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ON INTERPOLATION SPACES OF PIECEWISE POLYNOMIALS ON MIXED MESHES

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Abstract. We consider fractional Sobolev spaces H^θ, $\theta \in (0,1)$, on 2D domains and H^1-conforming discretizations by globally continuous piecewise polynomials on a mesh consisting of shape-regular triangles and quadrilaterals. We prove that the norm obtained from interpolating between the discrete space equipped with the L^2-norm on the one hand and the H^1-norm on the other hand is equivalent to the corresponding continuous interpolation Sobolev norm, and the norm-equivalence constants are independent of meshsize and polynomial degree. This characterization of the Sobolev norm is then used to show an inverse inequality between H^1 and H^θ.

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1. Introduction

Fractional Sobolev spaces arise frequently in both analysis and numerical analysis of partial differential or integral equations. As examples, we mention the classical trace space $H^{1/2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$ on the boundary of some domain Ω, which are basic function spaces in the analysis of boundary integral equations, or the more general spaces $H^\theta(\Omega)$ for $\theta \in (0,1)$, which arise, *e.g.*, in problems involving fractional diffusion processes. These spaces can be characterized as interpolation spaces between L^2 and H^1, *e.g.*,

$$H^\theta(\Omega) := [L^2(\Omega), H^1(\Omega)]_\theta := [L^2(\Omega), H^1(\Omega)]_{\theta,2},$$

where we use the definition of interpolation spaces *via* the K-method, *cf.* [7,31,33] and the details in Section 2.1 below. This characterization is especially convenient, as the so-called interpolation theorem allows one to extend mapping properties of linear operators $T : U^i \to V^i$ for $i = 0, 1$ to the case $T : U^\theta \to V^\theta$, where $U^\theta := [U^0, U^1]_\theta$ (same for V) is an interpolation space. In the numerical analysis of the problems mentioned above, in particular for Galerkin discretizations of partial differential or integral equations, discrete (*i.e.*, finite dimensional) subspaces $U_N \subset U^1$ are employed. We use the notation $U_N^\theta = (U_N, \|\cdot\|_{U^\theta})$, $\theta \in [0,1]$, to emphasize that these spaces can be equipped with different norms. A special case arises where mapping properties of a linear operator T can be derived exclusively on discrete spaces, *i.e.*, $T : U_N^i \to V_N^i$ for $i \in \{0,1\}$. Then, although one is

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ultimately interested in

$$T : U_N^\theta \to V_N^\theta,$$

the interpolation theorem only states

$$T : \left[U_N^0, U_N^1\right]_\theta \to \left[V_N^0, V_N^1\right]_\theta,$$

i.e., $\|Tu_N\|_{[V_N^0,V_N^1]_\theta} \lesssim \|u_N\|_{[U_N^0,U_N^1]_\theta}$. By definition of interpolation spaces, the left-hand side can be bounded from below by the norm of interest,

$$\|v_N\|_{V_N^\theta} \le \|v_N\|_{[V_N^0,V_N^1]_\theta} \qquad \text{for all } v_N \in V_N.$$

On the other hand, as U_N is finite dimensional, the estimate

$$\|u_N\|_{[U_N^0,U_N^1]_\theta} \le C\|u_N\|_{U_N^\theta} \qquad \text{for all } u_N \in U_N \tag{1.1}$$

is certainly true for some constant $C = C_N > 0$ depending on N. A natural question in this situation is whether C_N is in fact independent of N, or, in other words, if the norm obtained by interpolating a discrete space equipped with two norms is equivalent to the continuous interpolation norm, uniformly in the discretization parameter. From the exposition of the problem it is clear that finite element inverse estimates are an immediate application where such results are employed. We will prove certain inverse estimates in fractional order spaces in Section 3 below. In Section 1.3 we comment on various applications where results of this type are also employed.

One way to establish (1.1) with $C > 0$ independent of N is presented in [2]: Assuming that a projection $P_N : U^0 \to U_N$ is available that is simultaneously bounded in U^0 and U^1 uniformly in N, then (1.1) is valid. In the context of h-version discretizations, common quasi-interpolation operators can be used as P_N, *e.g.*, the Scott–Zhang projector [29] in the case of $U^0 = L^2(\Omega)$, $U^1 = H^1(\Omega)$, $U_N = \mathcal{S}^1(\mathcal{T}_h)$. An application of this argument to p-version (or spectral) discretizations would have to rely on the existence of projection operators that are simultaneously bounded in $L^2(\Omega)$ and $H^1(\Omega)$, uniformly in the polynomial degree. For the single-element case of tensor product elements, such operators can indeed be constructed, *cf.* [9,13]. An extension to elements not having tensor product structure or to multi-element settings is not immediate. In this connection, we also refer to the recent developments on the H^1-stability of the L^2-projection onto finite element spaces, *cf.* [5,16]. In the present work, we do not construct such a projection operator, but rely on the characterization of interpolation spaces as *trace spaces*, *cf.* Chapter 40 of [31], stating that $[U^0, U^1]_\theta$ is the space of traces at 0 of suitable Banach-space valued functions $U \in C\big([0,\infty), U^1\big) \cap C^1\big((0,\infty), U^0\big)$, in particular

$$\|U(0)\|^2_{[U^0,U^1]_\theta} \sim \int_0^\infty z^{1-2\theta}\big(\|U(z)\|^2_{U^1} + \|\partial_z U(z)\|^2_{U^0}\big)\,\mathrm{d}z.$$

Using this characterization to show (1.1) requires to construct a linear operator $\mathcal{L} : U^0 \to C\big([0,\infty], U^0\big)$ with the following 3 properties:

(i) **lifting:** $\mathcal{L}u_N(0) = u_N$,
(ii) **boundedness:** $\int_0^\infty z^{1-2\theta}\big(\|\mathcal{L}u(z)\|^2_{U^1} + \|\partial_z \mathcal{L}u(z)\|^2_{U^0}\big)\,\mathrm{d}z \lesssim \|u\|^2_{[U^0,U^1]_\theta}$,
(iii) **conformity:** $\mathcal{L} : U_N \to C([0,\infty], U_N)$.

In the single-element tensor-product case this avenue was successfully taken in [6,10,12,23]. We also refer to the exposition in [11] and to works considering stable polynomial trace lifting operators in the p and hp setting, *cf.* [4,26].

1.1. Contributions of the present work

We will take on the multi-element case of meshes $\mathcal{T}$ consisting of triangles and/or quadrilaterals, which we assume only to be admissible (*i.e.*, no hanging nodes) and shape-regular. We consider local polynomial degrees $\mathbf{p} = (p_K)_{K\in\mathcal{T}}$, and our discrete space will be the space $U_N = \mathcal{S}^{\mathbf{p},1}(\mathcal{T})$ of globally continuous, piecewise polynomial functions (possibly equipped with homogeneous Dirichlet boundary conditions, indicated by a tilde $\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$), and continuous spaces $U^0 = L^2(\Omega)$, $U^1 = H^1(\Omega)$ (possibly equipped with homogeneous Dirichlet boundary conditions $U^1 = \widetilde{H}^1(\Omega)$), respectively. The main result of this work is then the following.

Theorem 1.1. *Let $\mathcal{T}$ be a mesh of Ω that fulfills Assumption 2.2 and $\mathbf{p}$ be a degree distribution on $\mathcal{T}$ which fulfills Assumption 2.5. Then, for $\theta \in (0,1)$ there holds*

$$\left[\left(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}\right), \left(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{\widetilde{H}^1(\Omega)}\right)\right]_\theta = \left(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta}\right),$$
$$\left[\left(\mathcal{S}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}\right), \left(\mathcal{S}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{H^1(\Omega)}\right)\right]_\theta = \left(\mathcal{S}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{[L^2(\Omega),H^1(\Omega)]_\theta}\right)$$

with equivalent norms. The constants in the norm equivalences depend only on θ and the shape regularity constants of $\mathcal{T}$.

As mentioned in the beginning, interpolation spaces are defined *via* the K-method. Details are given in Section 2.1 below.

1.2. Construction of the lifting operator

The proof of Theorem 1.1 will be given in Section 5.3 below. It relies on the characterization of interpolation spaces as trace spaces as given above, and hence on the definition of an appropriate lifting operator $\mathcal{L}$ with the properties given above. We will construct the action of this lifting operator in several steps. In a first step we will construct a single-element lifting operator $\mathcal{A}$, mapping functions from the reference triangle $\widehat{T}$ to the reference tetrahedron $\widehat{T}^{3\mathrm{D}}$, *cf.* Lemma 5.2. This will be done by an ubiquitous averaging process, *cf.* [21], which goes back at least to [18]. This averaging process ensures the lifting property (i) and the boundedness property (ii) *locally.* In a second step, homogeneous Dirichlet boundary conditions on one or more edges $\widehat{\mathcal{E}}$ of $\partial\widehat{T}$ will be taken into account, leading to liftings $\mathcal{A}_{\widehat{\mathcal{E}}}$ that vanish on the associated faces of $\widehat{T}^{3\mathrm{D}}$, *cf.* Lemma 5.3. Finally, a Duffy transform will be used to transform the reference tetrahedron $\widehat{T}^{3\mathrm{D}}$ into a prism $\widehat{P} = \widehat{T}\times(0,1)$, which gives rise to the associated lifting operator $\mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}}$, *cf.* Lemma 5.4. Applying this operator elementwise, we can construct conforming liftings of "simple" discrete functions $u_{hp} \in \mathcal{S}^{\mathbf{p},1}(\mathcal{T})$, *i.e.*, functions with local support (elements, edge- or vertex-patches) which on every element of their support are copies of a certain symmetric reference function. Consequently, we can lift such simple discrete functions in a conforming way to ensure property (iii). It is obvious that $\mathcal{S}^{\mathbf{p},1}(\mathcal{T})$ can be decomposed on an algebraic level into such simple discrete functions by successively subtracting the degrees of freedom associated with vertices and edges. However, in order to combine the bounded local lifting operators into a globally bounded one, we need a decomposition which additionally is bounded in the norms of interest. To that end, we will employ a result of our recent work [22], *cf.* Lemma 4.5 below.

1.3. Applications

As already stated above, the result of Theorem 1.1 can be used to prove finite element inverse estimates on fractional order spaces, *cf.* Section 3. Such inverse estimates are widely employed in finite and boundary element analysis, *cf.* [19, 20]. Various other available results in the literature rely on our main result Theorem 1.1. We give a brief overview:

(i) In [3], inverse estimates for the classical boundary integral operators associated to the three-dimensional Laplacian are derived in the hp-setting based on the presently proved Theorem 1.1 (*cf.* [3], Cor. 3.2).

(ii) In [17], p-explicit bounds on the condition number of hp-boundary element methods are derived, with Theorem 1.1 proving crucial to obtain the needed estimates in fractional Sobolev norms.

(iii) When considering discretizations of parabolic problems, (analytic) semigroups are the natural setting. In this general theory, the regularity of the solution is governed by the regularity of the initial condition, as represented by different interpolation spaces.
Discretizations of such problems *via* the method of lines, *i.e.*, by first performing a discretization of the spatial variables, results in a semidiscrete semigroup ([32], Chapt. 9), and the corresponding interpolation spaces are between spaces of piecewise polynomial functions. Thus, when discretizing in time, studying these spaces becomes crucial to determine the speed of convergence; see, *e.g.*, [27].
This approach is used in [25], where an hp-finite element method with DG-Galerkin timestepping is derived for a parabolic equation with fractional Laplace operator in space, and exponential convergence in the number of degrees of freedom is shown. Theorem 1.1 provides the crucial regularity for the spatially discrete semigroup.

1.4. Structure of this work

In Section 2, we will introduce the functional spaces, reference elements, and discrete spaces. We will also give some results concerning the Duffy transformation (mapping between triangles and squares), as well as norm equivalences for functions extended by orthogonal projections. In Section 3 we will present a specific application of our main result. Next, in Section 4, we will collect some tools which will be needed in order to prove the main result. We will explain why and when these tools will be used. The final Section 5 is concerned with the proof of the main result. In Lemma 5.1, we will first show that interpolation spaces can be characterized as trace spaces, with corresponding estimates involving seminorms only. Next, in Lemma 5.2, we will analyze the basic lifting operator without boundary conditions. In Lemma 5.3, this operator will be equipped with boundary conditions. Then, in Lemma 5.4 we will use the Duffy transformation in order to define a lifting operator mapping to a prismatic volume element. Finally, subsection 5.3 contains two results concerning the liftings of the different parts of our decomposition result Lemma 4.5, and finally the proof of the main result.

2. Notation and preliminaries

The shorthand $a \lesssim b$ expresses $a \leq Cb$ for a constant $C > 0$ that does not depend on parameters of interest (in particular the mesh size h and the polynomial degree p). The notation $a \sim b$ is short for $a \lesssim b$ in conjunction with $b \lesssim a$.

2.1. Functional setting

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, or a Lipschitz-dissected subset of the boundary of a Lipschitz domain in $\Omega \subset \mathbb{R}^3$. Fow $\omega \subset \Omega$, a relatively open subset, the Sobolev spaces $L^2(\omega)$ and $H^1(\omega)$ are defined in the standard way, *cf.* [1,24,31]. For $\gamma \subset \partial\omega$ a relatively open subset of the boundary $\partial\omega$, we also define $\widetilde{H}^1_\gamma(\Omega)$ as functions in $H^1(\Omega)$ with vanishing trace on γ, and we use the standard notation $\widetilde{H}^1(\omega) = \widetilde{H}^1_{\partial\omega}(\omega)$. Fractional Sobolev spaces for $\theta \in (0,1)$ are defined in two ways. The first way is based on the K-method of interpolation, *cf.* [7,31,33]. If $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ are two Banach spaces with continuous embedding $X_1 \subset X_0$, define the K-functional $K^2(t,u) := \inf_{v\in X_1} \|u-v\|_0^2 + t^2\|v\|_1^2$ and the norm

$$\|u\|^2_{[X_0,X_1]_\theta} := \int_0^\infty t^{-2\theta} K^2(t,u) \frac{\mathrm{d}t}{t}\cdot$$

Then, we define the interpolation space

$$[X_0, X_1]_\theta := \big\{u \in X_0 \mid \|u\|_{[X_0,X_1]_\theta} < \infty\big\}.$$

We will use the spaces $[L^2(\omega), H^1(\omega)]_\theta$ and $[L^2(\omega), \widetilde{H}^1_\gamma(\omega)]_\theta$. We will also use seminorms defined by interpolation: if $\|\cdot\|_1^2 = \|\cdot\|_0^2 + |\cdot|_1^2$ with $|\cdot|_1$ being a seminorm, then let $k^2(t,u) := \inf_{v\in X_1} \|u-v\|_0^2 + t^2|v|_1^2$, and define

$$|u|^2_{[X_0,X_1]_\theta} := \int_0^\infty t^{-2\theta} k^2(t,u)\frac{\mathrm{d}t}{t}\cdot$$

The second way to define fractional order Sobolev spaces is by using Aronstein–Slobodeckij double integral norms. Define

$$\|u\|^2_{H^\theta(\omega)} := \|u\|^2_{L^2(\omega)} + |u|^2_{H^\theta(\omega)}, \qquad |u|^2_{H^\theta(\omega)} := \int_\omega\int_\omega \frac{|u(x)-u(y)|^2}{|x-y|^{2+2\theta}}\,\mathrm{d}x\,\mathrm{d}y,$$

and set

$$H^\theta(\omega) := \big\{u \in L^2(\omega) \mid \|u\|_{H^\theta(\omega)} < \infty\big\}.$$

Norms defined by interpolation and by double integrals are equivalent on fixed Lipschitz domains, *cf.* [24]:

Lemma 2.1. *Let $\widehat{\omega} \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\theta \in (0,1)$. Then,*

$$H^\theta(\widehat{\omega}) = \big[L^2(\widehat{\omega}), H^1(\widehat{\omega})\big]_\theta,$$

with equivalent norms.

2.2. Discrete setting

For finite sets M, we denote by $\#M$ the counting measure, *i.e.*, the number of elements in M. For geometric objects $M \subset \mathbb{R}^d$, we denote by h_M the Euclidean diameter of M, by d_M the Euclidean distance to M, and by $|M|$ the Lebesgue measure of M. We consider finite partitions (*meshes*) $\mathcal{T}$ of Ω into triangles and/or quadrilaterals K (*elements*), which we define to be open sets. We will use the reference triangle $\widehat{T}$ given by the vertices $\widehat{\mathbf{v}}_1 = (0, \frac{2}{\sqrt{3}})$, $\widehat{\mathbf{v}}_2 = (1, -\frac{1}{\sqrt{3}})$, $\widehat{\mathbf{v}}_3 = (-1, -\frac{1}{\sqrt{3}})$. The reference rectangle is given by

$$\widehat{S} := \Big\{(\xi,\eta) \mid -1 < \xi < 1, -1/\sqrt{3} < \eta < 2/\sqrt{3}\Big\}.$$

The set of all vertices of $\mathcal{T}$ is denoted by $\mathcal{V}$, and the set of all edges is denoted by $\mathcal{E}$. Additionally, we will use the set $\mathcal{V}^{\mathrm{int}}$ of inner vertices, *i.e.*, vertices not lying on the boundary of Ω (again the same definition for inner edges). Throughout the paper, we assume the following for the triangulation $\mathcal{T}$ of the 2-dimensional manifold Ω.

Assumption 2.2. (i) *For each element $K \in \mathcal{T}$, there exists $\widehat{K} \in \{\widehat{S}, \widehat{T}\}$ and an $F_K : \mathbb{R}^2 \to \mathbb{R}^d$ such that $F_K(\widehat{K}) = K$, where $d = 2$ if $\Omega \subset \mathbb{R}^2$ and $d = 3$ if Ω is a subset of the boundary of a Lipschitz domain in $\mathbb{R}^3$. We additionally assume that $F_K \in C^1(\overline{\widehat{K}})$.*

(ii) *The Gramian $G(x) := (F_K'(x))^\top F_K'(x)$ has two eigenvalues which fulfill*

$$\sup_{x\in\widehat{K}} \max\left(\frac{h_K^2}{\lambda_1(x)}, \frac{\lambda_1(x)}{h_K^2}, \frac{h_K^2}{\lambda_2(x)}, \frac{\lambda_2(x)}{h_K^2}\right) \le \gamma$$

for some fixed constant $\gamma > 0$.

(iii) *The intersection $\overline{K_1} \cap \overline{K_2}$ of two distinct elements is either empty, exactly one point, or exactly one edge. If the intersection is an edge $e = F_{K_1}(\widehat{e}_j) = F_{K_2}(\widehat{e}_k)$, then $F_{K_1}^{-1} \circ F_{K_2}|_{\widehat{e}_k} : \widehat{e}_k \to \widehat{e}_j$ is an affine bijection.*

(iv) *Each edge is either fully contained in Ω or in $\partial\Omega$.*

Remark 2.3. Assumption 2.2 allows for a large class of meshes while maintaining basic properties needed in finite element analysis. In particular, (i) states that elements are images of the reference square or triangle under smooth maps so that also triangulations of piecewise smooth surfaces are covered; (ii) states that the mesh is *locally quasi-uniform*, *i.e.*, constants in scaling arguments depend solely on the parameter γ, and (iii), (iv) and state that there are no hanging nodes.

Remark 2.4. In the remainder of this article, we will restrict, for notational convenience, to planar domains Ω. That is, we assume $\Omega \subset \mathbb{R}^2 \times \{0\}$ if appropriate, we will identify $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^2$ in the canonical way. However, all results are analogously valid for Sobolev spaces on surfaces as defined in Section 2.1.

