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STRUCTURE-PRESERVING FINITE ELEMENT METHODS FOR COMPUTING DYNAMICS OF ROTATING BOSE–EINSTEIN CONDENSATES

Meng Li[1], Junjun Wang[2,*], Zhen Guan[2] and Zhijie Du[3]

Abstract. This work is concerned with the construction and analysis of structure-preserving Galerkin methods for computing the dynamics of rotating Bose–Einstein condensate (BEC) based on the Gross–Pitaevskii equation with angular momentum rotation. Due to the presence of the rotation term, constructing finite element methods (FEMs) that preserve both mass and energy remains an unresolved issue, particularly in the context of nonconforming FEMs. Furthermore, in comparison to existing works, we provide a comprehensive convergence analysis, offering a thorough demonstration of the methods' optimal and high-order convergence properties. Finally, extensive numerical results are presented to check the theoretical analysis of the structure-preserving numerical method for rotating BEC, and the quantized vortex lattice's behavior is scrutinized through a series of numerical tests.

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1. Introduction

The phenomenon of Bose–Einstein condensate (BEC) occurs when a sparse gas of a particular type of bosons is confined by a potential and subsequently cooled to extremely low temperatures approaching the absolute minimum of 0 Kelvin. Since the pioneering experimental achievement of generating a quantized vortex in a gaseous BEC [1–4], there has been noteworthy advancement in both experimental and theoretical fronts in the field of research [5–19]. The topic of this work is to construct and analyze the structure-preserving finite element methods (FEMs) for a special case of BEC in a rotational framework. One notable characteristic of a BEC is its superfluid behavior. It is necessary to confirm the formation of vortices with quantized circulation for the purpose of distinguishing a superfluid from a normal fluid at the quantum level, which in experimental setups can be induced by rotating the condensate that can be achieved by applying laser beams to the magnetic trap to create a stirring potential. If the rotational speed is sufficiently high, these vortices become detectable (as indicated in Ref. [3]). Notably from the analytical proof in [20], the equilibrium velocity of the BEC no longer aligns with that of solid body rotation, and the breaking of rotational symmetry can be observed. The frequency of rotation significantly influences the number of vortices in a BEC, yet insufficient rotational speeds lead to a

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[1] School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, P.R. China.
Email: limeng@zzu.edu.cn

[2] School of Mathematics and Statistics, Pingdingshan University, Pingdingshan 467000, P.R. China.

[3] School of Science, Wuhan University of Technology, Wuhan 430070, P.R. China.

*Corresponding author: wjunjun8888@163.com

lack of vortices, and overly rapid speeds can cause the BEC's destruction by overpowering centrifugal forces. Refer to the related analytical and numerical results in [10, 11, 14, 21, 22] and references therein.

This paper is devoted to the Galerkin approximations to the Gross–Pitaevskii equation (GPE) with an angular momentum rotation term in two/three dimensions for modeling a rotating BEC [10, 23]

$$i\partial_t u(\mathbf{x},t) = \left[-\frac{1}{2}\Delta + V(\mathbf{x}) - \Omega L_z + \beta|u(\mathbf{x},t)|^2\right]u(\mathbf{x},t), \quad \mathbf{x}\in U\subset\mathbb{R}^d, \quad t>0, \tag{1.1}$$

with the homogeneous Dirichlet boundary condition

$$u(\mathbf{x},t) = 0, \quad \mathbf{x}\in\Gamma = \partial U, \quad t\geq 0, \tag{1.2}$$

and the initial condition

$$u(\mathbf{x},0) = u^0(\mathbf{x}), \quad \mathbf{x}\in U. \tag{1.3}$$

Here we assume that $U\subset\mathbb{R}^d$ $(d=2,3)$ is a bounded convex polyhedron that defines the computational domain, ∂U is the boundary of U, $\mathbf{x}=(x,y)$ in two dimensions (2D) and $\mathbf{x}=(x,y,z)$ in three dimensions (3D), $t\in[0,T]$ is time, $u := u(\mathbf{x},t)\in L^\infty([0,T],H_0^1(U))$ is the complex-valued wave function, $u^0(\mathbf{x})\in H_0^1(U)$ is a given complex-valued function, and Ω denotes a dimensionless constant corresponding to the angular speed of the laser beam in experiments. The parameter β is a dimensionless constant characterizing the interaction between particles in the rotating BEC that can be positive for repulsive interaction and negative for attractive interaction. When $\beta=0$, the model (1.1) is a linear GPE. $V(\mathbf{x})$ is a real-valued function corresponding to the external trap potential that can confines the BEC by adjusting $V(\mathbf{x})$ to some trap frequencies. Usually in numerical experiments, $V(\mathbf{x})$ can be selected as a harmonic potential, *i.e.*, a quadratic polynomial:

$$V(\mathbf{x}) = \begin{cases} (\gamma_x^2x^2+\gamma_y y^2)/2, & d=2,\\ (\gamma_x^2x^2+\gamma_y^2y^2+\gamma_z^2z^2)/2, & d=3, \end{cases} \tag{1.4}$$

with the constants γ_x, γ_y and γ_z. Obviously, we have $V(\boldsymbol{x})\in C(U)$. However, in the real world, $V(\boldsymbol{x})$ may be a potentially rough or discontinuous potential. For example, $V\in L^\infty(U)$ can be arbitrarily rough besides being bounded [24]. L_z denotes the z-component of the angular momentum:

$$L_z = -i(x\partial_y - y\partial_x) = -i\partial_\theta, \tag{1.5}$$

where (r,θ) and (r,θ,z) denote the polar coordinates in 2D and the cylindrical coordinates in 3D, respectively. For the derivation, well-posedness and dynamical properties of the GPE (1.1) with (*i.e.*, $\Omega\neq 0$) and without (*i.e.*, $\Omega=0$) an angular momentum rotation term, we can refer to [25–27] and the references therein. Obviously, the system (1.1)–(1.3) converses the total mass

$$N(u(\cdot,t)) := \int_U |u(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{x} \equiv N(u(\cdot,0)) = N(u^0), \quad t\geq 0, \tag{1.6}$$

and the total energy

$$\begin{aligned} E(u(\cdot,t)) :=& \int_U \left[\frac{1}{2}|\nabla u(\mathbf{x},t)|^2 + V(\mathbf{x})|u(\mathbf{x},t)|^2\frac{\beta}{2}|u(\mathbf{x},t)|^4 - \Omega\bar{u}(\mathbf{x},t)L_z u(\mathbf{x},t)\right]\mathrm{d}\mathbf{x} \equiv E(u(\cdot,0))\\ =& E(u^0), \quad t\geq 0, \end{aligned} \tag{1.7}$$

where $\bar{u}(\mathbf{x},t)$ represents the conjugate of $u(\mathbf{x},t)$.

As we all know, conservative schemes consistently outperform nonconservative ones. The crucial factor behind this lies in their ability to preserve certain invariant properties, allowing them to capture intricate details of physical processes. Viewed from this perspective, discrete schemes that maintain the invariant properties of the original continuous model serve as a criterion for assessing the success of numerical simulations. Compared with other spatial discretization methods, the usage of FEMs possesses at least the following two advantages: one is the ability to compute the GPE with a rough/discontinuous potential $V(\mathbf{x})$, as seen in [24, 28–30]; the other is that the FEMs offer an additional advantage by effortlessly integrating with mesh adaptivity, a feature that can be beneficial in effectively addressing localized vortices in BEC. There exist some well-known numerical works focusing on the conservative FEMs of the particular case of (1.1), without the angular momentum rotation term [31–33]. However, the coupling conditions between the time step size and the spatial mesh size cannot be removed in these references. Still for the above particular model, motivated by Wang [34], Henning and Peterseim [24] studied the unconditional error analysis for its modified Crank–Nicolson FEM, which has both the mass and energy conservations. When proving H^1-bounds of the discrete solutions, the analysis is limited to the repulsive interaction case (*i.e.*, $\beta > 0$). But actually, this condition does not significantly affect the error analysis. The only finite element (FE) works concerning the model (1.1) with the angular momentum rotation term can be documented in [16, 35] by Henning. But the scheme is only mass-conserving or only energy-conserving. In [36], Henning and Wärnegård considered the H^1-norm optimal error estimates for the Crank–Nicolson FEM for the GPE without the angular momentum rotation term. The detailed theoretical analysis of the classical mass- and energy-conserving scheme with Sanz-Serna in time, for the model (1.1), remains an unexplored issue and constitutes a key contribution in this paper.

