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NUMERICAL APPROXIMATION FOR STOCHASTIC NONLINEAR FRACTIONAL DIFFUSION EQUATION DRIVEN BY ROUGH NOISE

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Abstract. In this work, we are interested in building the fully discrete scheme for stochastic fractional diffusion equation driven by fractional Brownian sheet which is temporally and spatially fractional with Hurst parameters $H_1, \cdots, H_{d+1} \in (0, \frac{1}{2}]$ and $d = 1, 2$. We first provide the regularity of the solution. Then we employ the Wong–Zakai approximation to regularize the rough noise and discuss the convergence of the approximation. Next, the finite element and backward Euler convolution quadrature methods are used to discretize spatial and temporal operators for the obtained regularized equation, and the detailed error analyses are developed. Finally, some numerical examples are presented to confirm the theory.

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1. Introduction

The Brownian motion subordinated by inverse α-stable Lévy process is a powerful model for describing the subdiffusion phenomena [12]. In this paper, we are concerned with the Fokker–Planck equation (governing the probability density function of the subordinated Brownian motion) with nonlinear source term and external noise, *i.e.*, we present and analyze the fully discrete scheme for the following stochastic nonlinear fractional diffusion equation driven by fractional Brownian sheet noise:

$$\begin{cases} \partial_t u(\mathbf{x}, t) - {}_0\partial_t^{1-\alpha}\Delta u(\mathbf{x}, t) = f(u) + \beta\xi^{\mathbf{H}_s, H_{d+1}}(\mathbf{x}, t) & (\mathbf{x}, t) \in D \times (0, T], \\ u(\mathbf{x}, 0) = 0 & \mathbf{x} \in D, \\ u(\mathbf{x}, t) = 0 & (\mathbf{x}, t) \in \partial D \times (0, T], \end{cases} \tag{1.1}$$

where ${}_0\partial_t^{1-\alpha}$ with $\alpha \in (0, 1)$ is the Riemann–Liouville fractional derivative defined by Podlubny [29]

$${}_0\partial_t^{1-\alpha} u = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t - \xi)^{\alpha - 1} u(\xi)\, \mathrm{d}\xi;$$

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$f(u)$ is a nonlinear term satisfying the following assumptions

$$\begin{aligned} \|f(u)\|_{L^2(D)} &\le C(1+\|u\|_{L^2(D)}), \\ \|f(u)-f(v)\|_{L^2(D)} &\le C\|u-v\|_{L^2(D)} \end{aligned} \tag{1.2}$$

with C being a positive constant; β is a non-zero constant, and without loss of generality, we take $\beta = 1$ in our analyses; $\xi^{\mathbf{H}_s,H_{d+1}}$ is defined by

$$\xi^{\mathbf{H}_s,H_{d+1}}(\mathbf{x},t) = \frac{\partial^{d+1}W^{\mathbf{H}_s,H_{d+1}}(\mathbf{x},t)}{\partial x_1\cdots\partial x_d\partial t} \tag{1.3}$$

with $W^{\mathbf{H}_s,H_{d+1}}(\mathbf{x},t)$ being a fractional Brownian sheet on a stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ such that

$$\mathbb{E}\big[W^{\mathbf{H}_s,H_{d+1}}(\mathbf{x},t)W^{\mathbf{H}_s,H_{d+1}}(\mathbf{y},s)\big] = R_{H_{d+1}}(s,t)\prod_{k=1}^{d} R_{H_k}(x_k,y_k),$$

where $\mathbf{H}_s = \{H_1,\cdots,H_d\}$, $(\mathbf{x},t),(\mathbf{y},s)\in D\times[0,T]$, $\mathbf{x}=\{x_1,\cdots,x_d\}$, $\mathbf{y}=\{y_1,\cdots,y_d\}$ and

$$R_H(x,y) = \frac{|x|^{2H}+|y|^{2H}-|x-y|^{2H}}{2}. \tag{1.4}$$

Here $D=(0,l)^d\subset\mathbb{R}^d$ $(d=1,2)$ with l being a bounded constant, $H_1,\cdots,H_{d+1}\in(0,\frac{1}{2}]$ are Hurst parameters, and $\mathbb{E}$ denotes the expectation.

As we all know, fractional noises exist widely in the natural world, such as flows in porous media, the rough Hamiltonian systems and so on [6, 11]. In recent years, there have been many discussions about numerically solving stochastic partial differential equations driven by Brownian sheet (it can be called as "smoother noise") [4,19–22,27,30,32]. But for the fractional Brownian sheet with $H_i\in(0,\frac{1}{2}]$ and $\sum_{i=1}^{d+1}H_i<\frac{d+1}{2}$ (called as "rough noise"), the existing discussions seem to be few. In [9], the authors propose the regularity estimates and the corresponding numerical analyses about the stochastic evolution equation driven by fractional Brownian sheet with $H_1\in(0,\frac{1}{2})$ and $H_2=\frac{1}{2}$.

In this paper, we focus on the fractional diffusion equation driven by fractional Brownian sheet with Hurst parameters $H_1,\cdots,H_{d+1}\in(0,\frac{1}{2}]$ numerically, which are both rough in the temporal and spatial directions. Firstly, with the help of the obtained new estimate about stochastic integral with respect to $\xi^{\mathbf{H}_s,H_{d+1}}$ (for the details, see Thms. 2.5 and 5.1), we provide the regularity estimate of the solution, *i.e.*,

$$\mathbb{E}\|u(t)\|^2_{H^{2\sigma}(D)}\le C,\quad 2\sigma\in\left[0,\min\left\{\frac{2H_{d+1}}{\alpha}+\sum_{k=1}^{d}H_k-d,\sum_{k=1}^{d}H_k+2-d\right\}\right),$$

and

$$\mathbb{E}\left\|\frac{u(t)-u(t-\tau)}{\tau^\gamma}\right\|^2_{L^2(D)}\le C,\qquad 2\gamma\in\left[0,2H_{d+1}+\left(\sum_{k=1}^{d}H_k-d\right)\alpha\right).$$

Then the Wong–Zakai approximation [16,17] is used to regularize the fractional Brownian sheet noise $\xi^{\mathbf{H}_s,H_{d+1}}$. To our best knowledge, the existing convergence discussions for regularized solution rely on the Green function; see [9, 10, 25]. But for equation (1.1), its Green function is composed of Mittag–Leffler function, which makes the corresponding convergence analysis complicated. So, in this paper, a new approach based on approximation theory, operator theory, and the equivalence of different Sobolev spaces is built and the convergence rate of Wong–Zakai approximation is obtained. Next, we use the finite element method and backward Euler convolution quadrature method to build the fully discrete scheme of equation (1.1) and introduce some new techniques to obtain an $\mathcal{O}(\tau^{2H_{d+1}+(\sum_{k=1}^{d}H_k-d)\alpha-\epsilon})$ convergence rates in time.

The rest of this paper is organized as follows. In Sections 2–4, we mainly discuss equation (1.1) in one-dimensional case. To be specific, in Section 2, we first provide some properties about stochastic integral with respect to ξ^{H_1,H_2}, and then discuss the regularity of the solution; next, we consider the Wong–Zakai approximation of equation (1.1) and discuss its convergence in Section 3; in Section 4, we construct the numerical scheme for one-dimensional problem by finite element method and backward Euler convolution quadrature method, and provide the complete error estimates. In Section 5, we extend the corresponding analyses into two-dimensional case and show the corresponding regularity estimate and error estimate. In Section 6, a variety of numerical experiments are provided to verify the proposed theoretical results. At last, we conclude the paper with some discussions. Throughout the paper, C denotes a positive constant, whose value may vary from line to line, $\|\cdot\|$ denotes the operator norms from $L^2(D)$ to $L^2(D)$, $\epsilon,\epsilon_0>0$ are arbitrarily small quantities, and $\mathbb{E}$ denotes the expectation.

2. Regularity of the solution in one-dimensional case

In this section, we begin by discussing the properties of stochastic integrals with respect to fractional Brownian sheet noise in equation (1.1) with $d=1$. Also, the regularity of the solution is provided.

2.1. Some properties of stochastic integrals with respect to fractional Brownian sheet noise ξ^{H_1,H_2}

Lemma 2.1 ([23,24]). *Let D be a bounded domain in $\mathbb{R}^d$ $(d=1,2,3)$ and Λ_k the k-th eigenvalue of the Dirichlet boundary problem for the Laplace operator $-\Delta$ in D. Then, for all $k\geq 1$,*

$$\Lambda_k \geq \frac{C_d d}{d+2} k^{2/d} |D|^{-2/d},$$

where $C_d=(2\pi)^2 B_d^{-2/d}$, $|D|$ is the volume of D, and B_d means the volume of the unit d-dimensional ball.

Let $A=-\Delta$ be defined in $(0,l)$ with a zero Dirichlet boundary condition and $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\phi_k\}_{k=1}^{\infty}$ be its non-decreasing eigenvalues and L^2-norm normalized eigenfunctions, respectively. Then we present some fractional Sobolev spaces, which can refer to [1,2,7,13]. Introduce the operator A^q with $q\in[0,1]$ as

$$A^q u = \sum_{k=1}^{\infty} \lambda_k^q (u,\phi_k)\phi_k$$

and define $\hat{H}^{2q}(D)=\mathbb{D}(A^q)$ with norm $\|u\|_{\hat{H}^{2q}(D)}=\|A^q u\|_{L^2(D)}$. Here $\mathbb{D}(A^q)$ denotes the domain of A^q. It is easy to verify

$$\hat{H}^0(D)=L^2(D), \quad \hat{H}^1(D)=H_0^1(D).$$

For $s\in(0,1)$, we define the fractional Sobolev space $H^s(D)$ by

$$H^s(D)=\left\{u\in L^2(D): |u|_{H^s(D)}^2=\int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y<\infty\right\},$$

and its norm can be written as $\|\cdot\|_{H^s(D)}=\|\cdot\|_{L^2(D)}+|\cdot|_{H^s(D)}$.

Remark 2.2. According to [15], for $s\in(0,1)$ and $d=1$, the semi-norm of $H^s(D)$ can also be defined by

$$|u|_{H^s(D)}=\|{}_0\partial_x^s u\|_{L^2(D)},$$

and combining the fractional Poincáre inequality [13,15], we can also define the norm of $H^s(D)$ by

$$\|u\|_{H^s(D)}=\|{}_0\partial_x^s u\|_{L^2(D)},$$

where ${}_0\partial_x^s u$ is the Riemann–Liouville fractional derivative.

Moreover, for $s \in (0,1)$, another type of fractional Sobolev space [1,2,7] that we will use can be defined by

$$H_0^s(D) = \{u \in H^s(\mathbb{R}), \quad u = 0 \text{ in } D^c\}$$

with the norm

$$\begin{aligned}\|u\|^2_{H_0^s(D)} &= \|u\|^2_{L^2(D)} + |u|^2_{H_0^s(D)} \\ &= \|u\|^2_{L^2(D)} + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Remark 2.3. It is well-known that $H^s(D) = H_0^s(D)$ for $s \in [0, \frac{1}{2})$; see [2,7]. From [7], we have $\hat{H}^s(D) = H_0^s(D)$ for $s \in [0, \frac{3}{2})$.

For fractional Brownian sheet noise, we have the following Itô isometry.

Lemma 2.4 ([5]). *Let $g_1(x,t) = g_{1,1}(x)g_{1,2}(t)$ and $g_2(x,t) = g_{2,1}(x)g_{2,2}(t)$ satisfying $g_{1,1}(x), g_{2,1}(x) \in H_0^{\frac{1-2H_1}{2}}(D)$ and $g_{1,2}(t), g_{2,2}(t) \in H_0^{\frac{1-2H_2}{2}}((0,T))$. Then we have*

$$\begin{aligned}&\mathbb{E}\left(\int_0^T\int_D g_1(x,t)\xi^{H_1,H_2}(\mathrm{d}x,\mathrm{d}t)\int_0^T\int_D g_2(x,t)\xi^{H_1,H_2}(\mathrm{d}x,\mathrm{d}t)\right) \\ &\qquad = (\mathcal{L}_{H_2,t}g_{1,2}(t), g_{2,2}(t))_{\mathbb{R}}(\mathcal{L}_{H_1,x}g_{1,1}(x), g_{2,1}(x))_{\mathbb{R}},\end{aligned}$$

where

$$\mathcal{L}_{H_1,x}u(x) = \begin{cases} 2C_{H_1}\mathrm{P.V.}\int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{2-2H_1}}\,\mathrm{d}y & H_1 \in \left(0,\frac{1}{2}\right), \\ u(x) & H_1 = \frac{1}{2}, \end{cases}$$

and

$$\mathcal{L}_{H_2,t}u(t) = \begin{cases} 2C_{H_2}\mathrm{P.V.}\int_{\mathbb{R}} \frac{u(t)-u(r)}{|t-r|^{2-2H_2}}\,\mathrm{d}r & H_2 \in \left(0,\frac{1}{2}\right), \\ u(t) & H_2 = \frac{1}{2}. \end{cases}$$

Here $C_{H_i} = \frac{1}{2}H_i(1-2H_i)$, $i = 1,2$ and P.V. means the principal value integral.

Moreover, we can obtain following estimate about the above Itô isometry by the equivalence of different fractional Sobolev spaces.

Theorem 2.5. *Let $g_1(x,t) = g_{1,1}(x)g_{1,2}(t)$ and $g_2(x,t) = g_{2,1}(x)g_{2,2}(t)$ satisfying $g_{1,1}(x), g_{2,1}(x) \in H_0^{\frac{1-2H_1}{2}}(D)$ and $g_{1,2}(t), g_{2,2}(t) \in H_0^{\frac{1-2H_2}{2}}((0,T))$. Then one has*

$$\begin{aligned}&\mathbb{E}\left(\int_0^T\int_D g_1(x,t)\xi^{H_1,H_2}(\mathrm{d}x,\mathrm{d}t)\int_0^T\int_D g_2(x,t)\xi^{H_1,H_2}(\mathrm{d}x,\mathrm{d}t)\right) \\ &\quad \le C\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}g_{1,2}(t)\right\|_{L^2((0,T))}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}g_{2,2}(t)\right\|_{L^2((0,T))} \\ &\quad\quad \cdot \|g_{1,1}(x)\|_{H_0^{\frac{1-2H_1}{2}}(D)}\|g_{2,1}(x)\|_{H_0^{\frac{1-2H_1}{2}}(D)}.\end{aligned}$$

Here ${}_0\partial_t^\alpha$ is the Riemann–Liouville fractional derivative when $\alpha \in (0,1)$; and when $\alpha = 0$, it denotes an identity operator.

Proof. Here, we mainly prove the case that $H_1, H_2 \in (0, \frac{1}{2})$. As for the other cases, the desired results can be obtained similarly.

