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LOWEST-ORDER NONSTANDARD FINITE ELEMENT METHODS FOR TIME-FRACTIONAL BIHARMONIC PROBLEM

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Abstract. In this work, we consider an initial-boundary value problem for a time-fractional biharmonic equation in a bounded polygonal domain with a Lipschitz continuous boundary in $\mathbb{R}^2$ with clamped boundary conditions. After stating the well-posedness, we focus on some regularity results of the solution with respect to the regularity of the problem data. The spatially semidiscrete scheme covers several popular lowest-order piecewise-quadratic finite element schemes, namely, Morley, discontinuous Galerkin, and C^0 interior penalty methods, and includes both smooth and nonsmooth initial data. Optimal order error bounds with respect to the regularity assumptions on the data are proved for both homogeneous and nonhomogeneous problems. The numerical experiments validate the theoretical convergence rate results.

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1. Introduction

Let Ω be a bounded polygonal domain with a Lipschitz continuous boundary $\partial\Omega$ in $\mathbb{R}^2$ and T be a fixed positive real number. For $0 < \alpha < 1$, consider the following initial-boundary value problem for time-fractional biharmonic equation that seeks $u(t) \in H_0^2(\Omega)$ such that

$$\begin{aligned} \partial_t^\alpha u(t) + Au(t) &= f(t) \quad \text{in } \Omega,\ 0 < t \le T, \\ u(0) &= u_0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where A is the unbounded operator in $L^2(\Omega)$ corresponding to the biharmonic operator Δ^2 with clamped boundary conditions. For an absolutely continuous real function u over $[0,T]$, $\partial_t^\alpha u(t)$ denotes the Caputo fractional derivative defined by

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\,\mathrm{d}s = \mathcal{I}^{1-\alpha} u'(t), \quad 0 < \alpha < 1, \tag{1.2}$$

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where $u'(t) := \partial_t u(t)$ and $\mathcal{I}^\beta$, $0 < \beta < \infty$, denotes the Riemann–Liouville fractional integral defined by

$$\mathcal{I}^\beta v(t) = \int_0^t \kappa_\beta(t-s)v(s)\,\mathrm{d}s, \quad \kappa_\beta(t) := t^{\beta-1}/\Gamma(\beta), \tag{1.3}$$

with the gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t,\ \mathrm{Re}(z) > 0$. In Definition 2.4, we will use (1.2) for absolutely continuous functions with values in a Hilbert space V^* introduced later.

It is well-known that fractional order models describe physical phenomena more accurately compared to the usual integer-order models (*cf.* [35]). The equation in model (1.1) represents a particular case of the time-fractional Cahn–Hilliard equation [2, 18].

The main goal of this work is to study the convergence analysis of semidiscrete approximation with lowest-order nonstandard finite element methods (FEMs).

To present our contributions, we introduce below some standard notations. For any real r, $H^r(\Omega)$ denotes the standard Sobolev space associated with the Sobolev-Slobodeckii semi-norm $|\cdot|_{H^r(\Omega)}$ (*cf.* [15]). Let $H_0^r(\Omega)$ denote the closure of $\mathcal{D}(\Omega)$ in $H^r(\Omega)$, $V := H_0^2(\Omega)$, and let $H^{-r}(\Omega) = (H_0^r(\Omega))^*$ be the dual of $H_0^r(\Omega)$. The notations $(\cdot,\cdot)$ and $\|\cdot\|$ denote the L^2 scalar product and L^2 norm in Ω. For a Hilbert space H with norm $\|\cdot\|_H$, let $L^p(0,T;H)$, $1 \le p < \infty$ denote the standard Bochner space equipped with the norm $\|g\|_{L^p(0,T;H)} = (\int_0^T \|g(t)\|_H^p\,\mathrm{d}t)^{1/p}$. If $p = \infty$, the space $L^\infty(0,T;H)$ has the norm $\|g\|_{L^\infty(0,T;H)} = \inf\{C : \|g(t)\|_H \le C$ almost everywhere on $(0,T)\}$. In addition, for $m \ge 0$, $1 \le p \le \infty$, $W^{m,p}(0,T;H) := \{g : (0,T) \to H$ such that $\frac{\mathrm{d}^j g}{\mathrm{d}t^j} \in L^p(0,T;H)$ for $0 \le j \le m\}$ is equipped with the norm $\|g\|_{W^{m,p}(0,T;H)} = \sum_{j=0}^m \|\frac{\mathrm{d}^j g}{\mathrm{d}t^j}\|_{L^p(0,T;H)}$ (*cf.* [14]). Hereafter, $\gamma_0 \in (1/2, 1]$ is the elliptic regularity index of the biharmonic problem introduced in Section 2.1. Throughout the paper, $a \lesssim b$ denotes $a \le Cb$, where C is a positive generic constant that may depend on fractional order α and the final time T but is independent of the spatial mesh size.

In Theorem 2.6, we state that when initial data u_0 is simply an element of $L^2(\Omega)$ and source function f belongs to $W^{1,1}(0,T;V^*)$, problem (1.1) admits a unique weak solution with

$$\|u\|_{L^1(0,T;V)} + \|u'\|_{L^1(0,T;V^*)} \lesssim \|u_0\| + \|f\|_{W^{1,1}(0,T;V^*)}.$$

The main contributions of this work are summarized below.

(1) We derive regularity results for the solutions of both homogeneous and nonhomogeneous problems useful for the error analysis in Theorems 2.8 and 2.10.
(2) In (3.4), we introduce a novel Ritz projection $\mathcal{R}_h : V \to V_h$, where V_h denotes the space of lowest-order piecewise-quadratic polynomials for (1.1) with clamped boundary conditions in the proposed semidiscrete schemes and includes the popular Morley, discontinuous Galerkin (dG), and C^0 interior penalty (C^0IP) methods. We establish the quasi-optimal approximation property displayed below in Lemma 3.4.

$$\|v - \mathcal{R}_h v\| + h^\gamma \|v - \mathcal{R}_h v\|_h \lesssim h^{2\gamma}\|v\|_{H^{2+\gamma}(\Omega)},$$

for all $\gamma \in [0, \gamma_0]$ and for all $v \in H^{2+\gamma}(\Omega)$, where the regularity index γ_0 belongs to $(1/2, 1)$ if Ω is not convex and $\gamma_0 = 1$ if Ω is convex. These properties are crucial to obtaining optimal order error bounds for both smooth and nonsmooth initial data.
(3) We use an energy argument in a unified framework for the nonstandard FEM analysis with lowest-order piecewise-quadratic polynomials for (1.1). Since the solution of model (1.1) reflects a singular behavior near $t = 0$ (see below Thms. 2.8 and 2.10), a straightforward analysis of numerical scheme is problematic. This is addressed by multiplication by weights of type t^j, $j = 1, 2$ in the semidiscrete scheme. For initial data in $L^2(\Omega)$ and a source term equal to zero, in Theorem 4.2(i) we obtain the following error bound in the $L^2(\Omega)$ and energy norms for each fixed time $t \in (0, T]$,

$$\|u(t) - u_h(t)\| \lesssim \left(h^{2\gamma_0}t^{-\alpha} + h^2 t^{-(1+\alpha)/2}\right)\|u_0\|,$$

$$\|u(t)-u_h(t)\|_h \lesssim h^{\gamma_0} t^{-\alpha}\|u_0\| + \big(h^{2\gamma_0} t^{-3\alpha/2} + h^2 t^{-(\alpha+1/2)}\big)\|u_0\|,$$

where h is the mesh size of the FEM. Similar error bounds are also proved for the nonhomogeneous problem with an appropriate assumption on f and $u_0 \in D(A)$ in Theorem 4.2(ii) for $t \in (0,T]$.

(4) Numerical experiments that validate the theoretical results are presented for examples with smooth and nonsmooth initial data.

The article is organized as follows:

– In Section 2, we introduce some notations, state the well-posedness of (1.1), and derive regularity results of the solution of (1.1).
– This is followed by a discussion on the semidiscrete schemes and error analysis in Sections 3 and 4. Optimal order error bounds are proved for both smooth and nonsmooth initial data.
– Finally, Section 5 presents the results of the numerical implementations that validate the theoretical estimates.

Below, we compare our findings with those already known from the literature and review the existing results for (1.1).

– The well-posedness result is inspired by the work in [36] for second-order fractional diffusion-wave equations. For (1.1), the well-posedness result is summarized in appendix of [25]. However, in (A.1) and (A.2) of [25], the well-posedness and regularity results are stated with *higher* regularity assumptions on the problem data, for example, initial data $u_0 \in D(\Delta^{j+2})$ and source function f such that $\frac{\partial^l f(t)}{\partial t^l} \in D(\Delta^j)$, where $j \in \mathbb{N}$ and $l = 0,1,2,3$. These studies to (1.1) in our work are done comparatively with less smoother assumptions on the problem data (*cf.* Thms. 2.6, 2.8, and 2.10).
– The introduction of the Ritz projection for lowest-order piecewise-quadratic based FEM on $H^2_0(\Omega)\cap H^{2+\gamma}(\Omega)$ for all $\gamma \in [0,\gamma_0]$ and its quasi-optimal approximation properties on $H^{2+\gamma}(\Omega)$ in $L^2(\Omega)$ and energy norms are new. The idea of this operator is borrowed from [7] for the analysis of FEMs for biharmonic plates. In [25], the authors have introduced the Ritz projection on $H^2_0(\Omega)\cap H^{j+1}(\Omega)$, $j \geq 2$ for the analysis of virtual element methods, and have proved optimal projection errors which demand at least $H^3(\Omega)$ regularity of the continuous solution $u(t)$. We also have to mention that the idea of using a Ritz projection is suggested, without proof, in Remark 3.1 of [12], for the Morley finite element approximation of fourth-order nonlinear reaction-diffusion problems.
  There are technical difficulties that arise in the error analysis because (i) the space discretization is not conforming and (ii) the error estimates are established under lower regularity of the solution. To overcome these challenges, we introduce a novel Ritz projection $\mathcal{R}_h$ using a smoother (or companion) operator Q in (3.4). This leads to an error equation in (4.13) that is different from those obtained for the conforming second-order problems [22, 32]. The idea is also completely new in comparison to the available works for time-fractional fourth-order problems in the literature.
– The semidiscrete convergence analysis in this work is inspired by the technique in [22, 30, 32] for time-fractional parabolic problems with and without memory. The cited works discuss only second-order problems and the conforming case, where the discrete trial space is a finite-dimensional subspace of $H^1_0(\Omega)$. In contrast, the analysis of conforming FEMs for fourth-order problems demands C^1 continuity and requires a high polynomial degree, which imposes many conditions on the vertices and edges of an element, and hence, is prohibitively expensive. Most of the available literature for (1.1) discuss *simply supported boundary conditions* ($u = \Delta u = 0$) that enables a mixed formulation *via* an auxiliary variable $v = \Delta u$ and subsequently, the reduction of the problem to a system of two second-order equations.
  As far as we know, the semidiscrete error analysis with $u_0 \in L^2(\Omega)$ is a totally new result for the time-fractional biharmonic problem with clamped boundary conditions.
– The work in [11] deals with a compact finite difference scheme with clamped plate problem, where the author has discretized the spatial derivatives by the Stephenson scheme on uniform meshes, and fractional derivative

by L1 scheme on graded time meshes, and established the stability of the scheme and convergence analysis under the assumptions $\|\frac{\partial^{i+j}u(t)}{\partial x^i \partial y^j}\| \leq C$ with $0 \leq i+j \leq 8$ and $\|\frac{\partial^j u(t)}{\partial t^j}\| \leq C(1+t^{\alpha-j})$, $0 \leq j \leq 2$, for some positive constant C. In contrast, the semidiscrete finite element approximation in this work assumes a lower regularity on the solution $u(t)$ and includes the case $u_0 \in L^2(\Omega)$.

Several numerical methods to (1.1) and its nonlinear variants with *simply supported* boundary conditions (that is, $u = \Delta u = 0$) are studied in literature with stability and convergence analysis. These methods include, for example, Adomian decomposition method [20], finite difference method [41], weak Galerkin method [40], mixed FEMs [18, 19, 27–29], local discontinuous Galerkin methods [13, 39], Petrov–Galerkin method [1], and spline-based methods [16, 42]. The FEM based works in [18, 19, 27–29] for fourth-order problems are studied for simply supported boundary conditions. An auxiliary variable $v = \Delta u$ is introduced for the mixed formulation, and subsequently, the problem reduces to a system of two second-order equations with boundary conditions $u = v = 0$.

2. Well-posedness and regularity

We introduce function spaces and norms and present some useful properties of the Mittag–Leffler functions in Section 2.1. The well-posedness results for (1.1) are stated in Section 2.2. In Section 2.3, regularity results for the solutions of both homogeneous and nonhomogeneous problems are derived.

2.1. Preliminaries

The elliptic operator A

To study the well-posedness results, we start with the solution representation. The results are based on the eigenfunction expansion of the corresponding elliptic operator associated with homogeneous boundary conditions. Consider the following eigenvalue problem

$$\Delta^2\phi = \mu\phi \text{ in } \Omega,\ \phi = \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{2.1}$$

Here ν is the outward unit normal to $\partial\Omega$. It is well-known ([3], p. 761) that there exists a family $(\mu_j, \phi_j)_{j=1}^{\infty}$ such that (μ_j, ϕ_j) is solution of (2.1), $0 < \mu_1 \leq \mu_2 \leq \ldots$, $\mu_j \to \infty$ as $j \to \infty$, and the corresponding family of eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$. For further development of this work, define the unbounded operator $(A, D(A))$ in $L^2(\Omega)$ by

$$D(A) = \{\phi \in H_0^2(\Omega) \mid \Delta^2\phi \in L^2(\Omega)\} \text{ and } A\phi = \Delta^2\phi \quad \text{for all } \phi \in D(A).$$

It is known that [15]

$$D(A) \subset V \cap H^{2+\gamma^*}(\Omega) \text{ with } V = H_0^2(\Omega) \text{ and } \gamma^* \in (1/2, 2], \tag{2.2}$$

where the index γ^* belongs to $[1,2]$ if Ω is convex and $\gamma^* \in (1/2, 1)$ if Ω is not convex. The value of γ^* can be related to the greatest angle of the polygon. Define the domain of the fractional power A^r, $r \in \mathbb{R}^+$ ([34], Chap. 2), of the operator A, by

$$D(A^r) = \left\{ v = \sum_{j=1}^{\infty} v_j\phi_j \mid v_j \in \mathbb{R},\ \sum_{j=1}^{\infty} \mu_j^{2r}|v_j|^2 < \infty \right\},\ A^r v = \sum_{j=1}^{\infty} \mu_j^r v_j \phi_j = \sum_{j=1}^{\infty} \mu_j^r (v, \phi_j)\phi_j. \tag{2.3}$$

The space $D(A^r)$ is equipped with the norm

$$\|v\|_{D(A^r)} = \left(\sum_{j=1}^{\infty} \mu_j^{2r} |(v, \phi_j)|^2 \right)^{1/2}. \tag{2.4}$$

The orthonormality of the basis $\{\phi_j\}_{j=1}^{\infty}$ is used above and throughout in this section. In particular, for $r = 0$, we obtain $\|\cdot\|_{D(A^0)} = \|\cdot\|_{L^2(\Omega)} = \|\cdot\|$.

