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CONVERGENCE OF LATTICE BOLTZMANN METHODS WITH OVERRELAXATION FOR A NONLINEAR CONSERVATION LAW

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Abstract. We approximate a nonlinear multidimensional conservation law by Lattice Boltzmann Methods (LBM), based on underlying BGK type systems with finite number of velocities discretized by a transport-collision scheme. The collision part involves a relaxation parameter ω which value greatly influences the stability and accuracy of the method, as noted by many authors. In this article we clarify the relationship between ω and the parameters of the kinetic model and we highlight some new monotonicity properties which allow us to extend the previously obtained stability and convergence results. Numerical experiments are performed.

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1. Introduction

Consider a nonlinear conservation law

$$\partial_t u + \operatorname{div}_x A(u) = 0 \tag{1}$$

where $u(x,t) \in \mathcal{U}$. $\mathcal{U}$ is a real interval, $(x,t) \in \mathbb{R}^D \times \mathbb{R}$ and $A \in \mathrm{C}^1(\mathcal{U};\mathbb{R}^D)$. In what follows we assume that A is not constant and, without loss of generality, that $\mathcal{U} = \mathbb{R}$ and $A(0) = 0$. Numerical approximations of this equation can be designed by using a discrete kinetic system of BGK type, as introduced in [26,2]:

$$\forall \ell \in \{1,\dots,L\}, \quad \partial_t f_\ell^\varepsilon + \sum_{d=1}^{D} v_{\ell d}\partial_{x_d} f_\ell^\varepsilon = -\frac{1}{\varepsilon}(f_\ell^\varepsilon - \mathcal{M}_\ell(u^\varepsilon)), \quad u^\varepsilon = \sum_{\ell=1}^{L} f_\ell^\varepsilon. \tag{2}$$

Here, ε is a positive relaxation parameter, the $v_{\ell d}$ are fixed real coefficients. The equation (1) is purely hyperbolic, there is no diffusion included in it. As a consequence the velocities $v_{\ell d}$ do not depend on ε. The functions $\mathcal{M}_\ell$ are defined on $\mathbb{R}$. They are called Maxwellian by reference to the kinetic theory, or also equilibria in the lattice Boltzmann community. Moreover the following compatibility conditions are fulfilled:

$$\forall u \in \mathbb{R}, \quad \sum_{\ell=1}^{L} \mathcal{M}_\ell(u) = u, \quad \sum_{\ell=1}^{L} v_{\ell d}\mathcal{M}_\ell(u) = A_d(u), \quad d = 1,\dots,D. \tag{3}$$

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If f^ε is a solution of (2) then, denoting $w_j^\varepsilon = \sum_{\ell=1}^L v_{\ell j} f_\ell^\varepsilon$, we have

$$\begin{cases} \partial_t u^\varepsilon + \displaystyle\sum_{d=1}^D \partial_{x_d} w_d^\varepsilon = 0, \\ \partial_t w_j^\varepsilon + \displaystyle\sum_{d=1}^D \sum_{\ell=1}^L (v_{\ell d} v_{\ell j} \partial_{x_d} f_\ell^\varepsilon) = -\frac{1}{\varepsilon}\big(w_j^\varepsilon - A_j(u^\varepsilon)\big), \quad j = 1, \dots, D. \end{cases}$$

Hence formally if $f_\ell^\varepsilon \to f_\ell$ for all ℓ when ε tends to 0, then, denoting $w_j = \sum_{\ell=1}^L v_{\ell j} f_\ell$ and $u = \sum_{\ell=1}^L f_\ell$, one has $w_j = A_j(u)$ for all j, and u is a solution of equation (1). Actually, let $u_0 \in \mathrm{L}^1(\mathbb{R}^D) \cap \mathrm{L}^\infty(\mathbb{R}^D) \cap \mathrm{BV}(\mathbb{R}^D)$ be an initial data for (1):

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^D. \tag{4}$$

The convergence towards the unique weak entropy solution of the Cauchy problem (1)–(4) has been rigorously proved in [26]. Convergence also holds in the presence of boundary conditions, see [25]. An essential assumption for convergence is that the Maxwellian functions $\mathcal{M}_\ell$ are monotone nondecreasing:

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \mathcal{M}_\ell'(u) \geq 0, \quad \ell = 1, \dots, L \tag{5}$$

where

$$\mu_\infty = \|u_0\|_\infty. \tag{6}$$

In [2,3,1] we use system (2) to construct finite volume schemes for (1). The procedure is as follows:

(1) For fixed $\varepsilon > 0$, construct a numerical approximation of the BGK system (2) by splitting it into a set of linear transport equations which are solved by a monotone upwind scheme, and a system of ordinary differential equations which can be solved analytically.
(2) Make $\varepsilon \to 0$ in this approximation to obtain a finite volume "relaxed" scheme for (1).

Under condition (5), we established L^∞, L^1 and BV estimates which are uniform with respect to ε. We proved convergence for f^ε for all $\varepsilon > 0$ and also convergence of the relaxed scheme to the unique weak entropy solution of (1)–(4).

Systems of form (2) have also been used to construct lattice Boltzmann schemes for (1), see [18,12,13,20,4,6,5], and see [22] for a general overview on Lattice Boltzmann Methods (abbreviated LBM). The velocities $v^{(l)} = (v_{l1}, \dots, v_{lD})$ must be such that if x_α is a node of the lattice and Δt is the time step, then $x_\alpha - \Delta t v^{(l)}$ is also a node of the lattice, which we denote $x_{\alpha_\ell'}$. Splitting again system (2) into the linear transport part and an ordinary differential system, we study here a cartesian lattice Boltzmann scheme with space step $\Delta x = (\Delta x_d)_{1 \leq d \leq D}$:

$$x_\alpha = (x_{d,\alpha_d})_{1 \leq d \leq D} = (\alpha_d \Delta x_d)_{1 \leq d \leq D},$$
$$C_\alpha = \prod_{d=1}^D \left] x_{d,\alpha_d} - \frac{\Delta x_d}{2}, x_{d,\alpha_d} + \frac{\Delta x_d}{2} \right[, \qquad \alpha \in \mathbb{Z}^D.$$

We denote $\lambda = (\lambda_1, \dots, \lambda_D)$ the lattice velocity: $\lambda_d \frac{\Delta t}{\Delta x_d} = 1$ for $d = 1, \dots, D$. The set of characteristic velocities is such that

$$v_{\ell d} = j_{\ell d} \lambda_d, \quad j_{\ell d} \in \mathbb{Z}, \quad d = 1, \dots, D, \quad \ell = 1, \dots, L \tag{7}$$

and there exist ℓ and d such that $|j_{\ell d}| = 1$. Therefore

$$\forall d \in \{1, \dots, D\}, \quad \forall \ell \in \{1, \dots, L\}, \quad \alpha_{\ell,d}' = \alpha_d - j_{\ell d}$$

and, denoting $j_\ell = (j_{\ell d})_{1 \leq d \leq D}$,

$$\forall \ell \in \{1, \dots, L\}, \quad \alpha_\ell' = \alpha - j_\ell.$$

We denote $\mathcal{V} = \prod_{1 \leq d \leq D} \Delta x_d$ the volume of C_α.

Initialization

$$u_\alpha^0 = \frac{1}{\mathcal{V}} \int_{C_\alpha} u_0(x)\,\mathrm{d}x, \qquad \alpha \in \mathbb{Z}^D, \tag{8}$$

$$f_{\ell,\alpha}^0 = \mathcal{M}_\ell\big(u_\alpha^0\big), \qquad \alpha \in \mathbb{Z}^D, \quad \ell = 1, \dots, L. \tag{9}$$

Remark 1.1. The importance of the initialization step has been highlighted by other authors, see [9]. Here, the fact that $f_{\ell,\alpha}^0$ is at equilibrium is crucial for the convergence, see the proof of Proposition 4.9 below.

Then for $n \geq 0$ the scheme is as follows:

Step 1: stream phase. Each transport equation is solved exactly:

$$f_{\ell,\alpha}^{n+\frac{1}{2}} = f_{\ell,\alpha'_\ell}^n \quad \text{with} \quad \alpha'_\ell = \alpha - j_\ell, \quad \ell = 1, \dots, L, \tag{10}$$

and

$$u_\alpha^{n+\frac{1}{2}} = \sum_{\ell=1}^{L} f_{\ell,\alpha}^{n+\frac{1}{2}}. \tag{11}$$

Step 2: collision phase. For each node x_α we solve numerically on $[t_n, t_{n+1}]$ the system

$$(f_\ell^\varepsilon)' = -\frac{1}{\varepsilon}(f_\ell^\varepsilon - \mathcal{M}_\ell(u^\varepsilon)), \quad \ell = 1, \dots, L, \quad u^\varepsilon = \sum_{\ell=1}^{L} f_\ell^\varepsilon$$

with $f_\ell^\varepsilon(t_n) = f_{\ell,\alpha}^{n+\frac{1}{2}}$. Different methods of integrating this system exist, we refer to [15] for an enlightening study.

We drop the superscript ε for the sake of simplicity. By the compatibility conditions (3), we have

$$u(t) = u(t_n)$$

so that we set

$$u_\alpha^{n+1} = u_\alpha^{n+\frac{1}{2}}. \tag{12}$$

Note that the exact solution of the system is

$$f_{\ell,\alpha}(t_{n+1}) = \left(1 - \exp\left(-\frac{\Delta t}{\epsilon}\right)\right)\mathcal{M}_\ell\left(u_\alpha^{n+\frac{1}{2}}\right) + \exp\left(-\frac{\Delta t}{\epsilon}\right) f_{\ell,\alpha}^{n+\frac{1}{2}}, \quad \ell = 1, \dots, L \tag{13}$$

while the explicit Euler scheme gives

$$f_{\ell,\alpha}^{n+1} = \left(1 - \frac{\Delta t}{\epsilon}\right) f_{\ell,\alpha}^{n+\frac{1}{2}} + \frac{\Delta t}{\epsilon}\mathcal{M}_\ell\left(u_\alpha^{n+\frac{1}{2}}\right), \quad \ell = 1, \dots, L. \tag{14}$$

Here we choose a positive parameter ω and we set

$$f_{\ell,\alpha}^{n+1} = (1-\omega) f_{\ell,\alpha}^{n+\frac{1}{2}} + \omega\mathcal{M}_\ell\left(u_\alpha^{n+\frac{1}{2}}\right), \quad \ell = 1, \dots, L. \tag{15}$$

Note that we have then (12). With (15) we forget the interpretations (13) and (14) to deal with one parameter ω defining the scheme, relevant for our study. For a large class of models, it is known that for stability reasons we must have $\omega \in [0, 2]$, see [16].