Conforming and nonconforming FEMs are both widely employed in the numerical solution of partial differential equations. Unlike conforming FEMs, nonconforming FEMs relax the continuity constraint, allowing for inconsistencies at shared nodes between adjacent elements. This flexibility offers distinct advantages, particularly in managing complex geometries and capturing localized phenomena, and often imposes lower regularity requirements on the continuous solution. In this paper, we aim to develop a unified framework for solving the GPE model using both conforming and nonconforming FEMs. Our objective is to ensure that the proposed schemes preserve structural properties while achieving optimal and high-accuracy convergence results. However, constructing a conservative scheme for a nonconforming FEM poses a substantial challenge. Due to the presence of the rotation term and since the FE space is not included in the continuous Sobolev space $H_0^1(\Omega)$, the fully discrete scheme based on the Sanz-Serna temporal method cannot be proved to be conservative in the senses of both the discrete mass and energy. In this work, we introduce an innovative stabilizing term into the non-conservative scheme. This term does not affect the convergence rate, and the resulting numerical scheme successfully preserves both mass and energy conservation properties. This conservation-adjusting technique is the first application in developing the conservative scheme for the nonconforming FEM, which is another important contribution of this work. Additionally, this study offers a comprehensive proof of the boundedness of numerical solutions, utilizing the Gagliardo–Nirenberg inequality and a conforming companion operator.

The other main contribution of this paper is the establishment of unconditional error estimates for the proposed conservative schemes. These estimates hold without the existence of any time-space step coupling condition and include optimal L^2 and H^1 estimates, along with high-accuracy convergence rates in the H^1-norm (while maintaining computational complexity within reasonable limits). The theoretical analytical approach draws inspiration from the existing technique [24] for the nonlinear Schrödinger equation. However, a subtle distinct aspect of our approach is our ability to handle cases involving attractive interactions (*i.e.*, $\beta < 0$) when proving the well-posedness of the truncated time-discrete system, as we do not rely on the H^1-boundedness of the discrete solution. Additionally, the inclusion of the rotation term introduces significant complexity to the convergence analysis, and the nonconforming case exhibits notable differences from the literature [24]. Furthermore, different from [16, 24], we also achieve the high-order convergence rates in space for both numerical schemes. This contribution stands out as one of the key highlights of this work.

To summarize, the main innovations are as follows:

(1) Based on the Sanz-Serna method in time and FEMs in space, we develop a unified structure-preserving FE scheme for the GPE with an angular momentum rotation term. The primary innovation lies in the design of a conservation-adjusting technique, which can be effectively employed to develop conservative methods for

nonconforming FEMs. Another advantage of the conservation-adjusting technique proposed in this work is that it can be easily extended to other models and other nonconforming FEMs.

(2) For the conforming FEM, we can derive the H^1-norm boundedness using the Gagliardo–Nirenberg inequality and the conservation results. However, in the nonconforming FE space, the Gagliardo–Nirenberg inequality must be re-proved. By introducing a conforming companion operator, we prove the Gagliardo–Nirenberg inequality in nonconforming FE space. Then, we derive the H^1-norm boundedness of the discrete solution for both conforming and nonconforming FEMs without the requirement for $\beta \geq 0$.

(3) In terms of convergence analysis, we present a thorough and comprehensive proof for the convergence of the aforementioned structure-preserving numerical methods. Our unified convergence proof encompasses both conforming and nonconforming FEMs. It includes not only the optimal convergence in the L^2-norm but also the higher-order superconvergence analysis, all achieved without imposing any coupling conditions between the time step size and the spatial mesh size. For the structure-preserving FE scheme for the GPE with an angular momentum rotation term, there is currently no existing convergence proof in the literature for either conforming or nonconforming FEMs. This aspect of our work is innovative, as it provides a unified and rigorous convergence proof for these methods, thus addressing a gap in the current research.

(4) We present extensive numerical results to validate the theoretical analysis of the structure-preserving numerical methods for rotating BEC. These results thoroughly verify the accuracy, mass/energy conservation, and effectiveness of our approaches. Additionally, we conduct a series of detailed numerical experiments to investigate the behavior of the quantized vortex lattice, attempting to provide some insights into the dynamics of vortex structures under rotation.

Outline. In Section 2, we introduce the discretized schemes, encompassing both the time-discrete method and the fully discrete method. Detailed proofs are provided for the discrete mass and energy conservations inherent in the proposed schemes. Additionally, we establish the boundedness of the fully discrete solutions in the case of repulsive interaction by leveraging the Gagliardo–Nirenberg inequality and using a conforming companion operator. Section 3 presents the main convergence results of the fully discrete conforming and nonconforming schemes. Section 4 is devoted to the proof of the convergence results. Some numerical experiments are provided in Section 5 to check the theoretical results, and the tests also include the vortex lattice dynamics in rotating BEC by our numerical schemes. Some conclusions are drawn in Section 6. Finally, the brief convergence analysis for the nonconforming FEM is provided in appendix.

Notations. We denote the standard notations for Sobolev spaces $W^{s,p}(U)$ with $|\cdot|_{W^{s,p}}$ and $\|\cdot\|_{W^{s,p}}$ being its seminorm and norm, respectively. For the case of $p=2$, we denote the notations $H^s(U) = W^{s,2}(U)$ and $H_0^1(U) := \{v \in H^1(U) : v|_{\partial\Omega} = 0\}$. When $s = 0$, $L^p(U) = W^{0,p}(U)$. For any Banach space $\mathbb{X}$ and for a finite time interval $J := [0,T]$ with a positive constant T, let $L^p(J;\mathbb{X})$ be the space of all measurable function $\phi : J \to \mathbb{X}$ with the norm

$$\|\phi\|_{L^p(J;\mathbb{X})} := \begin{cases} \left[\int_0^T \|\phi(t)\|_{\mathbb{X}}^p \,\mathrm{d}t\right]^{\frac{1}{p}}, & \text{if } 1 \leq p < +\infty, \\ \operatorname{esssup}_{t\in J}\|\phi\|_{\mathbb{X}}, & \text{if } p = +\infty. \end{cases}$$

2. Discretized schemes

For the sake of simplicity in presentation, we only discuss the methods in 2D. The theoretical results in 3D and the corresponding analytical techniques are straightforward and similar to those in 2D.

2.1. Time-discrete method

Consider a family of admissible partitions of the time interval J denoted by $\{I_n; n \in \mathbb{N}, 0 \leq n \leq N-1\}$, where $I_n := (t_n, t_{n+1}]$ with $0 = t_0 < t_1 < \ldots < t_N = T$ and $\tau_n = |I_n|$. Additionally, we assume that the partitions are quasi-uniform, *i.e.*, $\tau := \max_{1\leq n\leq N}\{\tau_n\} \leq C_q \min_{1\leq n\leq N}\{\tau_n\}$ for any partitions, where C_q is a positive constant independent of the discretization. Subsequently, we consider the time-discrete Crank–Nicolson approximation for the GPE (1.1)–(1.3).

Definition 2.1 (Time-discrete method for GPE)**.** Let $u_\tau^0 := u^0$. Then for $n \geq 1$, we define the following time-discrete system, which is to find $u_\tau^{n+1} \in H_0^1(U)$, $0 \leq n \leq N-1$ such that

$$iD_\tau u_\tau^{n+\frac{1}{2}} = -\frac{1}{2}\Delta \hat{u}_\tau^{n+\frac{1}{2}} + V\hat{u}_\tau^{n+\frac{1}{2}} - \Omega L_z \hat{u}_\tau^{n+\frac{1}{2}} + \beta\frac{|u_\tau^n|^2 + |u_\tau^{n+1}|^2}{2}\hat{u}_\tau^{n+\frac{1}{2}}, \tag{2.1}$$

where $D_\tau u_\tau^{n+\frac{1}{2}} := (u_\tau^{n+1} - u_\tau^n)/\tau_n$ and $\hat{u}_\tau^{n+\frac{1}{2}} := (u_\tau^n + u_\tau^{n+1})/2$.

The well-posedness and convergence of the system (2.1) will be shown in Section 4. Furthermore, we can prove that the time-discrete scheme (2.1) keeps the mass and energy conservations as follows.