When $r = 1/2$, we have $D(A^{1/2}) = V$. From Part II, Chapter 1, Proposition 6.1 of [4], it follows that

$$D(A^r) = [L^2(\Omega), D(A)]_r \quad \text{for all } r \in [0,1].$$

Therefore, with (2.2) and interpolation, we can prove that

$$D(A^r) \subset V \cap H^{2+\gamma^*(2r-1)}(\Omega) \quad \text{for all } r \in [1/2, 1].$$

Now, for $r > 0$, let $D(A^{-r}) := (D(A^r))^*$ be the dual of $D(A^r)$, consisting of all bounded linear functionals on $D(A^r)$. Identifying the dual of $L^2(\Omega)$ with itself, we write $D(A^r) \subset L^2(\Omega) \subset D(A^{-r})$. The space $D(A^{-r})$ is a Hilbert space with the norm

$$\|g\|_{D(A^{-r})} = \left(\sum_{j=1}^{\infty} \frac{1}{\mu_j^{2r}} |\langle g, \phi_j \rangle|^2 \right)^{1/2}, \tag{2.5}$$

where for $g \in D(A^{-r})$ and $\phi \in D(A^r)$, the symbol $\langle g, \phi \rangle$ stands for the duality pairing of g with ϕ. Further, with $g \in L^2(\Omega)$ and $\phi \in D(A^r)$, we write $\langle g, \phi \rangle = (g, \phi)$ (*cf.* ([6], Chap. V)).

Remark 2.1. In addition to the index γ^* introduced in (2.2), we need to introduce the so-called regularity index (as defined in [5] or [7]) which is an index $\gamma_0 \in (1/2, 1]$ for which the operator A is an isomorphism from $H^{2+\gamma_0} \cap V$ into $H^{-2+\gamma_0}$. If Ω is convex, $\gamma_0 = 1$, while if Ω is not convex we have $\gamma_0 \in (1/2, 1)$. If Ω is not convex, we can choose γ_0 and γ^* such that $\gamma_0 = \gamma^*$, and the value of γ_0 can be related to the greatest angle of the polygon.

Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be the bilinear form associated with Δ^2 defined by

$$a(v, w) = \int_\Omega D^2 v : D^2 w \, \mathrm{d}x = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j} \, \mathrm{d}x,$$

where $D^2 v$ denotes the Hessian matrix of v. Then, we can verify that

$$(Av, w) = a(v, w) \quad \text{for all } v \in D(A) \text{ and for all } w \in V.$$

In the convergence analysis of the semidiscrete scheme, we need the following version of Gronwall's lemma (*cf.* ([9], Lem. 2.1)).

Lemma 2.2 (Gronwall's lemma)**.** *Assume that ϕ, ψ, and χ are three non-negative integrable functions on* $[0, T]$*. If ϕ satisfies*

$$\phi(t) \le \psi(t) + \int_0^t \chi(s)\phi(s) \, \mathrm{d}s \quad \text{for } t \in (0, T),$$

then,

$$\phi(t) \le \psi(t) + \int_0^t \psi(s)\chi(s) e^{\int_s^t \chi(\tau)\,\mathrm{d}\tau} \, \mathrm{d}s \quad \text{for } t \in (0, T).$$

Mittag–Leffler functions and their properties

The Mittag–Leffler functions play a very important role in the analysis of fractional differential equations. The two-parameter Mittag–Leffler function $E_{\alpha,\beta}(\cdot)$ is defined ([23], p. 42) by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha+\beta)}, \quad z \in \mathbb{C}, \quad \alpha > 0,\ \beta \in \mathbb{R}.$$

It is a generalization of the exponential function e^z in the sense that $E_{1,1}(z) = e^z$. Further, we can directly verify that it is an entire function in z. The following properties of the Mittag–Leffler function $E_{\alpha,\beta}(\cdot)$ ([23], (1.8.28)) or ([35], Thm. 1.4), and ([36], Lems. 3.2 and 3.3) are essential in Sections 2.2 and 2.3.

Lemma 2.3 (Properties of Mittag–Leffler functions)**.** *Let r be a real number with $\frac{\alpha\pi}{2} < r < \min\{\pi, \alpha\pi\}$, where $\alpha \in (0,2)$. Then for any real number β and $r \le |\arg z| \le \pi$,*

$$|E_{\alpha,\beta}(z)| \lesssim \begin{cases} \frac{1}{1+|z|^2}, & (\beta-\alpha) \in \mathbb{Z}^- \cup \{0\}, \\ \frac{1}{1+|z|}, & \text{otherwise.} \end{cases} \tag{2.6}$$

Further, for $\alpha, \mu > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}t^k} E_{\alpha,1}(-\mu t^\alpha) &= -\mu t^{\alpha-k} E_{\alpha,\alpha-k+1}(-\mu t^\alpha), && t \in (0,T], \\ E_{\alpha,\alpha}(-\mu) &\ge 0, && \alpha \in (0,1),\ \mu \ge 0. \end{aligned} \tag{2.7}$$

2.2. Well-posedness

Definition 2.4 (Weak solution)**.** Let us assume that $u_0 \in L^2(\Omega)$ and $f \in W^{1,1}(0,T;V^*)$. A function $u \in L^1(0,T;V) \cap W^{1,1}(0,T;V^*)$ is called a weak solution to (1.1) if the following identities hold.

$$\partial_t^\alpha \langle u(t), v\rangle + a(u(t), v) = \langle f(t), v\rangle \quad \text{ in } L^1(0,T) \text{ for all } v \in V, \tag{2.8}$$

and

$$\lim_{t\to 0} \|u(t) - u_0\|_{V^*} = 0.$$

Remark 2.5. Since we have

$$\begin{aligned} \|\partial_t^\alpha \langle u(t), v\rangle\|_{L^1(0,T)} &= \left\| \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \langle u'(s), v\rangle \,\mathrm{d}s \right\|_{L^1(0,T)} \lesssim \|\langle u'(\cdot), v\rangle\|_{L^1(0,T)} \\ &\lesssim \|u'\|_{L^1(0,T;V^*)} \|v\|_V \quad \text{ for all } v \in V, \end{aligned}$$

we can define $\partial_t^\alpha u$ in $L^1(0,T;V^*)$ by the equation

$$\langle \partial_t^\alpha u(t), v\rangle := \partial_t^\alpha \langle u(t), v\rangle \text{ for all } v \in V \text{ and all } t \in (0,T].$$

Theorem 2.6 (Well-posedness of (1.1))**.** *Assume that $u_0 \in L^2(\Omega)$ and $f \in W^{1,1}(0,T;V^*)$. Then, (1.1) admits a unique weak solution determined by*

$$u(t) = E(t)u_0 + \int_0^t F(t-s) f(s)\,\mathrm{d}s, \tag{2.9}$$

with the operators $E(t) \in \mathcal{L}(L^2(\Omega))$ and $F(t) \in \mathcal{L}(V^*)$ defined by

$$\begin{aligned} E(t)u_0 &= \sum_{j=1}^{\infty} (u_0, \phi_j) E_{\alpha,1}(-\mu_j t^\alpha) \phi_j(x), \\ F(t)f(\cdot) &= \sum_{j=1}^{\infty} \langle f(\cdot), \phi_j\rangle t^{\alpha-1} E_{\alpha,\alpha}(-\mu_j t^\alpha) \phi_j(x), \quad t \in (0,T]. \end{aligned} \tag{2.10}$$

Moreover, we have

$$\|u\|_{L^1(0,T;V)} + \|u'\|_{L^1(0,T;V^*)} \lesssim \|u_0\| + \|f\|_{W^{1,1}(0,T;V^*)}. \tag{2.11}$$

The following stability properties of the solution operator $F(t)$ of the nonhomogeneous problem are essential in the proof of Theorems 2.6 and 2.10. For related results of second-order time-fractional differential equations, we refer to Lemma 2.2 of [21].

Lemma 2.7 (Stability)**.** *For $q \geq -1$, $t \in (0,T]$, and $v \in D(A^{q/2})$, we have*

$$\|F(t)v\|_{D(A^{p/2})} \lesssim t^{-1+\left(1+\frac{(q-p)}{2}\right)\alpha}\|v\|_{D(A^{q/2})}, \quad 0 \leq p-q \leq 4.$$

Proof. In view of (2.10), (2.4) (or (2.5)), and (2.6), we obtain

$$\begin{aligned}\|F(t)v\|^2_{D(A^{p/2})} &= \sum_{j=1}^{\infty}\mu_j^p t^{2\alpha-2}|E_{\alpha,\alpha}(-\mu_j t^\alpha)|^2|\langle v,\phi_j\rangle|^2 \lesssim t^{2\alpha-2}\sum_{j=1}^{\infty}\frac{\mu_j^p}{(1+(\mu_j t^\alpha)^2)^2}|\langle v,\phi_j\rangle|^2\\ &= t^{2\alpha-2+(q-p)\alpha}\sum_{j=1}^{\infty}\frac{(\mu_j t^\alpha)^{p-q}}{(1+(\mu_j t^\alpha)^2)^2}\mu_j^q|\langle v,\phi_j\rangle|^2 \leq t^{2\alpha-2+(q-p)\alpha}\|v\|^2_{D(A^{q/2})},\end{aligned}$$

where we have used $\frac{(\mu_j t^\alpha)^{p-q}}{(1+(\mu_j t^\alpha)^2)^2} \leq 1$ for $0 \leq p-q \leq 4$ and $j \in \mathbb{N}$. This concludes the proof. □

Proof of Theorem 2.6. The statement follows from standard arguments using the stated properties of the Mittag–Leffler functions from Lemma 2.3 to show the claimed regularity $u \in L^1(0,T;V) \cap W^{1,1}(0,T;V^*)$. The details of the proof are omitted for the sake of brevity. For a complete proof, see [31]. □

2.3. Regularity results

The following regularity results for the solutions of both homogeneous and nonhomogeneous problems are employed in the error analysis.

Theorem 2.8 (Regularity for homogeneous case)**.** *Let u be the solution of (1.1) with $f = 0$. Then, for $t \in (0,T]$, the following results hold true:*

(i) *(Nonsmooth initial data) For $u_0 \in L^2(\Omega)$, we have*

$$\|u^{(i)}(t)\|_{D(A^p)} \lesssim t^{-(i+\alpha p)}\|u_0\| \quad \text{for } i \in \{0,1\}, 0 \leq p \leq 1, \tag{2.12}$$

$$\|u''(t)\| \lesssim t^{-2}\|u_0\|, \tag{2.13}$$

$$\|\partial_t^\alpha u(t)\| \lesssim t^{-\alpha}\|u_0\|. \tag{2.14}$$

In addition, $\|u\|_{C([0,T];L^2(\Omega))} \lesssim \|u_0\|$.

(ii) *(Smooth initial data) For $u_0 \in D(A)$,*

$$\|u(t)\|_{D(A)} \lesssim \|u_0\|_{D(A)} \tag{2.15}$$

$$\|u'(t)\|_{D(A^p)} \lesssim t^{-(1+\alpha(p-1))}\|u_0\|_{D(A)} \quad \text{for } 0 \leq p \leq 1, \tag{2.16}$$

$$\|u''(t)\| \lesssim t^{\alpha-2}\|u_0\|_{D(A)}, \tag{2.17}$$

$$\|\partial_t^\alpha u(t)\| \lesssim \|u_0\|_{D(A)}. \tag{2.18}$$

Furthermore, $\|u\|_{C([0,T];D(A))} \lesssim \|u_0\|_{D(A)}$.

Remark 2.9. In view of the estimates in Theorem 2.8(i) and (ii), by means of interpolation we obtain for time $t \in (0,T]$,

$$\begin{aligned} \|u(t)\|_{D(A^p)} &\lesssim t^{-\alpha(p-q)}\|u_0\|_{D(A^q)} && \text{for } 0 \le q \le p \le 1, \\ \|u'(t)\|_{D(A^p)} &\lesssim t^{-(1+\alpha(p-q))}\|u_0\|_{D(A^q)} && \text{for } 0 \le p, q \le 1, \\ \|u''(t)\| &\lesssim t^{q\alpha-2}\|u_0\|_{D(A^q)} && \text{for } 0 \le q \le 1, \text{ and,} \\ \|\partial_t^\alpha u(t)\| &\lesssim t^{-(1-q)\alpha}\|u_0\|_{D(A^q)} && \text{for } 0 \le q \le 1. \end{aligned}$$

Theorem 2.10 (Regularity for nonhomogeneous case with zero initial data)**.** *Let u be the solution of* (1.1) *with $u_0 = 0$ and $f \in W^{1,\infty}([0,T]; D(A^{q/2})) \cap W^{2,1}(0,T; L^2(\Omega))$, $q \in [-1,1]$. Then, for all $\epsilon \in (0,1)$, we have*

$$\|u(t)\|_{D(A^{q/2+1-\epsilon/2})} \lesssim \epsilon^{-1}t^{\alpha\epsilon/2}\|f\|_{L^\infty(0,t;D(A^{q/2}))}, \quad t \in [0,T], \tag{2.19}$$

$$\|u'(t)\|_{D(A^{q/2+1-\epsilon/2})} \lesssim \epsilon^{-1}t^{\alpha\epsilon/2}\|f'\|_{L^\infty(0,t;D(A^{q/2}))} + t^{-1+\alpha\epsilon/2}\|f(0)\|_{D(A^{q/2})}, \quad t \in (0,T], \tag{2.20}$$

$$\|u''(t)\| \lesssim \mathcal{I}^\alpha\|f''(\cdot)\|(t) + t^{\alpha-1}\|f'(0)\| + t^{\alpha-2}\|f(0)\|, \quad t \in (0,T], \tag{2.21}$$

$$\|\partial_t^\alpha u(t)\| \lesssim \epsilon^{-1}t^{\alpha\epsilon/2}\|f\|_{L^\infty(0,t;D(A^{\epsilon/2}))} + \|f(t)\|, \quad t \in [0,T]. \tag{2.22}$$

Remark 2.11. The combination of (2.16) with $\beta \in [1/2,1)$ and (2.20) with $q \in [2\beta - 2 + \epsilon, 1]$ results in the following estimate for the solution u of (1.1): For $t \in (0,T]$,

$$\|u'(t)\|_{D(A^\beta)} \lesssim t^{-1+\alpha(1-\beta)}\|u_0\|_{D(A)} + \epsilon^{-1}t^{\alpha\epsilon/2}\|f'\|_{L^\infty(0,t;D(A^{q/2}))} + t^{-1+\alpha\epsilon/2}\|f(0)\|_{D(A^{q/2})}.$$

Proofs of Theorems 2.8 and 2.10. The proofs of these assertions are based on the solution representation in (2.9), the properties of Mittag–Leffler functions in (2.6) and (2.7), and Lemma 2.7. For details, see [31]. ☐

3. Semidiscrete scheme

In this section, we describe the lowest-order finite element discretization schemes for the spatial variable in Section 3.1. The Ritz projection operator and its approximation properties are stated in Section 3.2.

3.1. Lowest-order finite element discretizations

Let $\mathcal{T}$ denote a shape regular triangulation of the polygonal Lipschitz domain into compact triangles. Associate its piecewise constant mesh-size $h_\mathcal{T} \in P_0(\mathcal{T})$ with $h_K := h_\mathcal{T}|_K := \mathrm{diam}(K) \approx |K|^{1/2}$ in any triangle $K \in \mathcal{T}$ of area $|K|$ and its maximal mesh-size $h := \max h_\mathcal{T}$. Let $\mathcal{V}$ (resp. $\mathcal{V}(\Omega)$ or $\mathcal{V}(\partial\Omega)$) denote the set of all (resp. interior or boundary) vertices in $\mathcal{T}$. Let $\mathcal{E}$ (resp. $\mathcal{E}(\Omega)$ or $\mathcal{E}(\partial\Omega)$) denote the set of all (resp. interior or boundary) edges. The length of an edge e is denoted by h_e. Let the Hilbert space $H^m(\mathcal{T}) \equiv \prod_{K\in\mathcal{T}} H^m(K)$. The edge-patch $\omega(e) := \mathrm{int}(K_+ \cup K_-)$ of the interior edge $e = \partial K_+ \cap \partial K_- \in \mathcal{E}(\Omega)$ is the interior of union $K_+ \cup K_-$ of the neighboring triangles K_+ and K_-; the jump and average of φ are defined by $[\![\varphi]\!] := \varphi|_{K_+} - \varphi|_{K_-}$ and $\{\!\{\varphi\}\!\} := \frac{1}{2}(\varphi|_{K_+} + \varphi|_{K_-})$ across the interior edge e of the adjacent triangles K_+ and $K_- \in \mathcal{T}$ in an order such that the unit normal vector $\nu_{K_+}|_e = \nu_e = -\nu_{K_-}|_e$ along the edge e has a fixed orientation and points outside K_+ and inside K_-; ν_K is the outward unit normal of K along ∂K. Further for $e \in \mathcal{E}(\partial\Omega)$, define $\omega(e) := \mathrm{int}(K)$, and the jump and average by $[\![\varphi]\!] := \varphi|_e$ and $\{\!\{\varphi\}\!\} := \varphi|_e$. For functions in $H_0^2(\Omega)$, the notation $|\!|\!| \cdot |\!|\!| := |\cdot|_{H^2(\Omega)}$ stands for the energy norm. The notation $|\!|\!| \cdot |\!|\!|_{\mathrm{pw}} := |\cdot|_{H^2(\mathcal{T})} := \|D^2_{\mathrm{pw}} \cdot\|$ refers to the piecewise energy norm with the piecewise Hessian D^2_{pw}. Define the piecewise polynomials space $P_r(\mathcal{T})$ of degree $r \in \mathbb{N}$ by $P_r(\mathcal{T}) = \{v \in L^2(\Omega) : v|_K \in P_r(K) \text{ for all } K \in \mathcal{T}\}$.