According to Lemma 2.4 and the definition of $H_0^s(D)$, after simple calculations, we have

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T\int_D g_1(x,t)\xi^{H_1,H_2}(\mathrm{d}x,\mathrm{d}t)\int_0^T\int_D g_2(x,t)\xi^{H_1,H_2}(\mathrm{d}x,\mathrm{d}t)\right)\\
&\quad= C_{H_1}C_{H_2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(g_{1,1}(x)-g_{1,1}(y))(g_{2,1}(x)-g_{2,1}(y))}{|x-y|^{2-2H_1}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\quad\cdot\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(g_{1,2}(t)-g_{1,2}(s))(g_{2,2}(t)-g_{2,2}(s))}{|t-s|^{2-2H_2}}\,\mathrm{d}t\,\mathrm{d}s\\
&\quad\leq C\|g_{1,2}(t)\|_{H_0^{\frac{1-2H_2}{2}}((0,T))}\|g_{2,2}(t)\|_{H_0^{\frac{1-2H_2}{2}}((0,T))}\\
&\quad\quad\cdot\|g_{1,1}(x)\|_{H_0^{\frac{1-2H_1}{2}}(D)}\|g_{2,1}(x)\|_{H_0^{\frac{1-2H_1}{2}}(D)}.
\end{aligned}$$

Combining further Remarks 2.2 and 2.3, one can arrive at the desired result. □

Remark 2.6. According to Lemma 2.4 and Theorem 2.5, we can convert the singular integral with respect to time in Itô isometry into the convolution form, which plays a crucial role in the following analyses.

2.2. Regularity of the solution

Before building the regularity of the solution of equation (1.1), we first give the representation of the solution. Introduce $\mathcal{G}(t,x,y)$ as

$$\mathcal{G}(t,x,y)=\sum_{k=1}^{\infty}\mathcal{G}_k(t,x,y), \tag{2.1}$$

where

$$\mathcal{G}_k(t,x,y)=E_k(t)\phi_k(x)\phi_k(y) \tag{2.2}$$

and

$$E_k(t)=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}}e^{zt}z^{\alpha-1}(z^\alpha+\lambda_k)^{-1}\,\mathrm{d}z. \tag{2.3}$$

Here $\Gamma_{\theta,\kappa}$ is defined by

$$\Gamma_{\theta,\kappa}=\{re^{-\mathbf{i}\theta}:r\geq\kappa\}\cup\{\kappa e^{\mathbf{i}\psi}:|\psi|\leq\theta\}\cup\{re^{\mathbf{i}\theta}:r\geq\kappa\},$$

where the circular arc is oriented counterclockwise and the two rays are oriented with an increasing imaginary part and $\mathbf{i}^2=-1$.

Thus the solution of equation (1.1) can be written as

$$u(x,t)=\int_0^t\int_D\mathcal{G}(t-s,x,y)f(u)\,\mathrm{d}s\,\mathrm{d}y+\int_0^t\int_D\mathcal{G}(t-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s). \tag{2.4}$$

For the convenience of analysis, we introduce the operator $\mathcal{R}$, which is defined by the Laplace transform, *i.e.*,

$$\tilde{\mathcal{R}}(z)=z^{\alpha-1}(z^\alpha+A)^{-1}.$$

It is easy to verify that

$$\mathcal{G}_k(t,x,y)=\mathcal{R}(t)\phi_k(x)\phi_k(y)$$

and

$$\mathcal{R}(t)u(x)=\int_D\mathcal{G}(t,x,y)u(y)\,\mathrm{d}y.$$

So the solution can also be written as

$$u(t) = \int_0^t \mathcal{R}(t-s)f(u(s))\,\mathrm{d}s + \int_0^t \int_D \mathcal{G}(t-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s). \tag{2.5}$$

Then we provide the spatial regularity estimate of solution.

Theorem 2.7. *Let $u(t)$ be the solution of equation (1.1) with $d=1$ and $f(u)$ satisfy the assumptions (1.2). Assume $2H_2 + (H_1-1)\alpha > 0$. Then there holds*

$$\mathbb{E}\|A^\sigma u(t)\|^2_{L^2(D)} \le C,$$

where $2\sigma \in [0, \min\{\frac{2H_2}{\alpha} + H_1 - 1, H_1 + 1\})$.

Proof. According to (2.5), we have

$$\begin{aligned}
\mathbb{E}\|A^\sigma u(t)\|^2_{L^2(D)} &\le C\mathbb{E}\left\|\int_0^t A^\sigma \mathcal{R}(t-s)f(u)\,\mathrm{d}s\right\|^2_{L^2(D)} \\
&\quad + C\mathbb{E}\left\|\int_0^t \int_D A^\sigma \mathcal{G}(t-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)} \\
&\le \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Consider I first. Using the resolvent estimate $\|(z^\alpha + A)^{-1}\| \le C|z|^{-\alpha}$ for $z \in \Sigma_\theta = \{z \in \mathbb{C} : z \ne 0, |\arg z| \le \theta\}$ [26], the Cauchy–Schwarz inequality, and the assumptions (1.2), we can obtain

$$\begin{aligned}
\mathrm{I} &\le C\mathbb{E}\left\|\int_{\Gamma_{\theta,\kappa}} e^{zt} A^\sigma z^{\alpha-1}(z^\alpha + A)^{-1}\tilde{f}(u)\,\mathrm{d}z\right\|^2_{L^2(D)} \\
&\le C\left(\int_0^t (t-s)^{-\sigma\alpha}\mathbb{E}\|f(u(s))\|_{L^2(D)}\,\mathrm{d}s\right)^2 \\
&\le C\int_0^t (t-s)^{-2\sigma\alpha+1-\epsilon}\mathbb{E}\|f(u(s))\|^2_{L^2(D)}\,\mathrm{d}s \\
&\le C\left(1 + \int_0^t (t-s)^{-2\sigma\alpha+1-\epsilon}\mathbb{E}\|u\|^2_{L^2(D)}\,\mathrm{d}s\right),
\end{aligned}$$

where $\tilde{f}(u)$ means the Laplace transform of $f(u)$ and $-2\sigma\alpha+1 > -1$ needs to be satisfied, *i.e.*, $\sigma < \min\{\frac{1}{\alpha}, 1+\epsilon\}$.

According to Theorem 2.5 and the definition of $\mathcal{G}$, one has

$$\begin{aligned}
\mathrm{II} &\le C\sum_{k=1}^\infty \int_0^t \left|{}_0\partial_t^{\frac{1-2H_2}{2}} \lambda_k^\sigma E_k(t)\right|^2 \|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)} \|\phi_k(x)\|^2_{L^2(D)}\,\mathrm{d}t \\
&\le C\sum_{k=1}^\infty \int_0^t \left|\int_{\Gamma_{\theta,\kappa}} e^{zt} z^{\alpha-1+\frac{1-2H_2}{2}} \lambda_k^\sigma (z^\alpha + \lambda_k)^{-1}\,\mathrm{d}z\right|^2 \\
&\quad \cdot \|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)} \|\phi_k(x)\|^2_{L^2(D)}\,\mathrm{d}t.
\end{aligned}$$

By the resolvent estimate, Remark 2.3, Lemma 2.1, and simple calculations, we have

$$\begin{aligned}
\mathrm{II} &\le C\sum_{k=1}^{\infty}\int_0^t\left(\int_{\Gamma_{\theta,\kappa}}|e^{zt}||z|^{\alpha-1+\frac{1-2H_2}{2}}|\lambda_k^{\sigma+\frac{1-2H_1}{4}}(z^\alpha+\lambda_k)^{-1}||\,\mathrm{d}z|\right)^2\mathrm{d}t\\
&\le C\sum_{k=1}^{\infty}\lambda_k^{-\frac{1}{2}-2\epsilon}\int_0^t\left(\int_{\Gamma_{\theta,\kappa}}|e^{zt}||z|^{\alpha-1+\frac{1-2H_2}{2}}|\lambda_k^{\sigma+\frac{1-H_1}{2}+\epsilon}(z^\alpha+\lambda_k)^{-1}||\,\mathrm{d}z|\right)^2\mathrm{d}t\\
&\le C\sum_{k=1}^{\infty}\lambda_k^{-\frac{1}{2}-2\epsilon}\int_0^t\left(\int_{\Gamma_{\theta,\kappa}}|e^{zt}||z|^{(\sigma+\frac{1-H_1}{2}+\epsilon)\alpha-1+\frac{1-2H_2}{2}}|\,\mathrm{d}z|\right)^2\mathrm{d}t,
\end{aligned}$$

where we need to require $(\sigma+\frac{1-H_1}{2}+\epsilon)\alpha-1+\frac{1-2H_2}{2}<-\frac{1}{2}$ and $\sigma+\frac{1-H_1}{2}<1$, *i.e.*, $2\sigma<\min\{\frac{2H_2}{\alpha}+H_1-1,H_1+1\}$. Combining the Grönwall inequality [14], the desired results can be reached. □

Also, the following temporal regularity estimate of solution can be obtained.

Theorem 2.8. *Let $u(t)$ be the solution of equation (1.1) with $d=1$ and $f(u)$ satisfy the assumptions (1.2). Assume $2H_2+(H_1-1)\alpha>0$. Then it holds*

$$\mathbb{E}\left\|\frac{u(t)-u(t-\tau)}{\tau^\gamma}\right\|_{L^2(D)}^2\le C, \tag{2.6}$$

where $2\gamma\in[0,2H_2+(H_1-1)\alpha)$.

Proof. According to (2.5), we have

$$\begin{aligned}
\mathbb{E}\left\|\frac{u(t)-u(t-\tau)}{\tau^\gamma}\right\|_{L^2(D)}^2 &\le C\mathbb{E}\left\|\frac{1}{\tau^\gamma}\left(\int_0^t\mathcal{R}(t-s)f(u)\,\mathrm{d}s-\int_0^{t-\tau}\mathcal{R}(t-\tau-s)f(u)\,\mathrm{d}s\right)\right\|_{L^2(D)}^2\\
&\quad+C\mathbb{E}\left\|\frac{1}{\tau^\gamma}\left(\int_0^t\int_D\mathcal{G}(t-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right.\right.\\
&\quad\left.\left.-\int_0^{t-\tau}\int_D\mathcal{G}(t-\tau-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right)\right\|_{L^2(D)}^2\\
&\le\vartheta_1+\vartheta_2.
\end{aligned}$$

As for ϑ_1, there holds

$$\begin{aligned}
\vartheta_1 &\le C\mathbb{E}\left\|\frac{1}{\tau^\gamma}\int_0^{t-\tau}(\mathcal{R}(t-s)-\mathcal{R}(t-\tau-s))f(u)\,\mathrm{d}s\right\|_{L^2(D)}^2\\
&\quad+C\mathbb{E}\left\|\frac{1}{\tau^\gamma}\int_{t-\tau}^t\mathcal{R}(t-s)f(u)\,\mathrm{d}s\right\|_{L^2(D)}^2\\
&\le\vartheta_{1,1}+\vartheta_{1,2}.
\end{aligned}$$

By the fact $\left|\frac{e^{z\tau}-1}{\tau^\gamma}\right|<C|z|^\gamma$ with $z\in\Gamma_{\theta,\kappa}$ and $\gamma\in[0,1]$ [18] and Theorem 2.7, we have

$$\begin{aligned}
\vartheta_{1,1} &\le C\left(\int_0^{t-\tau}\left\|\int_{\Gamma_{\theta,\kappa}}e^{z(t-s-\tau)}\frac{e^{z\tau}-1}{\tau^\gamma}z^{\alpha-1}(z^\alpha+A)^{-1}\,\mathrm{d}z\right\|\mathbb{E}\|f(u)\|_{L^2(D)}\,\mathrm{d}s\right)^2\\
&\le C\left(\int_0^{t-\tau}(t-\tau-s)^{-\gamma}(1+\mathbb{E}\|u\|_{L^2(D)})\,\mathrm{d}s\right)^2\\
&\le C,
\end{aligned}$$

where $\gamma \in [0,1)$. Similarly, for $\gamma \in [0,1)$, one can obtain

$$\vartheta_{1,2} \le C\tau^{1-2\gamma}\int_{t-\tau}^{t} \mathbb{E}\|f(u)\|_{L^2(D)}^2 \,\mathrm{d}s \le C.$$

For ϑ_2, we can split it into the following two parts:

$$\vartheta_2 \le C\mathbb{E}\left\|\frac{1}{\tau^\gamma}\left(\int_0^{t-\tau}\int_D (\mathcal{G}(t-s,x,y)-\mathcal{G}(t-\tau-s,x,y))\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right)\right\|_{L^2(D)}^2 + C\mathbb{E}\left\|\frac{1}{\tau^\gamma}\int_{t-\tau}^{t}\int_D \mathcal{G}(t-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|_{L^2(D)}^2 \le \vartheta_{2,1}+\vartheta_{2,2}.$$

Using Theorem 2.5 and Lemma 2.1 yields

$$\begin{aligned}
\vartheta_{2,1} &\le C\sum_{k=1}^{\infty}\int_0^{t-\tau}\frac{1}{\tau^{2\gamma}}\left|{}_0\partial_t^{\frac{1-2H_2}{2}}(E_k(t-s)-E_k(t-\tau-s))\right|^2 \|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\,\mathrm{d}s\\
&\le C\sum_{k=1}^{\infty}\int_0^{t-\tau}\left|\int_{\Gamma_{\theta,\kappa}} e^{z(t-\tau-s)}\frac{e^{z\tau}-1}{\tau^\gamma} z^{\frac{1-2H_2}{2}}\lambda_k^{\frac{1-2H_1}{4}} z^{\alpha-1}(z^\alpha+\lambda_k)^{-1}\,\mathrm{d}z\right|^2\mathrm{d}s\\
&\le C\sum_{k=1}^{\infty}\lambda_k^{-1/2-\epsilon}\int_0^{t-\tau}\left(\int_{\Gamma_{\theta,\kappa}}|e^{z(t-\tau-s)}||z|^{\gamma+\frac{1-2H_2}{2}+\frac{1-H_1}{2}\alpha-1+\frac{\alpha\epsilon}{2}}|\,\mathrm{d}z|\right)^2\mathrm{d}s\\
&\le C\sum_{k=1}^{\infty}\lambda_k^{-1/2-\epsilon}\int_0^{t-\tau} s^{(H_1-1)\alpha+2H_2-1-2\gamma-\alpha\epsilon}\,\mathrm{d}s,
\end{aligned}$$

where we need to require $2\gamma < 2H_2+(H_1-1)\alpha$. Similarly, for $\vartheta_{2,2}$, one has

$$\begin{aligned}
\vartheta_{2,2} &\le C\frac{1}{\tau^{2\gamma}}\sum_{k=1}^{\infty}\int_{t-\tau}^{t}\left({}_0\partial_t^{\frac{1-2H_2}{2}}\lambda_k^{\frac{1-2H_1}{4}}E_k(t-s)\right)^2\mathrm{d}s\\
&\le C\frac{1}{\tau^{2\gamma}}\sum_{k=1}^{\infty}\lambda_k^{-1/2-\epsilon}\int_{t-\tau}^{t}\left(\int_{\Gamma_{\theta,\kappa}}|e^{z(t-s)}||z|^{\frac{1-2H_2}{2}+\frac{1-H_1}{2}\alpha-1+\frac{\alpha\epsilon}{2}}|\,\mathrm{d}z|\right)^2\mathrm{d}s\\
&\le C\frac{1}{\tau^{2\gamma}}\sum_{k=1}^{\infty}\lambda_k^{-1/2-\epsilon}\int_0^{\tau} s^{(H_1-1)\alpha+2H_2-1-\alpha\epsilon}\,\mathrm{d}s,
\end{aligned}$$

where we need to require $2\gamma < 2H_2+(H_1-1)\alpha$. Collecting the above estimates leads to the desired result. □

3. Wong–Zakai approximation

In this section, we use the Wong–Zakai approximation [16, 17] to regularize the fractional Brownian sheet noise ξ^{H_1,H_2} and provide a systematic approach to prove the convergence of the Wong–Zakai approximation.

