ESAIM: M2AN 59 (2025) 137–166
https://doi.org/10.1051/m2an/2024057

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

POLYNOMIAL BOUNDS FOR THE SOLUTIONS OF PARAMETRIC TRANSMISSION PROBLEMS ON SMOOTH, BOUNDED DOMAINS

Simon Labrunie[1,*], Hassan Mohsen[1] and Victor Nistor[2]

Abstract. We consider a *family* $(P_\omega)_{\omega\in\Omega}$ of elliptic second order differential operators on a domain $U_0 \subset \mathbb{R}^m$ whose coefficients depend on the space variable $x \in U_0$ and on $\omega \in \Omega$, a parameter space. We allow the coefficients a_{ij} of P_ω to have jumps over a fixed interface $\Gamma \subset U_0$ (independent of $\omega \in \Omega$). We obtain estimates on the norm of the solution u_ω to the equation $P_\omega u_\omega = f$ with transmission and mixed boundary conditions that are *polynomial in the norms of the coefficients*. In particular, we show that, if f and the coefficients a_{ij} are smooth enough and follow a log-normal-type distribution, then the map $\Omega \ni \omega \mapsto \|u_\omega\|_{H^{k+1}(U_0)}$ is in $L^p(\Omega)$, for all $1 \le p < \infty$. The same is true for the norms of the inverses of the resulting operators. We also obtain similar integrability results for the parametric Finite Element approximations of the solution. We expect our estimates to be useful in Uncertainty Quantification.

Mathematics Subject Classification. 35J25, 35J75, 65N30.

Received September 25, 2023. Accepted July 11, 2024.

1. Introduction

We consider a *family* of second order elliptic problems $P_\omega u_\omega = f$ with $\omega \in \Omega$, a parameter space. We study the dependence of u_ω on ω with the goal of establishing suitable integrability results when Ω is a measure space. We start by taking the coefficients of our operators as parameters. Let us now explain our results in more detail.

1.1. Setting and statement of main result

We consider the second order transmission/mixed boundary value problem

$$\begin{cases} P^A u = f & \text{in } U_0 \\ u = \tilde{g} & \text{on } \partial_D U_0 \\ \partial_\nu^A u = g & \text{on } \partial_N U_0 \\ [[u]]_\Gamma = \tilde{h} & \text{on } \Gamma \\ [[\partial_\nu^A u]]_\Gamma = h & \text{on } \Gamma \end{cases} \tag{1}$$

Keywords and phrases. Parametric elliptic equations, strongly elliptic operators, transmission problems, Poincaré inequality, Sobolev spaces, finite element method, uncertainty quantification.

[1] Université de Lorraine, CNRS, IECL, 54000 Nancy, France.
[2] Université de Lorraine, CNRS, IECL, 57000 Metz, France.
[*]Corresponding author: simon.labrunie@univ-lorraine.fr

on a *bounded (open) domain* $U_0 \subset \mathbb{R}^m$. The symbols appearing in this problem have the following meaning. The matrix $A = [a_{ij}] \in M_{m+1}(L^\infty(U_0))$, $i, j = 0, \dots, m$, is the matrix of coefficients of

$$P^A u(x) := \sum_{i,j=0}^{m} \partial_i \big(a_{ij}(x) \partial_j u(x)\big), \tag{2}$$

where we have set $\partial_0 := id$, for convenience, and $\partial_j := \frac{\partial}{\partial x_j}$ for $j \geq 1$, as usual. We assume that the domain U_0 on which our problem is formulated is open and is *decomposed* into open subsets U_j, meaning that

$$\overline{U}_0 = \bigcup_{j=1}^{N} \overline{U}_j \tag{3}$$

where $U_j \subset U_0$, $j = 1, \dots, N$, are open and disjoint. We allow the coefficients a_{ij} of P^A to have jumps over a fixed interface

$$\Gamma := \bigcup_{j=1}^{N} \partial U_j \smallsetminus \partial U_0 \subset U_0. \tag{4}$$

Except in Section 2 and Subsection 5.1, we shall assume that U_0 is *bounded.* The expression $[[w]]_\Gamma$ denotes the *jump* $w_+ - w_-$ of w at the interface Γ, as usual (this is recalled in Def. 2.9 and the discussion preceeding it). We assume that

$$\partial U_0 = \partial_D U_0 \cup \partial_N U_0 \quad \text{(disjoint union)}, \tag{5}$$

with $\partial_D U_0$ and $\partial_N U_0$ both closed in ∂U_0 (they are hence both also open in ∂U_0). This is necessary in order to avoid the singularities due to the jump in the type of boundary conditions. The treatment of jumps in boundary conditions requires additional techniques, so we leave it for another project.

Our main result is a *polynomial estimate* in the norm of the coefficients A for the norm of the solution u of the problem (1), see Theorem 1.1 right below. To formulate this theorem, let

$$H^1_D(U_0) := \{v \in H^1(U_0) \mid v|_{\partial_D U_0} = 0\} \tag{6}$$

$\big($so $H^1_0(U_0) \subset H^1_D(U_0)$, but we have equality only if $\partial_D U_0 = \partial U_0\big)$ and

$$\mathcal{P}^A_0 : H^1_D(U_0) \to H^1_D(U_0)^* \tag{7}$$

be the operator associated to the Dirichlet form that yields also (2) (see the weak formulation of our problem, Def. 2.19). Let

$$\check{H}^k(U_0) := \bigoplus_{j=1}^{N} H^k(U_j) \quad \text{and} \quad \check{W}^{k,\infty}(U_0) := \bigoplus_{j=1}^{N} W^{k,\infty}(U_j) \tag{8}$$

be the *broken Sobolev spaces* associated to the decomposition of $U_0 \subset \mathbb{R}^m$ into the disjoint subdomains U_j. It is convenient to set $F := \Big(f, \tilde{g}, g, \tilde{h}, h\Big)$ for the data in problem (1) and

$$\begin{gathered} \|F\|_{tot,k} := \|f\|_{\check{H}^k} + \|\tilde{g}\|_{H^{k+\frac{3}{2}}} + \|\tilde{h}\|_{H^{k+\frac{3}{2}}} + \|g\|_{H^{k+\frac{1}{2}}} + \|h\|_{H^{k+\frac{1}{2}}} \quad \text{if} \\ F \in \check{H}^k(U_0) \oplus \check{H}^{k+3/2}(\partial_D U_0) \oplus \check{H}^{k+1/2}(\partial_N U_0) \oplus \check{H}^{k+3/2}(\Gamma) \oplus \check{H}^{k+1/2}(\Gamma). \end{gathered} \tag{9}$$

If E is a normed space, we shall typically write $\|\cdot\|_E$ for its norm. If $\xi \notin E$, we let $\|\xi\|_E := \infty$. We need one more ingredient to state our main theorem. Let $\phi_\alpha : W_\alpha \to W'_\alpha$ be coordinate changes that straighten (locally) the boundary or the interface. We can choose a finite covering of $\partial U_0 \cup \Gamma$ with such coordinate systems and we write the corresponding matrix of coefficients as $\big(a^\alpha_{ij}\big)$, with the straightened boundary being given by $x_m = 0$. We let then $\big|\big|\big|a^{-1}_{mm}\big|\big|\big|_{\mathcal{V},k}$ be the maximum norm of the $W^{k,\infty}$ norms of the coefficients $(a^\alpha_{mm})^{-1}$ (see Def. 3.10; the index $\mathcal{V}$ comes from the fact that this norm is defined in terms of a certain covering $\mathcal{V}$).

Theorem 1.1. *Let us assume that $U_0 \subset \mathbb{R}^m$ is bounded, that all boundaries of our domains ∂U_j, $j = 0, \ldots, N$, are smooth and that the Dirichlet part of the boundary $\partial_D U_0$, the Neumann part of the boundary $\partial_N U_0$, and the interface Γ are closed and disjoint. Then there exist $C_k > 0$, $k \in \mathbb{Z}_+$, with the following property. If the matrix of coefficients A of an invertible operator $\mathcal{P}_0^A : H_D^1(U_0) \to H_D^1(U_0)^*$ is such that $A = [a_{ij}] \in M_{m+1}\left(\check{W}^{k+1,\infty}(U_0)\right)$, then for every $F := \left(f, \tilde{g}, g, \tilde{h}, h\right)$, as in equation (9), the solution u of equation (1) satisfies*

$$\|u\|_{\check{H}^{k+2}(U_0)} \leq C_k \sum_{q=0}^{k+1} \left\| \left(\mathcal{P}_0^A\right)^{-1} \right\|^{q+1} |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)k+1} \|A\|_{\check{W}^{k+1,\infty}(U_0)}^{(q+1)(k+1)} \|F\|_{tot,k+1-q}.$$

The bound C_k of in this theorem *does not depend on the coefficients* A of the operator or on the forcing terms F (but may depend on k and on the geometry of the open sets $U_0, U_1, \ldots, U_N$). This is relevant since, for *a fixed operator,* (so no parameters) the result was known in the strongly elliptic case (it is a very classical result if there is no interface). Our result thus clarifies the dependence on the coefficients A of the regularity bounds in this classical results.

In particular, if $\mathcal{P}_0^A : H_D^1(U_0) \to H_D^1(U_0)^*$ is invertible, then the induced operator

$$\mathcal{P}_{k+1}^A : V_{k+1} := \check{H}^{k+2}(U_0) \cap H_D^1(U_0) \to V_{k+1}^- := \check{H}^k(U_0) \oplus H^{k+1/2}(\partial_N U_0 \cup \Gamma), \tag{10}$$

is also invertible ($k \geq 0$). This operator thus takes into account the Neumann boundary conditions and the jump at the interface. *What our theorem says then is that we can control polynomially the norm of $\left(\mathcal{P}_{k+1}^A\right)^{-1}$ in terms of the norm of $\left(\mathcal{P}_0^A\right)^{-1}$, of the norms of the coefficients A, and of the non-degeneracy of the coefficients at the boundary (the term $|||a_{mm}^{-1}|||_{\mathcal{V},k}$).* Note that we are not assuming that our operators are coercive, but we assume that $\mathcal{P}_0^A$ is invertible instead.

The polynomial bounds that we obtain are new even in the case of the Poisson problem (*i.e.* homogeneous Dirichlet boundary conditions and no interface). There is no significant difficulty in including an interface (*i.e.* studying transmission problems), as long as the strong ellipticity is preserved. In fact, as observed long time ago by Roitberg and Sheftel [55, 56], the study of strongly elliptic transmission problems problems is very similar to the usual elliptic ones. See also [42, 43, 45, 49–51] and the references therein for some more recent results. See [11, 13, 14, 18, 22, 52–54] for transmission problems that are *not* strongly elliptic.

1.2. The probabilistic setting

Let us assume that the coefficients a_{ij} depend on both the space variable $x \in U_0$ and on an additional variable $\omega \in \Omega$ in a probability space. This is a setting that appears in practice, because the coefficients a_{ij} represent properties of materials that are not always known exactly. Let $A(\omega) = [a_{ij}(\omega)] \in M_m(L^\infty(U_0))$. Our results then translate into estimates for the family $P^{A(\omega)}$ and the associated transmission/mixed boundary value problem. One of the reasons Theorem 1.1 is important is that it can be used to treat log-normal-type distributions for the coefficients. Let $u_\omega \in H_D^1(U_0)$ be the solution of $\mathcal{P}_0^{A(\omega)} u_\omega = f$. We then show that, if f and the coefficients a_{ij} are smooth enough and follow a log-normal-type distribution, then the map $\Omega \ni \omega \mapsto \|u_\omega\|_{H^{k+1}(U_0)}$ is in $L^p(\Omega)$, for all $0 \leq p < \infty$. The same is true for the norm of $\left(\mathcal{P}_k^{A(\omega)}\right)^{-1}$, see Theorems 5.5 and 5.8. We apply our results to obtain Finite Element estimates for Gaussian families and to prove that the error is integrable, Theorem 5.10, thus having the potential to yield optimal rates of convergence. We expect these estimates to be useful in Uncertainty Quantification [4–6, 23, 25, 32, 37, 39, 41, 57]. See [7, 24] for the importance of log-normal laws when studying problems in Uncertainty Quantification.

1.3. Contents

The paper is organized as follows. In Section 2 we introduce our setting: the domain $U_0 \subset \mathbb{R}^m$ and its decomposition in subdomains, the basic spaces (the broken Sobolev spaces $\check{H}^k$ and $\check{W}^{k,\infty}$, $V_k := \check{H}^{k+1}(U_0) \cap$

$H^1_D(U_0)$, for $k \geq 0$ and $V_k^- := \breve{H}^{k-1}(U_0) \oplus H^{k-1/2}(\Gamma \cup \partial_N U_0)$, for $k \geq 1$) and the operators $\mathcal{P}_k^A : V_k \to V_k^-$, $\mathcal{P}_k^A u = \big(P^A u, D_\nu^A u, \big[\big[D_\nu^A u\big]\big]\big)$. We also formulate Nirenberg's trick, which is used in "raising the regularity" in the induction step proof of our main theorem, Theorem 4.1. In Section 3 we localize our spaces and differential operators in order to be able to use the approach of [16, 47, 48]. Essentially, we "straighten the boundary or the interface," but in a more sophisticated way. More precisely, we introduce local bases $\big(X_j^\alpha\big)$, $j = 1, \dots, m$, of vector fields in local coordinate patches W_α and prove the needed technical results in terms of these vector fields. Section 4 contains the main estimates of this paper. It also contains some immediate applications, including the boundedness (and hence integrability) result for the norms of $\big(\mathcal{P}_k^A\big)^{-1}$ if both A and $\big(\mathcal{P}_0^A\big)^{-1}$ are bounded with respect to the parameters. In Section 5, we extend this integrability result to some situations when the norms of both A and $\big(\mathcal{P}_0^A\big)^{-1}$ are not bounded but are rather derived from some Gaussian variables. The operators of this section are assumed to be strongly elliptic. The last section includes some suggestions for future work.

1.4. Acknowledgements

We thank Patrick Ciarlet, Iulian Cîmpean, Mirela Kohr, Konstantin Pankrashkin, and Christoph Schwab for useful discussions. We are also deeply grateful to an anonymous referee who has very carefully read our paper and has made many suggestions for improvements.

2. The domain, the operator(s), and Nirenberg's trick

In this section, we formulate more precisely our mixed boundary value/transmission problem (1) and recall some useful results. We also introduce our domain U_0 and its decomposition into subdomains U_j and the basic spaces and operators. *Throughout this paper, U_0 will be an open subset of $\mathbb{R}^m$ with smooth boundary. Beginning with the next section, we shall assume that U_0 is bounded.*

We begin by introducing our bilinear form B^A and one of the basic differential operators $\mathcal{P}^A$, $\mathcal{P}_k^A$, and P^A that we will use. By $H^k(U_0)$ we denote the usual Sobolev spaces and by $L^p(U_0)$ the usual space of p-integrable functions [15, 33, 36, 44, 58].

2.1. The sesquilinear form B^A and the operator $\mathcal{P}^A$

Our main objects of study (forms, operators, ...) are based on the coefficient matrix

$$A := [a_{ij}] \in M_{m+1}(L^\infty(U_0)), \tag{11}$$

where, we recall, m is the dimension of the ambient space ($U_0 \subset \mathbb{R}^m$). We let $\partial_j := \frac{\partial}{\partial x_j}$, for $j = 1, 2, \dots, m$, as usual, and $\partial_0 := id$.

Definition 2.1. Let A be the $(m+1) \times (m+1)$ matrix of equation (11). We let $B^A : H^1(U_0) \times H^1(U_0) \to \mathbb{C}$ be the sesquilinear form

$$\begin{aligned} B^A(u, v) &:= \int_{U_0} \sum_{i,j=0}^{m} a_{ij}(x) \partial_j u(x) \partial_i \overline{v(x)} \, \mathrm{d}x \\ &= \int_{U_0} \left(\sum_{i,j=1}^{m} a_{ij} \partial_j u \partial_i \overline{v} + \sum_{j=1}^{m} a_{0j} (\partial_j u) \overline{v} + \sum_{i=1}^{m} a_{i0} u \partial_i \overline{v} + a_{00} u \overline{v} \right) \mathrm{d}x. \end{aligned}$$

Our operator $\mathcal{P}^A : H^1(\Omega) \to H^1(\Omega)^*$ and its restriction $P^A : H^1(\Omega) \to H^{-1}(\Omega)$ will be obtained from the sesquilinear form B^A through a weak formulation. To that end, we need to recall the following conventions and notation. Let V^* be the *conjugate linear* dual of some topological vector space V (in our applications, V will always be a Hilbert space). Let

$$\langle \cdot, \cdot \rangle_{V^*, V} : V^* \times V \to \mathbb{C}$$

be the duality between V and V^*, which is, hence, a sesquilinear form (just like B^A). We shall use this duality mostly for $V = H_0^1(U_0)$, for which $V^* = H^{-1}(U_0) := H_0^1(U_0)^*$. We can now introduce the two operators $\mathcal{P}^A$ and P^A associated to $A \in M_{m+1}(L^\infty(\Omega))$.