Theorem 2.1. *The time-discrete system* (2.1) *is conservative in the senses of the total mass and energy:*

$$N(u_\tau^n) = N(u^0), \quad E(u_\tau^n) = E(u^0), \quad 0 \leq n \leq N. \tag{2.2}$$

Proof. Taking the inner product of (2.1) with $\hat{u}_\tau^{n+\frac{1}{2}}$, and selecting the imaginary part of the resulting, the mass conservation is directly obtained. Furthermore, taking the inner product of (2.1) with $D_\tau u_\tau^{n+\frac{1}{2}}$, and selecting the real part of the resulting, one obtains

$$\begin{aligned}\frac{1}{4\tau_n}\left[\int_U \left(\nabla u_\tau^{n+1}\right)^2 \mathrm{d}\mathbf{x} - \int_U \left(\nabla u_\tau^n\right)^2 \mathrm{d}\mathbf{x}\right] + \frac{1}{2\tau_n}\left[\int_U V\left(u_\tau^{n+1}\right)^2 \mathrm{d}\mathbf{x} - \int_U V\left(u_\tau^n\right)^2 \mathrm{d}\mathbf{x}\right] - \Omega\mathrm{Re}\Big(L_z\hat{u}_\tau^{n+\frac{1}{2}}, D_\tau u_\tau^{n+\frac{1}{2}}\Big)\\ + \frac{\beta}{4\tau_n}\left[\int_U \left(u_\tau^{n+1}\right)^4 \mathrm{d}\mathbf{x} - \int_U \left(u_\tau^n\right)^4 \mathrm{d}\mathbf{x}\right] = 0.\end{aligned} \tag{2.3}$$

For the third term of (2.3), there holds

$$\begin{aligned}\mathrm{Re}\Big(L_z\hat{u}_\tau^{n+\frac{1}{2}}, D_\tau u_\tau^{n+\frac{1}{2}}\Big) &= -\mathrm{Re}\Big(i[x\partial_y - y\partial_x]\hat{u}_\tau^{n+\frac{1}{2}}, D_\tau u_\tau^{n+\frac{1}{2}}\Big)\\ &= \frac{1}{2\tau_n}\mathrm{Im}\big([x\partial_y - y\partial_x]\big(u_\tau^{n+1} + u_\tau^n\big), u_\tau^{n+1} - u_\tau^n\big)\\ &= \frac{1}{2\tau_n}\big[\mathrm{Im}\big([x\partial_y - y\partial_x]u_\tau^{n+1}, u_\tau^{n+1}\big) - \mathrm{Im}([x\partial_y - y\partial_x]u_\tau^n, u_\tau^n)\big]\\ &\quad + \frac{1}{2\tau_n}\big[\mathrm{Im}\big([x\partial_y - y\partial_x]u_\tau^n, u_\tau^{n+1}\big) - \mathrm{Im}\big([x\partial_y - y\partial_x]u_\tau^{n+1}, u_\tau^n\big)\big]\\ &=: \frac{1}{2\tau_n}\big[\mathrm{Im}\big([x\partial_y - y\partial_x]u_\tau^{n+1}, u_\tau^{n+1}\big) - \mathrm{Im}([x\partial_y - y\partial_x]u_\tau^n, u_\tau^n)\big] + \frac{1}{2\tau_n}A.\end{aligned} \tag{2.4}$$

By using integration by parts, we have

$$A = -\mathrm{Im}\big(u_\tau^n, [x\partial_y - y\partial_x]u_\tau^{n+1}\big) - \mathrm{Im}\big([x\partial_y - y\partial_x]u_\tau^{n+1}, u_\tau^n\big) = 0. \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3) gives that

$$\begin{aligned}&\frac{1}{2}\int_U (\nabla u_\tau^{n+1})^2 \mathrm{d}\mathbf{x} + \int_U V(u_\tau^{n+1})^2 \mathrm{d}\mathbf{x} - \Omega\mathrm{Im}\big([x\partial_y - y\partial_x]u_\tau^{n+1}, u_\tau^{n+1}\big) + \frac{\beta}{2}\int_U (u_\tau^{n+1})^4 \mathrm{d}\mathbf{x}\\ &\quad = \frac{1}{2}\int_U (\nabla u_\tau^n)^2 \mathrm{d}\mathbf{x} + \int_U V(u_\tau^n)^2 \mathrm{d}\mathbf{x} - \Omega\mathrm{Im}([x\partial_y - y\partial_x]u_\tau^n, u_\tau^n) + \frac{\beta}{2}\int_U (u_\tau^n)^4 \mathrm{d}\mathbf{x}.\end{aligned} \tag{2.6}$$

Since $\mathrm{Re}([x\partial_y - y\partial_x]u_\tau^m, u_\tau^m) = 0$ for any $m \in \mathbb{N}$, we can remove "Im" in (2.6). Hence, the energy conservation is obtained. Therefore, we have completed the proof. ∎

2.2. Fully discrete method

In this work, we build a unified framework of theoretical results in the conforming and nonconforming FEs. In what follows, we present a brief introduction of two types of elements. Assume that U is a rectangle in (x, y) plane with edges parallel to the coordinate axes. Let $\mathscr{T}_h$ be a quasi-uniform rectangular subdivision of U, and set $K \in \mathscr{T}_h$ and $h = \max_K diam(K)$. Let us consider

– Conforming FE space:

$$V_h^C := \{v \in H_0^1(U); v|_K \in Q_{11}(K), \forall K \in \mathscr{T}_h\}, \tag{2.7}$$

where $Q_{11}(K) = \text{span}\{1, x, y, xy\}$.

– Nonconforming FE space (EQ_1^{rot}):

$$V_h^{NC} := \left\{v_h \in L^2(U); v_h|_K \in \text{span}\{1, x, y, x^2, y^2\}, \int_F [v_h]\,\mathrm{d}s = 0, F \subset \partial K, \forall K \in \mathscr{T}_h\right\}, \tag{2.8}$$

where $[v_h]$ means the jump of v_h across the edge F if F is an internal edge, and v_h itself if F is a boundary edge.

Weak formulation. Given an initial data $u^0 \in H_0^1(U) := H_0^1(U;\mathbb{C})$, find a wave function $u \in L^\infty(J, H_0^1(U))$ with $\partial_t u \in L^\infty(J, H^{-1}(U))$, such that

$$i(\partial_t u(\mathbf{x}, t), \omega) = \frac{1}{2}(\nabla u(\cdot, t), \nabla\omega) + (Vu(\cdot, t), \omega) - \Omega(L_z u(\cdot, t), \omega) + \beta\big(|u(\mathbf{x}, t)|^2 u(\cdot, t), \omega\big), \quad \forall\omega \in H_0^1(U). \tag{2.9}$$

We refer to references [16, 25] for the well-posedness of the system. But only partial results regarding the well-posedness are available, and the general result is still an open problem. Moreover, it is obvious that the system (2.9) conserves the total mass and energy defined in (1.6) and (1.7).

For convenience, we denote V_h as a unified FE space, with $V_h = V_h^C$ and $V_h = V_h^{NC}$ for the conforming and nonconforming cases respectively. We define the inner product and corresponding norms piecewisely, given by

$$(\phi, \psi)_h := \sum_K \int_K \phi \cdot \bar{\psi}\,\mathrm{d}\mathbf{x}, \quad \|\phi\|_{1,h} := \left(\sum_K |\phi|_{1,K}^2\right)^{1/2},$$

where $\bar{\psi}$ means complex conjugation of a complex-value function ψ, and $|\cdot|_{1,K}$ denotes the H^1-seminorm on K. If the functions ϕ and ψ belong to the nonconforming FE space, we further define

$$\langle\phi, \psi\rangle := \sum_K \int_{\partial K} (\phi \cdot \bar{\psi})(\mathbf{x} \cdot \mathbf{n}^\perp)\,\mathrm{d}s \quad \text{with} \quad \mathbf{n}^\perp = (n_y, -n_x).$$

Here, the introduction of the definition $\langle\cdot, \cdot\rangle$ is for the convenience of constructing the conservative algorithm for the nonconforming FEM, as shown in the fully discrete method (2.13).

Define the Ritz projection $R_h : H_0^1(U) \to V_h$, such that

$$(\nabla v - \nabla R_h v, \nabla\omega_h)_h = 0, \quad \text{for all } \omega_h \in V_h. \tag{2.10}$$

There exists a generic h-independent constant C_{R_h}, such that [37]

$$\|v - R_h v\|_{L^2} + h\|v - R_h v\|_{1,h} \le C_{R_h} h^s \|v\|_{H^s}, \quad s = 1, 2, \quad \text{for all } v \in H_0^1(U) \cap H^2(U). \tag{2.11}$$

Let the four vertices and edges of an element K be a_i and $l_i = \overline{a_i a_{i+1}}, i = 1 \sim 4 \pmod 4$, respectively. Then the interpolation operator I_h is defined by

– For the case of $V_h = V_h^C$, for $u \in H^2(\Omega)$, define the associated interpolation operator I_h as

$$I_h : u \in H^2(\Omega) \to I_h u \in V_h,\ I_h|_K = I_K,\ I_K u(a_i) = u(a_i),\ i = 1 \sim 4 \pmod 4;$$

– For the case of $V_h = V_h^{NC}$, for $u \in H^1(\Omega)$, define the associated interpolation operator I_h as

$$\begin{aligned} &I_h : u \in H^1(\Omega) \to I_h u \in V_h,\ I_h|_K = I_K,\\ &\int_{l_i} (I_K u - u)\,\mathrm{d}s = 0,\ i = 1 \sim 4 \pmod 4 \quad \text{and} \quad \int_K (I_K u - u)\,\mathrm{d}X = 0. \end{aligned}$$

The interpolation operator I_h satisfies [37,38]

$$\|u - I_h u\|_{L^2} \le C_{I_h} h^2 \|u\|_{H^2}, \quad \text{and} \quad (\nabla(u - I_h u), \nabla v_h)_h = \begin{cases} O(h^2)\|u\|_{H^3}\|v_h\|_{1,h}, & \text{for } v_h \in V_h^C,\\ 0, & \text{for } v_h \in V_h^{NC}. \end{cases} \tag{2.12}$$

We note that the derivatives of functions $w_h \in V_h^{NC}$ are always understood to be defined piecewise over the elements of the mesh $\mathcal{T}_h$ (*i.e.*, $\mathcal{T}_h$-piecewise). With above preparation, we introduce the following new fully discrete Crank–Nicolson method, which is different from the existing works [16,39].