Here, we introduce the Wong–Zakai approximation first. Let $\tau = T/M$ and $h = l/N$. Denote $I_i = (t_i, t_{i+1}]$ and $D_j = (x_j, x_{j+1}]$ with $t_i = i\tau$ $(i = 0, 1, \ldots, M)$ and $x_j = jh$ $(j = 0, 1, \ldots, N)$. Then Wong–Zakai approximation of $\xi^{H_1,H_2}(x,t)$ can be written as

$$\xi_R^{H_1,H_2}(x,t) = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\left(\frac{1}{\tau h}\int_{I_i}\int_{D_j}\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right)\chi_{I_i\times D_j}(t,x), \tag{3.1}$$

where $\chi_{I_i\times D_j}(t,x)$ is the characteristic function on $I_i \times D_j$. Then we introduce $u_R(x,t)$ as the solution of the following regularized equation, *i.e.*,

$$\begin{cases} \partial_t u_R(x,t) + {}_0\partial_t^{1-\alpha} A u_R(x,t) = f(u_R) + \xi_R^{H_1,H_2}(x,t) & (x,t)\in D\times(0,T],\\ u_R(x,0)=0 & x\in D,\\ u_R(x,t)=0 & (x,t)\in\partial D\times(0,T]. \end{cases} \tag{3.2}$$

Simple calculations lead to

$$u_R(x,t) = \int_0^t \mathcal{R}(t-s) f(u_R)\,\mathrm{d}s + \int_0^t\int_D \mathcal{G}_R(t,s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s). \tag{3.3}$$

Here, $\mathcal{G}_R(t,r,x,y)$ is defined by

$$\mathcal{G}_R(t,r,x,y) = \sum_{k=1}^{\infty} E_{R,k}(t,r)\phi_k(x)\phi_{R,k}(y),$$

where

$$E_{R,k}(t,r) = \frac{1}{\tau}\sum_{i=0}^{M-1}\chi_{I_i}(r)\int_{I_i} E_k(t-\bar{r})\chi_{(0,t)}(t-\bar{r})\,\mathrm{d}\bar{r}, \qquad \phi_{R,k}(y) = \frac{1}{h}\sum_{j=0}^{N-1}\chi_{I_j}(y)\int_{D_j}\phi_k(y)\,\mathrm{d}y.$$

Before showing the regularity and convergence of the regularized solution, we first provide the property of $E_k(t)$.

Lemma 3.1. *Let $E_k(t)$ be defined in (2.3). For $k>0$, $\sigma\in[0,1]$, and $\gamma\ge 0$, if $\gamma+\sigma\alpha<\frac{1}{2}$, then there exists a uniform constant C such that*

$$\|\lambda_k^\sigma E_k(t)\|_{H^\gamma((0,T))} = \|{}_0\partial_t^\gamma \lambda_k^\sigma E_k(t)\|_{L^2((0,T))} \le C. \tag{3.4}$$

Proof. By the definition of $E_k(t)$ and the resolvent estimate [26], we have

$$\begin{aligned} \|{}_0\partial_t^\gamma \lambda_k^\sigma E_k(t)\|^2_{L^2((0,T))} &\le C\int_0^T \left(\int_{\Gamma_{\theta,\kappa}} |e^{zt}|\lambda_k^\sigma |z|^{\gamma+\alpha-1}|(z^\alpha+\lambda_k)|^{-1}|\,\mathrm{d}z|\right)^2 \mathrm{d}t\\ &\le C\int_0^T \left(\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{\gamma+\sigma\alpha-1}|\mathrm{d}z|\right)^2 \mathrm{d}t\\ &\le C\int_0^T t^{-2\gamma-2\sigma\alpha}\,\mathrm{d}t, \end{aligned}$$

where C is a positive constant independent of k. To preserve the boundedness of $\|{}_0\partial_t^\gamma\lambda_k^\sigma E_k(t)\|^2_{L^2((0,T))}$, we need to require $\gamma<\frac{1}{2}-\sigma\alpha$. □

In the following, we show regularity estimate of the regularized solution.

Theorem 3.2. *Let $u_R(t)$ be the solution of equation (3.2) and $f(u)$ satisfy the assumptions (1.2). Assume $2H_2+(H_1-1)\alpha>0$. Then we have*

$$\mathbb{E}\|A^\sigma u_R(t)\|^2_{L^2(D)} \le C,$$

where $2\sigma\in[0,\min\{\frac{2H_2}{\alpha}+H_1-1, H_1+1\})$.

Proof. Here we split $\mathbb{E}\|A^{\sigma}u_R(t)\|^2_{L^2(D)}$ into two parts, *i.e.*,

$$\begin{aligned}\mathbb{E}\|A^{\sigma}u_R(t)\|^2_{L^2(D)} &\leq C\mathbb{E}\left\|\int_0^t A^{\sigma}\mathcal{R}(t-s)f(u_R)\,\mathrm{d}s\right\|^2_{L^2(D)}\\ &\quad + C\mathbb{E}\left\|\int_0^t\int_D A^{\sigma}\mathcal{G}_R(t,s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)}\\ &\leq \mathrm{I}+\mathrm{II}.\end{aligned}$$

Similar to the proof of Theorem 2.7, one has

$$\mathrm{I}\leq C\left(1+\int_0^t (t-s)^{-2\sigma\alpha+1-\epsilon}\mathbb{E}\|u_R\|^2_{L^2(D)}\,\mathrm{d}s\right). \tag{3.5}$$

Using the standard approximation theory (one can refer to Thm. 4.4.4 in [8]) and $H_1,\ H_2\in(0,\frac{1}{2}]$, we have

$$\begin{aligned}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_{R,k}(t,r)\right\|^2_{L^2((0,t))} &\leq C\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}(E_{R,k}(t,r)-E_k(t-r)\chi_{(0,t)}(r))\right\|^2_{L^2((0,t))}\\ &\quad + C\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_k(t-r)\chi_{(0,t)}(r)\right\|^2_{L^2((0,t))}\\ &\leq C\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_k(t)\right\|^2_{L^2((0,T))}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|\phi_{R,k}(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)} &\leq C\|\phi_{R,k}(y)-\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}+C\|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\\ &\leq C\|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}.\end{aligned} \tag{3.7}$$

Thus, combining Theorem 2.5, one has

$$\mathrm{II}\leq C\sum_{k=1}^{\infty}\lambda_k^{2\sigma}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_k(t)\right\|^2_{L^2((0,T))}\|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}.$$

Similar to the proof of Theorem 2.7, we can get

$$\mathrm{II}\leq C$$

with $2\sigma\in[0,\min\{\frac{2H_2}{\alpha}+H_1-1,H_1+1\})$. Combining above estimates and using the Grönwall inequality [14,28] lead to the desired result. ☐

At last, we present the convergence of the regularized solution.

Theorem 3.3. *Let $u(t)$ and $u_R(t)$ be the solutions of equations (1.1) and (3.2), respectively. Assume $2H_2+(H_1-1)\alpha>0$. Then there exists a constant C such that*

$$\mathbb{E}\|u(t)-u_R(t)\|^2_{L^2(D)}\leq C\big(h^{2\sigma+2H_1-1}+\tau^{2H_2-\frac{\alpha}{2}-\epsilon}h^{2H_1-1}\big),$$

where $\sigma\in(\frac{1-2H_1}{2},\min\{\frac{2H_2}{\alpha}-\frac{1}{2},1+\epsilon\})$.

Proof. According to (2.5) and (3.3), one can get

$$\begin{aligned}\mathbb{E}\|u(t)-u_R(t)\|^2_{L^2(D)} \leq& C\mathbb{E}\left\|\int_0^t \mathcal{R}(t-s)(f(u)-f(u_R))\,\mathrm{d}s\right\|^2_{L^2(D)}\\ &+C\mathbb{E}\left\|\int_0^t\int_D(\mathcal{G}(t-s,x,y)-\mathcal{G}_R(t,s,x,y))\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)}\\ \leq& \mathrm{I}+\mathrm{II}.\end{aligned}$$

Assumptions (1.2) and the resolvent estimate lead to

$$\begin{aligned}\mathrm{I}\leq& C\mathbb{E}\left\|\int_{\Gamma_{\theta,\kappa}} e^{zt}z^{\alpha-1}(z^\alpha+A)^{-1}\left(\tilde{f}(u)-\tilde{f}(u_R)\right)\mathrm{d}z\right\|^2_{L^2(D)}\\ \leq& C\mathbb{E}\left(\int_0^t\int_{\Gamma_{\theta,\kappa}}|e^{z(t-s)}|\|z^{\alpha-1}(z^\alpha+A)^{-1}\||\,\mathrm{d}z|\|f(u(s))-f(u_R(s))\|_{L^2(D)}\,\mathrm{d}s\right)^2\\ \leq& C\int_0^t\mathbb{E}\|f(u)-f(u_R)\|^2_{L^2(D)}\,\mathrm{d}s\\ \leq& C\int_0^t\mathbb{E}\|u-u_R\|^2_{L^2(D)}\,\mathrm{d}s.\end{aligned}$$

As for II, using the following fact

$$\begin{aligned}E_k(t-r)\phi_k(y)-E_{R,k}(t,r)\phi_{R,k}(y)=&(E_k(t-r)\phi_k(y)-E_k(t-r)\phi_{R,k}(y))\\ &+(E_k(t-r)\phi_{R,k}(y)-E_{R,k}(t,r)\phi_{R,k}(y))\end{aligned}$$

and Theorem 2.5, one can get

$$\begin{aligned}\mathrm{II}\leq& C\sum_{k=1}^\infty\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_k(t)\right\|^2_{L^2((0,T))}\|\phi_k(y)-\phi_{R,k}(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\\ &+C\sum_{k=1}^\infty\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}(E_k(t-r)-E_{R,k}(t,r))\right\|^2_{L^2((0,t))}\|\phi_{R,k}(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\\ \leq& \mathrm{II}_1+\mathrm{II}_2.\end{aligned}$$

Combining Lemma 3.1 and standard approximation theory [8], one has

$$\begin{aligned}\mathrm{II}_1\leq& C\sum_{k=1}^\infty\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_k(t)\right\|^2_{L^2((0,T))}\|\phi_k(y)-\phi_{R,k}(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\\ \leq& C\sum_{k=1}^\infty\lambda_k^{-\frac{1}{2}-\epsilon}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\lambda_k^{\frac{1+2\epsilon}{4}}E_k(t)\right\|^2_{L^2((0,T))}\|\phi_k(y)-\phi_{R,k}(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\\ \leq& Ch^{2\sigma+2H_1-1}\sum_{k=1}^\infty\lambda_k^{-\frac{1}{2}-\epsilon}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\lambda_k^{\frac{1+2\epsilon+2\sigma}{4}}E_k(t)\right\|^2_{L^2((0,T))},\end{aligned}$$

where we need to require $\frac{1}{2}-\frac{1+2\epsilon+2\sigma}{4}\alpha>\frac{1-2H_2}{2}$ and $2\sigma+2H_1-1>0$ to preserve the boundedness of II_1, *i.e.*, $\sigma\in(\frac{1-2H_1}{2},\min\{\frac{2H_2}{\alpha}-\frac{1}{2},1+\epsilon\})$. Similarly, by the inverse estimate and projection theorem [8], we can also obtain

$$\begin{aligned}\mathrm{II}_2 &\le C\sum_{k=1}^{\infty}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}(E_k(t-r)-E_{R,k}(t,r))\right\|^2_{L^2((0,t))}\|\phi_{R,k}(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\\ &\le Ch^{2H_1-1}\sum_{k=1}^{\infty}\lambda_k^{-1/2-2\epsilon}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\lambda_k^{1/4+\epsilon}(E_k(t-r)-E_{R,k}(t,r))\right\|^2_{L^2((0,t))}\\ &\le C\tau^{2H_2-\frac{\alpha}{2}-\epsilon_0}h^{2H_1-1}.\end{aligned}$$

Thus the desired result can be achieved by the Grönwall inequality directly. □

Remark 3.4. When taking $H_2=\frac{1}{2}$ and $\alpha=1$ in Theorem 3.3, there holds

$$\mathbb{E}\|u(t)-u_R(t)\|^2_{L^2(D)}\le C\Big(h^{2H_1-\epsilon}+\tau^{\frac{1}{4}-\epsilon}h^{2H_1-1}\Big),$$

which can recover the result provided in [9].

4. Numerical scheme and error analysis

In this section, we construct the numerical scheme for the above regularized equation (3.2). Here we use the finite element and backward Euler convolution quadrature methods to discretize the spatial and temporal operators, respectively. At the same time, we provide the corresponding error analysis.