Definition 2.2. Let $A := [a_{ij}] \in M_{m+1}(L^\infty(U_0))$ and B^A be as in Definition 2.1. We define the *full operator* $\mathcal{P}^A : H^1(U_0) \to H^1(U_0)^*$ by

$$\langle \mathcal{P}^A u, \phi \rangle_{H^1(U_0)^*, H^1(U_0)} \;=\; B^A(u, \phi).$$

Similarly, the *"plain" differential operator* $P^A : H^1(U_0) \to H^{-1}(U_0)$ is

$$P^A u := \; - \sum_{j=1}^{m} \sum_{i=1}^{m} \partial_i (a_{ij} \partial_j u) - \sum_{j=1}^{m} \partial_j (a_{j0} u) + \sum_{i=1}^{m} a_{0i} \partial_i u + a_{00} u.$$

The definition of $\mathcal{P}^A$ justifies the use of the *conjugate* linear dual. In turn, this is required by the fact that B^A is sesquilinear, which, in turn, is convenient when using positivity (as in the Riesz or Lax-Milgram Lemma's). Let us continue with some remarks, the first one on the notation.

Remark 2.3. The choice of the operator $\partial_0 := id$, the identity operator, is as in [40, 47, 48], and is convenient for giving a compact form for the form B^A:

$$B^A(u, v) := \sum_{i,j=0}^{m} (a_{ij} \partial_j u, \partial_i v),$$

where $(f, g) := \int_\Omega f(x) \overline{g(x)} \, \mathrm{d}x$ is the scalar product on $L^2(\Omega)$. Then $\partial_j^* = -\partial_j$ for $j > 0$, but (obviously), $\partial_0^* = \partial_0$. Thus $P^A = \sum_{i,j=0}^m \partial_i^* a_{ij} \partial_j$. Let $\mathcal{C}_c^\infty(W)$ denote the set of smooth, compactly supported functions on some open set W of a smooth manifold. Note that the map $P^A : \mathcal{C}_c^\infty(U_0) \to \mathcal{C}_c^\infty(U_0)$ does not determine the coefficient matrix A (we can commute $\partial_i \partial_j = \partial_j \partial_i$ or move derivatives past coefficients).

The second remark concerns the important relation between $\mathcal{P}^A$ and P^A.

Remark 2.4. For any $\phi \in H_0^1(U_0)$ and $u \in H^1(U_0)$, the relations defining $\mathcal{P}^A$ and P^A (Def. 2.2) yield

$$\langle \mathcal{P}^A u, \phi \rangle := B^A(u, \phi) \;=\; (P^A u, \phi).$$

The inclusion $j : H_0^1(U_0) \to H^1(U_0)$ yields by duality a surjective restriction map $j^* : H^1(U_0)^* \to H^{-1}(U_0)$. We thus obtain that

$$P^A \;=\; j^* \circ \mathcal{P}^A. \tag{12}$$

The operator $\mathcal{P}^A$ thus contains more information than the differential operator P^A. We will see later that the additional information involves boundary and transmission conditions. In fact, our main interest is in some closely related operators $\mathcal{P}_k = \mathcal{P}_k^A : V_k \to V_k^-$, $k \geq 0$. The operator $\mathcal{P}_k^A$ is a restriction of $\mathcal{P}_0^A$ and, in turn, this operator, is somewhere "between" $\mathcal{P}^A$ and P^A, satisfying a relation similar to equation (12). The relation between the operators $P^A, \mathcal{P}^A$, and $\mathcal{P}_k^A$, $k \geq 0$, is discussed in Remarks 2.20 and 2.21.

The matrix $A = [a_{ij}] \in M_{m+1}(L^\infty(U_0))$ is typically fixed, so it will be often omitted from the notation.

2.2. Decomposition of the domain U_0

In applications, we shall assume that the coefficients $a_{ij} \in L^\infty(\Omega)$, $i, j = 0, \ldots, m$, of the matrix $A = [a_{ij}] \in M_{m+1}(L^\infty(U_0))$ are *piecewise regular.* To define exactly our assumptions on the coefficients a_{ij}, we need to specify more on the geometry of our domain U_0, namely, to introduce a decomposition of this domain and the resulting interface Γ. This is the purpose of this subsection. Recall that, throughout this paper, $U_0 \subset \mathbb{R}^m$ will be an open subset. For simplicity, we shall also assume that U_0 is connected. This is no loss of generality, since the general result can be obtained by studying one connected component at a time.

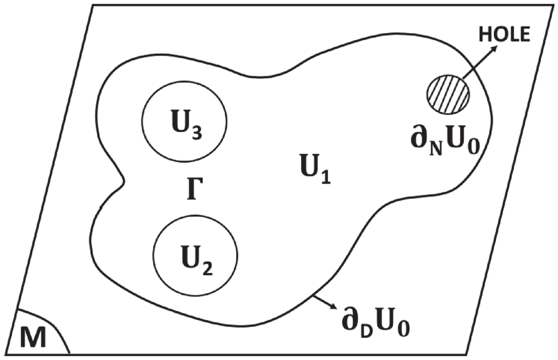

FIGURE 1. An illustration of the domains $U_j \subset M$ and of the interface Γ.

Remark 2.5. We recall that a tangent vector $X \in T_y\mathbb{R}^m$ is a derivation $X : \mathcal{C}^\infty(\mathbb{R}^m) \to \mathbb{C}$ such as $X(fg) = X(f)g(y) + f(y)X(g)$ and $X(\overline{f}) = \overline{X(f)}$ (since we are working with complex valued functions, but consider only vector fields with real coefficients). A tangent vector field to $\mathbb{R}^m$ is a family of tangent vectors $X(y) \in T_y\mathbb{R}^m$, for all $y \in \mathbb{R}^m$, which varies in a smooth way.

We will make the following assumptions (see [48] for a related, but different setting):

Assumptions 2.6. For any subset W of a topological space, we shall let ∂W denote the topological boundary of W. (Thus $\partial W := \overline{W} \smallsetminus W$ if W is open.)

(1) We are given N *disjoint, open* subsets $U_j \subset U_0$, $j = 1, \dots, N$, such that

$$\overline{U}_0 = \overline{U}_1 \cup \overline{U}_2 \cup \ldots \cup \overline{U}_N.$$

(2) Each U_j, $j = 0, \dots, N$ is connected with ∂U_j smooth.
(3) We are given a vector field $\nu^{[j]}$ that is smooth on $\overline{U}_j$ and that, on ∂U_j, is of unit length and outside pointing normal to ∂U_j.
(4) The *interface*

$$\Gamma := \left(\bigcup_{j=1}^{N} \partial U_j \right) \smallsetminus \partial U_0$$

is a compact, smooth submanifold of $\mathbb{R}^m$ (without boundary).
(5) We have fixed a vector field $\mathbf{N} : \Gamma \to \mathbb{R}^m$, unit normal to Γ at every point. (This is the same thing as choosing an orientation of Γ.)
(6) We are given $\partial_N U_0 \subset \partial U_0$ a union of connected components of ∂U_0 and we set

$$\partial_D U_0 := \partial U_0 \smallsetminus \partial_N U_0,$$

(which will hence also be closed).
(7) We will let $\nu : U_0 \smallsetminus \Gamma \to \mathbb{R}^m$ denote the vector field that on U_j coincides with $\nu^{[j]}$.

See Figure 1 for an illustration of our domains. Note that $\partial_D U_0$, $\partial_N U_0$, and Γ are all compact, smooth submanifolds of $\mathbb{R}^m$ and that they are *disjoint*. The fact that they are disjoint will prevent thus the appearance of singularities at the places where these sets would meet.

Note that the boundary ∂U_j of U_j is not assumed to be connected. We will impose Neumann boundary conditions on $\partial_N U_0$ and Dirichlet boundary conditions on $\partial_D U_0$. Also, as U_0 is open, we have

$$U_0 = U_1 \cup U_2 \cup \ldots \cup U_N \cup \Gamma \quad \text{(disjoint union).} \tag{13}$$

2.3. Function spaces: broken Sobolev spaces, jumps, and traces

Our hypotheses (Γ and all ∂U_j smooth) imply that Γ is closed and that $\Gamma \cap \partial U_0 = \emptyset$. In this framework, it is natural to consider the so-called "broken Sobolev spaces," as in other papers [43, 49].

Definition 2.7. Let $U_0 \subset \mathbb{R}^m$ be an open subset as in Assumption 2.6 and $\mathcal{O} \subset U_0$ be an open subset. We let

$$\check{H}^s(\mathcal{O}) := \bigoplus_{j=1}^{N} H^s(\mathcal{O} \cap U_j),$$
$$\check{W}^{s,\infty}(\mathcal{O}) := \bigoplus_{j=1}^{N} W^{s,\infty}(\mathcal{O} \cap U_j), \quad s \geq 0, \tag{14}$$

with the standard direct sum norms denoted $\|\cdot\|_{\check{H}^s(\mathcal{O})}$ and $\|\cdot\|_{\check{W}^{s,\infty}(\mathcal{O})}$.

We will omit the open set $\mathcal{O}$ in the notation of the norms on the above spaces if there is no danger of confusion (for instance, if $\mathcal{O} = U_0$), in particular, the norms on these spaces will be generically denoted $\|\cdot\|_{\check{H}^s}$ and $\|\cdot\|_{\check{W}^{s,\infty}}$ instead of $\|\cdot\|_{\check{H}^s(\mathcal{O})}$ and of $\|\cdot\|_{\check{W}^{s,\infty}(\mathcal{O})}$. All our function spaces will be complex vector spaces. The usual trace construction yields the "jump" on the interface, whose precise definition depends on our choices of orientations, as explained next.

Definition 2.8. Let $\Gamma_k \subset \partial U_j$ be a connected component of Γ and $p \in \Gamma_k \subset \overline{U}_j$. If $\mathbf{N}(p) = +\nu^{[j]}(p)$, we say that U_j is *to the right* of Γ in p. If $\mathbf{N}(p) = -\nu^{[j]}(p)$, we say that U_j is *to the left* of Γ in p.

We are now ready to introduce the jump over the interface Γ.

Definition 2.9. Using Definition 2.8, we define the jump $[[u]]_\Gamma \in H^{1/2}(\Gamma) = \bigoplus_k H^{1/2}(\Gamma_k)$ of $u \in \check{H}^1(U_0)$ (where Γ_k are the connected components of the interface Γ) by

$$[[u]]_{|\Gamma_k} := u^+|_{\Gamma_k} - u^-|_{\Gamma_k} \in H^{1/2}(\Gamma_k) \quad \text{and}$$
$$[[u]]_\Gamma := [[u]]_{|\Gamma_k} \text{ on } \Gamma_k \subset \Gamma\,,$$

where $u^+|_{\Gamma_k}$ and $u^-|_{\Gamma_k}$ be are the traces on Γ_k of the restrictions u^+ and u^- of u to the domain to the right and, respectively, to the left, of Γ_k.

We have the following simple remark

Remark 2.10. Let us assume that U_0 is bounded. Let $u \in \check{H}^1(U_0)$. We have $u \in H^1(U_0)$ if, and only if, $[[u]]_\Gamma = 0$. Moreover, the map $[[\cdot]]_\Gamma : \check{H}^1(U_0) \to H^{1/2}(\Gamma)$ is surjective. That is, we have an exact sequence

$$0 \longrightarrow H^1(U_0) \longrightarrow \check{H}^1(U_0) \xrightarrow{[[\cdot]]_\Gamma} H^{1/2}(\Gamma) \longrightarrow 0.$$

In particular, $\check{H}^1(U_0) \simeq H^1(U_0) \oplus H^{1/2}(\Gamma)$, but this isomorphism is not canonical (it depends on the choice of an extension $H^{1/2}(\Gamma) \to \check{H}^1(U_0)$). See [47] for a proof.

For us, a space more important than $H^1(U_0)$ will be the space

$$H^1_D(U_0) := \left\{u \in H^1(U_0) \mid u|_{\partial_D U_0} = 0\right\}.$$

Remark 2.11. By analogy with Remark 2.10, we have an exact sequence:

$$0 \longrightarrow H^1_D(U_0) \longrightarrow \check{H}^1(U_0) \cap \{w|_{\partial_D U_0} = 0\} \xrightarrow{[[\cdot]]_\Gamma} H^{1/2}(\Gamma) \longrightarrow 0.$$

In particular, $\check{H}^1(U_0) \cap \{w|_{\partial_D U_0} = 0\} \simeq H^1_D(U_0) \oplus H^{1/2}(\Gamma)$ non-canonically.

Below, we shall use without further comment the following identifications:

$$\check{H}^0(U_0) = H^0(U_0) = L^2(U_0) \text{ and } \check{W}^{0,\infty}(U_0) = W^{0,\infty}(U_0) = L^\infty(U_0).$$

2.4. The spaces V_k and V_k^- and the operators $\mathcal{P}_k^A$

As in [8, 16, 40, 47, 48] and in other papers, in order to study weak solutions, it is convenient to consider the spaces V_k and V_k^- introduced next:

Definition 2.12. Let $U_0 \subset \mathbb{R}^m$ satisfy Assumptions 2.6. For $k \geq 0$, we let:

$$\begin{aligned} V_k &:= \check{H}^{k+1}(U_0) \cap H^1_D(U_0) \\ V_k^- &:= \check{H}^{k-1}(U_0) \oplus H^{k-\frac{1}{2}}(\partial_N U_0 \cup \Gamma), \quad \text{if } k > 0, \text{ and} \\ V_0^- &:= V_0^* := H^1_D(U_0)^* . \end{aligned}$$

Remark 2.13. Explicitly, for $k \geq 0$, we have

$$V_k = \left\{ w \in \bigoplus_{i=1}^{N} H^{k+1}(U_i) \mid [[w]]_\Gamma = 0,\ w|_{\partial_D U_0} = 0 \right\}.$$

In particular, $V_0 = H^1_D(U_0)$. Similarly, for $k > 0$, we have that

$$V_k^- = \bigoplus_{i=1}^{N} H^{k-1}(U_i) \oplus H^{k-\frac{1}{2}}(\partial_N U_0) \oplus H^{k-\frac{1}{2}}(\Gamma).$$

Definition 2.14. Let $(u, v)_F := \int_F u(x)\overline{v(x)}\, \mathrm{d}S(x)$, where S is the induced volume form on $\Gamma \cup \partial U_0$ and let

$$I : V_1^- := L^2(U) \oplus H^{\frac{1}{2}}(\partial_N U_0) \oplus H^{\frac{1}{2}}(\Gamma) \to V_0^- := V_0^* := H^1_D(U_0)^*$$

be defined by

$$\langle I(f, g, h), v\rangle := (f, v) + (g, v)_{\partial_N U_0} + (h, v)_\Gamma,$$

for any $v \in H^1_D(U_0)$.

The content of the following remark is that the spaces V_k and V_k^- form an increasing scale of spaces.

Remark 2.15. Let $k \geq 0$. It follows immediately from the definitions that $V_{k+1} \subset V_k$. Similarly, $V_{k+1}^- \subset V_k^-$, for $k \geq 1$. Moreover, $I : V_1^- \to V_0^-$ of Definition 2.14 is also a continuous inclusion.

Before we introduce our operators of interest $\mathcal{P}_k^A$, let us introduce the conormal derivative D_ν^A.

Definition 2.16. Let $A := [a_{ij}] \in M_{m+1}\big(W^{1,\infty}(U_0)\big)$ and let ν be as in Assumptions 2.6. The *conormal derivative associated to* A is the operator $D_\nu^A : \check{H}^2(U_0) \to \check{H}^1(U_0)$,

$$D_\nu^A u := \sum_{i=1}^{m} \nu_i \left(\sum_{j=0}^{m} a_{ij} \partial_j u \right).$$

Since $\partial_0 = id$, we have the more explicit formula

$$D_\nu^A u := \sum_{i=1}^{m} \nu_i \left(\sum_{j=1}^{m} a_{ij} \partial_j u + a_{i0} u \right). \tag{15}$$

Remark 2.17. If $u \in \check{H}^2(U_0)$, then $D_\nu^A u \in \check{H}^1(U_0)$, so $D_\nu^A : \check{H}^2(U_0) \to \check{H}^1(U_0)$ is well defined because we have assumed that $\nu \in \check{W}^{1,\infty}(U_0)^m$. Therefore the jump $\big[\big[D_\nu^A u\big]\big]_\Gamma \in H^{\frac{1}{2}}(\Gamma)$ and the trace $D_\nu^A u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial U_0)$ are defined.

It is these last two terms of the above remark that are of greatest interest to us since they appear in the following simple integration by parts lemma, and then in the classical formulation of our transmission/boundary value problem, see Remark 2.22.

Lemma 2.18. *Let B^A and P^A be as in the Definition 2.2 and D_ν^A be as in Definition 2.16. If $A \in M_{m+1}\left(\check{W}^{1,\infty}(U_0)\right)$, then, for all $u \in \check{H}^2(U_0)$ and $v \in H^1(U_0)$, we have*

$$B^A(u,v) \,=\, \left(P^A u, v\right) + \left(D_\nu^A u, v\right)_{\partial U_0} + \left(\left[\left[D_\nu^A u\right]\right]_\Gamma, v\right)_\Gamma .$$

Proof. The proof is a standard integration by parts, as in [47, 48]. ☐

We are finally ready to introduce our operators of interest $\mathcal{P}_k^A$.