For a vertex $V \in \mathcal{V}$ we define the vertex patch ω_V to be the (open) domain covered by all elements having V as vertex,

$$\omega_V = \text{interior}\left(\bigcup_{V\in\overline{K}} \overline{K}\right),$$

and likewise we define edge patches. We stress that we will frequently abuse this notation and use patches as collection of the elements defining them. For $p \in \mathbb{N}$ we define polynomial spaces

$$\mathcal{P}^p := \text{span}\big\{x^i y^j \mid 0 \le i,j \text{ and } i+j \le p\big\},$$
$$\mathcal{Q}^p := \text{span}\big\{x^i y^j \mid 0 \le i,j \le p\big\}.$$

We also write $\widetilde{\mathcal{P}}^p$ and $\widetilde{\mathcal{Q}}^p$ to indicate spaces of polynomials vanishing on the boundary of $\widehat{K}$. We write

$$\mathcal{P}^p(\widehat{K}) := \begin{cases} \mathcal{P}^p & \text{if } \widehat{K} = \widehat{T}, \\ \mathcal{Q}^p & \text{if } \widehat{K} = \widehat{S}. \end{cases}$$

For each element $K \in \mathcal{T}$ we choose a polynomial degree $p_K \in \mathbb{N}$ and collect them in the family $\mathbf{p} := (p_K)_{K\in\mathcal{T}}$. We define spaces of piecewise polynomials as

$$\mathcal{S}^{\mathbf{p},0}(\mathcal{T}) := \Big\{u \in L^2(\Omega) \mid u \circ F_K \in \mathcal{P}^{p_K}(\widehat{K}) \text{ for all } K \in \mathcal{T}\Big\},$$
$$\mathcal{S}^{\mathbf{p},1}(\mathcal{T}) := \mathcal{S}^{\mathbf{p},0}(\mathcal{T}) \cap H^1(\Omega),$$
$$\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}) := \mathcal{S}^{\mathbf{p},0}(\mathcal{T}) \cap \widetilde{H}^1(\Omega).$$

For subpartitions $\mathcal{T}|_\omega \subset \mathcal{T}$ of elements belonging to the patch ω, we define $\mathcal{S}^{\mathbf{p},0}(\mathcal{T}|_\omega)$, $\mathcal{S}^{\mathbf{p},1}(\mathcal{T}|_\omega)$, and $\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_\omega)$ accordingly:

$$\mathcal{S}^{\mathbf{p},0}(\mathcal{T}|_\omega) := \Big\{u \in L^2(\omega) \mid u \circ F_K \in \mathcal{P}^{p_K}(\widehat{K}) \text{ for all } K \in \omega\Big\},$$
$$\mathcal{S}^{\mathbf{p},1}(\mathcal{T}|_\omega) := \mathcal{S}^{\mathbf{p},0}(\mathcal{T}|_\omega) \cap H^1(\omega),$$
$$\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_\omega) := \mathcal{S}^{\mathbf{p},0}(\mathcal{T}|_\omega) \cap \widetilde{H}^1(\omega).$$

In addition, we will assume that the polynomial degree distributions of our meshes satisfy the following assumption.

Assumption 2.5. *If a triangle T and a quadrilateral S share an edge e, then the corresponding polynomial degrees p_T and p_S satisfy*

$$p_T \le p_S \quad \text{or} \quad 2p_S \le p_T.$$

Remark 2.6. Assumption 2.5 is somewhat artificial, and is owed to the limitations of the current analysis. The same restriction is already present in [22]. Nevertheless, this assumption covers the most important cases of meshes and polynomial distributions, namely: meshes of pure quadrilaterals or triangles with arbitrary polynomial distributions as well as arbitrary mixed meshes of triangles and quadrilaterals equipped with a constant polynomial degree.

We will switch between $\widehat{T}$ and $\widehat{S}$ using the Duffy transformation

$$T_{\mathcal{D}}:\begin{cases}\widehat{S}\to\widehat{T},\\(\xi,\eta)\mapsto\left(\frac{\frac{2}{\sqrt{3}}-\eta}{\sqrt{3}}\xi,\eta,\right)\end{cases}$$

which collapses the upper edge of $\widehat{S}$ into the vertex $\widehat{\mathbf{v}}_1$. We will make use of the notation $\widehat{T}_\varepsilon := \{(\xi,\eta)\in\widehat{T} \mid d_{\widehat{\mathbf{v}}_1}(\xi,\eta)<\varepsilon\}$. Whenever $\widehat{T}_\varepsilon$ shows up, it is assumed implicitly that ε is small enough to ensure that $\widehat{T}_\varepsilon$ has positive distance to $\widehat{\mathbf{v}}_2,\widehat{\mathbf{v}}_3$. The proof of the following lemma follows from elementary considerations (*cf.* [22], Lem. 5.5).

Lemma 2.7. *Let the Duffy transformation $T_{\mathcal{D}}$ define the Duffy operator $\mathcal{D}: u\mapsto u\circ T_{\mathcal{D}}$. Then,*

$$\mathcal{D}:\left\{u \mid d_{\widehat{\mathbf{v}}_1}^{-1/2}u\in L^2(\widehat{T})\right\}\to L^2(\widehat{S})$$

is a linear and bounded operator, i.e.,

$$\|\mathcal{D}u\|_{L^2(\widehat{S})}\lesssim\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\right\|_{L^2(\widehat{T})}.$$

If $d_{\widehat{\mathbf{v}}_1}^{-1/2}\nabla u\in L^2(\widehat{T})$, then it holds

$$\|\nabla\mathcal{D}u\|_{L^2(\widehat{S})}\lesssim\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}\nabla u\right\|_{L^2(\widehat{T})}.$$

If additionally $u\in L^\infty(\widehat{T}_\varepsilon)$, then it holds

$$\|\mathcal{D}u\|_{L^2(\widehat{S})}\lesssim\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\right\|_{L^2(\widehat{T})}\lesssim\|u\|_{L^2(\widehat{T})}+\|u\|_{L^\infty(\widehat{T}_\varepsilon)},$$

where the constant in the second estimate depends on ε. Likewise, if additionally $\nabla u\in L^\infty(\widehat{T}_\varepsilon)$, then it holds

$$\|\nabla\mathcal{D}u\|_{L^2(\widehat{S})}\lesssim\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}\nabla u\right\|_{L^2(\widehat{T})}\lesssim\|\nabla u\|_{L^2(\widehat{T})}+\|\nabla u\|_{L^\infty(\widehat{T}_\varepsilon)},$$

where the constant in the second estimate depends on ε.

Proof. The fact that the linear operator $\mathcal{D}$ is bounded follows by substitution, taking into account that the Jacobian matrix $dT_{\mathcal{D}}$ is

$$\begin{pmatrix}\frac{\frac{2}{\sqrt{3}}-\eta}{\sqrt{3}} & -\frac{\xi}{\sqrt{3}}\\ 0 & 1\end{pmatrix},$$

whose determinant is $\frac{\frac{2}{\sqrt{3}}-\eta}{\sqrt{3}}$, which is equivalent to $d_{\widehat{\mathbf{v}}_1}$ on $\widehat{T}$. The second estimate follows in the same way, as $\nabla(u\circ T_{\mathcal{D}})=dT_{\mathcal{D}}^\top\nabla u\circ T_{\mathcal{D}}$ and $dT_{\mathcal{D}}$ is bounded uniformly on $\widehat{T}$. Finally, we note

$$\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\right\|_{L^2(\widehat{T})}^2=\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\right\|_{L^2(\widehat{T}\setminus\widehat{T}_\varepsilon)}^2+\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\right\|_{L^2(\widehat{T}_\varepsilon)}^2.$$

Now, $\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\|_{L^2(\widehat{T}\setminus\widehat{T}_\varepsilon)}\leq\varepsilon^{-1/2}\|u\|_{L^2(\widehat{T}\setminus\widehat{T}_\varepsilon)}$, as well as

$$\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}u\right\|_{L^2(\widehat{T}_\varepsilon)}\leq\|u\|_{L^\infty(\widehat{T}_\varepsilon)}\left\|d_{\widehat{\mathbf{v}}_1}^{-1/2}\right\|_{L^2(\widehat{T}_\varepsilon)}\lesssim\|u\|_{L^\infty(\widehat{T}_\varepsilon)},$$

where we use polar coordinates in the last estimate. ∎

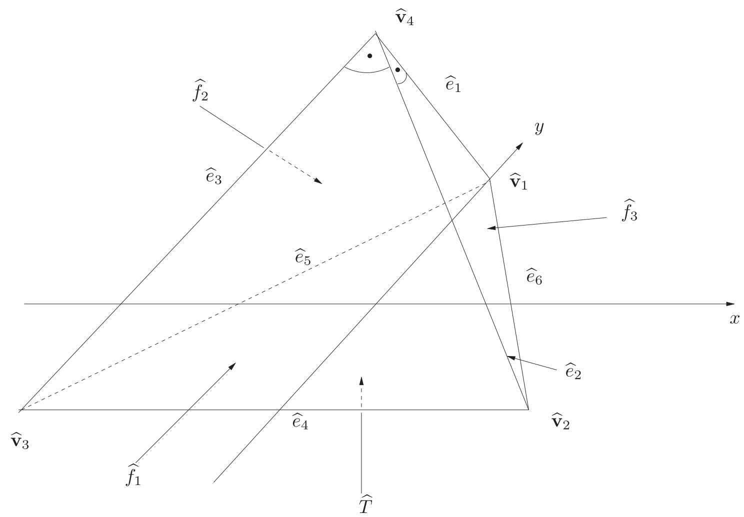

FIGURE 1. Reference tetrahedron $\widehat{T}^{3\mathrm{D}}$. The top vertex $\widehat{\mathbf{v}}_4$ is a right-angled corner.

We naturally embed $\mathbb{R}^2$ into $\mathbb{R}^3$ and identify $\widehat{T}$, $\widehat{S}$, $\widehat{\mathbf{v}}_1$, $\widehat{\mathbf{v}}_2$, $\widehat{\mathbf{v}}_3$ as objects in $\mathbb{R}^3$. We fix a reference tetrahedron $\widehat{T}^{3\mathrm{D}}$ with top vertex $\widehat{\mathbf{v}}_4 = (0,0,1)$ and bottom face $\widehat{T}$, *cf.* Figure 1. The lateral edge connecting $\widehat{\mathbf{v}}_j$ and $\widehat{\mathbf{v}}_4$ is called $\widehat{e}_j$, and the face opposite to $\widehat{\mathbf{v}}_j$ is called $\widehat{f}_j$. The edge that $\widehat{f}_j$ shares with $\widehat{T}$ is called $\widehat{e}_{3+j}$. For $\varepsilon > 0$, we denote $\widehat{T}^{3\mathrm{D}}_\varepsilon := \{(\mathbf{x}, z) \in \widehat{T}^{3\mathrm{D}} \mid z > \varepsilon\}$.

Let $\Pi_{\widehat{e}_k}, \Pi_{\widehat{f}_k} : \mathbb{R}^3 \to \mathbb{R}^3$ be the affine functions calculating the orthogonal projections onto the line spanned by $\widehat{e}_k$ and the plane spanned by $\widehat{f}_k$, respectively. Note that the lateral edges $\widehat{e}_j$, $j \in \{1,2,3\}$, intersect in $\widehat{\mathbf{v}}_4$ at right angles, which implies

$$\Pi_{\widehat{e}_j}|_{\widehat{f}_j} = \widehat{\mathbf{v}}_4 \qquad \text{for } j \in \{1,2,3\}. \tag{2.1}$$

Furthermore, for j, $k \in \{1,2,3\}$, $k \neq j$ and $\widehat{e}_{\ell_{j,k}}$ being the lateral edge shared by $\widehat{f}_j$ and $\widehat{f}_k$,

$$\Pi_{\widehat{f}_k}|_{\widehat{f}_j} = \Pi_{\widehat{e}_{\ell_{j,k}}}|_{\widehat{f}_j}. \tag{2.2}$$

The next result collects statements that will be needed later on.

Lemma 2.8. *For $j,k \in \{1,2,3\}$ and $j \neq k$ there holds for $\alpha, \beta \in \mathbb{R}$ with $-1/2 < \beta$,*

$$\left\| d^\alpha_{\widehat{\mathbf{v}}_j} d^\beta_{\widehat{T}} v \circ \Pi_{\widehat{e}_j} \right\|_{L^2(\widehat{T}^{3\mathrm{D}})} + \left\| d^\alpha_{\widehat{\mathbf{v}}_j} d^{1/2+\beta}_{\widehat{T}} v \circ \Pi_{\widehat{e}_j} \right\|_{L^2(\widehat{f}_k)} \sim \left\| d^{1+\alpha+\beta}_{\widehat{\mathbf{v}}_j} v \right\|_{L^2(\widehat{e}_j)}, \tag{2.3}$$

$$\left\| d^\alpha_{\widehat{e}_{3+k}} d^\beta_{\widehat{T}} v \circ \Pi_{\widehat{f}_k} \right\|_{L^2(\widehat{T}^{3\mathrm{D}})} \sim \left\| d^{1/2+\alpha+\beta}_{\widehat{e}_{3+k}} v \right\|_{L^2(\widehat{f}_k)}, \tag{2.4}$$

where we recall that $\widehat{e}_{3+k}$ is the edge that $\widehat{f}_k$ shares with $\widehat{T}$. Furthermore, we have for $k \in \{1,2,3\}$ and $-1/2 < \beta$ the equivalence

$$\left\| d^\beta_{\widehat{T}} v \circ \Pi_{\widehat{e}_k} \right\|_{L^2(\widehat{f}_k)} \sim |v(\widehat{\mathbf{v}}_4)|. \tag{2.5}$$

Proof. We may as well rotate and scale the setting to be able to use the standard Cartesian coordinate system and the orthogonal projection Π_x onto the x-axis to show

$$\begin{aligned}\left\|d^{\alpha}_{\widehat{\mathbf{v}}_j} d^{\beta}_{\widehat{T}} v\circ \Pi_{\widehat{e}_j}\right\|^2_{L^2(\widehat{T}^{3\mathrm{D}})} &\sim \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(1-x)^{2\alpha}(1-x-y-z)^{2\beta}\big(v\circ\Pi_x(x,y,z)\big)^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{x=0}^{1}(1-x)^{2\alpha}v(x,0,0)^2\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(1-x-y-z)^{2\beta}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{(2\beta+1)(2\beta+2)}\int_{x=0}^{1}v(x,0,0)^2(1-x)^{2\alpha+2\beta+2}\,\mathrm{d}x \sim \left\|d^{1+\alpha+\beta}_{\widehat{\mathbf{v}}_j}v\right\|^2_{L^2(\widehat{e}_j)}.\end{aligned}$$

The remaining estimates in (2.3) and (2.4) follow in a similar way. Equation (2.5) follows from the observation (2.1). ☐

3. Applications: Finite element inverse estimates

Inverse estimates are a common tool in the analysis of numerical methods. They allow for control of Sobolev norms of discrete functions via weighted versions of weaker norms. If the norms involved are of integer order (including dual ones), such estimates are usually proven locally using norm equivalence on finite dimensional spaces. Consequently, arguments are more involved in the case of fractional order norms, as those are non-local. A widely used approach here is to characterize fractional order Sobolev spaces as interpolation spaces, *cf.* Section 2.1. In this case it is necessary to know that the interpolation norm obtained from the discrete spaces is equivalent to the norm obtained by interpolating the continuous spaces. In the case of hp-finite element spaces, our main Theorem 1.1 applies.

For a positive, measurable function g on Ω we define $L^2(\Omega; g)$ to be the space of functions with finite norm $\|f\|^2_{L^2(\Omega;g)} := \int_\Omega f(x)^2 g(x)\,\mathrm{d}x$. The next result can be found in Lemma 23.1 of [31].

Proposition 3.1. *Let g_0, g_1 be positive, measurable functions on Ω. For $\theta \in (0,1)$ it holds that*

$$\big[L^2(\Omega; g_0), L^2(\Omega; g_1)\big]_\theta = L^2(\Omega; g_0^{1-\theta} g_1^{\theta})$$

with equivalent norms and equivalence constants depending only on θ.

For a mesh $\mathcal{T}$ on Ω and a space $\mathcal{S}^{\mathbf{p},1}(\mathcal{T})$, we define the mesh size function $h \in L^\infty(\Omega)$ and the polynomial degree function $p \in L^\infty(\Omega)$ elementwise by $h|_K = h_K$ and $p|_K = \mathbf{p}_K$.

Lemma 3.2. *For $\theta \in [0,1]$ there exists a constant $C > 0$ such that for any polynomial degree distribution $\mathbf{p}$ it holds*

$$\left\|h^{1-\theta} p^{-2(1-\theta)} \nabla u_h\right\|_{L^2(\Omega)} \le C\|u_h\|_{[L^2(\Omega), H^1(\Omega)]_\theta} \quad \text{for all} \quad u_h \in \mathcal{S}^{\mathbf{p},1}(\mathcal{T}).$$

Furthermore, this result is also valid if we use the discrete space $\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$ and the norm $\|\cdot\|_{[L^2(\Omega), \widetilde{H}^1(\Omega)]_\theta}$.

Proof. Obviously, $\|\nabla w_h\|_{L^2(\Omega)} \le \|w_h\|_{H^1(\Omega)}$ for all $w_h \in \mathcal{S}^{\mathbf{p},1}(\mathcal{T})$. On the other hand,

$$\|hp^{-2}\nabla w_h\|_{L^2(\Omega)} \le C\|w_h\|_{L^2(\Omega)}, \tag{3.1}$$

which follows from simple scaling arguments, combined with a p-explicit inverse estimate on the reference element (*cf.* [28], Thm. 4.76). Hence,

$$
\begin{aligned}
\left\| h^{1-\theta} p^{-2(1-\theta)} \nabla u_h \right\|^2_{L^2(\Omega)} &= \sum_{j=1}^{2} \left\| h^{1-\theta} p^{-2(1-\theta)} \partial_j u_h \right\|^2_{L^2(\Omega)} \\
&\overset{\text{Prop. 3.1}}{\lesssim} \sum_{j=1}^{2} \int_0^\infty t^{-2\theta-1} \inf_{v \in L^2(\Omega)} \|hp^{-2}(\partial_j u_h - v)\|^2_{L^2(\Omega)} + t^2 \|v\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&\leq \sum_{j=1}^{2} \int_0^\infty t^{-2\theta-1} \inf_{v_h \in \mathcal{S}^{\mathbf{p},1}(\mathcal{T})} \|hp^{-2}\partial_j(u_h - v_h)\|^2_{L^2(\Omega)} + t^2 \|\partial_j v_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&\overset{(3.1)}{\leq} \int_0^\infty t^{-2\theta-1} \inf_{v_h \in \mathcal{S}^{\mathbf{p},1}(\mathcal{T})} \|u_h - v_h\|^2_{L^2(\Omega)} + t^2 \|v_h\|^2_{H^1(\Omega)} \,\mathrm{d}t.
\end{aligned}
$$

On the right-hand side, we have the norm of u_h in the space $\left[(\mathcal{S}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}), (\mathcal{S}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{H^1(\Omega)})\right]_\theta$, which is equivalent to $\|u_h\|_{[L^2(\Omega),H^1(\Omega)]_\theta}$ according to Theorem 1.1. The last statement of the lemma clearly follows from the first one as $[L^2(\Omega), \widetilde{H}^1(\Omega)]_\theta \subset [L^2(\Omega), H^1(\Omega)]_\theta$. □

In the following Corollary 3.3, we assume a (globally) quasi-uniform mesh and constant polynomial degree distribution, that is

$$
\max_{K \in \mathcal{T}} h_K \leq C_{\mathrm{qu}} \min_{K \in \mathcal{T}} h_K
$$

for some constant $C_{\mathrm{qu}} > 0$, and $p_K = p$ for all $K \in \mathcal{T}$.

Corollary 3.3. *Let $\mathcal{T}$ be a quasi-uniform mesh and $\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$ have constant polynomial degree distribution. Let $0 \leq \theta \leq \mu \leq 1$. Then there exists a constant $C > 0$ (depending on C_{qu}) such that*

$$
\frac{h^{\mu-\theta}}{p^{2(\mu-\theta)}} \|u_h\|_{[L^2(\Omega),H^1(\Omega)]_\mu} \leq C \|u_h\|_{[L^2(\Omega),H^1(\Omega)]_\theta} \quad \text{for all } u_h \in \mathcal{S}^{\mathbf{p},1}(\mathcal{T}).
$$

Proof. According to Lemma 3.2 and obvious bounds, $h^{1-\theta} p^{-2(1-\theta)} \|u_h\|_{H^1(\Omega)} \leq C \|u_h\|_{[L^2(\Omega),H^1(\Omega)]_\theta}$. Note that the reiteration theorem ([7], Thm. 3.5.3) gives $[L^2(\Omega), H^1(\Omega)]_\mu = [H^\theta(\Omega), H^1(\Omega)]_s$ with $s = (\mu - \theta)/(1 - \theta)$. Hence, by common interpolation estimates (*cf.* [33], Thm. 1.3.3),

$$
\|u_h\|_{[L^2(\Omega),H^1(\Omega)]_\mu} \lesssim \|u_h\|^{1-s}_{[L^2(\Omega),H^1(\Omega)]_\theta} \|u_h\|^s_{H^1(\Omega)} \lesssim \|u_h\|_{[L^2(\Omega),H^1(\Omega)]_\theta} \frac{h^{\theta-\mu}}{p^{2(\theta-\mu)}}.
$$

□

4. Tools

In the present section, we collect different tools from the literature and our previous works, and explain exactly why and when each tool will be used.