The nonconforming Morley finite element space $\mathrm{M}(\mathcal{T})$ [10] is defined as

$$\begin{aligned} \mathrm{M}(\mathcal{T}) := \{\chi_h \in P_2(\mathcal{T}) :\ & \chi_h \text{ is continuous at the vertices and its normal derivative } \partial\chi_h/\partial\nu \text{ is} \\ & \text{continuous at the midpoints of interior edges, } \chi_h \text{ vanishes at the vertices on } \partial\Omega \\ & \text{and its normal derivative } \partial\chi_h/\partial\nu \text{ vanishes at the midpoints of boundary edges}\}. \end{aligned}$$

On a finite-dimensional space $V_h \subset H^2(\mathcal{T})$, define a mesh-dependent broken norm [8] by

$$\|\chi_h\|_h^2 = \sum_{K\in\mathcal{T}} |\chi_h|^2_{H^2(K)} + \sum_{e\in\mathcal{E}}\sum_{z\in\mathcal{V}(e)} h_e^{-2}\big|[\![\chi_h]\!](z)\big|^2 + \sum_{e\in\mathcal{E}}\left|\fint_e \left[\!\!\left[\frac{\partial \chi_h}{\partial \nu}\right]\!\!\right] \mathrm{d}s\right|^2, \tag{3.1}$$

where $\fint_e$ denotes the integral mean over the edge e. In particular, for $\chi_h \in \mathrm{M}(\mathcal{T})$, $\|\chi_h\|_h = |\!|\!|\chi_h|\!|\!|_{\mathrm{pw}}$ as the jump terms in (3.1) vanish. Further, the discrete bilinear form $a_h : (V_h + \mathrm{M}(\mathcal{T})) \times (V_h + \mathrm{M}(\mathcal{T})) \to \mathbb{R}$ in all the examples in this paper has the form

$$a_h(\cdot,\cdot) = a_{\mathrm{pw}}(\cdot,\cdot) + b_h(\cdot,\cdot) + c_h(\cdot,\cdot),$$

and satisfies **(H)** below.

(H) $a_h(\cdot,\cdot)$ is symmetric, positive-definite, and continuous on V_h with respect to the discrete norm $\|\cdot\|_h$, *i.e.*, $\exists$ constants $\beta_1, \beta_2 > 0$ independent of h such that, for all $w_h, \chi_h \in V_h$, $a_h(w_h,\chi_h) = a_h(\chi_h, w_h)$ and

$$a_h(\chi_h,\chi_h) \geq \beta_1 \|\chi_h\|_h^2 \text{ and } a_h(w_h,\chi_h) \leq \beta_2 \|w_h\|_h \|\chi_h\|_h. \tag{3.2}$$

The semidiscrete problem that corresponds to the weak formulation in Definition 2.4 seeks $u_h \in W^{1,1}(0,T;V_h)$ such that

$$\begin{aligned} \partial_t^\alpha (u_h(t),\chi_h) + a_h(u_h(t),\chi_h) &= (f(t),\chi_h) \text{ for all } \chi_h \in V_h, 0 < t \leq T,\\ u_h(0) &= P_h u_0 \in V_h. \end{aligned} \tag{3.3}$$

Here $P_h : L^2(\Omega) \to V_h$ denotes the L^2-projection defined by $(P_h v, \chi_h) = (v, \chi_h)$ for all $\chi_h \in V_h$.

Now we present Examples 3.1–3.3 below for which the discrete bilinear form $a_h(\cdot,\cdot)$ satisfies **(H)** ([7], Sect. 5).

Example 3.1 (Morley)**.** In this case, $V_h := \mathrm{M}(\mathcal{T})$, and for all $w_h, \chi_h \in \mathrm{M}(\mathcal{T})$,

$$a_h(w_h,\chi_h) := a_{\mathrm{pw}}(w_h,\chi_h) := \int_\Omega D^2_{\mathrm{pw}} w_h : D^2_{\mathrm{pw}}\chi_h \,\mathrm{d}x.$$

The discrete Morley norm on V_h is defined by $\|\cdot\|_h = |\!|\!|\cdot|\!|\!|_{\mathrm{pw}} = a_{\mathrm{pw}}(\cdot,\cdot)^{1/2}$. Notice that this norm is equivalent on V_h to that introduced in (3.1).

Example 3.2 (dG)**.** Choose $V_h := P_2(\mathcal{T})$, and for all $w_h, \chi_h \in V_h$, let

$$\begin{aligned} a_h(w_h,\chi_h) &:= a_{\mathrm{pw}}(w_h,\chi_h) + b_h(w_h,\chi_h) + c_{\mathrm{dG}}(w_h,\chi_h),\\ b_h(w_h,\chi_h) &:= -\mathcal{J}(w_h,\chi_h) - \mathcal{J}(\chi_h,w_h), \mathcal{J}(w_h,\chi_h) := \sum_{e\in\mathcal{E}}\int_e [\![\nabla w_h]\!]\cdot \{\!\!\{D^2_{\mathrm{pw}}\chi_h\}\!\!\}\nu \,\mathrm{d}s,\\ c_{\mathrm{dG}}(w_h,\chi_h) &:= \sum_{e\in\mathcal{E}}\left(\frac{\sigma^1_{\mathrm{dG}}}{h_e^3}\int_e [\![w_h]\!]\,[\![\chi_h]\!]\,\mathrm{d}s + \frac{\sigma^2_{\mathrm{dG}}}{h_e}\int_e \left[\!\!\left[\frac{\partial w_h}{\partial\nu}\right]\!\!\right]\left[\!\!\left[\frac{\partial \chi_h}{\partial\nu}\right]\!\!\right]\mathrm{d}s\right), \end{aligned}$$

where σ^1_{dG}, $\sigma^2_{\mathrm{dG}} > 0$ are the penalty parameters and $a_{\mathrm{pw}}(\cdot,\cdot)$ is as defined in Example 3.1. The dG norm $\|\cdot\|_{\mathrm{dG}}$ on $P_2(\mathcal{T})$ is defined by $\|w_h\|_{\mathrm{dG}} = (|\!|\!|w_h|\!|\!|^2_{\mathrm{pw}} + c_{\mathrm{dG}}(w_h,w_h))^{1/2}$, $w_h \in V_h$. As in the previous example, this norm is equivalent on V_h to that introduced in (3.1).

Example 3.3 (C^0IP)**.** Choose $V_h := P_2(\mathcal{T}) \cap H^1_0(\Omega)$ and for all $w_h, \chi_h \in V_h$, define

$$\begin{aligned} a_h(w_h,\chi_h) &:= a_{\mathrm{pw}}(w_h,\chi_h) + b_h(w_h,\chi_h) + c_{\mathrm{IP}}(w_h,\chi_h),\\ c_{\mathrm{IP}}(w_h,\chi_h) &:= \sum_{e\in\mathcal{E}} \frac{\sigma_{\mathrm{IP}}}{h_e}\int_e \left[\!\!\left[\frac{\partial w_h}{\partial\nu}\right]\!\!\right]\left[\!\!\left[\frac{\partial \chi_h}{\partial\nu}\right]\!\!\right]\mathrm{d}s, \end{aligned}$$

where σ_{IP} is a positive parameter, $a_{\mathrm{pw}}(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are as defined in Examples 3.1 and 3.2. The discrete norm $\|\cdot\|_{\mathrm{IP}}$ on the space V_h reads $\|w_h\|_{\mathrm{IP}} = (|\!|\!| w_h |\!|\!|^2_{\mathrm{pw}} + c_{\mathrm{IP}}(w_h, w_h))^{1/2}$ for $w_h \in V_h$. In this example as well, $\|\cdot\|_{\mathrm{IP}}$ is equivalent on V_h to that introduced in (3.1).

The equivalence of the common norm $\|\cdot\|_h$ with the norms defined in the Examples 3.1-3.3 (see [7]) is helpful in the proofs of the approximation properties of the Ritz projection in the next section.

3.2. Ritz projection and its approximation properties

We introduce the Ritz projection which is the elliptic projection for the biharmonic problem and state its approximation properties. The Ritz projection $\mathcal{R}_h : V \to V_h$ is defined by

$$a_h(\mathcal{R}_h v, \chi_h) = a(v, Q\chi_h) \quad \text{for all } \chi_h \in V_h \text{ and } v \in V, \tag{3.4}$$

where $Q := JI_{\mathrm{M}}$ is a smoother defined from $H^2(\mathcal{T})$ to V, with $I_{\mathrm{M}} : H^2(\mathcal{T}) \to \mathrm{M}(\mathcal{T})$ and $J : \mathrm{M}(\mathcal{T}) \to V$ denoting the extended Morley interpolation operator and the companion operator, respectively (see appendix of [31] for the details of the definitions and properties of the interpolation and companion operators). The Lax–Milgram lemma shows that the projection $\mathcal{R}_h$ is a well-defined operator on V.

In the semidiscrete error analysis discussed in Section 4, the error $u(t) - u_h(t)$ is split by introducing the Ritz projection $\mathcal{R}_h$ as

$$u(t) - u_h(t) = (u(t) - \mathcal{R}_h u(t)) + (\mathcal{R}_h u(t) - u_h(t)).$$

Given the approximation properties for the Ritz projection from Lemma 3.4, the main task will be to establish bounds for $u_h(t) - \mathcal{R}_h u(t)$ in the next section.

Lemma 3.4 (Approximation properties [7, 31])**.** *For any $v \in V$ and the Ritz projection $\mathcal{R}_h : V \to V_h$ defined in (3.4), the approximation properties in the energy and L^2 norms stated below hold.*

$$\|v - \mathcal{R}_h v\| + h^{\gamma}\|v - \mathcal{R}_h v\|_h \lesssim h^{2\gamma}\|v\|_{H^{2+\gamma}(\Omega)},$$

for all $\gamma \in [0, \gamma_0]$ and for all $v \in H^{2+\gamma}(\Omega)$, where $\gamma_0 \in (1/2, 1]$ is the regularity index introduced in Remark 2.1.

In the error analysis, we also need the following approximation property of the operator Q for piecewise quadratic functions.

Lemma 3.5 ([7], Thm. 4.5(d))**.** *For any $\chi_h \in P_2(\mathcal{T})$, the operator $Q = JI_{\mathrm{M}}$ satisfies*

$$\|\chi_h - Q\chi_h\|_{H^s(\mathcal{T})} \le C_1 h^{2-s} \min_{v \in V} \|v - \chi_h\|_h \text{ for any } 0 \le s \le 2 \text{ and a constant } C_1 > 0.$$

4. Semidiscrete error estimates

In this section, we derive error bounds for the semidiscrete scheme for both smooth and nonsmooth initial data. To start with, in Section 4.1, we state some important properties of fractional integrals that are relevant in the context. This is followed by the main results of this section and their proofs in Sections 4.2 and 4.3.

4.1. Properties of Riemann–Liouville fractional integrals

For all $\alpha \in (0, \infty)$, all $\beta \in (0, \infty)$ satisfying $\alpha + \beta \ge 1$, all $v \in L^1(0, T)$, the operators $\mathcal{I}^{\alpha}$ and $\mathcal{I}^{\beta}$ (*cf.* (1.3)) satisfy

$$\mathcal{I}^{\alpha}\mathcal{I}^{\beta}v(t) = \mathcal{I}^{\alpha+\beta}v(t), \text{ for almost all } t \in (0, T). \tag{4.1}$$

If $v \in C([0, T])$, then (4.1) is satisfied at all points $t \in [0, T]$ and $\alpha, \beta > 0$ (*cf.* ([37], p. 34)). Recall that $\kappa_{\alpha}(t) := t^{\alpha-1}/\Gamma(\alpha)$. In the rest of this subsection, we assume that $0 < \alpha < 1$, and so $0 < 1 - \alpha < 1$. The three identities below hold for $v_1(t) = tv(t)$ and $v_2(t) = t^2 v(t)$ for $t \in [0, T]$ (see Lem. 2 of [22] and Lem. 2.1 of [32] for a proof).

$$t\mathcal{I}^\alpha v(t) = \mathcal{I}^\alpha v_1(t) + \alpha \mathcal{I}^{\alpha+1} v(t), \tag{4.2}$$

$$t\mathcal{I}^\alpha v'(t) = \mathcal{I}^\alpha (v_1)'(t) + (\alpha - 1)\mathcal{I}^\alpha v(t) - t\kappa_\alpha(t) v(0), \tag{4.3}$$

$$t^2\mathcal{I}^\alpha v'(t) = \mathcal{I}^\alpha (v_2)'(t) + 2(\alpha - 1)\mathcal{I}^\alpha v_1(t) + \alpha(\alpha-1)\mathcal{I}^{\alpha+1} v(t) - t^2\kappa_\alpha(t) v(0). \tag{4.4}$$

For $\phi, v \in L^2(0, T; L^2(\Omega))$, since $\cos(\alpha\pi/2) - (1-\alpha) \geq 0$, the following continuity property with any positive ϑ holds for $\mathcal{I}^{1-\alpha}$ ([33], Lem. 3.1(iii)):

$$\int_0^t (\mathcal{I}^{1-\alpha}\phi, v)\,\mathrm{d}s \leq \vartheta \int_0^t (\mathcal{I}^{1-\alpha} v, v)\,\mathrm{d}s + \frac{1}{4\vartheta(1-\alpha)^2}\int_0^t (\mathcal{I}^{1-\alpha}\phi, \phi)\,\mathrm{d}s. \tag{4.5}$$

If $v : [0, T] \to L^2(\Omega)$ is a piecewise continuous function in time, then $\mathcal{I}^\alpha$ satisfies ([33], Lem. 3.1(ii))

$$\int_0^T (\mathcal{I}^\alpha v, v)\,\mathrm{d}t \geq \cos(\alpha\pi/2)\int_0^T \|\mathcal{I}^{\alpha/2} v\|^2\,\mathrm{d}t \geq 0. \tag{4.6}$$

In addition, if $v' : [0, T] \to L^2(\Omega)$ is a piecewise continuous function in time, then for $t \in (0, T]$, it follows from Lemma 2.1 of [24] that

$$\|v(t) - v(0)\|^2 \lesssim t^\alpha \int_0^t \|\mathcal{I}^{(1-\alpha)/2} v'\|^2\,\mathrm{d}s \lesssim t^\alpha \int_0^t (\mathcal{I}^{1-\alpha} v', v')\,\mathrm{d}s, \tag{4.7}$$

where the last inequality follows from (4.6).

