambda_h). \end{aligned}$$

Theorem 3.2 (The well-posedness of $(\mathbf{MP}_\varepsilon\textbf{-II})_h$). *There exists a unique solution* $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}) \in V_h \times Q_h$ *to* $(\mathbf{MP}_\varepsilon\textbf{-II})_h$ *with the boundedness*

$$\|\boldsymbol{\sigma}_{\varepsilon h}\|_V + \|u_{\varepsilon h}\|_Q + \frac{1}{\sqrt{\varepsilon}}\|[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda \le C\|f_h\|_Q. \tag{3.25}$$

Proof. The proof is a discrete analogy to that of Theorem 2.2. Define the operator $\mathcal{A}^{\mathrm{II}}_{\varepsilon h} : \mathring{V}_h \to \mathring{V}_h^*$ by

$${}_{\mathring{V}_h^*}\langle \mathcal{A}^{\mathrm{II}}_{\varepsilon h}\mathring{\boldsymbol{\sigma}}_h, \mathring{\boldsymbol{\tau}}_h\rangle_{V_h} := (\mathring{\boldsymbol{\sigma}}_h, \mathring{\boldsymbol{\tau}}_h) - \frac{1}{\varepsilon}([\mathring{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}]_-, \mathring{\boldsymbol{\tau}}_h\cdot\boldsymbol{n})_{\Gamma_S} \qquad (\forall \mathring{\boldsymbol{\sigma}}_h, \mathring{\boldsymbol{\tau}}_h \in \mathring{V}_h).$$

One can easily validate that:

(1) $\mathcal{A}^{\mathrm{II}}_{\varepsilon h}$ is bounded and continuous on V_h;
(2) $\mathcal{A}^{\mathrm{II}}_{\varepsilon h}$ is coercive on $\mathring{V}_h$;
(3) $\mathcal{A}^{\mathrm{II}}_{\varepsilon h}$ is strongly monotone on $\mathring{V}_h$.

Hence, there exists a unique solution to the problem $\mathcal{A}^{\mathrm{II}}_{\varepsilon h}\mathring{\boldsymbol{\sigma}}_{\varepsilon h} = \boldsymbol{\sigma}_{h,f}$ (see (3.7) for the definition of $\boldsymbol{\sigma}_{h,f}$), which is equivalent to the variational formulation:
$(\mathbf{MP}_\varepsilon\textbf{-II})_h^\circ$ Find $\mathring{\boldsymbol{\sigma}}_{\varepsilon h} \in \mathring{V}_h$ such that

$$(\mathring{\boldsymbol{\sigma}}_{\varepsilon h}, \mathring{\boldsymbol{\tau}}_h) - \frac{1}{\varepsilon}([\mathring{\boldsymbol{\sigma}}_{\varepsilon h}\cdot\boldsymbol{n}]_-, \mathring{\boldsymbol{\tau}}_h\cdot\boldsymbol{n})_{\Gamma_S} = (\boldsymbol{\sigma}_{h,f}, \mathring{\boldsymbol{\tau}}_h) \qquad (\forall \boldsymbol{\tau}_h \in \mathring{V}_h). \tag{3.26}$$

Setting $\boldsymbol{\sigma}_{\varepsilon h} := \mathring{\boldsymbol{\sigma}}_{\varepsilon h} - \boldsymbol{\sigma}_{h,f}$, and applying inf-sup condition (3.9) to (3.26), we asset the unique existence of $u_{\varepsilon h}$ such that $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ solves $(\mathbf{MP}_\varepsilon\textbf{-II})_h$. The estimate (3.25) can be easily obtained by substituting $(\boldsymbol{\tau}_h, v_h) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ into (3.24) and using the inf-sup condition (3.9). □

Below we give an iteration scheme to solve the discrete mixed penalty problem $(\mathbf{MP}_\varepsilon\textbf{-II})_h$. In particular, we adopt a penalty term to enforce $\boldsymbol{\sigma}^{(n)}_{\varepsilon h}\cdot\boldsymbol{n} \approx 0$ on the inactive set.

Algorithm 2. An iteration algorithm for $(\mathbf{MP}_\varepsilon\textbf{-II})_h$.

(Step 1) Find $(\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, u^{(0)}_{\varepsilon h}) \in V_h \times Q_h$ such that

$$(\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u^{(0)}_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), \tag{3.27a}$$

$$(\nabla\cdot\boldsymbol{\sigma}^{(0)}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{3.27b}$$

Set $n = 1$.

(Step 2) Divide Γ_S into two parts:

$$\Gamma_S^{1,(n)} := \{e \in \Gamma_S : \boldsymbol{\sigma}^{(n-1)}_{\varepsilon h}\cdot\boldsymbol{n} > 0\} \text{ (The active set)}, \quad \Gamma_S^{2,(n)} := \Gamma_S\backslash\Gamma_S^{1,(n)} \text{ (The inactive set)}.$$

Find $(\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, u^{(n)}_{\varepsilon h}) \in V_h \times Q_h$ such that

$$(\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, \boldsymbol{\tau}_h) + (u^{(n)}_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) + \frac{1}{\varepsilon}(\boldsymbol{\sigma}^{(n)}_{\varepsilon h}\cdot\boldsymbol{n}, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S^{2,(n)}} = 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), \tag{3.28a}$$

$$(\nabla\cdot\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, v_h) + (f_h, v_h) = 0 \qquad (\forall v_h \in Q_h). \tag{3.28b}$$

(Step 3) If $\Gamma_S^{1,(n)} = \Gamma_S^{1,(n-1)}$ $(n > 1)$, then stop iterating. Otherwise, $n \to n+1$ and go to (Step 2).

Proposition 3.2. *Assume that after* n_0 *iterations,* $\Gamma_S^{1,(n_0+1)} = \Gamma_S^{1,(n_0)}$. *Then we have: for all* $n \ge n_0$

$$\left(\boldsymbol{\sigma}^{(n+1)}_{\varepsilon h}, u^{(n+1)}_{\varepsilon h}\right) = \left(\boldsymbol{\sigma}^{(n)}_{\varepsilon h}, u^{(n)}_{\varepsilon h}\right) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}).$$

Proof. Under the assumption $\Gamma_S^{1,(n+1)} = \Gamma_S^{1,(n)}$, we have

$$\begin{aligned}
&\left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}, \boldsymbol{\tau}_h\right) + \left(u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}, \nabla\cdot\boldsymbol{\tau}_h\right) \\
&\qquad + \frac{1}{\varepsilon}\int_{\Gamma_S^{2,(n)}} \left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\right)\cdot\boldsymbol{n}(\boldsymbol{\tau}_h\cdot\boldsymbol{n})\,\mathrm{d}\Gamma = 0 \quad (\forall \boldsymbol{\tau}_h \in V_h),
\end{aligned} \tag{3.29a}$$

$$\nabla\cdot\left(\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\right) = 0. \tag{3.29b}$$

Substituting $\boldsymbol{\tau}_h = \boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}$ into (3.29a), we obtain $\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} = \boldsymbol{\sigma}_{\varepsilon h}^{(n)}$, which implies $u_{\varepsilon h}^{(n+1)} = u_{\varepsilon h}^{(n)}$. Note that (3.28a) implies

$$p_{\varepsilon h}^{(n)} = 0 \text{ on } \Gamma_S^{1,(n)} \text{ and } p_{\varepsilon h}^{(n)} = \frac{1}{\varepsilon}\boldsymbol{\sigma}_{\varepsilon h}^{(n)}\cdot\boldsymbol{n} \text{ on } \Gamma_S^{2,(n)}.$$

On the other hand, $\Gamma_S^{1,(n+1)} = \Gamma_S^{1,(n)}$ (also $\Gamma_S^{2,(n+1)} = \Gamma_S^{2,(n)}$) implies

$$\boldsymbol{\sigma}_{\varepsilon h}^{(n)}\cdot\boldsymbol{n} > 0 \text{ on } \Gamma_S^{1,(n)} \text{ and } \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\cdot\boldsymbol{n} \le 0 \text{ on } \Gamma_S^{2,(n)}.$$

Hence, $p_{\varepsilon h}^{(n)} = \frac{1}{\varepsilon}[\boldsymbol{\sigma}_{\varepsilon h}^{(n)}\cdot\boldsymbol{n}]_-$ on Γ_S, which means

$$(\boldsymbol{\sigma}_{\varepsilon h}^{(n)}, u_{\varepsilon h}^{(n)}) = (\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}).$$

□

4. The error analysis of the discrete mixed penalty problems

We will focus on the error analysis of the discrete mixed penalty problems $\mathbf{(MP_\varepsilon\text{-}I)}_h$ and $\mathbf{(MP_\varepsilon\text{-}II)}_h$ in this section. For briefness, we set the notations:

$$\begin{aligned}
&\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\varepsilon h} = (\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}) + (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\varepsilon h}) =: \boldsymbol{\rho}_h + \boldsymbol{\theta}_h, \\
&e_{u,\varepsilon h} := u - u_{\varepsilon h}, \quad e_{p,\varepsilon h} := p - p_{\varepsilon h}.
\end{aligned}$$

For error analysis, here and hereafter, we assume that $\boldsymbol{\sigma} \in H^1(\Omega)^d$ (see Rem. 2.2 for the discussion of regularity). By the Sobolev embedding $H^1(\Omega)^d \subset L^6(\Omega)^d$ $(d = 2, 3)$, the interpolation $\Pi_h\boldsymbol{\sigma}$ is well-defined.