4.1. Numerical scheme

Let $\mathcal{T}_h$ be a shape regular quasi-uniform partition of the domain D. Introduce $X_h\subset H_0^1(D)$ as the space of continuous piecewise linear function over the $\mathcal{T}_h$. Define the L^2 orthogonal projection $P_h:\ L^2(D)\to X_h$:

$$(P_hu,v_h)=(u,v_h)\qquad\forall u\in L^2(D)\ \ \forall v_h\in X_h,$$

Denote A_h as $(A_hu_h,v_h)=(\nabla u_h,\nabla v_h)$ with $u_h,v_h\in X_h$. The semi-discrete Galerkin scheme for (3.2) can be written as: find $u_h\in X_h$ such that

$$(\partial_tu_h,v_h)+\big({}_0\partial_t^{1-\alpha}A_hu_h,v_h\big)=(f(u_h),v_h)+\Big(\xi_R^{H_1,H_2},v_h\Big)$$

with $u_h(0)=0$. It can also be written as

$$\partial_tu_h+{}_0\partial_t^{1-\alpha}A_hu_h=P_hf(u_h)+P_h\xi_R^{H_1,H_2}. \tag{4.1}$$

Then we use the backward Euler convolution quadrature method to discretize the temporal derivative operator, *i.e.*, the fully discrete scheme can be written as

$$\frac{u_h^n-u_h^{n-1}}{\tau}+\sum_{i=0}^{n-1}d_i^{(1-\alpha)}A_hu_h^{n-i}=P_hf\big(u_h^{n-1}\big)+P_h\xi_{R,n}^{H_1,H_2}, \tag{4.2}$$

where $\xi_{R,n}^{H_1,H_2}=\xi_R^{H_1,H_2}(t_n)$ and

$$\sum_{i=0}^{\infty}d_i^{(\alpha)}\zeta^i=(\delta_\tau(\zeta))^\alpha,\quad \delta_\tau(\zeta)=\frac{1-\zeta}{\tau}. \tag{4.3}$$

Remark 4.1. In fact, when discretizing equation (1.1) directly, the fully discrete scheme can be written as

$$\frac{u_h^n - u_h^{n-1}}{\tau} + \sum_{i=0}^{n-1} d_i^{(1-\alpha)} A_h u_h^{n-i} = P_h f(u_h^{n-1}) + P_h \frac{\xi^{H_1,H_2}(x,t_n) - \xi^{H_1,H_2}(x,t_{n-1})}{\tau}. \tag{4.4}$$

From (4.4), we can find that the spatial direction of fractional Brownian sheet noise is approximated by the piecewise linear function. But if we discretize the regularization equation (see (4.2)), the spatial direction of noise is approximated by the piecewise constant function, which is the key to obtain the exact representation of solution for the fully discrete scheme. Moreover, it plays an important role in the following error analyses.

Next, we provide the spatial and temporal error analyses, respectively.

4.2. Spatial error analysis

Now, we suppose that $\bar{u}_h \in X_h$ is the solution of the following equation:

$$(\partial_t \bar{u}_h, v_h) + ({}_0\partial_t^{1-\alpha} A_h \bar{u}_h, v_h) = (f(\bar{u}_h), v_h) + \left(\xi_{RS}^{H_1,H_2}, v_h\right) \tag{4.5}$$

with $\bar{u}_h(0) = 0$ and

$$\xi_{RS}^{H_1,H_2}(t) = \sum_{j=0}^{N-1} \left(\frac{1}{h}\int_{D_j} \xi^{H_1,H_2}(\mathrm{d}y,t)\right)\chi_{D_j}(x).$$

Then equation (4.5) can also be written as

$$\partial_t \bar{u}_h + {}_0\partial_t^{1-\alpha} A_h \bar{u}_h = P_h f(\bar{u}_h) + P_h \xi_{RS}^{H_1,H_2}.$$

Introduce the Laplace transform of the operator $\mathcal{R}_h$ as

$$\tilde{\mathcal{R}}_h(z) = z^{\alpha-1}(z^\alpha + A_h)^{-1},$$

which leads to

$$\bar{u}_h(t) = \int_0^t \mathcal{R}_h(t-s) P_h f(\bar{u}_h(s))\,\mathrm{d}s + \int_0^t \mathcal{R}_h(t-s) P_h \xi_{RS}^{H_1,H_2}(\mathrm{d}s). \tag{4.6}$$

Let

$$\mathcal{G}_{R,h}(t,x,y) = \sum_{k=1}^{\infty} \mathcal{G}_{R,h,k}(t,x,y) \tag{4.7}$$

with

$$\mathcal{G}_{R,h,k}(t,x,y) = \mathcal{R}_h(t) P_h \phi_k(x) \phi_{R,k}(y). \tag{4.8}$$

Thus by (4.6), the solution $\bar{u}_h$ can also be written as

$$\bar{u}_h(t) = \int_0^t \mathcal{R}_h(t-s) P_h f(\bar{u}_h(s))\,\mathrm{d}s + \int_0^t \int_D \mathcal{G}_{R,h}(t-s,x,y) \xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s). \tag{4.9}$$

Below we first provide the properties of $\bar{u}_h(t)$.

Theorem 4.2. *Let $\bar{u}_h(t)$ be the solution of equation (4.5) and $f(u)$ satisfy (1.2). Assume $2H_2 + (H_1 - 1)\alpha > 0$. Then we have*

$$\mathbb{E}\|\bar{u}_h(t)\|_{L^2(D)}^2 \le C$$

and

$$\mathbb{E}\left\|\frac{\bar{u}_h(t) - \bar{u}_h(t-\tau)}{\tau^\gamma}\right\|_{L^2(D)}^2 \le C, \tag{4.10}$$

where $\gamma \in (0, 2H_2 + (H_1 - 1)\alpha)$.

Proof. According to (4.9), we have

$$\begin{aligned}\mathbb{E}\|\bar{u}_h(t)\|^2_{L^2(D)} &\le C\mathbb{E}\left\|\int_0^t \mathcal{R}_h(t-s)P_h f(\bar{u}_h(s))\,\mathrm{d}s\right\|^2_{L^2(D)} \\ &\quad + C\mathbb{E}\left\|\int_0^t\int_D \mathcal{G}_{R,h}(t-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)} \\ &\le \mathrm{I}+\mathrm{II}.\end{aligned}$$

Similar to the derivations of Theorem 2.7, one has

$$\mathrm{I} \le C\left(1+\int_0^t \mathbb{E}\|\bar{u}_h(t)\|^2_{L^2(D)}\,\mathrm{d}s\right).$$

As for II, Theorem 2.7, Lemma 3.1, and equation (3.7) show that

$$\begin{aligned}\mathrm{II} &\le C\sum_{k=1}^{\infty}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}E_k(t)\right\|^2_{L^2((0,T))}\|\phi_k(y)\|^2_{H_0^{\frac{1-2H_1}{2}}(D)} \\ &\le C.\end{aligned}$$

Thus we can get the first desired result by using the Grönwall inequality. As for the second estimate, using the stability of P_h, *i.e.*, $\|P_h u\|_{L^2(D)} \le \|u\|_{L^2(D)}$ [31] and the similar arguments in Theorem 2.8, we can obtain the desired result.

□

Here we begin to show spatial convergence of numerical scheme (4.2).

Theorem 4.3. *Let $u(t)$ and $\bar{u}_h(t)$ be the solutions of equations* (1.1) *and* (4.5)*, respectively. Assume that $f(u)$ satisfies* (1.2) *and $2H_2+(H_1-1)\alpha>0$. Then we have*

$$\mathbb{E}\|u(t)-\bar{u}_h(t)\|^2_{L^2(D)} \le Ch^{2\sigma+2H_1-1},$$

where $\sigma\in(\frac{1-2H_1}{2},\min\{\frac{2H_2}{\alpha}-\frac{1}{2},1+\epsilon\})$.

Proof. According to (2.5) and (4.6), one obtains

$$\begin{aligned}\mathbb{E}\|u(t)-\bar{u}_h(t)\|^2_{L^2(D)} &\le C\mathbb{E}\left\|\int_0^t \mathcal{R}(t-s)f(u(s))\,\mathrm{d}s-\int_0^t \mathcal{R}_h(t-s)P_h f(\bar{u}_h(s))\,\mathrm{d}s\right\|^2_{L^2(D)} \\ &\quad + C\mathbb{E}\left\|\int_0^t\int_D(\mathcal{G}(t-s,x,y)-\mathcal{G}_{R,h}(t-s,x,y))\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)} \\ &\le \mathrm{I}+\mathrm{II}.\end{aligned}$$

As for I, one has

$$\begin{aligned}\mathrm{I} &\le \mathbb{E}\left\|\int_0^t \mathcal{R}(t-s)f(u(s))\,\mathrm{d}s-\int_0^t \mathcal{R}_h(t-s)P_h f(u(s))\,\mathrm{d}s\right\|^2_{L^2(D)} \\ &\quad + \mathbb{E}\left\|\int_0^t \mathcal{R}_h(t-s)P_h f(u(s))\,\mathrm{d}s-\int_0^t \mathcal{R}_h(t-s)P_h f(\bar{u}_h(s))\,\mathrm{d}s\right\|^2_{L^2(D)} \\ &\le \mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

Using the assumptions (1.2), the fact $\|(\mathcal{R}(t-s)-\mathcal{R}_h(t-s)P_h)\| \le Ch^2(t-s)^{-\alpha}$ with $\alpha \in (0,1)$, and Theorem 2.7, we have

$$\begin{aligned}
\mathrm{I}_1 &\le \left(\int_0^t \|(\mathcal{R}(t-s)-\mathcal{R}_h(t-s)P_h)\|\mathbb{E}\|f(u(s))\|_{L^2(D)}\,\mathrm{d}s\right)^2 \\
&\le Ch^4\left(\int_0^t (t-s)^{-\alpha}(1+\mathbb{E}\|u(s)\|_{L^2(D)})\,\mathrm{d}s\right)^2 \\
&\le Ch^4.
\end{aligned}$$

As for I_2, the stability of P_h, the assumptions (1.2), and the Cauchy–Schwarz inequality imply

$$\begin{aligned}
\mathrm{I}_2 &\le C\mathbb{E}\left(\int_0^t \|\mathcal{R}_h(t-s)\|\|(f(u(s))-f(\bar{u}_h(s)))\|_{L^2(D)}\,\mathrm{d}s\right)^2 \\
&\le C\int_0^t \mathbb{E}\|u(s)-\bar{u}_h(s)\|^2_{L^2(D)}\,\mathrm{d}s.
\end{aligned}$$

As for II, one has

$$\begin{aligned}
\mathrm{II} &\le C\mathbb{E}\left\|\int_0^t\int_D (\mathcal{G}(t-s,x,y)-\mathcal{G}_{Rs}(t-s,x,y))\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)} \\
&\quad + C\mathbb{E}\left\|\int_0^t\int_D (\mathcal{G}_{Rs}(t-s,x,y)-\mathcal{G}_{R,h}(t-s,x,y))\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\right\|^2_{L^2(D)} \\
&\le \mathrm{II}_1+\mathrm{II}_2,
\end{aligned}$$

where

$$\mathcal{G}_{Rs}(t,x,y)=\sum_{k=1}^{\infty}\mathcal{R}(t)\phi_k(x)\phi_{R,k}(y).$$

Similar to the derivations of II_1 in the proof of Theorem 3.3, we have

$$\mathrm{II}_1 \le Ch^{2\sigma+2H_1-1},$$

where $\sigma \in (\frac{1-2H_1}{2}, \min\{\frac{2H_2}{\alpha}-\frac{1}{2}, 1+\epsilon\})$. As for II_2, the inverse estimate leads to

$$\begin{aligned}
\mathrm{II}_2 &\le C\sum_{k=1}^{\infty}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}(\mathcal{R}(t)-\mathcal{R}_h(t)P_h)\phi_k(x)\right\|^2_{L^2(0,t,L^2(D))}\|\phi_{R,k}(y)\|^2_{H^{\frac{1-2H_1}{2}}(D)} \\
&\le Ch^{2H_1-1-\epsilon}\sum_{k=1}^{\infty}\lambda_k^{-\frac{1}{2}-\epsilon/2}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}(\mathcal{R}(t)-\mathcal{R}_h(t)P_h)A^{\frac{1}{4}}\phi_k(x)\right\|^2_{L^2(0,t,L^2(D))}.
\end{aligned}$$

Using the fact $\|((z^\alpha+\mathcal{A})^{-1}-(z^\alpha+\mathcal{A}_h)^{-1}P_h)A^s\| \le Ch^{2-2s}$ for $s\in[0,\frac{1}{2}]$ [31] and the interpolation property [3], one can obtain

$$\begin{aligned}
&\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}(\mathcal{R}(t)-\mathcal{R}_h(t)P_h)A^{\frac{1}{4}}\phi_k(x)\right\|^2_{L^2(0,t,L^2(D))} \\
&\le C\int_0^t\left\|\int_{\Gamma_{\theta,\kappa}} e^{zt}z^{\frac{1-2H_2}{2}}z^{\alpha-1}\big((z^\alpha+A)^{-1}-(z^\alpha+A_h)^{-1}P_h\big)A^{\frac{1}{4}}\phi_k(x)\,\mathrm{d}z\right\|^2_{L^2(D)}\mathrm{d}t \\
&\le Ch^{\min\{\frac{4H_2}{\alpha}-1-2\epsilon,3\}}\int_0^t\left(\int_{\Gamma_{\theta,\kappa}}|e^{zt}||z|^{-\frac{1}{2}-\epsilon}|\mathrm{d}z|\right)^2\mathrm{d}t \\
&\le Ch^{\min\{\frac{4H_2}{\alpha}-1-2\epsilon,3\}},
\end{aligned}$$

which leads to

$$\mathrm{II}_2 \le Ch^{\min\{\frac{4H_2}{\alpha}-1-2\epsilon,3\}+2H_1-1-\epsilon}.$$

Therefore, we complete the proof. □

4.3. Temporal error analysis

In this subsection, we consider the temporal convergence of numerical scheme (4.2), that is, the estimate of $\mathbb{E}\|u_h^n - \bar{u}_h(t_n)\|_{L^2(D)}$.