The proofs of this section use the inequality below frequently. For $a, b \geq 0$, it holds that

$$(a+b)^2 \lesssim (a^2 + b^2) \lesssim (a+b)^2. \tag{4.8}$$

4.2. Main results

For the semidiscrete error analysis, we split the error $u(t) - u_h(t)$ by introducing the Ritz projection $\mathcal{R}_h$ from (3.4) as

$$u(t) - u_h(t) =: \rho(t) + \theta(t), \tag{4.9}$$

with

$$\rho(t) := u(t) - \mathcal{R}_h u(t) \text{ and } \theta(t) := \mathcal{R}_h u(t) - u_h(t).$$

Let

$$\Lambda_0(\epsilon, t) := \|u_0\|_{D(A)} + \epsilon^{-1} t^{\alpha\epsilon/2}\|f\|_{L^\infty(0,T;D(A^{\epsilon/2}))},\ \epsilon \in (0,1),\ t \in [0, T]. \tag{4.10}$$

Recall that $D(A) \subset V \cap H^{2+\gamma^*}(\Omega) \subset V \cap H^{2+\gamma_0}(\Omega)$. For the homogeneous problem, using (2.12) with $i = 0$, $p = 1$, and for the nonhomogeneous problem, applying (2.15) and (2.19) with $q = \epsilon$ we obtain

$$\|u(t)\|_{H^{2+\gamma_0}(\Omega)} \lesssim \begin{cases} t^{-\alpha}\|u_0\| \text{ for } \ u_0 \in L^2(\Omega) \text{ and } f = 0,\ t \in (0, T], \\ \Lambda_0(\epsilon, t) \text{ for } u_0 \in D(A) \text{ and } f \neq 0,\ t \in [0, T]. \end{cases}$$

This and Lemma 3.4 establish the estimates for $\rho(t)$ as given below.

$$\|\rho(t)\| + h^{\gamma_0}\|\rho(t)\|_h \lesssim \begin{cases} h^{2\gamma_0} t^{-\alpha}\|u_0\| \text{ for } u_0 \in L^2(\Omega) \text{ and } f = 0,\ t \in (0, T], \\ h^{2\gamma_0}\Lambda_0(\epsilon, t) \text{ for } u_0 \in D(A) \text{ and } f \neq 0,\ t \in [0, T]. \end{cases} \tag{4.11}$$

Hence the main task in the remaining part of this section is to bound $\theta(t)$ in the $L^2(\Omega)$ and energy norms. For $t \in [0,T]$ and an $\epsilon \in (0,1)$ (determined by the smoothness of f), set

$$
\begin{aligned}
\Lambda_1(\epsilon,t) &:= t^{\frac{3}{2}}\|u_0\|_{D(A)} + t^{\frac{3}{2}-\alpha\left(1-\frac{\epsilon}{2}\right)}\|f(0)\|_{D\left(A^{\frac{\epsilon}{2}}\right)} + t\|f\|_{L^2(0,t;L^2(\Omega))} \\
&\quad + \epsilon^{-1}t^{\frac{5}{2}-\alpha\left(1-\frac{\epsilon}{2}\right)}\|f'\|_{L^\infty\left(0,T;D\left(A^{\frac{\epsilon}{2}}\right)\right)} \\
\Lambda_2(\epsilon,t) &:= t^{\frac{3}{2}}\|u_0\|_{D(A)} + B\Big(\frac{\alpha\epsilon}{2},1-\alpha\Big)t^{\frac{3}{2}-\alpha\left(1-\frac{\epsilon}{2}\right)}\|f(0)\|_{D\left(A^{\frac{\epsilon}{2}}\right)} + t\|f\|_{L^2(0,t;L^2(\Omega))} \\
&\quad + \epsilon^{-1}t^{\frac{5}{2}-\alpha\left(1-\frac{\epsilon}{2}\right)}\|f'\|_{L^\infty\left(0,T;D\left(A^{\frac{\epsilon}{2}}\right)\right)} \\
\mathcal{B}_0(\epsilon,t) &:= \|u_0\|_{D(A)} + \epsilon^{-1}t^{\frac{\alpha\epsilon}{2}}\|f\|_{W^{1,\infty}(0,T;D(A^{\epsilon/2}))} \\
\mathcal{B}_1(\epsilon,t) &:= \Lambda_1(\epsilon,t) + \Lambda_2(\epsilon,t) + t^2\|f(t)\| + t\|f\|_{L^2(0,t;L^2(\Omega))} + t^2\|f'\|_{L^2(0,t;L^2(\Omega))} \\
&\quad + t^{\frac{3}{2}}(\|f(0)\| + t\|f'(0)\|) + t^{5/2}\|f''\|_{L^1(0,T;L^2(\Omega))},
\end{aligned}
\tag{4.12}
$$

where $B(\cdot,\cdot)$ denotes the standard beta function. Note that in the above expression and in the sequel, whenever the norm of the function f is dependent only on the space variable, we denote the dependence of f on t as $f(t)$; if the norm is space-time dependent, the arguments in f are omitted for notational brevity.

Theorem 4.1 (Estimates for $\theta(t)$)**.** *Let $u(t)$ and $u_h(t)$ solve (1.1) and (3.3), respectively. Let $\mathcal{R}_h u(t)$ denote the Ritz projection of $u(t)$ defined in (3.4). Then for $\theta(t) = \mathcal{R}_h u(t) - u_h(t)$, the estimates in (i)–(ii) below hold.*

(i) *(Nonsmooth initial data) For $u_0 \in L^2(\Omega)$ and $f = 0$,*

$$\|\theta(t)\| + t^{\alpha/2}\|\theta(t)\|_h \lesssim \big(h^{2\gamma_0}t^{-\alpha} + h^2 t^{-(1+\alpha)/2}\big)\|u_0\|, \quad t \in (0,T].$$

(ii) *(Smooth initial data) For $u_0 \in D(A)$, $f \in W^{1,\infty}([0,T];D(A^{\epsilon/2})) \cap W^{2,1}(0,T;L^2(\Omega))$, and for all $\epsilon \in (0,1)$,*

$$\|\theta(t)\| + t^{\alpha/2}\|\theta(t)\|_h \lesssim h^{2\gamma_0}\mathcal{B}_0(\epsilon,t) + h^2 t^{\alpha/2-2}\mathcal{B}_1(\epsilon,t), \quad t \in (0,T],$$

with $\mathcal{B}_0(\epsilon,t)$ and $\mathcal{B}_1(\epsilon,t)$ defined in (4.12).

A combination of the estimates for $\rho(t)$ from (4.11) and $\theta(t)$ from Theorem 4.1, shows the error estimates for the semidiscrete scheme for both smooth and nonsmooth initial data in Theorem 4.2.

Theorem 4.2 (Error estimates)**.** *Let $u(t)$ and $u_h(t)$ be solutions of the continuous and semidiscrete problems in (1.1) and (3.3), respectively.*

(i) *(Nonsmooth initial data) For $u_0 \in L^2(\Omega)$, source function $f = 0$, $u_h(0) = P_h u_0$, and $t \in (0,T]$, it holds that*

$$
\begin{aligned}
\|u(t) - u_h(t)\| &\lesssim \big(h^{2\gamma_0}t^{-\alpha} + h^2 t^{-(1+\alpha)/2}\big)\|u_0\|, \\
\|u(t) - u_h(t)\|_h &\lesssim h^{\gamma_0}t^{-\alpha}\|u_0\| + \big(h^{2\gamma_0}t^{-3\alpha/2} + h^2 t^{-(\alpha+1/2)}\big)\|u_0\|.
\end{aligned}
$$

(ii) *(Smooth initial data) For $u_0 \in D(A)$, $f \in W^{1,\infty}([0,T];D(A^{\epsilon/2})) \cap W^{2,1}(0,T;L^2(\Omega))$, $u_h(0) = P_h u_0$, $t \in (0,T]$, and for all $\epsilon \in (0,1)$, it holds that*

$$
\begin{aligned}
\|u(t) - u_h(t)\| &\lesssim h^{2\gamma_0}\Lambda_0(\epsilon,t) + h^{2\gamma_0}\mathcal{B}_0(\epsilon,t) + h^2 t^{\alpha/2-2}\mathcal{B}_1(\epsilon,t), \\
\|u(t) - u_h(t)\|_h &\lesssim h^{\gamma_0}\Lambda_0(\epsilon,t) + h^{2\gamma_0}t^{-\alpha/2}\mathcal{B}_0(\epsilon,t) + h^2 t^{-2}\mathcal{B}_1(\epsilon,t),
\end{aligned}
$$

where $\Lambda_0(\epsilon,t)$ and $\mathcal{B}_0(\epsilon,t)$, $\mathcal{B}_1(\epsilon,t)$ are defined, respectively, in (4.10) and (4.12).

Proof. The error decomposition (4.9), the triangle inequality, the first estimate in (4.11), and Theorem 4.1(i) lead to the proof of (i). The proof of (ii) follows from (4.9), the triangle inequality, the second estimate in (4.11), and Theorem 4.1(ii). □

4.3. Proof of Theorem 4.1

First of all, a key inequality is proved in Lemma 4.3, and bounds for the terms appearing in this lemma are established in Lemmas 4.5–4.8. A combination of these results establishes the proof of Theorem 4.1.

The following notations hold throughout this subsection.

$$\theta_1(t) := t\theta(t),\ \theta_2(t) := t^2\theta(t),\ \text{ and}$$
$$\widehat{v}(t) := \mathcal{I}v(t) = \int_0^t v(s)\,\mathrm{d}s,\ \widehat{\widehat{v}}(t) := \mathcal{I}^2 v(t) = \int_0^t \int_0^s v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Recall the smoother Q from Section 3.2. For each $\chi_h \in V_h$, test (2.8) with $Q\chi_h \in V$ and subtract (3.3) from (2.8) to obtain

$$(\partial_t^\alpha u(t), Q\chi_h) - (\partial_t^\alpha u_h(t), \chi_h) + (a(u(t), Q\chi_h) - a_h(u_h(t), \chi_h)) = (f(t), (Q-I)\chi_h).$$

Add and subtract $(\partial_t^\alpha u(t), \chi_h)$, utilize (3.4) and $\theta(t) := R_h u(t) - u_h(t)$ on the left-hand side of the above expression to obtain

$$(\partial_t^\alpha (u(t) - u_h(t)), \chi_h) + (\partial_t^\alpha u(t), (Q-I)\chi_h) + a_h(\theta(t), \chi_h) = (f(t), (Q-I)\chi_h).$$

Recall $u(t) - u_h(t) = \rho(t) + \theta(t)$ and the notation from (1.2), that yields $\partial_t^\alpha \theta(t) = \mathcal{I}^{1-\alpha}\theta'(t)$, $\partial_t^\alpha \rho(t) = \mathcal{I}^{1-\alpha}\rho'(t)$, and $\partial_t^\alpha u(t) = \mathcal{I}^{1-\alpha}u'(t)$. For ease of notation in the subsequent estimates, we denote $\varphi(t) := \mathcal{I}^{1-\alpha}u'(t)$. Then for all $\chi_h \in V_h$ and $t \in (0, T]$, the above displayed identity leads to the error equation in $\theta(t)$ as

$$(\mathcal{I}^{1-\alpha}\theta'(t), \chi_h) + a_h(\theta(t), \chi_h) = -(\mathcal{I}^{1-\alpha}\rho'(t), \chi_h) - (\varphi(t), (Q-I)\chi_h) + (f(t), (Q-I)\chi_h). \tag{4.13}$$

Recall the notations

$$\theta_i(t) = t^i\theta(t),\ \rho_i(t) = t^i\rho(t) \text{ for } i = 1, 2;\ \varphi_2(t) = t^2\varphi(t),\ f_2(t) = t^2 f(t), \widehat{\theta}(t) = \int_0^t \theta(s)\,\mathrm{d}s,$$
$$\widehat{\rho}(t) = \int_0^t \rho(s)\,\mathrm{d}s,\ \rho_2'(t) = (t^2\rho(t))',\ \varphi_2'(t) = (t^2\varphi(t))',\ f_2'(t) = (t^2 f(t))',\ \text{and}$$
$$\lambda(t) = -\rho_2'(t) + 2\alpha\rho_1(t) + \alpha(1-\alpha)\widehat{\rho}(t).$$

Lemma 4.3 (Key inequality)**.** *If θ satisfies (4.13), then for time $t \in (0, T]$, it holds that*

$$\|\theta(t)\|^2 + t^\alpha \|\theta(t)\|_h^2 \lesssim t^{\alpha-4} \int_0^t \Big((\mathcal{I}^{1-\alpha}\widehat{\theta}, \widehat{\theta}) + (\mathcal{I}^{1-\alpha}\theta_1, \theta_1) + (\mathcal{I}^{1-\alpha}\lambda, \lambda) \Big)\mathrm{d}s$$
$$+ h^4 t^{\alpha-4} \bigg(\|\varphi_2(t)\|^2 + \int_0^t (\|\varphi_2(s)\|^2 + \|\varphi_2'(s)\|^2)\,\mathrm{d}s + \|f_2(t)\|^2 + \int_0^t (\|f_2(s)\|^2 + \|f_2'(s)\|^2)\,\mathrm{d}s \bigg).$$

Remark 4.4. Note the dependency of the variables $\widehat{\theta}, \widehat{\theta}_1, \lambda$, etc. on the time variable is suppressed in the above and in the sequel for notational ease when there is no chance of confusion.

Proof. To prove the assertion, we first show that $\theta_2 \in W^{1,1}(0, T; V_h)$ and $\varphi_2 \in W^{1,1}(0, T; L^2(\Omega))$. For nonsmooth initial data, from (2.12), it follows that

$$\|u'(t)\|_V = \|u'(t)\|_{D(A^{1/2})} \lesssim t^{-(1+\alpha/2)}\|u_0\|.$$

From which, we deduce that

$$\|\mathcal{R}_h u'(t)\|_h \lesssim t^{-(1+\alpha/2)}\|u_0\|,$$

and that $t \mapsto t^2 \mathcal{R}_h u(t)$ belongs to $W^{1,1}(0,T;V_h)$. We already know that u_h belongs to $W^{1,1}(0,T;V_h)$. Thus θ_2 also belongs to $W^{1,1}(0,T;V_h)$. With (2.14), we have

$$\|\varphi(t)\| \lesssim t^{-\alpha}\|u_0\|.$$

Thus φ_2 belongs to $W^{1,1}(0,T;L^2(\Omega))$. A similar argument holds for smooth initial data.