Definition 2.19. Let $B^A : H^1(U_0)^2 \to \mathbb{C}$ and $P^A : H^1(U_0) \to H^{-1}(U_0)$ be as in Definition 2.2 and let $A := [a_{ij}] \in M_{m+1}\left(W^{k,\infty}(U_0)\right)$. We then define $\mathcal{P}_k^A : V_k \to V_k^-$ as follows:

(1) If $k = 0$, then $\mathcal{P}_0^A : V_0 \to V_0^- := V_0^*$ is given by duality by

$$\left\langle \mathcal{P}_0^A u, \phi \right\rangle_{V_0^*, V_0} := B^A(u, \phi).$$

(2) If $k \geq 1$, let D_ν^A be as in Definition 2.16, then

$$\mathcal{P}_k^A u := \left(P^A u, D_\nu^A u|_{\partial_N U_0}, \left[\left[D_\nu^A u\right]\right]_\Gamma\right).$$

We see that, essentially, $\mathcal{P}_k^A$ is (a restriction of) $\mathcal{P}_0^A$. The indices k may thus also be omitted. The following two remarks discus the relation between the various operators $P^A, \mathcal{P}^A$, and $\mathcal{P}_k^A$, $k \in \mathbb{Z}_+$, introduced so far.

Remark 2.20. Of course, for $k \geq 1$, the operator $\mathcal{P}_{k+1}^A$ is obtained from $\mathcal{P}_k^A$ by restriction to V_{k+1}, using to the inclusions $V_{k+1} \subset V_k$ and $V_{k+1}^- \subset V_k^-$. In particular, $\mathcal{P}_k^A u = \mathcal{P}_{k+1}^A u$ for $u \in V_{k+1}$ and $k \geq 1$. In fact, the same relation holds also for $k = 0$, using inclusion $I : V_1^- \to V_0^-$ defined in Remark 2.15. Indeed, Lemma 2.18 gives, for all $u \in V_1$ and $v \in V_0$:

$$\begin{aligned}
\left\langle \mathcal{P}_0^A u, v \right\rangle_{V_0^*, V_0} &=: B^A(u,v) \\
&= \left(P^A u, v\right) + \left(D_\nu^A u, v\right)_{\partial U_0} + \left(\left[\left[D_\nu^A u\right]\right]_\Gamma, v\right)_\Gamma \\
&= \left(P^A u, v\right) + \left(D_\nu^A u, v\right)_{\partial_N U_0} + \left(\left[\left[D_\nu^A u\right]\right]_\Gamma, v\right)_\Gamma \\
&=: \left\langle I\left(\mathcal{P}_1^A u\right), v \right\rangle_{V_0^*, V_0} .
\end{aligned}$$

Thus $I(\mathcal{P}_1^A u) = \mathcal{P}_0^A u$ for all $u \in V_1$. This relation is crucial, since the invertibility is easiest to prove for the operator $\mathcal{P}_0^A$, whereas is it needed in applications for $\mathcal{P}_k^A$, $k \geq 1$, which involves, in addition to the invertibility of $\mathcal{P}_0^A$, a *regularity* result.

Let us now discuss the relation between the operators $\mathcal{P}_k^A$, $k \geq 0$, and the operators $\mathcal{P}^A$ and P^A.

Remark 2.21. Let $j_0^* : H^1(U_0)^* \to H_D^1(U_0)^* =: V_0^-$ be the dual of the inclusions $V_0 := H_D^1(U_0) \subset H^1(U_0)$. Then $\mathcal{P}_0^A = j_0^* \circ \mathcal{P}^A|_{V_0}$. Thus, the operator $\mathcal{P}^A$ determines all the operators $\mathcal{P}_k^A$, $k \geq 0$. If there are no Dirichlet boundary conditions (*i.e.* $\partial_D U_0 = \emptyset$), then $\mathcal{P}_0^A = \mathcal{P}^A$, but, in general, this is not true. Of course, when Γ and ∂U_0 are empty, then $\mathcal{P}^A = P^A$, as in [48], but this case is excluded in this paper since we are assuming U_0 to be bounded (that would be not true if we worked with open subsets in manifolds, but, in this paper, we recall, $U_0 \subset \mathbb{R}^m$). In fact, in this paper, none of the operators $\mathcal{P}_k^A$ or $\mathcal{P}^A$ coincides with P^A, the "plain" differential operator associated to our problem.

The following remark makes the link between the objects we have introduced so far and the classical transmission/mixed boundary value problems (Neumann boundary conditions on $\partial_N U_0$ and Dirichlet boundary conditions on $\partial_D U_0$).

Remark 2.22. Let $F = (f,g,h) \in V_1^- = L^2(U_0) \oplus H^{1/2}(\partial_N U_0 \cup \Gamma)$. For $u \in V_1 \subset \check{H}^2(U_0)$, the equation $\mathcal{P}_1^A u = F$ is equivalent to the system (1) of the Introduction. Thus, if $\mathcal{P}_1^A : V_1 \to V_1^-$ is invertible, then the solution to this system is therefore $u := \left(\mathcal{P}_1^A\right)^{-1} F$, where $F := (f,g,h) \in V_1^-$. If $\mathcal{P}_0^A$ is invertible and the coefficients A are smooth enough, then it is known classically that $\mathcal{P}_k^A$ is also invertible, at least in the classical case of a trivial interface $\Gamma = \emptyset$, see [15,33,44,58]. See [43,49] and the references therein for the less classical case of a non-trivial interface Γ. Thus, if $F := (f,g,h) \in V_k^- \subset V_1^- \subset V_0^-$, then $\left(\mathcal{P}_k^A\right)^{-1} F = \left(\mathcal{P}_1^A\right)^{-1} F = \left(\mathcal{P}_0^A\right)^{-1} F$.

The purpose of this paper is to obtain a *polynomial estimate in* $\|A\|$ *and* $\|(\mathcal{P}_0^A)^{-1}\|$ for the norm of $\left(\mathcal{P}_k^A\right)^{-1} F$, where $F := (f,g,h) \in V_k^-$. To that end, we want to use the approach in [16,47,48]. Some significant modifications are needed.

2.5. Nirenberg's trick

We shall need a version of Nirenberg's lemma (or "trick"), as it is formalized in [16, 48]. Let us first recall some classical definitions. We let $\mathcal{L}(\mathcal{X};\mathcal{Y})$ denote the set of continuous, linear maps $\mathcal{X} \to \mathcal{Y}$ between two normed spaces.

Definition 2.23. Let $\mathcal{X}$ and $\mathcal{Y}$ be two normed spaces and $T_t \in \mathcal{L}(\mathcal{X};\mathcal{Y})$, $t \geq 0$. We say that T_t *converges strongly* to T in $\mathcal{L}(\mathcal{X};\mathcal{Y})$ for $t \searrow 0$ (*i.e.* $t \to 0$, $t > 0$), if

$$\forall x \in \mathcal{X}, \quad \lim_{t \searrow 0} \|T_t x - Tx\|_{\mathcal{Y}} = 0.$$

We shall need also the following well-known concept [1].

Definition 2.24. A family of operators $(S(t))_{t\geq 0} \in \mathcal{L}(X) := \mathcal{L}(\mathcal{X};\mathcal{X})$ is a *strongly continuous semigroup* on $\mathcal{X}$ if:

(i) $S(0) = id_{\mathcal{X}}$ (the identity map).
(ii) For all $t, t' \geq 0$, we have $S(t+t') = S(t)S(t')$.
(iii) $S(t)$ converges strongly to $S(0) = id_{\mathcal{X}}$ in $\mathcal{L}(\mathcal{X})$ for $t \searrow 0$.

Definition 2.25. Let $S(t) \in \mathcal{L}(\mathcal{X})$ be a strongly continuous semigroup on a Banach space. Its *infinitesimal generator* is the (possibly unbounded) operator

$$L_S x := \lim_{t\to 0} t^{-1}(S(t)x - x)$$

with domain $\mathcal{D}(L_S) \subset \mathcal{X}$ the set of $x \in \mathcal{X}$ for which this limit exists.

We shall make essential use of the following lemma. (It is a form of "Nirenberg's trick".)

Lemma 2.26. *Let $S_{\mathcal{X}}(t) \in \mathcal{L}(\mathcal{X})$ and $S_{\mathcal{Y}}(t) \in \mathcal{L}(\mathcal{Y})$ be two strongly continuous semi-groups of operators on Banach spaces with infinitesimal generators $L_{\mathcal{X}}$ and $L_{\mathcal{Y}}$, respectively. Let us assume that:*

(i) $T \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ is invertible,
(ii) for all $t \geq 0$, there exists $T_t \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that $T_t S_{\mathcal{X}}(t) = S_{\mathcal{Y}}(t)T$, and
(iii) $t^{-1}(T_t - T)$ converges strongly to $Q \in \mathcal{L}(\mathcal{X};\mathcal{Y})$ for $t \searrow 0$.

Then $T^{-1}(\mathcal{D}(L_{\mathcal{Y}})) \subset \mathcal{D}(L_{\mathcal{X}})$, and, most importantly, for all $v \in \mathcal{D}(L_{\mathcal{Y}})$, we have

$$L_{\mathcal{X}}\left(T^{-1}v\right) = T^{-1}L_{\mathcal{Y}}v - T^{-1}QT^{-1}v.$$

See [16, 48] for a proof. It also follows from the proof that the induced operator $\tilde{T} := T|_{\mathcal{D}(L_{\mathcal{X}})} : \mathcal{D}(L_{\mathcal{X}}) \to \mathcal{D}(L_{\mathcal{Y}})$ is well-defined, continuous, and bijective.

3. Localization, Sobolev spaces, and diffeomorphism groups

In order to be able to use the approach in [16, 47, 48], as explained in the last section, we want to localize our functions and operators to coordinate neighborhood patches. This is the usual approach when dealing with regularity questions. We are faced however, with the following problem. Although we can localize the definitions of Sobolev spaces and of the operator $\mathcal{P}_0$, we cannot do the same for $\mathcal{P}_0^{-1}$. So, we will need a localization construction for derivatives (or vector fields) that will replace the canonical derivatives ∂_j (coming from coordinates) with other vector fields. This accounts for one of the greatest technical differences between this paper and [47,48]. It is linked to the fact that we must use general vector fields to study our Sobolev spaces (instead of the standard derivatives ∂_ℓ, as in [40]).

We assume, from now on, that U_0 is bounded.

3.1. Localization of vector fields and of norms

Let $\overline{U}_0 = \bigcup_{j=1}^N \overline{U}_j$, as before, see Assumptions 2.6. Recall that all boundaries ∂U_j are smooth, $j = 0, 1, \ldots, N$. (This implies that the boundary ∂U_0 of our domain and the interface $\Gamma := \left(\bigcup_{j=1}^N \partial U_j\right) \smallsetminus \partial U_0$ are smooth and closed.) We let $\mathbb{R}^m_\pm := \{\pm x_m \geq 0\}$ be the two half-spaces defined by the hyperplane $\{x_m = 0\}$. The following objects and assumptions will remain in place throughout the rest of this paper (some of which have already been used).

Notation 3.1. We shall use the following notation for the following fixed from now on objects:

(1) $\mathcal{V} := (V_\alpha)_{\alpha \in I}$ is a finite, open covering of the compact set $\overline{U}_0$ (so $V_\alpha \subset \mathbb{R}^m$ is open and $\overline{U}_0 \subset \bigcup_\alpha V_\alpha$).
(2) $\phi_\alpha : W_\alpha \to W'_\alpha \subset \mathbb{R}^m$ are some fixed from now on diffeomorphisms (coordinate charts) of open subsets with $\overline{V}_\alpha \subset W_\alpha$.
(3) Γ_k are the connected components of Γ (so $\Gamma = \bigcup \Gamma_k$, a disjoint union).

In addition to this specific notation, we shall let $\mathcal{C}^\infty_{\rm c}(W)$ denote the set of smooth, compactly supported functions on an open set W, as usual. The objects introduced in Notation 3.1 satisfy the following properties.

Assumptions 3.2. In addition to Assumptions 2.6, we assume that the objects introduced in Notation 3.1 satisfy the following properties:

(1) If $W_\alpha \cap \partial U_0 \neq \emptyset$, then $W_\alpha \cap \Gamma = \emptyset$ and

$$\phi_\alpha(W_\alpha \cap U_0) \subset \mathbb{R}^m_+ \quad \text{ and } \quad \phi_\alpha(W_\alpha \cap \partial U_0) \subset \mathbb{R}^{m-1} \, = \, \partial \mathbb{R}^m_+.$$

(2) If $W_\alpha \cap \Gamma \neq \emptyset$, then $W_\alpha \cap \partial U_0 = \emptyset$, there is only one k such that $W_\alpha \cap \Gamma_k \neq \emptyset$, and

$$\phi_\alpha(W_\alpha \cap \Gamma_k) \subset \mathbb{R}^{m-1} \, = \, \{x_m = 0\}.$$

Moreover, if $\Gamma_k \subset \partial U_j$, then, for some $\epsilon \in \{+, -\}$, we have

$$\phi_\alpha(W_\alpha \cap U_j) \subset \mathbb{R}^{m-1}_\epsilon := \{\epsilon x_m \geq 0\}.$$

(3) There is $\psi_\alpha \in \mathcal{C}^\infty_{\rm c}(W_\alpha)$ such that $\psi_\alpha = 1$ on V_α.
(4) We assume that both ϕ_α and ϕ_α^{-1} have components in $W^{\infty,\infty}$.

Thus each of the sets W_α intersects at most one of ∂U_0 or Γ_k. Since $\overline{V}_\alpha \subset W_\alpha$, it follows that $\mathcal{W} := (W_\alpha)_{\alpha \in I}$ is also an open covering of $\overline{U}_0$. Such finite coverings $\mathcal{V} := (V_\alpha)_{\alpha \in I}$ and $\mathcal{W} := (W_\alpha)_{\alpha \in I}$ and the functions ψ_α exist by standard topology results since U_0 is bounded and all boundaries ∂U_j, $j = 0, \ldots, N$, were assumed to be smooth, see Assumptions 2.6.

3.1.1. Localizing Sobolev spaces

We localize the definitions of our Sobolev spaces in the standard way.

Remark 3.3. We continue to use the notation of Notation 3.1 and Assumptions 3.2. The subdomains $(U_j)_{j=1}^N$ of U_0 and the given coordinate charts (diffeomorphisms) $\phi_\alpha : W_\alpha \to W'_\alpha$ induce decompositions on $W'_\alpha = \phi_\alpha(W_\alpha)$, which hence define broken Sobolev spaces on these domains. Let $V'_\alpha := \phi_\alpha(V_\alpha)$. Then the finite, open cover $\mathcal{V} = (V_\alpha)_{\alpha\in I}$, the coordinate charts ϕ_α, and the functions ψ_α define equivalent norms on the broken Sobolev spaces $\check{H}^s(U_0) := \bigoplus_{j=1}^N H^s(U_j)$ and $\check{W}^{s,\infty}(U_0) := \bigoplus_{j=1}^N W^{s,\infty}(U_j)$:

$$\begin{aligned} |u|_{\mathcal{V},s} &:= \sum_{\alpha\in I} \left\| u\circ\phi_\alpha^{-1} \right\|_{\check{H}^s(V'_\alpha)} \\ |||u|||_{\mathcal{V},s} &:= \max_{\alpha\in I} \left\| u\circ\phi_\alpha^{-1} \right\|_{\check{W}^{s,\infty}(V'_\alpha)}, \quad s\geq 0. \end{aligned}$$

The fact that these norms are equivalent to the original ones is a very standard calculation using the fact that we have a *finite covering* (by the compactness of $\overline{U}_0$) and that each diffeomorphism $\phi_\alpha|_{V_\alpha} : V_\alpha \to \phi_\alpha(V_\alpha) \subset W'_\alpha$ extends (obviously) to the larger open subset W_α containing $\overline{V}_\alpha$. Very similar proofs can be found, for instance, in [2, 34, 60] for infinite coverings in the framework of bounded geometry.

In order to define similar equivalent norms for the coefficient matrices, it will be convenient to use the following matrix norm:

Definition 3.4. Let $\mathcal{O} \subset U_0$. For a matrix $A = [a_{ij}] \in M_{m+1}\left(\check{W}^{k,\infty}(\mathcal{O})\right)$, we let

$$\|A\|_{\check{W}^{k,\infty}(\mathcal{O})} := \max_{0\leq i\leq m} \left\{ \sum_{j=0}^m \|a_{ij}\|_{\check{W}^{k,\infty}(\mathcal{O})}, \sum_{j=0}^m \|a_{ji}\|_{\check{W}^{k,\infty}(\mathcal{O})} \right\}.$$

We next define analogous norms to those introduced in Remark 3.3. Let ψ_α be as in Assumptions 3.2 (and in the definition of the vector fields X_j^α, see Def. 3.6) and the comments following it.

Remark 3.5. Let $A^\alpha \in M_{m+1}\left(W^{k,\infty}(W'_\alpha)\right)$ be the coefficients corresponding to the form induced by B^A on $\mathcal{C}_c^\infty(W'_\alpha)$. (The precise definition of A^α, while standard, is technical and hence is recalled after the following definition.) We let

$$|||A|||_{\mathcal{V},k} := \max_{\alpha\in I} \|A^\alpha\|_{\check{W}^{k,\infty}(W'_\alpha)}.$$

As in Remark 3.3 the norms $|||\cdot|||_{\mathcal{V},k}$ and $\|\cdot\|_{\check{W}^{k,\infty}}$ are equivalent on $M_{m+1}\left(\check{W}^{k,\infty}(U_0)\right)$.