Theorem 4.1. *Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$ and $(\boldsymbol{\sigma}, u, p)$ be the solutions of* $\mathbf{(MP_\varepsilon\text{-}I)}_h$ *and* **(MLM-I)** *respectively. Assume that $\boldsymbol{\sigma} \in H^1(\Omega)^d$, $\nabla\cdot\boldsymbol{\sigma} \in H^1(\Omega)$, $u \in H^1(\Omega)$, $p \in H^1(\Gamma_S)$ and $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in H^1(\Gamma_S)$. Then we have the error estimates:*

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_V + \|e_{u,\varepsilon h}\|_Q + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\cdot\boldsymbol{n}\|_\Lambda \le C(h + \sqrt{\varepsilon}), \tag{4.1a}$$

$$\|e_{p,\varepsilon h}\|_\Lambda \le C(\sqrt{\varepsilon} + \sqrt{h}). \tag{4.1b}$$

Remark 4.1. To obtain the $O(h)$-convergence in Theorems 4.1 and 4.3, we need the regularity assumptions $\boldsymbol{\sigma} \in H^1(\Omega)^d$, $\nabla\cdot\boldsymbol{\sigma} = -f \in H^1(\Omega)$, $u \in H^1(\Omega)$, $p \in H^1(\Gamma_S)$ and $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in H^1(\Gamma_S)$. In view of $(\boldsymbol{\sigma}, p) = (\nabla u, u|_{\Gamma_S})$, the above assumptions can be fulfilled if $u \in H^{\frac{5}{2}}(\Omega)$ and $f \in H^1(\Omega)$. However, the regularity theory of the Signorini problem has not been fully investigated, especially for the 3D case (see [3,13,29]). If Ω is a C^2-smooth or convex polygonal domain and $f \in L^2(\Omega)$, we have $u \in H^2(\Omega)$ [9,13]. The Sobolev regularity and Hölder continuity of the solution in a smooth or polygonal domain in $\mathbb{R}^2$ are investigated in [13,29] comments that the $H^{\frac{5}{2}}$-regularity cannot be passed beyond in general.

In our finite element analysis, to avoid the issue of domain perturbation in triangulation, we assume that Ω is a polygon or polyhedron. Therefore, we cannot expect $u \in H^{\frac{5}{2}}(\Omega)$ generally. Nevertheless, in numerical experiments, we test the scheme on a square domain and a ring domain, and we observe $O(h)$-convergence in both cases (see Sects. 5.1 and 5.3). Moreover, we also investigate the experimental convergence rate for the case with an L-shaped domain, which cannot achieve the first order convergence due to the bad regularity of the solution (see Sect. 5.2).

Proof of Theorem 4.1. From (3.4) and (2.2), we obtain

$$
\begin{aligned}
(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}, \boldsymbol{\tau}_h) + (e_{u,\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) - (e_{p,\varepsilon h}, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S} &= 0 \qquad (\forall \boldsymbol{\tau}_h \in V_h), &\text{(4.2a)}\\
(\nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}, v_h) &= 0 \qquad (\forall v_h \in Q_h). &\text{(4.2b)}
\end{aligned}
$$

Substituting $\boldsymbol{\tau}_h = \boldsymbol{\theta}_h$ into (4.2a) yields

$$
(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}, \boldsymbol{\theta}_h) + (e_{u,\varepsilon h}, \nabla\cdot\boldsymbol{\theta}_h) - (e_{p,\varepsilon h}, \boldsymbol{\theta}_h\cdot\boldsymbol{n})_{\Gamma_S} = 0. \tag{4.3}
$$

The left-hand side is calculated as follows.

$$
\begin{aligned}
(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}, \boldsymbol{\theta}_h) &= \|\boldsymbol{\theta}_h\|_Q^2 + (\boldsymbol{\rho}_h, \boldsymbol{\theta}_h), &\text{(4.4a)}\\
(e_{u,\varepsilon h}, \nabla\cdot\boldsymbol{\theta}_h) &= (u - P_h u, \nabla\cdot\boldsymbol{\theta}_h) + (P_h u - u_{\varepsilon h}, \nabla\cdot\boldsymbol{\theta}_h)\\
&= 0 + (P_h u - u_{\varepsilon h}, \nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h} - \nabla\cdot\boldsymbol{\rho}_h) \quad (\text{by } (3.1)) &\text{(4.4b)}\\
&= (P_h u - u_{\varepsilon h}, \nabla\cdot\boldsymbol{\rho}_h) \quad (\text{by } (4.2\text{b})),\\
(e_{p,\varepsilon h}, \boldsymbol{\theta}_h\cdot\boldsymbol{n})_{\Gamma_S} &= (p - p_{\varepsilon h}, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n})_{\Gamma_S}\\
&= (p - P_{\Lambda_h}p + P_{\Lambda_h}p, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} - (p, \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} - (p, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n})_{\Gamma_S}\\
&\quad - ([p_{\varepsilon h}]_+ - [p_{\varepsilon h}]_-, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n})_{\Gamma_S}\\
&= (p - P_{\Lambda_h}p, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} - \left(p, \frac{1}{\varepsilon}[p_{\varepsilon h}]_-\right)_{\Gamma_S} - ([p_{\varepsilon h}]_+, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} &\text{(4.4c)}\\
&\quad + \left([p_{\varepsilon h}]_+, \frac{1}{\varepsilon}[p_{\varepsilon h}]_-\right)_{\Gamma_S} + (\varepsilon\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n})_{\Gamma_S}\\
&\qquad (\text{by } (3.2),\ (p, \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} = 0 \text{ and } (3.4\text{c}))\\
&\le (p - P_{\Lambda_h}p, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} - \varepsilon\|\boldsymbol{\theta}_h\cdot\boldsymbol{n}\|_\Lambda^2 + \varepsilon(\Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n}, \boldsymbol{\theta}_h\cdot\boldsymbol{n})_{\Gamma_S}\\
&\qquad (\text{by } p \ge 0, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} \ge 0 \text{ and } [s]_+[s]_- = 0).
\end{aligned}
$$

Inserting (4.4) into (4.3), we arrive

$$
\begin{aligned}
\|\boldsymbol{\theta}_h\|_Q^2 + \varepsilon\|\boldsymbol{\theta}_h\cdot\boldsymbol{n}\|_\Lambda^2 &\le -(\boldsymbol{\rho}_h, \boldsymbol{\theta}_h) + (P_h u - u_{\varepsilon h}, \nabla\cdot\boldsymbol{\rho}_h)\\
&\quad + (p - P_{\Lambda_h}p, \boldsymbol{\rho}_h\cdot\boldsymbol{n})_{\Gamma_S} + \varepsilon(\Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n}, \boldsymbol{\theta}_h\cdot\boldsymbol{n})_{\Gamma_S}\\
&\le \frac{1}{2}\|\boldsymbol{\rho}_h\|_Q^2 + \frac{1}{2}\|\boldsymbol{\theta}_h\|_Q^2 + \|P_h u - u_{\varepsilon h}\|_Q Ch\|\nabla\cdot\boldsymbol{\sigma}\|_{H^1(\Omega)}\\
&\quad + Ch^2\|p\|_{H^1(\Gamma_S)}\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)} + \frac{\varepsilon}{2}\|\boldsymbol{\theta}_h\cdot\boldsymbol{n}\|_\Lambda^2 + \frac{\varepsilon}{2}\|\Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_\Lambda^2,
\end{aligned} \tag{4.5}
$$

which yields

$$
\begin{aligned}
\|\boldsymbol{\theta}_h\|_Q^2 + \varepsilon\|\boldsymbol{\theta}_h\cdot\boldsymbol{n}\|_\Lambda^2 &\le C(h^2+\varepsilon)\Big(\|\boldsymbol{\sigma}\|_{H^1(\Omega)}^2 + \|p\|_{H^1(\Gamma_S)}^2 + \|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)}^2\Big)\\
&\quad + Ch\|P_h u - u_{\varepsilon h}\|_Q\|\nabla\cdot\boldsymbol{\sigma}\|_{H^1(\Omega)}.
\end{aligned} \tag{4.6}
$$

Substituting $v_h = \nabla\cdot\boldsymbol{\theta}_h \in Q_h$ into (4.2b), we get

$$
\|\nabla\cdot\boldsymbol{\theta}_h\|_Q^2 = -(\nabla\cdot\boldsymbol{\rho}_h, \nabla\cdot\boldsymbol{\theta}_h) \le Ch\|\nabla\cdot\boldsymbol{\sigma}\|_{H^1(\Omega)}\|\nabla\cdot\boldsymbol{\theta}_h\|_Q,
$$

which results

$$
\|\nabla\cdot\boldsymbol{\theta}_h\|_Q \le Ch\|\nabla\cdot\boldsymbol{\sigma}\|_{H^1(\Omega)}. \tag{4.7}
$$