Definition 2.2 (Fully discrete method for GPE)**.** Let $u_{h,\tau}^0$ be a suitable interpolation of u^0. Then for $n \ge 1$, we define the following time-discrete system, which is to find $u_{h,\tau}^{n+1} \in V_h$, $0 \le n \le N-1$ such that

$$\begin{aligned} i\Big(D_\tau u_{h,\tau}^{n+\frac12}, \omega_h\Big) = &\frac12\Big(\nabla \hat{u}_{h,\tau}^{n+\frac12}, \nabla\omega_h\Big)_h + \Big(V\hat{u}_{h,\tau}^{n+\frac12}, \omega_h\Big) - \Omega\Big(L_z \hat{u}_{h,\tau}^{n+\frac12}, \omega_h\Big)_h \\ &+ \beta\left(\frac{|u_{h,\tau}^n|^2 + |u_{h,\tau}^{n+1}|^2}{2}\hat{u}_{h,\tau}^{n+\frac12}, \omega_h\right) + \big\langle S^{n+1}, \omega_h\big\rangle, \end{aligned} \tag{2.13}$$

for any $\omega_h \in V_h$, where $\langle S^{n+1}, \omega_h\rangle$ is defined by

$$\big\langle S^{n+1}, \omega_h\big\rangle := \begin{cases} 0, & \text{for conforming case,}\\ -i\frac{\Omega}{2}\mathrm{Re}\Big\langle \hat{u}_{h,\tau}^{n+\frac12}, \omega_h\Big\rangle + \frac{\Omega}{2}\mathrm{Im}\Big\langle u_{h,\tau}^{n+1}, \omega_h\Big\rangle, & \text{for nonconforming case.} \end{cases} \tag{2.14}$$

We will supply the well-posedness and convergence of the system (2.13) in Section 5. In the following theorem, we prove that the fully discrete scheme (2.13) keeps the total mass and energy conservations.

Theorem 2.2 (Mass and energy conservations)**.** *The fully discrete system* (2.13) *is conservative in the senses of the total mass and energy:*

$$N(u_{h,\tau}^n) = N(u_{h,\tau}^0), \quad E_h^n = E_h^0, \quad 0 \le n \le N, \tag{2.15}$$

where the discrete energy E_h^n is defined by

$$E_h^n := \frac12\|u_{h,\tau}^n\|_{1,h}^2 + \big(Vu_{h,\tau}^n, u_{h,\tau}^n\big) + \frac{\beta}{2}\|u_{h,\tau}^n\|_{L^4}^4 - \Omega\mathrm{Re}\big(L_z u_{h,\tau}^n, u_{h,\tau}^n\big)_h. \tag{2.16}$$

Proof. The results are directly obtained for the conforming case, and thus we only show some proof details for the nonconforming case. Setting $\omega_h = \hat{u}_{h,\tau}^{n+\frac12}$ in (2.13), and selecting the imaginary part, we have

$$\frac{\|u_{h,\tau}^{n+1}\|_{L^2}^2 - \|u_{h,\tau}^n\|_{L^2}^2}{2\tau_n} = -\Omega\mathrm{Im}\Big(L_z\hat{u}_{h,\tau}^{n+\frac12}, \hat{u}_{h,\tau}^{n+\frac12}\Big)_h + \mathrm{Im}\Big\langle S^{n+1}, \hat{u}_{h,\tau}^{n+\frac12}\Big\rangle. \tag{2.17}$$

For the first term at the right-hand side of (2.17), by virtue of the integration by parts, it holds

$$
\begin{aligned}
&-\Omega\mathrm{Im}\Big(L_z\hat{u}_{h,\tau}^{n+\frac12},\hat{u}_{h,\tau}^{n+\frac12}\Big)_h=\Omega\mathrm{Re}\Big([x\partial_y-y\partial_x]\hat{u}_{h,\tau}^{n+\frac12},\hat{u}_{h,\tau}^{n+\frac12}\Big)_h\\
&\quad=\frac{\Omega}{2}\Big[\Big([x\partial_y-y\partial_x]\hat{u}_{h,\tau}^{n+\frac12},\hat{u}_{h,\tau}^{n+\frac12}\Big)_h+\Big(\hat{u}_{h,\tau}^{n+\frac12},[x\partial_y-y\partial_x]\hat{u}_{h,\tau}^{n+\frac12}\Big)_h\Big]=\frac{\Omega}{2}\Big\langle\hat{u}_{h,\tau}^{n+\frac12},\hat{u}_{h,\tau}^{n+\frac12}\Big\rangle.
\end{aligned}\tag{2.18}
$$

Moreover, for the second term at the right-hand side of (2.17), we have

$$
\mathrm{Im}\Big\langle S^{n+1},\hat{u}_{h,\tau}^{n+\frac12}\Big\rangle=-\frac{\Omega}{2}\Big\langle\hat{u}_{h,\tau}^{n+\frac12},\hat{u}_{h,\tau}^{n+\frac12}\Big\rangle.\tag{2.19}
$$

Substituting (2.18) and (2.19) into (2.17), we derive the mass conservation.

Setting $\omega_h=D_\tau u_{h,\tau}^{n+\frac12}$ in (2.13), and selecting the real part of the resulting, we have

$$
\mathscr{D}+\mathscr{E}=0,\tag{2.20}
$$

where

$$
\mathscr{D}:=\frac12\mathrm{Re}\Big(\nabla\hat{u}_{h,\tau}^{n+\frac12},\nabla D_\tau u_{h,\tau}^{n+\frac12}\Big)_h+\mathrm{Re}\Big(V\hat{u}_{h,\tau}^{n+\frac12},D_\tau u_{h,\tau}^{n+\frac12}\Big)+\beta\mathrm{Re}\Big(\frac{|u_\tau^n|^2+|u_\tau^{n+1}|^2}{2}\hat{u}_\tau^{n+\frac12},D_\tau u_{h,\tau}^{n+\frac12}\Big)
$$

and

$$
\mathscr{E}:=-\Omega\mathrm{Re}\Big(L_z\hat{u}_{h,\tau}^{n+\frac12},D_\tau u_{h,\tau}^{n+\frac12}\Big)_h+\mathrm{Re}\Big\langle S^{n+1},D_\tau u_{h,\tau}^{n+\frac12}\Big\rangle.
$$

The term $\mathscr{D}$ is equivalent to

$$
\mathscr{D}=\frac{\|u_{h,\tau}^{n+1}\|_{1,h}^2-\|u_{h,\tau}^{n}\|_{1,h}^2}{4\tau_n}+\frac{(Vu_{h,\tau}^{n+1},u_{h,\tau}^{n+1})-(Vu_{h,\tau}^{n},u_{h,\tau}^{n})}{2\tau_n}+\frac{\beta}{4\tau_n}\Big(\|u_{h,\tau}^{n+1}\|_{L^4}^4-\|u_{h,\tau}^{n}\|_{L^4}^4\Big).\tag{2.21}
$$