In the case where the equation (1) is linear, a Von Neumann stability analysis of the LBM is possible, see [18,10,8]. In [18], the author proves that the D1Q2[1] scheme is L^2-stable for $\omega \in [0,2]$. In [10,8] a more general analysis is performed *via* a multistep finite difference interpretation. Still in the linear framework, L^1 convergence of lattice Boltzmann schemes is proved for hyperbolic and parabolic equations in [27, 28].

For general hyperbolic non linear equations, one can distinguish two cases. If $\omega \in]0,1]$ the LBM (10)–(15) can be interpreted as (10)–(13) by setting $\omega = 1 - \exp\left(-\frac{\Delta t}{\epsilon}\right)$, the asymptotic limit $\varepsilon = 0$ being the case $\omega = 1$. Therefore all the estimates of [3] apply. Giving a fixed value to the ratio $\Delta t/\varepsilon$ means that Δt (and therefore Δx) and ε tend simultaneously to 0. This case has been treated in [11] for the D1Q2 approximation and convergence has been proved.

The overrelaxation case $\omega > 1$ cannot be treated by the same method. Several numerical experiments show that the LBM is more accurate for $\omega > 1$, see [13, 20] for example. One of the reasons of this better accuracy is the fact that the numerical diffusion – when keeping all the parameters fixed except ω – is proportional to $1/\omega - 1/2$, see [7], Theorem 3.7, thus decreases as ω approaches two. In the general nonlinear scalar case, as far as we know, the only result of convergence with $\omega > 1$ is due to T. Bellotti [5,6] in the case of the D1Q2 model: the LBM is interpreted as a multistep finite difference method, and by monotonicity convergence is proved for $\omega \in]1,2[$.

In the present paper, we highlight the fact that monotonicity properties can be retrieved in the case $\omega > 1$, for a large class of multidimensional models, thus allowing to prove convergence of the LBM. We do not follow the multistep interpretation of [6] but rather keep the scheme under the form (10)–(15). We show that the choice of the velocities $v_{\ell d}$ dictates the range of values of ω for which monotonicity holds. To give an idea of our approach, consider the D1Q2 model (detailed below). The monotonicity condition (5) imposes a first constraint: $\lambda > |A'(u)|$, which is the well known subcharacteristic condition. It does not depend on ω. What is new is the following condition on ω, generalised in Proposition 2.1:

$$\omega \leq \frac{2}{1 + \frac{|A'(u)|}{\lambda}}$$

and our convergence result of the LBM to the unique entropy solution of (1)–(4) under those conditions.

The plan of the paper is the following. In Section 2 we present the LBM in detail and study monotonicity conditions. In Section 3, some 1D and 2D examples are given. Section 4 is devoted to the proof of convergence. Some numerical tests illustrate our result in Section 5.

2. Detailed presentation of the LBM and monotonicity conditions

We remark that except in the case $\omega = 1$ the scheme (10)–(15) cannot be made explicit as a one-step finite volume approximation of (1). We have to deal with the f_ℓ.

For a given initial data $u_0 \in \mathrm{L}^\infty\left(\mathbb{R}^D\right)$ we define the $f^0_{\ell,\alpha}$ by (8)–(9). For $n \geq 0$ the scheme can be written as:

$$f^{n+1}_{\ell,\alpha} = (1-\omega) f^n_{\ell,\alpha'_\ell} + \omega \mathcal{M}_\ell\left(\sum_{k=1}^{L} f^n_{k,\alpha'_k}\right), \quad \ell = 1, \ldots, L$$

and

$$u^{n+1}_\alpha = \sum_{k=1}^{L} f^n_{k,\alpha'_k}.$$

We consider here the most usual case where

$$\mathcal{M}_\ell(u) = a_\ell u + \sum_{d=1}^{D} b_{\ell d} A_d(u), \quad \ell = 1, \ldots, L, \tag{16}$$

[1]DDQL means that the scheme is constructed for $x \in \mathbb{R}^D$ with L equations for the BGK model (2).

where a_ℓ and $b_{\ell d}$ are real coefficients. In order to satisfy the compatibility conditions (3), we have to impose

$$\sum_{\ell=1}^{L} a_\ell = 1, \qquad \sum_{\ell=1}^{L} b_{\ell d} = 0, \quad \sum_{\ell=1}^{L} v_{\ell d} a_\ell = 0, \qquad \sum_{\ell=1}^{L} v_{\ell d} b_{\ell j} = \delta_{dj}. \tag{17}$$

Note that $\mathcal{M}(0) = 0$, $\mathcal{M}$ being defined by $\mathcal{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_L)$.

We can write the scheme as follows:

$$f_{\ell,\alpha}^{n+1} = \mathcal{S}_\ell\Big(f_{1,\alpha_1'}^{n}, \ldots, f_{L,\alpha_\ell'}^{n}\Big) \quad (\ell = 1, \ldots, L), \quad \text{and} \quad u_\alpha^{n+1} = \sum_{k=1}^{L} f_{k,\alpha}^{n+1} \tag{18}$$

with

$$\forall f = (f_1, \ldots, f_L) \in \mathbb{R}^L, \quad \mathcal{S}_\ell(f) = (1-\omega) f_\ell + \omega \mathcal{M}_\ell\left(\sum_{k=1}^{L} f_k\right), \quad \ell = 1, \ldots, L,$$

or

$$\mathcal{S}_\ell(f) = (1 - \omega(1 - a_\ell)) f_\ell + \omega a_\ell \sum_{k \neq \ell} f_k + \omega \sum_{d=1}^{D} b_{\ell d} A_d\left(\sum_{k=1}^{L} f_k\right), \quad \ell = 1, \ldots, L. \tag{19}$$

It is useful to remark that by the compatibility conditions (3)

$$\forall u \in \mathbb{R}, \quad \mathcal{S}_\ell(\mathcal{M}(u)) = \mathcal{M}_\ell(u), \quad \ell = 1, \ldots, L \tag{20}$$

that is for all u, $\mathcal{M}(u)$ is an eigenfunction of the collision operator $\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_L)$. Using notation (6), we define

$$\mathbf{m}_\ell = \mathcal{M}_\ell(-\mu_\infty), \quad \mathbf{M}_\ell = \mathcal{M}_\ell(\mu_\infty), \quad V = \prod_{\ell=1}^{L} [\mathbf{m}_\ell, \mathbf{M}_\ell]. \tag{21}$$

We denote $f = (f_1, \ldots, f_\ell)$. We define the following monotonicity condition:

$$\forall k \in \{1, \ldots, L\}, \quad \forall \ell \in \{1, \ldots, L\}, \quad \forall f \in V, \quad \partial_{f_k} \mathcal{S}_\ell(f) \geq 0. \tag{22}$$

For linear equations, this condition means that the positivity of approximate solutions is preserved, see [17] for a study in that case.

Proposition 2.1. *Suppose that (5) is satisfied. For $f \in V$, we denote $u = \sum_{\ell=1}^{L} f_\ell$. Then $u \in [-\mu_\infty, \mu_\infty]$,*

$$0 \leq \mathcal{M}_\ell'(u) = a_\ell + \sum_{d=1}^{D} b_{\ell d} A_d'(u) \leq 1, \quad \ell = 1, \ldots, L, \tag{23}$$

and condition (22) is satisfied if and only if the following condition is satisfied:

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \forall \ell \in \{1, \ldots, L\}, \quad \omega \leq \frac{1}{1 - a_\ell - \sum_{d=1}^{D} b_{\ell d} A_d'(u)}. \tag{24}$$

Moreover if (5) and (22) are satisfied, then $\omega \in]0, 2[$.

Note that the value $\omega = 2$ is not allowed. This fact is not surprising since the LBM can then be second order accurate and, at least for finite differences, a monotone method is at most first order accurate.

Proof. Suppose that (5) is satisfied. If $f \in V$ we set $u = \sum_{k=1}^{L} f_k$. The compatibility conditions (3) imply that $u \in [-\mu_\infty, \mu_\infty]$, so for $k \neq \ell$

$$\partial_{f_k} \mathcal{S}_\ell(f) = \omega \mathcal{M}'_\ell(u) \geq 0. \tag{25}$$

Consequently condition (22) is satisfied if and only if $\partial_{f_\ell} \mathcal{S}_\ell \geq 0$ on V for all ℓ.

By (3), $\displaystyle\sum_{\ell=1}^{L} \mathcal{M}'_\ell(u) = 1$ which with (5) proves (23). Now we have

$$\begin{aligned} \partial_{f_\ell} \mathcal{S}_\ell(f) &= 1 - \omega(1 - \mathcal{M}'_\ell(u)) \\ &= 1 - \omega\left(1 - a_\ell - \sum_{d=1}^{D} b_{\ell d} A'_d(u)\right) \end{aligned} \tag{26}$$

which gives condition (24).

We prove now that $\omega \in]0, 2[$. As $\sum_{\ell=1}^{L} \mathcal{M}'_\ell(u) = 1$, if $L > 2$ then there exists ℓ and u such that $\mathcal{M}'_\ell(u) < \frac{1}{2}$, hence $\omega < 2$. If $L = 2$ we are in one space dimension. If $\mathcal{M}'_\ell(u) \geq \frac{1}{2}$ for all ℓ and u, then $\mathcal{M}'_\ell(u) = \frac{1}{2}$ for all ℓ and u. This happens only in the trivial case of a constant flux A. Therefore $\omega < 2$. □

In view of these properties, we can proceed as follows:

- u_0 being given, we choose Δx.
- We fix the lattice velocity λ (and hence Δt) and ω such that conditions (5) and (24) are satisfied.

We now illustrate the method by some 1D and 2D examples.

3. Examples in 1D and 2D

3.1. The D1Q2 model

In 1D, the minimal number of velocities is 2, and in that case system (2) is the well-known Jin and Xin's model [21]:

$$v_2 = -v_1 = \lambda > 0,$$

and

$$\mathcal{M}_1(u) = \frac{1}{2}\left(u - \frac{A(u)}{\lambda}\right), \quad \mathcal{M}_2(u) = \frac{1}{2}\left(u + \frac{A(u)}{\lambda}\right).$$

Condition (5) is satisfied if and only if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad |A'(u)| \leq \lambda.$$

Condition (24) is satisfied if and only if

$$\omega \leq \frac{2}{1 + \max_{u \in [-\mu_\infty, \mu_\infty]} \frac{|A'(u)|}{\lambda}}. \tag{27}$$

Note that this condition appears in T. Bellotti's proof of convergence [6].