3.1.2. Localizing vector fields

The role of the new coordinates ϕ_α, $\alpha \in I$, is to recover the setting considered in [47, 48]. That setting has already been dealt with in a uniform, comprehensive way. Most importantly, it allows us to replace the partial derivatives ∂_j [47, 48] with some other vector fields, which we introduce next.

Definition 3.6. See Notation 3.1 and Assumptions 3.2 for the notation and let, for each $\alpha \in I$, $Y_0^\alpha = X_0^\alpha := id$ and

$$\begin{aligned} Y_j^\alpha &:= \phi_{\alpha*}^{-1}(\partial_j) \quad \text{and} \\ X_j^\alpha &:= \tilde{\psi}_\alpha Y_j^\alpha = \tilde{\psi}_\alpha \phi_{\alpha*}^{-1}(\partial_j), \end{aligned}$$

where $\phi_{\alpha*}^{-1}(\partial_j)$ is the push-forward of the standard vector field $\partial_j := \frac{\partial}{\partial x_j}$, $j = 1, \ldots, m$, *via* the differentiable map ϕ_α^{-1}.

More explicitly, $X_j^\alpha = \tilde{\psi}_\alpha \sum_{i=1}^m \partial_j\left(\phi_{\alpha i}^{-1}\right)\partial_i$, where $\phi_{\alpha j}^{-1}$ are the components of ϕ^{-1}. We also obtain that X_j^α is tangent to the boundary of U_k for all $j < m$ and all k. Also, all vector fields X_j^α, $j = 1, \ldots, m$, $\alpha \in I$, are defined *everywhere* on $\overline{U}_0$ (unlike the vector fields Y_j^α, which will play an auxiliary role). We have defined $X_0^\alpha = id$ for the same reason for which we have introduced $\partial_0 := id$, because this will simplify some formulas.

3.1.3. Localizing the coefficients A

Let $P^A := \sum_{i,j=0}^m \partial_i^* a_{ij} \partial_j$ as before (see also Rem. 2.3 for notation and further explanations). For each $\alpha \in I$, the diffeomorphism $\phi_\alpha : W_\alpha \to W'_\alpha$ induces an isomorphism $\phi_\alpha^* : \mathcal{C}_c^\infty(W'_\alpha) \to \mathcal{C}_c^\infty(W_\alpha)$ by $\phi_\alpha^*(f) := f \circ \phi_\alpha$. A vector field X on W_α can be defined as a derivation $X : \mathcal{C}_c^\infty(W_\alpha) \to \mathcal{C}_c^\infty(W_\alpha)$, see Remark 2.5. As such, it can uniquely be written as $X = \sum_{j=1}^m X_j \partial_j$, with $X_j \in \mathcal{C}^\infty(W_\alpha)$, and it induces a vector field

$$\phi_{\alpha*}(X) := \phi_\alpha^{*-1} \circ X \circ \phi_\alpha^* =: Y = \sum_{j=1}^m Y_j \partial_j \tag{16}$$

on W'_α. We can apply this construction to any differential operator $Q : \mathcal{C}_c^\infty(W'_\alpha) \to \mathcal{C}_c^\infty(W'_\alpha)$ to obtain a differential operator

$$\phi_{\alpha*}(Q) := \phi_\alpha^{*-1} \circ Q \circ \phi_\alpha^* : \mathcal{C}_c^\infty(W'_\alpha) \to \mathcal{C}_c^\infty(W'_\alpha) \tag{17}$$

In particular, we can do that for our differential operator P^A (or, more precisely, to its restriction to $\mathcal{C}_c^\infty(W_\alpha)$) to obtain a differential operator $\phi_{\alpha*}(P^A) : \mathcal{C}_c^\infty(W'_\alpha) \to \mathcal{C}_c^\infty(W'_\alpha)$. It is, of course of the form $P^{A^\alpha} \in M_{m+1}(W'_\alpha)$, but A^α is not uniquely determined (see Rem. 2.3). However, by using that $\phi_{\alpha*}$ is an algebra morphism, the unique writing of equation (16) applied to each of the vector fields $\phi_{\alpha*}(\partial_j)$, $j = 1, \ldots, m$, and that $\phi_{\alpha*}(\partial_0) = \partial_0 = id$, we obtain a *canonical* $A^\alpha = \left[a_{ij}^\alpha\right]$:

$$\begin{aligned} \phi_{\alpha*}(P^A) &= \sum_{i,j=0}^m \phi_{\alpha*}(\partial_i^*) \left(a_{ij} \circ \phi_\alpha^{-1}\right) \phi_{\alpha*}(\partial_j) \\ &= \sum_{i,j=0}^m \partial_i^* \, a_{ij}^\alpha \, \partial_j \; =: P^{A^\alpha}. \end{aligned}$$

In particular, A^α depends linearly on A. Indeed, let $\phi_\alpha(\partial_i) = \sum_{j=1}^m Z_{ij} \partial_j$ and $\phi_\alpha(\partial_i^*) = \sum_{j=1}^m Z_{ij}^* \partial_j$. Let $Z_{00} = Z_{00}^*$ and $Z_{0i} = Z_{0i}^* = Z_{j0} = Z_{j0}^*$ if $i, j \geq 1$. Then

$$a_{kl}^\alpha = \sum_{ij} Z_{ik}^* a_{ij} \circ \phi_\alpha^{-1} Z_{jl}. \tag{18}$$

By applying the inverse morphism $\phi_{\alpha*}$ to the above equation, we obtain

Lemma 3.7. *Recall that $Y_j^\alpha := \phi_{\alpha*}^{-1}(\partial_j)$, $j = 1, \ldots, m$. Let $Y_j^{\alpha*} := -Y_j^\alpha$, if $j = 1, \ldots, m$. Let also $Y_0^\alpha = Y_0^{\alpha*} = id$. Then, the canonical matrix $A^\alpha = [a_{ij}^\alpha]$ of equation (18) satisfies $A^\alpha \in M_{m+1}\left(\check{W}^{k,\infty}(W'_\alpha)\right)$ and*

$$P^A = \sum_{i,j=0}^m Y_i^{\alpha*} \left(a_{ij}^\alpha \circ \phi_\alpha\right) Y_j^\alpha,$$

an equality of differential operators on $\mathcal{C}_c^\infty(W_\alpha)$.

3.2. Sobolev spaces and vector fields

It will be convenient to use the following shorthand notation for the following two norms that often appear in our results.

Notation 3.8. We let

$$\|T_1\|_k := \|T_1\|_{\mathcal{L}\left(V_k; V_k^-\right)} \quad \text{and} \quad |||T_2|||_k := \|T_2\|_{\mathcal{L}\left(V_k^-; V_k\right)}.$$

We shall write $a \lesssim b$ if there is a C that depends only on the order of the Sobolev spaces, on $\mathcal{V}$, and m such that $a \le Cb$. In particular, in these inequalities, C is independent of the variables belonging to function spaces and to spaces of operators.

Lemma 3.9. *We use Notation 3.8. Let $k \ge 0$, $a \in \check{W}^{k,\infty}(U_0)$, and $f \in \check{H}^k(U_0)$. Then $af \in \check{H}^k(U_0)$ and $\|af\|_{\check{H}^k} \lesssim \|a\|_{\check{W}^{k,\infty}} \|f\|_{\check{H}^k}$. Consequently,*

$$\left\|\mathcal{P}_k^A\right\|_k \lesssim \|A\|_{\check{W}^{k,\infty}} \quad \textit{and} \quad \left\|\mathcal{P}_k^A\right\|_k \lesssim |||A|||_{\mathcal{V},k}\,.$$

Proof. The first part is well known. The second part is a consequence of the first part. The third part is a consequence of the first part and of the definitions of the norms in local coordinates. Indeed

$$\left\|\mathcal{P}_k^A u\right\|_{V_k^-} \le \left\|\mathcal{P}_k^A\right\|_k \|u\|_{V_k} \lesssim \|A\|_{W^{k,\infty}} \|u\|_{V_k} \lesssim |||A|||_{\mathcal{V},k} \|u\|_{V_k},$$

where the last inequality is due to the equivalence of the norms $\|\cdot\|_{\check{W}^{k,\infty}}$ and $|||\cdot|||_{\mathcal{V},k}$ of Remark 3.5. □

We next localize the operators $\mathcal{P}_k$.

Definition 3.10. Let $A \in M_{m+1}\left(\check{W}^{k,\infty}(U_0)\right)$ and $A^\alpha = \left[a_{ij}^\alpha\right] \in M_{m+1}\left(\check{W}^{k,\infty}\left(W'_\alpha\right)\right)$ be as in Remark 3.5. We then let

$$\left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} := \max_\alpha \left\|\left(a_{mm}^\alpha\right)^{-1}\right\|_{\check{W}^{k,\infty}(W'_\alpha)}.$$

Note also that we may have $\left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} := \infty$, which is the same as saying that a_{mm}^α is not invertible everywhere on all W_α.

We will need the following results, which give a new description of the broken Sobolev spaces.

Lemma 3.11. *Let $A \in M_{m+1}\left(\check{W}^{k,\infty}(U_0)\right)$ and assume that $\mathcal{P}_k^A : V_k \to V_k^-$ is invertible (see Sect. 2.4). Let $A^\alpha = \left[a_{ij}^\alpha\right] \in M_{m+1}\left(\check{W}^{k,\infty}\left(W'_\alpha\right)\right)$ be as in Remark 3.5. Then*

$$1 \lesssim \left|\left|\left|(\mathcal{P}_k^A)^{-1}\right|\right|\right|_k |||A|||_{\mathcal{V},k} \quad \textit{and} \quad 1 \le \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} |||A|||_{\mathcal{V},k} \lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k,\infty}}.$$

Proof. The submultiplicativity of the norms, Lemma 3.9, and the equivalence of the norms of A in Remark 3.5 (in this order) give that

$$1 \le \left|\left|\left|(P^A)^{-1}\right|\right|\right|_k \left\|P^A\right\|_k \lesssim \left|\left|\left|\left(\mathcal{P}_0^A\right)^{-1}\right|\right|\right|_k \|A\|_{\check{W}^{k,\infty}} \lesssim \left|\left|\left|\left(\mathcal{P}_0^A\right)^{-1}\right|\right|\right|_k |||A|||_{\mathcal{V},k}\,.$$

The second part is similar. □

The following lemma is a crucial computational step for our estimates, it generalizes Lemma 5.1 of [48]. In the following, we include the domain U_0 in the notation for the norms on U_0, unlike in most of the other formulas. Recall that U_0 is assumed to be bounded.

Lemma 3.12. *Recall the vector fields X_j^α, $j = 1, \ldots, m$, $\alpha \in I$ of Definition 3.6 and that $X_0^\alpha = id$. We use also the notation of Lemma 3.11.*

(1) An equivalent norm on $\check{H}^{k+2}(U_0)$ is given by

$$\|u\|' := \sum_{\alpha \in I} \sum_{j=0}^{m} \left\|X_j^\alpha u\right\|_{\check{H}^{k+1}(U_0)}.$$

(2) Similarly, an equivalent norm on $\check{H}^{k+2}(U_0)$ is given by

$$\|u\|'' := \sum_{\alpha\in I}\sum_{i,j=0}^{m} \left\|X_i^\alpha X_j^\alpha u\right\|_{\check{H}^k(U_0)}.$$

(3) Let A and $P^A := \sum_{i,j=0}^m \partial_i^* a_{ij}\partial_j$ be as in Definition 2.2 and $\left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k}$ be as in Definition 3.10. Then

$$\|u\|_{\check{H}^{k+2}(U_0)} \lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k}\left(\|P^A u\|_{\check{H}^k(U_0)} + \|A\|_{\check{W}^{k+1,\infty}(U_0)}\sum_{j=0}^{m-1}\left\|X_j^\alpha u\right\|_{\check{H}^{k+1}(U_0)}\right).$$

The point of (3) is that we are not using the vector fields X_m^α.

Proof. The property (1) is true because it is true (well known) in the Euclidean case and because we have the norm equivalences of Remark 3.3. We include the details for the convenience of the reader. Let $V_\alpha' := \phi_\alpha(V_\alpha)$, where ϕ_α are as in Notation 3.1 and Assumptions 3.2. We then have $\left\|X_j^\alpha u\right\|_{\check{H}^{k+1}} \lesssim \|u\|_{\check{H}^{k+2}}$, for all $\alpha\in I$ and $j = 0,\dots,m$, and hence $\|u\|' \lesssim \|u\|_{\check{H}^{k+2}}$. To prove the other inequality, we proceed as follows:

$$\begin{aligned}
\|u\|_{\check{H}^{k+2}(U_0)} \lesssim |u|_{\mathcal{V},k+2} &:= \sum_{\alpha\in I}\left\|u\circ\phi_\alpha^{-1}\right\|_{\check{H}^{k+2}(V_\alpha')}\\
&\lesssim \sum_{\alpha\in I}\sum_{j=0}^{m}\left\|\partial_j\left(u\circ\phi_\alpha^{-1}\right)\right\|_{\check{H}^{k+1}(V_\alpha')}\\
&\lesssim \sum_{\alpha\in I}\sum_{j=0}^{m}\left\|\left(X_j^\alpha u\right)\circ\phi_\alpha^{-1}\right\|_{\check{H}^{k+1}(V_\alpha')}\\
&\lesssim \sum_{\alpha\in I}\sum_{j=0}^{m}\left|X_j^\alpha u\right|_{\mathcal{V},k+1} \lesssim \sum_{\alpha\in I}\sum_{j=0}^{m}\left\|X_j^\alpha u\right\|_{\check{H}^{k+1}} =: \|u\|',
\end{aligned}$$

where in the second inequality we have used the result in the Euclidean case (in which case it is simple, classical, and well known, as we have mentioned already).

The proof of (2) is completely similar, but this time comparing with

$$\sum_{\alpha\in I}\sum_{i,j=0}^{m}\left\|\partial_i\partial_j\left(u\circ\phi_\alpha^{-1}\right)\right\|_{\check{H}^{k+1}(V_\alpha')}.$$

Let us now prove (3). We may assume $A\in\check{W}^{k+1,\infty}(U_0)$ (otherwise the result is trivial, because the right hand term is infinite). For $u\in\mathcal{C}_c^\infty(V_\alpha)$, Lemma 3.7 gives

$$P^A u = \sum_{i,j=1}^{m} a_{ij}^\alpha X_i^\alpha X_j^\alpha u + R^\alpha u \tag{19}$$

where $R^\alpha := \sum_{j=0}^m r_j^\alpha\partial_j u$ is a differential operator of order one, with coefficients $r_j^\alpha \in \check{W}^{k,\infty}(U_0)$ and $\left\|r_j^\alpha\right\|_{\check{W}^{k,\infty}} \lesssim \|A\|_{\check{W}^{k+1,\infty}}$. Hence

$$\|R^\alpha u\|_{\check{H}^k} \lesssim \|A\|_{\check{W}^{k+1,\infty}}\|u\|_{\check{H}^{k+1}} \tag{20}$$

by Lemma 3.9. Moreover $a_{ij}^\alpha\in\check{W}^{k+1,\infty}(U_0)$ satisfy $\left\|a_{ij}^\alpha\right\|_{\check{W}^{k+1,\infty}} \lesssim \|A\|_{\check{W}^{k+1,\infty}}$.

Let Y_j^α be as defined in Definition 3.6. Let us write $\widetilde{\sum}_{i+j<2m}$ for the sum over all pairs (i,j) such that $i,j \geq 1$ and $i+j<2m$. For $u \in \mathcal{C}_c^\infty(W_\alpha)$, Lemma 3.7 gives

$$Y_m^\alpha Y_m^\alpha u \;=\; (a_{mm}^\alpha)^{-1}\left(R^\alpha u - \widetilde{\sum_{i+j<2m}} a_{ij}^\alpha Y_i^\alpha Y_j^\alpha u - P^A u\right). \tag{21}$$

Since $X_j^\alpha := \psi_\alpha Y_j^\alpha$ with $\psi_\alpha \in \mathcal{C}_c^\infty(W_\alpha)$, the inequality $1 \lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k+1,\infty}}$ of Lemma 3.11, the inequality (20), Lemma 3.9, equation (21), and the relation $Y_i^\alpha Y_j^\alpha = Y_j^\alpha Y_i^\alpha$ (which is true by the definition of Y_i^α and Y_j^α as the pull-backs of commuting vector fields) imply that

$$\begin{aligned}
\|X_m^\alpha X_m^\alpha u\|_{\check{H}^k} &\lesssim \|\psi_\alpha Y_m^\alpha Y_m^\alpha u\|_{\check{H}^k(W_\alpha)} + \|u\|_{\check{H}^{k+1}(W_\alpha)}\\
&\lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k+1,\infty}} \left(\|u\|_{\check{H}^{k+1}} + \widetilde{\sum_{i+j<2m}} \left\|\psi_\alpha Y_i^\alpha Y_j^\alpha u\right\|_{\check{H}^k}\right) + \left\|a_{mm}^{-1} P u\right\|_{\check{H}^k}\\
&\lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k+1,\infty}} \left(\|u\|_{\check{H}^{k+1}} + \sum_{i=1}^{m-1} \|X_i^\alpha(u)\|_{\check{H}^{k+1}}\right) + \left\|a_{mm}^{-1} P u\right\|_{\check{H}^k}\\
&= \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k+1,\infty}} \sum_{i=0}^{m-1} \|X_i^\alpha(u)\|_{\check{H}^{k+1}} + \left\|a_{mm}^{-1} P u\right\|_{\check{H}^k}.
\end{aligned}$$

The last inequality and part (2) then give that

$$\begin{aligned}
\|u\|_{\check{H}^{k+2}} &\lesssim \sum_{\alpha\in I} \sum_{i,j=0}^{m} \left\|X_i^\alpha X_j^\alpha u\right\|_{\check{H}^k}\\
&\lesssim \sum_{\alpha\in I} \left(\|X_m^\alpha X_m^\alpha u\|_{\check{H}^k} + \widetilde{\sum_{i+j\leq 2m-1}} \left\|X_i^\alpha X_j^\alpha u\right\|_{\check{H}^k} + \|u\|_{\check{H}^{k+1}}\right)\\
&\lesssim \sum_{\alpha\in I} \left(\|X_m^\alpha X_m^\alpha u\|_{\check{H}^k} + \sum_{i=1}^{m-1} \|X_i^\alpha u\|_{\check{H}^{k+1}} + \|u\|_{\check{H}^{k+1}}\right)\\
&\lesssim \sum_{\alpha\in I} \left(\left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k+1,\infty}} \sum_{i=0}^{m-1} \|X_i^\alpha(u)\|_{\check{H}^{k+1}} + \left\|a_{mm}^{-1} P u\right\|_{\check{H}^k}\right),
\end{aligned}$$

where, for the last equation, we used again the relation $1 \lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|A\|_{\check{W}^{k+1,\infty}}$ of Lemma 3.11. A final application of Lemma 3.9 gives $\left\|a_{mm}^{-1} P u\right\|_{\check{H}^k} \leq \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k} \|Pu\|_{\check{H}^k}$, and hence (3). □

3.3. Groups of diffeomorphisms

Let X be a vector field on $\mathbb{R}^m$ (see Rem. 2.5). We have that $X = \sum_{j=1}^m a_j \partial_j$, with the functions a_j smooth, real valued.