The term $\mathscr{E}$ can be rewritten as

$$
\begin{aligned}
\mathscr{E}=&-\frac{\Omega}{2\tau_n}\Big[\mathrm{Re}\Big(L_zu_{h,\tau}^{n+1},u_{h,\tau}^{n+1}\Big)_h-\mathrm{Re}\big(L_zu_{h,\tau}^{n},u_{h,\tau}^{n}\big)_h\Big]+\frac{\Omega}{2\tau_n}\Big[\mathrm{Re}\Big(L_zu_{h,\tau}^{n+1},u_{h,\tau}^{n}\Big)_h-\mathrm{Re}\Big(L_zu_{h,\tau}^{n},u_{h,\tau}^{n+1}\Big)_h\Big]\\
&+\mathrm{Re}\Big\langle S^{n+1},D_\tau u_{h,\tau}^{n+\frac12}\Big\rangle=:-\frac{\Omega}{2\tau_n}\Big[\mathrm{Re}\Big(L_zu_{h,\tau}^{n+1},u_{h,\tau}^{n+1}\Big)_h-\mathrm{Re}\big(L_zu_{h,\tau}^{n},u_{h,\tau}^{n}\big)_h\Big]+\mathscr{E}_1.
\end{aligned}\tag{2.22}
$$

For the term $\mathscr{E}_1$ in (2.22), using integration by parts, we have

$$
\begin{aligned}
\mathscr{E}_1&=\frac{\Omega}{2\tau_n}\Big[\mathrm{Re}\Big(L_zu_{h,\tau}^{n+1},u_{h,\tau}^{n}\Big)_h-\mathrm{Re}\Big(L_zu_{h,\tau}^{n},u_{h,\tau}^{n+1}\Big)_h\Big]+\mathrm{Re}\Big\langle S^{n+1},D_\tau u_{h,\tau}^{n+\frac12}\Big\rangle\\
&=\frac{\Omega}{2\tau_n}\mathrm{Im}\Big\{\Big([x\partial_y-y\partial_x]u_{h,\tau}^{n+1},u_{h,\tau}^{n}\Big)_h-\Big([x\partial_y-y\partial_x]u_{h,\tau}^{n},u_{h,\tau}^{n+1}\Big)_h-\Big\langle u_{h,\tau}^{n+1},u_{h,\tau}^{n}\Big\rangle\Big\}\\
&=-\frac{\Omega}{2\tau_n}\mathrm{Im}\Big\{\Big(u_{h,\tau}^{n+1},[x\partial_y-y\partial_x]u_{h,\tau}^{n}\Big)_h+\Big([x\partial_y-y\partial_x]u_{h,\tau}^{n},u_{h,\tau}^{n+1}\Big)_h\Big\}=0.
\end{aligned}\tag{2.23}
$$

Substituting (2.21)–(2.23) into (2.20) obtains the energy conservation. □

Remark 2.3. From the proof, we can observe that the reason for introducing the term $\langle S^{n+1},\omega_h\rangle$ in the construction of a structure-preserving numerical scheme based on nonconforming FEM is that, in the nonconforming FE space,

$$
\frac{\Omega}{2\tau_n}\Big[\mathrm{Re}\Big(L_zu_{h,\tau}^{n+1},u_{h,\tau}^{n}\Big)_h-\mathrm{Re}\Big(L_zu_{h,\tau}^{n},u_{h,\tau}^{n+1}\Big)_h\Big]\neq0.
$$

The challenging point is that, even though we introduce this term, we can still obtain the desired convergence results as presented in the following sections.

In order to obtain the boundedness of the numerical solution, we should first prove the following lemma.

Lemma 2.4. *For any $v_h \in V_h$, no matter for the conforming or nonconforming FE spaces, there holds*

$$\|v_h\|_{L^4} \le C_{gn}\|v_h\|_{L^2}^{\frac{1}{2}}\|v_h\|_{1,h}^{\frac{1}{2}}, \tag{2.24}$$

where C_{gn} is a bounded and positive constant.

Proof. For the case of the conforming FE space, we can obtain (2.24) by directly using the Gagliardo–Nirenberg inequality [40, 41]. When $V_h = V_h^{NC}$ that means $V_h \subset H_0^1(U)$, the Gagliardo–Nirenberg inequality is not applicable straightforwardly. We introduce a conforming companion operator $\wedge_h v_h \in H_0^1(U)$ on $\mathscr{T}_h$, where $v_h \in V_h^{NC}$. It holds that ([42], Lem. 3.3)

$$\|v_h - \wedge_h v_h\|_{L^2} + h\|v_h - \wedge_h v_h\|_{1,h} \le Ch\|v_h\|_{1,h}. \tag{2.25}$$

Then, by virtue of the inverse inequality and (2.25), and for $\wedge_h v_h \in H_0^1(U)$, we have

$$\begin{aligned}
\|v_h\|_{L^4} &\le \|v_h - \wedge_h v_h\|_{L^4} + \|\wedge_h v_h\|_{L^4}\\
&\le Ch^{-\frac{1}{2}}\|v_h - \wedge_h v_h\|_{L^2} + \|\wedge_h v_h\|_{L^4}\\
&= Ch^{-\frac{1}{2}}\sqrt{\|v_h - \wedge_h v_h\|_{L^2}}\sqrt{\|v_h - \wedge_h v_h\|_{L^2}} + \|\wedge_h v_h\|_{L^4}\\
&\le Ch^{-\frac{1}{2}}\sqrt{h\|v_h\|_{1,h}}\sqrt{\|v_h\|_{L^2}} + C\|v_h\|_{L^2}^{\frac{1}{2}}\|v_h\|_{1,h}^{\frac{1}{2}}\\
&\le C_{gn}\|v_h\|_{L^2}^{\frac{1}{2}}\|v_h\|_{1,h}^{\frac{1}{2}},
\end{aligned}$$

where we have used

$$\|\wedge_h v_h\|_{L^2} \le \|v_h - \wedge_h v_h\|_{L^2} + \|v_h\|_{L^2} \le Ch\|v_h\|_{1,h} + \|v_h\|_{L^2} \le C\|v_h\|_{L^2} + \|v_h\|_{L^2} \le C\|v_h\|_{L^2}, \tag{2.26}$$

$$\|\wedge_h v_h\|_{1,h} \le \|v_h - \wedge_h v_h\|_{1,h} + \|v_h\|_{1,h} \le C\|v_h\|_{1,h} + \|v_h\|_{1,h} \le C\|v_h\|_{1,h}. \tag{2.27}$$

We have completed the proof. ☐

Theorem 2.5 (Boundedness)**.** *Assume that one of the following conditions holds,*

(a) $\beta \ge 0$,
(b) $\beta < 0$ *and* $\frac{1}{4} + \frac{\beta(C_{gn})^2}{2}\|u_{h,\tau}^0\|_{L^2}^2 \ge (C_0)^{-1} > 0$.

Then, it follows that

$$\|u_{h,\tau}^n\|_{L^2} \le M_1, \quad \|u_{h,\tau}^n\|_{1,h} \le M_2, \quad 1 \le n \le N, \tag{2.28}$$

where M_1 and M_2 are positive constants.

Proof. The L^2-norm boundedness of $u_{h,\tau}^n$ can be directly derived by the mass conservation (2.15). To prove the boundedness of $\|u_{h,\tau}^n\|_{1,h}$, we consider the following two cases.

If $\beta \ge 0$, from the conservation property (2.15), we can directly obtain

$$\begin{aligned}
\frac{1}{2}\|u_{h,\tau}^n\|_{1,h}^2 + \big(Vu_{h,\tau}^n, u_{h,\tau}^n\big) &\le E_h^0 + \Omega\mathrm{Im}\big(x\partial_y u_{h,\tau}^n - y\partial_x u_{h,\tau}^n, u_{h,\tau}^n\big)\\
&\le |E_h^0| + C|\Omega|\big(\|\partial_y u_{h,\tau}^n\|_{L^2}\|u_{h,\tau}^n\|_{L^2} + \|\partial_x u_{h,\tau}^n\|_{L^2}\|u_{h,\tau}^n\|_{L^2}\big)\\
&\le |E_h^0| + C|\Omega|\|u_{h,\tau}^n\|_h\|u_{h,\tau}^n\|_{L^2} = |E^0| + C|\Omega|\|u_{h,\tau}^n\|_h\|u_{h,\tau}^0\|_{L^2}\\
&\le |E_h^0| + \frac{1}{4}\|u_{h,\tau}^n\|_{1,h}^2 + C\|u_{h,\tau}^0\|_{L^2}^2,
\end{aligned} \tag{2.29}$$

which implies the boundedness of $\|u_{h,\tau}^n\|_{1,h}$.