As a consequence, ω can take all values in $]0, 2[$, provided that λ is large enough. The numerical scheme can be written as

$$\begin{cases} f_{1,\alpha}^{n+1} = (1 - \omega) f_{1,\alpha+1}^{n} + \dfrac{\omega}{2}\left(u_\alpha^{n+1} - \dfrac{A\left(u_\alpha^{n+1}\right)}{\lambda}\right) \\ f_{2,\alpha}^{n+1} = (1 - \omega) f_{1,\alpha-1}^{n} + \dfrac{\omega}{2}\left(u_\alpha^{n+1} + \dfrac{A\left(u_\alpha^{n+1}\right)}{\lambda}\right) \\ u_\alpha^{n+1} = f_{1,\alpha+1}^{n} + f_{2,\alpha-1}^{n}. \end{cases}$$

Remark 3.1. We can express the variables $u = \sum_{\ell=1}^{L} f_\ell$ and $w = \sum_{\ell=1}^{L} v_\ell f_\ell$, showing that u is constant in the collision phase while w is not at its asymptotic equilibrium $A(u)$ except for $\omega = 1$. Using the notation (7) we have

$$\begin{cases} u_\alpha^{n+\frac{1}{2}} = \displaystyle\sum_{\ell=1}^{L} f_{\ell,\alpha-j_\ell}^n \\ w_\alpha^{n+\frac{1}{2}} = \displaystyle\sum_{\ell=1}^{L} v_\ell f_{\ell,\alpha-j_\ell}^n \end{cases} \quad \text{and} \quad \begin{cases} u_\alpha^{n+1} = u_\alpha^{n+\frac{1}{2}} \\ w_\alpha^{n+1} = w_\alpha^{n+\frac{1}{2}} - \omega\Big(w_\alpha^{n+\frac{1}{2}} - A\Big(u_\alpha^{n+\frac{1}{2}}\Big)\Big). \end{cases}$$

Remark 3.2. We could also consider a D1Q3 model with a parameter $\theta \in \mathbb{R}$, $v_3 = -v_1 = \lambda > 0$, $v_2 = 0$,

$$\mathcal{M}_1(u) = \frac{1}{2}\left(\frac{u}{\theta} - \frac{A(u)}{\lambda}\right), \quad \mathcal{M}_2(u) = \left(1 - \frac{1}{\theta}\right)u, \quad \mathcal{M}_3(u) = \frac{1}{2}\left(\frac{u}{\theta} + \frac{A(u)}{\lambda}\right).$$

Condition (5) is satisfied if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad |A'(u)| \le \frac{\lambda}{\theta} \quad \text{and} \quad \theta \ge 1. \tag{28}$$

The case $\theta = 1$ leads us to the D1Q2 model, so we consider $\theta > 1$. Condition (24) is then satisfied if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \omega \le \min\left(\theta, \frac{2\theta}{2\theta - 1 + \frac{\theta|A'(u)|}{\lambda}}\right).$$

It is easy to see that $\omega \in \left[0, \frac{3}{2}\right]$ necessarily. For all $\theta > 1$ and λ satisfying (28), there exist ω satisfying (24). The maximal value of ω is $\frac{3}{2+\max_{u\in[-\mu_\infty,\mu_\infty]}\frac{|A'(u)|}{\lambda}}$, obtained for $\theta = \frac{3}{2+\max_{u\in[-\mu_\infty,\mu_\infty]}\frac{|A'(u)|}{\lambda}}$. In the purely hyperbolic context considered in the present work, we did not find a case for which this D1Q3 model is better than the D1Q2 one.

3.2. A D1Q4 model

We consider the following model:

$$-v_1 = v_4 = 2\lambda, \quad -v_2 = v_3 = \lambda \tag{29}$$

where $\lambda > 0$, and

$$\begin{cases} \mathcal{M}_1(u) = \dfrac{u}{4} - \dfrac{A(u)}{6\lambda}, \\ \mathcal{M}_2(u) = \dfrac{u}{4} - \dfrac{A(u)}{6\lambda}, \\ \mathcal{M}_3(u) = \dfrac{u}{4} + \dfrac{A(u)}{6\lambda}, \\ \mathcal{M}_4(u) = \dfrac{u}{4} + \dfrac{A(u)}{6\lambda}. \end{cases} \tag{30}$$

Condition (5) is satisfied if and only if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad |A'(u)| \le \frac{3}{2}\lambda.$$

Condition (24) is here

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \omega \le \frac{1}{\frac{3}{4} + \frac{|A'(u)|}{6\lambda}}.$$

The maximal value for ω is then $\frac{4}{3}\cdot$ The numerical scheme can be written as

$$\begin{cases} f_{1,\alpha}^{n+1} = (1-\omega) f_{1,\alpha+2}^{n} + \omega\left(\dfrac{u_\alpha^{n+1}}{4} - \dfrac{A(u_\alpha^{n+1})}{6\lambda}\right) \\ f_{2,\alpha}^{n+1} = (1-\omega) f_{2,\alpha+1}^{n} + \omega\left(\dfrac{u_\alpha^{n+1}}{4} - \dfrac{A(u_\alpha^{n+1})}{6\lambda}\right) \\ f_{3,\alpha}^{n+1} = (1-\omega) f_{3,\alpha-1}^{n} + \omega\left(\dfrac{u_\alpha^{n+1}}{4} + \dfrac{A(u_\alpha^{n+1})}{6\lambda}\right) \\ f_{4,\alpha}^{n+1} = (1-\omega) f_{4,\alpha-2}^{n} + \omega\left(\dfrac{u_\alpha^{n+1}}{4} + \dfrac{A(u_\alpha^{n+1})}{6\lambda}\right) \\ u_\alpha^{n+1} = f_{1,\alpha+2}^{n} + f_{2,\alpha+1}^{n} + f_{3,\alpha-1}^{n} + f_{4,\alpha-2}^{n}. \end{cases}$$

3.3. A D2Q4 model

This model is the direct extension of the D1Q2 model in 2D:

$$v^{(1)} = \lambda_1(-1,0), \quad v^{(2)} = \lambda_2(0,-1), \quad v^{(3)} = \lambda_1(1,0), \quad v^{(4)} = \lambda_2(0,1), \tag{31}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and

$$\begin{cases} \mathcal{M}_1(u) = \dfrac{u}{4} - \dfrac{A_1(u)}{2\lambda_1}, \\ \mathcal{M}_2(u) = \dfrac{u}{4} - \dfrac{A_2(u)}{2\lambda_2}, \\ \mathcal{M}_3(u) = \dfrac{u}{4} + \dfrac{A_1(u)}{2\lambda_1}, \\ \mathcal{M}_4(u) = \dfrac{u}{4} + \dfrac{A_2(u)}{2\lambda_2}\cdot \end{cases} \tag{32}$$

Condition (5) is satisfied if and only if

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad 2|A_d'(u)| \le \lambda_d, \quad d = 1,2.$$

Condition (24) reads as

$$\omega \le \frac{4}{3 + 2\frac{|A_d'(u)|}{\lambda_d}}, \quad d = 1,2.$$

The maximal value of ω is $\omega = \frac{4}{3}\cdot$

3.4. A D2Q8 model

One can consider more velocities: for $\lambda_1 > 0$, $\lambda_2 > 0$ we define

$$\begin{aligned} v^{(1)} &= (-\lambda_1, 0) = -v^{(5)}, \quad v^{(2)} = (-\lambda_1, -\lambda_2) = -v^{(6)} \\ v^{(3)} &= (0, -\lambda_2) = -v^{(7)}, \quad v^{(4)} = (\lambda_1, -\lambda_2) = -v^{(8)}. \end{aligned} \tag{33}$$

and

$$\begin{cases} \mathcal{M}_1(u) = \dfrac{u}{8} - \dfrac{A_1(u)}{6\lambda_1} & \mathcal{M}_2(u) = \dfrac{u}{8} - \dfrac{A_1(u)}{6\lambda_1} - \dfrac{A_2(u)}{6\lambda_2}, \\ \mathcal{M}_3(u) = \dfrac{u}{8} - \dfrac{A_2(u)}{6\lambda_2} & \mathcal{M}_4(u) = \dfrac{u}{8} + \dfrac{A_1(u)}{6\lambda_1} - \dfrac{A_2(u)}{6\lambda_2}, \\ \mathcal{M}_5(u) = \dfrac{u}{8} + \dfrac{A_1(u)}{6\lambda_1} & \mathcal{M}_6(u) = \dfrac{u}{8} + \dfrac{A_1(u)}{6\lambda_1} + \dfrac{A_2(u)}{6\lambda_2}, \\ \mathcal{M}_7(u) = \dfrac{u}{8} + \dfrac{A_2(u)}{6\lambda_2} & \mathcal{M}_8(u) = \dfrac{u}{8} - \dfrac{A_1(u)}{6\lambda_1} + \dfrac{A_2(u)}{6\lambda_2}. \end{cases} \tag{34}$$

Proposition 3.3. *We denote* $c_d = \max_{u\in[-\mu_\infty,\mu_\infty]} \frac{8|A'_d(u)|}{3\lambda_d}$. *A sufficient condition for* (5) *to be satisfied is that* $c_d \leq 1$ *for* $d = 1, 2$. *In this case condition* (24) *is verified if* $\omega \leq \dfrac{8}{7 + \frac{1}{2}(c_1 + c_2)}$.

Proof. For all ℓ we have

$$a_\ell + \sum_{d=1}^{2} b_{\ell d} A'_d(u) = \frac{1}{8}\left(1 + \frac{1}{2}\sum_{d=1}^{2} \epsilon_d \frac{8A'_d(u)}{3\lambda_d}\right), \quad \epsilon_d \in \{-1, 0, 1\}.$$

Hence

$$0 \leq a_\ell + \sum_{d=1}^{2} b_{\ell d} A'_d(u) \leq \frac{1}{8}\left(1 + \frac{1}{2}(c_1 + c_2)\right) \leq \frac{1}{4}$$

so condition (5) is satisfied. Moreover

$$a_\ell + \sum_{d=1}^{2} b_{\ell d} A'_d(u) \geq \frac{1}{8}\left(1 - \frac{1}{2}(c_1 + c_2)\right)$$

so that condition (24) is satisfied if $\omega \leq \dfrac{8}{7 + \frac{1}{2}(c_1 + c_2)}$. ☐

The above examples show that monotonicity can hold for $\omega > 1$, and that the maximal value of this parameter depends on λ. The space step being fixed, when $|\lambda|$ increases, Δt decreases. A BGK model being chosen, conditions (5) and (24) have been useful to fix the lattice velocity λ and the parameter ω. As proved in Section 4, the method then converges.