4.1. Interpolation spaces

The first result of the following proposition can be found in Theorem 6 of [6]. It is a special case of our main result, Theorem 1.1, on the reference square $\widehat{S}$. As laid out in Section 1.2, we will construct lifting operators only for the reference triangle $\widehat{T}$ and rely on the first result of the following proposition in the case of quadrilateral elements. The second result is concerned with the stability of the Gauss-Lobatto interpolation operator and is a consequence of results available in [8], *cf.* also Lemma 5.6 of [22]. We will use this operator in the case of quadrilateral elements in order to halve polynomial orders in a stable way, which is a technical subtlety in the proof in our decomposition result Lemma 4.5 from [22].

Proposition 4.1. (i) *There holds for* $\theta \in (0,1)$ *and* $p \in \mathbb{N}_0$

$$\left[\left(\mathcal{Q}^p, \|\cdot\|_{L^2(\widehat{S})}\right), \left(\mathcal{Q}^p, \|\cdot\|_{H^1(\widehat{S})}\right)\right]_\theta = \left(\mathcal{Q}^p, \|\cdot\|_{[L^2(\widehat{S}),H^1(\widehat{S})]_\theta}\right) \qquad \text{(equivalent norms)},$$
$$\left[\left(\widetilde{\mathcal{Q}}^p, \|\cdot\|_{L^2(\widehat{S})}\right), \left(\widetilde{\mathcal{Q}}^p, \|\cdot\|_{\widetilde{H}^1(\widehat{S})}\right)\right]_\theta = \left(\widetilde{\mathcal{Q}}^p, \|\cdot\|_{[L^2(\widehat{S}),\widetilde{H}^1(\widehat{S})]_\theta}\right) \qquad \text{(equivalent norms)}.$$

(ii) *Let* $i_p : C(\overline{\widehat{S}}) \to \mathcal{Q}^p$ *be the tensor-product Gauß-Lobatto interpolation operator. Then for every* $\theta \in [0,1]$ *there exists* $C > 0$ *such that for all* p, $q \in \mathbb{N}_0$ *the following stability estimate holds for the operator* i_p*:*

$$\|i_p\|_{(\mathcal{Q}^p,\|\cdot\|_{H^\theta(\widehat{S})})\leftarrow(\mathcal{Q}^q,\|\cdot\|_{H^\theta(\widehat{S})})} \le C(1+q/(p+1))^{2-\theta}.$$

Due to their scaling properties, it is often preferable to work with seminorms instead of full norms. When working with interpolation spaces, the interpolation between seminorms inherits these advantageous properties as is stated in the following proposition, taken from Lemma 4.1 of [22]. This result will be used in Lemma 5.1 to show that interpolation spaces can indeed be characterized as trace spaces, with corresponding bounds for the seminorms.

Proposition 4.2 ([22], Lem. 4.1)**.** *Let* $X_1 \subseteq X_0$ *be two continuously embedded Banach spaces with norms* $\|\cdot\|_0$ *and* $\|\cdot\|_1 := H^{-1}\|\cdot\|_0 + |\cdot|_1$*, where* $|\cdot|_1$ *is a seminorm and* $H > 0$*. Introduce the following two* K*-functionals:*

$$K(u,t)^2 := \inf_{v\in X_1} \|u-v\|_0^2 + t\|v\|_1^2, \qquad k(u,t)^2 := \inf_{v\in X_1} \|u-v\|_0^2 + t|v|_1^2.$$

For $\theta \in (0,1)$ *introduce the seminorm* $|\cdot|_\theta$ *and the norms* $\|\cdot\|_\theta$ *and* $\|\cdot\|_{\tilde\theta}$ *by*

$$|u|_\theta^2 = \int_{t=0}^\infty t^{-2\theta} k^2(u,t)\frac{\mathrm{d}t}{t},$$
$$\|u\|_\theta^2 = \int_{t=0}^\infty t^{-2\theta} K^2(u,t)\frac{\mathrm{d}t}{t},$$
$$\|u\|_{\tilde\theta}^2 = H^{-2\theta}\|u\|_0^2 + |u|_\theta^2.$$

Then there exists $C > 0$*, which depends solely on* θ *(in particular, it is independent of* H*) such that*

$$C^{-1}\|u\|_\theta \le \|u\|_{\tilde\theta} \le C\|u\|_\theta.$$

As laid out in Section 1.2, we will exploit homogeneous Dirichlet boundary conditions on one or several edges of the reference triangle $\widehat{T}$ in order to make the lifting vanish on the corresponding lateral faces of the reference tetrahedron $\widehat{T}^{3\mathrm{D}}$. This procedure includes weight functions which measure the distance to the boundary, *cf.* Lemma 5.3, Step 2, and these weight functions will then naturally show up in the boundedness results of our lifting operators, *cf.* Lemma 5.3(v). At some point it will be necessary to get back to interpolation norms including spaces with essential boundary conditions, and this is the task of Lemma 4.4. It will be convenient to construct first a function realizing a quasi-optimal decomposition as required in the K-functional for the pair $L^2(\Omega)$ and $H^1(\Omega)$, with additional local stability properties.

Lemma 4.3. *Let* $\omega \subset \mathbb{R}^2$ *be a bounded Lipschitz domain,* $\theta \in (0,1)$*, and* $C > 0$*. For* $u \in [L^2(\omega), H^1(\omega)]_\theta$*, there is a function* $U : (0,\infty) \to H^1(\omega)$ *such that*

$$\int_0^\infty t^{-2\theta}\left(\|u-U(t)\|_{L^2(\omega)}^2 + t^2\|U(t)\|_{H^1(\omega)}^2\right)\frac{\mathrm{d}t}{t} \lesssim \|u\|_{[L^2(\omega),H^1(\omega)]_\theta}^2. \qquad (4.1)$$

Additionally, if $\omega' \subset \omega$ *with* $\operatorname{dist}(\omega', \partial\omega) > C$*, then*

$$\|U(t)\|_{L^2(\omega')} \lesssim \|u\|_{L^2(\{x\in\omega | d_{\omega'}(x)<t\})}. \qquad (4.2)$$

Proof. Let $\rho_t(\cdot) = t^{-2}\rho(\cdot/t)$, where ρ is a mollifier, *i.e.*, a smooth function supported in the unit ball and integrating to 1. It is known that (*cf.* [15], Sect. 2.5, Lem. 10),

$$\begin{aligned} \|\rho_t \star f\|_{H^k(S)} &\lesssim t^{\ell-k}\|f\|_{H^\ell(\{x\in\mathbb{R}^2 | d_S(x)<t\})}, \qquad k,\ell\in\{0,1\}, k\geq \ell,\\ \|f-\rho_t\star f\|_{L^2(S)} &\lesssim t\|f\|_{H^1(\{x\in\mathbb{R}^2 | d_S(x)<t\})}, \end{aligned}$$

for any measurable open set $S\subset\mathbb{R}^2$ with implied constants not depending on f or S. We will employ Stein's extension operator E (*cf.* [30], Ch. VI, Thm. 5), which extends functions from ω to $\mathbb{R}^2$. This linear operator is bounded from $L^2(\omega)$ to $L^2(\mathbb{R}^2)$ and from $H^1(\omega)$ to $H^1(\mathbb{R}^2)$, and hence from $[L^2(\omega), H^1(\omega)]_\theta$ to $[L^2(\mathbb{R}^2), H^1(\mathbb{R}^2)]_\theta$. We will show that the regularized function

$$U(t) := \chi_{[0,C/4]}(t)\cdot\rho_t\star Eu,$$

$\chi_{[a,b]}:\mathbb{R}\to\mathbb{R}$ being the characteristic function of the interval $[a,b]$, fulfills

$$\int_0^\infty t^{-2\theta}\Big(\|Eu-U(t)\|^2_{L^2(\mathbb{R}^2)} + t^2\|U(t)\|^2_{H^1(\mathbb{R}^2)}\Big)\frac{\mathrm{d}t}{t} \lesssim \|Eu\|^2_{[L^2(\mathbb{R}^2),H^1(\mathbb{R}^2)]_\theta}.$$

The boundedness properties of the extension operator E then imply (4.1), while the local boundedness properties (4.2) follow in conjunction with the properties of the convolution with ρ_t given above. For the rest of this proof, K denotes the K-functional corresponding to the pair $L^2(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$. Let $t\in(0,\infty)$ be fixed. Choose a function $v\in H^1(\mathbb{R}^2)$ such that

$$\|Eu-v\|^2_{L^2(\mathbb{R}^2)} + t^2\|v\|^2_{H^1(\mathbb{R}^2)} \leq 2K(t,Eu)^2.$$

Note that

$$\|v-\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{L^2(\mathbb{R}^2)} \lesssim \begin{cases} \|v\|_{L^2(\mathbb{R}^2)} \leq \frac{4}{C}t\|v\|_{H^1(\mathbb{R}^2)} & \text{for } t>\frac{C}{4}\\ t\|v\|_{H^1(\mathbb{R}^2)} & \text{for } t\leq \frac{C}{4}. \end{cases}$$

Then,

$$\begin{aligned} \|Eu-\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{L^2(\mathbb{R}^2)} &+ t\|\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{H^1(\mathbb{R}^2)}\\ &\lesssim \|Eu-v\|_{L^2(\mathbb{R}^2)} + \|v-\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{L^2(\mathbb{R}^2)} + t\|v\|_{H^1(\mathbb{R}^2)}\\ &\lesssim \|Eu-v\|_{L^2(\mathbb{R}^2)} + t\|v\|_{H^1(\mathbb{R}^2)}. \end{aligned}$$

Hence,

$$\begin{aligned} \|Eu-\chi_{[0,C/4]}(t)\cdot\rho_t\star Eu\|_{L^2(\mathbb{R}^2)} &\leq \|Eu-\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{L^2(\mathbb{R}^2)} + \|\chi_{[0,C/4]}(t)\cdot\rho_t\star(v-Eu)\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \|Eu-\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{L^2(\mathbb{R}^2)} + \|v-Eu\|_{L^2(\mathbb{R}^2)}\\ &\lesssim \|Eu-v\|_{L^2(\mathbb{R}^2)} + t\|v\|_{H^1(\mathbb{R}^2)} \end{aligned}$$

as well as

$$\begin{aligned} \|\chi_{[0,C/4]}(t)\cdot\rho_t\star Eu\|_{H^1(\mathbb{R}^2)} &\leq \|\chi_{[0,C/4]}(t)\cdot\rho_t\star v\|_{H^1(\mathbb{R}^2)} + \|\chi_{[0,C/4]}(t)\cdot\rho_t\star(v-Eu)\|_{H^1(\mathbb{R}^2)}\\ &\lesssim \|v\|_{H^1(\mathbb{R}^2)} + t^{-1}\|v-Eu\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Hence,

$$\|Eu-\chi_{[0,C/4]}(t)\cdot\rho_t\star Eu\|^2_{L^2(\mathbb{R}^2)} + t^2\|\chi_{[0,C/4]}(t)\cdot\rho_t\star Eu\|^2_{H^1(\mathbb{R}^2)} \lesssim K(t,Eu)^2.$$

Multiplying this last estimate by $t^{-2\theta-1}$ and integrating in t shows the result. □

The next result states certain equivalences between fractional order norms defined by interpolation and by double integrals, on spaces including partial homogeneous boundary conditions.

Lemma 4.4. *Let* $\theta \in (0,1)$.

(i) *Let* $\widehat{\mathcal{E}} \subset \{\widehat{e}_4, \widehat{e}_5, \widehat{e}_6\}$ *be a subset of the edges of* $\widehat{T}$. *Then, there holds for all* $v \in [L^2(\widehat{T}), \widetilde{H}^1_{\widehat{\mathcal{E}}}(\widehat{T})]_\theta$

$$|v|_{H^\theta(\widehat{T})} + \left\| d_{\widehat{\mathcal{E}}}^{-\theta} v \right\|_{L^2(\widehat{T})} \lesssim \|v\|_{[L^2(\widehat{T}), \widetilde{H}^1_{\widehat{\mathcal{E}}}(\widehat{T})]_\theta}.$$

(ii) *There holds for all* $v \in H^\theta(\widehat{S})$ *with* $d_{\partial\widehat{S}}^{-\theta} v \in L^2(\widehat{S})$

$$\|v\|_{[L^2(\widehat{S}), \widetilde{H}^1(\widehat{S})]_\theta} \lesssim |v|_{H^\theta(\widehat{S})} + \left\| d_{\partial\widehat{S}}^{-\theta} v \right\|_{L^2(\widehat{S})}.$$

Proof. First, note that

$$|v|_{H^\theta(\widehat{T})} \leq \|v\|_{H^\theta(\widehat{T})} \lesssim \|v\|_{[L^2(\widehat{T}), H^1(\widehat{T})]_\theta} \leq \|v\|_{[L^2(\widehat{T}), \widetilde{H}^1_{\widehat{\mathcal{E}}}(\widehat{T})]_\theta}.$$

Here, the first estimate follows by definition, the second from Lemma 2.1, and the last one from $\widetilde{H}^1_{\widehat{\mathcal{E}}}(\widehat{T}) \subset H^1(\widehat{T})$. Next, let $\widehat{e}_j \in \widehat{\mathcal{E}}$. For $v|_{\widehat{e}_j} = 0$, Hardy's inequality (*cf.* [22], Lem. 4.4(iii)), shows

$$\left\| d_{\widehat{e}_j}^{-1} v \right\|_{L^2(\widehat{T})} \lesssim \|v\|_{H^1(\widehat{T})}.$$

Interpolating this estimate with $\|v\|_{L^2(\widehat{T})} \leq \|v\|_{L^2(\widehat{T})}$ (*cf.* Prop. 3.1) shows

$$\left\| d_{\widehat{e}_j}^{-\theta} v \right\|_{L^2(\widehat{T})} \lesssim \|v\|_{[L^2(\widehat{T}), \widetilde{H}^1_{\widehat{\mathcal{E}}}(\widehat{T})]_\theta},$$

and the obvious estimate $d_{\widehat{\mathcal{E}}}^{-1} \leq \sum_{\widehat{e}_j \in \widehat{\mathcal{E}}} d_{\widehat{e}_j}^{-1}$ concludes the proof of (i). To show (ii), we extend the function v given on $\widehat{S}$ to a function v on a rectangle ω such that the boundaries of $\widehat{S}$ and ω have positive distance $\operatorname{dist}(\widehat{S}, \partial\omega) > 0$. This extension is done by mirroring symmetrically along certain edges, *cf.* Figure 2. We note that this extension procedure is linear and bounded from $L^2(\widehat{S})$ to $L^2(\omega)$, as well as from $H^1(\widehat{S})$ to $H^1(\omega)$. The interpolation theorem then gives

$$\|v\|_{[L^2(\omega), H^1(\omega)]_\theta} \lesssim \|v\|_{[L^2(\widehat{S}), H^1(\widehat{S})]_\theta}.$$

Next, we apply Lemma 4.3 with $C = \operatorname{dist}(\widehat{S}, \partial\omega)$ to obtain a function V such that

$$\int_0^\infty t^{-2\theta} \Big(\|v - V(t)\|^2_{L^2(\omega)} + t^2 \|V(t)\|^2_{H^1(\omega)} \Big) \frac{\mathrm{d}t}{t} \lesssim \|v\|^2_{[L^2(\omega), H^1(\omega)]_\theta}.$$

Note that the preceding two estimates imply

$$\int_0^\infty t^{-2\theta} \Big(\|v - V(t)\|^2_{L^2(\widehat{S})} + t^2 \|V(t)\|^2_{H^1(\widehat{S})} \Big) \frac{\mathrm{d}t}{t} \lesssim \|v\|^2_{[L^2(\widehat{S}), H^1(\widehat{S})]_\theta}. \tag{4.3}$$

We introduce, for $t > 0$, strips $S_t = \{x \in \widehat{S} | d_{\partial\widehat{S}}(x) < t\}$ and see

$$\|V(t)\|_{L^2(S_t)} \lesssim \|v\|_{L^2(\{x \in \omega | d_{S_t}(x) < t\})} \lesssim \|v\|_{L^2(S_{2t})},$$

where the first estimate follows from Lemma 4.3, and the last estimate follows from the local stability of the specific extension of v, *cf.* Figure 2. Choose a smooth cut-off function χ_t with $\|\chi_t\|_{L^\infty} \leq 1$, $\widehat{S} \cap \operatorname{supp} \chi_t \subset S_t$, $\chi_t|_{S_{t/2}} = 1$, and $\|\nabla \chi_t\|_{L^\infty(\widehat{S})} \lesssim t^{-1}$. Define $\widetilde{V}(t) := (1 - \chi_t) V(t)$, note $\widetilde{V}(t) \in \widetilde{H}^1(\widehat{S})$ as well as

$$\begin{aligned} \left\| v - \widetilde{V}(t) \right\|_{L^2(\widehat{S})} &\leq \|v - V(t)\|_{L^2(\widehat{S})} + \|V(t)\|_{L^2(S_t)} \leq \|v - V(t)\|_{L^2(\widehat{S})} + \|v\|_{L^2(S_{2t})}, \\ t \left\| \widetilde{V}(t) \right\|_{H^1(\widehat{S})} &\lesssim t \|V(t)\|_{H^1(\widehat{S})} + \|V(t)\|_{L^2(S_t)} \leq t \|V(t)\|_{H^1(\widehat{S})} + \|v\|_{L^2(S_{2t})}. \end{aligned} \tag{4.4}$$

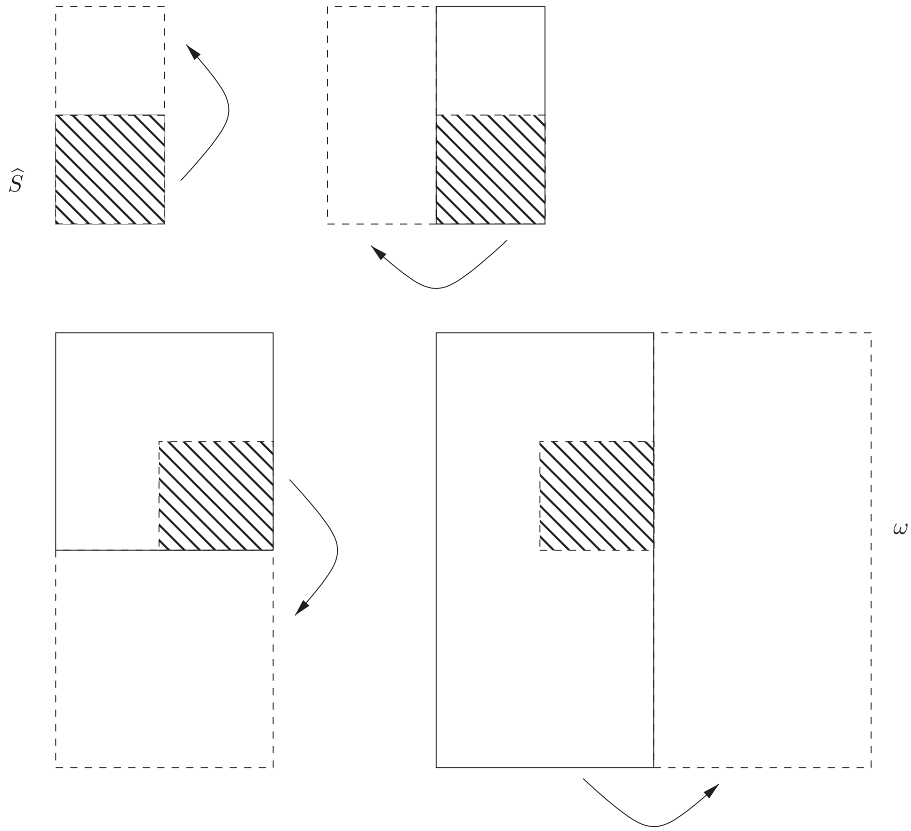

FIGURE 2. Extension procedure in Lemma 4.4 for the function v from $\widehat{S}$ to ω.