In the following, we provide the representation of u_h^n first. Multiplying ζ^n on both sides of equation (4.2) and summing n from 1 to ∞, we have

$$\sum_{n=1}^{\infty} \frac{u_h^n - u_h^{n-1}}{\tau}\zeta^n + \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} d_i^{(1-\alpha)} A_h u_h^{n-i}\zeta^n = \sum_{n=1}^{\infty} P_h f(u_h^{n-1})\zeta^n + \sum_{n=1}^{\infty} P_h \xi_{R,n}^{H_1,H_2}\zeta^n.$$

Combining the definition of $d_i^{(1-\alpha)}$, one has

$$\begin{aligned}\sum_{n=1}^{\infty} u_h^n\zeta^n = &(\delta_\tau(\zeta))^{\alpha-1}((\delta_\tau(\zeta))^{\alpha} + A_h)^{-1} P_h \sum_{n=1}^{\infty} f(u_h^{n-1})\zeta^n \\ &+ (\delta_\tau(\zeta))^{\alpha-1}((\delta_\tau(\zeta))^{\alpha} + A_h)^{-1} P_h \sum_{n=1}^{\infty} \xi_{R,n}^{H_1,H_2}\zeta^n.\end{aligned}$$

Using Cauchy's integral formula and doing simple calculations lead to

$$\begin{aligned}u_h^n = &\frac{\tau}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^{\alpha} + A_h)^{-1} P_h \sum_{j=1}^{\infty} f(u_h^{j-1}) e^{-zj\tau}\,\mathrm{d}z \\ &+ \frac{\tau}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt_n}(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^{\alpha} + A_h)^{-1} P_h \sum_{j=1}^{\infty} \xi_{R,j}^{H_1,H_2} e^{-zj\tau}\,\mathrm{d}z,\end{aligned} \tag{4.11}$$

where $\Gamma_{\theta,\kappa}^{\tau} = \{z \in \mathbb{C} : \kappa \le |z| \le \frac{\pi}{\tau\sin(\theta)}, |\arg z| = \theta\} \cup \{z \in \mathbb{C} : |z| = \kappa, |\arg z| \le \theta\}$. According to the definition of $\xi_R^{H_1,H_2}$, it holds

$$\sum_{n=1}^{\infty} \xi_{R,n}^{H_1,H_2} e^{-zt_n} = \frac{z}{e^{z\tau}-1}\tilde{\xi}_R^{H_1,H_2}.$$

Introduce $\bar{F}(t)$ as

$$\bar{F}(t) = \begin{cases} 0 & t = t_0, \\ f\left(u_h^{j-1}\right) & t \in (t_{j-1}, t_j], \end{cases}$$

and $F(t) = f(u_h(t))$. In what follows, we abbreviate $P_h F(t)$ and $P_h\bar{F}(t)$ as $F_h(t)$ and $\bar{F}_h(t)$.

Simple calculations lead to

$$\sum_{n=1}^{\infty} \bar{F}_h(t_n) e^{-zt_n} = \frac{z}{e^{z\tau}-1}\tilde{\bar{F}}_h(z).$$

Thus there holds

$$\begin{aligned}u_h^n = &\int_0^{t_n} \bar{\mathcal{R}}_h(t_n - s)\bar{F}_h(s)\,\mathrm{d}s \\ &+ \int_0^{t_n}\int_D \bar{\mathcal{G}}_{R,h}(t-s,x,y)\xi_R^{H_1,H_2}(\mathrm{d}y,s)\,\mathrm{d}s,\end{aligned}$$

where

$$\bar{\mathcal{G}}_{R,h}(t,x,y)=\sum_{k=1}^{\infty}\bar{\mathcal{G}}_{R,h,k}(t,x,y),\qquad \bar{\mathcal{G}}_{R,h,k}(t,x,y)=\bar{\mathcal{R}}_h(t)P_h\phi_k(x)\phi_{R,k}(y),\tag{4.12}$$

and

$$\bar{\mathcal{R}}_h(t)=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^{\tau}_{\theta,\kappa}}e^{zt}(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^{\alpha}+A_h)^{-1}\frac{z\tau}{e^{z\tau}-1}\,\mathrm{d}z.\tag{4.13}$$

Let $\{\lambda_{k,h}\}_{k=1}^{N-1}$ and $\{\phi_{k,h}\}_{k=1}^{N-1}$ be the non-decreasing eigenvalues and L^2-norm normalized eigenfunctions of operator A_h. Then $\bar{\mathcal{R}}_h(t)$ can also be written as, for $u_h\in X_h$,

$$\bar{\mathcal{R}}_h(t)u_h=\sum_{k=1}^{N-1}E_{k,h}(t)(u_h,\phi_{k,h})\phi_{k,h},$$

where

$$E_{k,h}(t)=\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^{\tau}_{\theta,\kappa}}e^{zt}(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^{\alpha}+\lambda_{k,h})^{-1}\frac{z\tau}{e^{z\tau}-1}\,\mathrm{d}z.\tag{4.14}$$

Similar to the proof of Lemma 3.1, one can obtain the following property of $E_{k,h}(t)$.

Lemma 4.4. *Let $E_{k,h}(t)$ be defined in (4.14). For $k=1,2,\dots,N-1$, $\sigma\in[0,1]$, and $\gamma\geq 0$, if $\gamma+\sigma\alpha<\frac{1}{2}$, then there exists a uniform constant C such that*

$$\|\lambda^{\sigma}_{k,h}E_{k,h}(t)\|_{H^{\gamma}((0,T))}=\|{}_0\partial^{\gamma}_t\lambda^{\sigma}_{k,h}E_{k,h}(t)\|_{L^2((0,T))}\leq C.\tag{4.15}$$

In the rest of paper, we take $\kappa\leq\frac{\pi}{t_n\sin(\theta)}$. Then we provide the temporal error estimate.

Theorem 4.5. *Let $\bar{u}_h$ and u^n_h be the solutions of (4.5) and (4.2), respectively. Let $f(u)$ satisfy the assumptions (1.2). Assume $2H_2+(H_1-1)\alpha>0$. Then we have*

$$\mathbb{E}\|\bar{u}_h(t_n)-u^n_h\|^2_{L^2(D)}\leq C\tau^{2H_2+(H_1-1)\alpha-\epsilon}$$

with $0<\epsilon<2H_2+(H_1-1)\alpha$.

Proof. Subtracting (4.11) from (4.9), we have

$$\begin{aligned}
\mathbb{E}\|\bar{u}_h(t_n)-u^n_h\|^2_{L^2(D)}\leq{}& C\mathbb{E}\bigg\|\int_0^{t_n}\mathcal{R}_h(t_n-s)F(s)-\bar{\mathcal{R}}_h(t_n-s)\bar{F}_h(s)\,\mathrm{d}s\bigg\|^2_{L^2(D)}\\
&+C\mathbb{E}\bigg\|\int_0^{t_n}\int_D\mathcal{G}_{R,h}(t_n-s,x,y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\\
&-\int_0^{t_n}\int_D\bar{\mathcal{G}}_{R,h}(t_n-s,x,y)\xi^{H_1,H_2}_R(\mathrm{d}y,s)\,\mathrm{d}s\bigg\|^2_{L^2(D)}\\
\leq{}& C\mathbb{E}\bigg\|\int_0^{t_n}\mathcal{R}_h(t_n-s)F(s)-\bar{\mathcal{R}}_h(t_n-s)\bar{F}_h(s)\,\mathrm{d}s\bigg\|^2_{L^2(D)}\\
&+C\mathbb{E}\bigg\|\int_0^{t_n}\int_D\big(\mathcal{G}_{R,h}(t_n-s,x,y)-\bar{\mathcal{G}}_{R,h}(t_n-s,x,y)\big)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\bigg\|^2_{L^2(D)}\\
&+C\mathbb{E}\bigg\|\int_0^{t_n}\int_D\bar{\mathcal{G}}_{R,h}(t_n-s,x,y)\Big(\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)-\xi^{H_1,H_2}_R(\mathrm{d}y,s)\,\mathrm{d}s\Big)\bigg\|^2_{L^2(D)}\\
\leq{}&\vartheta_1+\vartheta_2+\vartheta_3.
\end{aligned}$$

For ϑ_1, one can split it into three parts, *i.e.*,

$$\begin{aligned}\vartheta_1 \le\ & C\mathbb{E}\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\mathcal{R}_h(t_n-s)(F_h(s)-F_h(t_{i-1}))\,\mathrm{d}s\right\|_{L^2(D)}^2\\ &+C\mathbb{E}\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}(\mathcal{R}_h(t_n-s)-\bar{\mathcal{R}}_h(t_n-s))F_h(t_{i-1})\,\mathrm{d}s\right\|_{L^2(D)}^2\\ &+C\mathbb{E}\left\|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(t_n-s)(F_h(t_{i-1})-\bar{F}_h(s))\,\mathrm{d}s\right\|_{L^2(D)}^2\\ \le\ & \vartheta_{1,1}+\vartheta_{1,2}+\vartheta_{1,3}.\end{aligned}$$

Using Theorem 4.2, one has

$$\vartheta_{1,1}\le C\tau^{2H_2+(H_1-1)\alpha-\epsilon}.$$

We first consider the estimate of $\|\mathcal{R}_h(t_n-s)-\bar{\mathcal{R}}_h(t_n-s)\|$. According to the definitions of $\mathcal{R}_h(t_n-s)$ and $\bar{\mathcal{R}}_h(t_n-s)$, there holds

$$\begin{aligned}&\|\mathcal{R}_h(t_n-s)-\bar{\mathcal{R}}_h(t_n-s)\|\\ &\quad\le C\left\|\int_{\Gamma_{\theta,\kappa}}e^{z(t_n-s)}z^{\alpha-1}(z^\alpha+A_h)^{-1}\,\mathrm{d}z\right.\\ &\qquad\left.-\int_{\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-s)}(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^\alpha+A_h)^{-1}\frac{z\tau}{e^{z\tau}-1}\,\mathrm{d}z\right\|\\ &\quad\le C\left\|\int_{\Gamma_{\theta,\kappa}\setminus\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-s)}z^{\alpha-1}(z^\alpha+A_h)^{-1}\,\mathrm{d}z\right\|\\ &\qquad+C\left\|\int_{\Gamma^\tau_{\theta,\kappa}}e^{z(t_n-s)}\left(z^{\alpha-1}(z^\alpha+A_h)^{-1}-(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^\alpha+A_h)^{-1}\frac{z\tau}{e^{z\tau}-1}\right)\mathrm{d}z\right\|.\end{aligned}$$

By the fact $|z-\delta_\tau(e^{-z\tau})|\le C\tau|z|^2$ for $z\in\Gamma^\tau_{\theta,\kappa}$ [18], one has

$$\left\|z^{\alpha-1}(z^\alpha+A_h)^{-1}-(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^\alpha+A_h)^{-1}\frac{z\tau}{e^{z\tau}-1}\right\|\le C\tau,$$

which yields

$$\left\|\mathcal{R}_h(t_n-s)-\bar{\mathcal{R}}_h(t_n-s)\right\|\le C\left(\tau^{1-\epsilon}\int_{\Gamma_{\theta,\kappa}\setminus\Gamma^\tau_{\theta,\kappa}}|e^{z(t_n-s)}||z|^\epsilon|\,\mathrm{d}z|+\tau\int_{\Gamma_{\theta,\kappa}}|e^{z(t_n-s)}||\,\mathrm{d}z|\right).$$

Thus combining assumptions (1.2) and Theorem 4.2 and using the Cauchy–Schwarz inequality, one can obtain

$$\begin{aligned}\vartheta_{1,2} \leq & C\tau^{2-2\epsilon}\int_0^{t_n}(t_n-s)^{1-\epsilon}\left(\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|e^{z(t_n-s)}||z|^{-\epsilon}|\mathrm{d}z|\right)^2\mathbb{E}\|u(s)\|^2_{L^2(D)}\,\mathrm{d}s\\ &+C\tau^2\int_0^{t_n}(t_n-s)^{1-\epsilon}\left(\int_{\Gamma^\tau_{\theta,\kappa}}|e^{z(t_n-s)}||\mathrm{d}z|\right)^2\mathbb{E}\|u(s)\|^2_{L^2(D)}\,\mathrm{d}s\\ \leq & C\tau^{2-2\epsilon}\int_0^{t_n}(t_n-s)^{\epsilon-1}\,\mathrm{d}s\\ &+C\tau^2\int_0^{t_n}(t_n-s)^{1-\epsilon}\int_{\Gamma^\tau_{\theta,\kappa}}|e^{2z(t_n-s)}||z|^{1-2\epsilon}|\,\mathrm{d}z|\int_{\Gamma^\tau_{\theta,\kappa}}|z|^{2\epsilon-1}|\,\mathrm{d}z|\,\mathrm{d}s\\ \leq & C\tau^{2-2\epsilon}.\end{aligned}$$

As for $\vartheta_{1,3}$, we arrive at

$$\vartheta_{1,3}\leq C\tau\sum_{i=1}^{n-1}\|\bar{u}_h(t_i)-u_h^i\|^2_{L^2(D)}.$$

The fact $\|A_h^{s/2}P_hA^{-s/2}\|\leq C$ for $s\in[0,1]$ [31] and Theorem 2.5 lead to

$$\begin{aligned}\vartheta_2\leq & C\sum_{k=1}^\infty\int_0^{t_n}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\big(\mathcal{R}_h(s)-\bar{\mathcal{R}}_h(s)\big)P_h\phi_k(y)\right\|^2_{L^2(D)}\|\phi_k\|^2_{H_0^{\frac{1-2H_1}{2}}(D)}\,\mathrm{d}s\\ \leq & C\sum_{k=1}^\infty\lambda_k^{-\frac{1}{2}-\epsilon}\int_0^{t_n}\left\|\lambda_k^{\frac{1-H_1}{2}+\frac{\epsilon}{2}}{}_0\partial_t^{\frac{1-2H_2}{2}}\big(\mathcal{R}_h(s)-\bar{\mathcal{R}}_h(s)\big)P_h\phi_k\right\|^2_{L^2(D)}\mathrm{d}s\\ \leq & C\sum_{k=1}^\infty\lambda_k^{-\frac{1}{2}-\epsilon}\int_0^{t_n}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\big(\mathcal{R}_h(s)-\bar{\mathcal{R}}_h(s)\big)P_hA^{\frac{1-H_1}{2}+\frac{\epsilon}{2}}\phi_k\right\|^2_{L^2(D)}\mathrm{d}s\\ \leq & C\int_0^{t_n}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\big(\mathcal{R}_h(s)-\bar{\mathcal{R}}_h(s)\big)A_h^{\frac{1-H_1}{2}+\frac{\epsilon}{2}}\right\|^2\mathrm{d}s.\end{aligned}$$