By inf-sup condition (3.9) and (4.2a),

$$\begin{aligned}\|P_h u - u_{\varepsilon h}\|_Q &\le C_4 \sup_{\boldsymbol{\tau}_h \in V_{h,\Gamma_S}} \frac{(P_h u - u_{\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_V} \\ &= C_4 \sup_{\boldsymbol{\tau}_h \in V_{h,\Gamma_S}} \frac{-(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}, \boldsymbol{\tau}_h) - (u - P_h u, \nabla\cdot\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_V} \\ &\le C\big(h\|\boldsymbol{\sigma}\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h\|_Q + h\|u\|_{H^1(\Omega)}\big).\end{aligned} \tag{4.8}$$

Combining (4.6)–(4.8), we conclude (4.1a). Applying inf-sup condition (3.13) to (4.2a), we derive

$$\begin{aligned}\|P_{\Lambda_h} p - p_{\varepsilon h}\|_\Lambda &\le C_3 \sup_{\boldsymbol{\tau}_h \in V_h} \frac{(P_{\Lambda_h} p - p_{\varepsilon h}, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S}}{\|\boldsymbol{\tau}_h\|_V} \\ &= C_3 \sup_{\boldsymbol{\tau}_h \in V_h} \frac{(\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}, \boldsymbol{\tau}_h) + (e_{u,\varepsilon h}, \nabla\cdot\boldsymbol{\tau}_h) - (p - P_{\Lambda_h} p, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S}}{\|\boldsymbol{\tau}_h\|_V} \\ &\le C_3\Big(\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_Q + \|e_{u,\varepsilon h}\|_Q + C_{T1} h^{-\frac{1}{2}}\|p - P_{\Lambda_h} p\|_\Lambda\Big) \quad (\text{by } (3.5)) \\ &\le C\Big(h\|\boldsymbol{\sigma}\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h\|_Q + h\|u\|_{H^1(\Omega)} + \sqrt{h}\|p\|_{H^1(\Gamma_S)}\Big)\end{aligned} \tag{4.9}$$

which implies (4.1b). $\square$

Remark 4.2. We have shown the $O(h)$-convergence for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$ and $\|u - u_{\varepsilon h}\|_Q$ by using RT_0/P_0 element. For the mixed method of the Signorini problem using RT_k/P_k element ($k \ge 1$), as far as we know, the error analysis has not been fully investigated. One may improve the rate of convergence by utilizing the higher-order polynomials. It is worth mentioning that, for the Laplace problem with the Dirichlet boundary condition, when using the mixed method with RT_k/P_k element (which satisfy the inf-sup condition), to obtain error estimate $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_V + \|u - u_h\|_Q \le Ch^{k+1}$, one need the regularity assumptions $u \in H^{k+1}(\Omega)$, $(\nabla u =)\boldsymbol{\sigma} \in H^{k+1}(\Omega)$ and $\nabla\cdot\boldsymbol{\sigma} \in H^{k+1}(\Omega)$ (see [32], Thm. 13.12).

Next, we study the errors of $(\mathbf{MP}_\varepsilon\textbf{-I})_h$ and $(\mathbf{MP}_\varepsilon\textbf{-I})$. For simplicity, we introduce the following notations:

$$\begin{aligned}\boldsymbol{e}_{\boldsymbol{\sigma},h} &:= \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h} = (\boldsymbol{\sigma}_\varepsilon - \Pi_h\boldsymbol{\sigma}_\varepsilon) + (\Pi_h\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}), \\ e_{u,h} &:= u_\varepsilon - u_{\varepsilon h}, \quad e_{p,h} := p_\varepsilon - p_{\varepsilon h}.\end{aligned}$$

Theorem 4.2. *Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h}, p_{\varepsilon h})$ and $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon, p_\varepsilon)$ be the solutions of* $(\mathbf{MP}_\varepsilon\textbf{-I})_h$ *and* $(\mathbf{MP}_\varepsilon\textbf{-I})$*, respectively. Assume that*

$$\boldsymbol{\sigma}_\varepsilon \in H^1(\Omega)^d,\ \nabla\cdot\boldsymbol{\sigma}_\varepsilon \in H^1(\Omega),\ u_\varepsilon \in H^1(\Omega),\ p_\varepsilon \in H^1(\Gamma_S),\ \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n} \in H^1(\Gamma_S). \tag{4.10}$$

We have the error estimates:

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},h}\|_V + \|e_{u,h}\|_Q + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},h}\cdot\boldsymbol{n}\|_\Lambda \le C_\varepsilon h, \tag{4.11a}$$

$$\|e_{p,h}\|_\Lambda \le C_\varepsilon\sqrt{h}, \tag{4.11b}$$

where $C_\varepsilon := C(\|\boldsymbol{\sigma}_\varepsilon\|_{H^1} + \|\nabla\cdot\boldsymbol{\sigma}_\varepsilon\|_{H^1} + \|\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)} + \|p_\varepsilon\|_{H^1(\Gamma_S)} + \|u_\varepsilon\|_{H^1})$.

Proof. It follows from (3.4a) and (1.5a) that

$$(\boldsymbol{e}_{\boldsymbol{\sigma},h}, \boldsymbol{\tau}_h) + (e_{u,h}, \nabla\cdot\boldsymbol{\tau}_h) - (e_{p,h}, \boldsymbol{\tau}_h\cdot\boldsymbol{n})_{\Gamma_S} = 0 \quad (\forall\boldsymbol{\tau}_h \in V_h), \tag{4.12}$$

Testing (4.12) by $\boldsymbol{\tau}_h = \Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}$, we see that

$$(\boldsymbol{e}_{\boldsymbol{\sigma},h}, \Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}) + (e_{u,h}, \nabla \cdot (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})) - (e_{p,h}, (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}) \cdot \boldsymbol{n})_{\Gamma_S} = 0. \tag{4.13}$$

Similar to (4.4a) and (4.4b), we have

$$(\boldsymbol{e}_{\boldsymbol{\sigma},h}, \Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}) = \|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q^2 + (\boldsymbol{\sigma}_\varepsilon - \Pi_h \boldsymbol{\sigma}_\varepsilon, \Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}), \tag{4.14a}$$
$$(e_{u,h}, \nabla \cdot (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})) = (P_h u_\varepsilon - u_{\varepsilon h}, \nabla \cdot (\boldsymbol{\sigma}_\varepsilon - \Pi_h \boldsymbol{\sigma}_\varepsilon)). \tag{4.14b}$$

The third term of (4.13) is calculated as follows.

$$\begin{aligned}
(e_{p,h}, (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}) \cdot \boldsymbol{n})_{\Gamma_S} &= (p_\varepsilon - p_{\varepsilon h}, \Pi_h \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}) + (p_\varepsilon - p_{\varepsilon h}, \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{\sigma}_{\varepsilon h} \cdot \boldsymbol{n})_{\Gamma_S} \\
&= (p_\varepsilon - p_{\varepsilon h}, \Pi_h \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}) + \left(p_\varepsilon - p_{\varepsilon h}, \frac{1}{\varepsilon}[p_\varepsilon]_- - \frac{1}{\varepsilon}[p_{\varepsilon h}]_-\right)_{\Gamma_S} \\
&\le (p_\varepsilon - p_{\varepsilon h}, \Pi_h \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}) - \frac{1}{\varepsilon}\|[p_\varepsilon]_- - [p_{\varepsilon h}]_-\|_\Lambda^2 \quad \text{(by (2.18))} \\
&= (p_\varepsilon - P_{\Lambda_h} p_\varepsilon, \Pi_h \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}) - \varepsilon \|\boldsymbol{e}_{\boldsymbol{\sigma},h} \cdot \boldsymbol{n}\|_\Lambda^2 \quad \text{(by (3.2))}.
\end{aligned}$$

As a result, we obtain from (4.13) that

$$\begin{aligned}
\|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q^2 + \varepsilon \|\boldsymbol{e}_{\boldsymbol{\sigma},h} \cdot \boldsymbol{n}\|_\Lambda^2 &\le -(\boldsymbol{\sigma}_\varepsilon - \Pi_h \boldsymbol{\sigma}_\varepsilon, \Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}) + (P_h u_\varepsilon - u_{\varepsilon h}, \nabla \cdot (\boldsymbol{\sigma}_\varepsilon - \Pi_h \boldsymbol{\sigma}_\varepsilon)) \\
&\quad + (p_\varepsilon - P_{\Lambda_h} p_\varepsilon, \Pi_h \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}) \\
&\le \frac{1}{2}\|\boldsymbol{\sigma}_\varepsilon - \Pi_h \boldsymbol{\sigma}_\varepsilon\|_Q^2 + \frac{1}{2}\|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q^2 + \|P_h u_\varepsilon - u_{\varepsilon h}\|_Q Ch \|\nabla \cdot \boldsymbol{\sigma}_\varepsilon\|_{H^1} \\
&\quad + Ch^2 \|p_\varepsilon\|_{H^1(\Gamma_S)} \|\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}\|_{H^1(\Gamma_S)} \quad \text{(by (3.3f))},
\end{aligned}$$

which implies

$$\begin{aligned}
\|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q^2 + \varepsilon \|\boldsymbol{e}_{\boldsymbol{\sigma},h} \cdot \boldsymbol{n}\|_\Lambda^2 &\le Ch^2 (\|\boldsymbol{\sigma}_\varepsilon\|_{H^1}^2 + \|p_\varepsilon\|_{H^1(\Gamma_S)}^2 + \|\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}\|_{H^1(\Gamma_S)}^2) \\
&\quad + Ch \|P_h u_\varepsilon - u_{\varepsilon h}\|_Q \|\nabla \cdot \boldsymbol{\sigma}_\varepsilon\|_{H^1}.
\end{aligned} \tag{4.15}$$