If $\beta < 0$, from Lemma 2.4 and similar as (2.29), we have

$$\begin{aligned}\frac{1}{2}\|u_{h,\tau}^n\|_{1,h}^2 + \left(Vu_{h,\tau}^n, u_{h,\tau}^n\right) &= E_h^0 - \frac{\beta}{2}\|u_{h,\tau}^n\|_{L^4}^4 + \Omega \mathrm{Im}\left(x\partial_y u_{h,\tau}^n - y\partial_x u_{h,\tau}^n, u_{h,\tau}^n\right)\\ &\le |E_h^0| - \frac{\beta(C_{gn})^2}{2}\|u_{h,\tau}^n\|_{L^2}^2\|u_{h,\tau}^n\|_h^2 + C|\Omega|\big(\|\partial_y u_{h,\tau}^n\|_{L^2}\|u_{h,\tau}^n\|_{L^2}\\ &\quad + \|\partial_x u_{h,\tau}^n\|_{L^2}\|u_{h,\tau}^n\|_{L^2}\big)\\ &\le |E_h^0| - \frac{\beta(C_{gn})^2}{2}\|u_{h,\tau}^0\|_{L^2}^2\|u_{h,\tau}^n\|_h^2 + C|\Omega|\|u_{h,\tau}^n\|_h\|u_{h,\tau}^0\|_{L^2}\\ &\le |E_h^0| + \left(\frac{1}{4} - \frac{\beta(C_{gn})^2}{2}\|u_{h,\tau}^0\|_{L^2}^2\right)\|u_{h,\tau}^n\|_{1,h}^2 + C\|u_{h,\tau}^0\|_{L^2}^2, \end{aligned} \tag{2.30}$$

which further gives that

$$\left(\frac{1}{4} + \frac{\beta(C_{gn})^2}{2}\|u_{h,\tau}^0\|_{L^2}^2\right)\|u_{h,\tau}^n\|_{1,h}^2 \le |E_h^0| + C\|u_{h,\tau}^0\|_{L^2}^2. \tag{2.31}$$

Thus from (2.31), if the condition (b) holds, we can derive

$$\|u_{h,\tau}^n\|_{1,h}^2 \le C_0|E_h^0| + CC_0\|u_{h,\tau}^0\|_{L^2}^2, \tag{2.32}$$

which means the boundedness of $u_{h,\tau}^n$ in the sense of piecewise H^1-norm. □

Remark 2.6. If $\Omega = 0$, the model (1.1) simplifies to the classical NLS. Even in this special case, we have expanded upon the findings presented in reference [24]. Notably, Henning and Peterseim [24] exclusively addresses the scenario of $\beta \ge 0$ and within the confines of conforming FEM. From this point of view, our extension covers a broader range of cases. Under specific conditions on the initial values, the boundedness of the numerical solutions $u_{h,\tau}^n$ can be directly derived from Theorem 2.5, *i.e.*, if the condition (a) or (b) holds, we have $\|u_{h,\tau}^n\|_{1,h} \le M_2$. Hence, taking the nonlinear term $f(u) = |u|^2u$ as example, using the Gagliardo–Nirenberg inequality, we can obtain

$$\|u^n - u_{h,\tau}^n\|_{L^4}^4 \le C_{gn}\|u^n - u_{h,\tau}^n\|_{L^2}^2\|u^n - u_{h,\tau}^n\|_{1,h}^2 \le C\|u^n - u_{h,\tau}^n\|_{L^2}^2. \tag{2.33}$$

By using above inequality, as the method given in [24], we can also obtain the optimal convergence without any coupling conditions between the time step size and the spatial mesh size. But, we cannot say the result is unconditional, since it requires (a) or (b) holds. Indeed, the time-space error splitting method from [24] can be directly used to analyze unconditional convergence without the requirement of (a) and (b).

Remark 2.7. Compared with references [24, 32], we in this work consider a more complex rotating Gross–Pitaevskii equation. By cleverly constructing numerical schemes, we are able to ensure both the mass and energy conservations for *conforming and nonconforming* FEMs. In fact, the presence of the rotation term poses inherent challenges to the conservation properties of nonconforming FEMs. Traditional approaches fail to achieve both the mass and energy conservation in this case.

Remark 2.8. In what follows, we will provide a more comprehensive framework for the unconditional error analysis of the FE schemes, removing any restrictions on the parameter β and eliminating the need for specific requirements on the initial values. In this context, the term "unconditional" takes on a more general connotation, as we relax the continuity requirements on the FE spaces, impose no limitations on the parameter and initial value, and also remove the classical restrictions of the time-space mesh ratio.

3. Main results

In this section, we present the main convergence results, including optimal and high-order error estimates, and the proof will be given in the subsequent section. We in this work assume that the solution to (1.1) uniquely exists and satisfies that

$$\|u^0\|_{H^2} + \|u\|_{L^\infty((0,T);H^2)} + \|u_t\|_{L^2((0,T);H^2)} + \|u_{tt}\|_{L^2((0,T);H^4)} + \|u_{ttt}\|_{L^2((0,T);H^2)} \leq C_u. \tag{3.1}$$

In the subsequent analysis, for the sake of simplicity, we assume that $\tau_n = \tau$.

Theorem 3.1. *Let $u^n_{h,\tau} \in V_h$ be the unique solution of the fully discrete scheme (2.13). Then, under the regularity assumption (3.1), there hold*

(a) *(the L^∞-norm boundedness)*

$$\sup_{0\leq n\leq N} \|u^n_{h,\tau}\|_{L^\infty} \leq M, \tag{3.2}$$

(b) *(the optimal L^2-norm error estimate)*

$$\sup_{0\leq n\leq N} \|u^n - u^n_{h,\tau}\|_{L^2} \leq C(h^2+\tau^2), \tag{3.3}$$

(c) *(the optimal H^1-norm error estimate)*

$$\sup_{0\leq n\leq N} \|u^n - u^n_{h,\tau}\|_{1,h} \leq C(h+\tau^2), \tag{3.4}$$

(d) *(the high-order H^1-norm error estimates)*

$$\sup_{0\leq n\leq N} \|I_h u^n - u^n_{h,\tau}\|_{1,h} \leq C(h^2+\tau^2), \tag{3.5}$$

$$\sup_{0\leq n\leq N} \|u^n - I_{2h} u^n_{h,\tau}\|_{1,h} \leq C(h^2+\tau^2), \tag{3.6}$$

where $C>0$ is a constant independent of h and τ and I_h is the interpolation operator satisfying (2.12).

The proof of Theorem 3.1 will be given in the following section.

4. Error analysis for the FEM

In this section, we provide an elaborate proof of Theorem 3.1.

4.1. Error analysis for the time-discrete method

In this subsection, our focus is on investigating the convergence and boundedness of the solution in the context of the time-discrete method (2.1). To this end, we first consider a truncated auxiliary problem of the time-discrete system.

4.1.1. A truncated auxiliary problem

Under the regularity assumption (3.1), we define

$$K_0 := \sup_{M\in\mathbb{N}} \left\{ \max_{0\leq m\leq M} \|u^m\|_{L^\infty} \right\} + 1. \tag{4.1}$$

The following truncated function plays a pivotal role in convergence analysis.

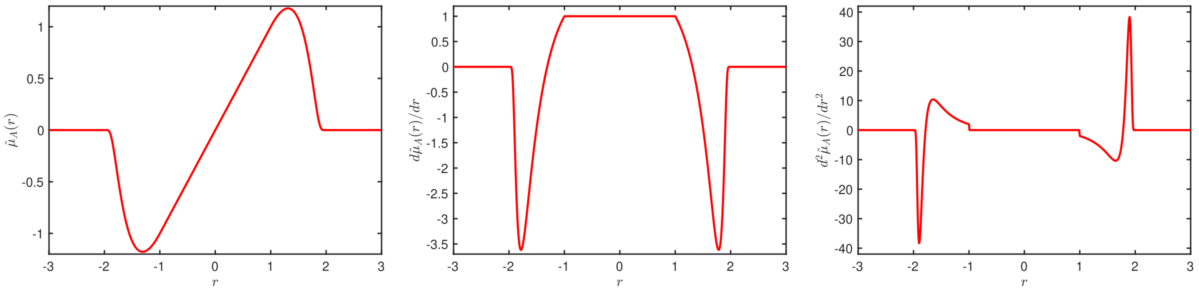

FIGURE 1. Figures of the functions $\hat{\mu}_A(r)$, $\mathrm{d}\hat{\mu}_A(r)/\mathrm{d}r$ and $\mathrm{d}^2\hat{\mu}_A(r)/\mathrm{d}r^2$.

Definition 4.1 (Truncated function)**.** Define the truncated function as

$$\mu_A(s) = s \cdot \chi(s \cdot K_0^{-2}), \tag{4.2}$$

where

$$\chi(x) = \begin{cases} 0, & |x| \in [2, +\infty), \\ \exp\Big(1 + \frac{1}{(|x|-1)^2-1}\Big), & |x| \in [1, 2), \\ 1, & |x| \in [0, 1). \end{cases} \tag{4.3}$$

The truncated function $\mu_A(\cdot)$ exhibits the following characteristics.