Remark 3.13. Let X be a compactly supported smooth vector field on $\mathbb{R}^m$ which is *tangent* to ∂U_0. Since X is compactly supported and tangent to the boundary of U_0, there is a one-parameter subgroup of diffeomorphisms $\phi_t : \overline{U}_0 \to \overline{U}_0$ that integrates X, in the sense that, for every $y \in \overline{U}_0$ and every smooth function $f : \overline{U}_0 \to \mathbb{R}$, the derivative of $\phi_t(y)$ at $t = 0$ is $X(y)$:

$$\lim_{t\to 0} t^{-1}(f(\phi_t(y)) - f(y)) \;=\; X(y)f.$$

We also have the group property: $\phi_t \circ \phi_s = \phi_{t+s}$, for all $t, s \in \mathbb{R}$. If moreover X is *tangent* at all boundaries ∂U_k, $k = 0, 1, \ldots, N$, then $\phi_t(U_k) = U_k$ for all $k = 0, 1, \ldots, N$.

Proposition 3.14. *Let X be a compactly supported, smooth vector field on $\mathbb{R}^m$ that is tangent to all boundaries ∂U_j, $j = 0, 1, \ldots, N$. We let $(\tau_t)_{t\in\mathbb{R}}$ denote the group of isomorphisms generated by X, that is, $\tau_t(f) := f \circ \phi_t$, where ϕ_t is as in Remark 3.13. Let $k \geq 0$ and $\mathcal{P}_k^A$ and the V_k and V_k^- be as in Section 2.4. Then*

(1) $(\tau_t)_{t\in\mathbb{R}}$ defines a strongly continuous group $(S(t))_{t\in\mathbb{R}}$ of operators on V_k such that $V_{k+1} \subset \mathcal{D}(L_S)$ and $L_S = X$ on V_{k+1}.
(2) Similarly, $(\tau_t)_{t\in\mathbb{R}}$ defines a strongly continuous group $S_-(t)$ of operators on V_k^- such that $V_{k+1}^- \subset \mathcal{D}(L_{S_-})$ and $L_{S_-} = X$ on V_{k+1}^-.
(3) The maps $X : V_{k+1} \to V_k$ and $X : V_{k+1}^- \to V_k^-$ induced by X are well-defined and continuous.
(4) $(\tau_t)_{t\in\mathbb{R}}$ also lifts to a group of isomorphisms (still denoted $(\tau_t)_{t\in\mathbb{R}}$) of $M_{m+1}\big(\check{W}^{k,\infty}(U_0)\big)$ by the relation $S(t)\mathcal{P}_k^A = \mathcal{P}_k^{\tau_t(A)} S_-(t)$ for $t \in \mathbb{R}$ and $A \in M_{m+1}\big(\check{W}^{k,\infty}(U_0)\big)$.
(5) If $A \in M_{m+1}\big(\check{W}^{k+1,\infty}(U_0)\big)$, then there exists $X(A) \in M_{m+1}\big(\check{W}^{k,\infty}(U_0)\big)$ (loss of one derivative!) such that

$$t^{-1}\left(\mathcal{P}_k^{\tau_t(A)} - \mathcal{P}_k^A\right) \to \mathcal{P}_k^{X(A)} =: X\left(\mathcal{P}_k^A\right)$$

strongly in $\mathcal{L}(V_k; V_k^-)$.

The precise formula for $X(A)$ is

$$X(A) = \sum_{i,j=0}^{m} \left([X, \partial_i^*]\, a_{ij}\partial_j + \partial_i^* X(a_{ij})\partial_j + \partial_i^* a_{ij}\, [X, \partial_j]\right).$$

We also note that the semigroup defined by $(\tau_t)_{t\in\mathbb{R}}$ on $\check{W}^{k,\infty}(U_0)$ is *not* strongly continuous. See Section 2.4 for the notation not explained in the statement.

Proof. The assumption that $\partial_D U_0$, $\partial_N U_0$, and Γ are all closed and disjoint and the fact that the problem is local show that we can assume that that $X = \psi\partial_\ell$, with $\ell < m$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^m)$ and that

- either $N = 1$ (so no interface), $U_0 = \mathbb{R}^m_+$, and $X = \psi\partial_\ell$
- or that $N = 2$ (so two subdomains), $U_0 = \mathbb{R}^m$, $U_1 = \mathbb{R}^m_+$, and $U_2 = \mathbb{R}^m_-$.

Except for the factor ψ, this is the case treated in [48]. We thus repeat the proof there with the small modifications warranted by the introduction of the factor ψ. As in [48], it suffices to show that we have a continuous inclusion $V_1^- \subset D(L_S)$ and the points (4) and (5), because the rest is an immediate consequence of the definitions.

To show the continuous inclusion $V_1^- \subset D(L_S)$, we will use the inclusion $I : V_1^- \to V_0^-$ of Remark 2.15. For all $v \in H^1(U_0)$ and $(f, g, h) \in V_1^- := L^2(U_0) \oplus H^{1/2}(\partial_N U_0) \oplus H^{1/2}(\Gamma)$, we have

$$\begin{aligned}\left\langle t^{-1}(S(t) - 1)I(f,g,h), v\right\rangle &= \left\langle I(f,g,h), t^{-1}(S(t)^* - 1)v\right\rangle \\ &:= (f, t^{-1}(S^*(t) - 1)\, v) + (g \oplus h, t^{-1}(S^*(t) - 1)v)_{\partial_N U_0 \cup \Gamma} \\ &\to -(f, \partial_\ell(\psi v)) - \langle g, \partial_\ell(\psi v|_{\partial_N U_0})\rangle_{\psi\partial_N U_0} - \langle h, \partial_\ell(\psi v|_\Gamma)\rangle_\Gamma \\ &=: \langle \psi\partial_\ell I(f,g,h), v\rangle = \langle \psi\partial_\ell f, v\rangle_{U_0} + \langle \psi\partial_\ell g, v\rangle_{\psi\partial_N U_0} + \langle \psi\partial_\ell h, v\rangle_\Gamma\,,\end{aligned}$$

where $\langle\cdot,\cdot\rangle_\Sigma$ is the duality between distributions and test-functions on the set Σ (which is an extension of the scalar product $(\cdot,\cdot)_\Sigma$, according to our conventions). So $I(f, g, h)$ is in the domain of the infinitesimal generator L_{S_-} of S_-,

$$L_{S_-} I(f,g,h) = \psi\partial_\ell I(f,g,h) = XI(f,g,h),$$

and the map $X = \psi\partial_\ell : V_1^- \to V_0^-$ is still well defined and continuous.

The point (4) follows from Lemma 3.7. Let us now prove the point (5). To simplify the notation, we will assume that $P = \partial_i(a_{ij}\partial_j)$, $a_{ij} \in \check{W}^{k+1,\infty}(U_0)$ (just one non-zero coefficient). Let then

$$g_t(x) := t^{-1}(a_{ij}(x + t\psi e_\ell) - a_{ij}(x)) - \psi\partial_\ell(a_{ij}(x)).$$

Then the $g_t \in W^{k,\infty}$ are uniformly bounded by $2^k(1 + \|\psi\|_{W^{k+1,\infty}})\|a_{ij}\|_{\check{W}^{k+1,\infty}}$ and therefore $\lim_{t\to 0} g_t w = 0$ in $\check{H}^k(U_0)$ for all $w \in \check{H}^k(U_0)$. By taking $w = \partial_j u$ for all j, we obtain (5). □

Now we state the corollary below which gives a version of Nirenberg's trick using vector fields.

Corollary 3.15. *We use the notation of Proposition 3.14. Suppose that $A \in M_{m+1}\left(\check{W}^{k+1,\infty}(U_0)\right)$ and that $\mathcal{P}_k = \mathcal{P}_k^A : V_k \to V_k^-$ is bijective. Let $X(\mathcal{P}_k^A) := \mathcal{P}_k^{X(A)}$, as in item (5) of that Proposition. Then, for all $F \in V_{k+1}^-$ we have*

$$X\left(\mathcal{P}_k^{-1}F\right) = \mathcal{P}_k^{-1}(XF) - \mathcal{P}_k^{-1}X(\mathcal{P}_k)\mathcal{P}_k^{-1}F.$$

The corollary gives then that

$$\left\|X\left(\mathcal{P}_k^{-1}F\right)\right\|_{\check{H}^{k+1}} = \left\|X\left(\mathcal{P}_k^{-1}F\right)\right\|_{V_k} \leq \left\|\mathcal{P}_k^{-1}XF\right\|_{V_k} + \left\|\mathcal{P}_k^{-1}X(\mathcal{P}_k)\mathcal{P}_k^{-1}F\right\|_{V_k}. \tag{22}$$

Proof. This is a direct and immediate consequence of Lemma 2.26 and of Proposition 3.14. Indeed, let's take $\mathcal{X} = V_k$, $\mathcal{Y} = V_k^-$, $T := \mathcal{P}_k = \mathcal{P}_k^A$ and the groups of diffeomorphisms S and S_-. It was assumed that $T := \mathcal{P}_k := \mathcal{P}_k^A$ is bijective. Proposition 3.14 (4) and (5) shows that the other two hypotheses of Lemma 2.26 are satisfied with $Q := X(\mathcal{P}_k) = \mathcal{P}_k^{X(A)}$. That lemma then gives that, for all F in $V_{k+1}^- \subset D(L_{S_-})$, we have $X(\mathcal{P}_k^{-1}F) = \mathcal{P}_k^{-1}(Xf) - \mathcal{P}_k^{-1}X(\mathcal{P}_k)\mathcal{P}_k^{-1}F$, therefore $\left\|X\left(\mathcal{P}_k^{-1}F\right)\right\|_{V_k} \leq \left\|\mathcal{P}_k^{-1}XF\right\|_{V_k} + \left\|\mathcal{P}_k^{-1}X(\mathcal{P}_k)\mathcal{P}_k^{-1}F\right\|_{V_k}$. □

4. Uniform estimates for families of operators

We now prove our main result, Theorem 4.1 and provide some applications. We consider the setting already introduced. In particular, recall that all our domains U_j, $j = 0, \ldots, N$, are bounded with smooth boundary and that $\partial_D U_0$, $\partial_N U_0$ and the interface $\Gamma := \left(\bigcup_{j=1}^N \partial U_j\right) \smallsetminus \partial U_0$ are compact, smooth, disjoint submanifolds of $\mathbb{R}^m$. Also, recall that the sets U_j, $j = 1, \ldots, N$ are open and that $U_0 = \bigcup_{j=1}^N U_j \cup \Gamma$ is a disjoint union.

See Sections 2.2–2.4 for notation and assumptions.

4.1. Estimates for the norm of $\left(\mathcal{P}_k^A\right)^{-1}$

Theorem 5.4 is essentially equivalent to the following theorem, for whose proof we shall repeatedly use Lemma 3.11, usually without further comment. Also, recall that we write $\|\xi\|_E = \infty$ if $\xi \notin E$, where $\|\cdot\|_E$ is the norm on E.

In the following theorem, we are assuming, as usual, that U_j are as in Assumptions 2.6. In that theorem, we are also using the resulting spaces $\check{W}^{k,\infty}(U_0)$ and $\check{H}^k(U_0)$ introduced in Definition 2.7, the spaces V_n and V_n^- introduced in Definition 2.12, and the operators $\mathcal{P}_k^A$ introduced in Definition 2.19. Moreover, $|||a_{mm}^{-1}|||_{\mathcal{V},n+k}$ was introduced in Definition 3.10.

Theorem 4.1. *Let $k, n \geq 0$, let $A \in M_{m+1}\left(\check{W}^{n+k+1,\infty}(U_0)\right)$, and suppose that the induced operator $\mathcal{P}_n = \mathcal{P}_n^A : V_n \to V_n^-$ is invertible, where the notation was recalled in the paragraph above. Given $F \in V_{n+k+1}^-$ we have*

$$\left\|\mathcal{P}_n^{-1}F\right\|_{V_{n+k+1}} \lesssim \sum_{q=0}^{k+1} |||\mathcal{P}_n^{-1}|||_n^{q+1}\ |||a_{mm}^{-1}|||_{\mathcal{V},n+k}^{(q+1)k+1}\ \|A\|_{\check{W}^{n+k+1,\infty}}^{(q+1)(k+1)}\ \|F\|_{V_{n+k+1-q}^-}. \tag{$\mathcal{J}_k$}$$

Typically, we shall have $n = 0$, in which case, we recall that $|||\mathcal{P}_0^{-1}|||_0$ is the norm of the operator $\mathcal{P}_0^{-1} = \left(\mathcal{P}_0^A\right)^{-1} : V_0^- := H_D^1(\mathbb{R}^m)^* \to V_0 := H_D^1(U_0)$.

Proof. The proof is by induction on k as in [40,47,48]. The proof is the same for all n, so we shall assume $n = 0$, for simplicity.

Step 0: initial verification. The estimate $(\mathcal{J}_{-1})$ is true by the definition of the norm $|||\mathcal{P}_0^{-1}|||_0 := |||\mathcal{P}_0^{-1}|||_{\mathcal{L}(V_0^-,V_0)}$. Indeed, the relation $(\mathcal{J}_{-1})$ simply reads

$$\left\|\mathcal{P}_0^{-1}F\right\|_{V_0} \leq |||\mathcal{P}_0^{-1}|||_0 \, \|F\|_{V_0^-}.$$

Let us proceed now to the *induction property,* that is, prove the relation $(\mathcal{J}_{k+1})$ assuming the relations $(\mathcal{J}_j)$ for $j = -1, 0, 1, 2, \ldots, k$. The proof of the induction property will consist of several steps.

Step 1: Estimate for $\left\|\mathcal{P}_0^{-1}F\right\|_{V_{k+1}}$. The induction hypothesis for k, *i.e.* for $F \in V_{k+2}^- \subset V_{k+1}^-$, says that the relation $(\mathcal{J}_k)$ is true for F, so

$$\begin{aligned}
\left\|\mathcal{P}_0^{-1}F\right\|_{V_{k+1}} &\lesssim \sum_{q=0}^{k+1} |||\mathcal{P}_0^{-1}|||_0^{q+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)k+1} \, \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+1)} \, \|F\|_{V_{k+1-q}^-} \\
&= \sum_{q=0}^{k+1} |||\mathcal{P}_0^{-1}|||_0^{q+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)(k+1)-q} \, \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+2)-1-q} \, \|F\|_{V_{k+1-q}^-} \\
&\lesssim \sum_{q=0}^{k+1} |||\mathcal{P}_0^{-1}|||_0^{q+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)(k+1)} \, \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+2)-1} \, \|F\|_{V_{k+1-q}^-}, \qquad (23)
\end{aligned}$$

where, for the last step, we also used Lemma 3.11.