Taking $\boldsymbol{\tau}_h \in V_{h,\Gamma_S}$ into (4.12), similar to the derivation of (4.8), one can obtain

$$\|P_h u_\varepsilon - u_{\varepsilon h}\|_Q \le Ch (\|\boldsymbol{\sigma}_\varepsilon\|_{H^1(\Omega)} + \|u_\varepsilon\|_{H^1(\Omega)}) + C \|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q. \tag{4.16}$$

Together with (4.15), we conclude that

$$\|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q \le Ch (\|\boldsymbol{\sigma}_\varepsilon\|_{H^1} + \|p_\varepsilon\|_{H^1(\Gamma_S)} + \|\boldsymbol{\sigma}_\varepsilon \cdot \boldsymbol{n}\|_{H^1(\Gamma_S)} + \|u_\varepsilon\|_{H^1(\Omega)} + \|\nabla \cdot \boldsymbol{\sigma}_\varepsilon\|_{H^1}).$$

As with (4.7), we also have

$$\|\nabla \cdot (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})\|_Q^2 = -(\nabla \cdot (\boldsymbol{\sigma}_\varepsilon - \Pi_h \boldsymbol{\sigma}_\varepsilon), \nabla \cdot (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})) \le Ch \|\nabla \cdot \boldsymbol{\sigma}_\varepsilon\|_{H^1(\Omega)} \|\nabla \cdot (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})\|_Q,$$

which implies

$$\|\nabla \cdot (\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})\|_Q \le Ch \|\nabla \cdot \boldsymbol{\sigma}_\varepsilon\|_{H^1(\Omega)}. \tag{4.17}$$

Via a similar calculation of (4.9), we obtain from (4.12) that

$$\|P_{\Lambda_h} p_\varepsilon - p_{\varepsilon h}\|_\Lambda \le Ch (\|\boldsymbol{\sigma}_\varepsilon\|_{H^1(\Omega)} + \|u_\varepsilon\|_{H^1(\Omega)}) + C\sqrt{h} \|p_\varepsilon\|_{H^1(\Gamma_S)} + C \|\Pi_h \boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q. \tag{4.18}$$

Hence, we conclude (4.11). $\square$

Now, we deal with the error estimates of $\mathbf{(MP_\varepsilon\text{-II})}_h$.

Theorem 4.3. *Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ and $(\boldsymbol{\sigma}, u, p)$ be the solutions of $\mathbf{(MP_\varepsilon\text{-II})}_h$ and* **(MLM-II)** *respectively. Assume that $\boldsymbol{\sigma} \in H^1(\Omega)^d,\ \nabla\cdot\boldsymbol{\sigma} \in H^1(\Omega),\ u \in H^1(\Omega),\ p \in H^1(\Gamma_S)$ and $\boldsymbol{\sigma}\cdot\boldsymbol{n} \in H^1(\Gamma_S)$. Then we have the error estimate*

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_V + \|e_{u,\varepsilon h}\|_Q \le C(h + \sqrt{\varepsilon}), \tag{4.19a}$$

$$\|e_{p,\varepsilon h}\|_\Lambda \le C(\sqrt{\varepsilon} + \sqrt{h}). \tag{4.19b}$$

Proof. Note that (4.2a)–(4.4b) of the proof of Theorem 4.1 still hold. We only need to recalculate the third term at the left-hand side of (4.3). Similar to (4.4c), we have

$$\begin{aligned}(e_{p,\varepsilon h}, \boldsymbol{\theta}_h\cdot\boldsymbol{n})_{\Gamma_S} &= \left(p - \frac{1}{\varepsilon}[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}\right)_{\Gamma_S}\\ &= (p - P_{\Lambda_h}p + P_{\Lambda_h}p, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} + (p, \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} - (p, [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_+ - [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-)_{\Gamma_S}\\ &\quad - \frac{1}{\varepsilon}([\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} - \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda^2\\ &\le (p - P_{\Lambda_h}p, \Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} + (p, [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-)_{\Gamma_S} - \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda^2,\end{aligned} \tag{4.20}$$

where we have utilized (3.2), $(p, \boldsymbol{\sigma}\cdot\boldsymbol{n})_{\Gamma_S} = 0$, and $p \ge 0$, $\Pi_h\boldsymbol{\sigma}\cdot\boldsymbol{n} \ge 0$ on Γ_S. As with (4.5), we derive

$$\begin{aligned}\|\boldsymbol{\theta}_h\|_Q^2 + \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda^2 &\le -(\boldsymbol{\rho}_h, \boldsymbol{\theta}_h) + (P_hu - u_{\varepsilon h}, \nabla\cdot\boldsymbol{\rho}_h)\\ &\quad + (p - P_{\Lambda_h}p, \boldsymbol{\rho}_h\cdot\boldsymbol{n})_{\Gamma_S} + (p, [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-)_{\Gamma_S}\\ &\le \frac{1}{2}\|\boldsymbol{\rho}_h\|_Q^2 + \frac{1}{2}\|\boldsymbol{\theta}_h\|_Q^2 + \|P_hu - u_{\varepsilon h}\|_Q Ch\|\nabla\cdot\boldsymbol{\sigma}\|_{H^1(\Omega)}\\ &\quad + Ch^2\|p\|_{H^1(\Gamma_S)}\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)} + \frac{1}{2\varepsilon}\|[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda^2 + \frac{\varepsilon}{2}\|p\|_\Lambda^2,\end{aligned} \tag{4.21}$$

which yields

$$\begin{aligned}\|\boldsymbol{\theta}_h\|_Q^2 + \varepsilon^{-1}\|[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda^2 &\le C(h^2 + \varepsilon)(\|\boldsymbol{\sigma}\|_{H^1(\Omega)}^2 + \|p\|_{H^1(\Gamma_S)}^2 + \|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)}^2)\\ &\quad + Ch\|P_hu - u_{\varepsilon h}\|_Q\|\nabla\cdot\boldsymbol{\sigma}\|_{H^1(\Omega)}.\end{aligned}$$

Since (4.7) and (4.9) still hold in this case, we conclude (4.19). ∎

Theorem 4.4. *Let $(\boldsymbol{\sigma}_{\varepsilon h}, u_{\varepsilon h})$ and $(\boldsymbol{\sigma}_\varepsilon, u_\varepsilon)$ be the solutions of $\mathbf{(MP_\varepsilon\text{-II})}_h$ and $\mathbf{(MP_\varepsilon\text{-II})}$, respectively. Under the same assumption of Theorem 4.2, we have the error estimates:*

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},h}\|_V + \|e_{u,h}\|_Q \le C_\varepsilon h, \tag{4.22a}$$

$$\|e_{p,h}\|_\Lambda \le C_\varepsilon\sqrt{h}. \tag{4.22b}$$

where $C_\varepsilon := C(\|\boldsymbol{\sigma}_\varepsilon\|_{H^1} + \|\nabla\cdot\boldsymbol{\sigma}_\varepsilon\|_{H^1} + \|\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)} + \|p_\varepsilon\|_{H^1(\Gamma_S)} + \|u_\varepsilon\|_{H^1})$.

Proof. Note that (4.13) and (4.14) in the proof of Theorem 4.2 still hold. In this case, $p_\varepsilon = \frac{1}{\varepsilon}[\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_-$ and $p_{\varepsilon h} = \frac{1}{\varepsilon}[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-$. We only need to re-estimate the third term on the left-hand of (4.13).

$$\begin{aligned}&\left(\frac{1}{\varepsilon}[\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_- - \frac{1}{\varepsilon}[\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-, (\Pi_h\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h})\cdot\boldsymbol{n}\right)_{\Gamma_S}\\ &\quad = \frac{1}{\varepsilon}([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_- - [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-, \Pi_h\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n})_{\Gamma_S} + \frac{1}{\varepsilon}([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_- - [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-, \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n} - \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n})_{\Gamma_S}\\ &\quad \le \frac{1}{\varepsilon}([\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_- - [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-, \Pi_h\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n})_{\Gamma_S} - \frac{1}{\varepsilon}\|[\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}]_- - [\boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}]_-\|_\Lambda^2\\ &\quad = (p_\varepsilon - P_{\Lambda_h}p_\varepsilon, \Pi_h\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n} - \boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n})_{\Gamma_S} - \frac{1}{\varepsilon}\|[\boldsymbol{e}_{\boldsymbol{\sigma},h}\cdot\boldsymbol{n}]_-\|_\Lambda^2,\end{aligned}$$

where we have utilized (2.18) and (3.2). As with (4.15), we derive

$$\|\Pi_h\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_{\varepsilon h}\|_Q^2 + \frac{1}{\varepsilon}\|[\boldsymbol{e}_{\boldsymbol{\sigma},h}\cdot\boldsymbol{n}]_-\|_\Lambda^2 \le Ch^2\Big(\|\boldsymbol{\sigma}_\varepsilon\|_{H^1}^2 + \|p_\varepsilon\|_{H^1(\Gamma_S)}^2 + \|\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)}^2\Big)$$
$$+ Ch\|P_hu_\varepsilon - u_{\varepsilon h}\|_Q\|\nabla\cdot\boldsymbol{\sigma}_\varepsilon\|_{H^1}.$$