4. Convergence of the LBM

In the remaining of the paper, we consider the cartesian D-dimensional ($D \geq 1$) LBM with equilibrium functions under the form (16). We take an initial function $u_0 \in \mathrm{L}^1(\mathbb{R}^D) \cap \mathrm{L}^\infty(\mathbb{R}^D) \cap \mathrm{BV}(\mathbb{R}^D)$ and initialize the scheme by (8) and (9). Then the scheme is given by (18) with (19).

Denoting χ_α^n the characteristic function of $C_\alpha \times [t_n, t_{n+1}[$ and χ_α the characteristic function of C_α we define

$$\begin{aligned} f_\alpha^n &= \left(f_{1,\alpha}^n, \dots, f_{L,\alpha}^n\right), & f^n &= (f_\alpha^n)_{\alpha\in\mathbb{Z}^D} \\ f_\Delta(x,t) &= \sum_{n=0}^{\infty} \sum_{\alpha\in\mathbb{Z}^D} f_\alpha^n \chi_\alpha^n(x,t), & u_\Delta(x,t) &= \sum_{\ell=1}^{L} f_{\Delta,\ell}(x,t) \\ f_\Delta^n(x) &= \sum_{\alpha\in\mathbb{Z}^D} f_\alpha^n \chi_\alpha(x), & u_\Delta^n(x) &= \sum_{\ell=1}^{L} f_{\Delta,\ell}^n(x). \end{aligned}$$

Note that $f_\Delta(x,t) \in \mathbb{R}^L$ and $f_\Delta^n(x) \in \mathbb{R}^L$.

Our goal is to prove the convergence of u_Δ to the unique weak entropy solution of (1)–(4) when Δx (and hence Δt) tends to zero.

The initial value u_0 being given, we fix the $v_{\ell d}$ and ω such that conditions (5) and (24) are satisfied. Therefore the ratios $\Delta t/\Delta x_d$ are constant and when we make Δt vary, then Δx varies with the same proportion.

We establish L^∞, L^1, BV and entropy estimates in the spirit of Crandall and Majda [14]. In this reference, the authors consider monotone schemes in conservative form for (1). They prove convergence to the unique weak entropy solution of the Cauchy problem. Our framework is different, as we cannot express u_α^{n+1} as a function of some u_β^n where the C_β are some neighbour cells of C_α. Nevertheless, we prove some properties on the f_α^n and u_α^n which allow us to use the same mathematical tools.

Remark 4.1. It is shown in [10] that u_α^{n+1} depends upon u_β^n, $u_\beta^{n-1}, \ldots$, that is the scheme is multi-step, with all the difficulties that come with this (see [5,6]). Moreover, this multi-step scheme on u is highly dependent on the choice of discrete velocities that one has done, thus it would be difficult to prove our result in this way.

We recall that

$$\|u_\Delta^n\|_\infty = \sup_{\alpha\in\mathbb{Z}^D} |u_\alpha^n|,$$

$$\|u_\Delta^n\|_1 = \mathcal{V} \sum_{\alpha\in\mathbb{Z}^D} |u_\alpha^n|,$$

$$\mathrm{TV}(u_\Delta^n) = \sum_{\alpha\in\mathbb{Z}^D} \sum_{d=1}^{D} \left(\frac{\mathcal{V}}{\Delta x_d} \left| u_{\alpha+e_d}^n - u_\alpha^n \right| \right)$$

where we denoted $(e_1, \ldots, e_D)$ the canonical basis of $\mathbb{R}^D$. Also:

$$\|f_\Delta^n\|_\infty = \sup_{\alpha\in\mathbb{Z}^D} \sum_{\ell=1}^{L} |f_{\ell,\alpha}^n|,$$

$$\|f_\Delta^n\|_1 = \mathcal{V} \sum_{\alpha\in\mathbb{Z}^D} \sum_{\ell=1}^{L} |f_{\ell,\alpha}^n|,$$

$$\mathrm{TV}(f_\Delta^n) = \sum_{\alpha\in\mathbb{Z}^D} \sum_{d=1}^{D} \sum_{\ell=1}^{L} \left(\frac{\mathcal{V}}{\Delta x_d} \left| f_{\ell,\alpha+e_d}^n - f_{\ell,\alpha}^n \right| \right).$$

Lemma 4.2. *The discretization (8) and (9) of the initial data satisfies the following properties:*

$$\left\|u_\Delta^0\right\|_\infty \le \mu_\infty, \quad \left\|u_\Delta^0\right\|_1 \le \|u_0\|_1, \quad \mathrm{TV}\big(u_\Delta^0\big) \le \mathrm{TV}(u_0) \tag{35}$$

and

$$\left\|f_\Delta^0\right\|_\infty = \left\|u_\Delta^0\right\|_\infty, \quad \left\|f_\Delta^0\right\|_1 = \left\|u_\Delta^0\right\|_1, \quad \mathrm{TV}\big(f_\Delta^0\big) = \mathrm{TV}\big(u_\Delta^0\big). \tag{36}$$

Proof. The two first inequalities in (35) are clear, the last one is proved in [14]. To prove (36) we first use the fact that $\mathcal{M}(0) = 0$:

$$\left\|f_\Delta^0\right\|_\infty = \sup_{\alpha\in\mathbb{Z}^D} \sum_{\ell=1}^{L} \left| \mathcal{M}_\ell\big(u_\alpha^0\big) - \mathcal{M}_\ell(0) \right|.$$

The monotonicity property (5) implies that the sign of $\mathcal{M}_\ell(u_\alpha^0) - \mathcal{M}_\ell(0)$ is the same for all ℓ:

$$\left\|f_\Delta^0\right\|_\infty = \sup_{\alpha\in\mathbb{Z}^D} \left| \sum_{\ell=1}^{L} \big(\mathcal{M}_\ell(u_\alpha^0) - \mathcal{M}_\ell(0) \big) \right|.$$

The first compatibility condition in (3) then implies that $\left\|f^0_\Delta\right\|_\infty = \left\|u^0_\Delta\right\|_\infty$. The proof is the same for the L^1 norm.

$$\mathrm{TV}\left(f^0_\Delta\right) = \sum_{\alpha\in\mathbb{Z}^D}\sum_{d=1}^{D}\sum_{\ell=1}^{L}\left(\frac{\mathcal{V}}{\Delta x_d}\left|\mathcal{M}_\ell\left(u^0_{\alpha+e_d}\right) - \mathcal{M}_\ell\left(u^0_\alpha\right)\right|\right)$$

and we conclude with the same arguments. □

4.1. Convergence to a weak solution

Here we obtain the same kind of estimates as in [3]: supremum norm bound, L^1 contraction, TVD property and convergence to equilibrium, in order to obtain convergence of u^n_Δ to a weak solution of the problem (1)–(4).

Proposition 4.3. *Suppose that conditions* (5) *and* (24) *are satisfied. Then for all* $n \geq 0$,

$$\forall\alpha\in\mathbb{Z}^D,\quad u^n_\alpha \in [-\mu_\infty,\mu_\infty] \quad\text{and}\quad f^n_\alpha \in V \tag{37}$$

where V *is defined in* (21).

Proof. We apply Lemma 4.2: $u^0_\alpha \in [-\mu_\infty,\mu_\infty]$, which is the domain where the $\mathcal{M}_\ell$ are monotone. Therefore $f^0_\alpha \in V$. By recurrence, suppose that (37) is true for a given $n \geq 0$. We remark that by (20)

$$\mathcal{S}_\ell(\mathbf{m}_1,\ldots,\mathbf{m}_L) = \mathbf{m}_\ell,\quad \mathcal{S}_\ell(\mathbf{M}_1,\ldots,\mathbf{M}_L) = \mathbf{M}_\ell.$$

Denoting $f^{n+\frac{1}{2}}_\alpha = \left(f^n_{1,\alpha'_1},\ldots,f^n_{L,\alpha'_L}\right)$, $\mathbf{m} = (\mathbf{m}_1,\ldots,\mathbf{m}_L)$:

$$\begin{aligned} f^{n+1}_{\ell,\alpha} - \mathbf{m}_\ell &= \mathcal{S}_\ell\left(f^{n+\frac{1}{2}}_\alpha\right) - \mathcal{S}_\ell(\mathbf{m}) \\ &= \int_0^1 \sum_{k=1}^{L} \partial_{f_k}\mathcal{S}_\ell\left(\mathbf{m} + \theta\left(f^{n+\frac{1}{2}}_\alpha - \mathbf{m}\right)\right)\left(f^n_{k,\alpha'_k} - \mathbf{m}_k\right)\mathrm{d}\theta \geq 0 \end{aligned}$$

by (22). In the same way, $\mathbf{M}_\ell - f^{n+1}_{\ell,\alpha} \geq 0$ so that $f^{n+1}_\alpha \in V$ and hence $u^{n+1}_\alpha \in [-\mu_\infty,\mu_\infty]$. □

Corollary 4.4. *Suppose that conditions* (5) *and* (24) *are satisfied.*

$$\|u_\Delta\|_\infty \leq \mu_\infty \quad\text{and}\quad \|f_\Delta\|_\infty \leq \sum_{\ell=1}^{L}\max(|\mathbf{m}_\ell|,|\mathbf{M}_\ell|). \tag{38}$$

Lemma 4.5. *Suppose that conditions* (5) *and* (24) *are satisfied. For all* $f \in V$, $g \in V$*:*

$$\sum_{\ell=1}^{L}|\mathcal{S}_\ell(g) - \mathcal{S}_\ell(f)| \leq \sum_{\ell=1}^{L}|g_\ell - f_\ell|. \tag{39}$$

Proof. We denote $u = \sum_{k=1}^{L} f_k$, $v = \sum_{k=1}^{L} g_k$.

$$\begin{aligned} \mathcal{S}_\ell(g) - \mathcal{S}_\ell(f) &= (1-\omega)(g_\ell - f_\ell) + \omega(\mathcal{M}_\ell(v) - \mathcal{M}_\ell(u)) \\ &= (1-\omega)(g_\ell - f_\ell) + \omega\int_0^1 \mathcal{M}'_\ell(u + \theta(v-u))\sum_{k=1}^{L}(g_k - f_k)\,\mathrm{d}\theta. \end{aligned}$$

Denote $w_\theta = u + \theta(v-u)$. $w_\theta \in [-\mu_\infty, \mu_\infty]$ and

$$\mathcal{S}_\ell(g) - \mathcal{S}_\ell(f) = (g_\ell - f_\ell)\int_0^1 (1-\omega(1-\mathcal{M}'_\ell(w_\theta)))\,\mathrm{d}\theta + \omega \sum_{k\neq \ell}(g_k - f_k)\int_0^1 \mathcal{M}'_\ell(w_\theta)\,\mathrm{d}\theta.$$