Step 2: Estimate for $\left\|X_\ell^\alpha\left(\mathcal{P}_0^{-1}F\right)\right\|_{V_{k+1}}$, $\ell < m$ *for* $F \in V_{k+2}^-$. First, since $F \in V_{k+2}^-$ and $\ell < m$, Proposition 3.14 (3) gives us that, for all α, we have $X_\ell^\alpha F \in V_{k+1}^-$. So Corollary 3.15 for $\mathcal{P}_k = \mathcal{P}_{k+1}^A : V_{k+1} \to V_{k+1}^-$ and for the group of diffeomorphisms of generator X_ℓ^α gives

$$\left\|X_\ell^\alpha\left(\mathcal{P}_0^{-1}F\right)\right\|_{V_{k+1}} \leq \left\|\mathcal{P}_0^{-1}\left(X_\ell^\alpha F\right)\right\|_{V_{k+1}} + \|QF\|_{V_{k+1}}, \qquad (24)$$

with $Q := \mathcal{P}_0^{-1}X_\ell^\alpha(\mathcal{P}_0)\mathcal{P}_0^{-1} : V_{k+1}^- \to V_{k+1}$. We will now estimate the last two terms in the last equation. First, equation (23) for F replaced with $X_\ell^\alpha F \in V_{k+1}^-$ gives:

$$\left\|\mathcal{P}_0^{-1}\left(X_\ell^\alpha F\right)\right\|_{V_{k+1}} \lesssim \sum_{q=0}^{k+1} |||\mathcal{P}_0^{-1}|||_0^{q+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)(k+1)} \, \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+2)-1} \, \|F\|_{V_{k+2-q}^-}. \qquad (25)$$

On the other hand, for $G := X_\ell^\alpha(\mathcal{P}_0)\mathcal{P}_0^{-1}F \in V_{k+1}^-$, the recurrence hypothesis $(\mathcal{J}_k)$ gives us, again, that

$$\|QF\|_{V_{k+1}} := \left\|\mathcal{P}_0^{-1}(G)\right\|_{V_{k+1}} \lesssim \sum_{q=0}^{k+1} |||\mathcal{P}_0^{-1}|||_0^{q+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)k+1} \, \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+1)} \, \|G\|_{V_{k+1-q}^-}. \qquad (26)$$

In addition, for q fixed in $\{0, 1, \ldots, k+1\}$ we have, still by the hypothesis of induction, that

$$\begin{aligned}
\|G\|_{V_{k+1-q}^-} &:= \left\|X_\ell^\alpha(\mathcal{P}_0)\mathcal{P}_0^{-1}F\right\|_{V_{k+1-q}^-} \leq \|X_\ell^\alpha(A))\|_{\check{W}^{k+1-q,\infty}} \left\|\mathcal{P}_0^{-1}(F)\right\|_{V_{k+1-q}} \\
&\lesssim \sum_{s=0}^{k-q+1} |||\mathcal{P}_0^{-1}|||_0^{s+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k-q}^{(s+1)(k-q)+1} \, \|A\|_{\check{W}^{k+2-q,\infty}}^{(s+1)(k-q+1)+1} \, \|F\|_{V_{k+1-q-s}^-}. \qquad (27)
\end{aligned}$$

Let $t := q + s + 1$ and $c(q, s) := sq + 2q + s \geq 0$ (because $s, q \geq 0$). Subsequently, using the equations (26) and (27) as well as Lemma 3.11, we have:

$$\|QF\|_{V_{k+1}} \lesssim \sum_{q=0}^{k+1} \sum_{s=0}^{k+1-q} |||\mathcal{P}_0^{-1}|||_0^{t+1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(t+1)(k+1)-c(q,s)} \, \|A\|_{\check{W}^{k+2,\infty}}^{(t+1)(k+2)-c(q,s)-1} \, \|F\|_{V_{k+1-q-s}^-}$$

$$\lesssim \sum_{t=1}^{k+2} \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0^{t+1} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k}^{(t+1)(k+1)} \|A\|_{\check{W}^{k+2,\infty}}^{(t+1)(k+2)-1} \|F\|_{V_{k+2-t}^-}. \tag{28}$$

Finally, by replacing equations (25) and (28) in (24), for $\ell < m$, we find that:

$$\left\|X_\ell^\alpha \left(\mathcal{P}_0^{-1}F\right)\right\|_{V_{k+1}} \lesssim \sum_{q=0}^{k+2} \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0^{q+1} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k}^{(q+1)(k+1)} \|A\|_{\check{W}^{k+2,\infty}}^{(q+1)(k+2)-1} \|F\|_{V_{k+2-q}^-}. \tag{29}$$

Step 3: Induction property estimate. We now finally turn to the proof of the induction property. So, let us assume that the relation $(\mathcal{J}_j)$ is true for $-1 \le j \le k$ and prove $(\mathcal{J}_{k+1})$ if $F \in V_{k+2}^-$ and $A \in M_{m+1}\left(\check{W}^{k+2,\infty}(U_0)\right)$.

First, it follows from Lemma 3.11 that:

$$\left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1} \|f\|_{V_{k+2}^-} \le \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0 \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1} \|A\|_{\check{W}^{k+2,}}^{k+1} \|F\|_{V_{k+2}^-}. \tag{30}$$

Let $F = (f,g,h)$ and $u := \mathcal{P}_0^{-1}F$. Thus $Pu = f$. Next, Lemma 3.12 (for $u = \mathcal{P}_0^{-1}F$ and k replaced with $k+1$) and equations (30), (23), and (29) imply (recall that for all α, we set $X_0^\alpha = id$):

$$\begin{aligned}
\left\|\mathcal{P}_0^{-1}F\right\|_{\check{H}^{k+3}} &\lesssim \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1} \left(\|f\|_{\check{H}^{k+1}} + \|A\|_{\check{W}^{k+2,\infty}} \sum_{\alpha\in I}\sum_{\ell=0}^{m-1} \|X_\ell^\alpha(u)\|_{\check{H}^{k+2}}\right)\\
&\lesssim \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0 \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1}^{k+1} \|A\|_{\check{W}^{k+2,\infty}}^{k+1} \|F\|_{V_{k+2}^-}\\
&\quad + \sum_{q=0}^{k+1} \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0^{q+1} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1}^{(q+1)(k+1)+1} \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+2)} \|F\|_{V_{k+1-q}^-}\\
&\quad + \sum_{q=0}^{k+2} \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0^{q+1} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1}^{(q+1)(k+1)+1} \|A\|_{\check{W}^{k+2,\infty}}^{(q+1)(k+2)} \|F\|_{V_{k+2-q}^-}\\
&\lesssim \sum_{q=0}^{k+2} \left|\left|\left|\mathcal{P}_0^{-1}\right|\right|\right|_0^{q+1} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},k+1}^{(q+1)(k+1)+1} \|A\|_{\check{W}^{k+2,\infty}}^{(q+1)(k+2)} \|F\|_{V_{k+2-q}^-}.
\end{aligned}$$

The induction step is thus verified. This completes the proof of the relation $(\mathcal{J}_{k+1})$. □

Remark 4.2. The invertibility of $\mathcal{P}_0^A$ can be obtained using positivity (more precisely, the strong ellipticity) of suitable operators. This will be used in the next section. A generalization of the strong ellipticity is "T-coercivity" [12], which also may yield the invertibility of $\mathcal{P}_0^A$. See also [10,11,13,14,18,22] for related work. A different approach is in [52,53]. In these cases, we have $n = 0$ in our theorem. Yet another method of proving the invertibility of $\mathcal{P}_n^A$, for $n = 1$, is to use the self-adjointness of our operator (when this is the case).

The main result stated in the introduction, Theorem 1.1, follows right away from Theorem 4.1, as explained next:

Proof of Theorem 1.1. Since all boundaries ∂U_j are smooth (so the interface Γ is also smooth) and because U_0 is bounded, we obtain that there exists an extension constant C_{U_0} such that, for all $\tilde{g} \in H^{k+3/2}(\partial_D U_0)$ and all $\tilde{h} \in H^{k+3/2}(\Gamma)$, there exists $u_0 \in \check{H}^{k+2}(U_0)$ satisfying $u_0|_{\partial_D U_0} = \tilde{g}$, $[[u_0]]_\Gamma = \tilde{h}$, and

$$\|u_0\|_{\check{H}^{k+2}} \le C_{U_0}\left(\|\tilde{g}\|_{H^{k+1/2}(\partial_D U_0)} + \left\|\tilde{h}\right\|_{H^{k+3/2}(\Gamma)}\right).$$

Our last theorem (Thm. 4.1) applied to $F := \left(f - P^A u_0, g - D_\nu^A u_0, h - \left[\left[D_\nu^A u_0\right]\right]\right)$ then yields immediately Theorem 1.1. □

4.2. Automatic invertibility

This subsection is devoted to some consequences of the following corollary that is based on Theorem 4.1, but does not use its full force. We use the notation recalled before the statement of Theorem 4.1.

Corollary 4.3. *Let $k, n \geq 0$, let $A \in M_{m+1}\left(\check{W}^{n+k,\infty}(U_0)\right)$, and suppose that the operator $\mathcal{P}_n^A : V_n \to V_n^-$ is invertible. Then the operator $\mathcal{P}_{n+k}^A : V_{n+k} \to V_{n+k}^-$ induced by restriction is also invertible.*

Proof. We replace $k+1$ with k in Theorem 4.1, in what follows, for convenience. We have that $\mathcal{P}_{n+k}^A : V_{n+k} \to V_{n+k}^-$ is well defined and continuous by Lemma 3.9 since $A \in M_{m+1}\left(\check{W}^{n+k,\infty}(U_0)\right)$. The hypothesis that $\mathcal{P}_n$ is injective implies that its restriction $\mathcal{P}_{n+k}^A$ is also injective. Finally, Theorem 4.1 implies that it is also surjective, and hence an isomorphism, as claimed. □

Our results also imply a regularity result.

Remark 4.4. Under the assumptions of Theorem 4.1 (especially $F \in V_{n+k}^-$ and A with coefficients in $\check{W}^{n+k,\infty}(U_0)$), we obtain, in particular, that $\left(\mathcal{P}_n^A\right)^{-1} F \in V_{n+k}$. This is a *regularity result.* In the strongly elliptic case, the oldest statement we know of such a regularity result is in the work of Roitberg and Sheftel [55, 56]. See also [43]. See [42, 49–51] for the case of polygons, where, we stress, the results may be very different.

If E and F are two normed spaces, recall that $\mathcal{L}(E; F)$ denotes the set of continuous, linear maps $E \to F$. Moreover, we shall write $\mathcal{L}(E; F)^{-1}$ for the set of continuous, *invertible* linear maps $E \to F$. We shall need also the well known fact that the map $\mathcal{L}(E; F)^{-1} \ni T \mapsto T^{-1} \in \mathcal{L}(F; E)$ is analytic (hence continuous, hence measurable) [15, 26]. This gives the following consequences.

Corollary 4.5. *Let $n, k \in \mathbb{Z}_+$ and $A : \Theta \to M_{(m+1)}\left(\check{W}^{n+k,\infty}(U_0)\right)$. We assume that, for all $\theta \in \Theta$, $\mathcal{P}_n^{A(\theta)} : V_n \to V_n^-$ is invertible, and hence that the function*

$$\left(\mathcal{P}_{n+k}^A\right)^{-1} : \Theta \to \mathcal{L}(V_{n+k}^-; V_{n+k}),$$

$\left(\mathcal{P}_{n+k}^A\right)^{-1}(\theta) := \left(\mathcal{P}_{n+k}^{A(\theta)}\right)^{-1}$*, is well-defined and we have the following:*

(i) If Θ is a measurable space and A is measurable, then $\left(\mathcal{P}_{n+k}^A\right)^{-1}$ is measurable.

(ii) If Θ is a topological space and A is continuous, then $\left(\mathcal{P}_{n+k}^A\right)^{-1}$ is continuous.

(iii) If Θ is an open subspace in a locally convex, topological vector space and A is analytic, then $\left(\mathcal{P}_{n+k}^A\right)^{-1}$ is analytic.

Proof. Let us prove (i), since the other two points are completely similar. Since $M_{(m+1)}\left(\check{W}^{n+k,\infty}(U_0)\right) \ni B \mapsto \mathcal{P}_{n+k}^B \in \mathcal{L}\left(V_{n+k}^-; V_{n+k}\right)$ is continuous and linear (Lem. 3.9), it follows that the map $\mathcal{P}_{n+k}^A : \Theta \to \mathcal{L}\left(V_{n+k}^-; V_{n+k}\right)$ is also measurable. By Corollary 4.3, we know that $\mathcal{P}_{n+k}^A : \Theta \to \mathcal{L}\left(V_{n+k}^-; V_{n+k}\right)$ has values invertible elements. Since $T \to T^{-1}$ is measurable (even analytic!), the result follows from the fact that the composition of two measurable maps is measurable. □

4.3. Integrability from uniform boundedness

We now include a result that uses the full force of Theorem 4.1. The main notation was recalled before the statement of Theorem 4.1, and we continue to use it. First we refine the invertibility result of Corollary 4.3 as follows:

Corollary 4.6. *See the paragraph before Theorem 4.1 for notation. Let* $A \in M_{m+1}\left(\check{W}^{n+k+1,\infty}(U_0)\right)$*,* $k, n \geq 0$*,* $\mathcal{P}_k := \mathcal{P}_k^A$*, and suppose that the operator* $\mathcal{P}_n : V_n \to V_n^-$ *is invertible. Then* $\mathcal{P}_{n+k+1} : V_{n+k+1} \to V_{n+k+1}^-$ *is also invertible and its inverse has norm*

$$\left|\left|\left|\mathcal{P}_{n+k+1}^{-1}\right|\right|\right|_{n+k+1} \lesssim \left|\left|\left|\mathcal{P}_n^{-1}\right|\right|\right|_n^{k+2} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},n+k}^{(k+1)^2} \|A\|_{\check{W}^{n+k+1,\infty}}^{(k+2)(k+1)}.$$

Proof. Lemma 3.11 implies that $\left|\left|\left|\mathcal{P}_n^{-1}\right|\right|\right|_n^{q+1} \left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},n+k}^{(q+1)k+1} \|A\|_{\check{W}^{n+k+1,\infty}}^{(q+1)(k+1)}$ is increasing in q. Since $\|F\|_{V_{k+2-q}^-}$ is decreasing in q, the result follows. □

The above result is obviously true also for $k = -1$, as long as one defines $\left|\left|\left|a_{mm}^{-1}\right|\right|\right|_{\mathcal{V},n+k}$ for $n = 0$ (actually, one can simply ignore that term for $n = 0$ and $k = -1$). In order to avoid this discussion, it was convenient to state the above result (as well as Thm. 4.1) in the given range of k (*i.e.* $k \geq 0$). We now shift back and replace $k + 1$ with k. We obtain the following consequence.

Proposition 4.7. *Let* $n, k \in \mathbb{Z}_+$*. Suppose that*

(i) $A : \Theta \to M_{(m+1)}\left(\check{W}^{n+k,\infty}(U_0)\right)$ *is bounded;*
(ii) $\mathcal{P}_n^{A(\theta)} : V_n \to V_n^-$ *is invertible for all* $\theta \in \Theta$*; and*
(iii) the function $\theta \to \left(\mathcal{P}_n^{A(\theta)}\right)^{-1} \in \mathcal{L}\left(V_n^-; V_n\right)$ *is bounded.*

Then the function $\left(\mathcal{P}_{n+k}^A\right)^{-1}(\theta) := \left(\mathcal{P}_{n+k}^{A(\theta)}\right)^{-1} \in \mathcal{L}\left(V_{n+k}^-; V_{n+k}\right)$ *is also bounded. Consequently, if* Θ *is a probability space and* A *is measurable, then the functions* $\left(\mathcal{P}_{n+k}^A\right)^{-1}$ *and* $\left\|\left(\mathcal{P}_{n+k}^A\right)^{-1}\right\|$ *are integrable on* Θ*.*

Proof. The first part follows right away from Corollary 4.6. The second part follows by combining the first part with Corollary 4.5(i), since bounded, measurable functions on probability spaces are integrable. □

5. Estimates for strongly elliptic operators and integrability

In this section, we include some further applications, most notably, an integrability result that extends Proposition 4.7 to the case when the norms of the operators $\mathcal{P}_n^{A(\theta)} \in \mathcal{L}\left(V_n; V_n^-\right)$ are not uniformly bounded. We use the same notation and assumptions as in the last sections. In particular, $U_0 \subset \mathbb{R}^m$ is bounded with smooth boundary and is endowed with a decomposition into subdomains U_j along a smooth interface $\Gamma := \left(\bigcup_{j=1}^N \partial U_j\right) \smallsetminus \partial U_0$. We also allow $U_j = (a_j, b_j) \times \mathbb{R}^{m-1}$ in Subsection 5.1, in which case the results of the previous sections are replaced with the results in [47] or [48].

Notation 5.1. See Sections 2.2–2.4 for notation and assumptions. More precisely, the sets U_j are as in Assumptions 2.6, the resulting spaces $\check{W}^{k,\infty}(U_0)$ and $\check{H}^k(U_0)$ are as in Definition 2.7, the spaces V_n and V_n^- are as in Definition 2.12, and the operators $\mathcal{P}_k^A$ are as in Definition 2.19. In particular, $\overline{U}_0 = \left(\bigcup_{j=1}^N U_j\right) \cup \Gamma \cup \partial_D U_0 \cup \partial_N U_0$ is a disjoint union with U_j all open and Γ, $\partial_D U_0$, and $\partial_N U_0$ compact, smooth submanifolds of $\mathbb{R}^m$. All the necessary notation was also introduced in the Introduction.

5.1. General case

We will need the following standard lemma (see, for example, Chap. 5 of [47] or one of the following papers [16, 40, 48]). Recall that in this subsection, we also allow $U_j = (a_j, b_j) \times \mathbb{R}^{m-1}$, in which case the results of the previous sections are replaced with the results in Chaps. 2 and 3 of [47], or with the results in [48].