It is easy to check that (4.16)–(4.18) remain valid in this case. Therefore, we conclude (4.4). □

Remark 4.3. Combining (2.25) (resp. (2.33)) and (4.11) (resp. (4.22)), together with the triangle inequality, we obtain the following convergence results:

$$\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_V \le \|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon}\|_V + \|\boldsymbol{e}_{\boldsymbol{\sigma},h}\|_V \le C(\varepsilon + C_\varepsilon h),$$
$$\|e_{u,\varepsilon h}\|_Q \le \|e_{u,\varepsilon}\|_Q + \|e_{u,h}\|_Q \le C(\varepsilon + C_\varepsilon h),$$
$$\|e_{p,\varepsilon h}\|_\Lambda \le \|e_{p,\varepsilon}\|_\Lambda + \|e_{p,h}\|_\Lambda \le C(\varepsilon + C_\varepsilon\sqrt{h}).$$

Notice that the constant C_ε depends on the norms $\|\boldsymbol{\sigma}_\varepsilon\|_{H^1}$, $\|\nabla\cdot\boldsymbol{\sigma}_\varepsilon\|_{H^1}$, $\|\boldsymbol{\sigma}_\varepsilon\cdot\boldsymbol{n}\|_{H^1(\Gamma_S)}$, $\|p_\varepsilon\|_{H^1(\Gamma_S)}$ and $\|u_\varepsilon\|_{H^1}$, which may depend on the reciprocal of ε. As a result, at this stage, we cannot assert the convergence rate of ε is $O(\varepsilon)$.

For the penalty problem without using the mixed method, the error estimate $\|u - u_{\varepsilon h}\|_{H^1(\Omega)} \le C(h|\ln h|^{\frac{1}{2}} + (\varepsilon h)^{\frac{1}{2}} + \varepsilon)$ is obtained by Theorem 3.2 from [11]. In Theorem 4.1, we show that $\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_V + \|e_{u,\varepsilon h}\|_Q + \sqrt{\varepsilon}\|\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\cdot\boldsymbol{n}\|_\Lambda \le C(h+\sqrt{\varepsilon})$, which results $O(h)$-convergence (slightly better than $O(h|\ln h|^{\frac{1}{2}})$) when taking $\varepsilon = O(h^2)$. Note that taking smaller h increases the degree of freedom (DoF), but the penalty parameter ε is unrelated to the DoF. Another benefit of the mixed method is that it offers an error estimate of $\|\nabla\cdot\boldsymbol{e}_{\boldsymbol{\sigma},\varepsilon h}\|_{L^2(\Omega)}$.

Remark 4.4. In the continuous case, we have shown the $O(\varepsilon)$-convergence for **(MP$_\varepsilon$-I)** and **(MP$_\varepsilon$-II)** (see Thms. 2.3 and 2.4). For the discrete penalty problems **(MP$_\varepsilon$-I)$_h$** and **(MP$_\varepsilon$-II)$_h$**, we have proved the error bound $C(\sqrt{\varepsilon}+h)$ for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{\varepsilon h}\|_V$ and $\|u-u_{\varepsilon h}\|_Q$, and the error bound $C(\sqrt{\varepsilon}+\sqrt{h})$ for $\|p-p_{\varepsilon h}\|_\Lambda$. Nevertheless, our numerical experiments reveal the $O(h)$-convergence and $O(\varepsilon)$-convergence (see Figs. 4, 8 and 12). We find that the experimental errors have a better convergence behavior than $O(\sqrt{\varepsilon})$. Meanwhile, the experimental convergence rate of $\|p-p_{\varepsilon h}\|_\Lambda$ is better than $O(\sqrt{h})$.

5. Numerical experiments

To investigate the rates of convergence for discretization, penalty, and iteration algorithms, we conducted three numerical examples using various geometric domains.

5.1. Example 1

Consider the Signorini problem in the square domain $\Omega = (0,1)^2$, with the Signorini boundary $\Gamma_S = \Gamma_{S1}\cup\Gamma_{S2}$, $\Gamma_{S1} = (0,1)\times\{1\}$, $\Gamma_{S2} = \{0\}\times(0,1)$, the Dirichlet boundary $\Gamma_D = \partial\Omega\setminus\Gamma_S$. Set $f = 100(\frac{1}{2}-x)(\frac{1}{2}-y)$.

We conducted a simulation using Algorithms 1 and 2, which resulted in nearly identical numerical solutions. Therefore, we only present the numerical solution $(\boldsymbol{\sigma}_h, u_h)$ of Algorithm 1 with $(h,\varepsilon) = (2^{-8}, 10^{-8})$ in Figure 1. In addition, for Algorithms 1 and 2, we plot the values of u_h and $\boldsymbol{\sigma}_h\cdot\boldsymbol{n}$ on the boundary Γ_S, along with the Lagrange multiplier p_h with $(h,\varepsilon) = (2^{-8}, 2^{-5})$ and a smaller ε $(h,\varepsilon) = (2^{-8}, 10^{-8})$, which indicates that $u_h|_{\Gamma_S} \approx p_h$, and the numerical results accurately approximate the Signorini conditions $\boldsymbol{\sigma}_h\cdot\boldsymbol{n} \ge 0$, $p_h \ge 0$ and $p_h(\boldsymbol{\sigma}_h\cdot\boldsymbol{n}) = 0$ (see Figs. 2 and 3, respectively).

To investigate the experimental convergence rates of discretization, we take the reference solution $(\boldsymbol{\sigma}_{\varepsilon ref}, u_{\varepsilon ref}, p_{\varepsilon ref})$ with $(h,\varepsilon) = (2^{-8}, 10^{-8})$ and apply Algorithms 1 and 2 for simulation. First, we fix $\varepsilon = 10^{-8}$ and perform the simulation under different mesh sizes. The errors are plotted in Figure 4a, which indicates the $O(h)$-convergence. Then, we fix $h = 2^{-8}$, and conduct the simulations using the penalty parameter $\varepsilon = 2^{-5}$, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}. We plot the errors in Figure 4b, which shows $O(\varepsilon)$-convergence for $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$

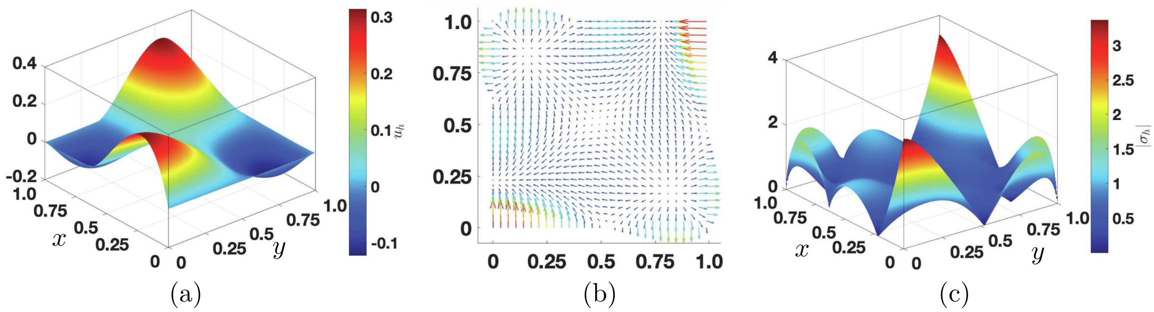


FIGURE 1. Example 1: the numerical solution $(\boldsymbol{\sigma}_h, u_h)$. (a) u_h. (b) $\boldsymbol{\sigma}_h$. (c) $|\boldsymbol{\sigma}_h|$.