Lemma 4.1. *The truncated function $\mu_A(s)$ belongs to $C^\infty(\mathbb{R})$, and there hold*

$$\|\mu_A(w)\|_{L^\infty} \le C_M, \quad |\mu_A(w_1) - \mu_A(w_2)| \le C_\mu |w_1 - w_2|, \tag{4.4a}$$

$$\big|\mu_A(|w_1|^2) - \mu_A(|w_2|^2)\big| \le C_\mu |w_1 - w_2|, \tag{4.4b}$$

where C_M and C_μ are positive and bounded constants.

Proof. Based on the simple elementary properties of functions, we can directly obtain the conclusion of this lemma. □

Remark 4.2. Let $r = s \cdot K_0^{-2}$ in (4.2), then

$$\mu_A = K_0^2 r \cdot \chi(r) =: K_0^2\, \hat{\mu}_A(r).$$

To study the properties (4.4) of the function $\mu_A(s)$, we only need to consider the function $\hat{\mu}_A(r)$. For giving a vivid description, we plot the figures of $\hat{\mu}_A(r)$, $\mathrm{d}\hat{\mu}_A(r)/\mathrm{d}r$ and $\mathrm{d}^2\hat{\mu}_A(r)/\mathrm{d}r^2$ in Figure 1. As depicted in Figure 1, we can clearly observe the boundedness of the function $\hat{\mu}_A(r)$ and its first-order and second-order derivatives.

By using the truncated function $\mu_A(\cdot)$, we introduce the following truncated time-discrete scheme.

Definition 4.2 (Time-discrete method with truncation)**.** Let $u_\tau^{T,0} := u^0$. Then for $n \ge 1$, we define the truncated time-discrete method, to find $u_\tau^{T,n+1} \in H_0^1(U)$ as the solution to

$$iD_\tau u_\tau^{T,n+\frac{1}{2}} = -\frac{1}{2}\Delta \hat{u}_\tau^{T,n+\frac{1}{2}} + V\hat{u}_\tau^{T,n+\frac{1}{2}} - \Omega L_z \hat{u}_\tau^{T,n+\frac{1}{2}} + \beta \frac{\mu_A\big(|u_\tau^{T,n}|^2\big) + \mu_A\big(|u_\tau^{T,n+1}|^2\big)}{2} \hat{u}_\tau^{T,n+\frac{1}{2}}. \tag{4.5}$$

4.1.2. Existence of truncated time-discrete method

To begin our investigation into the properties of solutions to the truncated system (4.5), it is crucial to demonstrate that there is at least one solution. To accomplish this, we can rely on the following Brouwer fixed point theorem.

Lemma 4.3 (Brouwer Fixed Point Theorem)**.** *Denote* $\overline{S_1(0)} = \{\boldsymbol{\alpha} \in \mathbb{C}^N; |\boldsymbol{\alpha}| \leq 1\}$ *as an unit disk in* $\mathbb{C}^N$*. Then, every continuous function* $g : \mathbb{C}^N \to \mathbb{C}^N$ *with* $\mathrm{Re}(g(\boldsymbol{\alpha}), \boldsymbol{\alpha}) \geq 0$ *with* $\boldsymbol{\alpha} \in \partial S_1(0)$ *has a zero in* $\overline{S_1(0)}$*, i.e., a point* $\boldsymbol{\alpha}_0 \in \overline{S_1(0)}$ *such that* $g(\boldsymbol{\alpha}_0) = 0$.

In addition, we will utilize the following lemma, which corresponds to a specific scenario of the Vitali convergence theorem [43].

Lemma 4.4 (Vitali Convergence Theorem)**.** *A sequence* $(g_k)_{k\in\mathbb{N}} \subset L^2(U)$ *convergences strongly to the function* $g \in L^2(U)$ *if and only if*

(1) $(g_k)_{k\in\mathbb{N}}$ *locally converges to* g *in measure;*
(2) $(g_k)_{k\in\mathbb{N}}$ *is 2-equi-integrable, which means that for every* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *such that* $\|g_k\|_{L^2(S)} < \varepsilon$ *for any measurable subset* $S \subset U$ *with the measure* $\mu(S) \leq \delta_\varepsilon$.

We prove the truncated time-discrete system (4.5) has at least one solution in $H_0^1(U)$. The system (4.5) is indeed equivalent to

$$\hat{u}_\tau^{T,n+\frac{1}{2}} - u_\tau^{T,n} - \frac{\Omega\tau}{2}[x\partial_y - y\partial_x]\hat{u}_\tau^{T,n+\frac{1}{2}} + \tau i\Phi\Big(\hat{u}_\tau^{T,n+\frac{1}{2}}\Big) = 0, \tag{4.6}$$

where $\Phi(\hat{u}_\tau^{T,n+\frac{1}{2}}) = \Phi(w)\big|_{w=\hat{u}_\tau^{T,n+\frac{1}{2}}}$ with

$$\Phi(w) := -\frac{1}{4}\Delta w + \frac{1}{2}Vw + \frac{\beta}{4}\big[\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|2w - u_\tau^{T,n}|^2)\big]w.$$

Lemma 4.5. *There exists at least one solution* $u_\tau^{M,n} \in H_0^1(U)$*,* $n \geq 1$ *for the truncated system* (4.5).

Proof. We only need to prove the existence of the solution to

$$\mathcal{G}(w) := w - u_\tau^{T,n} - \frac{\Omega\tau}{2}[x\partial_y - y\partial_x]w + \tau i\Phi(w) = 0, \quad w \in H_0^1(U). \tag{4.7}$$

The proof will be built in the following three steps.

Step 1. We first show the existence in a finite-dimensional subspace

$$\mathscr{X}_N := \{\phi_m; m \in \mathbb{N}\}, \tag{4.8}$$

which denotes a countable basis of the space $H_0^1(U)$. For given $u_{h,\tau}^{T,n} \in H_0^1(U)$, we want to check whether there exists $X_N \in \mathscr{X}_N$, such that

$$\mathcal{G}(X_N) = X_N - u_\tau^{T,n} - \frac{\Omega\tau}{2}[x\partial_y - y\partial_x]X_N + \tau i\Phi(X_N) = 0. \tag{4.9}$$

Taking the inner product of (4.9) with X_N gives that

$$(\mathcal{G}(X_N), X_N) = \|X_N\|_{L^2}^2 - (u_\tau^{T,n}, X_N) - \frac{\Omega\tau}{2}([x\partial_y - y\partial_x]X_N, X_N) + \tau i(\Phi(X_N), X_N) = 0. \tag{4.10}$$

It is obvious that

$$\operatorname{Im}(\Phi(X_N), X_N) = 0. \tag{4.11}$$

Additionally, since

$$([x\partial_y - y\partial_x]X_N, X_N) = -(X_N, [x\partial_y - y\partial_x]X_N), \tag{4.12}$$

we have

$$\operatorname{Re}([x\partial_y - y\partial_x]X_N, X_N) = 0. \tag{4.13}$$

Extracting the real part of (4.10), and using (4.11) and (4.13), we get

$$\begin{aligned}\operatorname{Re}(\mathcal{G}(X_N), X_N) &= \|X_N\|_{L^2}^2 - \operatorname{Re}(u_\tau^{T,n}, X_N) \geq \|X_N\|_{L^2}^2 - \|u_\tau^{T,n}\|_{L^2}\|X_N\|_{L^2} \\ &= \big(\|X_N\|_{L^2} - \|u_\tau^{T,0}\|_{L^2}\big)\|X_N\|_{L^2},\end{aligned} \tag{4.14}$$

where the last equality is due to the mass conservation that can be derived similar as (2.1). Take sufficiently large $\|X_N\|_{L^2}$ such $\operatorname{Re}(\mathcal{G}(X_N), X_N) \geq 0$, then the existence of the solutions to (4.9), provided that $u_\tau^{T,n}$ exists. Therefore, the solution $Z_N = 2X_N - u_\tau^{T,n}$ exists, and $Z_N \in \mathscr{X}_N$ is indeed the solution to

$$\begin{aligned} i\frac{Z_N - u_\tau^{T,n}}{\tau_n} &= -\frac{\Delta Z_N + \Delta u_\tau^{T,n}}{4} + V\frac{Z_N + u_\tau^{T,n}}{2} \\ &\quad - \Omega\frac{L_z Z_N + L_z u_\tau^{T,n}}{2} + \beta\frac{\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|Z_N|^2)}{2}\frac{Z_N + u_\tau^{T,n}}{2}. \end{aligned} \tag{4.15}$$

Step 2. We build the estimate of (4.15) and further demonstrate the boundedness of Z_N under the H^2-norm. The corresponding results are shown as follows:

$$\big\|Z_N - u_\tau^{T,n+1}\big\|_{H^1} + \tau^2\big\|Z_N - u_\tau^{T,n+1}\big\|_{H^2} \leq C_Z\tau^2, \tag{4.16}$$

where $C_Z > 0$ is a constant independent of τ. This implies the boundedness of Z_N in the H^2-norm, which, in turn, guarantees its boundedness in the L^∞-norm by employing the Sobolev embedding inequality. The estimate (4.16) follows a similar approach as the error analysis of the time-discrete system in the next subsection, obviating the need to present its proof here.