Lemma 5.2. *There exists* $c_1 > 0$ *with the following property. Let* $b \in \check{W}^{k,\infty}(U_0)$ *(see Def. 2.7) be such that* $b^{-1} \in L^\infty(U_0)$*. Then* $b^{-1} \in \check{W}^{k,\infty}(U_0)$ *and*

$$\|b^{-1}\|_{\check{W}^{k,\infty}} \leq c_1 \left\|b^{-1}\right\|_{L^\infty}^{k+1} \|b\|_{\check{W}^{k,\infty}}^k.$$

We will denote $\operatorname{Re} A := \frac{1}{2}(A + A^*)$, for any matrix $A \in M_{m+1}(\mathbb{C})$ (with A^* the adjoint of A, *i.e.* its transposed conjugate). We write $A \geq \gamma I_{m+1}$ if, for any complex vector $\xi \in \mathbb{C}^{m+1}$ on which A acts, we have $(A\xi, \xi) \geq \gamma \|\xi\|^2$. (Thus, I_{m+1} is the identity matrix of $M_{m+1}(\mathbb{C})$.) We write $A > 0$ if $A \geq 0$ and A is invertible. If $A \in M_{m+1}(L^\infty(U_0))$, the inequality $A \geq \gamma I_{m+1}$ means that $A(x) \geq \gamma I_{m+1}$ for almost all $x \in U_0$. In the following, we will also omit A in the notation of the operators P^A and $\mathcal{P}_k^A$ when there is no risk of confusion (for instance, when the result is about a single operator).

Lemma 5.3. *See Notation 5.1 for a review of the notation. Let $A \in M_{m+1}(L^\infty(U_0))$. If $\operatorname{Re} A \geq \gamma I_{m+1}$, then $\mathcal{P}_0 = \mathcal{P}_0^A : V_0 \to V_0^-$ is invertible with $|||\mathcal{P}_0^{-1}|||_0 \leq \gamma^{-1}$.*

Proof. Let $\mathcal{P}_0 := \mathcal{P}_0^A$. As $\gamma > 0$, we obtain for $u \in V_0 = H^1(U_0)$,

$$\begin{aligned}\operatorname{Re}(\mathcal{P}_0 u, u) := \operatorname{Re} B^A(u,u) &= \operatorname{Re} \int_{U_0} \sum_{i,j=0}^m a_{ij} \partial_i u \partial_j \overline{u} \, \mathrm{d}x \\ &\geq \gamma \int_{U_0} \sum_{i=0}^m |\partial_i u|^2 \, \mathrm{d}x =: \gamma \|u\|_{H^1}^2 .\end{aligned}$$

(Recall that $\partial_0 = id$.) So $\mathcal{P}_0$ is invertible and $|||\mathcal{P}_0^{-1}|||_0 \leq \gamma^{-1}$, by Lax–Milgram's lemma. ☐

Recall that $|||a_{mm}^{-1}|||_{\mathcal{V},k}$ was introduced in Definition 3.10.

Theorem 5.4. *Suppose that $U_0 = \bigcup_{k=1}^N U_k \subset \mathbb{R}^m$ is bounded with ∂U_k smooth or that $U_j = (a_j, b_j) \times \mathbb{R}^{m-1}$. Let $A \in M_{m+1}(\check{W}^{k+1,\infty}(U_0))$ be such that $A \geq \gamma I_{m+1}$, with $\gamma > 0$. Let $\mathcal{P}_j := \mathcal{P}_j^A : V_j \to V_j^-$ for all j and $k \geq -1$. (See Notation 5.1 for a review of the notation.) Then, given $F \in V_{k+1}^-$, we have*

$$\|\mathcal{P}_0^{-1} F\|_{V_{k+1}} \lesssim \sum_{q=0}^{k+1} \gamma^{-q-1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^{(q+1)k+1} \, \|A\|_{\check{W}^{k+1,\infty}}^{(q+1)(k+1)} \, \|F\|_{V_{k+1-q}^-}.$$

In particular, $\mathcal{P}_{k+1}$ is invertible. Let $K := (k+2)\left(k^2 + k + 1\right) + k$. Then

$$|||\mathcal{P}_{k+1}^{-1}|||_{k+1} \leq \gamma^{-K-1} \, \|A\|_{\check{W}^{k+1,\infty}}^K .$$

Proof. The hypothesis of the theorem imply that $\mathcal{P}_0 = \mathcal{P}_0^A : V_0 := H_D^1(U_0) \to V_0^- =: H_D^1(U_0)^*$ is invertible and $|||\mathcal{P}_0^{-1}|||_0 \leq \gamma^{-1}$, according to Lemma 5.3. Moreover, we also have $\gamma \lesssim a_{mm}$, so $\|a_{mm}^{-1}\|_{L^\infty} \lesssim \gamma^{-1}$. It follows from Lemma 5.2 that

$$|||a_{mm}^{-1}|||_{\mathcal{V},k} \leq \gamma^{-k-1} \, |||a_{mm}^{-1}|||_{\mathcal{V},k}^k \leq \gamma^{-k-1} \|A\|_{\check{W}^{k+1,\infty}}^k .$$

The rest is a consequence of Theorem 4.1. ☐

We obtain the following consequence for the case where the coefficients are random Gaussian variables defined on a measure space Ω.

Theorem 5.5. *Let U_j be as in Theorem 5.4. Let $X = (X_1, X_2, \ldots, X_q) : \Omega \to \mathbb{R}^q$ be a Gaussian vector random variable with covariance $\sigma = (\sigma_{ij}) > 0$ and let $\gamma > 0$. Let $A_1, A_2, \ldots, A_q \in M_{m+1}(\check{W}^{k+1,\infty}(U_0))$ satisfying $\operatorname{Re} A_\ell \geq 0$ for all $1 \leq \ell \leq q$ and such that, for all $x \in U_0$, there is ℓ with $\operatorname{Re} A_\ell(x) \geq \gamma I_{m+1}$. We let*

$$A(\omega) := \sum_{\ell=1}^q e^{X_\ell(\omega)} A_\ell$$

and $\gamma(\omega)$ be the largest constant such that $\gamma(\omega)I \le \operatorname{Re} A(\omega)$. Then, for all $r, s \in \mathbb{R}$ and $0 \le p < \infty$, the function

$$\Omega \ni \omega \ \mapsto \ \gamma(\omega)^{-s} \, \|A(\omega)\|^r_{\breve{W}^{k+1}} \left|\left|\left| \left(\mathcal{P}^{A(\omega)}_{k+1}\right)^{-1} \right|\right|\right|^p_{k+1} \in (0, \infty)$$

is integrable.

Proof. The given function is measurable by Corollary 4.5(i). We have

$$\operatorname{Re} A(\omega) := \ \operatorname{Re} \sum_{\ell=1}^{q} e^{X_\ell(\omega)} A_j \ge \gamma \ \min_{\ell=1}^{q} \left\{ e^{X_\ell(\omega)} \right\} \, I_{m+1}, \tag{31}$$

where I_{m+1} is the identity matrix of $M_{m+1}(\mathbb{C})$, as before. Hence $\gamma(\omega) \ge \gamma \ \min_{\ell=1}^{q} \left\{ e^{X_\ell(\omega)} \right\}$. Similarly, we have the estimate

$$\gamma(\omega) \le \|A(\omega)\|_{\breve{W}^{k+1,\infty}} \ \le \ \sum_{j=1}^{q} e^{X_\ell(\omega)} \|A_j\|_{\breve{W}^{k+1,\infty}}. \tag{32}$$

We can thus assume $s, r \ge 0$. By Theorem 5.4, we obtain

$$\begin{aligned} \gamma(\omega)^{-s/p} \, \|A(\omega)\|^{r/p}_{\breve{W}^{k+1}} \left|\left|\left| \left(\mathcal{P}^{A(\omega)}_{k+1}\right)^{-1} \right|\right|\right|_{k+1} &\le \left(\gamma \min_{\ell=1}^{q} \left\{ e^{X_\ell(\omega)} \right\} \right)^{-K-s/p-1} \left(\sum_{\ell=1}^{q} e^{X_\ell(\omega)} \|A_\ell\|_{\breve{W}^{k,\infty}} \right)^{K+r/p} \\ &\le C \left(\sum_{\ell=1}^{q} e^{|X_\ell(\omega)|} \right)^{2K+r/p+s/p+1} . \end{aligned}$$

Hence

$$\begin{aligned} \int_\Omega \gamma(\omega)^{-s/p} \, \|A(\omega)\|^r_{\breve{W}^{k+1}} \left|\left|\left| (\mathcal{P}^{A(\omega)}_{k+1})^{-1} \right|\right|\right|^p_{k+1} \mathrm{d}\omega &\le \ C \int_\Omega \left(\sum_{\ell=1}^{q} e^{|X_\ell(\omega)|} \right)^{2K+r/p+s/p+1} \mathrm{d}\omega \\ &= \ C \int_{\mathbb{R}^q} \left(\sum_{\ell=1}^{q} e^{|x_\ell|} \right)^{a} e^{-(\sigma^{-1}x,x)} \, \mathrm{d}x \ < \ \infty \end{aligned}$$

since, for all $a > 0$, $\left(\sum_{\ell=1}^{q} e^{|x_\ell|}\right)^a < \left(q + e^{|x_1|+\ldots+|x_q|}\right)^a$, which is integrable with respect to the measure of density $e^{-\left(\sigma^{-1}x,x\right)} \le e^{-\epsilon\|x\|^2}$ on $\mathbb{R}^q$. □

5.2. Poincaré inequality

We shall say that the Poincaré inequality is satisfied by the functions in $H^1_D(U_0) := \left\{ u \in H^1(U_0) \mid u|_{\partial_D U_0} = 0 \right\}$ with constant η_U if, for all $u \in H^1_D(U_0)$, we have

$$|u|^2_{H^1(U_0)} \ := \ \int_{U_0} |\nabla u(x)| \, \mathrm{d}x \ \ge \ \eta_U^2 \big\|u\big\|^2_{H^1(U_0)}. \tag{33}$$

Since we have assumed that U_0 is connected and that $\partial_D U_0$ is a union of connected components of the boundary, this assumption (the Poincaré inequality) is satisfied if, and only if, $\partial_D U_0 \ne \emptyset$ (recall that U_0 is either bounded or that $U_j = (a_j, b_j) \times \mathbb{R}^{m-1}$). If this assumption is satisfied, then we get stronger results as follows. Let $A = [a_{ij}] \in M_{m+1}\left(L^\infty(U_0)\right)$ and P^A and $\mathcal{P}^A_k$ be the associated operators, as before. Recall then that A (or P^A, or, yet, $\mathcal{P}^A_k$) is γ–*strongly elliptic* if, for all $x \in U_0$ and all $\xi \in \mathbb{C}^m$, we have

$$\sum_{i,j=1}^{m} a_{ij}(x) \xi_i \overline{\xi}_j \ge \gamma |\xi|^2. \tag{34}$$

The largest γ with this property will be denoted $\gamma(A)$ and is called the *coercivity constant* of A. Of course, $\gamma(A) = \left\|(A + A^*)^{-1}\right\|^{-1}/2$, if A is coercive. Let $I' := 0 \oplus I_m \in M_{m+1}(\mathbb{C})$ be the projection with entries $(I')_{ij}$ given by

$$\begin{cases} (I')_{ii} = 1 & \text{for } 1 \leq i \leq m \\ (I')_{ij} = 0 & \text{otherwise.} \end{cases} \tag{35}$$

(Thus I' differs from the identity matrix $I_{m+1} \in M_{m+1}(\mathbb{C})$ in only one entry.) The strong ellipticity condition then becomes $I'AI' \geq \gamma I'$. If $a_{0j} = a_{i0} = 0$ for all $0 \leq i, j \leq m$, then $I'AI' = A$ and the strong ellipticity condition becomes $A \geq \gamma I'$. The interest of the concept of strong ellipticity is that operators satisfying this condition have the following properties.

Lemma 5.6. *Let $\gamma > 0$ and $A = [a_{ij}] \in M_{m+1}\left(L^\infty(U_0)\right)$ be such that $a_{i0} = a_{0j} = 0$ for all $1 \leq i, j \leq m$ and $\operatorname{Re} A \geq 0$. Suppose that $A \geq \gamma I'$ (i.e. A is γ–strongly elliptic) and that the Poincaré inequality is satisfied on $H^1_D(U_0)$ with constant $\eta_U > 0$. Then $\mathcal{P}_0^A : V_0 \to V_0^-$ is invertible with norm $\left|\left|\left|\left(\mathcal{P}_0^A\right)^{-1}\right|\right|\right|_0 \leq (\eta_U \gamma)^{-1}$.*

Proof. We have

$$\operatorname{Re}\left(\mathcal{P}_0^A u, u\right) \;=\; B^A(u,u) \geq \gamma^2 \, |u|^2_{H^1(U_0)} + a_{00}\|u\|_{L^2} \geq (\eta_U\gamma)^2 \, \|u\|^2_{H^1(U_0)} \,,$$

then the Lax–Milgram's Lemma gives the results in view of the definitions of $V_0 := H^1_D(U_0)$ and $V_0^- := V_0^*$. □

We immediately obtain the following consequence, with essentially the same proof as that of Theorem 5.4. See Notation 5.1 for a review of the notation.

Theorem 5.7. *Let U_j be as in Theorem 5.4. Suppose that $A \in \check{W}^{k+1,\infty}(U_0)$ satisfies the hypotheses of Lemma 5.6. Let $\mathcal{P}_j := \mathcal{P}_j^A : V_j \to V_j^-$ for all j and $k \geq -1$. Then, given $F \in V_{k+1}^-$ we have*

$$\left\|\mathcal{P}_0^{-1}F\right\|_{V_{k+1}} \lesssim \sum_{q=0}^{k+1} \gamma^{-q-1} \; \left|\left|\left|a_{mm}^{-1}\right|\right|\right|^{(q+1)k+1}_{\mathcal{V},k} \; \|A\|^{(q+1)(k+1)}_{\check{W}^{k+1,\infty}} \; \|F\|_{V^-_{k+1-q}}.$$

In particular, $\mathcal{P}_{k+1}$ is invertible. Let $K := (k+2)\left(k^2+k+1\right) + k$. Then

$$\left|\left|\left|\mathcal{P}_{k+1}^{-1}\right|\right|\right|_{k+1} \lesssim \gamma^{-K-1} \; \|A\|^K_{\check{W}^{k+1,\infty}} \,.$$

We obtain the following consequence (with the same proof as Thm. 5.5). We use the hypotheses of the Theorems 5.5 and 5.7 (listed below):

Theorem 5.8. *Let U_j be as in Theorem 5.4. We continue to use Notation 5.1. Let $\gamma > 0$, $k \in \mathbb{Z}_+$, and $A_1, A_2, \ldots, A_q \in M_{m+1}\left(\check{W}^{k+1,\infty}\left(U_0\right)\right)$ be matrices such that*

(i) $(A_\ell)_{i0} = (A_\ell)_{0j} = 0$, for all $1 \leq i, j \leq m$ and all $1 \leq \ell \leq q$,
(ii) $\operatorname{Re} A_\ell \geq 0$ for all $1 \leq \ell \leq q$, and
(iii) for each $x \in U_0$, there is $1 \leq \ell \leq q$ such that $\operatorname{Re} A_\ell(x) \geq \gamma I'$, where $I' := 0 \oplus I_m$ (see Eq. (35)).

We assume that the Poincaré inequality is satisfied on $H^1_D(U_0)$. Let $X : \Omega \to \mathbb{R}^q$ be Gaussian as in Theorem 5.5 and $A(\omega) := \sum_{\ell=1}^q e^{X_\ell(\omega)} A_\ell$. Then $A(\omega)$ is coercive with coercivity constant $\gamma(\omega) := \gamma(A(\omega))$ such that, for all $s, r \in \mathbb{R}$ and $0 \leq p < \infty$, the function

$$\Omega \ni \omega \;\mapsto\; \gamma(\omega)^{-s} \, \|A(\omega)\|^r_{\check{W}^{k+1}} \left|\left|\left|\left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1}\right|\right|\right|^p_{k+1} \in (0,\infty)$$

is integrable.

The conditions $(A_\ell)_{i0} = (A_\ell)_{0j} = 0$, for all $0 \le i, j \le m$ and all $1 \le \ell \le q$ model homogeneous second order operators (such as "sign-changing Laplacians"). Our operators are slightly more general since we allow $a_{00} \ge 0$. They can be replaced with some less restrictive conditions, but then the results are more technical to state. The condition (iii) means, of course, that the operator associated to A_ℓ is strongly elliptic at x and hence that the sum $A(\omega) := \sum_{\ell=1}^{q} e^{X_\ell(\omega)} A_\ell$ is strongly elliptic everywhere.