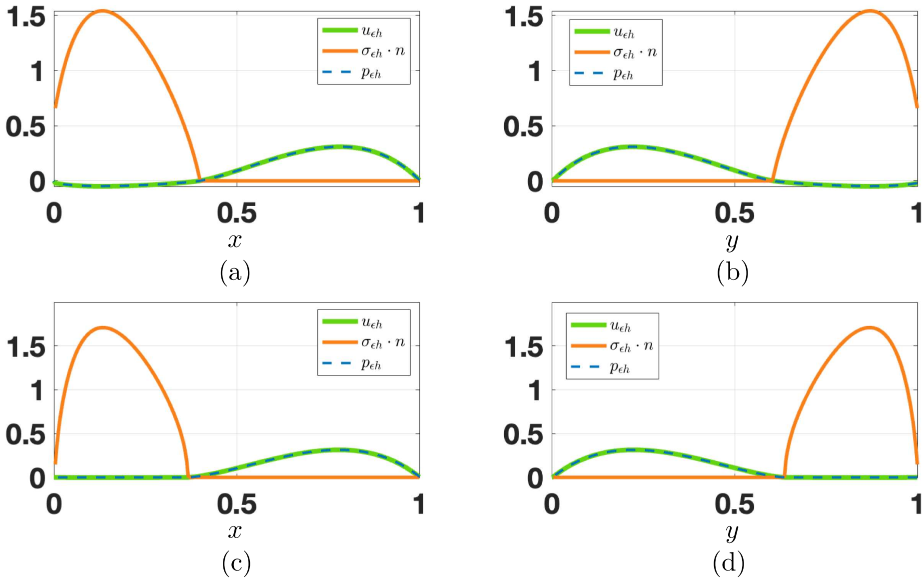


FIGURE 2. Example 1: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_S with $(h, \varepsilon) = (2^{-8}, 2^{-5})$ (see (a), (b)) and $(h, \varepsilon) = (2^{-8}, 10^{-8})$ (see (c), (d)) of $(\mathbf{MP}_\varepsilon\textbf{-I})_h$ by using Algorithm 1. (a) $\varepsilon = 2^{-5}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S1}. (b) $\varepsilon = 2^{-5}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S2}. (c) $\varepsilon = 10^{-8}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S1}. (d) $\varepsilon = 10^{-8}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S2}.

and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$. In Figure 4, we plot the error data using logarithmic scales on the x-axis and the y-axis. The line $y = x$ represents the first order convergence. The same treatment applies to the below Examples (see Figs. 8 and 12 in Sects. 5.2 and 5.3, respectively).

Furthermore, we explore the efficiency of Algorithms 1 and 2. We compute the errors $\|\boldsymbol{\sigma}_h^{(n+1)} - \boldsymbol{\sigma}_h^{(n)}\|_V$ and $\|u_h^{(n+1)} - u_h^{(n)}\|_Q$ $(n = 1, 2, \ldots)$ at each iteration step for the two algorithms with $(h, \varepsilon) = (2^{-8}, 10^{-8})$ and display in Figures 5a and 5b, respectively.

5.2. Example 2

We consider an L-shaped region $\Omega = (0,2)^2 \backslash [1,2]^2$ with Signorini boundary $\Gamma_S = \Gamma_{S1} \cup \Gamma_{S2}$, $\Gamma_{S1} = (0,2) \times \{0\}$, $\Gamma_{S2} = \{0\} \times (0,2)$ and Dirichlet boundary $\Gamma_D = \partial\Omega \setminus \Gamma_S$, and set $f = 6\sin(\pi(x - \frac{1}{3}))\cos(\frac{y}{3})$.

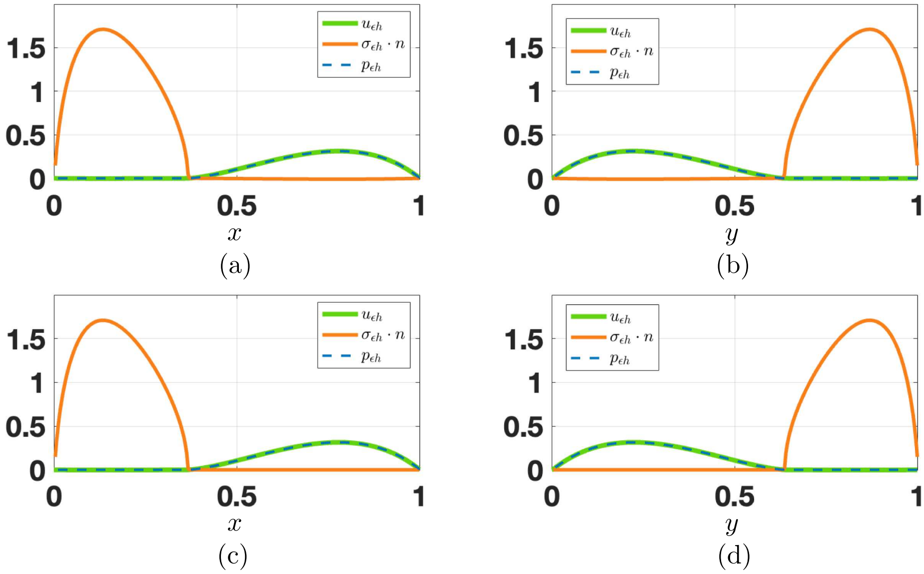


FIGURE 3. Example 1: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_S with $(h,\varepsilon) = (2^{-8}, 2^{-5})$ (see (a), (b)) and $(h,\varepsilon) = (2^{-8}, 10^{-8})$ (see (c), (d)) of $\mathbf{(MP_\varepsilon\text{-}II)}_h$ by using Algorithm 2. (a) $\varepsilon = 2^{-5}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S1}. (b) $\varepsilon = 2^{-5}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S2}. (c) $\varepsilon = 10^{-8}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S1}. (d) $\varepsilon = 10^{-8}$: $(u_{\varepsilon h}, \boldsymbol{\sigma}_{\varepsilon h}\cdot\boldsymbol{n}, p_{\varepsilon h})$ on Γ_{S2}.

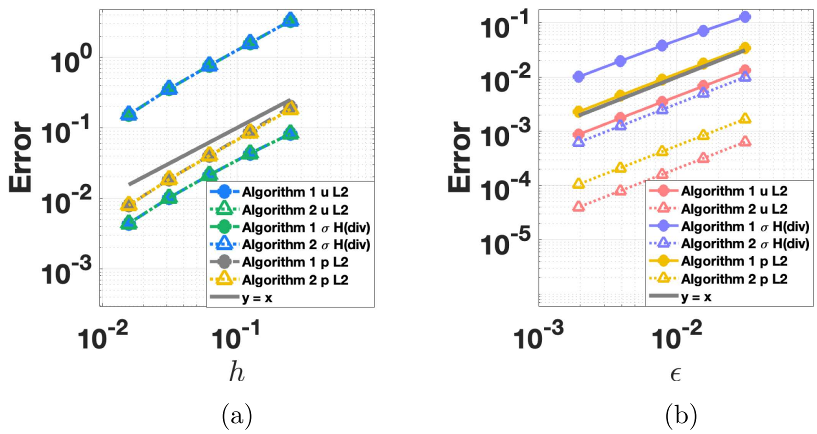
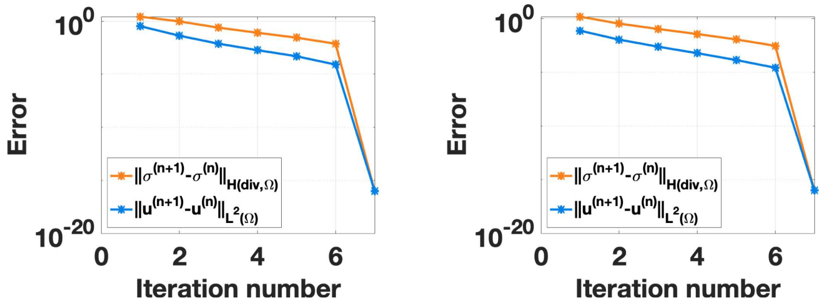


FIGURE 4. Example 1: the errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ computed by Algorithms 1 and 2. (a) Fixed $\varepsilon = 10^{-8}$. (b) Fixed $h = 2^{-8}$.

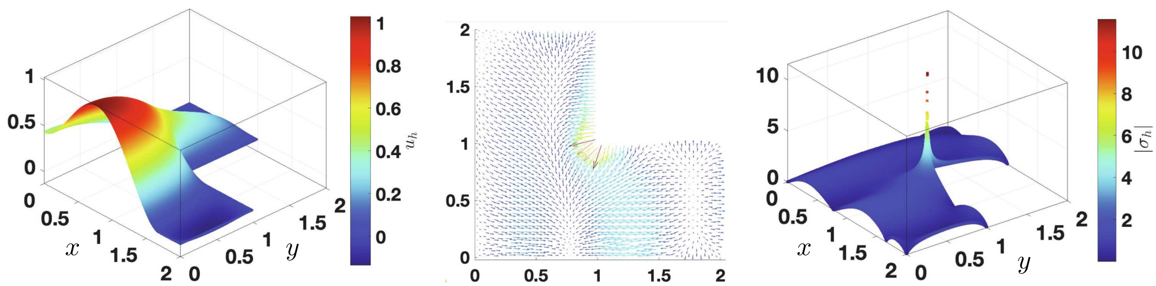
The numerical solution is plotted in Figure 6 which is computed by Algorithm 1 with $(h,\varepsilon) = (2^{-8}, 10^{-8})$, as well as the figures of u_h, $\boldsymbol{\sigma} \cdot_h \boldsymbol{n}$ and the Lagrange multiplier p_h on Γ_{S1} and Γ_{S2} are displayed in Figure 7. We observe from Figures 6b and 6c that $\boldsymbol{\sigma}_h$ has a singularity at the vertex $(1,1)$.