Multiply both sides of (4.13) by t^2, apply (4.4) (twice) to rewrite both $t^2\mathcal{I}^{1-\alpha}\theta'$ and $t^2\mathcal{I}^{1-\alpha}\rho'$, utilize $(\rho(0)+\theta(0),\chi_h)=0$ from the second identity in (3.3), and the definition of λ to obtain

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\theta_2',\chi_h)+a_h(\theta_2,\chi_h) = 2\alpha(\mathcal{I}^{1-\alpha}\theta_1,\chi_h)+\alpha(1-\alpha)(\mathcal{I}^{1-\alpha}\widehat{\theta},\chi_h)+(\mathcal{I}^{1-\alpha}\lambda,\chi_h)\\ -(\varphi_2,(Q-I)\chi_h)+(f_2,(Q-I)\chi_h) \text{ for all } \chi_h\in V_h.\end{aligned}$$

Substitute $\chi_h=\theta_2'(t)$ in the last displayed equation, integrate over $(0,t)$, and apply (4.5) for the first three terms on the right-hand side with the choice of ϑ as $\frac{1}{2(2\alpha+\alpha(1-\alpha)+1)}$ to obtain

$$\begin{aligned}\int_0^t(\mathcal{I}^{1-\alpha}\theta_2',\theta_2')\,\mathrm{d}s+\int_0^t a_h(\theta_2,\theta_2')\,\mathrm{d}s \le \frac{1}{2}\int_0^t(\mathcal{I}^{1-\alpha}\theta_2',\theta_2')\,\mathrm{d}s + C\int_0^t\Big((\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\\ +(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)+(\mathcal{I}^{1-\alpha}\lambda,\lambda)\Big)\,\mathrm{d}s-\int_0^t(\varphi_2,(Q-I)\theta_2')\,\mathrm{d}s+\int_0^t(f_2,(Q-I)\theta_2')\,\mathrm{d}s.\end{aligned}$$

The constant C in the above inequality depends on α. The symmetry of $a_h(\cdot,\cdot)$ from **(H)** shows $2a_h(\theta_2(t),\theta_2'(t))=\frac{\mathrm{d}}{\mathrm{d}t}a_h(\theta_2(t),\theta_2(t))$. This with (3.2), (4.7), and $\theta_2(0)=0$ on the left-hand side of the above inequality establishes

$$\begin{aligned}t^{-\alpha}\|\theta_2(t)\|^2+\beta_1\|\theta_2(t)\|_h^2 \lesssim \int_0^t\Big((\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})+(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)+(\mathcal{I}^{1-\alpha}\lambda,\lambda)\Big)\,\mathrm{d}s\\ +\left|\int_0^t(\varphi_2,(Q-I)\theta_2')\,\mathrm{d}s\right|+\left|\int_0^t(f_2,(Q-I)\theta_2')\,\mathrm{d}s\right|. \qquad (4.14)\end{aligned}$$

Now, the task is to bound the last two terms on the right-hand side of (4.14). Since θ_2 and φ_2 belong to $W^{1,1}(0,T;L^2(\Omega))$, with an integration by parts, we have

$$T_1 := \int_0^t(\varphi_2(s),(Q-I)\theta_2'(s))\,\mathrm{d}s = (\varphi_2(t),(Q-I)\theta_2(t))-\int_0^t(\varphi_2'(s),(Q-I)\theta_2(s))\,\mathrm{d}s.$$

The Hölder inequality and Lemma 3.5 show

$$\begin{aligned}|T_1| &\le C_1h^2\Big(\|\varphi_2(t)\|\|\theta_2(t)\|_h+\int_0^t\|\varphi_2'(s)\|\|\theta_2(s)\|_h\,\mathrm{d}s\Big)\\ &\le \frac{\beta_1}{4}\|\theta_2(t)\|_h^2+C_1^2\beta_1^{-1}h^4\|\varphi_2(t)\|^2+\frac{1}{2}\int_0^t\|\theta_2(s)\|_h^2\,\mathrm{d}s+\frac{C_1^2h^4}{2}\int_0^t\|\varphi_2'(s)\|^2\,\mathrm{d}s,\end{aligned}$$

with an application of Young's inequality in the last step above. A similar approach is applied to bound the term $T_2 := \int_0^t(f_2,(Q-I)\theta_2')\,\mathrm{d}s$ and leads to

$$\begin{aligned}|T_2| &\le C_1h^2\Big(\|f_2(t)\|\|\theta_2(t)\|_h+\int_0^t\|f_2'(s)\|\|\theta_2(s)\|_h\,\mathrm{d}s\Big)\\ &\le \frac{\beta_1}{4}\|\theta_2(t)\|_h^2+C_1^2\beta_1^{-1}h^4\|f_2(t)\|^2+\frac{1}{2}\int_0^t\|\theta_2(s)\|_h^2\,\mathrm{d}s+\frac{C_1^2h^4}{2}\int_0^t\|f_2'(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Substitute these bounds of $|T_1|$ and $|T_2|$ in (4.14) to obtain

$$t^{-\alpha}\|\theta_2(t)\|^2+\beta_1\|\theta_2(t)\|_h^2 \lesssim \int_0^t \Big((\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})+(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)+(\mathcal{I}^{1-\alpha}\lambda,\lambda)\Big)\,\mathrm{d}s \\ + h^4\Big(\|\varphi_2(t)\|^2+\int_0^t\|\varphi_2'(s)\|^2\,\mathrm{d}s+\|f_2(t)\|^2+\int_0^t\|f_2'(s)\|^2\,\mathrm{d}s\Big)+\int_0^t\|\theta_2(s)\|_h^2\,\mathrm{d}s.$$

Apply Gronwall's lemma in Lemma 2.2 with

$$\phi(t)=t^{-\alpha}\|\theta_2(t)\|^2+\beta_1\|\theta_2(t)\|_h^2,\ \chi(t)=C,\ \text{and}$$
$$\psi(t)=C\int_0^t\big((\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})+(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)+(\mathcal{I}^{1-\alpha}\lambda,\lambda)\big)\,\mathrm{d}s \\ +Ch^4\Big(\|\varphi_2(t)\|^2+\int_0^t\|\varphi_2'(s)\|^2\,\mathrm{d}s+\|f_2(t)\|^2+\int_0^t\|f_2'(s)\|^2\,\mathrm{d}s\Big),$$

utilize (4.6), $\int_0^t\int_0^s g(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\lesssim\int_0^t g(s)\,\mathrm{d}s$ for $\int_0^t g(s)\,\mathrm{d}s>0$ $(g(t)\geq 0)$ and $t\in(0,T]$ for the terms corresponding to a double integral in time, and recall $\theta_2(t)=t^2\theta(t)$ to conclude the proof. □

Lemmas 4.5–4.8 bound each term on the right-hand side of the estimate in Lemma 4.3.

The notations $\widehat{v}(t)=\int_0^t v(s)\,\mathrm{d}s$ for the choices $v=\theta,\rho,\varphi,f$ and $\widehat{\widehat{v}}(t)=\int_0^t\int_0^s v(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$ for the choices $v=\theta,\varphi,f$ are used in the next lemma.

Lemma 4.5 (Estimate for $\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s$)**.** *For $t\in(0,T]$ and θ satisfying (4.13), the bounds stated below hold.*

$$\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s+\|\widehat{\widehat{\theta}}(t)\|_h^2\lesssim t^{4-\alpha}\Big(h^{2\gamma_0}t^{-\alpha}+h^2t^{-(\alpha+1)/2}\Big)^2\|u_0\|^2,$$

if $u_0\in L^2(\Omega)$ and $f=0$, and

$$\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s+\|\widehat{\widehat{\theta}}(t)\|_h^2\lesssim t^{4-\alpha}\Big(h^{2\gamma_0}\Lambda_0(\epsilon,t)+h^2t^{\alpha/2-2}\Lambda_1(\epsilon,t)\Big)^2,$$

for all $\epsilon\in(0,1)$, if $u_0\in D(A)$ and f is such that $\Lambda_0(\epsilon,t)$ and $\Lambda_1(\epsilon,t)$, defined in (4.10) and (4.12), are finite.

Proof. The proof is split into four steps. The first step derives an intermediate bound, and *Steps 2–4* estimate the terms derived in *Step 1*.

Step 1 (An intermediate bound). The definition of $\mathcal{I}^\beta(\cdot)$ in (1.3) and (4.1) establish

$$\mathcal{I}^{2-\alpha}v'(t)=\mathcal{I}^{1-\alpha}\mathcal{I}(v'(t))=\mathcal{I}^{1-\alpha}(v(t)-v_0)=\mathcal{I}^{1-\alpha}v(t)-\kappa_{2-\alpha}(t)v(0).$$

Integrate (4.13) over $(0,t)$, utilize the above identity twice, and $(\rho(0)+\theta(0),\chi_h)=0$ for all $\chi_h\in V_h$ from the second identity in (3.3) to obtain

$$(\mathcal{I}^{1-\alpha}\theta,\chi_h)+a_h(\widehat{\theta},\chi_h)=-(\mathcal{I}^{1-\alpha}\rho,\chi_h)-(\widehat{\varphi},(Q-I)\chi_h)+(\widehat{f},(Q-I)\chi_h)\quad\text{for all }\chi_h\in V_h.$$

Integrate the above identity in time from 0 to t and choose $\chi_h=\widehat{\theta}(t)$ to establish

$$(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})+a_h(\widehat{\widehat{\theta}},\widehat{\theta})=-(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\theta})-(\widehat{\widehat{\varphi}},(Q-I)\widehat{\theta})+(\widehat{\widehat{f}},(Q-I)\widehat{\theta}).$$

Integrate the above identity once again in time from 0 to t, apply (4.5) with $\vartheta=1/2$ to $\int_0^t(\mathcal{I}^{1-\alpha}(-\widehat{\rho}),\widehat{\theta})\,\mathrm{d}s$, utilize $2a_h(\widehat{\widehat{\theta}},\widehat{\theta})=\frac{\mathrm{d}}{\mathrm{d}t}a_h(\widehat{\widehat{\theta}},\widehat{\widehat{\theta}})$, and (3.2) to obtain

$$\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s+\beta_1\|\widehat{\widehat{\theta}}\|_h^2\lesssim\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\,\mathrm{d}s+\Big|\int_0^t(\widehat{\widehat{\varphi}},(Q-I)\widehat{\theta})\,\mathrm{d}s\Big|+\Big|\int_0^t(\widehat{\widehat{f}},(Q-I)\widehat{\theta})\,\mathrm{d}s\Big|.$$

The bounds for the terms $\left|\int_0^t(\widehat{\widehat{\varphi}},(Q-I)\widehat{\theta})\,\mathrm{d}s\right|$ and $\left|\int_0^t(\widehat{\widehat{f}},(Q-I)\widehat{\theta})\,\mathrm{d}s\right|$ can be established in an analogous way following the steps of the derivation of the bounds for $|T_1|$ and $|T_2|$ in Lemma 4.3. In this case, we obtain

$$\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s+\|\widehat{\widehat{\theta}}(t)\|_h^2\lesssim\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\,\mathrm{d}s+h^4\Bigg(\|\widehat{\widehat{\varphi}}(t)\|^2+\int_0^t\|\widehat{\varphi}(s)\|^2\,\mathrm{d}s$$
$$+\|\widehat{\widehat{f}}(t)\|^2+\int_0^t\|\widehat{f}(s)\|^2\,\mathrm{d}s\Bigg)+\int_0^t\|\widehat{\widehat{\theta}}(s)\|_h^2\,\mathrm{d}s. \tag{4.15}$$

Apply Gronwall's lemma in (4.15) to obtain

$$\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s+\|\widehat{\widehat{\theta}}(t)\|_h^2\lesssim\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\,\mathrm{d}s+h^4\Bigg(\|\widehat{\widehat{\varphi}}(t)\|^2+\int_0^t(\|\widehat{\varphi}(s)\|^2+\|\widehat{\widehat{\varphi}}(s)\|^2)\,\mathrm{d}s$$
$$+\|\widehat{\widehat{f}}(t)\|^2+\int_0^t(\|\widehat{f}(s)\|^2+\|\widehat{\widehat{f}}(s)\|^2)\,\mathrm{d}s\Bigg). \tag{4.16}$$

Step 2 (Estimate for the term $\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\,\mathrm{d}s$ on the right-hand side of (4.16)). An application of (4.11) results in

$$\|\widehat{\rho}(s)\|\le\int_0^s\|\rho(\tau)\|\,\mathrm{d}\tau\lesssim\begin{cases}h^{2\gamma_0}\|u_0\|\int_0^s\tau^{-\alpha}\,\mathrm{d}\tau\lesssim h^{2\gamma_0}s^{1-\alpha}\|u_0\| \text{ for } f=0 \text{ and } u_0\in L^2(\Omega),\\ h^{2\gamma_0}s\,\Lambda_0(\epsilon,s) \text{ for } f\neq 0 \text{ and } u_0\in D(A).\end{cases}$$

This bound, the definition of $\mathcal{I}^{1-\alpha}$, and $\tau\le s$ in the computation of the integral in the second step below lead to

$$\|\mathcal{I}^{1-\alpha}\widehat{\rho}(s)\|\lesssim\int_0^s(s-\tau)^{-\alpha}\|\widehat{\rho}(\tau)\|\,\mathrm{d}\tau\lesssim\begin{cases}h^{2\gamma_0}s^{2-2\alpha}\|u_0\| \text{ for } f=0 \text{ and } u_0\in L^2(\Omega),\\ h^{2\gamma_0}s^{2-\alpha}\Lambda_0(\epsilon,s) \text{ for } f\neq 0 \text{ and } u_0\in D(A).\end{cases}$$

The Hölder inequality and a combination of the last two displayed bounds show

$$\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\,\mathrm{d}s\le\int_0^t\|\mathcal{I}^{1-\alpha}\widehat{\rho}\|\|\widehat{\rho}\|\,\mathrm{d}s\lesssim\begin{cases}h^{4\gamma_0}t^{4-3\alpha}\|u_0\|^2 \text{ for } f=0 \text{ and } u_0\in L^2(\Omega),\\ h^{4\gamma_0}t^{4-\alpha}\Lambda_0(\epsilon,t)^2 \text{ for } f\neq 0 \text{ and } u_0\in D(A).\end{cases}$$

Step 3 (Estimate for the term $(\|\widehat{\widehat{\varphi}}(t)\|^2+\int_0^t(\|\widehat{\varphi}(s)\|^2+\|\widehat{\widehat{\varphi}}(s)\|^2)\,\mathrm{d}s)$ on the right-hand side of (4.16)). The approach for the estimates for the nonsmooth and smooth initial data differs here as direct bounds for the required estimates offer challenges due to the singularity factor t^{-1} (see (2.12) with $i=1,p=0$) when $u_0\in L^2(\Omega)$. However, this issue is resolved below with the help of (4.1).

Nonsmooth initial data ($f=0$ and $u_0\in L^2(\Omega)$). Recall $\varphi=\mathcal{I}^{1-\alpha}u'$ and apply (4.1) to observe

$$\widehat{\varphi}=\mathcal{I}(\mathcal{I}^{1-\alpha}u'(t))=\mathcal{I}^{1-\alpha}\mathcal{I}(u'(t))=\mathcal{I}^{1-\alpha}(u(t)-u_0),\ \ \widehat{\widehat{\varphi}}(t)=\mathcal{I}^{2-\alpha}(u(t)-u_0).$$

The definition of $\mathcal{I}^{2-\alpha}$ (*cf.* (1.3)), $(t-s)\le t$, and (2.12) with $i=0,p=0$ lead to

$$\|\widehat{\widehat{\varphi}}(t)\|^2=\left\|\int_0^t\frac{(t-s)^{1-\alpha}}{\Gamma(2-\alpha)}(u(s)-u_0)\,\mathrm{d}s\right\|^2\lesssim\left(t^{1-\alpha}\int_0^t(\|u(s)\|+\|u_0\|)\,\mathrm{d}s\right)^2\lesssim t^{4-2\alpha}\|u_0\|^2.$$

This yields

$$\int_0^t \|\widehat{\widehat{\varphi}}(s)\|^2\,\mathrm{d}s \lesssim \int_0^t s^{4-2\alpha}\|u_0\|^2\,\mathrm{d}s = \frac{t^{5-2\alpha}}{5-2\alpha}\|u_0\|^2 \lesssim t^{4-2\alpha}\|u_0\|^2 \tag{4.17}$$

with $t \le T$ (used once) in the last step. The definition of $\mathcal{I}^{1-\alpha}$ (*cf.* (1.3)) and (2.12) with $i=0, p=0$ establish

$$\|\widehat{\varphi}(s)\|^2 = \left\|\int_0^s \frac{(s-\tau)^{-\alpha}}{\Gamma(1-\alpha)}(u(\tau)-u_0)\,\mathrm{d}\tau\right\|^2 \lesssim \left(\int_0^s (s-\tau)^{-\alpha}(\|u(\tau)\|+\|u_0\|)\,\mathrm{d}\tau\right)^2 \lesssim s^{2-2\alpha}\|u_0\|^2,$$

and this shows $\int_0^t \|\widehat{\varphi}(s)\|^2\,\mathrm{d}s \lesssim t^{3-2\alpha}\|u_0\|^2$. Altogether, we obtain

$$\|\widehat{\widehat{\varphi}}(t)\|^2 + \int_0^t (\|\widehat{\varphi}(s)\|^2 + \|\widehat{\widehat{\varphi}}(s)\|^2)\,\mathrm{d}s \lesssim t^{3-2\alpha}(t+1)\|u_0\|^2.$$

Smooth initial data ($f \neq 0$ and $u_0 \in D(A)$). Using $\widehat{\widehat{\varphi}}(t) = \mathcal{I}^{3-\alpha}u'(t)$, $(t-s) \le t$, and (2.16) with $p=0$ and (2.20) with $q=\epsilon$, we have

$$\begin{aligned}\|\widehat{\widehat{\varphi}}(t)\|^2 &\lesssim \left(\int_0^t (t-s)^{2-\alpha}\|u'(s)\|\,\mathrm{d}s\right)^2 \lesssim \Big(t^{2-\alpha}\int_0^t \big(s^{\alpha-1}\|u_0\|_{D(A)} + s^{-1+\frac{\alpha\epsilon}{2}}\|f(0)\|_{D(A^{\epsilon/2})} \\ &\qquad + \epsilon^{-1}s^{\frac{\alpha\epsilon}{2}}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\big)\,\mathrm{d}s\Big)^2 \\ &\lesssim \left(t^2\|u_0\|_{D(A)} + t^{2-\alpha(1-\epsilon/2)}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}t^{3-\alpha(1-\epsilon/2)}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\right)^2.\end{aligned}$$

An integration from 0 to t, (4.8), elementary algebraic manipulations, and $t \le T$ (used once) as in (4.17), lead to a similar bound for $\int_0^t \|\widehat{\widehat{\varphi}}(s)\|^2\,\mathrm{d}s$. Analogous arguments with elementary manipulations lead to

$$\int_0^t \|\widehat{\varphi}(s)\|^2\,\mathrm{d}s \lesssim \left(t^{\frac{3}{2}}\|u_0\|_{D(A)} + t^{\frac{3}{2}-\alpha\left(1-\frac{\epsilon}{2}\right)}\|f(0)\|_{D(A^{\frac{\epsilon}{2}})} + \epsilon^{-1}t^{\frac{5}{2}-\alpha\left(1-\frac{\epsilon}{2}\right)}\|f'\|_{L^\infty(0,T;D(A^{\frac{\epsilon}{2}}))}\right)^2.$$

A combination of the last two displayed inequalities establishes the required estimate in this step.