Simple calculations imply

$$\begin{aligned}\left\|{}_0\partial_t^{\frac{1-2H_2}{2}}\big(\mathcal{R}_h(s)-\bar{\mathcal{R}}_h(s)\big)A_h^{\frac{1-H_1}{2}+\frac{\epsilon}{2}}\right\|\leq & C\left\|\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}e^{zs}z^{\frac{1-2H_2}{2}}z^{\alpha-1}(z^\alpha+A_h)^{-1}A_h^{\frac{1-H_1}{2}+\frac{\epsilon}{2}}\,\mathrm{d}z\right\|\\ &+C\left\|\int_{\Gamma^\tau_{\theta,\kappa}}e^{zs}z^{\frac{1-2H_2}{2}}\bigg(z^{\alpha-1}(z^\alpha+A_h)^{-1}\right.\\ &\left.-\big(\delta_\tau(e^{-z\tau})\big)^{\alpha-1}\big((\delta_\tau(e^{-z\tau}))^\alpha+A_h\big)^{-1}\frac{z\tau}{e^{z\tau}-1}\bigg)A_h^{\frac{1-H_1}{2}+\frac{\epsilon}{2}}\,\mathrm{d}z\right\|\\ \leq & C\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|e^{zs}||z|^{\frac{1-2H_2}{2}+(\frac{1-H_1}{2}+\frac{\epsilon}{2})\alpha-1}|\,\mathrm{d}z|\\ &+C\tau\int_{\Gamma^\tau_{\theta,\kappa}}|e^{zs}||z|^{\frac{1-2H_2}{2}+(\frac{1-H_1}{2}+\frac{\epsilon}{2})\alpha}|\,\mathrm{d}z|,\end{aligned}$$

which yields

$$
\begin{aligned}
\vartheta_2 \leq & C\int_0^{t_n}\left(\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|e^{zs}||z|^{\frac{1-2H_2}{2}+\left(\frac{1-H_1}{2}+\frac{\epsilon}{2}\right)\alpha-1}|\,\mathrm{d}z|\right)^2\mathrm{d}s\\
&+C\tau^2\int_0^{t_n}\left(\int_{\Gamma^\tau_{\theta,\kappa}}|e^{zs}||z|^{\frac{1-2H_2}{2}+\left(\frac{1-H_1}{2}+\frac{\epsilon}{2}\right)\alpha}|\,\mathrm{d}z|\right)^2\mathrm{d}s\\
\leq & C\int_0^{t_n}\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|e^{2zs}||z|^{-\epsilon}|\,\mathrm{d}z|\int_{\Gamma_{\theta,\kappa}\backslash\Gamma^\tau_{\theta,\kappa}}|z|^{1-2H_2+(1-H_1+\epsilon)\alpha-2+\epsilon}\,\mathrm{d}s\\
&+C\tau^2\int_0^{t_n}\int_{\Gamma^\tau_{\theta,\kappa}}|e^{2zs}||z|^{-\epsilon}|\,\mathrm{d}z|\int_{\Gamma^\tau_{\theta,\kappa}}|z|^{1-2H_2+(1-H_1+\epsilon)\alpha+\epsilon}|\,\mathrm{d}z|\,\mathrm{d}s\\
\leq & C\tau^{2H_2+(H_1-1)\alpha-\epsilon_0}.
\end{aligned}
$$

Here $0<\epsilon_0<2H_2+(H_1-1)\alpha$. As for ϑ_3, we have

$$
\begin{aligned}
\vartheta_3 \leq & C\mathbb{E}\Bigg\|\sum_{k=1}^\infty\int_0^{t_n}\int_D\left(\bar{\mathcal{R}}_h(t_n-s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{(t_{i-1},t_i]}(s)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(t_n-r)\,\mathrm{d}r\right)\\
&\cdot P_h\phi_k(x)\phi_k(y)\xi^{H_1,H_2}(\mathrm{d}y,\mathrm{d}s)\Bigg\|^2_{L^2(D)}\\
\leq & C\sum_{k=1}^\infty\int_0^{t_n}\left\|{}_0\partial_s^{\frac{1-2H_2}{2}}\left(\bar{\mathcal{R}}_h(s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{[t_{i-1},t_i)}(s)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(r)\,\mathrm{d}r\right)P_h\phi_k(x)\right\|^2_{L^2(D)}\\
&\cdot\|\phi_k(y)\|^2_{H^{\frac{1-2H_1}{2}}(D)}\,\mathrm{d}s\\
\leq & C\sum_{k=1}^\infty\lambda_k^{-\frac{1}{2}-2\epsilon}\\
&\cdot\int_0^{t_n}\left\|{}_0\partial_s^{\frac{1-2H_2}{2}}\left(\bar{\mathcal{R}}_h(s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{[t_{i-1},t_i)}(s)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(r)\,\mathrm{d}r\right)P_h\lambda_k^{\frac{1-H_1}{2}+\epsilon}\phi_k(x)\right\|^2_{L^2(D)}\mathrm{d}s\\
\leq & C\int_0^{t_n}\left\|{}_0\partial_s^{\frac{1-2H_2}{2}}\left(\bar{\mathcal{R}}_h(s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{[t_{i-1},t_i)}(s)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(r)\,\mathrm{d}r\right)A_h^{\frac{1-H_1}{2}+\epsilon}\right\|^2_{X_h\to X_h}\mathrm{d}s.
\end{aligned}
$$

Introduce $v_h=\sum_{k=1}^{N-1}a_k\phi_{k,h}$ with $\|v_h\|^2_{X_h}=(v_h,v_h)=\sum_{k=1}^{N-1}a_k^2$. Thus

$$
\begin{aligned}
&\left\|{}_0\partial_s^{\frac{1-2H_2}{2}}\left(\bar{\mathcal{R}}_h(s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{[t_{i-1},t_i)}(s)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(r)\,\mathrm{d}r\right)A_h^{\frac{1-H_1}{2}+\epsilon}\right\|^2_{X_h\to X_h}\\
&\leq \sup_{v_h\in X_h,\|v_h\|_{X_h}=1}\left\|{}_0\partial_s^{\frac{1-2H_2}{2}}\left(\bar{\mathcal{R}}_h(s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{[t_{i-1},t_i)}(s)\int_{t_{i-1}}^{t_i}\bar{\mathcal{R}}_h(r)\,\mathrm{d}r\right)A_h^{\frac{1-H_1}{2}+\epsilon}v_h\right\|^2_{X_h}\\
&\leq C\sup_{v_h\in X_h,\|v_h\|_{X_h}=1}\sum_{k=1}^{N-1}a_k^2\left|{}_0\partial_s^{\frac{1-2H_2}{2}}\left(E_{k,h}(s)-\frac{1}{\tau}\sum_{i=1}^n\chi_{[t_{i-1},t_i)}(s)\int_{t_{i-1}}^{t_i}E_{k,h}(r)\,\mathrm{d}r\right)\lambda_{k,h}^{\frac{1-H_1}{2}+\epsilon}\right|^2.
\end{aligned}
$$

Combining Lemma 4.4 and the similar arguments of Theorem 3.3, we obtain

$$
\vartheta_3\leq C\tau^{2H_2+(H_1-1)\alpha-\epsilon}.
$$

After gathering the above estimates and using the discrete Grönwall inequality [31], the desired result can be reached. □

5. Further discussions in two-dimensional case

In this section, we begin to discuss equation (1.1) in two-dimensional case. Here, we first introduce some notations. Let $\mathcal{A} = -\Delta$ be defined on $D = (0, l)^2$ with a zero Dirichlet boundary condition. Thus the eigenvalues $\{\lambda_{k_1,k_2}\}_{k_1,k_2=1}^{\infty}$ and eigenfunctions $\{\phi_{k_1,k_2}\}_{k_1,k_2=1}^{\infty}$ of $\mathcal{A}$ can be represented by

$$\{\lambda_{k_1,k_2}, \phi_{k_1,k_2}\}_{k_1,k_2=1}^{\infty} = \{\lambda_{k_1} + \lambda_{k_2}, \phi_{k_1}\phi_{k_2}\}_{k_1,k_2=1}^{\infty},$$

where $\{\lambda_k\}_{k=1}^{\infty}$ are the non-decreasing eigenvalues and $\{\phi_k\}_{k=1}^{\infty}$ are L^2-norm normalized eigenfunctions of operator $A = (-\Delta)$ defined on $(0, l)$ with a zero Dirichlet boundary condition.

5.1. Regularity of solution in two dimensions

Similar to the proof of Theorem 2.5, one can obtain the estimate of Itô isometry for fractional Brownian sheet in two-dimensional case.

Theorem 5.1. *Let* $g_1(x, y, t) = g_{1,1}(x)g_{1,2}(y)g_{1,3}(t)$ *and* $g_2(x, y, t) = g_{2,1}(x)g_{2,2}(y)g_{2,3}(t)$ *satisfying* $g_{1,1}(x), g_{2,1}(x) \in H_0^{\frac{1-2H_1}{2}}((0, l))$, $g_{1,2}(y), g_{2,2}(y) \in H_0^{\frac{1-2H_2}{2}}((0, l))$ *and* $g_{1,3}(t), g_{2,3}(t) \in H_0^{\frac{1-2H_3}{2}}((0, T))$. *Then we have*

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \int_D g_1(x, y, t)\xi^{H_1,H_2,H_3}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}t) \int_0^T \int_D g_2(x, y, t)\xi^{H_1,H_2,H_3}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}t)\right)\\
&\quad \le C\left\|{}_0\partial_t^{\frac{1-2H_3}{2}} g_{1,3}(t)\right\|_{L^2((0,T))}\left\|{}_0\partial_t^{\frac{1-2H_3}{2}} g_{2,3}(t)\right\|_{L^2((0,T))}\\
&\qquad \cdot \|g_{1,1}(x)\|_{H_0^{\frac{1-2H_1}{2}}((0,l))}\|g_{2,1}(x)\|_{H_0^{\frac{1-2H_1}{2}}((0,l))}\\
&\qquad \cdot \|g_{1,2}(y)\|_{H_0^{\frac{1-2H_2}{2}}((0,l))}\|g_{2,2}(y)\|_{H_0^{\frac{1-2H_2}{2}}((0,l))}.
\end{aligned}$$

Here ${}_0\partial_t^{\alpha}$ *is the Riemann–Liouville fractional derivative when* $\alpha \in (0, 1)$; *and when* $\alpha = 0$, *it denotes an identity operator.*

Next, we introduce

$$\mathcal{G}(t, x, y, \hat{x}, \hat{y}) = \sum_{k_1,k_2=1}^{\infty} \mathcal{G}_{k_1,k_2}(t, x, y, \hat{x}, \hat{y}), \tag{5.1}$$

where

$$\mathcal{G}_{k_1,k_2}(t, x, y, \hat{x}, \hat{y}) = E_{k_1,k_2}(t)\phi_{k_1}(x)\phi_{k_1}(\hat{x})\phi_{k_2}(y)\phi_{k_2}(\hat{y}) \tag{5.2}$$

and

$$E_{k_1,k_2}(t) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\theta,\kappa}} e^{zt}z^{\alpha-1}(z^{\alpha} + \lambda_{k_1,k_2})^{-1}\,\mathrm{d}z. \tag{5.3}$$

Thus the solution of equation (1.1) with $d = 2$ can be written as

$$u(x, y, t) = \int_0^t \int_D \mathcal{G}(t-s, x, y, \hat{x}, \hat{y})f(u)\,\mathrm{d}s\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y} + \int_0^t \int_D \mathcal{G}(t-s, x, y, \hat{x}, \hat{y})\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x}, \mathrm{d}\hat{y}, \mathrm{d}s). \tag{5.4}$$

For the convenience of analysis, we introduce the operator $\mathcal{R}$, which is defined by the Laplace transform, *i.e.*,

$$\tilde{\mathcal{R}}(z) = z^{\alpha-1}(z^{\alpha} + \mathcal{A})^{-1}.$$

It is easy to verify

$$\mathcal{G}_{k_1,k_2}(t,x,y,\hat{x},\hat{y})=\mathcal{R}(t)\phi_{k_1}(x)\phi_{k_1}(\hat{x})\phi_{k_2}(y)\phi_{k_2}(\hat{y})$$

and

$$\mathcal{R}(t)u(x,y)=\int_D \mathcal{G}(t,x,y,\hat{x},\hat{y})u(\hat{x},\hat{y})\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}.$$

Therefore, the solution of equation (1.1) with $d=2$ can also be written as

$$u(t)=\int_0^t \mathcal{R}(t-s)f(u(s))\,\mathrm{d}s+\int_0^t\int_D \mathcal{G}(t-s,x,y,\hat{x},\hat{y})\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x},\mathrm{d}\hat{y},\mathrm{d}s). \tag{5.5}$$

Then, we give the regularity estimate of the solution to equation (1.1) with $d=2$.

Theorem 5.2. *Let $u(t)$ be the solution of equation (1.1) with $d=2$ and $f(u)$ satisfy the assumptions (1.2). Assume $2H_3+(H_1+H_2-2)\alpha>0$. Then there holds*

$$\mathbb{E}\|\mathcal{A}^\sigma u(t)\|^2_{L^2(D)}\le C,$$

where $2\sigma\in[0,\min\{\frac{2H_3}{\alpha}+H_1+H_2-2,H_1+H_2\})$. Moreover, one has

$$\mathbb{E}\left\|\frac{u(t)-u(t-\tau)}{\tau^\gamma}\right\|^2_{L^2(D)}\le C,$$

where $2\gamma\in[0,2H_3+(H_1+H_2-2)\alpha)$.

Proof. According to (5.5), we have

$$\begin{aligned}\mathbb{E}\|\mathcal{A}^\sigma u(t)\|^2_{L^2(D)}\le{}& C\mathbb{E}\left\|\int_0^t \mathcal{A}^\sigma\mathcal{R}(t-s)f(u)\,\mathrm{d}s\right\|^2_{L^2(D)}\\&+C\mathbb{E}\left\|\int_0^t\int_D \mathcal{A}^\sigma\mathcal{G}(t-s,x,y,\hat{x},\hat{y})\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x},\mathrm{d}\hat{y},\mathrm{d}s)\right\|^2_{L^2(D)}\\\le{}&\mathrm{I}+\mathrm{II}.\end{aligned}$$

Similar to the proof of Theorem 2.7, one has

$$\mathrm{I}\le C\left(1+\int_0^t (t-s)^{-2\sigma\alpha+1-\epsilon}\mathbb{E}\|u\|^2_{L^2(D)}\,\mathrm{d}s\right),$$

where $-2\sigma\alpha+1>-1$ needs to be satisfied, *i.e.*, $\sigma<\frac{1}{\alpha}$.