We adopt Algorithms 1 and 2 to solve $\mathbf{(MP_\varepsilon\text{-}I)}_h$ and $\mathbf{(MP_\varepsilon\text{-}II)}_h$ respectively. In Figure 8a, we calculate the errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ with fixed $\varepsilon = 10^{-8}$ and different h. We observe the first-order convergence for $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$, while the convergence rate of $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$ is less than 1. Under fixed $h = 2^{-8}$, the errors displayed in Figure 8b with $\varepsilon = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$, which implies



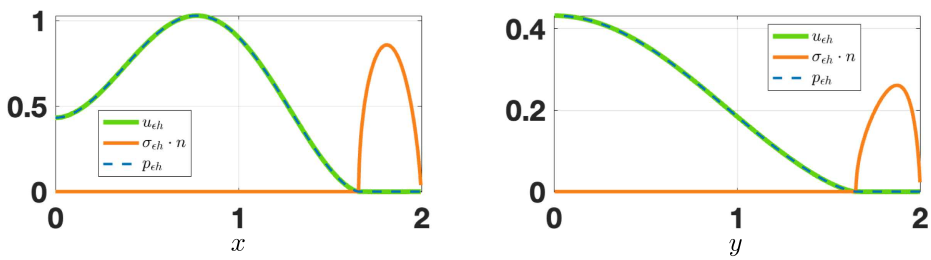
(a) (b)

FIGURE 5. Example 1: the iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}\|_Q$ of Algorithms 1 and 2 with $(h, \varepsilon) = (2^{-8}, 10^{-8})$. (a) Algorithm 1. (b) Algorithm 2.



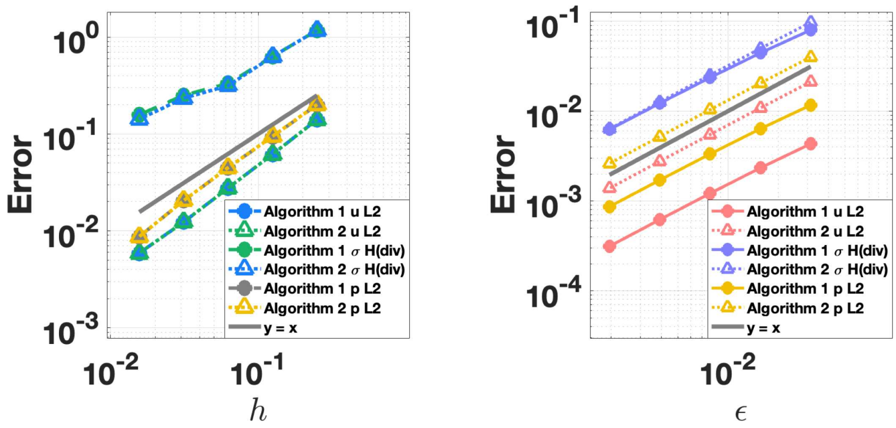
(a) (b) (c)

FIGURE 6. Example 2: the numerical solution $(\boldsymbol{\sigma}_h, u_h)$. (a) u_h. (b) $\boldsymbol{\sigma}_h$. (c) $|\boldsymbol{\sigma}_h|$.



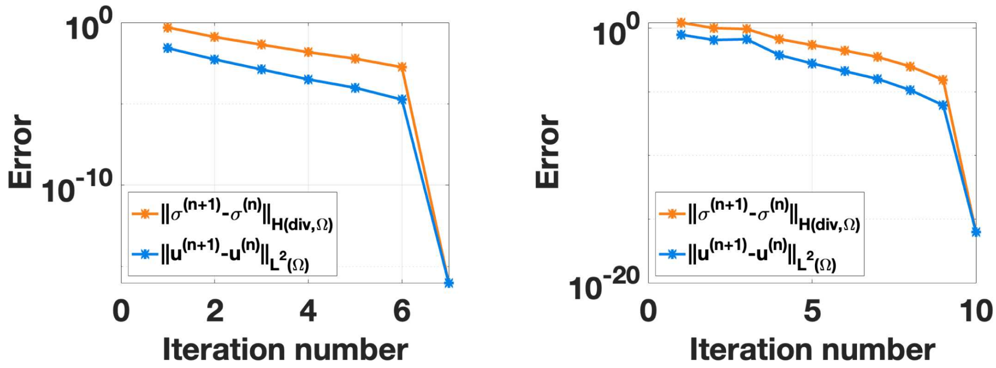
(a) (b)

FIGURE 7. Example 2: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_S. (a) $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_{S1}. (b) $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_{S2}.



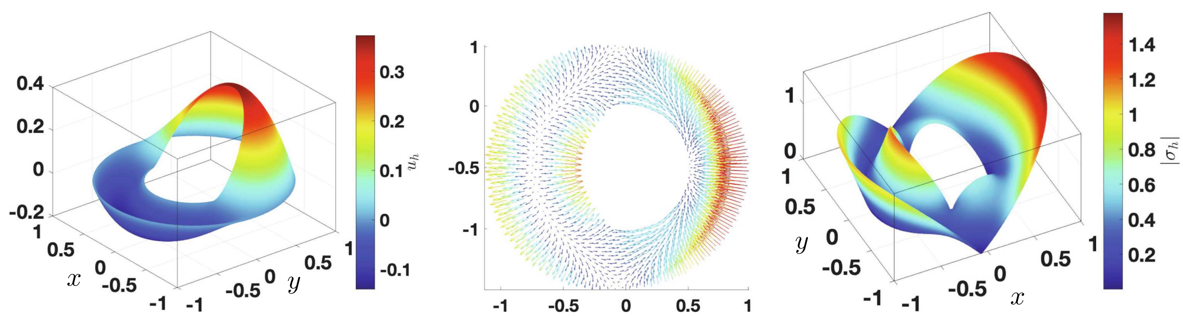
(a) (b)

FIGURE 8. Example 2: the errors $\|\boldsymbol{\sigma}_{\varepsilon ref}-\boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref}-u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref}-p_{\varepsilon h}\|_\Lambda$ computed by Algorithms 1 and 2. (a) Fixed $\varepsilon = 10^{-8}$. (b) Fixed $h = 2^{-8}$.



(a) (b)

FIGURE 9. Example 2: the iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)}-\boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)}-u_{\varepsilon h}^{(n)}\|_Q$ of Algorithms 1 and 2 with $(h,\varepsilon) = (2^{-8}, 10^{-8})$. (a) Algorithm 1. (b) Algorithm 2.



(a) (b) (c)

FIGURE 10. Example 3: the numerical solution $(\boldsymbol{\sigma}_h, u_h)$. (a) u_h. (b) $\boldsymbol{\sigma}_h$. (c) $|\boldsymbol{\sigma}_h|$.

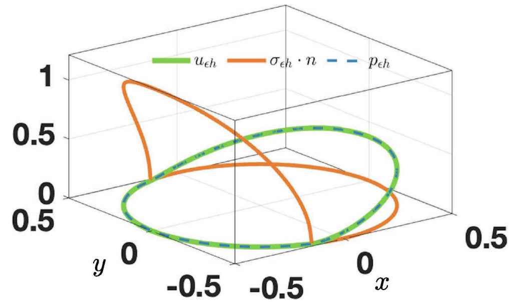
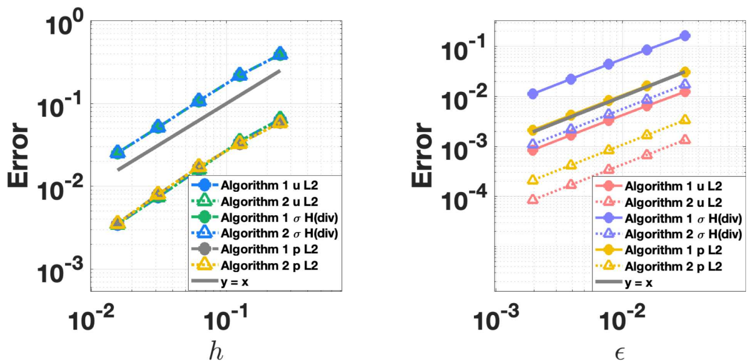
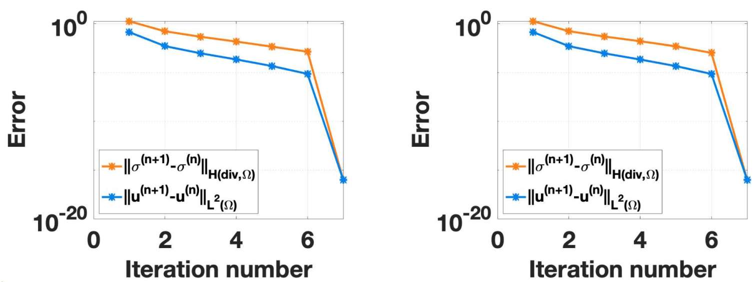


FIGURE 11. Example 3: $(u_h, \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, p_h)$ on Γ_S.



(a) (b)

FIGURE 12. Example 3: the errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ computed by Algorithms 1 and 2. (a) Fixed $\varepsilon = 10^{-8}$. (b) Fixed $h = 2^{-8}$.



(a) (b)

FIGURE 13. Example 3: the iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}\|_Q$ of Algorithms 1 and 2 with $(h, \varepsilon) = (2^{-8}, 10^{-8})$. (a) Algorithm 1. (b) Algorithm 2.