We apply Proposition 2.1: the terms in the integrals are nonnegative (see (26)) so

$$\sum_{\ell=1}^L |\mathcal{S}_\ell(g) - \mathcal{S}_\ell(f)| \le \sum_{\ell=1}^L |g_\ell - f_\ell| \int_0^1 (1-\omega(1-\mathcal{M}'_\ell(w_\theta)))\,\mathrm{d}\theta + \omega \sum_{\ell=1}^L \left[\sum_{k\neq\ell} |g_k - f_k| \int_0^1 \mathcal{M}'_\ell(w_\theta)\,\mathrm{d}\theta\right]$$

which can be written as

$$\sum_{\ell=1}^L |\mathcal{S}_\ell(g) - \mathcal{S}_\ell(f)| \le \sum_{\ell=1}^L |g_\ell - f_\ell|(1-\omega) + \omega \sum_{\ell=1}^L \left[\sum_{k=1}^L |g_k - f_k| \int_0^1 \mathcal{M}'_\ell(w_\theta) d\theta\right].$$

As $\sum_{\ell=1}^L \mathcal{M}'_\ell(w_\theta) = 1$ we obtain the desired inequality. □

Proposition 4.6. *Suppose that conditions (5) and (24) are satisfied. Let f^n_α and g^n_α be two numerical solutions, with initial data u_0 and v_0 respectively, $u_0, v_0 \in \mathrm{L}^1(\mathbb{R}^D) \cap \mathrm{L}^\infty(\mathbb{R}^D) \cap \mathrm{BV}(\mathbb{R}^D)$ satisfying $\|u_0\|_\infty \le \mu_\infty$, $\|v_0\|_\infty \le \mu_\infty$. Then for all $n \ge 0$*

$$\left\|g^{n+1}_\Delta - f^{n+1}_\Delta\right\|_1 \le \|g^n_\Delta - f^n_\Delta\|_1 \le \|v_0 - u_0\|_1 \tag{40}$$

and there exists $C > 0$ such that

$$\left\|f^{n+1}_\Delta - f^n_\Delta\right\|_1 \le C\,\Delta t\,\mathrm{TV}(u_0). \tag{41}$$

Proof.

$$\left\|g^{n+1}_\Delta - f^{n+1}_\Delta\right\|_1 = \mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|g^{n+1}_{\ell,\alpha} - f^{n+1}_{\ell,\alpha}\right|.$$

We apply inequality (39):

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|g^{n+1}_{\ell,\alpha} - f^{n+1}_{\ell,\alpha}\right| &= \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|\mathcal{S}_\ell\big(g^n_{1,\alpha-j_1},\dots,g^n_{L,\alpha-j_L}\big) - \mathcal{S}_\ell\big(f^n_{1,\alpha-j_1},\dots,f^n_{L,\alpha-j_L}\big)\right| \\ &\le \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|g^n_{\ell,\alpha-j_\ell} - f^n_{\ell,\alpha-j_\ell}\right| = \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|g^n_{\ell,\alpha} - f^n_{\ell,\alpha}\right|.\end{aligned}$$

Moreover

$$\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|g^0_{\ell,\alpha} - f^0_{\ell,\alpha}\right| = \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|\mathcal{M}_\ell(v^0_\alpha) - \mathcal{M}_\ell(u^0_\alpha)\right| = \sum_{\alpha\in\mathbb{Z}^D}\left|\sum_{\ell=1}^L \big(\mathcal{M}_\ell(v^0_\alpha) - \mathcal{M}_\ell(u^0_\alpha)\big)\right|$$

by condition (5). Then by (3) and (8)

$$\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^L \left|g^0_{\ell,\alpha} - f^0_{\ell,\alpha}\right| = \sum_{\alpha\in\mathbb{Z}^D}\left|v^0_\alpha - u^0_\alpha\right| \le \frac{1}{\mathcal{V}}\|v_0 - u_0\|_1$$

which proves (40). Let us prove the second inequality.

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\Big|f_{\ell,\alpha}^{n+1}-f_{\ell,\alpha}^{n}\Big| &= \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\Big|\mathcal{S}_\ell\big(f_{1,\alpha-j_1}^{n},\dots,f_{L,\alpha-j_L}^{n}\big)-\mathcal{S}_\ell\Big(f_{1,\alpha-j_1}^{n-1},\dots,f_{L,\alpha-j_L}^{n-1}\Big)\Big|\\ &\le \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\Big|f_{\ell,\alpha-j_\ell}^{n}-f_{\ell,\alpha-j_\ell}^{n-1}\Big|\\ &\le \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\Big|f_{\ell,\alpha}^{n}-f_{\ell,\alpha}^{n-1}\Big|\\ &\le \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|f_{\ell,\alpha}^{1}-\mathcal{M}_\ell\big(u_\alpha^0\big)\big|.\end{aligned}$$

Moreover

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|f_{\ell,\alpha}^{1}-\mathcal{M}_\ell\big(u_\alpha^0\big)\big| &= \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|\mathcal{S}_\ell\big(f_{1,\alpha-j_1}^{0},\dots,f_{L,\alpha-j_L}^{0}\big)-\mathcal{S}_\ell\big(\mathcal{M}_1\big(u_\alpha^0\big),\dots,\mathcal{M}_L\big(u_\alpha^0\big)\big)\big|\\ &\le \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|\mathcal{M}_\ell\big(u_{\alpha-j_\ell}^0\big)-\mathcal{M}_\ell\big(u_\alpha^0\big)\big|\\ &\le \sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|u_{\alpha-j_\ell}^0-u_\alpha^0\big|\end{aligned}$$

by (23). Consequently

$$\big\|f_\Delta^{n+1}-f_\Delta^{n}\big\|_1 \le C\max_d \Delta x_d\,\mathrm{TV}(u_0)$$

which gives (41). □

Corollary 4.7. *Suppose that conditions* (5) *and* (24) *are satisfied.*

$$\forall t'>t>0,\quad \|f_\Delta(\cdot,t')-f_\Delta(\cdot,t)\|_1 \le C(t'-t+\Delta t)\mathrm{TV}(u_0). \tag{42}$$

The inequality (39) gives a BV estimate. Proceeding as in the proof of Proposition 4.6 we obtain

Proposition 4.8. *Suppose that conditions* (5) *and* (24) *are satisfied. Then for all* $n\ge 0$

$$\mathrm{TV}\big(f_\Delta^{n+1}\big)\le \mathrm{TV}\big(f_\Delta^{n}\big)\le \mathrm{TV}(u_0) \tag{43}$$

and

$$\mathrm{TV}(u_\Delta^n)\le \mathrm{TV}(f_\Delta^n)\le \mathrm{TV}(u_0). \tag{44}$$

We now estimate the distance between f^n and the equilibrium.

Proposition 4.9. *Suppose that conditions* (5) *and* (24) *are satisfied. There exists a constant* $C>0$ *such that*

$$\forall n\ge 0,\quad \|\mathcal{M}(u_\Delta^n)-f_\Delta^n\|_1 \le C\,\mathrm{TV}(u_0)\frac{|1-\omega|}{1-|1-\omega|}\Delta t. \tag{45}$$

Remark that if $\omega = 1$ then the collision part of the scheme is just the projection on equilibrium: $\mathcal{M}_\ell(u^n_\alpha) = f^n_{\ell,\alpha}$. In this case the inequality (45) is trivially satisfied and one has

$$f^{n+1}_{\ell,\alpha} = f^{n+\frac{1}{2}}_{\ell,\alpha} + \Big(\mathcal{M}_\ell\Big(u^{n+\frac{1}{2}}_\alpha\Big) - f^{n+\frac{1}{2}}_{\ell,\alpha}\Big).$$

For $\omega = 2$ condition (24) cannot be satisfied. Actually, (45) is meaningless. Remark that in this case the collision phase reads as a flipped version of the one for $\omega = 1$:

$$f^{n+1}_{\ell,\alpha} = \mathcal{M}_\ell\Big(u^{n+\frac{1}{2}}_\alpha\Big) + \Big(\mathcal{M}_\ell\Big(u^{n+\frac{1}{2}}_\alpha\Big) - f^{n+\frac{1}{2}}_{\ell,\alpha}\Big).$$

Proof. We denote $E^n = \|\mathcal{M}(u^n_\Delta) - f^n_\Delta\|_1$.

$$\begin{aligned}
E^{n+1} &= \mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\Big|\mathcal{M}_\ell(u^{n+1}_\alpha) - f^{n+1}_{\ell,\alpha}\Big| \\
&= \mathcal{V}|1-\omega|\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|\mathcal{M}_\ell(u^{n+1}_\alpha) - f^{n}_{\ell,\alpha-j_\ell}\big| \\
&\le \mathcal{V}|1-\omega|\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big(\big|\mathcal{M}_\ell(u^{n}_{\alpha-j_\ell}) - f^{n}_{\ell,\alpha-j_\ell}\big| + \big|\mathcal{M}_\ell(u^{n+1}_\alpha) - \mathcal{M}_\ell(u^{n}_{\alpha-j_\ell})\big|\big) \\
&\le |1-\omega|\left(E^n + \mathcal{V}\sum_{\alpha\in\mathbb{Z}^D}\sum_{\ell=1}^{L}\big|u^{n+1}_\alpha - u^n_{\alpha-j_\ell}\big|\right) \\
&\le |1-\omega|(E^n + C\,\mathrm{TV}(u_0)\Delta t).
\end{aligned}$$

By recurrence, as $E^0 = 0$, we obtain the result. □

The Propositions 4.3, 4.6, 4.8, 4.9 allow us to prove the following theorem.

Theorem 4.10. *Suppose that conditions* (5) *and* (24) *are satisfied. Let* f^n_α *be the numerical solution given by* (8), (9) *and* (18) *with* (19), *with initial data* $u_0 \in \mathrm{L}^1(\mathbb{R}^D)\cap\mathrm{L}^\infty(\mathbb{R}^D)\cap\mathrm{BV}(\mathbb{R}^D)$ *satisfying* $\|u_0\|_\infty \le \mu_\infty$. *Let* (Δt_i) *a sequence of time steps tending to 0, the lattice velocity* λ *being kept constant. There exists a subsequence* (Δt_{i_k}) *and a function* $f : [0,+\infty[\to \mathrm{L}^1(\mathbb{R}^D)$ *such that*

$$\lim_{k\to\infty}\max_{0\le t\le T}\|f_{\Delta_k}(\cdot,t) - f(\cdot,t)\|_1 = 0 \quad \text{for} \quad T>0 \tag{46}$$

and, denoting $u = \sum_{\ell=1}^L f_\ell$,

$$\lim_{k\to\infty}\max_{0\le t\le T}\|u_{\Delta_k}(\cdot,t) - u(\cdot,t)\|_1 = 0 \quad \text{for} \quad T>0.$$

Moreover, $f = \mathcal{M}(u)$ *and* u *is a weak solution of the problem* (1)–(4).