The definition of the interpolation norm, the preceding estimates, and (4.3) show

$$\begin{aligned}\|v\|^2_{[L^2(\widehat{S}),\widetilde{H}^1(\widehat{S})]_\theta} &\le \int_0^\infty t^{-2\theta}\left(\left\|v-\widetilde{V}(t)\right\|^2_{L^2(\widehat{S})}+t^2\left\|\widetilde{V}(t)\right\|^2_{H^1(\widehat{S})}\right)\frac{\mathrm{d}t}{t}\\ &\le \int_0^\infty t^{-2\theta}\left(\|v-V(t)\|^2_{L^2(\widehat{S})}+t^2\|V(t)\|^2_{H^1(\widehat{S})}\right)\frac{\mathrm{d}t}{t}+\int_0^\infty t^{-2\theta}\|v\|^2_{L^2(S_{2t})}\frac{\mathrm{d}t}{t}\\ &\overset{(4.3)}{\lesssim} \|v\|^2_{[L^2(\widehat{S}),H^1(\widehat{S})]_\theta}+\int_0^\infty t^{-2\theta}\|v\|^2_{L^2(S_{2t})}\frac{\mathrm{d}t}{t}.\end{aligned}$$

For the last term on the right-hand, the substitution $\tau = 2t$ shows for any $\delta > 0$

$$\begin{aligned}\int_0^\infty t^{-2\theta}\|v\|^2_{L^2(S_{2t})}\frac{\mathrm{d}t}{t} &= 4^\theta\int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau)}\frac{\mathrm{d}\tau}{\tau}+4^\theta\int_\delta^\infty \tau^{-2\theta}\|v\|^2_{L^2(S_\tau)}\frac{\mathrm{d}\tau}{\tau}\\ &\le 4^\theta\int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau)}\frac{\mathrm{d}\tau}{\tau}+4^\theta\|v\|^2_{L^2(\widehat{S})}\int_\delta^\infty \tau^{-2\theta}\frac{\mathrm{d}\tau}{\tau}\\ &= 4^\theta\int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau)}\frac{\mathrm{d}\tau}{\tau}+4^\theta\|v\|^2_{L^2(\widehat{S})}\frac{\delta^{-2\theta}}{2\theta}.\end{aligned}$$

Lemma 2.1 and the trivial bound $\|v\|_{L^2(\widehat{S})} \lesssim \|d_{\partial\widehat{S}}^{-\theta} v\|_{L^2(\widehat{S})}$ then give

$$\|v\|^2_{[L^2(\widehat{S}),\widetilde{H}^1(\widehat{S})]_\theta} \lesssim |v|^2_{H^\theta(\widehat{S})} + \left\|d_{\partial\widehat{S}}^{-\theta} v\right\|^2_{L^2(\widehat{S})} + \int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau)}\frac{\mathrm{d}\tau}{\tau}$$

with a constant depending only on δ and θ. It remains to bound the last integral on the right-hand side by $\|d_{\partial\widehat{S}}^{-\theta} v\|_{L^2(\widehat{S})}$. To that end, choose $\delta > 0$ small enough so that $S_\delta \neq \widehat{S}$ and consider for $\tau \leq \delta$ the (overlapping) decomposition $S_\tau = S_\tau^{(1)} \cup S_\tau^{(2)} \cup S_\tau^{(3)} \cup S_\tau^{(4)}$ given by

$$S_\tau^{(1)} = (-1, -1+\tau) \times (-1/\sqrt{3}, 2/\sqrt{3}), \qquad S_\tau^{(2)} = (1-\tau, 1) \times (-1/\sqrt{3}, 2/\sqrt{3})$$
$$S_\tau^{(3)} = (-1, 1) \times (-1/\sqrt{3}, -1/\sqrt{3}+\tau), \qquad S_\tau^{(4)} = (-1, 1) \times (2/\sqrt{3}-\tau, 2/\sqrt{3})$$

and note that

$$\int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau)}\frac{\mathrm{d}\tau}{\tau} \leq \sum_{j=1}^4 \int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau^{(j)})}\frac{\mathrm{d}\tau}{\tau}.$$

We will bound only the right-hand side term for $j = 1$ since the three remaining terms can be bounded similarly. Using Fubini, we see

$$\begin{aligned}\int_0^\delta \tau^{-2\theta}\|v\|^2_{L^2(S_\tau^{(1)})}\frac{\mathrm{d}\tau}{\tau} &= \int_0^\delta \tau^{-2\theta-1}\int_0^\tau\int_{-1/\sqrt{3}}^{2/\sqrt{3}} |v(-1+x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \int_0^\delta\int_x^\delta \tau^{-2\theta-1}\int_{-1/\sqrt{3}}^{2/\sqrt{3}} |v(-1+x,y)|^2\,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}x \lesssim \int_0^\delta x^{-2\theta}\int_{-1/\sqrt{3}}^{2/\sqrt{3}} |v(-1+x,y)|^2\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

We conclude the proof by noting that for $(x, y) \in S_\delta$ there holds $x^{-\theta} \leq \mathrm{dist}_{\partial\widehat{S}}^{-\theta}(x, y)$. □

4.2. Decomposition of FEM spaces

The following Lemma 4.5 summarizes the main results of [22]. It establishes that $\mathcal{S}^{\mathbf{p},1}(\mathcal{T})$ (possibly constrained by Dirichlet boundary conditions) can be decomposed as indicated in Section 1.2, *i.e.*, into $\mathcal{S}^{\mathbf{p},1}(\mathcal{T})$-functions supported on vertex and edges patches and on every element of their support, these functions are copies of a certain "symmetric" reference function. These reference functions themselves are polynomials on the reference triangle $\widehat{T}$ with corresponding boundary conditions and symmetry properties. The corresponding spaces are defined in (4.6) below. The operators that copy these functions onto the corresponding patches (*push-forward* operators) are denoted by T_ω, with ω denoting the corresponding patch.

Lemma 4.5. *Let $\mathcal{T}$ be a mesh of Ω that fulfills Assumption 2.2 and $\mathbf{p} = (p_K)_{K\in\mathcal{T}}$ be a degree distribution on $\mathcal{T}$ that fulfills Assumption 2.5. For all vertices $V \in \mathcal{V}^{\mathrm{int}}$ let*

$$q_V := \min_{K'\in\omega_V} p_{K'}.$$

For all edges $e \in \mathcal{E}^{\mathrm{int}}$, let $p_e = \min_{K'\in\omega_e} p_{K'}$

$$q_e := \begin{cases} p_e & \text{if there exists a triangle } K \in \omega_e \text{ with } p_K = p_e, \\ 2p_e & \text{if there does not exist a triangle } K \in \omega_e \text{ with } p_K = p_e. \end{cases} \tag{4.5}$$

Set

$$\begin{aligned} X_{\omega_V} &:= \{\widetilde{u} \in \mathcal{P}^{q_V}(\widehat{T}) \mid \widetilde{u}|_{\widehat{e}_6} = 0, \widetilde{u} \text{ symmetric w.r.t. the line passing through } \widehat{\mathbf{v}}_3 \text{ and the origin}\}, \\ X_{\omega_e} &:= \{\widetilde{u} \in \mathcal{P}^{q_e}(\widehat{T}) \mid \widetilde{u}|_{\widehat{e}_5\cup\widehat{e}_6} = 0\}. \end{aligned} \tag{4.6}$$

Then, for all patches $\omega \in \{\omega_V \mid V \in \mathcal{V}^{\rm int}\} \cup \{\omega_e \mid e \in \mathcal{E}^{\rm int}\}$ *there exist push-forward operators* $T_\omega : X_\omega \to \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_\omega)$ *such that for any* $\varepsilon > 0$ *sufficiently small there hold the mapping properties*

$$h_\omega^{-1}\|T_\omega \widetilde{u}\|_{L^2(\omega)} \lesssim \|\widetilde{u}\|_{L^2(\widehat{T})} + \|\widetilde{u}\|_{L^\infty(\widehat{T}_\varepsilon)}, \tag{4.7a}$$

$$\|\nabla T_\omega \widetilde{u}\|_{L^2(\omega)} \lesssim \|\nabla \widetilde{u}\|_{L^2(\widehat{T})} + \|\nabla \widetilde{u}\|_{L^\infty(\widehat{T}_\varepsilon)}, \tag{4.7b}$$

for all polynomials $\widetilde{u} \in X_\omega$*, with constants depending only on* ε*. Furthermore, every function* $u \in \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$ *can be written as*

$$u = u_1 + \sum_{V\in\mathcal{V}^{\rm int}} T_{\omega_V}(\widetilde{u}_V) + \sum_{e\in\mathcal{E}^{\rm int}} T_{\omega_e}(\widetilde{u}_e) + \sum_{K\in\mathcal{T}} u_K, \tag{4.8}$$

for suitable functions $u_1 \in \widetilde{\mathcal{S}}^{1,1}(\mathcal{T})$, $\widetilde{u}_V \in X_{\omega_V}$, $\widetilde{u}_e \in X_{\omega_e}$*, and* $u_K \in \mathcal{P}^{p_K} \cap \widetilde{H}^1(K)$*, all of them depending linearly on* u*, and such that for* $\theta \in (0,1)$ *and* $\delta > 0$ *sufficiently small there holds*

$$\|u_1\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta} \lesssim \|u\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta},$$

$$\sum_{V\in\mathcal{V}^{\rm int}} h_{\omega_V}^{2-2\theta}\Big(|\widetilde{u}_V|^2_{H^\theta(\widehat{T})} + \|d_{\widehat{e}_6}^{-\theta}\widetilde{u}_V\|^2_{L^2(\widehat{T})} + \|\widetilde{u}_V\|^2_{W^{1,\infty}(\widehat{T}_\delta)}\Big) \lesssim \|u\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta},$$

$$\sum_{e\in\mathcal{E}^{\rm int}} h_{\omega_e}^{2-2\theta}\Big(|\widetilde{u}_e|^2_{H^\theta(\widehat{T})} + \|d_{\widehat{e}_5\cup\widehat{e}_6}^{-\theta}\widetilde{u}_e\|^2_{L^2(\widehat{T})} + \|\widetilde{u}_e\|^2_{W^{1,\infty}(\widehat{T}_\delta)}\Big) \lesssim \|u\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta},$$

$$\sum_{K\in\mathcal{T}} |u_K|^2_{H^\theta(K)} + \|d_{\partial K}^{-\theta} u_K\|^2_{L^2(K)} \lesssim \|u\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta}.$$

The implied constants depend only on θ *and* δ*. The result remains true if we use* $\mathcal{S}^{\mathbf{p},1}(\mathcal{T})$ *instead of* $\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$ *and* $H^1(\Omega)$ *instead of* $\widetilde{H}^1(\Omega)$*. In this case, all vertices* $V \in \mathcal{V}$ *and edges* $e \in \mathcal{E}$ *have to be taken into account.*

Proof. The operators T_ω are defined depending on the type of the underlying patch. For vertex patches $\omega = \omega_V$ and edge patches $\omega = \omega_e$ with q_e being defined by the first case in (4.5), we define

$$(T_\omega \widetilde{u})|_{K'} := \begin{cases} \widetilde{u} \circ F_{K'}^{-1} & \text{if } \widehat{K'} = \widehat{T}, \\ (\mathcal{D}\widetilde{u}) \circ F_{K'}^{-1} & \text{if } \widehat{K'} = \widehat{S}. \end{cases}$$

Here, we assume $F_{K'}(\widehat{\mathbf{v}}_3) = V$ in the case of a vertex patch ω_V. In the case of an edge patch ω_e, let V denote a node of e and assume $F_{K'}(\widehat{e}_4) = e$ and $F_{K'}(\widehat{\mathbf{v}}_2) = V$. By scaling arguments and Lemma 2.7, these operators fulfill the stipulated mapping properties (4.7). For edge patches ω_e with q_e being defined by the second case in (4.5), we define

$$(T_{\omega_e}\widetilde{u})|_{K'} := \begin{cases} \big(\mathcal{A}^{\widehat{e}_4}_{\mathbf{v}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\big) \circ F_{K'}^{-1} & \text{if } \widehat{K'} = \widehat{T}, \\ \big(i_{p_e} \circ \mathcal{D} \circ \mathcal{A}^{\widehat{e}_4}_{\mathbf{v}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\big) \circ F_{K'}^{-1} & \text{if } \widehat{K'} = \widehat{S}, \end{cases}$$

where we again assume that V denotes a node of e and $F_{K'}(\widehat{e}_4) = e$ and $F_{K'}(\widehat{\mathbf{v}}_2) = V$. The operator $\mathcal{R}$ denotes the operator restricting a function on $\widehat{S}$ to a function on $\widehat{T}$. Here, $\mathcal{A}^{\widehat{e}_4}_{\mathbf{v}_1}$ is the operator from Lemma 5.10 of [22] on the reference element $\widehat{T}$. We estimate for $\widehat{K'} = \widehat{S}$

$$\begin{aligned} \Big\| i_{p_e} \circ \mathcal{D} \circ \mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\Big\|_{L^2(\widehat{S})} &\lesssim \|\mathcal{D} \circ \mathcal{A}^{\widehat{e}_4}_{\mathbf{v}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{L^2(\widehat{S})} \\ &\lesssim \Big\| d_{\mathbf{v}_1}^{-1/2} \mathcal{A}^{\widehat{e}_4}_{\mathbf{v}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\Big\|_{L^2(\widehat{T})} \\ &\lesssim \|i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{L^2(\widehat{S})}, \end{aligned} \tag{4.9}$$

where in the three estimates we use Proposition 4.1(ii), Lemma 2.7, and Lemma 5.10(vii) of [22]. For $\widehat{K'} = \widehat{T}$, Lemma 5.10(i) of [22] shows

$$\|\mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{L^2(\widehat{T})} \lesssim \|i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{L^2(\widehat{S})}. \tag{4.10}$$

Applying the L^2 bounds of Proposition 4.1(ii) and Lemma 2.7 to the right-hand sides of (4.9) and (4.10) shows the L^2-bound on T_{ω_e}. To get the H^1-bounds, we start using Lemma 5.10(ii) of [22],

$$\left\|\nabla \mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{T})} \lesssim \|i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{H^1(\widehat{S})} + \left\|d^{-1}_{\widehat{\mathbf{v}}_2} i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})} + \left\|d^{-1}_{\widehat{\mathbf{v}}_3} i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})} \tag{4.11}$$

as well as

$$\begin{aligned} \left\|\nabla i_{p_e} \circ \mathcal{D} \circ \mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})} &\lesssim \left\|\mathcal{D} \circ \mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{H^1(\widehat{S})} \\ &\lesssim \left\|d^{-1/2}_{\widehat{\mathbf{v}}_1} \mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{T})} + \left\|d^{-1/2}_{\widehat{\mathbf{v}}_1} \nabla \mathcal{A}^{\widehat{e}_4}_{\widehat{\mathbf{v}}_1} \circ \mathcal{R} \circ i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{T})} \\ &\lesssim \|i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{H^1(\widehat{S})} + \left\|d^{-1}_{\widehat{\mathbf{v}}_2} i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})} + \left\|d^{-1}_{\widehat{\mathbf{v}}_3} i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})}. \end{aligned} \tag{4.12}$$

Note that an argument based on Hardy's inequality (*cf.* [22], Lem. 4.4(i)) shows that

$$\left\|d^{-1}_{\widehat{\mathbf{v}}_2} i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})} + \left\|d^{-1}_{\widehat{\mathbf{v}}_3} i_{p_e} \circ \mathcal{D}(\widetilde{u})\right\|_{L^2(\widehat{S})} \lesssim \|i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{H^1(\widehat{S})}.$$

Hence, the right-hand sides of (4.11) and (4.12) can be bounded by

$$\|i_{p_e} \circ \mathcal{D}(\widetilde{u})\|_{H^1(\widehat{S})} \lesssim \|\mathcal{D}(\widetilde{u})\|_{H^1(\widehat{S})} \lesssim \|\nabla \mathcal{D}(\widetilde{u})\|_{L^2(\widehat{S})}, \tag{4.13}$$

where the last estimate follows by a Poincaré inequality and the fact that $\widetilde{u}$, and consequently $\mathcal{D}\widetilde{u}$, vanish on an edge of $\widehat{S}$. To conclude the bound, we estimate the last term in (4.13) further with Lemma 2.7 to produce the H^1-bounds on T_{ω_e}.

Next, Theorem 2.6 of [22] states that

$$u = u_1 + \sum_{V \in \mathcal{V}^{\text{int}}} u_V + \sum_{e \in \mathcal{E}^{\text{int}}} u_e + \sum_{K \in \mathcal{T}} u_K,$$

where $u_1 = u - I_h u$ with I_h being the Scott–Zhang projection operator. The proofs of Theorem 2.6, Corollary 6.2, Lemma 6.1 from [22] reveal that $u_V = T_{\omega_V}(\widetilde{u}_V)$ for some function $\widetilde{u}_V \in X_{\omega_V}$ and that

$$\sum_{V \in \mathcal{V}^{\text{int}}} h^{2-2\theta}_{\omega_V} \|\widetilde{u}_V\|^2_{[L^2(\widehat{T}), \widetilde{H}^1_{\widehat{e}_6}(\widehat{T})]_\theta} + \|\widetilde{u}_V\|^2_{W^{1,\infty}(\widehat{T}_\delta)} \lesssim \|u\|^2_{[L^2(\Omega), \widetilde{H}^1(\Omega)]_\theta}.$$

The proofs of Theorem 2.6, Corollary 6.4, Lemma 6.3 from [22] reveal that $u_e = T_{\omega_e}(\widetilde{u}_e)$ for some function $\widetilde{u}_e \in X_{\omega_e}$ and

$$\sum_{e \in \mathcal{E}^{\text{int}}} h^{2-2\theta}_{\omega_e} \|\widetilde{u}_e\|^2_{[L^2(\widehat{T}), \widetilde{H}^1_{\widehat{e}_5 \cup \widehat{e}_6}(\widehat{T})]_\theta} + \|\widetilde{u}_e\|^2_{W^{1,\infty}(\widehat{T}_\delta)} \lesssim \|u\|^2_{[L^2(\Omega), \widetilde{H}^1(\Omega)]_\theta}.$$

The proof of Theorem 2.6 from [22] reveals

$$\sum_{K \in \mathcal{T}} |u_K|^2_{H^\theta(K)} + \left\|d^{-\theta}_{\partial K} u_K\right\|^2_{L^2(K)} \lesssim \|u\|^2_{[L^2(\Omega), \widetilde{H}^1(\Omega)]_\theta}.$$

Finally, Lemma 4.4(i) shows the stipulated estimates. □

5. Proofs of the main results

5.1. Interpolation spaces as trace spaces

We recall that Lemma 40.1 of [31] states the following: $u \in [X_0, X_1]_\theta$ if and only if there exists a function $v : (0, \infty) \to X_1$ with $t^{1-\theta}\|v\|_1 \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ whose derivative v' exists and satisfies $t^{1-\theta}\|v'\|_0 \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ and $\lim_{t\to 0} v(t) = v(0) = u$ (in X_0). The following lemma shows that a similar characterization can be achieved for the $|\cdot|_\theta$-seminorm.

Lemma 5.1. *Under the hypotheses of Proposition 4.2 there holds the following for* $\theta \in (0, 1)$*:*

(i) *Let* $u \in [X_0, X_1]_\theta$*. Then, there exists a function* $v : (0, \infty) \to X_1$ *with* $t^{1-\theta}|v|_1 \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ *whose derivative* v' *satisfies* $t^{1-\theta}\|v'\|_0 \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ *and* $\lim_{t\to 0} v(t) = u$ *(convergence in* X_0*). Moreover, for a* $C > 0$ *that depends solely on* θ

$$\int_{t=0}^{\infty} t^{2(1-\theta)}|v(t)|_1^2 \frac{\mathrm{d}t}{t} + \int_{t=0}^{\infty} t^{2(1-\theta)}\|v'(t)\|_0^2 \frac{\mathrm{d}t}{t} \le C|u|_\theta^2.$$

(ii) *Let* $v : (0, \infty) \to X_1$ *be such that* $t^{1-\theta}|v|_1 \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$ *and* $t^{1-\theta}\|v'\|_0 \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{t})$*. Then* $\lim_{t\to 0} v(t)$ *exists (in* X_0*) and* $v(0) \in [X_0, X_1]_\theta$*. Moreover, for a* $C > 0$ *depending solely on* θ

$$|v(0)|_\theta^2 \le \int_{t=0}^{\infty} t^{2(1-\theta)}|v(t)|_1^2 \frac{\mathrm{d}t}{t} + \int_{t=0}^{\infty} t^{2(1-\theta)}\|v'(t)\|_0^2 \frac{\mathrm{d}t}{t}.$$

Proof. We modify the arguments presented in Lemma 40.1 from [31].