$O(\varepsilon)$-convergence. We obtain the reference solution $(\boldsymbol{\sigma}_{\varepsilon ref}, u_{\varepsilon ref}, p_{\varepsilon ref})$ by solving the numerical solution on a fine mesh with $(h, \varepsilon) = (2^{-8}, 10^{-8})$. Moreover, the iteration errors $\|\boldsymbol{\sigma}_{\varepsilon h}^{(n+1)} - \boldsymbol{\sigma}_{\varepsilon h}^{(n)}\|_V$ and $\|u_{\varepsilon h}^{(n+1)} - u_{\varepsilon h}^{(n)}\|_Q$ are shown in Figures 9a and 9b with $(h, \varepsilon) = (2^{-8}, 10^{-8})$.

5.3. Example 3

We conduct a test on the numerical schemes by solving a problem involving the Signorini condition on a curved boundary. The computational domain is $\Omega = \{(x,y) : \frac{1}{2} < \sqrt{x^2+y^2} < 1\}$, with the Signorini boundary $\Gamma_S = \{(x,y) : \sqrt{x^2+y^2} = \frac{1}{2}\}$ and the Dirichlet boundary $\Gamma_D = \partial\Omega \setminus \Gamma_S$. We set $f = 5\sin(\frac{1}{2}\pi x)$.

The numerical solution of Algorithm 1 is depicted in Figure 10, while Figure 11 shows the plots of u_h, $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ and the Lagrange multiplier p_h on Γ_S which satisfies the Signorini conditions.

We apply Algorithms 1 and 2 for simulation, similar to Examples 1 and 2. We first take the reference solution $(\boldsymbol{\sigma}_{\varepsilon ref}, u_{\varepsilon ref}, p_{\varepsilon ref})$ with $(h,\varepsilon) = (2^{-8}, 10^{-8})$ and display the errors $\|\boldsymbol{\sigma}_{\varepsilon ref} - \boldsymbol{\sigma}_{\varepsilon h}\|_V$, $\|u_{\varepsilon ref} - u_{\varepsilon h}\|_Q$ and $\|p_{\varepsilon ref} - p_{\varepsilon h}\|_\Lambda$ for fixed $\varepsilon = 10^{-8}$ in Figure 12a. Next, we fix $h = 2^{-8}$ and carry out the simulation for different penalty parameter $\varepsilon = 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}, 2^{-9}$ (see Fig. 12b). The experimental errors exhibit first-order convergence in both Figures 12a and 12b. Finally, we plot the iteration errors of Algorithms 1 and 2 in Figures 13a and 13b.

This example demonstrates the effectiveness of the mixed formulation even in cases with curved contact boundaries.

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Data availability

Data/code are available on request from the authors.

References

[1] J.M. Aitchison and M.W. Poole, A numerical algorithm for the solution of Signorini problems. *J. Comput. Appl. Math.* **94** (1998) 55–67.

[2] J.M. Aitchison, A.A. Lacey and M. Shillor, A model for an electropaint process. *IMA J. Appl. Math.* **33** (1984) 17–31.

[3] T. Apel and S. Nicaise, Regularity of the solution of the scalar Signorini problem in polygonal domains. *Results Math.* **75** (2020) 75.

[4] L. Baillet and T. Sassi, Mixed finite element methods for the Signorini problem with friction. *Numer. Methods Partial Differ. Equ.* **22** (2006) 1489–1508.

[5] H.J. Barbosa and T.J.R. Hughes, Circumventing the Babuška–Brezzi condition in mixed finite element approximations of elliptic variational inequalities. *Comput. Methods Appl. Mech. Eng.* **97** (1992) 193–210.

[6] F.B. Belgacem, Numerical simulation of some variational inequalities arisen from unilateral contact problems by the finite element methods. *SIAM J. Numer. Anal.* **37** (2000) 1198–1216.

[7] F.B. Belgacem and Y. Renard, Hybrid finite element methods for the Signorini problem. *Math. Comput.* **72** (2003) 1117–1145.

[8] D. Boffi, F. Brezzi and M. Fortin, Mixed Finite Element Methods and Application. Springer, Heidelberg (2013).

[9] H. Brézis, Problémes unilatéraus. *J. Math. Pures Appl.* **51** (1972) 1–168.

[10] F. Brezzi, W.W. Hager and P.A. Raviart, Error estimates for the finite element solution of variational inequalities, Part II: mixed methods. *Numer. Math.* **31** (1978) 1–16.

[11] F. Chouly and P. Hild, A Nitsche-based method for unilateral contact problems: numerical analysis. *SIAM J. Numer. Anal.* **51** (2013) 1295–1307.

[12] F. Chouly and P. Hild, On convergence of the penalty method for unilateral contact problems. *Appl. Numer. Math.* **65** (2013) 27–40.

[13] F. Chouly, P. Hild and Y. Renard, Finite Element Approximation of Contact and Friction in Elasticity. Springer (2023).

[14] I. Dione, Optimal convergence analysis of the unilateral contact problem with and without Tresca friction conditions by the penalty method. *J. Math. Anal. Appl.* **472** (2019) 266–284.

[15] G. Drouet and P. Hild, Optimal convergence for discrete variational inequalities modeling Signorini contact in 2D and 3D without additional assumptions on the unknown contact set. *SIAM J. Numer. Anal.* **53** (2015) 1488–1507.

[16] T. Führer, N. Heuer and E.P. Stephan, On the DPG method for Signorini problems. *IMA J. Numer. Anal.* **38** (2018) 1893–1926.

[17] G.N. Gatica, M. Maischak and E.P. Stephan, Numerical analysis of a transmission problem with Signorini contact using mixed-FEM and BEM. *ESAIM:M2AN* **45** (2011) 779–802.

[18] R. Glowinski, Numerical Methods for Nonlinear Variational Problems. Springer-Verlag (1984).

[19] W. Han and B.D. Reddy, On the finite element method for mixed variational inequalities arising in elastoplasticity. *SIAM J. Numer. Anal.* **32** (1995) 1778–1807.

[20] J. Haslinger and T. Sassi, Mixed finite element approximation of 3D contact problems with given friction: error analysis and numerical realization. *ESAIM:M2AN* **38** (2004) 563–578.

[21] P. Hild and Y. Renard, An improved a priori error analysis for finite element approximations of Signorini's problem. *SIAM J. Numer. Anal.* **50** (2012) 2400–2419.

[22] D. Hua and L. Wang, A mixed finite element method for the unilateral contact problem in elasticity. *Sci. China Ser. A* **49** (2006) 513–524.

[23] K. Ito and K. Kunisch, Semi-smooth newton methods for variational inequalities of the first kind. *ESAIM:M2AN* **37** (2003) 41–62.

[24] K. Ito and K. Kunisch, Semi-smooth Newton methods for the Signorini problem. *Appl. Math.* **53** (2008) 455–468.

[25] T. Kashiwabara, Finite element method for Stokes equations under leak boundary condition of friction type. *SIAM J. Numer. Anal.* **51** (2013) 2448–2469.

[26] N. Kikuchi and Y.J. Song, Remarks on relations between penalty and mixed finite element methods for a class of variational inequalities. *Int. J. Numer. Methods Eng.* **15** (1980) 1557–1561.

[27] K. Kunisch and G. Stadler, Generalized Newton methods for the 2D-Signorini contact problem with friction in function space. *ESAIM:M2AN* **39** (2005) 827–854.

[28] J. Lions and E. Magenes, Non-homogeneous Boundary Value Problems and Applications. Vol. I. Springer, Berlin (1972).

[29] M. Moussaoui and K. Khodja, Régularité des solutions d'un problème mêlé Dirichlet–Signorini dans un domaine polygonal plan. *Commun. Partial Differ. Equ.* **17** (1992) 805–826.

[30] J.T. Oden and S.J. Kim, Interior penalty methods for finite element approximations of the Signorini problem in elastostatics. *Comput. Math. Appl.* **8** (1982) 35–56.

[31] B.D. Reddy, Mixed variational inequalities arising in elastoplasticity. *Nonlinear Anal. TMA* **19** (1992) 1071–1089.

[32] J.E. Roberts and J.-M. Thomas, Mixed and hybrid methods, in Handbook of Numerical Analysis. *Finite Elemen Methods (Part 1)*, edited by P.G. Ciarlet and J.L. Lions. Vol. II. Elsevier, North-Holland (1991) 523–639.

[33] N. Saito, Y. Sugitani and G. Zhou, Unilateral problem for the Stokes equations: the well-posedness and finite element approximation. *Appl. Numer. Math.* **105** (2016) 124–147.

[34] A. Signorini, Sopra akune questioni di elastostatica. *Atti della Societa Italiana per il Progresso delle Scienze* **21** (1933) 143–148.

[35] L. Slimane, A. Bendali and P. Laborde, Mixed formulations for a class of variational inequalities. *ESAIM:M2AN* **38** (2004) 177–201.

[36] E. Zeidler, Applied Functional Analysis: Main Principles and Their Applications. Vol. 109 of *Applied Mathematical Sciences*. Springer, New York (1995).

[37] G. Zhou and N. Saito, The Navier–Stokes equations under a unilateral boundary condition of Signorini's type. *J. Math. Fluid Mech.* **18** (2016) 481–510.