Proof. The first step is to prove that f_Δ converges to a function f. We proceed as in [14]: using the fact that bounded subsets of $\mathrm{L}^1(\mathbb{R}^D)\cap\mathrm{BV}(\mathbb{R}^D)$ are precompact in L^1_{loc} and the time equicontinuity (42), we have the convergence of a subsequence (f_{Δ_k}) (46). The convergence of (u_{Δ_k}) is immediate. The fact that $f = \mathcal{M}(u)$ is the consequence of the estimate (45).

In a second step, in order to prove that u is a weak solution of the problem, we rewrite the scheme in the framework of conservative difference schemes.

We remark that the scheme can be written as

$$\begin{cases} f_{\ell,\alpha}^{n+\frac{1}{2}} = f_{\ell,\alpha}^{n} - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d}\Big(F_{\ell,\alpha+\frac{e_d}{2}}^{n} - F_{\ell,\alpha-\frac{e_d}{2}}^{n}\Big), \\ u_\alpha^{n+1} = u_\alpha^{n+\frac{1}{2}} = \sum_{\ell=1}^{L} f_{\ell,\alpha}^{n+\frac{1}{2}}, \\ f_{\ell,\alpha}^{n+1} = (1-\omega) f_{\ell,\alpha}^{n+\frac{1}{2}} + \omega \mathcal{M}_\ell\big(u_\alpha^{n+1}\big). \end{cases} \tag{47}$$

The numerical flux functions are given by

$$F_{\ell,\alpha+\frac{e_d}{2}} = \left| \begin{array}{rll} -\lambda_d \sum_{i=j_{\ell d}}^{-1} f_{\ell,\alpha-ie_d} & \text{if} & v_{\ell d} < 0, \\ 0 & \text{if} & v_{\ell d} = 0, \\ \lambda_d \sum_{i=0}^{j_{\ell d}-1} f_{\ell,\alpha-ie_d} & \text{if} & v_{\ell d} > 0. \end{array} \right. \tag{48}$$

We recall that $v_{\ell d} = \lambda_d j_{\ell d}$ and $\lambda_d \frac{\Delta t}{\Delta x_d} = 1$. Therefore

$$F_{\ell,\alpha+\frac{e_d}{2}} = \left| \begin{array}{rll} F_{\ell d}\big(f_{\ell,\alpha-j_{\ell d}e_d}^{n}, \dots, f_{\ell,\alpha-e_d}^{n}\big) & \text{if} & v_{\ell d} < 0, \\ 0 & \text{if} & v_{\ell d} = 0, \\ F_{\ell d}\Big(f_{\ell,\alpha}^{n}, \dots, f_{\ell,\alpha-(j_{\ell d}-1)e_d}^{n}\Big) & \text{if} & v_{\ell d} > 0, \end{array} \right.$$

with

$$\forall g_\ell \in \mathbb{R}, \quad F_{\ell d}(g_\ell, \dots, g_\ell) = v_{\ell d} g_\ell. \tag{49}$$

As a consequence, if $g_\ell = \mathcal{M}_\ell(u)$ then by (3)

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \sum_{\ell=1}^{L} F_{\ell d}(\mathcal{M}_\ell(u), \dots, \mathcal{M}_\ell(u)) = A_d(u). \tag{50}$$

For the unknown u_α^n we have

$$u_\alpha^{n+1} = u_\alpha^n - \sum_{d=1}^{D} \frac{\Delta t}{\Delta x_d}\Big(\mathcal{A}_{\alpha+\frac{e_d}{2}}^n - \mathcal{A}_{\alpha-\frac{e_d}{2}}^n\Big)$$

with $\mathcal{A}_{\alpha+\frac{e_d}{2}}^n = \sum_{\ell=1}^{L} F_{\ell,\alpha+\frac{e_d}{2}}^n$.

$$\mathcal{A}_{\alpha+\frac{e_d}{2}}^n = \sum_{\ell/v_{\ell d}<0} F_{\ell d}\big(f_{\ell,\alpha-j_{\ell d}e_d}^n, \dots, f_{\ell,\alpha-e_d}^n\big) + \sum_{\ell/v_{\ell d}>0} F_{\ell d}\Big(f_{\ell,\alpha}^n, \dots, f_{\ell,\alpha-(j_{\ell d}-1)e_d}^n\Big).$$

So

$$\mathcal{A}_{\alpha+\frac{e_d}{2}}^n = \mathcal{A}_d\bigg(\Big(f_{\ell,\alpha+|j_{\ell d}|e_d}^n, \dots, f_{\ell,\alpha-|j_{\ell d}|e_d-e_d}^n\Big)_{1\le\ell\le L}\bigg)$$

with

$$\forall (g_1, \dots, g_\ell) \in \mathbb{R}^L, \quad \mathcal{A}_d((g_\ell, \dots, g_\ell)_{1\le\ell\le L}) = \sum_{\ell=1}^{L} v_{\ell d} g_\ell.$$

and

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \mathcal{A}_d\Big((\mathcal{M}_\ell(u), \ldots, \mathcal{M}_\ell(u))_{1\le \ell \le L}\Big) = A_d(u). \tag{51}$$

We then follow the same steps as the Lax-Wendroff theorem [24,14] but here when Δt tends to zero, $\Delta t/\Delta x_d$ being kept constant we take also into account the fact that u_Δ tends to equilibrium, so that the difference of flux tends to $\partial_{x_d} A_d(u)$. □

4.2. Convergence to the entropy solution

We recall that the Cauchy problem (1) and (4) admits a unique weak entropy solution which is characterized by [23]:

$$\int\int \left\{ |u-c|\partial_t \varphi + \operatorname{sgn}(u-c)\sum_{d=1}^{D}(A_d(u)-A_d(c))\partial_{x_d}\varphi \right\} \mathrm{d}x\,\mathrm{d}t \ge 0 \tag{52}$$

for any $c \in \mathbb{R}$ and $\varphi \in \mathrm{C}_0^\infty\big(\mathbb{R}^D \times (0,T)\big)$, $\varphi \ge 0$, and, for any interval I of $\mathbb{R}^D$:

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_I |u(x,t)-u_0(x)|\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{53}$$

As in [26], we associate to $\eta_c(u) = |u-c|$ the kinetic entropy-entropy flux pair:

$$\mathcal{H}_{\ell,c}(g_\ell) = |g_\ell - \mathcal{M}_\ell(c)|, \quad \mathcal{G}_{\ell,c}(g_\ell) = v_{\ell d}|g_\ell - \mathcal{M}_\ell(c)|, \quad d = 1,\ldots,D. \tag{54}$$

As the transport part of the scheme is monotone, the following discrete entropy inequality holds for every $l \in \{1,\ldots,L\}$:

$$\frac{\mathcal{H}_{\ell,c}\Big(f^{n+\frac12}_{\ell,\alpha}\Big) - \mathcal{H}_{\ell,c}\Big(f^n_{\ell,\alpha}\Big)}{\Delta t} + \sum_{d=1}^{D} \frac{\mathcal{Q}^n_{\ell,c,\alpha+\frac{e_d}{2}} - \mathcal{Q}^n_{\ell,c,\alpha-\frac{e_d}{2}}}{\Delta x_d} \le 0 \tag{55}$$

with

$$\mathcal{Q}_{\ell,c,\alpha+\frac{e_d}{2}} = \left| \begin{array}{rl} -\lambda_d \displaystyle\sum_{i=j_{\ell d}}^{-1} |f_{\ell,\alpha-ie_d} - \mathcal{M}_\ell(c)| & \text{if} \quad v_{\ell d} < 0, \\ 0 & \text{if} \quad v_{\ell d} = 0, \\ \lambda_d \displaystyle\sum_{i=0}^{j_{\ell d}-1} |f_{\ell,\alpha-ie_d} - \mathcal{M}_\ell(c)| & \text{if} \quad v_{\ell d} > 0. \end{array} \right. \tag{56}$$

$$\mathcal{Q}_{\ell,c,\alpha+\frac{e_d}{2}} = \left| \begin{array}{rl} \mathcal{Q}_{\ell,c,d}\big(f^n_{\ell,\alpha-j_{\ell d}e_d}, \ldots, f^n_{\ell,\alpha-e_d}\big) & \text{if} \quad v_{\ell d} < 0, \\ 0 & \text{if} \quad v_{\ell d} = 0, \\ \mathcal{Q}_{\ell,c,d}\Big(f^n_{\ell,\alpha}, \ldots, f^n_{\ell,\alpha-(j_{\ell d}-1)e_d}\Big) & \text{if} \quad v_{\ell d} > 0, \end{array} \right.$$

with

$$\forall g_\ell \in \mathbb{R}, \quad \mathcal{Q}_{\ell,c,d}(g_\ell,\ldots,g_\ell) = v_{\ell d}|g_\ell - \mathcal{M}_\ell(c))|. \tag{57}$$

As a consequence, if $g_\ell = \mathcal{M}_\ell(u)$, as $\mathcal{M}_\ell$ is non decreasing and by (3)

$$\forall u \in [-\mu_\infty, \mu_\infty], \quad \sum_{\ell=1}^{L} \mathcal{Q}_{\ell,c,d}(\mathcal{M}_\ell(u),\ldots,\mathcal{M}_\ell(u)) = \operatorname{sgn}(u-c)(A_d(u)-A_d(c)). \tag{58}$$

Lemma 4.11. *Define*

$$\forall f \in \mathbb{R}^L, \quad H_c(f) = \sum_{\ell=1}^{L} \mathcal{H}_{\ell,c}(f_\ell), \quad Q^n_{c,\alpha+\frac{e_d}{2}} = \sum_{\ell=1}^{L} \mathcal{Q}^n_{\ell,c,\alpha+\frac{e_d}{2}}, \quad (1 \leq d \leq D). \tag{59}$$

The following inequality holds:

$$\frac{H_c\left(f_\alpha^{n+1}\right) - H_c(f_\alpha^n)}{\Delta t} + \sum_{d=1}^{D} \frac{Q^n_{c,\alpha+\frac{e_d}{2}} - Q^n_{c,\alpha-\frac{e_d}{2}}}{\Delta x_d} \leq 0. \tag{60}$$

Proof. We use Lemma 4.5. Noting that $\mathcal{M}_\ell(c) = \mathcal{S}_\ell(\mathcal{M}(c))$ we can write

$$\begin{aligned}\sum_{\ell=1}^{L}\left|f_{\ell,\alpha}^{n+1} - \mathcal{M}_\ell(c)\right| &= \sum_{\ell=1}^{L}\left|\mathcal{S}_\ell\left(f_\alpha^{n+\frac{1}{2}}\right) - \mathcal{S}_\ell(\mathcal{M}(c))\right| \\ &\leq \sum_{\ell=1}^{L} |f_{\ell,\alpha}^{n+\frac{1}{2}} - \mathcal{M}_\ell(c)| \\ &\leq \sum_{\ell=1}^{L} \mathcal{H}_{\ell,c}\left(f_{\ell,\alpha}^{n+\frac{1}{2}}\right).\end{aligned}$$

Then by (55) we obtain the result. □

As a consequence of Proposition 4.9, when $\Delta t \to 0$, $f_{\Delta,\ell} \to \mathcal{M}_\ell(u)$ where u is limit of $\sum_\ell f_{\Delta,\ell}$. Hence, by monotonicity $H_c(f_{\Delta,\ell}) \to \eta_c(u)$. Consequently we can proceed again as in the proof of Lax-Wendroff theorem [24,14] and we can now state our result.