Proof of (i). Due to Proposition 4.2 we know that $|u|_\theta < \infty$. For every $n \in \mathbb{Z}$ pick $v_n \in X_1$ such that

$$\|u - v_n\|_0 + e^n|v_n|_1 \le 2k(u, e^n) \tag{5.1}$$

and define on $(0, \infty)$ the function v as the piecewise linear interpolant with values v_n at the knots $t_n = e^n$. Note that for $n \to -\infty$ there holds $k(u, e^n) \to 0$, and hence also $v_n \to u$ in X_0. We conclude that $\lim_{t\to 0} v(t) = u$ in X_0. Next, in view of

$$k(u, \lambda t) \le \max\{1, \lambda\}k(u, t), \qquad \lambda > 0, \tag{5.2}$$

we have for $t \in (e^n, e^{n+1})$

$$|v(t)|_1 \le \max\{|v_n|_1, |v_{n+1}|_1\} \le 2\max\Big\{e^{-n}k(u, e^n), e^{-(n+1)}k\big(u, e^{n+1}\big)\Big\} \le 2e^{-n}k(u, e^n). \tag{5.3}$$

Therefore, by exploiting (5.2) we get

$$\int_{t=0}^{\infty} t^{2(1-\theta)}|v(t)|_1^2 \frac{\mathrm{d}t}{t} \lesssim \sum_{n\in\mathbb{Z}} e^{2(1-\theta)n}e^{-2n}k^2(u, e^n) \sim \int_{t=0}^{\infty} t^{-2\theta}k^2(u, t)\frac{\mathrm{d}t}{t} = |u|_\theta^2. \tag{5.4}$$

For v', we note that on the interval (e^n, e^{n+1}) we have $v'(t) = \frac{v_{n+1}-v_n}{e^{n+1}-e^n}$ and therefore for $t \in (e^n, e^{n+1})$

$$\|v'(t)\|_0 \le \frac{1}{e-1}e^{-n}(\|v_{n+1} - u\|_0 + \|v_n - u\|_0) \le \frac{2}{e-1}e^{-n}(k(u, e^{n+1}) + k(u, e^n)) \le \frac{2(1+e)}{e-1}e^{-n}k(u, e^n),$$

so that

$$\int_{t=0}^{\infty} t^{2(1-\theta)}\|v'(t)\|_0^2 \frac{\mathrm{d}t}{t} \lesssim \sum_{n\in\mathbb{Z}} e^{2(1-\theta)n}e^{-2n}k^2(u, e^n) \sim \int_{t=0}^{\infty} t^{-2\theta}k^2(u, t)\frac{\mathrm{d}t}{t} = |u|_\theta^2.$$

Proof of (ii). From $v(t)-v(\varepsilon)=\int_\varepsilon^t v'(s)\,\mathrm{d}s$, we infer for $t\ge\varepsilon$ the estimate

$$\|v(t)-v(\varepsilon)\|_0 \le \sqrt{\int_{s=0}^t s^{2(1-\theta)}\|v'(s)\|_0^2\,\frac{\mathrm{d}s}{s}}\sqrt{\int_\varepsilon^t s^{-1+2\theta}\,\mathrm{d}s} \le CZ(v,t)t^\theta,$$

where $Z(v,t)^2:=\int_{s=0}^t s^{2(1-\theta)}\|v'(s)\|_0^2\,\frac{\mathrm{d}s}{s}$. By assumption, $\sup_{t>0}Z(v,t)<\infty$. This shows that $\lim_{t\to 0}v(t)=: v(0)$ exists (convergence in X_0). Next, we estimate $k(v(0),t)=\inf_{w\in X_1}\|v(0)-w\|_0+t|w|_1\le\|v(0)-v(t)\|_0+t|v(t)|_1$. The observation $v(t)-v(0)=\int_{s=0}^t v'(s)\,\mathrm{d}s$ implies with Hardy's inequality

$$\int_{t=0}^\infty t^{-2\theta}\|v(t)-v(0)\|_0^2\frac{\mathrm{d}t}{t}\le\int_{t=0}^\infty t^{-2\theta+2}\left|\frac{1}{t}\int_{s=0}^t\|v'(s)\|_0\,\mathrm{d}s\right|^2\frac{\mathrm{d}t}{t}\lesssim\int_{t=0}^\infty t^{2(1-\theta)}\|v'(t)\|_0^2\,\frac{\mathrm{d}t}{t}$$

so that

$$\begin{aligned}|v(0)|_\theta^2&=\int_{t=0}^\infty t^{-2\theta}k^2(v(0),t)\frac{\mathrm{d}t}{t}\le\int_{t=0}^\infty t^{-2\theta}\|v(t)-v(0)\|_0^2+t^{2-2\theta}|v(t)|_1^2\frac{\mathrm{d}t}{t}\\&\lesssim\int_{t=0}^\infty t^{2(1-\theta)}\|v'\|_0^2+t^{2-2\theta}|v(t)|_1^2\frac{\mathrm{d}t}{t}.\end{aligned}$$

This concludes the argument. □

5.2. Liftings from a triangle to a tetrahedron and prism

As laid out in Section 1.2, we present now the lifting operator $\mathcal{A}$ from the reference triangle to the reference tetrahedron. This operator is the first building block of our overall lifting procedure.

Lemma 5.2. *There exists a linear operator $\mathcal{A}: L^1_{loc}(\widehat{T})\to C^\infty(\widehat{T}^{3\mathrm{D}})$ with the following properties:*

(i) *If u is a polynomial of degree $p\ge 0$, then $\mathcal{A}u$ is a polynomial of degree p.*
(ii) *If u is continuous at a point $\mathbf{x}\in\widehat{T}$, then $(\mathcal{A}u)(\mathbf{x},0)=u(\mathbf{x})$.*
(iii) *For every $\gamma>-1/2$ there is a constant C_γ such that*

$$\left\|d^\gamma_{\widehat{T}\times\{0\}}\mathcal{A}u\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}\le C_\gamma\|u\|_{L^2(\widehat{T})}.$$

(iv) *Let $\widehat{f}\ne\widehat{T}\times\{0\}$ be a face of $\widehat{T}^{3\mathrm{D}}$ and $\widehat{e}$ be the edge shared by $\widehat{f}$ and $\widehat{T}\times\{0\}$. Then, for every $\gamma\in\mathbb{R}$ there is a constant $C_\gamma>0$ such that*

$$\left\|d^\gamma_{\widehat{e}}\mathcal{A}u\right\|_{L^2(\widehat{f})}\le C_\gamma\left\|d^\gamma_{\widehat{e}}u\right\|_{L^2(\widehat{T})}.$$

(v) *Let $\widehat{e}$ be an edge of $\widehat{T}^{3\mathrm{D}}$ from the top $(0,0,1)$ to the vertex $\widehat{V}\ne(0,0,1)$. Then, for every $\gamma<3/2$ there is a constant $C_\gamma>0$ such that*

$$\left\|d^\gamma_{\widehat{V}}\mathcal{A}u\right\|_{L^2(\widehat{e})}\le C_\gamma\left\|d^{\gamma-1/2}_{\widehat{V}}u\right\|_{L^2(\widehat{T})}.$$

(vi) *Let $\widehat{f}\ne\widehat{T}\times\{0\}$ be a face of $\widehat{T}^{3\mathrm{D}}$ and $\widehat{e}$ be an edge of $\widehat{T}^{3\mathrm{D}}$ from the top $(0,0,1)$ to the vertex $\widehat{V}\ne(0,0,1)$. For every $\theta\in(0,1)$ and $k\in\mathbb{N}$ there is a constant $C_{\theta,k}>0$ such that for $\mathbf{k}\in\mathbb{N}_0^3$ with $|\mathbf{k}|=k\ge 1$ there holds*

$$\left\|d^{k-1/2-\theta}_{\widehat{T}\times\{0\}}\partial^{\mathbf{k}}\mathcal{A}u\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}+\left\|d^{k-\theta}_{\widehat{T}\times\{0\}}\partial^{\mathbf{k}}\mathcal{A}u\right\|_{L^2(\widehat{f})}+\left\|d^{k+1/2-\theta}_{\widehat{V}}\partial^{\mathbf{k}}\mathcal{A}u\right\|_{L^2(\widehat{e})}\le C_{\theta,k}|u|_{H^\theta(\widehat{T})}.$$

(vii) *For every* $\varepsilon > 0$ *and* $j \in \mathbb{N} \cup \{0\}$*, there is* $C_{\varepsilon,j} > 0$ *such that*

$$\|\mathcal{A}u\|_{W^{j,\infty}(\widehat{T}^{3D}_{\varepsilon})} \leq C_{\varepsilon,j}\|u\|_{L^2(\widehat{T})}.$$

(viii) *For* $\varepsilon > 0$ *consider the set* $\widehat{T}_{\varepsilon}$*, and for* $z \in [0,1]$ *the scaled versions* $(1-z)\widehat{T}_{\varepsilon}$*. Then, for arbitrary* δ *with* $\delta > \varepsilon$ *and* $\mathbf{k} \in \mathbb{N}_0^3$ *with* $|\mathbf{k}| \leq 1$ *there holds*

$$\left\|\partial^{\mathbf{k}}\mathcal{A}u(\cdot,z)\right\|_{L^\infty((1-z)\widehat{T}_{\varepsilon})} \lesssim \|u\|_{L^2(\widehat{T})} + \|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_{\delta})},$$

with a constant depending only on the difference $\delta - \varepsilon$*.*

Proof. The lifting operator $\mathcal{A}$ will be defined by an averaging process. To that end, define for $(\mathbf{x}, z) \in \widehat{T}^{3D}$ the mapping

$$F_{(\mathbf{x},z)} : \begin{cases} \widehat{T} \to \widehat{T} \\ \boldsymbol{\xi} \mapsto \mathbf{x} + \frac{z}{2}\boldsymbol{\xi}. \end{cases}$$

Fix a mollifier $\rho \in C^\infty(\mathbb{R}^2)$ with $\operatorname{supp}(\rho) \subset \widehat{T}$ and $\int_{\widehat{T}} \rho(\mathbf{y})\,\mathrm{d}\mathbf{y} = 1$ and define

$$(\mathcal{A}u)(\mathbf{x}, z) := \int_{\widehat{T}} \rho(\boldsymbol{\xi})u(F_{(\mathbf{x},z)}(\boldsymbol{\xi}))\,\mathrm{d}\boldsymbol{\xi} = \frac{4}{z^2}\int_{\mathbb{R}^2} \rho\left(\frac{\mathbf{s}-\mathbf{x}}{z/2}\right)u(\mathbf{s})\,\mathrm{d}\mathbf{s}. \tag{5.5}$$

This formula is well defined as $(\mathbf{x}, z) \in \widehat{T}^{3D}$ implies $F_{(\mathbf{x},z)}(\widehat{T}) \subset \widehat{T}$. We note that $\mathcal{A}u \in C^\infty(\widehat{T}^{3D})$. We calculate for $j = 1, 2$ (*cf.* [21], Lem. 1.4.1.4), for any constant $c \in \mathbb{R}$

$$\begin{aligned}(\partial_j\mathcal{A}u)(\mathbf{x}, z) &= \frac{2}{z}\int_{\widehat{T}} \partial_j\rho(\boldsymbol{\xi})[c - u(F_{(\mathbf{x},z)}(\boldsymbol{\xi}))]\,\mathrm{d}\boldsymbol{\xi}, \\ (\partial_3\mathcal{A}u)(\mathbf{x}, z) &= -\frac{2}{z}\int_{\widehat{T}} \rho(\boldsymbol{\xi})[c - u(F_{(\mathbf{x},z)}(\boldsymbol{\xi}))]\,\mathrm{d}\boldsymbol{\xi} - \frac{1}{z}\int_{\widehat{T}} \nabla\rho(\boldsymbol{\xi})\cdot\boldsymbol{\xi}\cdot[c - u(F_{(\mathbf{x},z)}(\boldsymbol{\xi}))]\,\mathrm{d}\boldsymbol{\xi},\end{aligned}$$

and inductively we can conclude for $|\mathbf{k}| = k \in \mathbb{N}$, $k \geq 1$, the basic estimates

$$|\mathcal{A}u(\mathbf{x}, z)| \lesssim \frac{1}{z^2}\int_{\mathbf{x}+\frac{z}{2}\widehat{T}} |u(\mathbf{s})|\,\mathrm{d}\mathbf{s}, \tag{5.6}$$

$$|\partial^{\mathbf{k}}\mathcal{A}u(\mathbf{x}, z)| \leq C_k\frac{1}{z^{2+k}}\min_{c\in\mathbb{R}}\int_{\mathbf{x}+\frac{z}{2}\widehat{T}} |c - u(\mathbf{s})|\,\mathrm{d}\mathbf{s}. \tag{5.7}$$

Proof of (i). This follows at once as $(\mathbf{x}, z) \mapsto F_{(\mathbf{x},z)}(\boldsymbol{\xi})$ is affine for fixed $\boldsymbol{\xi}$.
Proof of (ii). This follows by inspection.
Proof of (iii). The estimate (5.6) and Cauchy–Schwarz imply

$$|\mathcal{A}u(\mathbf{x}, z)|^2 \lesssim \frac{1}{z^2}\int_{\frac{z}{2}\widehat{T}} |u(\mathbf{x}+\mathbf{s})|^2\,\mathrm{d}\mathbf{s} \leq \frac{1}{z^2}\int_{z\widehat{T}} |u(\mathbf{x}+\mathbf{s})|^2\,\mathrm{d}\mathbf{s}.$$

Using Fubini, we get

$$\left\|d^{\gamma}_{\widehat{T}\times\{0\}}\mathcal{A}u\right\|^2_{L^2(\widehat{T}^{3D})} \lesssim \int_{z=0}^1 \frac{1}{z^{2-2\gamma}}\int_{z\widehat{T}}\int_{(1-z)\widehat{T}} |u(\mathbf{x}+\mathbf{s})|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{s}\,\mathrm{d}z.$$

As $\widehat{T}$ is convex, $\mathbf{x} \in (1-z)\widehat{T}$ and $\mathbf{s} \in z\widehat{T}$ imply $\mathbf{x}+\mathbf{s} \in \widehat{T}$, and we conclude

$$\left\|d^{\gamma}_{\widehat{T}\times\{0\}}\mathcal{A}u\right\|^2_{L^2(\widehat{T}^{3D})} \lesssim \|u\|^2_{L^2(\widehat{T})}\int_{z=0}^1 \frac{1}{z^{2-2\gamma}}\int_{z\widehat{T}} \mathrm{d}\mathbf{s}\,\mathrm{d}z \leq C_\gamma\|u\|^2_{L^2(\widehat{T})}.$$

Proof of (iv). Let $T' = \{\mathbf{x} \in \widehat{T} \mid \exists z \text{ such that } (\mathbf{x}, z) \in \widehat{f}\}$. Consider z as a function of $\mathbf{x}$. Note that $\mathbf{x} \mapsto z(\mathbf{x})$ is affine and has the form $z(u,v) = 1 + uz_x + vz_y$ for some $z_x, z_y \in \mathbb{R}$. For $\mathbf{x} \in T'$, it holds $d_{\widehat{e}}(\mathbf{x}) \sim z(\mathbf{x})$ and hence also

$$d_{\widehat{e}}(\mathbf{s}) \sim z(\mathbf{x}) \quad \text{ for all } \mathbf{s} \in \mathbf{x} + \frac{z(\mathbf{x})}{2}\widehat{T}.$$

We conclude

$$\begin{aligned}\left\|d_{\widehat{e}}^{\gamma}\mathcal{A}u\right\|_{L^2(\widehat{f})}^2 &\sim \int_{T'} z(\mathbf{x})^{2\gamma}|\mathcal{A}u(\mathbf{x}, z(\mathbf{x}))|^2 \,\mathrm{d}\mathbf{x} \lesssim \int_{T'} z(\mathbf{x})^{2\gamma-2}\int_{\mathbf{x}+\frac{z(\mathbf{x})}{2}\widehat{T}}|u(\mathbf{s})|^2 \,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{x} \\ &\lesssim \int_{T'} z(\mathbf{x})^{-2}\int_{\mathbf{x}+\frac{z(\mathbf{x})}{2}\widehat{T}} d_{\widehat{e}}(\mathbf{s})^{2\gamma}|u(\mathbf{s})|^2 \,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{x} \lesssim \int_{T'}\int_{\widehat{T}} f(\mathbf{x}+z(\mathbf{x})\boldsymbol{\xi}/2)\,\mathrm{d}\boldsymbol{\xi}\,\mathrm{d}\mathbf{x},\end{aligned}$$

where we wrote $f(\mathbf{s}) := d_{\widehat{e}}(\mathbf{s})^{2\gamma}|u(\mathbf{s})|^2$ and used the substitution $\mathbf{s} = \mathbf{x} + z(\mathbf{x})\boldsymbol{\xi}/2$. We apply Fubini and the substitution $\mathbf{x}' = \mathbf{x} + z(\mathbf{x})\boldsymbol{\xi}/2$. Note that for $\mathbf{x} \in T'$ and $\boldsymbol{\xi} \in \widehat{T}$ it holds $\mathbf{x}' \in \widehat{T}$, so that

$$\left\|d_{\widehat{e}}^{\gamma}\mathcal{A}u\right\|_{L^2(\widehat{f})}^2 \lesssim \int_{\widehat{T}}\int_{T'} f(\mathbf{x}+z(\mathbf{x})\boldsymbol{\xi}/2)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\boldsymbol{\xi} \lesssim \int_{\widehat{T}}\int_{\widehat{T}} \frac{f(\mathbf{x}')}{|1+\xi_1 z_x/2+\xi_2 z_y/2|}\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \lesssim \|d_{\widehat{e}}^{\gamma}u\|_{L^2(\widehat{T})}^2.$$

The last estimate follows from the fact that $|1+\xi_1 z_x/2+\xi_2 z_y/2| = z(\boldsymbol{\xi}/2)$, which is bounded from below away from zero uniformly in $\boldsymbol{\xi} \in \widehat{T}$.

Proof of (v). Parametrize $\widehat{e}$ by $z \in (0,1) \mapsto (\mathbf{x}(z), z) = (\mathbf{x}, z)$ and note $d_{\widehat{V}}(\mathbf{x}, z) \sim z$ on $\widehat{e}$. Furthermore, there is a constant $\alpha > 0$ such that, extending u by zero outside $\widehat{T}$,

$$|\mathcal{A}u(\mathbf{x}, z)| \lesssim \frac{1}{z^2}\int_{B_z(0)} |u(\widehat{V}+\alpha\mathbf{s})|\,\mathrm{d}\mathbf{s} = \frac{1}{z^2}\int_0^z\int_0^{2\pi} |u(\widehat{V}+\alpha(r\cos\phi, r\sin\phi))|r\,\mathrm{d}\phi\,\mathrm{d}r.$$

The weighted Hardy inequality ([34], Thm. I.9.16) for $2\gamma - 2 < 1$ and Hölder show

$$\begin{aligned}\left\|d_{\widehat{V}}^{\gamma}\mathcal{A}u\right\|_{L^2(\widehat{e})}^2 &\lesssim \int_0^1 z^{2\gamma-2}\left(z^{-1}\int_0^z\int_0^{2\pi} |u(\widehat{V}+\alpha(r\cos\phi, r\sin\phi))|r\,\mathrm{d}\phi\,\mathrm{d}r\right)^2 \mathrm{d}z \\ &\lesssim \int_0^1 z^{2\gamma}\int_0^{2\pi} |u(\widehat{V}+\alpha(z\cos\phi, z\sin\phi))|^2\,\mathrm{d}\phi\,\mathrm{d}z \lesssim \|d_{\widehat{V}}^{\gamma-1/2}u\|_{L^2(\widehat{T})}^2.\end{aligned}$$

Proof of (vi). We follow Lemma 1.4.1.4 of [21] and note that by (5.7) and Cauchy–Schwarz

$$\begin{aligned}\left\|d_{\widehat{T}\times\{0\}}^{k-1/2-\theta}\partial^{\mathbf{k}}\mathcal{A}u\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}^2 &\lesssim \int_{\widehat{T}^{3\mathrm{D}}} z^{-3-2\theta}\int_{\mathbf{x}+\frac{z}{2}\widehat{T}} |u(\mathbf{x})-u(\mathbf{s})|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}(\mathbf{x}, z) \\ &\lesssim \int_{\widehat{T}}\int_{\widehat{T}} |u(\mathbf{x})-u(\mathbf{s})|^2\int_{|\mathbf{x}-\mathbf{s}|/2}^{\infty} z^{-3-2\theta}\,\mathrm{d}z\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{s} \lesssim |u|_{H^\theta(\widehat{T})}^2.\end{aligned}$$

To treat the second term on the left-hand side of (vi), we use (5.7) and the notation introduced in the proof of (iv). With Cauchy–Schwarz we calculate

$$\begin{aligned}\left\|d_{\widehat{T}\times\{0\}}^{k-\theta}\partial^{\mathbf{k}}\mathcal{A}u\right\|_{L^2(\widehat{f})}^2 &\sim \int_{T'} z(\mathbf{x})^{-2-2\theta}\int_{\mathbf{x}+\frac{z(\mathbf{x})}{2}\widehat{T}} |u(\mathbf{x})-u(\mathbf{s})|^2\,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{x} \\ &\lesssim \int_{T'}\int_{\mathbf{x}+\frac{z(\mathbf{x})}{2}\widehat{T}} \frac{|u(\mathbf{x})-u(\mathbf{s})|^2}{|\mathbf{x}-\mathbf{s}|^{2+2\theta}}\,\mathrm{d}\mathbf{s}\,\mathrm{d}\mathbf{x} \lesssim |u|_{H^\theta(\widehat{T})}^2.\end{aligned}$$

To treat the third term on the left-hand side of (vi), suppose that $\widehat{e}$ is an edge of the lateral face $\widehat{f}$. From the one-dimensional trace inequality

$$x|v(x,0)|^2 \lesssim \int_0^x |v(x,y)|^2 \,\mathrm{d}y + x^2 \int_0^x |\partial_y v(x,s)|^2 \,\mathrm{d}s$$

we conclude the trace inequality

$$\left\| d_{\widehat{V}}^{k+1/2-\theta} \partial^{\mathbf{k}} \mathcal{A}u \right\|_{L^2(\widehat{e})}^2 \lesssim \left\| d_{\widehat{T}\times\{0\}}^{k-\theta} \partial^{\mathbf{k}} \mathcal{A}u \right\|_{L^2(\widehat{f})}^2 + \sum_{|\mathbf{k}'|=k+1} \left\| d_{\widehat{T}\times\{0\}}^{(k+1)-\theta} \partial^{\mathbf{k}'} \mathcal{A}u \right\|_{L^2(\widehat{f})}^2,$$

and the result follows using the estimate for the second term on the left.