Proof. Equation (31) of the proof of Theorem 5.5 becomes

$$\mathrm{Re}\, A(\omega) := \mathrm{Re} \sum_{\ell=1}^{q} e^{X_\ell(\omega)} A_j \ge \gamma \min_{\ell=1}^{q} \left\{ e^{X_\ell} \right\} I'.$$

We also have that $\mathrm{Re}\, A(\omega) \ge 0$ and that $(A(\omega))_{i0} = (A(\omega))_{0j}$ for all $1 \le i, j \le m$. Thus we can use Theorem 5.7 (instead of Thm. 5.4) to complete the proof as in the proof of Theorem 5.5. ☐

5.3. Application to the finite element method

Let $S_N \subset H^1_D$ be a sequence of finite dimensional vector subspaces. We let Q_N denote the projection of $u \in H^1_D$ on S_N in the norm $\|\cdot\|_{H^1}$. We suppose that there exist $k > 0$, $C_{rate} > 0$, and $\mu > 0$ such that, for all $u \in \check{H}^{k+1}(U_0)$, we have

$$\|u - Q_N u\|_{H^1(U_0)} \le C_{rate} \dim(S_N)^{-\mu} \|u\|_{\check{H}^{k+1}(U_0)}. \tag{36}$$

See [45] for an example of such subspaces using the Generalized Finite Element spaces and [8] for an example of such subspaces using anisotropically graded meshes on 3D polyhedral domains. Because of this condition, we now assume again that U_0 is bounded.

Let B^A and P^A be as in Definition 2.2. Let us denote by Q^A_N the projection onto S_N in the inner product B^A on $H^1_D(U_0)$ whenever $A = A^*$ and A satisfies the hypotheses of Lemma 5.6 (this guarantees then that B^A defines an inner product on $H^1_D(U_0)$). Thus $u_N := Q^A_N u$ is the Finite Element approximation of the solution u of the equation $P^A u = f$, for all f (with the given boundary conditions: Dirichlet on $\partial_D U_0$ and Neumann on $\partial_N U_0$). See also Remark 5.12.

Proposition 5.9. *We keep the notation and assumptions of Lemma 5.6 and of equation (36). Let $A = A^*$ and Q^A_N be the projection onto S_N in the inner product B^A. Then*

$$\left\| u - Q^A_N u \right\|_{H^1(U_0)} \le C_{rate} (\gamma \eta_U)^{-1/2} \|A\|^{1/2}_{L^\infty} \dim(S_N)^{-\mu} \|u\|_{\check{H}^{k+1}(U_0)}.$$

Proof. Let $\mathrm{dist}_{(\,,\,)}$ be the distance in some inner product $(\,,\,)$ (or in some norm induced from an inner product). We have

$$\gamma \eta_U \|u\|^2_{H^1} \le \gamma \|\nabla u\|^2_{L^2} \le B^A(u, u) \le \|A\|_{L^\infty} \|u\|^2_{H^1}.$$

Hence we have successively

$$\begin{aligned}
\left\| u - Q^A_N u \right\|_{H^1(U_0)} &\le (\gamma \eta_U)^{-1/2} B^A \left(u - Q^A_N u, u - Q^A_N u \right)^{1/2} \\
&= (\gamma \eta_U)^{-1/2} \,\mathrm{dist}_{B^A}(u, S_N) \\
&\le (\gamma \eta_U)^{-1/2} \|A\|^{1/2}_{L^\infty} \,\mathrm{dist}_{\|\cdot\|_{H^1}}(u, S_N) \\
&= (\gamma \eta_U)^{-1/2} \|A\|^{1/2}_{L^\infty} \|u - Q_N u\|_{H^1(U_0)} \\
&\le C_{rate} (\gamma \eta_U)^{-1/2} \|A\|^{1/2}_{L^\infty} \dim(S_N)^{-\mu} \|u\|_{\check{H}^{k+1}(U_0)},
\end{aligned}$$

where the last inequality is by the assumption (36). ☐

We then obtain the following theorem.

Theorem 5.10. *We use the assumptions and notation of Theorem 5.8 and the condition (36). In particular, $0 \leq p < \infty$, Then, there exists $C_{X,p}$ such that, for all $F \in V_k^-$, we have*

$$\int_\Omega \left\| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F - Q_N \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F \right\|_{H^1}^p \mathrm{d}\omega \ \leq \ C_{p,X} \dim(S_N)^{-p\mu} \|F\|_{V_k^-}^p .$$

If $A_\ell = A_\ell^*$ for all ℓ, then there exists also $C_{X,p}^{FEM}$ such that, for all $F \in V_k^-$,

$$\int_\Omega \left\| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F - Q_N^{A(\omega)} \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F \right\|_{H^1}^p \mathrm{d}\omega \ \leq \ C_{p,X}^{FEM} \dim(S_N)^{-p\mu} \|F\|_{V_k^-}^p .$$

Proof. Equation (36) for $u := \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F$ immediately gives:

$$\begin{aligned} \left\| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F - Q_N \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F \right\|_{H^1} &\leq C \dim(S_N)^{-\mu} \left\| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F \right\|_{\check{H}^{k+1}} \\ &\leq C \dim(S_N)^{-\mu} \left|\left|\left| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} \right|\right|\right|_k \|F\|_{V_k^-}. \end{aligned}$$

So we may just choose $C_{p,X} := C^p \int_\Omega \left|\left|\left| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} \right|\right|\right|_k^p \mathrm{d}\omega$, which is finite, by Theorem 5.8. For the second estimate, we proceed similarly, but using Proposition 5.9 instead of equation (36) and

$$C_{p,X}^{FEM} := C^p \int_\Omega \gamma(\omega)^{-p/2} \|A(\omega)\|_{\check{W}^{k+1}}^{p/2} \left|\left|\left| \left(\mathcal{P}_{k+1}^{A(\omega)}\right)^{-1} F \right|\right|\right|_k^p \mathrm{d}\omega,$$

which is finite by the same theorem. □

Remark 5.11. In all of the above results, we may choose $F = F(X_1, X_2, \ldots, X_q)$ to depend on $X_\ell(\omega)$ as well, $1 \leq \ell \leq q$, as long as the norm growth of $F(x_1, \ldots, x_q)$ in x is at most exponential.

Remark 5.12. The last part of the last theorem gives an estimate for the average error in the Finite Element Method approximation for all the equations $\mathcal{P}_0^{A(\omega)} u = F$, since $Q_N^{A(\omega)} \left(\mathcal{P}_0^{A(\omega)}\right)^{-1} F \in S_n$ is the Finite Element approximation of the solution u in the discretization space S_N.

6. Extensions and future work

An important research direction is to extend the results above to systems and to polygonal or polyhedral domains and interfaces. We expect that the right framework for this extension is that of Babuška-Kondratiev spaces [3, 17, 29–31, 35, 50, 51]. (These spaces are also called "weighted Sobolev spaces.") In the same vein, it would be useful to develop the approach to these problems using layer potentials [27, 28, 46], including the approach using pseudodifferential operators [38, 59]. Recently, the method of layer potentials for domains with conical points was studied using pseudodifferential operators in [20, 21]. A related approach is *via* boundary triples [9, 19, 54].

As discussed in Remark 4.2, the invertibility of $\mathcal{P}_0$ in our main result, Theorem 4.1, follows from strong ellipticity. In the case of non-definite coefficients, it may follow from [10–14, 18, 22, 52, 53]. The case of non-definite coefficients ("sign changing problems") is very interesting and important, but still presents many challenges.

References

[1] H. Amann, Linear and quasilinear parabolic problems. In Vol. 89 of *Monographs in Mathematics. Vol. I.* Birkhäuser Boston Inc., Boston (1995). Abstract linear theory.

[2] B. Ammann, N. Große and V. Nistor, Well-posedness of the Laplacian on manifolds with boundary and bounded geometry. *Math. Nachr.* **292** (2019) 1213–1237.

[3] I. Babuška, R.B. Kellogg and J. Pitkäranta, Direct and inverse error estimates for finite elements with mesh refinements. *Numer. Math.* **33** (1979) 447–471.

[4] I. Babuška, F. Nobile and R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Rev.* **52** (2010) 317–355.

[5] I. Babuska, R. Tempone and G.E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.* **42** (2004) 800–825.

[6] M. Bachmayr, A. Cohen, D. Dũng and C. Schwab, Fully discrete approximation of parametric and stochastic elliptic PDEs. *SIAM J. Numer. Anal.* **55** (2017) 2151–2186.

[7] M. Bachmayr, A. Cohen and G. Migliorati, Representations of Gaussian random fields and approximation of elliptic PDEs with lognormal coefficients. *J. Fourier Anal. Appl.* **24** (2018) 621–649.

[8] C. Bacuta, V. Nistor and L. Zikatanov, Improving the rate of convergence of high-order finite elements on polyhedra. II. Mesh refinements and interpolation. *Numer. Funct. Anal. Optim.* **28** (2007) 775–824.

[9] J. Behrndt and D. Krejcirik, An indefinite Laplacian on a rectangle. Preprint `arXiv:1407.7802` (2014).

[10] A.-S. Bonnet-Ben Dhia and K. Ramdani, Mathematical analysis of conducting and superconducting transmission lines. *SIAM J. Appl. Math.* **60** (2000) 2087–2113.

[11] A.-S. Bonnet-Bendhia, M. Dauge and K. Ramdani, Analyse spectrale et singularités d'un problème de transmission non coercif. *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999) 717–720.

[12] A.S. Bonnet-Ben Dhia, P. Ciarlet, Jr. and C.M. Zwölf, Two- and three-field formulations for wave transmission between media with opposite sign dielectric constants. *J. Comput. Appl. Math.* **204** (2007) 408–417.

[13] A.-S. Bonnet-Ben Dhia, L. Chesnel and P. Ciarlet, Jr., T-coercivity for scalar interface problems between dielectrics and metamaterials. *ESAIM:M2AN* **46** (2012) 1363–1387.

[14] A.-S. Bonnet-Ben Dhia, L. Chesnel and P. Ciarlet, Jr., $\mathtt{T}$-coercivity for the Maxwell problem with sign-changing coefficients. *Commun. Partial Differ. Equ.* **39** (2014) 1007–1031.

[15] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, in Universitext. Springer, New York (2011).

[16] C. Băcuţă, H. Li and V. Nistor, Differential operators on domains with conical points: precise uniform regularity estimates. *Rev. Roum. Math. Pures Appl.* **62** (2017) 383–411.

[17] A. Buffa, M. Costabel and M. Dauge, Anisotropic regularity results for Laplace and Maxwell operators in a polyhedron. *C. R. Math. Acad. Sci. Paris* **336** (2003) 565–570.

[18] R. Bunoiu, K. Ramdani and C. Timofte, T-coercivity for the asymptotic analysis of scalar problems with sign-changing coefficients in thin periodic domains. *Electron. J. Differ. Equ.* **2021** (2021) 22.

[19] C. Cacciapuoti, K. Pankrashkin and A. Posilicano, Self-adjoint indefinite Laplacians. Preprint `arXiv:1611.00696v3` (2016).

[20] C. Carvalho and Y. Qiao, Layer potentials C^*-algebras of domains with conical points. *Cent. Eur. J. Math.* **11** (2013) 27–54.

[21] C. Carvalho, R. Côme and Y. Qiao, Gluing action groupoids: Fredholm conditions and layer potentials. *Rev. Roum. Math. Pures Appl.* **64** (2019) 113–156.

[22] L. Chesnel and P. Ciarlet, Jr., $\mathtt{T}$-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients. *Numer. Math.* **124** (2013) 1–29.

[23] A. Chkifa, A. Cohen and C. Schwab, Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs. *J. Math. Pures Appl.* **103** (2015) 400–428.

[24] A. Cohen and G. Migliorati, Near-optimal approximation methods for elliptic PDEs with lognormal coefficients. *Math. Comput.* **92** (2023) 1665–1691.

[25] A. Cohen, R. Devore and C. Schwab, Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's. *Anal. Appl. (Singap.)* **9** (2011) 11–47.

[26] J. Conway, A course in functional analysis, 2nd edition. In Vol. 96 *Graduate Texts in Mathematics Springer*. New York (1990).

[27] M. Costabel, Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19** (1988) 613–626.

[28] M. Costabel and E. Stephan, A direct boundary integral equation method for transmission problems. *J. Math. Anal. Appl.* **106** (1985) 367–413.

[29] M. Costabel, M. Dauge and S. Nicaise, Corner singularities and analytic regularity for linear elliptic systems. Part I: smooth domains (2010). Preprint arXiv:hal-00453934v2.

[30] M. Costabel, M. Dauge and S. Nicaise, Analytic regularity for linear elliptic systems in polygons and polyhedra. *Math. Models Methods Appl. Sci.* **22** (2012) 1250015, 63.

[31] M. Dauge, Elliptic boundary value problems on corner domains. In Vol. 1341 *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (1988). Smoothness and asymptotics of solutions.

[32] D. Dũng, V. Nguyen, C. Schwab and J. Zech, Analyticity and sparsity in uncertainty quantification for PDEs with Gaussian random field inputs. Preprint https://arxiv.org/abs/2201.01912 (2022).

[33] L. Evans, Partial differential equations, 2nd edition. In Vol. 19 *Graduate Studies in Mathematics*. American Mathematical Society, Providence (2010).

[34] N. Große and C. Schneider, Sobolev spaces on Riemannian manifolds with bounded geometry: general coordinates and traces. *Math. Nachr.* **286** (2013) 1586–1613.

[35] B. Guo and I. Babuška, Regularity of the solutions for elliptic problems on nonsmooth domains in $\mathbf{R}^3$. I. Countably normed spaces on polyhedral domains. *Proc. R. Soc. Edinb. Sect. A* **127** (1997) 77–126.

[36] E. Hebey and F. Robert, Sobolev spaces on manifolds. In: *Handbook of Global Analysis*. Elsevier Sci. B. V., Amsterdam (2008) 375–415.

[37] L. Herrmann, M. Keller and C. Schwab, Quasi-Monte Carlo Bayesian estimation under Besov priors in elliptic inverse problems. *Math. Comput.* **90** (2021) 1831–1860.

[38] G.C. Hsiao and W.L. Wendland, Boundary Integral Equations, 2nd edition. Springer Nature Switzerland, Cham (2021).

[39] K. Klioba and C. Seifert, Approximation of random evolution equations. Preprint arXiv:2404.07660 [math.FA] (2024).

[40] M. Kohr, S. Labrunie, H. Mohsen and V. Nistor, Polynomial estimates for solutions of parametric elliptic equations on complete manifolds. *Stud. Univ. Babes-Bolyai Math.* **67** (2022) 369–382.

[41] O.P. Le Maître and O. Knio, Spectral Methods for Uncertainty Quantification: With Applications to Computational Fluid Dynamics. Springer, Dordrecht (2010).

[42] H. Li, A. Mazzucato and V. Nistor, Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains. *Electron. Trans. Numer. Anal.* **37** (2010) 41–69.

[43] H. Li, V. Nistor and Y. Qiao Uniform shift estimates for transmission problems and optimal rates of convergence for the parametric finite element method, in Numerical Analysis and Its Applications. In Vol. 8236 of *Lecture Notes in Compututer Science*. Springer, Heidelberg (2013) 12–23.

[44] J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications. Vol. I. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg (1972).

[45] A. Mazzucato, V. Nistor and Q. Qu, A nonconforming generalized finite element method for transmission problems. *SIAM J. Numer. Anal.* **51** (2013) 555–576.

[46] M. Mitrea and M. Taylor, Boundary layer methods for Lipschitz domains in Riemannian manifolds. *J. Funct. Anal.* **163** (1999) 181–251.

[47] H. Mohsen, *Estimations uniformes pour des problèmes de transmission à changement de signe: Liens avec les triplets de frontière et la quantification de l'incertitude*, Ph.D. thesis, Lorraine University (2022).

[48] H. Mohsen, S. Labrunie and V. Nistor, Estimations polynomiales pour les solutions des problèmes de transmission sur cylindres. Preprint, to appear in Tunisian J. Math. (2022).

[49] S. Nicaise, Polygonal Interface Problems. Peter Lang, Frankfurt/Main (1993).

[50] S. Nicaise and A.-M. Sändig, General interface problems. I. *Math. Methods Appl. Sci.* **17** (1994) 395–429.

[51] S. Nicaise and A.-M. Sändig, General interface problems. II. *Math. Methods Appl. Sci.* **17** (1994) 431–450.

[52] H.-M. Nguyen, Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients. *J. Math. Pures Appl.* **106** (2016) 342–374.

[53] H.-M. Nguyen and S. Sil, Limiting absorption principle and well-posedness for the time-harmonic Maxwell equations with anisotropic sign-changing coefficients. *Commun. Math. Phys.* **379** (2020) 145–176.

[54] K. Pankrashkin, On self-adjoint realizations of sign-indefinite Laplacians. *Rev. Roum. Math. Pures Appl.* **64** (2019) 345–372.

[55] Y.A. Roĭtberg and Z.G. Sheftel, Elliptic equations with discontinuous coefficients. *Sov. Math. Dokl.* **146** (1962) 1491–1494.

[56] Y.A. Roĭtberg and Z.G. Sheftel, General boundary-value problems for elliptic equations with discontinuous coefficients. *Sov. Math. Dokl.* **4** (1963) 231–234.

[57] T.J. Sullivan, Introduction to Uncertainty Quantification, Vol. 63. Springer, Cham (2015).

[58] M.E. Taylor, Partial differential equations. I. In Vol. 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York (1996). Basic theory.

[59] M. Taylor, Partial differential equations II. Qualitative studies of linear equations, 2nd edition. In Vol. 116 of *Applied Mathematical Sciences*. Springer, New York (2011).

[60] H. Triebel, Characterizations of function spaces on a complete Riemannian manifold with bounded geometry. *Math. Nachr.* **130** (1987) 321–346.