Theorem 4.12. *Suppose that conditions* (5) *and* (24) *are satisfied. Let* f_α^n *be the numerical solution given by* (8), (9) *and* (18) *with* (19), *with initial data* $u_0 \in \mathrm{L}^1\left(\mathbb{R}^D\right) \cap \mathrm{L}^\infty\left(\mathbb{R}^D\right) \cap \mathrm{BV}\left(\mathbb{R}^D\right)$ *satisfying* $\|u_0\|_\infty \leq \mu_\infty$. *Let* (Δt_i) *a sequence of time steps tending to 0, the lattice velocity* λ *being kept constant.* $(u_{\Delta t_i}(\cdot,t))$ *converge in* $\mathrm{L}^1\left(\mathbb{R}^D\right)$, *uniformly on all* $[0,T]$, $T > 0$, *to* $u(\cdot,t)$, *where* u *is the unique weak entropy solution of the Cauchy problem* (1)–(4).

Remark 4.13. The condition $u_0 \in \mathrm{L}^1\left(\mathbb{R}^D\right)$ seems restrictive but it is sufficient in practice because numerically we always compute the solution on a compact set $K \times [0,T]$ of $\mathbb{R}^D \times \mathbb{R}$. Because of the finite propagation speed property, the solution on this compact set depends only on the values of the initial data in a larger compact set $K(T)$ of $\mathbb{R}^D$. Consequently one may choose $u_0 \in \mathrm{L}^1_{loc}\left(\mathbb{R}^D\right)$.

5. Numerical experiments

5.1. One-dimensional computations

We approximate the entropy solution of the Burgers equation $\partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0$ with initial condition

$$u_0(x) = \begin{cases} 1, & \text{if } x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\forall t > 0, \quad u(x,t) = \begin{cases} 1, & \text{if } x < \dfrac{t}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

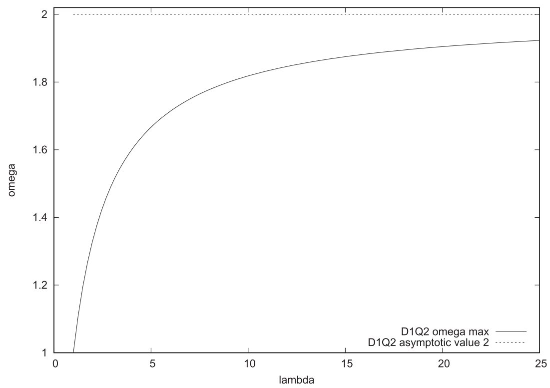

FIGURE 1. Maximal values of ω imposed by condition (24) for the D1Q2 model with our data.

The computational domain is $[-1, 1]$. We do not add any order increasing procedure because our goal is to study the influence of the conditions of monotonicity (5) and (24) on the behaviour of the solution. As the choice of λ determines the number of time steps, we fix λ and make ω vary.

For the D1Q2 model presented in Section 3, condition (5) gives

$$\lambda \geq 1 = \|A'(u_0)\|_\infty.$$

The maximal value of ω as a function of λ is plotted in Figure 1. We set $\lambda = 5$, so that the maximal value $\omega_{\max}$ of ω is not close to its asymptotic value 2, and the lost of monotonicity can be studied for $\omega > \omega_{\max}$. Condition (24) is satisfied if

$$\omega \leq \omega_0 = \frac{5}{3}\cdot$$

For the D1Q4 model presented in Section 3, condition (5) gives

$$\lambda \geq \frac{2}{3}\cdot$$

The maximal value of ω as a function of λ is plotted in Figure 2. We set $\lambda = 5$. Condition (24) is satisfied if

$$\omega \leq \omega_1 = \frac{60}{47}\cdot$$

We have $1.276 < \omega_1 < 1.277$. With $\Delta x = \frac{1}{50}$ and a final time of computation $T_{\max} = 0.8$, the supremum bound of the solution takes the values indicated in Table 1. We can see that for the D1Q2 model, if $\omega > \omega_0$ then $\|u(\cdot, T_{\max})\|_\infty > 1$: the scheme does not preserve the extrema. The D1Q4 model is very diffusive so that the value 1 is not reached for $\omega \leq 1.70$.

In Figure 3, we show the solution as a function of x at the final time $T_{\max} = 0.8$, for some values of ω. The shock is more and more accurately approximated when ω increases but the maximum of the solution is not preserved. This is classical in the context of hyperbolic conservation laws: accuracy and stability are often contradictory. This indicates that convergence in a weaker sense than the one proved here could hold. As recalled in the introduction L^2 results are available in the linear case. For example the D1Q2 model is stable for λ satisfying (5) and $\omega \in [0, 2]$, see [18]. The L^2 stability in the non linear case is an open problem. On the left picture we show the D1Q2 results, while the D1Q4 results are depicted on the right. The D1Q4 model computes a more diffusive solution than the D1Q2 one. It is also the case with a 1000 points mesh, as shown in Figure 4 left if we take the same value of ω for both models. In Figure 4 right, we take $\omega = 1.67$ for the D1Q2 model in

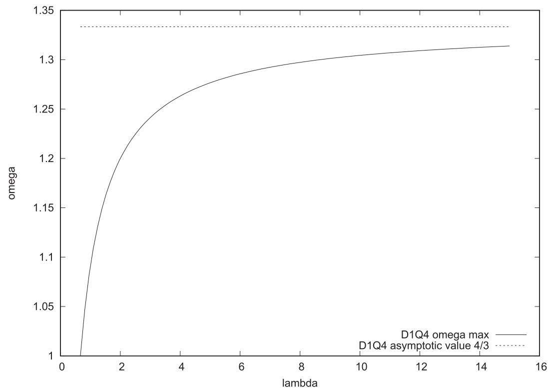

FIGURE 2. Maximal values of ω imposed by condition (24) for the D1Q4 model with our data.

TABLE 1. The values of $\|u\|_\infty$ when ω varies, at time $T_{\max} = 0.8$ with 100 points on $[-1, 1]$, $\lambda = 5$.

ω	D1Q2 $\|u(\cdot, T_{\max})\|_\infty$	D1Q4 $\|u(\cdot, T_{\max})\|_\infty$
1.28	1.0000000000000000	0.99999972434153028
1.30	1.0000000000000000	0.99999986264749974
1.67	1.0000000000000004	0.99999999999999889
1.70	1.0000814222675634	0.99999999999997691
1.80	1.0169571099362162	1.0003640299470273
1.90	1.0362991157537209	1.0734866415961333

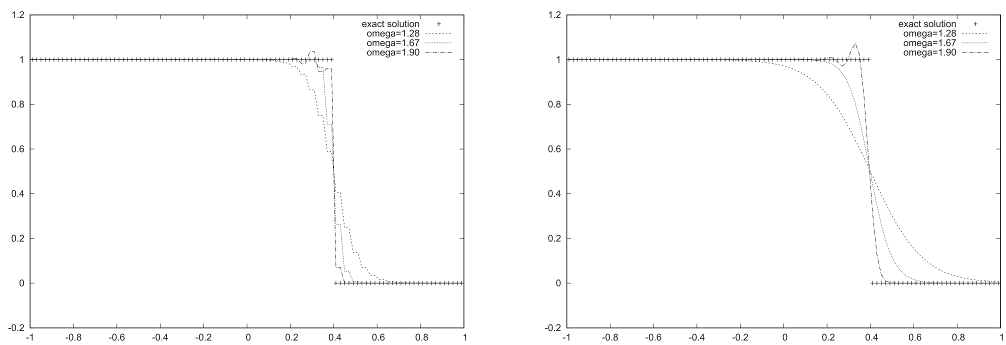

FIGURE 3. Shock solution of Burgers equation at time $T_{\max} = 0.8$ with 100 points on $[-1, 1]$, $\lambda = 5$. *Left*: D1Q2 model. *Right*: D1Q4 model.

order to keep the monotonicity property but for D1Q4 we can take $\omega = 1.8$ and we obtain the same accuracy, without loss of stability. The extended stencil seems to bring more stability than expected by the monotonicity conditions.

5.2. Two-dimensional computations

Let $\overline{u}$ be a solution of the 1D equation $\partial_t \overline{u} + \partial_y A(\overline{u}) = 0$. For $\nu = (\cos\theta, \sin\theta)$ fixed, we define for $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$:

$$u(x, t) = \overline{u}(x \cdot \nu, t).$$

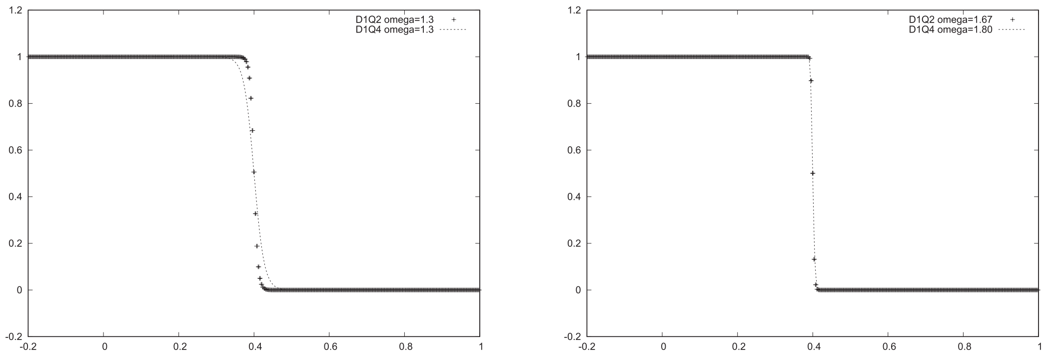

FIGURE 4. Shock solution of Burgers equation at time $T_{\max} = 0.8$ with 1000 points on $[-1, 1]$, $\lambda = 5$. *Left*: $\omega = 1.3$ for both D1Q2 and D1Q4 models. *Right*: D1Q2 model with $\omega = 1.67$ and D1Q4 model with $\omega = 1.80$.