Proof of (vii). For $j=0$ this follows immediately from formula (5.6), and for $j \geq 1$ from formula (5.7).

Proof of (viii). The formulas (5.6) and (5.7) and Hölder's inequality show that

$$|\partial^{\mathbf{k}} \mathcal{A}u(\mathbf{x},z)| \lesssim z^{-1-|\mathbf{k}|} \|u\|_{L^2(\mathbf{x}+\frac{z}{2}\widehat{T})}. \tag{5.8}$$

Using the Bramble–Hilbert lemma (Lemma 4.3.8 of [14]), and the Hölder inequality we can even conclude from formula (5.7)

$$|\nabla \mathcal{A}u(\mathbf{x},z)| \lesssim z^{-3} \min_{c\in\mathbb{R}} \int_{\mathbf{x}+\frac{z}{2}\widehat{T}} |c-u(\mathbf{s})| \,\mathrm{d}\mathbf{s} \lesssim z^{-2} \int_{\mathbf{x}+\frac{z}{2}\widehat{T}} |\nabla u(\mathbf{s})| \,\mathrm{d}\mathbf{s} \lesssim z^{-1} \|\nabla u\|_{L^2(\mathbf{x}+\frac{z}{2}\widehat{T})}. \tag{5.9}$$

For z sufficiently small (depending on $\delta-\varepsilon$) and $\mathbf{x} \in (1-z)\widehat{T}_\varepsilon$ we have $F_{(\mathbf{x},z)}(\widehat{T}) = \mathbf{x} + \frac{z}{2}\widehat{T} \subset \widehat{T}_\delta$, and we conclude with the previous estimates ((5.8) for $|\mathbf{k}|=0$ and (5.9) for $|\mathbf{k}|=1$)

$$\left\| \partial^{\mathbf{k}} \mathcal{A}u(\cdot,z) \right\|_{L^\infty((1-z)\widehat{T}_\varepsilon)} \lesssim \|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_\delta)}. \tag{5.10}$$

The combination of (5.10) for z sufficiently small and (5.8) for z not sufficiently small proves the desired estimates. □

The operator $\mathcal{A}$ can be modified to vanish on lateral faces of $\widehat{T}^{3\mathrm{D}}$ if u vanishes on the corresponding bottom edges of $\widehat{T}$.

Lemma 5.3. *Let $\widehat{\mathcal{E}} \subset \{\widehat{e}_4, \widehat{e}_5, \widehat{e}_6\}$ and let $\widehat{\mathcal{F}} = \{\widehat{f}_{j-3} \mid \widehat{e}_j \in \widehat{\mathcal{E}}\}$ be all lateral faces with edge in $\widehat{\mathcal{E}}$. There exists a linear operator $\mathcal{A}_{\widehat{\mathcal{E}}} : L^1_{loc}(\widehat{T}) \to C^\infty(\widehat{T}^{3\mathrm{D}})$ with the following properties:*

(i) *If u is continuous at a point $\mathbf{x} \in \widehat{T}$, then $(\mathcal{A}_{\widehat{\mathcal{E}}}u)(\mathbf{x},0) = u(\mathbf{x})$.*

(ii) *For every $\gamma > -1/2$ there is a constant C_γ such that*

$$\left\| d_{\widehat{T}\times\{0\}}^{\gamma} \mathcal{A}_{\widehat{\mathcal{E}}} u \right\|_{L^2(\widehat{T}^{3\mathrm{D}})} \leq C_\gamma \|u\|_{L^2(\widehat{T})}.$$

(iii) *The function $\mathcal{A}_{\widehat{\mathcal{E}}}u$ vanishes on all faces in $\widehat{\mathcal{F}}$.*

(iv) *If u is a polynomial of degree $p \geq \#\widehat{\mathcal{E}}$ that vanishes on all edges in $\widehat{\mathcal{E}}$, then $\mathcal{A}_{\widehat{\mathcal{E}}}u$ is a polynomial of degree p.*

(v) *For every $s \in (0,1)$ there is a constant $C_s > 0$ such that*

$$\left\| d_{\widehat{T}\times\{0\}}^{1/2-s} \nabla \mathcal{A}_{\widehat{\mathcal{E}}} u \right\|_{L^2(\widehat{T}^{3\mathrm{D}})} \leq C_s \Big(|u|_{H^s(\widehat{T})} + \|d_{\widehat{\mathcal{E}}}^{-s} u\|_{L^2(\widehat{T})} \Big).$$

(vi) *For every $\varepsilon > 0$ and $j \in \mathbb{N} \cup \{0\}$, there is $C_{\varepsilon,j} > 0$ such that*

$$\|\mathcal{A}_{\widehat{\mathcal{E}}} u\|_{W^{j,\infty}(\widehat{T}_\varepsilon^{3\mathrm{D}})} \leq C_{\varepsilon,j} \|u\|_{L^2(\widehat{T})}.$$

(vii) *For* $\varepsilon > 0$ *consider the set* $\widehat{T}_\varepsilon$ *and for* $z \in [0,1]$ *the scaled versions* $(1-z)\widehat{T}_\varepsilon$*. Then, for sufficiently small* $\varepsilon > 0$ *there is a* $\delta > \varepsilon$ *depending only on* ε *such that for* $\mathbf{k} \in \mathbb{N}_0^3$ *with* $|\mathbf{k}| \le 1$ *there holds for any* u *vanishing on* $\widehat{\mathcal{E}}$ *that*

$$\left\|\partial^{\mathbf{k}} \mathcal{A}_{\widehat{\mathcal{E}}} u(\cdot, z)\right\|_{L^\infty((1-z)\widehat{T}_\varepsilon)} \lesssim \|u\|_{L^2(\widehat{T})} + \|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_\delta)},$$

with a constant depending only on ε.

Proof. If $\widehat{\mathcal{E}} = \emptyset$, then we set $\mathcal{A}_{\widehat{\mathcal{E}}} = \mathcal{A}$ the operator from Lemma 5.2. If $\widehat{\mathcal{E}}$ is not empty, the construction will be carried out in several steps.

Step 1. Define

$$u_1(x,y,z) = \mathcal{A}u(x,y,z) - z\mathcal{A}u(\widehat{\mathbf{v}}_4),$$

note that $u_1(\widehat{\mathbf{v}}_4) = 0$ and therefore also

$$f_k \in \widehat{\mathcal{F}} \implies u_1 \circ \Pi_{\widehat{e}_k}|_{\widehat{f}_k} = u_1(\widehat{\mathbf{v}}_4) = 0. \tag{5.11}$$

Step 2. We will subtract edge contributions corresponding to all lateral edges of all faces in $\widehat{\mathcal{F}}$. To that end, define the corresponding indices $N = \{j \in \{1,2,3\} \mid \widehat{e}_j \text{ is lateral edge of } \widehat{f} \in \widehat{\mathcal{F}}\}$. For $j \in N$, let $p_j(x,y,z) = 0$ be the affine equation of the hyperplane orthogonal to $\widehat{e}_j$ and passing through the point $\widehat{\mathbf{v}}_j$, and for convenience let p_j be positive on $\widehat{T}^{3\mathrm{D}}$. We claim that there is a constant $c_j \neq 0$ such that for $(x,y,z) \in \widehat{e}_j$ it holds $p_j(x,y,z) = c_j z$. Indeed, write $p_j(x,y,z) = \ell_j(x,y,z) + d_j$ with ℓ_j linear and $d_j \in \mathbb{R}$. Parametrize $\widehat{e}_j$ by $z \mapsto \widehat{\mathbf{v}}_j + z\mathbf{n}_j$ and calculate $p_j(\widehat{\mathbf{v}}_j + z\mathbf{n}_j) = \ell_j(\widehat{\mathbf{v}}_j) + z\ell_j(\mathbf{n}_j) + d_j = z\ell_j(\mathbf{n}_j)$. Define

$$u_2(x,y,z) = u_1(x,y,z) - \sum_{j\in N} c_j \frac{z}{p_j(x,y,z)} u_1(\Pi_{\widehat{e}_j}(x,y,z)),$$

where $\Pi_{\widehat{e}_j}$ is the affine function calculating the orthogonal projection onto the line spanned by $\widehat{e}_j$. Note that $p_j(x,y,z)$ is proportional to the distance of $(x,y,z) \in \widehat{T}^{3\mathrm{D}}$ to the hyperplane $p_j = 0$, and hence

$$p_j(x,y,z) \sim d_{\widehat{\mathbf{v}}_j}(x,y,z) > z \qquad \text{for } (x,y,z) \in \widehat{T}^{3\mathrm{D}}. \tag{5.12}$$

Clearly, $|\nabla p_j(x,y,z)| \sim 1$, and this shows that

$$\left|\nabla \frac{z}{p_j(x,y,z)}\right| \lesssim \frac{1}{d_{\widehat{\mathbf{v}}_j}(x,y,z)} \qquad \text{for } (x,y,z) \in \widehat{T}^{3\mathrm{D}}. \tag{5.13}$$

Step 3. We will subtract face contributions corresponding to all faces in $\widehat{\mathcal{F}}$. Let $\widehat{f} \in \widehat{\mathcal{F}}$ be contained in the plane given by the affine equation $p_{\widehat{f}}(x,y) - z = 0$, and for convenience let $p_{\widehat{f}}(x,y) - z$ be positive on $\widehat{T}^{3\mathrm{D}}$. Let $\Pi_{\widehat{f}}$ be the affine function calculating the orthogonal projection onto this plane. Define

$$\mathcal{A}_{\widehat{\mathcal{E}}} u(x,y,z) = u_2(x,y,z) - \sum_{\widehat{f}\in\widehat{\mathcal{F}}} \frac{z}{p_{\widehat{f}}(x,y)} u_2\Big(\Pi_{\widehat{f}}(x,y,z)\Big).$$

For $\widehat{f}_k \in \widehat{\mathcal{F}}$, as $p_{\widehat{f}_k}(x,y) - z$ is positive on $\widehat{T}^{3\mathrm{D}}$, it holds

$$\frac{z}{p_{\widehat{f}_k}(x,y)} < 1. \tag{5.14}$$

For a point $(x,y,z) \in \widehat{T}^{3\mathrm{D}}$, consider angles $\sin\alpha = z/d_{\widehat{e}_{k+3}}(x,y,z)$ and $\sin\beta = d_{\widehat{f}_k}(x,y,z)/d_{\widehat{e}_{k+3}}(x,y,z)$. Recall that $\widehat{e}_{k+3}$ is the edge that $\widehat{f}_k$ shares with the base $\widehat{T}\times\{0\}$ of the tetrahedron, and hence $\alpha+\beta$ is constant. Furthermore, $p_{\widehat{f}_k}(x,y) - z$ is proportional to the distance of $(x,y,z) \in \widehat{T}^{3\mathrm{D}}$ to the plane spanned by $\widehat{f}_k$, and hence

$$p_{\widehat{f}_k}(x,y) \geq C d_{\widehat{f}_k}(x,y,z) + z = d_{\widehat{e}_{k+3}}(x,y,z)(C\sin\beta + \sin\alpha) \gtrsim d_{\widehat{e}_{k+3}}(x,y,z). \tag{5.15}$$

Now, with (5.14),

$$\left|\partial_x \frac{z}{p_{\widehat{f}_k}(x,y)}\right| + \left|\partial_y \frac{z}{p_{\widehat{f}_k}(x,y)}\right| = \left|\frac{z\partial_x p_{\widehat{f}_k}(x,y)}{p_{\widehat{f}_k}(x,y)^2}\right| + \left|\frac{z\partial_y p_{\widehat{f}_k}(x,y)}{p_{\widehat{f}_k}(x,y)^2}\right| \lesssim \frac{1}{p_{\widehat{f}_k}(x,y)},$$

and also

$$\left|\partial_z \frac{z}{p_{\widehat{f}_k}(x,y)}\right| = \frac{1}{p_{\widehat{f}_k}(x,y)}.$$

We conclude, using (5.15),

$$\left|\nabla \frac{z}{p_{\widehat{f}_k}(x,y)}\right| \lesssim \frac{1}{d_{\widehat{e}_{k+3}}(x,y,z)}. \tag{5.16}$$

The operator $\mathcal{A}_{\widehat{\mathcal{E}}}$ is clearly linear and $\mathcal{A}_{\widehat{\mathcal{E}}} u \in C^\infty(\widehat{T}^{3\mathrm{D}})$.

Proof of (i). This follows by construction, as $\mathcal{A}u(\mathbf{x}) = u(\mathbf{x})$ due to Lemma 5.2(i), and the correction terms are all multiplied by z.

Proof of (ii). Due to Lemma 5.2(iii), (iv), (v), and (vii), it follows for $-1/2 < \gamma$

$$\left\|d^{\gamma}_{\widehat{T}\times\{0\}} u_1\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} + \left\|d^{1/2+\gamma}_{\widehat{e}_{3+k}} u_1\right\|_{L^2(\widehat{f}_k)} + \left\|d^{1+\gamma}_{\widehat{\mathbf{v}}_j} u_1\right\|_{L^2(\widehat{e}_j)} \lesssim \|u\|_{L^2(\widehat{T})} \tag{5.17}$$

for $j \in N$ and $\widehat{f}_k \in \widehat{\mathcal{F}}$. Then,

$$\left\|d^{\gamma}_{\widehat{T}\times\{0\}} u_2\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} \lesssim \left\|d^{\gamma}_{\widehat{T}\times\{0\}} u_1\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} + \sum_{j\in N}\left\|d^{\gamma}_{\widehat{T}\times\{0\}} u_1 \circ \Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} \lesssim \|u\|_{L^2(\widehat{T})}, \tag{5.18}$$

where we used (5.12) in the first inequality, and (2.3) (with $\alpha = 0, \beta = \gamma$) together with (5.17) in the second one. On $\widehat{f}_k$ there holds $d_{\widehat{e}_{3+k}} \sim d_{\widehat{T}}$, so that we obtain

$$\begin{aligned}\left\|d^{1/2+\gamma}_{\widehat{e}_{3+k}} u_2\right\|_{L^2(\widehat{f}_k)} &\lesssim \left\|d^{1/2+\gamma}_{\widehat{e}_{3+k}} u_1\right\|_{L^2(\widehat{f}_k)} + \sum_{\substack{j\in N\\ j\neq k}}\left\|d^{1/2+\gamma}_{\widehat{e}_{3+k}} u_1 \circ \Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{f}_k)} \\ &\lesssim \left\|d^{1/2+\gamma}_{\widehat{e}_{3+k}} u_1\right\|_{L^2(\widehat{f}_k)} + \sum_{\substack{j\in N\\ j\neq k}}\left\|d^{1/2+\gamma}_{\widehat{T}} u_1 \circ \Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{f}_k)} \lesssim \|u\|_{L^2(\widehat{T})},\end{aligned} \tag{5.19}$$

where we used (5.11) and (5.12) in the first inequality, and (2.3) (with $\alpha = 0, \beta = \gamma$) together with (5.17) in the last one. We conclude with (5.14), (2.4) and (5.18), (5.19)

$$\begin{aligned}\left\|d^{\gamma}_{\widehat{T}\times\{0\}}\mathcal{A}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} &\lesssim \left\|d^{\gamma}_{\widehat{T}\times\{0\}}u_2\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} + \sum_{\widehat{f}_k\in\widehat{\mathcal{F}}}\left\|d^{\gamma}_{\widehat{T}\times\{0\}}u_2\circ\Pi_{\widehat{f}_k}\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}\\ &\lesssim \left\|d^{\gamma}_{\widehat{T}\times\{0\}}u_2\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} + \sum_{\widehat{f}_k\in\widehat{\mathcal{F}}}\left\|d^{1/2+\gamma}_{\widehat{e}_{3+k}}u_2\right\|_{L^2(\widehat{f}_k)} \lesssim \|u\|_{L^2(\widehat{T})}.\end{aligned}$$

Proof of (iii). According to Step 1, u_1 vanishes in $\widehat{\mathbf{v}}_4$. We will now show that u_2 vanishes on all edges with indices in N. To that end, let $j\in N$. For $(x,y,z)\in\widehat{e}_j$ it holds according to Step 2 that $c_jz=p_j(x,y,z)$ as well as $u_1(\Pi_{\widehat{e}_k}(x,y,z))=u_1(\widehat{\mathbf{v}}_4)=0$ for $k\neq j\in N$. Hence, for $(x,y,z)\in\widehat{e}_j$,

$$\begin{aligned}u_2(x,y,z) &= u_1(x,y,z)-c_j\frac{z}{p_j(x,y)}u_1\circ\Pi_{\widehat{e}_j}(x,y,z)+\sum_{\substack{k\in N\\k\neq j}}c_k\frac{z}{p_k(x,y)}u_1\circ\Pi_{\widehat{e}_k}(x,y,z)\\ &= u_1(x,y,z)-u_1(x,y,z)-\sum_{\substack{k\in N\\k\neq j}}c_k\frac{z}{p_k(x,y)}u_1(\widehat{\mathbf{v}}_4)=0.\end{aligned}\tag{5.20}$$

Next, we will show that $\mathcal{A}_{\widehat{\mathcal{E}}}u$ vanishes on all faces in $\widehat{\mathcal{F}}$. To that end, let $\widehat{f}_j\in\widehat{\mathcal{F}}$. For $(x,y,z)\in\widehat{f}_j$ it holds $z=p_{\widehat{f}_j}(x,y)$. Furthermore, if $\widehat{f}_k\in\widehat{\mathcal{F}}$ with $k\neq j$, then $\ell_{j,k}\in N$ for the lateral edge $\widehat{e}_{\ell_{j,k}}$ that is shared by $\widehat{f}_k$ and $\widehat{f}_j$. Hence, for $(x,y,z)\in\widehat{f}_j$, (2.2) implies $u_2(\Pi_{\widehat{f}_k}(x,y,z))=u_2(\Pi_{\widehat{e}_{\ell_{j,k}}}(x,y,z))=0$, as we have already demonstrated that u_2 vanishes on all edges with indices in N. Hence, for $(x,y,z)\in\widehat{f}_j\in\widehat{\mathcal{F}}$,

$$\begin{aligned}\mathcal{A}_{\widehat{\mathcal{E}}}u(x,y,z) &= u_2(x,y,z)-\frac{z}{p_{\widehat{f}_j}(x,y)}u_2(\Pi_{\widehat{f}_j}(x,y,z))-\sum_{\substack{\widehat{f}_k\in\widehat{\mathcal{F}}\\k\neq j}}\frac{z}{p_{\widehat{f}_k}(x,y)}u_2(\Pi_{\widehat{f}_k}(x,y,z))\\ &= u_2(x,y,z)-u_2(x,y,z)-\sum_{\substack{\widehat{f}_k\in\widehat{\mathcal{F}}\\k\neq j}}\frac{z}{p_{\widehat{f}_k}(x,y)}u_2(\Pi_{\widehat{e}_{\ell_{j,k}}}(x,y,z))=0.\end{aligned}$$

Proof of (iv). If u is polynomial of degree p, then so is $\mathcal{A}u$, and hence also u_1. Furthermore, $\mathcal{A}u$, and hence u_1, vanish on $\widehat{\mathcal{E}}$ if u does.