Step 4 (Estimate for the term $\left(\|\widehat{\widehat{f}}(t)\|^2 + \int_0^t(\|\widehat{f}(s)\|^2 + \|\widehat{\widehat{f}}(s)\|^2)\,\mathrm{d}s\right)$ on the right-hand side of (4.16)). For *nonsmooth* initial data, since f is chosen as zero, the term is zero. For *smooth* initial data, the definitions of $\widehat{f}$ and $\widehat{\widehat{f}}$ show

$$\|\widehat{\widehat{f}}(t)\|^2 + \int_0^t (\|\widehat{f}(s)\|^2 + \|\widehat{\widehat{f}}(s)\|^2)\,\mathrm{d}s \lesssim \left(t^{3/2}\|f\|_{L^2(0,t;L^2(\Omega))} + t\|f\|_{L^2(0,t;L^2(\Omega))}\right)^2.$$

A combination of *Steps 2–4* in (4.16) and algebraic manipulations conclude the proof. ☐

The notations $\upsilon_1(t) = t\upsilon(t)$ for $\upsilon = \theta, \rho, \varphi, f$ and $\widehat{\upsilon}(t) = \int_0^t \upsilon(s)\,\mathrm{d}s$ for $\upsilon = \theta, \theta_1, \rho, \varphi_1, f_1$ are used in the next lemma.

Lemma 4.6 (Estimate for $\int_0^t (\mathcal{I}^{1-\alpha}\theta_1, \theta_1)\,\mathrm{d}s$)**.** *For θ that satisfies (4.13) and $t \in (0,T]$, the bounds stated below hold.*

$$\int_0^t (\mathcal{I}^{1-\alpha}\theta_1, \theta_1)\,\mathrm{d}s + \|\widehat{\theta_1}(t)\|_h^2 \lesssim t^{4-\alpha}\left(h^{2\gamma_0}t^{-\alpha} + h^2 t^{-(\alpha+1)/2}\right)^2\|u_0\|^2,$$

if $u_0 \in L^2(\Omega)$ and $f = 0$, and

$$\int_0^t (\mathcal{I}^{1-\alpha}\theta_1, \theta_1)\,\mathrm{d}s + \|\widehat{\theta_1}(t)\|_h^2 \lesssim t^{4-\alpha}\left(h^{2\gamma_0}\Lambda_0(\epsilon,t) + h^2 t^{\alpha/2-2}\Lambda_2(\epsilon,t)\right)^2,$$

for all $\epsilon \in (0,1)$, if $u_0 \in D(A)$ and f is such that $\Lambda_0(\epsilon,t)$ and $\Lambda_2(\epsilon,t)$, defined in (4.10) and (4.12), are finite.

Proof. The structure of the proof is similar to that of Lemma 4.5. However, the majorizations are different and for the readability of the paper, we give the necessary details. The proof is presented in four steps. To prove the desired bounds, we proceed following Lemma 4.3 as follows.

Step 1 (An intermediate bound). Multiply (4.13) by t, use (4.3), and $(\rho(0)+\theta(0),\chi_h)=0$ for all $\chi_h\in V_h$ to arrive at

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\theta_1',\chi_h)+a_h(\theta_1,\chi_h)&=\alpha(\mathcal{I}^{1-\alpha}\theta,\chi_h)-(\mathcal{I}^{1-\alpha}\rho_1',\chi_h)+\alpha(\mathcal{I}^{1-\alpha}\rho,\chi_h)\\&\quad-(\varphi_1(t),(Q-I)\chi_h)+(f_1(t),(Q-I)\chi_h)\quad\text{ for all } \chi_h\in V_h.\end{aligned}$$

Integrate the above equality over $(0,t)$, and next choose $\chi_h=\theta_1(t)$ to obtain

$$\begin{aligned}(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)+a_h(\widehat{\theta_1},\theta_1)&=\alpha(\mathcal{I}^{1-\alpha}\widehat{\theta},\theta_1)-(\mathcal{I}^{1-\alpha}\rho_1,\theta_1)+\alpha(\mathcal{I}^{1-\alpha}\widehat{\rho},\theta_1)\\&\quad-(\widehat{\varphi_1}(t),(Q-I)\theta_1)+(\widehat{f_1}(t),(Q-I)\theta_1).\end{aligned}$$

An integration over $(0,t)$ once again, an application of (4.5) to the first three terms on the right-hand side, and the choice $\vartheta=\frac{1}{2(2\alpha+1)}$ reveal

$$\begin{aligned}\int_0^t(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)\,\mathrm{d}s+\int_0^t a_h(\widehat{\theta_1},\theta_1)\,\mathrm{d}s&\le\frac12\int_0^t(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)\,\mathrm{d}s+C\int_0^t\Big[(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})+(\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\\&\quad+(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\Big]\,\mathrm{d}s+\Big|\int_0^t(\widehat{\varphi_1}(s),(Q-I)\theta_1)\,\mathrm{d}s\Big|+\Big|\int_0^t(\widehat{f_1}(s),(Q-I)\theta_1)\,\mathrm{d}s\Big|.\end{aligned}$$

Apply $2a_h(\widehat{\theta_1}(t),\theta_1(t))=\frac{\mathrm{d}}{\mathrm{d}t}a_h(\widehat{\theta_1}(t),\widehat{\theta_1}(t))$, (3.2), approach of *Step 1* of Lemma 4.5 to bound the last two terms of the above displayed estimate, and Gronwall's lemma to obtain

$$\begin{aligned}\int_0^t(\mathcal{I}^{1-\alpha}\theta_1,\theta_1)\,\mathrm{d}s+\|\widehat{\theta_1}(t)\|_h^2&\lesssim\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\theta},\widehat{\theta})\,\mathrm{d}s+\int_0^t(\mathcal{I}^{1-\alpha}\widehat{\rho},\widehat{\rho})\,\mathrm{d}s+\int_0^t(\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,\mathrm{d}s\\&\quad+h^4\Big(\|\widehat{\varphi_1}(t)\|^2+\int_0^t(\|\varphi_1(s)\|^2+\|\widehat{\varphi_1}(s)\|^2)\,\mathrm{d}s+\|\widehat{f_1}(t)\|^2+\int_0^t(\|f_1(s)\|^2+\|\widehat{f_1}(s)\|^2)\,\mathrm{d}s\Big).\end{aligned}\tag{4.18}$$

Observe that the bounds of the first two terms on the right-hand side of (4.18) are available from the statement and *Step 2* of Lemma 4.5. Hence in the steps below, we bound the remaining terms.

Step 2 (Estimate for the term $\int_0^t(\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,\mathrm{d}s$ on the right-hand side of (4.18)). In view of (4.11), we obtain

$$\|\rho_1(s)\|=s\|\rho(s)\|\lesssim\begin{cases}h^{2\gamma_0}s^{1-\alpha}\|u_0\| \text{ for } f=0 \text{ and } u_0\in L^2(\Omega),\\ h^{2\gamma_0}s\,\Lambda_0(\epsilon,s) \text{ for } f\neq 0 \text{ and } u_0\in D(A).\end{cases}$$

Employ this to establish

$$\|\mathcal{I}^{1-\alpha}\rho_1(s)\|\lesssim\int_0^s(s-\tau)^{-\alpha}\|\rho_1(\tau)\|\,\mathrm{d}\tau\lesssim\begin{cases}h^{2\gamma_0}s^{2-2\alpha}\|u_0\| \text{ for } f=0 \text{ and } u_0\in L^2(\Omega),\\ h^{2\gamma_0}s^{2-\alpha}\Lambda_0(\epsilon,s) \text{ for } f\neq 0 \text{ and } u_0\in D(A).\end{cases}$$

The Hölder inequality plus the last two displayed bounds reveal

$$\int_0^t(\mathcal{I}^{1-\alpha}\rho_1,\rho_1)\,\mathrm{d}s\le\int_0^t\|\mathcal{I}^{1-\alpha}\rho_1\|\|\rho_1\|\,\mathrm{d}s\lesssim\begin{cases}h^{4\gamma_0}t^{4-3\alpha}\|u_0\|^2 \text{ for } f=0 \text{ and } u_0\in L^2(\Omega),\\ h^{4\gamma_0}t^{4-\alpha}\Lambda_0(\epsilon,t)^2 \text{ for } f\neq 0 \text{ and } u_0\in D(A).\end{cases}$$

Step 3 (Estimate for the term $\big(\|\widehat{\varphi_1}(t)\|^2+\int_0^t(\|\varphi_1(s)\|^2+\|\widehat{\varphi_1}(s)\|^2)\,\mathrm{d}s\big)$ on the right-hand side of (4.18)). Since the nonsmooth data stability estimate reflects a singularity factor t^{-1} (*cf.* (2.12) with $i=1,p=0$), we proceed as follows for this case.

Nonsmooth initial data ($f = 0$ and $u_0 \in L^2(\Omega)$). Recall that $\varphi = \mathcal{I}^{1-\alpha} u'$ and apply (4.3) to observe

$$\begin{aligned}\varphi_1(s) = s\varphi(s) = s\mathcal{I}^{1-\alpha}u'(s) &= \mathcal{I}^{1-\alpha}(su(s))' - \alpha\mathcal{I}^{1-\alpha}u(s) - s\kappa_{1-\alpha}(s)u_0 \\ &= \mathcal{I}^{1-\alpha}(u(s) + su'(s)) - \alpha\mathcal{I}^{1-\alpha}u(s) - s\kappa_{1-\alpha}(s)u_0.\end{aligned}$$

The definition of $\mathcal{I}^{1-\alpha}$ in (1.3) and repeated applications of (2.12) with $i = 0, p = 0$ and $i = 1, p = 0$ yield $\|\varphi_1(s)\| \lesssim s^{1-\alpha}\|u_0\|$ and hence $\|\widehat{\varphi_1}(t)\|^2 \lesssim t^{4-2\alpha}\|u_0\|^2$. Utilize this bound to arrive at

$$\int_0^t \|\widehat{\varphi_1}(s)\|^2 \,\mathrm{d}s \lesssim \int_0^t s^{4-2\alpha}\|u_0\|^2 \,\mathrm{d}s \lesssim t^{4-2\alpha}\|u_0\|^2,$$

with $t \le T$ (used once) in the last step. Analogous arguments show $\int_0^t \|\varphi_1(s)\|^2 \,\mathrm{d}s \lesssim t^{3-2\alpha}\|u_0\|^2$. A combination of these bounds shows

$$\|\widehat{\varphi_1}(t)\|^2 + \int_0^t (\|\varphi_1(s)\|^2 + \|\widehat{\varphi_1}(s)\|^2) \,\mathrm{d}s \lesssim t^{3-2\alpha}(t+1)\|u_0\|^2.$$

Smooth initial data ($f \neq 0$ and $u_0 \in D(A)$). Apply the definition of $\mathcal{I}^{1-\alpha}$ in (1.3), $s \le t$, and (2.16) with $p = 0$ and (2.20) with $q = \epsilon$ to obtain

$$\begin{aligned}\|\widehat{\varphi_1}(t)\|^2 &= \left\|\int_0^t s\mathcal{I}^{1-\alpha}u'(s)\,\mathrm{d}s\right\|^2 \lesssim \left(t\int_0^t\int_0^s (s-\tau)^{-\alpha}\|u'(\tau)\|\,\mathrm{d}\tau\,\mathrm{d}s\right)^2 \\ &\lesssim \left(t\int_0^t\int_0^s (s-\tau)^{-\alpha}\big(\tau^{\alpha-1}\|u_0\|_{D(A)} + \tau^{-1+\frac{\alpha\epsilon}{2}}\|f(0)\|_{D(A^{\epsilon/2})}\right. \\ &\qquad \left. + \epsilon^{-1}\tau^{\frac{\alpha\epsilon}{2}}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\big)\,\mathrm{d}\tau\,\mathrm{d}s\right)^2.\end{aligned} \tag{4.19}$$

For $\alpha \in (0,1)$, recall the following identities involving the beta function $B(\cdot,\cdot)$:

$$\begin{aligned}&\int_0^t (t-s)^{-\alpha}s^{\alpha-1}\,\mathrm{d}s = \int_0^1 (1-s)^{\alpha-1}s^{-\alpha}\,\mathrm{d}s = B(\alpha, 1-\alpha), \\ &\int_0^t (t-s)^{-\alpha}s^{\frac{\alpha\epsilon}{2}-1}\,\mathrm{d}s = t^{-\alpha(1-\epsilon/2)}\int_0^1 (1-s)^{\frac{\alpha\epsilon}{2}-1}s^{-\alpha}\,\mathrm{d}s = B(\alpha\epsilon/2, 1-\alpha)t^{-\alpha(1-\epsilon/2)}.\end{aligned} \tag{4.20}$$

Substitute (4.20) in (4.19) to obtain

$$\|\widehat{\varphi_1}(t)\|^2 \lesssim \left(t^2\|u_0\|_{D(A)} + B(\alpha\epsilon/2, 1-\alpha)t^{2-\alpha(1-\epsilon/2)}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}t^{3-\alpha(1-\epsilon/2)}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\right)^2.$$

Utilize $t \le T$ to conclude that an analogous bound holds for the term $\int_0^t \|\widehat{\varphi_1}(s)\|^2 \,\mathrm{d}s$. Further, similar calculations as for $\|\widehat{\varphi_1}(t)\|^2$ above and algebraic manipulations with (4.8) reveal

$$\begin{aligned}\int_0^t \|\varphi_1(s)\|^2\,\mathrm{d}s &= \int_0^t \|s\mathcal{I}^{1-\alpha}u'(s)\|^2\,\mathrm{d}s \lesssim t^2\int_0^t\left(\int_0^s (s-\tau)^{-\alpha}\|u'(\tau)\|\,\mathrm{d}\tau\right)^2\mathrm{d}s \lesssim \Big(t^{3/2}\|u_0\|_{D(A)} \\ &\quad + B(\alpha\epsilon/2, 1-\alpha)t^{3/2-\alpha(1-\epsilon/2)}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}t^{5/2-\alpha(1-\epsilon/2)}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\Big)^2.\end{aligned}$$

Step 4 (Estimate for the term $\big(\|\widehat{f}_1(t)\|^2 + \int_0^t (\|f_1(s)\|^2 + \|\widehat{f}_1(s)\|^2)\,\mathrm{d}s\big)$ on the right-hand side of (4.18)). Since $f = 0$ for *nonsmooth* initial data, the estimate is trivial. For the case of *smooth* initial data, the definitions of $f_1, \widehat{f}_1$ lead to

$$\|\widehat{f}_1(t)\|^2 + \int_0^t (\|f_1(s)\|^2 + \|\widehat{f}_1(s)\|^2)\,\mathrm{d}s \lesssim \left(t^{3/2}\|f\|_{L^2(0,t;L^2(\Omega))} + t\|f\|_{L^2(0,t;L^2(\Omega))}\right)^2.$$

A combination of the estimates from *Steps 2–4* with (4.18) concludes the proof. □

Lemma 4.7 (Estimate for $\int_0^t (\mathcal{I}^{1-\alpha}\lambda, \lambda)\,\mathrm{d}s$)**.** *For $\lambda = -\rho_2' + 2\alpha\rho_1 + \alpha(1-\alpha)\widehat{\rho}$ with $\rho_2'(t) = (t^2\rho(t))'$, $\rho_1(t) = t\rho(t)$, $\widehat{\rho}(t) = \int_0^t \rho(s)\,\mathrm{d}s$, and $t \in (0, T]$, the following bounds hold.*

$$\int_0^t (\mathcal{I}^{1-\alpha}\lambda, \lambda)\,\mathrm{d}s \lesssim t^{4-\alpha}\Big(h^{2\gamma_0} t^{-\alpha}\|u_0\|\Big)^2,$$

for $u_0 \in L^2(\Omega)$ and $f = 0$, and

$$\int_0^t (\mathcal{I}^{1-\alpha}\lambda, \lambda)\,\mathrm{d}s \lesssim t^{4-\alpha}\Big(h^{2\gamma_0}\mathcal{B}_0(\epsilon, t)\Big)^2,$$

for all $\epsilon \in (0,1)$, if $u_0 \in D(A)$ and f such that $\mathcal{B}_0(\epsilon, t)$ defined in (4.12) is finite.