This defines a solution of the two-dimensional equation

$$\partial_t u + \partial_{x_1}(A(u)\cos\theta) + \partial_{x_2}(A(u)\sin\theta) = 0.$$

We take the same initial value as for the one-dimensional tests:

$$\forall x \in \mathbb{R}^2, \quad u_0(x) = 1 \quad \text{if} \quad x \cdot \nu < 0, \quad u_0(x) = 0 \quad \text{else}.$$

The rotation angle is $\theta = \pi/12$. The final simulation time is equal to $T_{\max} = 0.8$. The test is performed on a 100×100 uniform mesh of $[-1, 1] \times [-1, 1]$:

$$\Delta x_1 = \Delta x_2 = \frac{1}{50}\cdot$$

As a consequence $\lambda_1 = \lambda_2 = \lambda > 0$. We test the D2Q4 and D2Q8 models presented in Section 3.

For the D2Q4 model the condition (5) reads as

$$2\max(\cos\theta, \sin\theta) \leq \lambda.$$

For $\theta = \pi/12$ we obtain the condition $\lambda > 2\cos\theta$, that is approximately $\lambda > 1.93$. The condition (24) can be written as

$$\omega \leq \omega_2 = \min\left(\frac{4}{3 + 2\frac{\cos\theta}{\lambda}}, \frac{4}{3 + 2\frac{\sin\theta}{\lambda}}\right)\cdot$$

Here we choose $\lambda = 10$, so that $1.252 < \omega_2 < 1.253$.

For the D2Q8 model the condition (5) reads as

$$\frac{4}{3}(\cos\theta + \sin\theta) \leq \lambda$$

that is approximately $\lambda > 1.63$. We choose $\lambda = 10$. Then condition (24) is satisfied if

$$\omega \leq \omega_3 = \frac{8}{7 + 4\frac{\cos\theta + \sin\theta}{3\lambda}}$$

that is $1.116 < \omega_3 < 1.117$.

TABLE 2. The values of $\|u\|_\infty$ when ω varies, at time $T_{\max} = 0.8$ with 100×100 points on $[-1, 1] \times [-1, 1]$, $\lambda = 10$.

ω	D2Q4 $\|u(\cdot, T_{\max})\|_\infty$	D2Q8 $\|u(\cdot, T_{\max})\|_\infty$
1.11	0.99999999999999400	0.99999999764886827
1.25	0.99999999999999933	0.99999999999974676
1.30	1.0000000000000000	0.99999999999999500
1.40	1.0000000000000004	1.0000000000055236
1.60	1.0000000000000004	1.0000000000013374
1.90	1.0795668257759483	1.0559666526035698

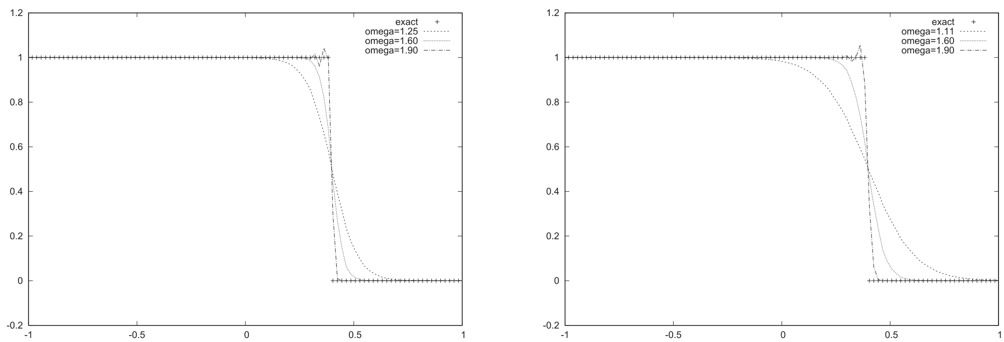

FIGURE 5. Shock solution of Burgers equation with rotated data in 2D, at time $T_{\max} = 0.8$ with 100×100 points on $[-1, 1] \times [-1, 1]$, $\lambda = 10$. Solution along the axis containing $(0, 0)$ and orthogonal to the direction of propagation of the shock. *Left*: D2Q4 model. *Right*: D2Q8 model.

The supremum bound of the solution takes the values indicated in Table 2. We observe that the L^∞ bound of the solution is preserved for $\omega \le 1.30$.

In Figure 5 we represent the solution along the axis which contains $(0, 0)$ and is orthogonal to the direction of propagation of the shock for several values of ω, beginning with ω_2 for D2Q4 and ω_3 for D2Q8. We also represent the two-dimensional isovalues for $\omega = 1.60$ in Figure 6. We remark that the D2Q8 model is more diffusive than the D2Q4 one.

Remark 5.1. One could think naively that a model with many velocities provides better results than a minimal model, because when dealing with the physical BGK equation, one has to discretize the space of velocities with enough points. But we are not in this context here and actually, the numerical tests show that the D1Q2 model better approximates shocks than the D1Q4 one, while in two space dimensions the D2Q4 model is more efficient than the D2Q8 one. Moreover, when dealing with L velocities, at least in our examples, our condition (24) gives the asymptotic maximal value $\dfrac{1}{1 - \frac{1}{L}}$ for ω, which tends to 1 when L tends to infinity, hence the monotonicity interval of ω shrinks to $[0, 1]$.

6. CONCLUSION

In this article the monotonicity properties of the considered Lattice Boltzmann Method allowed us to obtain its convergence for $\omega > 1$, for a scalar multidimensional conservation law. In particular, for the D1Q2 model, $\omega \in]0, 2[$ being given, one can find λ such that the scheme converges. The one and two dimensional tests show that the L^∞ norm of the solution can exceed the L^∞ norm of the initial data for values of ω greater than the

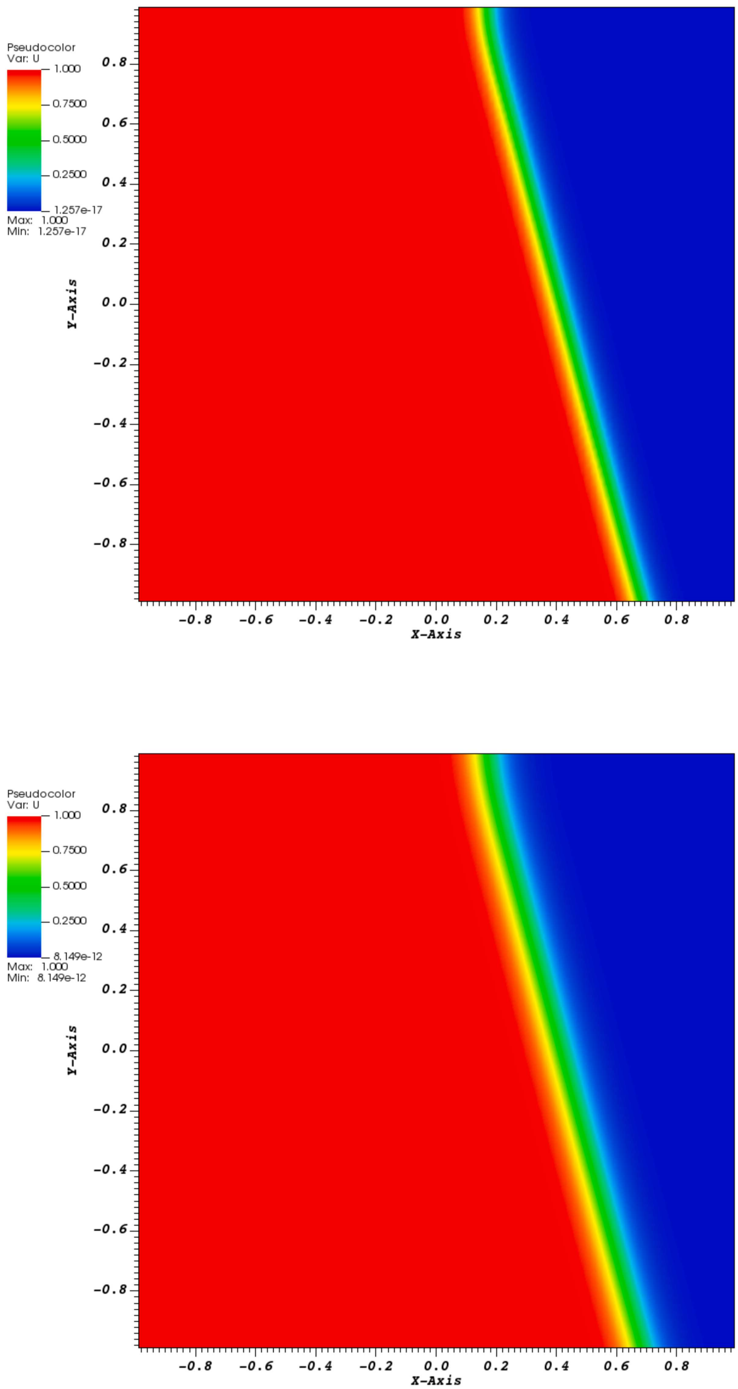

FIGURE 6. Shock solution of Burgers equation with rotated data in 2D, at time $T_{\max} = 0.8$ with 100×100 points on $[-1, 1] \times [-1, 1]$, $\lambda = 10$, $\omega = 1.60$. *Top*: isovalues for the D2Q4 model. *Bottom*: isovalues for the D2Q8 model.

theoritical value given by (24). Except for the D1Q2 model, the L^∞ norm of the solution can also be preserved on a larger interval, as shown in Tables 1 and 2. In order to better understand the processes involved, a work entirely devoted to comparisons between the different choices of models and numerical parameters has to be carried out, on smooth and non-smooth solutions, including in particular the study of equivalent equations of the LBM.

Hyperbolic systems of conservation laws can be approximated by the same scheme, but monotonicity does not hold anymore for systems, so that the method presented here cannot work. Some investigations were already done for systems, with enlightening results in the linear case, see [19] and references in this paper.

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