As $\Pi_{\widehat{e}_j}$ is affine, $p_j(x,y,z)=0$ implies $u_1(\Pi_{\widehat{e}_j}(x,y,z))=u_1(\widehat{\mathbf{v}}_j)=0$. Polynomial division shows that u_2 is indeed a polynomial of degree p, and due to construction it vanishes on $\widehat{\mathcal{E}}$, *cf.* (5.20).

Finally, note that $\Pi_{\widehat{f}_j}$ is affine, and $p_{\widehat{f}_j}(x,y)=0$ implies $(x,y,0)\in\widehat{f}_j$ and hence $u_2(\Pi_{\widehat{f}_j}(x,y,0))=u_2(x,y,0)=0$. Polynomial division shows that $A_{\widehat{\mathcal{E}}}u$ is indeed a polynomial of degree p.

Proof of (v). At first, various applications of Lemma 5.2 give for $j\in N$ and $\widehat{f}_k\in\widehat{\mathcal{F}}$

$$\begin{aligned}\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}+\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}u_1\right\|_{L^2(\widehat{e}_j)}+\left\|d^{3/2-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{e}_j)} &\lesssim |u|_{H^s(\widehat{T})}+\left\|d^{-s}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T})}\\ \left\|d^{-s}_{\widehat{T}\times\{0\}}u_1\right\|_{L^2(\widehat{f}_k)}+\left\|d^{1-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{f}_k)} &\lesssim |u|_{H^s(\widehat{T})}+\left\|d^{-s}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T})}.\end{aligned}\tag{5.21}$$

Specifically, the terms without derivatives on u_1 are bounded by Lemma 5.2(iv) and (v), while the terms containing ∇u_1 are bounded by Lemma 5.2(vi). Note that $\Pi_{\widehat{e}_j}$ is affine, so that

$$|\nabla(u_1\circ\Pi_{\widehat{e}_j})(x,y,z)|\sim|(\nabla u_1)\circ\Pi_{\widehat{e}_j}(x,y,z)|.\tag{5.22}$$

We conclude with (2.3) (with $\alpha=-1,\beta=1/2-s$ as well as $\alpha=0,\beta=1/2-s$), (5.21), and (5.12), (5.13)

$$\begin{aligned}\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}\nabla u_2\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} &\le \left\|d^{1/2-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}\\ &\quad+\sum_{j\in N}\left(\left\|d^{-1}_{\widehat{\mathbf{v}}_j}d^{1/2-s}_{\widehat{T}\times\{0\}}u_1\circ\Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}+\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}(\nabla u_1)\circ\Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}\right)\\ &\le \left\|d^{1/2-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}+\sum_{j\in N}\left(\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}u_1\right\|_{L^2(\widehat{e}_j)}+\left\|d^{3/2-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{e}_j)}\right)\\ &\overset{(5.21)}{\lesssim} |u|_{H^s(\widehat{T})}+\left\|d^{-s}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T})}.\end{aligned}\tag{5.23}$$

Furthermore, for $(x,y,z)\in\widehat{T}^{3\mathrm{D}}$ we have, *cf.* (5.12),

$$d^{-s}_{\widehat{T}\times\{0\}}\frac{z}{p_j(x,y,z)}\sim d^{1-s}_{\widehat{T}\times\{0\}}(x,y,z)d^{-1}_{\widehat{\mathbf{v}}_j}(x,y,z).$$

Together with (5.11), (2.3) (with $\alpha=-1,\beta=1/2-s$), and (5.21) we then conclude

$$\left\|d^{-s}_{\widehat{T}\times\{0\}}u_2\right\|_{L^2(\widehat{f}_k)}\le\left\|d^{-s}_{\widehat{T}\times\{0\}}u_1\right\|_{L^2(\widehat{f}_k)}+\sum_{\substack{j\in N\\ j\neq k}}\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}u_1\right\|_{L^2(\widehat{e}_j)}\lesssim|u|_{H^s(\widehat{T})}+\left\|d^{-s}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T})}.\tag{5.24}$$

Note that, according to Lemma 5.2(vii),

$$|\nabla u_1(\widehat{\mathbf{v}}_4)|\le|\nabla\mathcal{A}u(\widehat{\mathbf{v}}_4)|+|\mathcal{A}u(\widehat{\mathbf{v}}_4)|\lesssim\|u\|_{L^2(\widehat{T})}.$$

As $d^{-1}_{\widehat{\mathbf{v}}_k}\lesssim 1$ on $\widehat{f}_k$, we conclude with (5.13), (5.22), (2.3) (with $\alpha=-1,\beta=1/2-s$ as well as $\alpha=0,\beta=1/2-s$), (2.5) (with $\beta=1-s$), and (5.21)

$$\begin{aligned}\left\|d^{1-s}_{\widehat{T}\times\{0\}}\nabla u_2\right\|_{L^2(\widehat{f}_k)}&\lesssim\left\|d^{1-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{f}_k)}\\ &\quad+\sum_{\substack{j\in N\\ j\neq k}}\left(\left\|d^{-1}_{\widehat{\mathbf{v}}_j}d^{1-s}_{\widehat{T}\times\{0\}}u_1\circ\Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{f}_k)}+\left\|d^{1-s}_{\widehat{T}\times\{0\}}(\nabla u_1)\circ\Pi_{\widehat{e}_j}\right\|_{L^2(\widehat{f}_k)}\right)\\ &\quad+\left\|d^{-1}_{\widehat{\mathbf{v}}_k}d^{1-s}_{\widehat{T}\times\{0\}}u_1\circ\Pi_{\widehat{e}_k}\right\|_{L^2(\widehat{f}_k)}+\left\|d^{1-s}_{\widehat{T}\times\{0\}}(\nabla u_1)\circ\Pi_{\widehat{e}_k}\right\|_{L^2(\widehat{f}_k)}\\ &\lesssim\left\|d^{1-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{f}_k)}\\ &\quad+\sum_{\substack{j\in N\\ j\neq k}}\left(\left\|d^{1/2-s}_{\widehat{T}\times\{0\}}u_1\right\|_{L^2(\widehat{e}_j)}+\left\|d^{3/2-s}_{\widehat{T}\times\{0\}}\nabla u_1\right\|_{L^2(\widehat{e}_j)}\right)+|\nabla u_1(\widehat{\mathbf{v}}_4)|+\underbrace{|u_1(\widehat{\mathbf{v}}_4)|}_{=0}\\ &\lesssim|u|_{H^s(\widehat{T})}+\left\|d^{-s}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T})}.\end{aligned}\tag{5.25}$$

As before,

$$\left|\nabla\left(u_2\circ\Pi_{\widehat{f}_j}\right)(x,y,z)\right|\sim\left|(\nabla u_2)\circ\Pi_{\widehat{f}_j}(x,y,z)\right|.\tag{5.26}$$

We employ (5.14), (5.16) (2.4) (with $\alpha=-1,\beta=1/2-s$ as well as $\alpha=0,\beta=1/2-s$) and conclude with (5.23)–(5.25)

$$\begin{aligned}\left\|d_{\widehat{T}\times\{0\}}^{1/2-s}\nabla\mathcal{A}_{\widehat{\mathcal{E}}}u\right\|_{L^2(\widehat{T}^{3\mathrm{D}})} &\lesssim \left\|d_{\widehat{T}\times\{0\}}^{1/2-s}\nabla u_2\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}\\ &\quad+\sum_{\widehat{f}_k\in\widehat{\mathcal{F}}}\left(\left\|d_{\widehat{e}_{3+k}}^{-1}d_{\widehat{T}\times\{0\}}^{1/2-s}u_2\circ\Pi_{\widehat{f}_k}\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}+\left\|d_{\widehat{T}\times\{0\}}^{1/2-s}(\nabla u_2)\circ\Pi_{\widehat{f}_k}\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}\right)\\ &\lesssim \left\|d_{\widehat{T}\times\{0\}}^{1/2-s}\nabla u_2\right\|_{L^2(\widehat{T}^{3\mathrm{D}})}+\sum_{\widehat{f}_k\in\widehat{\mathcal{F}}}\left(\left\|d_{\widehat{T}\times\{0\}}^{-s}u_2\right\|_{L^2(\widehat{f}_k)}+\left\|d_{\widehat{T}\times\{0\}}^{1-s}\nabla u_2\right\|_{L^2(\widehat{f}_k)}\right)\\ &\lesssim |u|_{H^s(\widehat{T})}+\left\|d_{\widehat{\mathcal{E}}}^{-s}u\right\|_{L^2(\widehat{T})}.\end{aligned}$$

Proof of (vi). This follows by the consecutive construction of $\mathcal{A}_{\widehat{\mathcal{E}}}u$ *via* u_1 and u_2, using orthogonal projections onto lateral edges and faces and Lemma 5.2(vii).

Proof of (vii). First, let $\delta > \widetilde{\varepsilon} > 0$. Using Lemma 5.2(vii) and (viii), we conclude

$$\left\|\partial^{\mathbf{k}}u_1(\cdot,z)\right\|_{L^\infty((1-z)\widehat{T}_{\widetilde{\varepsilon}})} \lesssim \|u\|_{L^2(\widehat{T})}+\|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_\delta)} \tag{5.27}$$

with an implied constant depending only on $\delta-\widetilde{\varepsilon}$. We proceed to consider u_2. Using the triangle inequality and estimate (5.27), it suffices to consider the correction term

$$t_j(x,y,z) := \frac{z}{p_j(x,y,z)}u_1(\Pi_{\widehat{e}_j}(x,y,z))$$

for $j\in N$ and (x,y,z) with $(x,y)\in(1-z)\widehat{T}_{\widetilde{\varepsilon}}$. With (5.12), (5.13), and (5.22) we conclude

$$|t_j(x,y,z)| \lesssim |u_1(\Pi_{\widehat{e}_j}(x,y,z))| \quad\text{and}\quad |\nabla t_j(x,y,z)| \lesssim \frac{|u_1(\Pi_{\widehat{e}_j}(x,y,z))|}{d_{\widehat{\mathbf{v}}_j}(x,y,z)}+|(\nabla u_1)(\Pi_{\widehat{e}_j}(x,y,z))|. \tag{5.28}$$

We distinguish two cases for $j\in N$:

- $j\in\{2,3\}$. In this case, note that for $(x,y,z)\in\widehat{T}^{3\mathrm{D}}$ with $(x,y)\in(1-z)\widehat{T}_{\widetilde{\varepsilon}}$ it holds $\Pi_{\widehat{e}_j}(x,y,z)\in\overline{\widehat{T}^{3\mathrm{D}}_{\varepsilon_1}}$ for some ε_1 depending only on $\widetilde{\varepsilon}$, as well as $d_{\widehat{\mathbf{v}}_j}(x,y,z)\gtrsim 1$. We conclude with (5.28)

$$\left|\partial^{\mathbf{k}}t_j(x,y,z)\right| \lesssim \|u_1\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}^{3\mathrm{D}}_{\varepsilon_1})},$$

  with a constant depending only on $\widetilde{\varepsilon}$. To bound further the right-hand side, we may use point (vi) of the present lemma in the particular case of $\widehat{\mathcal{E}}=\emptyset$ as in this case $u_1=\mathcal{A}_\emptyset u$ so that

$$\|u_1\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}^{3\mathrm{D}}_{\varepsilon_1})}=\|\mathcal{A}_\emptyset u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}^{3\mathrm{D}}_{\varepsilon_1})}\lesssim\|u\|_{L^2(\widehat{T})}.$$

- $j=1$. For $|\mathbf{k}|=0$, we use the first estimate of (5.28) and (5.27), as for $(x,y,z)\in\widehat{T}^{3\mathrm{D}}$ with $(x,y)\in(1-z)\widehat{T}_{\widetilde{\varepsilon}}$ it holds

$$|t_1(x,y,z)|\lesssim\|u_1\|_{L^\infty(\widehat{e}_1)}\le\sup_{z\in[0,1]}\|u_1(\cdot,z)\|_{L^\infty((1-z)\widehat{T}_{\widetilde{\varepsilon}})}\lesssim\|u\|_{L^2(\widehat{T})}+\|u\|_{L^\infty(\widehat{T}_\delta)}.$$

  If $|\mathbf{k}|=1$, we note that our assumptions yield $u(\widehat{\mathbf{v}}_1)=0$, and Poincaré's inequality shows

$$|u_1(\Pi_{\widehat{e}_1}(x,y,z))|\lesssim d_{\widehat{\mathbf{v}}_1}(\Pi_{\widehat{e}_1}(x,y,z))\|\nabla u_1\|_{L^\infty(\widehat{e}_1)}$$

  as well as $d_{\widehat{\mathbf{v}}_1}(\Pi_{\widehat{e}_1}(x,y,z))\le d_{\widehat{\mathbf{v}}_1}(x,y,z)$. Finally, we conclude with the second estimate of (5.28) for $(x,y,z)\in\widehat{T}^{3\mathrm{D}}$ with $(x,y)\in(1-z)\widehat{T}_{\widetilde{\varepsilon}}$ and (5.27)

$$\left|\partial^{\mathbf{k}}t_j(x,y,z)\right|\lesssim\|\nabla u_1\|_{L^\infty(\widehat{e}_1)}\le\sup_{z\in[0,1]}\|\nabla u_1(\cdot,z)\|_{L^\infty((1-z)\widehat{T}_{\widetilde{\varepsilon}})}\lesssim\|u\|_{L^2(\widehat{T})}+\|u\|_{W^{1,\infty}(\widehat{T}_\delta)}.$$

We arrive at

$$\left\|\partial^{\mathbf{k}} u_2(\cdot,z)\right\|_{L^\infty((1-z)\widehat{T}_{\widetilde{\varepsilon}})} \lesssim \|u\|_{L^2(\widehat{T})} + \|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_\delta)} \tag{5.29}$$

for any $\widetilde{\varepsilon}$ and $\delta > \widetilde{\varepsilon}$, with a constant depending on $\widetilde{\varepsilon}$ as well as $\delta - \widetilde{\varepsilon}$. Finally, let $\varepsilon > 0$. For $\widehat{f}_k \in \widehat{\mathcal{F}}$ consider the correction term

$$r_k(x,y,z) := \frac{z}{p_{\widehat{f}_k}(x,y)} u_2\Big(\Pi_{\widehat{f}_k}(x,y,z)\Big)$$

for (x,y,z) with $(x,y) \in (1-z)\widehat{T}_\varepsilon$. With (5.14), (5.16), and (5.26) we conclude that

$$|r_k(x,y,z)| \lesssim |u_2(\Pi_{\widehat{f}_k}(x,y,z))| \quad \text{and} \quad |\nabla r_k(x,y,z)| \lesssim \frac{|u_2(\Pi_{\widehat{f}_k}(x,y,z))|}{d_{\widehat{e}_{k+3}}(x,y,z)} + |(\nabla u_2)(\Pi_{\widehat{f}_k}(x,y,z))|.$$

Again we distinguish two cases.

– $k = 1$. In this case, note that for $(x,y,z) \in \widehat{T}^{3\mathrm{D}}$ with $(x,y) \in (1-z)\widehat{T}_\varepsilon$ it holds $\Pi_{\widehat{f}_k}(x,y,z) \in \overline{\widehat{T}^{3\mathrm{D}}_{\varepsilon_2}}$ for some ε_2 depending only on ε, as well as as well as $d_{\widehat{e}_{k+3}}(x,y,z) \gtrsim 1$. Hence, using again point (vi) of the present lemma, we conclude

$$\left|\partial^{\mathbf{k}} r_k(x,y,z)\right| \lesssim \|u_2\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}^{3\mathrm{D}}_{\varepsilon_2})} \lesssim \|u\|_{L^2(\widehat{T})},$$

with a constant depending only on ε.

– $k \in \{2,3\}$. In this case, note that for ε sufficiently small there is some $\widetilde{\varepsilon} > \varepsilon$ such that for $(x,y,z) \in \widehat{T}^{3\mathrm{D}}$ with $(x,y) \in (1-z)\widehat{T}_\varepsilon$ it holds for $(\widetilde{x},\widetilde{y},\widetilde{z}) = \Pi_{\widehat{f}_k}(x,y,z)$ that $(\widetilde{x},\widetilde{y}) \in \overline{(1-\widetilde{z})\widehat{T}_{\widetilde{\varepsilon}}}$. For $|\mathbf{k}| = 1$, given that u_2 vanishes on $\widehat{e}_k$, Poincaré's inequality shows

$$\left|u_2(\Pi_{\widehat{f}_k}(x,y,z))\right| \lesssim d_{\widehat{e}_{k+3}}\Big(\Pi_{\widehat{f}_k}(x,y,z)\Big) \max_{\widetilde{z}\in[0,1]} \|\nabla u_2\|_{L^\infty\left(\widehat{f}_k \cap \overline{(1-\widetilde{z})\widehat{T}_{\widetilde{\varepsilon}}}\right)}$$

as well as $d_{\widehat{e}_{k+3}}(\Pi_{\widehat{f}_k}(x,y,z)) \le d_{\widehat{e}_{k+3}}(x,y,z)$. Finally, for $|\mathbf{k}| \in \{0,1\}$, we conclude with (5.29)

$$\left|\partial^{\mathbf{k}} r_k(x,y,z)\right| \lesssim \max_{\widetilde{z}\in[0,1]} \left\|\nabla^{|\mathbf{k}|} u_2\right\|_{L^\infty\left(\widehat{f}_k \cap \overline{(1-\widetilde{z})\widehat{T}_{\widetilde{\varepsilon}}}\right)} \lesssim \|u\|_{L^2(\widehat{T})} + \|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_\delta)}.$$

Together with triangle inequality, this shows the stipulated estimate. □

To end this section, we change the target of our liftings from the reference tetrahedron to the reference prism $\widehat{P} := \widehat{T} \times (0,1)$, where $\widehat{T}$ is the reference triangle. This is in line with the requirements in the trace-space characterization of interpolation spaces Lemma 5.1.

Lemma 5.4. *Let $\widehat{\mathcal{E}} \subset \{\widehat{e}_4, \widehat{e}_5, \widehat{e}_6\}$. There exists a linear operator $\mathcal{A}^{\widehat{P}}_{\widehat{\mathcal{E}}} : L^1_{loc}(\widehat{T}) \to C^\infty(\widehat{P})$ with the following properties:*

(i) *If u is continuous at a point $\mathbf{x} \in \widehat{T}$, then $(\mathcal{A}^{\widehat{P}}_{\widehat{\mathcal{E}}} u)(\mathbf{x}, 0) = u(\mathbf{x})$.*
(ii) *The operator $\mathcal{A}^{\widehat{P}}_{\widehat{\mathcal{E}}} : L^2(\widehat{T}) \to L^2(\widehat{P})$ is bounded.*
(iii) *If u vanishes on an edge $\widehat{e} \in \widehat{\mathcal{E}}$, then $\mathcal{A}^{\widehat{P}}_{\widehat{\mathcal{E}}} u$ vanishes on the face $\widehat{e} \times (0,1)$.*
(iv) *If u is a polynomial of degree $p \ge |\widehat{\mathcal{E}}|$ that vanishes on all edges in $\widehat{\mathcal{E}}$, then $\mathcal{A}^{\widehat{P}}_{\widehat{\mathcal{E}}}(\cdot,\cdot,z)$ is a polynomial of degree p for fixed z.*
(v) *$\mathcal{A}^{\widehat{P}}_{\widehat{\mathcal{E}}} u$ vanishes on the top face $\widehat{T} \times 1$.*

(vi) *For every $\theta \in (0,1)$ there is a constant $C_\theta > 0$ such that*

$$\left\| d_{\widehat{T}\times\{0\}}^{1/2-\theta} \nabla \mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} u \right\|_{L^2(\widehat{P})} \le C_\theta \left(|u|_{H^\theta(\widehat{T})} + \left\| d_{\widehat{\mathcal{E}}}^{-\theta} u \right\|_{L^2(\widehat{T})} \right).$$

(vii) *For $\varepsilon > 0$ sufficiently small there is a $\delta > 0$ with $\delta > \varepsilon$ such that for $\mathbf{k} \in \mathbb{N}_0^3$ with $|\mathbf{k}| \le 1$ there holds for any u vanishing on $\widehat{\mathcal{E}}$ that*

$$\left\| \partial^{\mathbf{k}} \mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} u(\cdot, z) \right\|_{L^\infty(\widehat{T}_\varepsilon)} \lesssim \|u\|_{L^2(\widehat{T})} + \|u\|_{W^{|\mathbf{k}|,\infty}(\widehat{T}_\delta)} \qquad \text{for all } z \in [0,1],$$

with implied constant depending only on ε.