Proof. Step 1 (Nonsmooth data). For $f = 0$ and $u_0 \in L^2(\Omega)$, Lemma 3.4 and (2.12) (with $i = 0, p = 1$ and $i = p = 1$) result in

$$\begin{aligned}\|\lambda(s)\| &\le \|\widehat{\rho}(s)\| + 3\|\rho_1(s)\| + s^2\|\rho'(s)\| \lesssim h^{2\gamma_0}s^{1-\alpha}\|u_0\| + h^{2\gamma_0}s^2\|u'(s)\|_{H^{2+\gamma_0}(\Omega)}\\ &\lesssim h^{2\gamma_0}s^{1-\alpha}\|u_0\|.\end{aligned}$$

This bound leads to

$$\|\mathcal{I}^{1-\alpha}\lambda(s)\| \lesssim \int_0^s (s-\tau)^{-\alpha}\|\lambda(\tau)\|\,\mathrm{d}\tau \lesssim h^{2\gamma_0}s^{2-2\alpha}\|u_0\|.$$

The Hölder inequality and the two bounds displayed above establish

$$\int_0^t (\mathcal{I}^{1-\alpha}\lambda, \lambda)\,\mathrm{d}s \le \int_0^t \|\mathcal{I}^{1-\alpha}\lambda\|\|\lambda\|\,\mathrm{d}s \lesssim h^{4\gamma_0}t^{4-3\alpha}\|u_0\|^2.$$

Step 2 (Smooth data). For $f \ne 0$ and $u_0 \in D(A)$, apply Lemma 3.4, (2.15) and (2.19) with $q = \epsilon$, (2.16) with $p = 1$ and (2.20) with $q = \epsilon$ to obtain

$$\begin{aligned}\|\lambda(s)\| &\le \|\widehat{\rho}(s)\| + 3\|\rho_1(s)\| + s^2\|\rho'(s)\| \lesssim h^{2\gamma_0}s\Lambda_0(\epsilon, s) + h^{2\gamma_0}s^2\|u'(s)\|_{H^{2+\gamma_0}(\Omega)}\\ &\lesssim h^{2\gamma_0}s\Lambda_0(\epsilon, s) + h^{2\gamma_0}\Big(s\|u_0\|_{D(A)} + s^{1+\frac{\alpha\epsilon}{2}}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}s^{2+\frac{\alpha\epsilon}{2}}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\Big).\end{aligned}$$

An application of this bound in the definition of $\mathcal{I}^{1-\alpha}$ shows

$$\begin{aligned}\|\mathcal{I}^{1-\alpha}\lambda(s)\| &\lesssim \int_0^s (s-\tau)^{-\alpha}\|\lambda(\tau)\|\,\mathrm{d}\tau \lesssim h^{2\gamma_0}s^{2-\alpha}\Lambda_0(\epsilon, s)\\ &\quad + h^{2\gamma_0}\Big(s^{2-\alpha}\|u_0\|_{D(A)} + s^{2-\alpha(1-\epsilon/2)}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}s^{3-\alpha(1-\epsilon/2)}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\Big).\end{aligned}$$

The Hölder inequality and the two bounds displayed above establish

$$\begin{aligned}\int_0^t (\mathcal{I}^{1-\alpha}\lambda, \lambda)\,\mathrm{d}s &\le \int_0^t \|\mathcal{I}^{1-\alpha}\lambda\|\|\lambda\|\,\mathrm{d}s \lesssim h^{4\gamma_0}t^{4-\alpha}\Big(\Lambda_0(\epsilon, t)\\ &\qquad + t^{\alpha\epsilon/2}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}t^{1+\alpha\epsilon/2}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\Big)^2.\end{aligned}$$

This concludes the proof. $\square$

Lemma 4.8 (Estimate for the last term in Lem. 4.3). *For $t \in (0,T]$ and for all $\epsilon \in (0,1)$, it holds that*

$$h^4\Big(\|\varphi_2(t)\|^2+\int_0^t(\|\varphi_2(s)\|^2+\|\varphi_2'(s)\|^2)\,\mathrm{d}s+\|f_2(t)\|^2+\int_0^t(\|f_2(s)\|^2+\|f_2'(s)\|^2)\,\mathrm{d}s\Big)$$
$$\lesssim t^{4-\alpha}\begin{cases}\Big(h^2t^{-(\alpha+1)/2}\|u_0\|\Big)^2, & \text{if } u_0\in L^2(\Omega), f=0,\\ \Big(h^2t^{\alpha/2-2}\mathcal{B}_1(\epsilon,t)\Big)^2, & \text{if } u_0\in D(A), f\neq 0,\end{cases}$$

where $\varphi_2(t)=t^2\varphi(t)$, $\varphi_2'(s)=(s^2\varphi(s))'$, $f_2(t)=t^2f(t)$, $f_2'(s)=(s^2f(s))'$, and f is such that $\mathcal{B}_1(\epsilon,t)$ defined in (4.12) is finite.

Proof. Step 1 (Nonsmooth data). For $f=0$ and $u_0\in L^2(\Omega)$, we recall (4.4) to arrive at

$$\begin{aligned}\varphi_2(t)&=t^2\mathcal{I}^{1-\alpha}u'(t)=\mathcal{I}^{1-\alpha}(u_2)'(t)-2\alpha\mathcal{I}^{1-\alpha}(u_1)(t)-\alpha(1-\alpha)\mathcal{I}^{2-\alpha}(u(t))-t^2\kappa_{1-\alpha}(t)u_0\\&=\mathcal{I}^{1-\alpha}(2tu(t)+t^2u'(t))-2\alpha\mathcal{I}^{1-\alpha}(tu(t))-\alpha(1-\alpha)\mathcal{I}^{2-\alpha}(u(t))-t^2\kappa_{1-\alpha}(t)u_0.\end{aligned}$$

Applications of (1.3) and (2.12) with $i=p=0$ and $i=1,p=0$ in the above equality yield $\|\varphi_2(t)\|\lesssim t^{2-\alpha}\|u_0\|$ and this leads to

$$\|\varphi_2(t)\|^2+\int_0^t\|\varphi_2(s)\|^2\,\mathrm{d}s\lesssim t^{4-2\alpha}\|u_0\|^2.$$

To bound the term $\varphi_2'(s)$, we see that

$$\varphi_2'(s)=2s\varphi(s)+s^2\varphi'(s)\quad\text{ with }\ \varphi(s)=\mathcal{I}^{1-\alpha}u'(s). \tag{4.21}$$

Differentiating $\varphi(s)$ with respect to s we get

$$\varphi'(s)=\frac{1}{\Gamma(1-\alpha)}\int_0^s\tau^{-\alpha}u''(s-\tau)\,\mathrm{d}s+\frac{s^{-\alpha}u'(0)}{\Gamma(1-\alpha)}=\mathcal{I}^{1-\alpha}u''(s)+\kappa_{1-\alpha}(s)u'(0).$$

Thus, in view of (4.4) we arrive at

$$\begin{aligned}s^2\varphi'(s)&=\mathcal{I}^{1-\alpha}(s^2u')'-2\alpha\mathcal{I}^{1-\alpha}(su')-\alpha(1-\alpha)\mathcal{I}^{2-\alpha}(u')-s^2\kappa_{1-\alpha}(s)u'(0)+s^2\kappa_{1-\alpha}(s)u'(0)\\&=\mathcal{I}^{1-\alpha}(s^2u')'-2\alpha\mathcal{I}^{1-\alpha}(su')-\alpha(1-\alpha)\mathcal{I}^{2-\alpha}(u').\end{aligned}$$

Hence $\|s^2\varphi'(s)\|\lesssim\|\mathcal{I}^{1-\alpha}(s^2u')'\|+\|\mathcal{I}^{1-\alpha}(su')\|+\|\mathcal{I}^{2-\alpha}(u')\|$. The definition of $\mathcal{I}^{1-\alpha}$ and (2.12) first with $i=1$, $p=0$, next with $i=0$, $p=0$, and (2.13) lead to

$$\|s^2\varphi'(s)\|\lesssim\int_0^s(s-\tau)^{-\alpha}\big(\|\tau u'(\tau)\|+\|\tau^2u''(\tau)\|+\|u(\tau)\|+\|u_0\|\big)\,\mathrm{d}\tau\lesssim s^{1-\alpha}\|u_0\|.$$

This and *Step 3* of Lemma 4.6 in (4.21) show

$$\|\varphi_2'(s)\|\lesssim\|s^2\varphi'(s)\|+\|s\varphi(s)\|\lesssim s^{1-\alpha}\|u_0\|.$$

This shows $\int_0^t \|\varphi_2'(s)\|^2\,\mathrm{d}s \lesssim t^{3-2\alpha}\|u_0\|^2$.
Step 2 (Smooth data). For $f \neq 0$ and $u_0 \in D(A)$, follow the steps to bound $\|\widehat{\varphi_1}(t)\|^2$ for smooth data in the *Step 3* in Lemma 4.6 and recall (4.20) to arrive at

$$\|\varphi_2(t)\|^2 \lesssim \Big(t^2\|u_0\|_{D(A)} + B(\alpha\epsilon/2, 1-\alpha)t^{2-\alpha(1-\epsilon/2)}\|f(0)\|_{D(A^{\epsilon/2})} + \epsilon^{-1}t^{3-\alpha(1-\epsilon/2)}\|f'\|_{L^\infty(0,T;D(A^{\epsilon/2}))}\Big)^2.$$

It is easy to establish a similar bound for $\int_0^t \|\varphi_2(s)\|^2\,\mathrm{d}s$. The approach in *Step 1* of this lemma and the stability properties in (2.16) with $p = 0$, (2.17), (2.20) with $q = \epsilon$, and (2.21) show

$$\int_0^t \|\varphi_2'(s)\|^2\,\mathrm{d}s \lesssim \Big(h^2 t^{\alpha/2-2} B_1(\epsilon, t)\Big)^2.$$

The definition of f_2 and (4.8) establish

$$\|f_2(t)\|^2 + \int_0^t (\|f_2(s)\|^2 + \|f_2'(s)\|^2)\,\mathrm{d}s \lesssim \Big(t^2\|f(t)\| + t\|f\|_{L^2(0,t;L^2(\Omega))} + t^2\|f'\|_{L^2(0,t;L^2(\Omega))}\Big)^2.$$

A combination of *Steps* 1 and 2 establishes the assertion. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. A substitution of the bounds from Lemmas 4.5–4.8 in the key inequality from Lemma 4.3 and algebraic manipulations establish

$$\|\theta(t)\| + t^{\alpha/2}\|\theta(t)\|_h \lesssim \begin{cases} h^{2\gamma_0}t^{-\alpha}\|u_0\| + h^2 t^{-(\alpha+1)/2}\|u_0\|, & \text{for } u_0 \in L^2(\Omega), f = 0, \\ h^{2\gamma_0}\mathcal{B}_0(\epsilon, t) + h^2 t^{\alpha/2-2}\mathcal{B}_1(\epsilon, t), & \text{for } u_0 \in D(A), f \neq 0, \end{cases}$$

and this concludes the proof. □

5. Numerical illustrations

The numerical experiments in this section validate the theoretical orders of convergences (OCs) for the semidiscrete solution established in Section 4. Let EOC denote the expected orders of convergence with respect to the space variable. The numerical experiments are performed using Freefem++ [17] with the following two sets of problem data on $\Omega = (0,1)^2$. To compute the discrete solution, we first triangulate $\overline{\Omega}$, and then consider a uniform partition of $[0, t]$ with grid points $t_n = nk$, $n = 0, 1 \ldots, N$, where $k = t/N$ is the time step size and t is the time of interest. The Caputo fractional derivative $\partial_t^\alpha u(t)$ at $t = t_n$ is approximated by the L1 scheme (*cf.* [26, 38]) as shown below:

$$\begin{aligned}
\partial_t^\alpha u(t_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - s)^{-\alpha} \frac{\partial u(s)}{\partial s}\,\mathrm{d}s \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{u(t_{j+1}) - u(t_j)}{k} \int_{t_j}^{t_{j+1}} (t_n - s)^{-\alpha}\,\mathrm{d}s = \sum_{j=0}^{n-1} l_j \frac{u(t_{n-j}) - u(t_{n-j-1})}{k^\alpha} \\
&= k^{-\alpha}\left(l_0 u(t_n) - l_{n-1}u(t_0) + \sum_{j=1}^{n-1}(l_j - l_{j-1})u(t_{n-j})\right),
\end{aligned}$$

where the weights l_j are given by $l_j = ((j+1)^{1-\alpha} - j^{1-\alpha})/\Gamma(2-\alpha)$, $\;j = 0, \ldots, N-1$.