According to Theorem 5.1 and the definition of $\mathcal{G}$, it holds

$$\begin{aligned}\mathrm{II}\le{}& C\sum_{k_1,k_2=1}^{\infty}\int_0^t\left|{}_0\partial_t^{\frac{1-2H_3}{2}}\lambda^\sigma_{k_1,k_2}E_{k_1,k_2}(t)\right|^2\|\phi_{k_1}\|^2_{H_0^{\frac{1-2H_1}{2}}((0,l))}\|\phi_{k_1}\|^2_{L^2((0,l))}\\&\cdot\|\phi_{k_2}\|^2_{H_0^{\frac{1-2H_2}{2}}((0,l))}\|\phi_{k_2}\|^2_{L^2((0,l))}\,\mathrm{d}t\\\le{}& C\sum_{k_1,k_2=1}^{\infty}\int_0^t\left|\int_{\Gamma_{\theta,\kappa}}e^{zt}z^{\alpha-1+\frac{1-2H_3}{2}}\lambda^\sigma_{k_1,k_2}(z^\alpha+\lambda_{k_1,k_2})^{-1}\,\mathrm{d}z\right|^2\|\phi_{k_1}\|^2_{H_0^{\frac{1-2H_1}{2}}((0,l))}\|\phi_{k_1}\|^2_{L^2((0,l))}\\&\cdot\|\phi_{k_2}\|^2_{H_0^{\frac{1-2H_2}{2}}((0,l))}\|\phi_{k_2}\|^2_{L^2((0,l))}\,\mathrm{d}t.\end{aligned}$$

By the fact that $\frac{a^{\sigma_1}b^{\sigma_2}}{(a+b)^{\sigma_1+\sigma_2}} = \frac{a^{\sigma_2}b^{\sigma_2}}{(a+b)^{2\sigma_2}}\frac{a^{\sigma_1-\sigma_2}}{(a+b)^{\sigma_1-\sigma_2}} \leq C$ for $a, b > 0$ and $\sigma_1 \geq \sigma_2 > 0$, the resolvent estimate, Remark 2.3, Lemma 2.1, and simple calculations, we have

$$\begin{aligned}
\mathrm{II} \leq & C \sum_{k_1,k_2=1}^{\infty} \int_0^t \left(\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{\alpha-1+\frac{1-2H_3}{2}} |\lambda_{k_1,k_2}^{\sigma+\frac{1-H_1-H_2}{2}} (z^\alpha + \lambda_{k_1,k_2})^{-1}||\,\mathrm{d}z| \right)^2 \mathrm{d}t \\
\leq & C \sum_{k_1,k_2=1}^{\infty} \lambda_{k_1,k_2}^{-1-2\epsilon} \int_0^t \left(\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{\alpha-1+\frac{1-2H_3}{2}} |\lambda_{k_1,k_2}^{\sigma+\frac{2-H_1-H_2}{2}+\epsilon} (z^\alpha + \lambda_{k_1,k_2})^{-1}||\,\mathrm{d}z| \right)^2 \mathrm{d}t \\
\leq & C \sum_{k_1,k_2=1}^{\infty} \lambda_{k_1,k_2}^{-1-2\epsilon} \int_0^t \left(\int_{\Gamma_{\theta,\kappa}} |e^{zt}||z|^{(\sigma+\frac{2-H_1-H_2}{2}+\epsilon)\alpha-1+\frac{1-2H_3}{2}} |\,\mathrm{d}z| \right)^2 \mathrm{d}t,
\end{aligned}$$

where we need to require $(\sigma + \frac{2-H_1-H_2}{2} + \epsilon)\alpha - 1 + \frac{1-2H_3}{2} < -\frac{1}{2}$ and $\sigma + \frac{2-H_1-H_2}{2} < 1$, *i.e.*, $2\sigma < \min\{\frac{2H_3}{\alpha} + H_1 + H_2 - 2, H_1 + H_2\}$. Combining the Grönwall inequality [14], the first desired result can be obtained. As for the temporal regularity estimate, one can obtain it by using similar techniques of Theorem 2.8. □

5.2. Numerical scheme and error analyses in two dimensions

To construct the numerical scheme, we introduce the mesh size $h = l/N$ first. Denote $D_{i,j} = (x_i, x_{i+1}) \times (y_j, y_{j+1})$ $(i, j \in \mathbb{N})$ with $x_i = y_i = ih$. Introduce $\tau = T/M$ and $I_k = (t_k, t_{k+1}]$ with $t_k = k\tau$. Similar to the one-dimensional case, we can discretize equation (1.1) with $d = 2$ by finite element method and backward Euler method, and the fully discrete scheme can be written as: find $u_h^n \in X_h$ satisfying

$$\frac{u_h^n - u_h^{n-1}}{\tau} + \sum_{i=0}^{n-1} d_i^{(1-\alpha)} \mathcal{A}_h u_h^{n-i} = P_h f(u_h^{n-1}) + P_h \xi_{R,n}^{H_1,H_2,H_3}, \tag{5.6}$$

where $\mathcal{A}_h$ is defined by $(\mathcal{A}_h u_h, v_h) = (\nabla u_h, \nabla v_h)$ with $u_h, v_h \in X_h$, $d_i^{(\alpha)}$ is defined in (4.3), $\xi_{R,n}^{H_1,H_2,H_3} = \xi_R^{H_1,H_2,H_3}(t_n)$ and

$$\xi_R^{H_1,H_2,H_3}(x,y,t) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{M-1} \left(\frac{1}{\tau h^2} \int_{I_k}\int_{D_{i,j}} \xi^{H_1,H_2,H_3}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}s) \right) \chi_{D_{i,j}\times I_k}(x,y,t).$$

Similar to the derivation of one-dimensional case, one can write the solution of equation (5.6) as

$$u_h^n = \int_0^{t_n} \bar{\mathcal{R}}_h(t_n - s) P_h \bar{F}_h(s)\,\mathrm{d}s + \int_0^{t_n}\int_D \bar{\mathcal{G}}_{R,h}(t-s,x,y,\hat{x},\hat{y}) \xi_R^{H_1,H_2,H_3}(\mathrm{d}\hat{x}, \mathrm{d}\hat{y}, s)\,\mathrm{d}s, \tag{5.7}$$

where $\bar{F}_h(t)$ is

$$\bar{F}_h(t) = \begin{cases} 0 & t = t_0, \\ f\left(u_h^{j-1}\right) & t \in (t_{j-1}, t_j]. \end{cases}$$

Here, $\bar{\mathcal{G}}_{R,h}(t,x,y,\hat{x},\hat{y})$ is defined by

$$\begin{aligned}
\bar{\mathcal{G}}_{R,h}(t,x,y,\hat{x},\hat{y}) &= \sum_{k_1,k_2=1}^{\infty} \bar{\mathcal{G}}_{R,h,k_1,k_2}(t,x,y,\hat{x},\hat{y}), \\
\bar{\mathcal{G}}_{R,h,k_1,k_2}(t,x,y,\hat{x},\hat{y}) &= \bar{\mathcal{R}}_h(t) P_h(\phi_{k_1}(x)\phi_{k_2}(y))\phi_{R,k_1}(\hat{x})\phi_{R,k_2}(\hat{y}),
\end{aligned} \tag{5.8}$$

and

$$\bar{\mathcal{R}}_h(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}^\tau} e^{zt} (\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^\alpha + \mathcal{A}_h)^{-1} \frac{z\tau}{e^{z\tau}-1}\,\mathrm{d}z. \tag{5.9}$$

At last, we show the convergence of numerical scheme (5.6).

Theorem 5.3. *Let $u(t)$ and u_h^n be the solutions of equations (1.1) and (5.6), respectively. Assume that $f(u)$ satisfies (1.2) and $2H_3 + (H_1 + H_2 - 2)\alpha > 0$. Then we have*

$$\mathbb{E}\|u(t) - u_h^n\|_{L^2(D)}^2 \le Ch^{4\sigma} + C\tau^{2\gamma},$$

where $2\sigma \in [0, \min\{\frac{2H_3}{\alpha} + H_1 + H_2 - 2, H_1 + H_2\})$ and $2\gamma \in [0, 2H_3 + (H_1 + H_2 - 2)\alpha)$.

Proof. Similar to the one-dimensional case, we introduce $\bar{u}^n$ as the solution of the following auxiliary equation

$$\frac{\bar{u}^n - \bar{u}^{n-1}}{\tau} + \sum_{i=0}^{n-1} d_i^{(1-\alpha)} \mathcal{A}\bar{u}^{n-i} = f(\bar{u}^{n-1}) + \xi_{RT,n}^{H_1,H_2,H_3} \tag{5.10}$$

with $\bar{u}_h(0) = 0$, $\xi_{RT,n}^{H_1,H_2,H_3} = \xi_{RT}^{H_1,H_2,H_3}(t_n)$, and

$$\xi_{RT}^{H_1,H_2,H_3}(t) = \sum_{j=0}^{M-1} \left(\frac{1}{\tau} \int_{I_j} \xi^{H_1,H_2,H_3}(x, y, \mathrm{d}t) \right) \chi_{I_j}(t),$$

where $\chi_{I_j}(t)$ is the characteristic function on I_j. Then the solution of equation (5.10) can also be written as

$$\bar{u}^n = \int_0^{t_n} \bar{\mathcal{R}}(t_n - s)\bar{F}(s)\,\mathrm{d}s + \int_0^t \int_D \bar{\mathcal{G}}_R(t - s, x, y, \hat{x}, \hat{y})\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x}, \mathrm{d}\hat{y}, \mathrm{d}s), \tag{5.11}$$

where

$$\bar{\mathcal{R}}(t) = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\theta,\kappa}^{\tau}} e^{zt}(\delta_\tau(e^{-z\tau}))^{\alpha-1}((\delta_\tau(e^{-z\tau}))^\alpha + \mathcal{A})^{-1} \frac{z\tau}{e^{z\tau} - 1}\,\mathrm{d}z,$$

$$\bar{\mathcal{G}}_R(t, x, y, \hat{x}, \hat{y}) = \sum_{k_1,k_2=1}^{\infty} \bar{\mathcal{G}}_{R,k_1,k_2}(t, x, y, \hat{x}, \hat{y}),$$

$$\bar{\mathcal{G}}_{R,k_1,k_2}(t, x, y, \hat{x}, \hat{y}) = \bar{\mathcal{R}}(t)\phi_{k_1}(x)\phi_{k_1}(\hat{x})\phi_{k_2}(y)\phi_{k_2}(\hat{y}).$$

Here, $\bar{F}(t)$ is

$$\bar{F}(t) = \begin{cases} 0 & t = t_0, \\ f(\bar{u}^{j-1}) & t \in (t_{j-1}, t_j]. \end{cases}$$

As for temporal error estimate of (5.6), comparing the solutions of equations (1.1) and (5.10) and using similar proof techniques for Theorem 4.5, one can obtain it. Then we begin to show spatial error estimate. According to (5.7) and (5.11), there exists

$$\begin{aligned}
\mathbb{E}\|\bar{u}^n - u_h^n\|_{L^2(D)}^2 &\le C\mathbb{E}\left\| \int_0^{t_n} \bar{\mathcal{R}}(t_n - s)\bar{F}(s)\,\mathrm{d}s - \int_0^{t_n} \bar{\mathcal{R}}_h(t_n - s)P_h\bar{F}_h(s)\,\mathrm{d}s \right\|_{L^2(D)}^2 \\
&\quad + C\mathbb{E}\left\| \int_0^{t_n} \int_D \big(\bar{\mathcal{G}}_R(t_n - s, x, y, \hat{x}, \hat{y}) - \bar{\mathcal{G}}_{R,h}(t_n - s, x, y, \hat{x}, \hat{y})\big)\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x}, \mathrm{d}\hat{y}, \mathrm{d}s) \right\|_{L^2(D)}^2 \\
&\le \mathrm{I} + \mathrm{II}.
\end{aligned}$$

By using approximation theory, it holds

$$\mathrm{I} \le Ch^4 + C\tau \sum_{k=0}^{n-1} \mathbb{E}\big\|\bar{u}^k - u_h^k\big\|_{L^2(D)}^2.$$

As for II, one has

$$\begin{aligned}\mathrm{II} \leq & C\mathbb{E}\bigg\|\int_0^{t_n}\int_D(\bar{\mathcal{G}}_R(t_n-s,x,y,\hat{x},\hat{y})-\bar{\mathcal{G}}_{Rs}(t_n-s,x,y,\hat{x},\hat{y}))\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x},\mathrm{d}\hat{y},\mathrm{d}s)\bigg\|^2_{L^2(D)}\\ &+C\mathbb{E}\bigg\|\int_0^{t_n}\int_D(\bar{\mathcal{G}}_{Rs}(t_n-s,x,y,\hat{x},\hat{y})-\bar{\mathcal{G}}_{R,h}(t_n-s,x,y,\hat{x},\hat{y}))\xi^{H_1,H_2,H_3}(\mathrm{d}\hat{x},\mathrm{d}\hat{y},\mathrm{d}s)\bigg\|^2_{L^2(D)}\\ \leq & \mathrm{II}_1+\mathrm{II}_2,\end{aligned}$$

where

$$\bar{\mathcal{G}}_{Rs}(t,x,y,\hat{x},\hat{y})=\sum_{k_1,k_2=1}^{\infty}\bar{\mathcal{R}}(t)\phi_{k_1}(x)\phi_{R,k_1}(\hat{x})\phi_{k_2}(y)\phi_{R,k_2}(\hat{y}).$$

Using the following fact

$$\begin{aligned}\phi_{k_1}(\hat{x})\phi_{k_2}(\hat{y})-\phi_{R,k_1}(\hat{x})\phi_{R,k_2}(\hat{y})=&(\phi_{k_1}(\hat{x})-\phi_{R,k_1}(\hat{x}))(\phi_{k_2}(\hat{y})-\phi_{R,k_2}(\hat{y}))\\ &+\phi_{R,k_1}(\hat{x})(\phi_{k_2}(\hat{y})-\phi_{R,k_2}(\hat{y}))+(\phi_{k_1}(\hat{x})-\phi_{R,k_1}(\hat{x}))\phi_{R,k_2}(\hat{y})\end{aligned}$$

and doing simple calculations yield

$$\begin{aligned}\mathrm{II}_1\leq & C\sum_{k_1,k_2=1}^{\infty}\Big\|{}_0\partial_t^{\frac{1-2H_3}{2}}\bar{\mathcal{R}}(t)\phi_{k_1}(x)\phi_{k_2}(y)\Big\|^2_{L^2(0,t_n,L^2(D))}\|\phi_{k_1}(\hat{x})-\phi_{R,k_1}(\hat{x})\|^2_{H^{\frac{1-2H_1}{2}}((0,l))}\\ &\cdot\|\phi_{k_2}(\hat{y})-\phi_{R,k_2}(\hat{y})\|^2_{H^{\frac{1-2H_2}{2}}((0,l))}\\ &+C\sum_{k_1,k_2=1}^{\infty}\Big\|{}_0\partial_t^{\frac{1-2H_3}{2}}\bar{\mathcal{R}}(t)\phi_{k_1}(x)\phi_{k_2}(y)\Big\|^2_{L^2(0,t_n,L^2(D))}\|\phi_{R,k_1}(\hat{x})\|^2_{H^{\frac{1-2H_1}{2}}((0,l))}\\ &\cdot\|\phi_{k_2}(\hat{y})-\phi_{R,k_2}(\hat{y})\|^2_{H^{\frac{1-2H_2}{2}}((0,l))}\\ &+C\sum_{k_1,k_2=1}^{\infty}\Big\|{}_0\partial_t^{\frac{1-2H_3}{2}}\bar{\mathcal{R}}(t)\phi_{k_1}(x)\phi_{k_2}(y)\Big\|^2_{L^2(0,t_n,L^2(D))}\|\phi_{k_1}(\hat{x})-\phi_{R,k_1}(\hat{x})\|^2_{H^{\frac{1-2H_1}{2}}((0,l))}\\ &\cdot\|\phi_{R,k_2}(\hat{y})\|^2_{H^{\frac{1-2H_2}{2}}((0,l))}\\ \leq & Ch^{4\sigma}\sum_{k_1,k_2=1}^{\infty}\lambda_{k_1,k_2}^{-1-2\epsilon}\Big\|\lambda_{k_1,k_2}^{\sigma+\frac{2-H_1-H_2}{2}+\epsilon}{}_0\partial_t^{\frac{1-2H_3}{2}}\bar{\mathcal{R}}(t)\phi_{k_1}(x)\phi_{k_2}(y)\Big\|^2_{L^2(0,t_n,L^2(D))},\end{aligned}$$

where $2\sigma\in[0,\min\{\frac{2H_3}{\alpha}+H_1+H_2-2,H_1+H_2\})$. As for II_2, the inverse estimate gives

$$\begin{aligned}\mathrm{II}_2\leq & C\sum_{k_1,k_2=1}^{\infty}\Big\|{}_0\partial_t^{\frac{1-2H_3}{2}}(\bar{\mathcal{R}}(t)-\bar{\mathcal{R}}_h(t)P_h)\phi_{k_1}(x)\phi_{k_2}(y)\Big\|^2_{L^2(0,t_n,L^2(D))}\|\phi_{R,k_1}(\hat{x})\|^2_{H^{\frac{1-2H_1}{2}}((0,l))}\\ &\cdot\|\phi_{R,k_2}(\hat{y})\|^2_{H^{\frac{1-2H_2}{2}}((0,l))}\\ \leq & Ch^{2(H_1+H_2-1)-2\epsilon}\sum_{k_1,k_2=1}^{\infty}\lambda_{k_1,k_2}^{-1}\lambda_{k_1}^{-\epsilon}\lambda_{k_2}^{-\epsilon}\Big\|{}_0\partial_t^{\frac{1-2H_3}{2}}(\bar{\mathcal{R}}(t)-\bar{\mathcal{R}}_h(t)P_h)\mathcal{A}^{\frac{1}{2}}\phi_{k_1}(x)\phi_{k_2}(y)\Big\|^2_{L^2(0,t_n,L^2(D))}.\end{aligned}$$