Proof. Denote by

$$T_{\mathcal{D}}^{3\mathrm{D}} : \widehat{P} \to \widehat{T}^{3\mathrm{D}}, \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x(1-z) \\ y(1-z) \\ z \end{pmatrix}$$

the Duffy transformation and note $|\det dT_{\mathcal{D}}^{3\mathrm{D}}| = (1-z)^2$. Let $\mathcal{A}_{\widehat{\mathcal{E}}}$ be the operator of Lemma 5.3 and define

$$\left(\mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} u \right)(x,y,z) := (1-z)(\mathcal{A}_{\widehat{\mathcal{E}}} u) \circ T_{\mathcal{D}}^{3\mathrm{D}}(x,y,z).$$

Proof of (i). This follows immediately from Lemma 5.3(i).

Proof of (ii). This follows from Lemma 5.3(ii) by substitution.

Proof of (iii). This follows from Lemma 5.3(iii) and the fact that the Duffy transformation maps the face of $\widehat{T}^{3\mathrm{D}}$ where $\mathcal{A}_{\widehat{\mathcal{E}}} u$ vanishes to $\widehat{e} \times (0,1)$.

Proof of (iv). This follows from Lemma 5.3(iv) and the definition of $T_{\mathcal{D}}^{3\mathrm{D}}$.

Proof of (v). This follows by construction.

Proof of (vi). By the product and the chain rule,

$$\nabla \mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} u = \begin{pmatrix} 0 \\ 0 \\ -\mathcal{A}_{\widehat{\mathcal{E}}} u \circ T_{\mathcal{D}}^{3\mathrm{D}} \end{pmatrix} + (1-z)\left(dT_{\mathcal{D}}^{3\mathrm{D}} \right)^\top (\nabla \mathcal{A}_{\widehat{\mathcal{E}}} u) \circ T_{\mathcal{D}}^{3\mathrm{D}}. \tag{5.30}$$

To bound the first term, we choose some $\varepsilon > 0$ and calculate

$$\begin{aligned} \int_{\widehat{P}} z^{1-2\theta} |\mathcal{A}_{\widehat{\mathcal{E}}} u \circ T_{\mathcal{D}}^{3\mathrm{D}}(x,y,z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\widehat{T}^{3\mathrm{D}}} z^{1-2\theta} |\mathcal{A}_{\widehat{\mathcal{E}}} u(x,y,z)|^2 \frac{1}{(1-z)^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\lesssim \int_{\widehat{T}^{3\mathrm{D}} \setminus \widehat{T}_\varepsilon^{3\mathrm{D}}} z^{1-2\theta} |\mathcal{A}_{\widehat{\mathcal{E}}} u(x,y,z)|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \|\mathcal{A}_{\widehat{\mathcal{E}}} u\|_{L^\infty(T_\varepsilon^{3\mathrm{D}})}^2 \int_{\widehat{T}_\varepsilon^{3\mathrm{D}}} \frac{1}{(1-z)^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\lesssim \|u\|_{L^2(\widehat{T})}^2, \end{aligned}$$

where the last estimate follows from Lemma 5.3(ii) and (vi). To bound the second term in (5.30), we use $\|dT_{\mathcal{D}}^{3\mathrm{D}}\|_2 \lesssim 1$ and substitution

$$\begin{aligned} \int_{\widehat{P}} z^{1-2\theta} (1-z)^2 \left| (\nabla \mathcal{A}_{\widehat{\mathcal{E}}} u) \circ T_{\mathcal{D}}^{3\mathrm{D}}(x,y,z) \right|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &\lesssim \int_{\widehat{T}^{3\mathrm{D}}} z^{1-2\theta} \left| \nabla \mathcal{A}_{\widehat{\mathcal{E}}} u(x,y,z) \right|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\lesssim |u|_{H^\theta(\widehat{T})}^2 + \left\| d_{\widehat{\mathcal{E}}}^{-\theta} u \right\|_{L^2(\widehat{T})}^2, \end{aligned}$$

where the last estimate follows from Lemma 5.3(v).

Proof of (vii). For $|\mathbf{k}| = 0$, this follows immediately from Lemma 5.3(vii), taking into account that

$$\left\|\mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} u(\cdot, z)\right\|_{L^\infty(\widehat{T}_\varepsilon)} = (1-z)\left\|\mathcal{A}_{\widehat{\mathcal{E}}} u(\cdot, z)\right\|_{L^\infty((1-z)\widehat{T}_\varepsilon)}.$$

For $|\mathbf{k}| = 1$, we additionally use the formula (5.30). □

5.3. Liftings for decomposed FEM spaces

We now construct the liftings for different locally supported functions, before moving on to global functions. We start with functions supported on a single element.

Lemma 5.5. *Let $K \in \mathcal{T}$ be an element and $u_K \in \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_K)$. Then there exists a function $v : [0,\infty) \to \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_K)$ such that $v(0) = u_K$ and, for $\theta \in (0,1)$,*

$$\int_0^\infty t^{2(1-\theta)} \Big(\|\nabla v(t)\|_{L^2(K)}^2 + \|v'(t)\|_{L^2(K)}^2\Big) \frac{\mathrm{d}t}{t} \lesssim |u_K|_{H^\theta(K)}^2 + \left\|d_{\partial K}^{-\theta} u_K\right\|_{L^2(K)}^2.$$

Proof. Set $\widehat{u} = u_K \circ F_K$ and $p = p_K$. First, suppose that K is a square, *i.e.*, $\widehat{K} = \widehat{S}$, such that $\widehat{u} \in \widetilde{\mathcal{Q}}^p$. We apply Lemma 5.1(i) with $X_0 = (\widetilde{\mathcal{Q}}^p, \|\cdot\|_{L^2(\widehat{S})})$ and $X_1 = (\widetilde{\mathcal{Q}}^p, \|\cdot\|_{\widetilde{H}^1(\widehat{S})})$. Accordingly, there exists a function $\widehat{v} : [0,\infty] \to \widetilde{\mathcal{Q}}^p$ such that

$$\begin{aligned}\int_0^\infty t^{2(1-\theta)} \Big(\|\nabla \widehat{v}(t)\|_{L^2(\widehat{S})}^2 + \|\widehat{v}'(t)\|_{L^2(\widehat{S})}^2\Big) \frac{\mathrm{d}t}{t} &\lesssim |\widehat{u}|_{[X_0,X_1]_\theta}^2 \le \|\widehat{u}\|_{[X_0,X_1]_\theta}^2 \lesssim \|\widehat{u}\|_{[L^2(\widehat{S}),\widetilde{H}^1(\widehat{S})]_\theta}^2 \\ &\lesssim |\widehat{u}|_{H^\theta(\widehat{S})}^2 + \left\|d_{\partial\widehat{S}}^{-\theta}\widehat{u}\right\|_{L^2(\widehat{S})}^2.\end{aligned} \tag{5.31}$$

where the penultimate estimate follows from Proposition 4.1(i), and the last estimate from Lemma 4.4(ii). If K is a triangle, *i.e.*, $\widehat{K} = \widehat{T}$, then in particular $p \ge 3$, and we use Lemma 5.4 to define $\widehat{v} := \mathcal{A}_{\partial\widehat{T}}^{\widehat{P}}\widehat{u}$ on $\widehat{P}$. Note that $\widehat{v}$ vanishes at $t = 1$, and we extend it by zero on $\widehat{T} \times [1,\infty)$. Due to Lemma 5.4(iii) and (iv) it holds $\widehat{v} : [0,\infty] \to \widetilde{\mathcal{P}}^p$. Then, due to Lemma 5.4(vi),

$$\int_0^\infty t^{2(1-\theta)} \Big(\|\nabla \widehat{v}(t)\|_{L^2(\widehat{T})}^2 + \|\widehat{v}'(t)\|_{L^2(\widehat{T})}^2\Big) \frac{\mathrm{d}t}{t} = \left\|d_{\widehat{T}\times\{0\}}^{1/2-\theta}\nabla\mathcal{A}_{\partial\widehat{T}}^{\widehat{P}}\widehat{u}\right\|_{L^2(\widehat{P})}^2 \lesssim |\widehat{u}|_{H^\theta(\widehat{T})}^2 + \left\|d_{\partial\widehat{T}}^{-\theta}\widehat{u}\right\|_{L^2(\widehat{T})}^2. \tag{5.32}$$

Independently of the shape of K we define $v(t) := \widehat{v}(t/h_K) \circ F_K^{-1}$, so that $v : [0,\infty) \to \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_K)$. Scaling arguments transform the left hand sides of (5.31) and (5.32) into

$$h_K^{2\theta-2}\int_0^\infty t^{2(1-\theta)} \Big(\|\nabla v(t)\|_{L^2(K)}^2 + \|v'(t)\|_{L^2(K)}^2\Big) \frac{\mathrm{d}t}{t},$$

while the right-hand sides transform into

$$h^{2\theta-2}\Big(|u_K|_{H^\theta(K)}^2 + \left\|d_{\partial K}^{-\theta} u_K\right\|_{L^2(K)}^2\Big).$$

The stated estimate follows. □

Next, we consider a certain class of patch-local functions of simple structure.

Lemma 5.6. *Let ω be a vertex or an edge patch in $\mathcal{T}$. If ω is a vertex patch, set $\widehat{\mathcal{E}} = \{\widehat{e}_6\}$. If ω is an edge patch, set $\widehat{\mathcal{E}} = \{\widehat{e}_5, \widehat{e}_6\}$. With the notation introduced in Lemma 4.5, let $\widetilde{u} \in X_\omega$. Then there exists a function $v : [0,\infty) \to \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_\omega)$ such that $v(0) = T_\omega \widetilde{u}$ and, for $\theta \in (0,1)$,*

$$h_\omega^{-2+2\theta} \int_0^\infty t^{2(1-\theta)} \Big(\|\nabla v(t)\|_{L^2(\omega)}^2 + \|v'(t)\|_{L^2(\omega)}^2 \Big) \frac{\mathrm{d}t}{t} \lesssim |\widetilde{u}|_{H^\theta(\widehat{T})}^2 + \Big\| d_{\widehat{\mathcal{E}}}^{-\theta} \widetilde{u} \Big\|_{L^2(\widehat{T})}^2 + \|\widetilde{u}\|_{W^{1,\infty}(\widehat{T}_\delta)}^2,$$

where $\delta > 0$ is chosen according to Lemma 5.4(vii).

Proof. Let $\varepsilon > 0$ be chosen according to Lemma 5.4. Define a function v from $[0,1]$ into the space of functions on $\widehat{T}$ by $\widetilde{v}(t)(x,y) := (\mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} \widetilde{u})(x,y,t)$. Note that $\widetilde{v}(1)(x,y) = 0$, and we extend v by zero to $[1,\infty)$. Then set $v(t) := T_\omega(\widetilde{v}(t/h_\omega))$. We have $v : [0,\infty) \to \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}|_\omega)$ and $v(0) = T_\omega((\mathcal{A}_{\widehat{\mathcal{E}}}^{P} \widetilde{u})(0)) = T_\omega \widetilde{u}$. With Lemma 5.4, we conclude

$$\begin{aligned}
h_\omega^{2-2\theta} & \int_0^\infty t^{2(1-\theta)} \Big(\|\nabla v(t)\|_{L^2(\omega)}^2 + \|v'(t)\|_{L^2(\omega)}^2 \Big) \frac{\mathrm{d}t}{t} \\
&= \int_0^1 t^{2(1-\theta)} \Big(\|\nabla T_\omega \widetilde{v}(t)\|_{L^2(\omega)}^2 + h_\omega^{-2} \|T_\omega \widetilde{v}'(t)\|_{L^2(\omega)}^2 \Big) \frac{\mathrm{d}t}{t} \\
&\lesssim \int_0^1 t^{2(1-\theta)} \Big(\|\nabla \widetilde{v}(t)\|_{L^2(\widehat{T})}^2 + \|\nabla \widetilde{v}(t)\|_{L^\infty(\widehat{T}_\varepsilon)}^2 + \|\widetilde{v}'(t)\|_{L^2(\widehat{T})}^2 + \|\widetilde{v}'(t)\|_{L^\infty(\widehat{T}_\varepsilon)}^2 \Big) \frac{\mathrm{d}t}{t} \\
&= \Big\| d_{\widehat{T}\times\{0\}}^{1/2-\theta} \nabla \mathcal{A}_{\widehat{\mathcal{E}}}^{\widehat{P}} \widetilde{u} \Big\|_{L^2(\widehat{P})}^2 + \int_0^1 t^{2(1-\theta)} \|\nabla \widetilde{v}(t)\|_{L^\infty(\widehat{T}_\varepsilon)}^2 + \|\widetilde{v}'(t)\|_{L^\infty(\widehat{T}_\varepsilon)}^2 \frac{\mathrm{d}t}{t} \\
&\lesssim |\widetilde{u}|_{H^\theta(\widehat{T})}^2 + \Big\| d_{\widehat{\mathcal{E}}}^{-\theta} \widetilde{u} \Big\|_{L^2(\widehat{T})}^2 + \int_0^1 t^{2(1-\theta)} \|\nabla \widetilde{v}(t)\|_{L^\infty(\widehat{T}_\varepsilon)}^2 + \|\widetilde{v}'(t)\|_{L^\infty(\widehat{T}_\varepsilon)}^2 \frac{\mathrm{d}t}{t} \\
&\lesssim |\widetilde{u}|_{H^\theta(\widehat{T})}^2 + \Big\| d_{\widehat{\mathcal{E}}}^{-\theta} \widetilde{u} \Big\|_{L^2(\widehat{T})}^2 + \|\widetilde{u}\|_{L^2(\widehat{T})}^2 + \|\nabla \widetilde{u}\|_{L^\infty(\widehat{T}_\delta)}^2 + \|\widetilde{u}\|_{L^\infty(\widehat{T}_\delta)}^2 .
\end{aligned}$$

Here, the first estimate follows from (4.7) in Lemma 4.5, the second one from Lemma 5.4(vi), and the last one from Lemma 5.4(vii). ☐

Finally, we combine the previous liftings for local functions with the decomposition result of Lemma 4.5 to see that arbitrary global functions admit a stable lifting. This culminates in:

Proof of Theorem 1.1. We will only treat the case of homogeneous Dirichlet boundary conditions, the general case follows along the same lines. The canonical continuous embeddings

$$\Big(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)} \Big) \subset L^2(\Omega), \qquad \Big(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}), \|\cdot\|_{\widetilde{H}^1(\Omega)} \Big) \subset \widetilde{H}^1(\Omega)$$

immediately yield for all $u \in \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$ the estimate

$$\|u\|_{[(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta} \gtrsim \|u\|_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta}.$$

It therefore remains to show the converse estimate. To that end, we employ Lemma 4.5 and write

$$u = u_1 + \sum_{V \in \mathcal{V}^{\mathrm{int}}} T_{\omega_V}(\widetilde{u}_V) + \sum_{e \in \mathcal{E}^{\mathrm{int}}} T_{\omega_e}(\widetilde{u}_e) + \sum_{K \in \mathcal{T}} u_K.$$

The map $u \mapsto u_1$ is linear and bounded in $L^2(\Omega)$ and $\widetilde{H}^1(\Omega)$, and hence also in $[L^2(\Omega), \widetilde{H}^1(\Omega)]_\theta$. As it maps into $\widetilde{\mathcal{S}}^{1,1}(\mathcal{T})$, we moreover obtain

$$\|u_1\|_{[(\widetilde{\mathcal{S}}^{1,1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{1,1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta} \lesssim \|u\|_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta},$$

and this also yields

$$\|u_1\|_{[(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta} \lesssim \|u\|_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta}.$$

We conclude that it remains to show that

$$\|u-u_1\|_{[(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta} \lesssim \|u\|_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta}.$$

For $K \in \mathcal{T}$, let v_K be the functions constructed from u_K by Lemma 5.5. For $V \in \mathcal{V}^{\rm int}$ and $e \in \mathcal{E}^{\rm int}$ denote by v_V and v_e the functions constructed from u_V and u_e by Lemma 5.6. Define

$$v = \sum_{V\in\mathcal{V}^{\rm int}} v_V + \sum_{e\in\mathcal{E}^{\rm int}} v_e + \sum_{K\in\mathcal{T}} v_K$$

and note that $v : [0,\infty) \to \widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T})$ as well as $v(0) = u - u_1$. Furthermore,

$$\begin{aligned}
&\int_0^\infty t^{2(1-\theta)}\Big(\|\nabla v(t)\|^2_{L^2(\Omega)} + \|v'(t)\|^2_{L^2(\Omega)}\Big)\frac{\mathrm{d}t}{t}\\
&\lesssim \sum_{V\in\mathcal{V}^{\rm int}} \int_0^\infty t^{2(1-\theta)}\Big(\|\nabla v_V(t)\|^2_{L^2(\omega_V)} + \|v_V'(t)\|^2_{L^2(\omega_V)}\Big)\frac{\mathrm{d}t}{t}\\
&\quad + \sum_{e\in\mathcal{E}^{\rm int}} \int_0^\infty t^{2(1-\theta)}\Big(\|\nabla v_e(t)\|^2_{L^2(\omega_e)} + \|v_e'(t)\|^2_{L^2(\omega_e)}\Big)\frac{\mathrm{d}t}{t}\\
&\quad + \sum_{K\in\mathcal{T}} \int_0^\infty t^{2(1-\theta)}\Big(\|\nabla v_K(t)\|^2_{L^2(K)} + \|v_K'(t)\|^2_{L^2(K)}\Big)\frac{\mathrm{d}t}{t}\\
&\lesssim \sum_{V\in\mathcal{V}^{\rm int}} h_{\omega_V}^{2-2\theta}\Big(|\widetilde{u}_V|^2_{H^\theta(\widehat{T})} + \|d_{\widehat{e}_6}^{-\theta}\widetilde{u}_V\|^2_{L^2(\widehat{T})} + \|\widetilde{u}_V\|^2_{W^{1,\infty}(\widehat{T}_\delta)}\Big)\\
&\quad + \sum_{e\in\mathcal{E}^{\rm int}} h_{\omega_e}^{2-2\theta}\Big(|\widetilde{u}_e|^2_{H^\theta(\widehat{T})} + \|d_{\widehat{e}_5\cup\widehat{e}_6}^{-\theta}\widetilde{u}_e\|^2_{L^2(\widehat{T})} + \|\widetilde{u}_e\|^2_{W^{1,\infty}(\widehat{T}_\delta)}\Big)\\
&\quad + \sum_{K\in\mathcal{T}} |u_K|^2_{H^\theta(K)} + \|d_{\partial K}^{-\theta} u_K\|^2_{L^2(K)}\\
&\lesssim \|u\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta}.
\end{aligned}$$

Here, the first estimate follows using the finite overlap of the supports of the involved patches and a coloring argument, the second one from Lemmas 5.5 and 5.6, and the last one from Lemma 4.5. Hence, with Lemma 5.1(ii) we conclude that

$$|u-u_1|^2_{[(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta} \leq \|u\|^2_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta},$$

and Proposition 4.2 shows

$$\begin{aligned}
\|u-u_1\|_{[(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta} &\lesssim \|u-u_1\|_{L^2(\Omega)} + |u-u_1|_{[(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{L^2(\Omega)}),(\widetilde{\mathcal{S}}^{\mathbf{p},1}(\mathcal{T}),\|\cdot\|_{\widetilde{H}^1(\Omega)})]_\theta}\\
&\lesssim \|u\|_{[L^2(\Omega),\widetilde{H}^1(\Omega)]_\theta},
\end{aligned}$$

which concludes the proof. ☐

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