TABLE 1. The L^2-OC and energy-OC for the Morley method in case (i) for $\alpha = 0.25, 0.50, 0.75$ at $t = 0.1$ with $k = 0.001$.

h	α	Errors in L^2-norm	L^2-OC	Errors in energy norm	Energy-OC
1/16	0.25	4.50057e−05	–	9.30188e−03	–
1/32		1.18977e−05	1.91942	4.69592e−03	0.98611
1/64		3.07517e−06	1.95195	2.34989e−03	0.99881
1/128		7.63686e−07	2.00961	1.17378e−03	1.00143
1/16	0.50	5.59721e−05	–	1.14995e−02	–
1/32		1.47909e−05	1.92000	5.80462e−03	0.98630
1/64		3.82260e−06	1.95208	2.90464e−03	0.99884
1/128		9.49291e−07	2.00963	1.45088e−03	1.00142
1/16	0.75	5.12578e−05	–	1.02222e−02	–
1/32		1.35183e−05	1.92286	5.15662e−03	0.98720
1/64		3.49206e−06	1.95276	2.58010e−03	0.99899
1/128		8.67171e−07	2.00969	1.28878e−03	1.00142
EOC			2.0		1.0

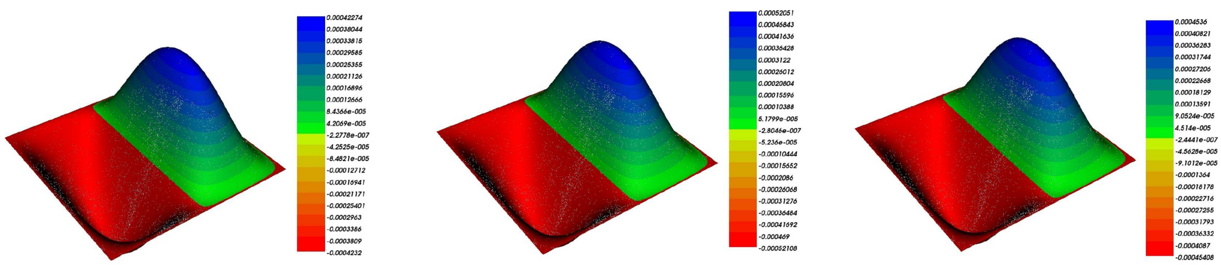

FIGURE 1. Numerical solutions of dG method on 256×256 mesh with $\alpha = 0.25, 0.5, 0.75$.

5.1. Examples

(i) Choose the nonsmooth initial data as

$$u_0(x,y) = \begin{cases} 1, & x \in (0,1/2],\ y \in (0,1), \\ -1, & x \in (1/2,1),\ y \in (0,1), \end{cases}$$

and $f(x,y,t) = 0$. In this case, the exact solution $u(x,y,t)$ is not known.

(ii) The manufactured exact solution u given by $u(x,y,t) = (t^{\alpha+1}+1)(x(1-x)y(1-y))^2$ leads to the smooth initial data $u_0(x,y) = (x(1-x)y(1-y))^2$ and source function $f(x,y,t) = \Gamma(\alpha+2)t(x(1-x)y(1-y))^2 + (t^{\alpha+1}+1)(24(x^2-2x^3+x^4)+24(y^2-2y^3+y^4)+2(2-12x+12x^2)(2-12y+12y^2))$. Note that the initial data is in $D(A)$ and the source function belongs to the appropriate function space mentioned in Theorem 4.2(ii) for any $\epsilon \in (0,1/4)$.

For the dG method, the numerical computations are performed with the choice of penalty parameters as $\sigma^1_{\rm dG} = \sigma^2_{\rm dG} = 2$, whereas for the C^0IP, we choose $\sigma_{\rm IP} = 8$.

TABLE 2. The L^2-OC and energy-OC for the C^0IP method in case (i) for $\alpha = 0.25, 0.50, 0.75$ at $t = 0.1$ with $k = 0.001$ and $\sigma_{\mathrm{IP}} = 8$.

h	α	Errors in L^2-norm	L^2-OC	Errors in energy-norm	Energy-OC
1/16	0.25	5.51766e−06	–	3.13001e−03	–
1/32		1.56160e−06	1.82103	1.59248e−03	0.97489
1/64		4.02967e−07	1.95429	7.94704e−04	1.00278
1/128		1.05072e−07	1.93928	4.02570e−04	0.98117
1/16	0.50	6.79515e−06	–	3.85380e−03	–
1/32		1.92315e−06	1.82103	1.96072e−03	0.97489
1/64		4.96266e−07	1.95429	9.78472e−04	1.00278
1/128		1.29399e−07	1.93929	4.95661e−04	0.98117
1/16	0.75	5.92698e−06	–	3.35774e−03	–
1/32		1.67746e−06	1.82102	1.70834e−03	0.97490
1/64		4.32868e−07	1.95428	8.52520e−04	1.00279
1/128		1.12867e−07	1.93930	4.31861e−04	0.98116
EOC			2.0		1.0

TABLE 3. The L^2-OC and energy-OC for the Morley method in case (ii) for $\alpha = 0.25, 0.50, 0.75$ at $t = 0.1$ with $k = 0.001$.

h	α	Errors in L^2-norm	L^2-OC	Errors in energy-norm	Energy-OC
1/12	0.25	2.03729e−04	–	2.06933e−02	–
1/24		5.22317e−05	1.96365	1.04746e−02	0.98226
1/48		1.31474e−05	1.99015	5.25437e−03	0.99531
1/96		3.29262e−06	1.99747	2.62936e−03	0.99880
1/12	0.50	1.98919e−04	–	2.02103e−02	–
1/24		5.09999e−05	1.96361	1.02305e−02	0.98222
1/48		1.28375e−05	1.99013	5.13193e−03	0.99530
1/96		3.21502e−06	1.99746	2.56809e−03	0.99880
1/12	0.75	1.96278e−04	–	1.99395e−02	–
1/24		5.03223e−05	1.96363	1.00933e−02	0.98223
1/48		1.26670e−05	1.99013	5.06308e−03	0.99530
1/96		3.17243e−06	1.99741	2.53364e−03	0.99880
EOC			2.0		1.0

5.2. Order of spatial convergence for nonsmooth initial data

Since the exact solution is not known in this case, the OC is calculated by the following formula

$$\mathrm{OC} = \log(\|W_{2h} - W_h\|_* / \|W_h - W_{h/2}\|_*)/\log 2,$$

where $\|\cdot\|_*$ denotes either the $L^2(\Omega)$- or energy norm and W_h is the discrete solution with mesh size h. The spatial numerical experiments are performed for case (i) with the mesh size $h = \{1/16, 1/32, 1/64, 1/128, 1/256\}$, and fractional order $\alpha = 0.25, 0.50, 0.75$ keeping the time step size k fixed at $k = 0.001$.

For the Morley method, the numerical results are illustrated in Table 1. The solution plots for the dG method are displayed in Figure 1 on 256×256 mesh with $\alpha = 0.25, 0.5, 0.75$ with fixed $k = 0.001$. Finally, for C^0IP method, the empirical results are shown in Table 2. The energy norm and L^2 norm errors demonstrate linear

TABLE 4. The L^2-OC and energy-OC for the dG method in case (ii) for $\alpha = 0.25, 0.50, 0.75$ at $t = 0.1$ with $k = 0.001$ and $\sigma^1_{\mathrm{dG}} = \sigma^2_{\mathrm{dG}} = 2$.

h	α	Errors in L^2-norm	L^2-OC	Errors in energy-norm	Energy-OC
1/12	0.25	1.03592e−03	–	6.45313e−02	–
1/24		2.92835e−04	1.82275	3.05090e−02	1.08076
1/48		7.85626e−05	1.89817	1.41902e−02	1.10434
1/96		2.03336e−05	1.94998	6.97876e−03	1.02385
1/12	0.50	1.01130e−03	–	6.30133e−02	–
1/24		2.85915e−04	1.82256	2.97960e−02	1.08054
1/48		7.67092e−05	1.89812	1.38593e−02	1.10426
1/96		1.98541e−05	1.94996	6.81611e−03	1.02383
1/12	0.75	9.97939e−04	–	6.21740e−02	–
1/24		2.82122e−04	1.82263	2.93972e−02	1.08063
1/48		7.56902e−05	1.89814	1.36735e−02	1.10430
1/96		1.95904e−05	1.94996	6.72469e−03	1.02384
EOC			2.0		1.0

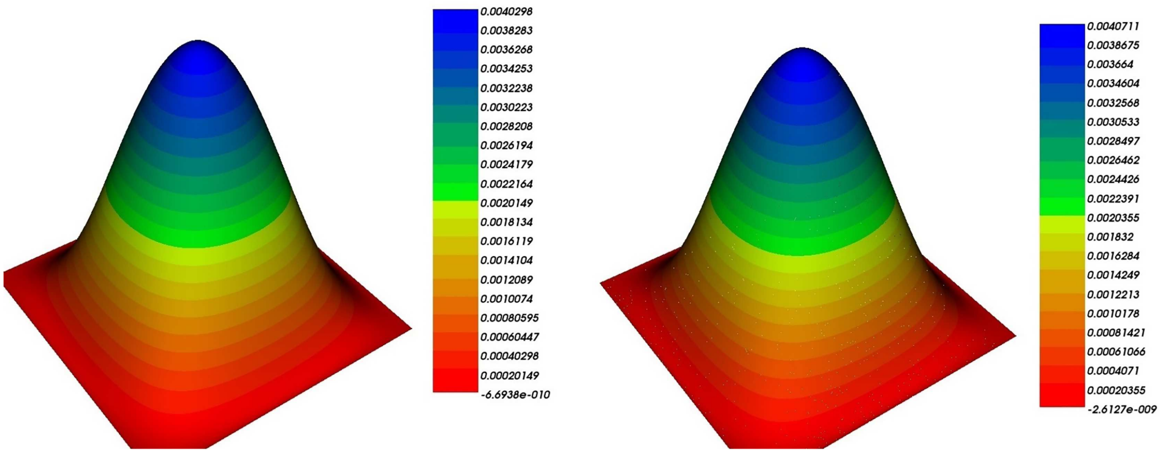

FIGURE 2. Exact (*left*) and numerical (*right*) solutions of dG method on 96×96 mesh, $\alpha = 0.5$.

and quadratic orders of convergence, respectively. These findings agree with the theoretical convergence rates established in Theorem 4.2(i).

5.3. Order of spatial convergence for smooth data

The spatial numerical experiments are performed with the mesh size $h = \{\frac{1}{12}, \frac{1}{24}, \frac{1}{48}, \frac{1}{96}\}$, and fractional order $\alpha = 0.25, 0.50, 0.75$ with a fixed time step size $k = 0.001$. For the Morley method, the results are presented in Table 3. The convergence results for the dG scheme are demonstrated in Table 4. Further, the exact and numerical solutions plots are also shown in Figure 2. Table 5 and Figure 3 display the numerical results and the exact and numerical solutions for the C^0IP method. From the experiments, we observe that the empirical results are independent of the fractional order α. The energy norm and L^2 norm errors for the Morley, dG, and C^0IP methods are plotted in a log–log scale in Figure 4. This validates the theoretical OCs of Theorem 4.2(ii).

Table 5. The L^2-OC and energy-OC for the C^0IP method in case (ii) for $\alpha = 0.25, 0.50, 0.75$ at $t = 0.1$ with $k = 0.001$ and $\sigma_{\mathrm{IP}} = 8$.

h	α	Errors in L^2-norm	L^2-OC	Errors in energy-norm	Energy-OC
1/12	0.25	1.15038e−04	–	1.72387e−02	–
1/24		3.29403e−05	1.80419	8.87466e−03	0.95788
1/48		8.67545e−06	1.92484	4.47807e−03	0.98681
1/96		2.21587e−06	1.96907	2.24645e−03	0.99522
1/12	0.50	1.12328e−04	–	1.68369e−02	–
1/24		3.21638e−05	1.80421	8.66785e−03	0.95788
1/48		8.47091e−06	1.92485	4.37372e−03	0.98681
1/96		2.16363e−06	1.96906	2.19411e−03	0.99522
1/12	0.75	1.10834e−04	–	1.66111e−02	–
1/24		3.17360e−05	1.80421	8.55157e−03	0.95788
1/48		8.35815e−06	1.92487	4.31505e−03	0.98681
1/96		2.13470e−06	1.96915	2.16467e−03	0.99522
EOC			2.0		1.0

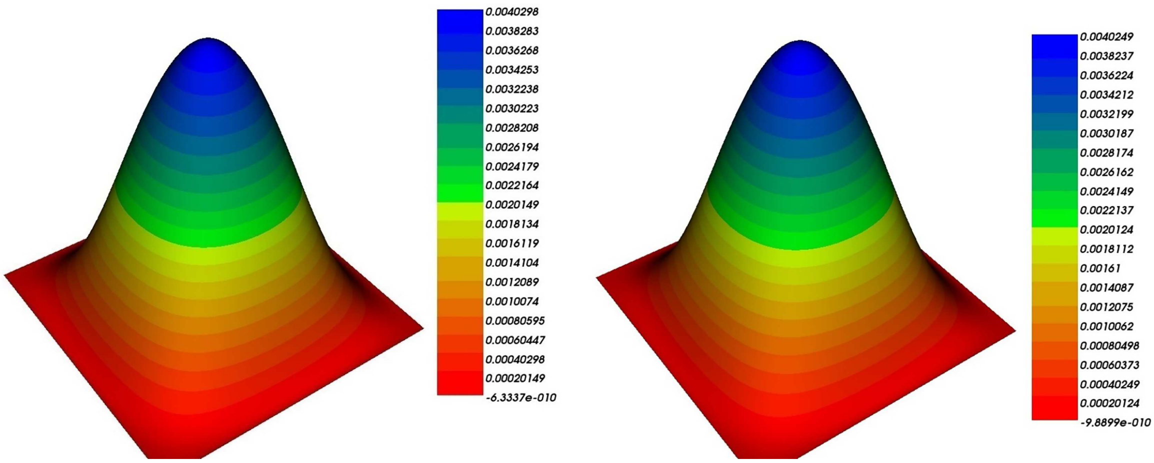

Figure 3. Exact (*left*) and numerical (*right*) solutions of C^0IP method on 96×96 mesh, $\alpha = 0.5$.

6. Concluding remarks

In this paper, we have studied an initial-boundary value problem for a time-fractional biharmonic problem with clamped boundary conditions in a bounded polygonal domain with a Lipschitz continuous boundary in $\mathbb{R}^2$. After defining a weak solution, we have stated the well-posedness result of the problem and derived several regularity results of the solutions of both homogeneous and nonhomogeneous problems which are useful in the error analysis. Using an energy argument a spatially semidiscrete scheme that covers various popular lowest-order nonstandard piecewise quadratic finite element schemes (*e.g.*, the Morley, discontinuous Galerkin, and C^0 interior penalty) is developed and analyzed. The convergence analysis is carried out for smooth and nonsmooth initial data cases, including the initial data $u_0 \in L^2(\Omega)$. Numerical results are provided to validate the theoretical convergence rates of the discrete solution. One future direction is the convergence analysis for time discretization, which will be addressed in another work.

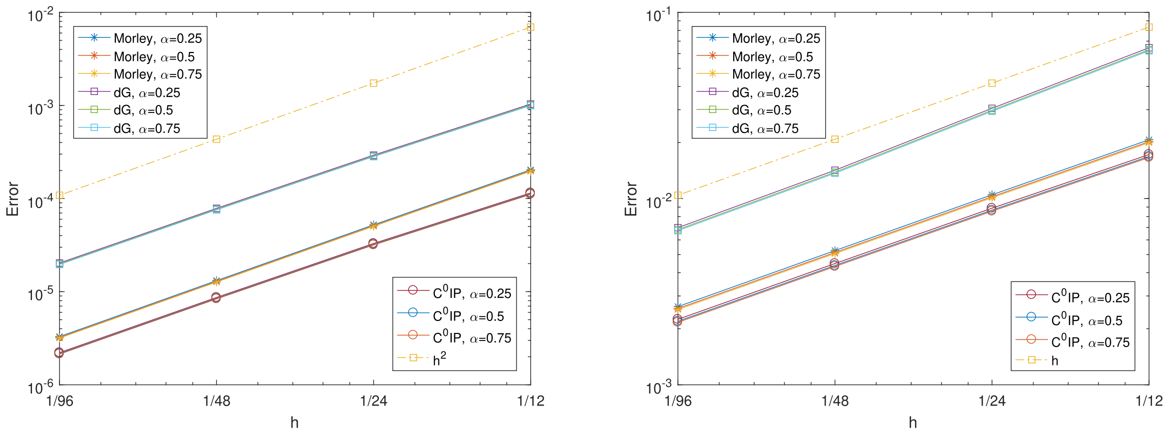

FIGURE 4. L^2 errors (*left*) and energy errors (*right*) plots for case (ii) for the Morley, dG, and C^0IP schemes at $t = 0.1$ with $k = 0.001$.

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