Using the fact $\|((z^\alpha+\mathcal{A})^{-1}-(z^\alpha+\mathcal{A}_h)^{-1}P_h)\mathcal{A}^s\|\le Ch^{2-2s}$ for $s\in[0,\frac{1}{2}]$, $z\in\Gamma_{\theta,\kappa}$ [31], and the interpolation property [3], one obtains

$$\begin{aligned}
&\left\|{}_0\partial_t^{\frac{1-2H_3}{2}}(\bar{\mathcal{R}}(t)-\bar{\mathcal{R}}_h(t)P_h)\mathcal{A}^{\frac{1}{2}}\phi_{k_1}(x)\phi_{k_2}(y)\right\|^2_{L^2(0,t_n,L^2(D))}\\
&\le C\int_0^{t_n}\left\|\int_{\Gamma_{\theta,\kappa}}e^{zt}z^{\frac{1-2H_3}{2}}z^{\alpha-1}\big((z^\alpha+\mathcal{A})^{-1}-(z^\alpha+\mathcal{A}_h)^{-1}P_h\big)\mathcal{A}^{\frac{1}{2}}\phi_{k_1}(x)\phi_{k_2}(y)\,\mathrm{d}z\right\|^2_{L^2(D)}\mathrm{d}t\\
&\le Ch^{\min\{\frac{4H_3}{\alpha}-2-\epsilon,2\}}\int_0^{t_n}\left(\int_{\Gamma_{\theta,\kappa}}|e^{zt}||z|^{-\frac{1}{2}-\epsilon}|\mathrm{d}z|\right)^2\mathrm{d}t\\
&\le Ch^{\min\{\frac{4H_3}{\alpha}-2-\epsilon,2\}},
\end{aligned}$$

which leads to

$$\mathrm{II}_2\le Ch^{\min\{\frac{4H_3}{\alpha}-2-\epsilon,2\}+2(H_1+H_2-1)-2\epsilon}.$$

Therefore, the spatial error estimate of (5.6) can be obtained by using the discrete Grönwall inequality. □

6. Numerical experiments

Here we present some numerical examples in one and two dimensions to show the effectiveness of the numerical methods and confirm the theoretical results. We take m trajectories $\{\omega_j\}_{j=1}^m$ to calculate the solution. Since the exact solutions are unknown in the numerical experiments, we measure the convergence rates by

$$\mathrm{Rate}=\frac{\ln(e_h/e_{h/2})}{\ln(2)},\quad \mathrm{Rate}=\frac{\ln(e_\tau/e_{\tau/2})}{\ln(2)},$$

where

$$e_h=\left(\frac{1}{m}\sum_{i=1}^m\|u_h^M(\omega_i)-u_{h/2}^M(\omega_i)\|^2_{L^2(D)}\right)^{1/2},\quad e_\tau=\left(\frac{1}{m}\sum_{i=1}^m\|u_\tau(\omega_i)-u_{\tau/2}(\omega_i)\|^2_{L^2(D)}\right)^{1/2}$$

with u_h^M and u_τ being the solutions at time $T=t_M$ with mesh size h and time step size τ, respectively.

Example 6.1 (One-dimensional case)**.** In this example, to show the temporal convergence, we take $m=200$, $\beta=1$, $T=0.5$, $l=0.5$, $f(u)=\sin(u)$, and $h=l/512$ to solve equation (1.1) with $d=1$. The numerical results with different α, H_1, and H_2 are given in Table 1, where the numbers in the bracket in the last column denote the theoretical rates predicted by Theorem 4.5. As it can be seen, the numerical convergence rates can achieve $\mathcal{O}(\tau^{H_2+(H_1-1)\alpha/2-\epsilon})$, which agree well with the theoretical results of Theorem 4.5.

Example 6.2 (One-dimensional case)**.** To show the spatial convergence rates, we choose $\beta=10$ and $f(u)=\frac{1}{50}\sin(u)$ in equation (1.1). Here, we take $m=100$, $T=0.01$, $l=0.1$, and $\tau=T/2048$. We show the corresponding errors and convergence rates with different α, H_1, and H_2 in Table 2. All the numerical convergence rates well agree with the predicted ones stated in Theorem 4.3, *i.e.*, $\mathcal{O}(h^{\sigma+H_1-1/2})$ with $\sigma=\min\{\frac{2H_2}{\alpha}-\frac{1}{2},1+\epsilon\}-\epsilon$.

Example 6.3 (Two-dimensional case)**.** Here we provide a numerical example to verify the temporal convergence rates of (5.6). We take $f(u)=\sin(u)$, $m=200$, $T=0.3$, $D=(0,0.5)^2$, $\beta=1$, and choose $h=0.5/128$ to avoid the influence of spatial errors on temporal convergence. The errors and convergences rates with different H_1, H_2, H_3, and α are shown in Table 3, where the numbers in the bracket in the last column denote the theoretical rates predicted by Theorem 5.3. It can be noted that all the convergence rates agree with the results provided in Theorem 5.3, *i.e.*, $\mathcal{O}(\tau^{H_3+(H_1+H_2-2)\alpha/2-\epsilon})$.

TABLE 1. Temporal errors and convergence rates.

$(\alpha, H_1, H_2)\backslash T/\tau$	16	32	64	128	Rate
$(0.3, 0.2, 0.2)$	4.002E−05	4.191E−05	4.155E−05	3.478E−05	≈0.0675(0.08)
$(0.3, 0.3, 0.5)$	1.012E−05	8.111E−06	5.976E−06	4.188E−06	≈0.4243(0.395)
$(0.5, 0.2, 0.3)$	6.808E−05	5.872E−05	5.805E−05	5.069E−05	≈0.1419(0.1)
$(0.5, 0.4, 0.3)$	5.808E−05	5.563E−05	4.244E−05	4.497E−05	≈0.1230(0.15)
$(0.7, 0.4, 0.5)$	3.886E−05	3.354E−05	2.822E−05	2.126E−05	≈0.2901(0.29)
$(0.7, 0.5, 0.2)$	1.339E−04	1.351E−04	1.482E−04	1.268E−04	≈0.0264(0.025)

TABLE 2. Spatial errors and convergence rates.

$(\alpha, H_1, H_2)\backslash l/h$	8	16	32	64	Rate
$(0.3, 0.2, 0.5)$	1.385E−01	7.186E−02	5.079E−02	3.173E−02	≈0.7087(0.7)
$(0.3, 0.5, 0.5)$	3.188E−02	1.554E−02	7.991E−03	4.002E−03	≈0.9979(1)
$(0.5, 0.2, 0.3)$	9.038E−01	6.727E−01	4.861E−01	3.072E−01	≈0.5190(0.4)
$(0.5, 0.5, 0.4)$	1.533E−01	7.697E−02	4.082E−02	2.042E−02	≈0.9695(1)
$(0.7, 0.4, 0.4)$	4.977E−01	3.482E−01	2.396E−01	1.468E−01	≈0.5871(0.5429)
$(0.7, 0.3, 0.4)$	7.310E−01	5.083E−01	3.810E−01	2.546E−01	≈0.5073(0.4429)

TABLE 3. Temporal errors and convergence rates.

$(\alpha, H_1, H_2, H_3)\backslash T/\tau$	8	16	32	64	Rate
$(0.3, 0.5, 0.4, 0.5)$	3.836E−02	3.050E−02	2.384E−02	1.959E−02	≈0.3233(0.335)
$(0.4, 0.3, 0.5, 0.5)$	7.303E−02	7.141E−02	6.104E−02	4.166E−02	≈0.2699(0.26)
$(0.5, 0.5, 0.3, 0.4)$	2.558E−01	2.439E−01	2.121E−01	2.105E−01	≈0.0937(0.1)
$(0.5, 0.3, 0.5, 0.4)$	2.595E−01	2.265E−01	2.176E−01	2.205E−01	≈0.0783(0.1)
$(0.6, 0.5, 0.3, 0.5)$	2.620E−01	2.111E−01	1.902E−01	1.865E−01	≈0.1636(0.14)
$(0.6, 0.4, 0.5, 0.5)$	2.261E−01	1.977E−01	1.857E−01	1.595E−01	≈0.1677(0.17)

TABLE 4. Spatial errors and convergence rates.

$(\alpha, H_1, H_2, H_3)\backslash l/h$	8	16	32	64	Rate
$(0.3, 0.4, 0.3, 0.5)$	9.571E−02	6.150E−02	4.260E−02	2.826E−02	≈0.5866(0.7)
$(0.4, 0.3, 0.3, 0.4)$	2.733E−01	1.947E−01	1.483E−01	1.014E−01	≈0.4769(0.6)
$(0.5, 0.3, 0.3, 0.4)$	4.384E−01	3.496E−01	2.439E−01	1.745E−01	≈0.4431(0.4)
$(0.5, 0.4, 0.3, 0.4)$	3.269E−01	2.293E−01	1.640E−01	1.085E−01	≈0.5303(0.6)
$(0.6, 0.3, 0.3, 0.5)$	4.063E−01	3.219E−01	2.365E−01	1.704E−01	≈0.4179(0.5333)
$(0.6, 0.5, 0.5, 0.4)$	2.403E−01	1.522E−01	8.644E−02	4.904E−02	≈0.7643(0.6667)

Example 6.4 (Two-dimensional case)**.** In this example, a numerical example is presented to verify spatial convergence of (5.6). We take $f(u) = \sin(u)$, $m = 200$, $T = 0.1$, $D = (0, 0.5)^2$, and $\beta = 1$. Here we take $\tau = T/256$ to decrease the influence of temporal errors on spatial convergence. The errors and convergence rates with different H_1, H_2, H_3, and α are shown in Table 4. All the convergence rates are consistent with the predicted results in Theorem 5.3, *i.e.*, $\mathcal{O}(h^{2\sigma})$ with $2\sigma = \min\{\frac{2H_3}{\alpha} + H_1 + H_2 - 2, H_1 + H_2\} - \epsilon$.

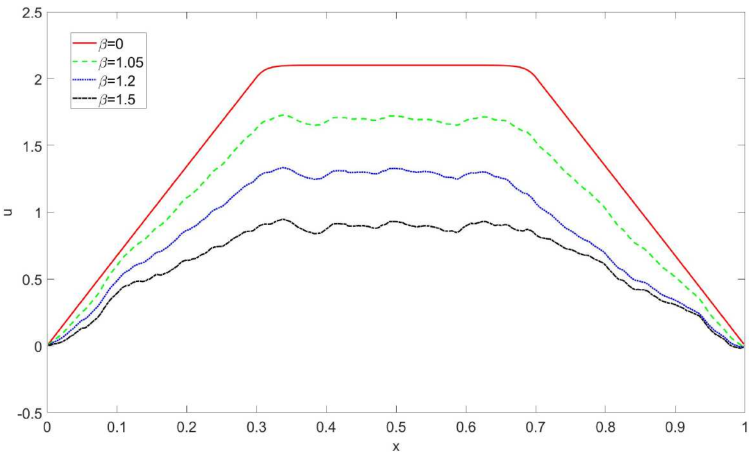

FIGURE 1. The solution u with different β.

Example 6.5. To verify the feasibility of our numerical scheme, we try to use the scheme (4.2) to capture the transition of the particle's density function. Here we take $D = (0, 1)$, $\alpha = 0.8$ and

$$u(x, 0) = 0.01x + 2, \quad f(x, u) = \chi_{(0.3,0.7)}(x)(\max\{-u, \cos(u\pi/0.2)\}).$$

It's easy to verify that the solution $u(x, t)$ satisfies

$$u(x, t) \approx 0.4k + 0.1 \qquad x \in (0.3, 0.7),$$

where k is an integer. We take $T = 0.5$, $\tau = T/M = T/4096$, $h = 1/512$ and $H_1 = 0.4$, $H_2 = 0.5$ to simulate the solution of equation (1.1) with different β. The corresponding numerical solutions $u_h^M(x)$ are shown in Figure 1. It's easy to find that the larger β is, the smaller the solution $u_h^M(x)$ $(x \in (0.3, 0.7))$ will be, which means that the greater the noise intensity, the greater the impact on the solution.

7. Conclusions

Anomalous diffusions are ubiquitous in the nature world. The Brownian motion subordinated by inverse α-stable Lévy process can effectively model the subdiffusion. In this paper, we introduce its Fokker–Planck equation with nonlinear source term and external fractional noise in one- and two-dimensional domains, and put all our efforts on the numerical methods of the equation. That is, we approximate the stochastic nonlinear fractional diffusion equation driven by the fractional Brownian sheet noise. After providing the regularity of the solution and regularizing the rough noise by Wong–Zakai approximation, we build the fully discrete scheme by backward Euler convolution quadrature and finite element methods. Moreover, the complete error analyses are also developed. Finally, the numerical experiments validate the effectiveness of the designed algorithm.

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