Step 3. Thanks to the boundedness of of Z_N in H^2-norm, and applying the Vitali Convergence Theorem shown in Lemma 4.4, we can derive iteratively the existence of a solution $u_\tau^{T,n+1} \in H_0^1(U)$ to the truncated system (4.5). Here, the nonlinear part in (4.15) strongly converges to the corresponding nonlinear term in (4.5) using Vitalis theorem, as demonstrated in a similar proof in [24].

□

4.1.3. Uniform L^∞-boundedness of the truncated approximation

Define $e_\tau^{T,n} = u^n - u_\tau^{T,n}$ for $n \geq 0$. Then, from (4.5) and (1.1), we have

$$iD_\tau e_\tau^{T,n+\frac12} = -\frac12\Delta\hat e_\tau^{T,n+\frac12} + V\hat e_\tau^{T,n+\frac12} - \Omega L_z\hat e_\tau^{T,n+\frac12} + \mathcal{N}_\tau^{T,n+\frac12} + \mathcal{R}_\tau^{T,n+\frac12}, \tag{4.17}$$

where

$$\mathcal{N}_\tau^{T,n+\frac12} = \beta\left[\frac{|u^n|^2 + |u^{n+1}|^2}{2}\hat u^{n+\frac12} - \frac{\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|u_\tau^{T,n+1}|^2)}{2}\hat u_\tau^{T,n+\frac12}\right], \tag{4.18}$$

and

$$\mathcal{R}_\tau^{T,n+\frac{1}{2}} = i\Big[D_\tau u^{n+\frac{1}{2}} - \partial_t u^{n+\frac{1}{2}}\Big] + \frac{1}{2}\Big[\Delta \hat{u}^{n+\frac{1}{2}} - \Delta u^{n+\frac{1}{2}}\Big] + \Omega\Big[L_z \hat{u}^{n+\frac{1}{2}} - L_z u^{n+\frac{1}{2}}\Big] + \beta\Big[|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}} - \frac{|u^n|^2 + |u^{n+1}|^2}{2}\hat{u}^{n+\frac{1}{2}}\Big]. \tag{4.19}$$

Based on the regularity assumption (3.1), we readily derive

$$|\mathcal{R}_\tau^{T,n+\frac{1}{2}}|_{H^2} \le C_R \tau^2, \tag{4.20}$$

where C_R is a positive constant independent of τ. Moreover, utilizing the definition of the truncated function in (4.2), we obtain

$$\begin{aligned}\mathcal{N}_\tau^{T,n+\frac{1}{2}} &= \beta\Big[\frac{|u^n|^2 + |u^{n+1}|^2}{2}\hat{u}^{n+\frac{1}{2}} - \frac{\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|u_\tau^{T,n+1}|^2)}{2}\hat{u}_\tau^{T,n+\frac{1}{2}}\Big] \\ &= \beta\Big[\frac{\mu_A(|u^n|^2) + \mu_A(|u^{n+1}|^2)}{2}\hat{u}^{n+\frac{1}{2}} - \frac{\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|u_\tau^{T,n+1}|^2)}{2}\hat{u}_\tau^{T,n+\frac{1}{2}}\Big] \\ &= \beta\Big[\frac{\mu_A(|u^n|^2) + \mu_A(|u^{n+1}|^2)}{2} - \frac{\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|u_\tau^{T,n+1}|^2)}{2}\Big]\hat{u}^{n+\frac{1}{2}} \\ &\quad + \beta\frac{\mu_A(|u_\tau^{T,n}|^2) + \mu_A(|u_\tau^{T,n+1}|^2)}{2}\hat{e}^{n+\frac{1}{2}}.\end{aligned} \tag{4.21}$$

Then, using (4.4) in (4.21) gives

$$\mathcal{N}_\tau^{T,n+\frac{1}{2}} \le \frac{|\beta|(K_0-1)}{2}C_\mu\big(|e_\tau^{T,n}| + |e_\tau^{T,n+1}|\big) + \frac{|\beta|}{2}C_M\big(|e_\tau^{T,n}| + |e_\tau^{T,n+1}|\big),$$

which implies that

$$|\mathcal{N}_\tau^{T,n+\frac{1}{2}}|_{H^s} \le C_{\mathcal{N},\tau}\big(|e_\tau^{T,n}|_{H^s} + |e_\tau^{T,n+1}|_{H^s}\big), \quad s = 0, 1, \tag{4.22}$$

where $C_{\mathcal{N},\tau}$ is a positive constant independent of h and τ.

Taking the inner product of the error equation (4.17) with $\hat{e}_\tau^{T,n+\frac{1}{2}}$ and $\Delta\hat{e}_\tau^{T,n+\frac{1}{2}}$ respectively, and selecting the imaginary part of the resulting, we can easily derive

$$\|e_\tau^{T,n+1}\|_{H^1} \le C_{E,\tau}\tau^2, \tag{4.23}$$

provided that $\tau \le \tau_1$ with a positive constant τ_1. Obviously, we have $\|D_\tau e_\tau^{T,n+\frac{1}{2}}\|_{L^2} \le 2C_{E,\tau}\tau$ and $\|\Delta\hat{e}_\tau^{T,n+\frac{1}{2}}\|_{L^2} \le C_{E,\tau}\tau$. Thus, we have $\|\Delta e_\tau^{T,n+1}\|_{L^2} \le CC_{E,\tau}$. Hence, it holds

$$\|e_\tau^{T,n+1}\|_{H^2} = \big(\|e_\tau^{T,n+1}\|_{L^2}^2 + \|\nabla e_\tau^{T,n+1}\|_{L^2}^2 + \|\Delta e_\tau^{T,n+1}\|_{L^2}^2\big)^{\frac{1}{2}} \le CC_{E,\tau}. \tag{4.24}$$

By virtue of the Gagliardo–Nirenberg inequality, we obtain

$$\|e_\tau^{T,n+1}\|_{L^\infty} \le C\|e_\tau^{T,n+1}\|_{H^2}^{\frac{3}{4}}\|e_\tau^{T,n+1}\|_{L^2}^{\frac{1}{4}} + C\|e_\tau^{T,n+1}\|_{L^2} \le CC_{E,\tau}\tau^{\frac{1}{2}} \le \tau^{\frac{1}{4}}. \tag{4.25}$$

Here, we reduce the order of convergence to eliminate the constant $CC_{E,\tau}$, thereby avoiding the possible accumulation of constants later on. From (4.25), if $\tau \le 1$, it follows that

$$\|u_\tau^{T,n+1}\|_{H^2} \le \|e_\tau^{T,n+1}\|_{H^2} + \|u^{n+1}\|_{H^2} \le C_{E,\tau} + C_u \le C_{u,\tau}, \tag{4.26}$$

$$\|u_\tau^{T,n+1}\|_{L^\infty} \le \|e_\tau^{T,n+1}\|_{L^\infty} + \|u^{n+1}\|_{L^\infty} \le K_0, \tag{4.27}$$

with a positive constant $C_{u,\tau}$.

The proof for the uniform L^∞-boundedness of the truncated approximation is now concluded. Utilizing the definition of the function $\mu_A(\cdot)$ in (4.2), we establish the relationship:

$$\mu_A(u_\tau^{T,n}) = u_\tau^{T,n}, \quad \text{for all } n \ge 0. \tag{4.28}$$

This implies that the time-truncated system (4.5) is indeed equivalent to the time-discrete method (2.1). Consequently, the L^∞-norm boundedness of u_τ^n naturally follows.

4.1.4. Convergence and boundedness of the time-discrete method

From above analysis, we have derived some theoretical results of the time-discrete method, which are given in the following lemma.

Lemma 4.6. *Assume u be the solution of the model (1.1) satisfying the regularized condition (3.1). Then the time-discrete system (2.1) has the unique solution u_τ^n. Meanwhile, there exists $\tau_0 = \min\{\tau_1, 1\} > 0$ such that when $\tau \le \tau_0$, it holds*

$$\sup_{0\le n\le N} \|u_\tau^n\|_{H^2} \le C_{u,\tau}, \qquad \sup_{0\le n\le N} \|u_\tau^n\|_{L^\infty} \le K_0, \tag{4.29}$$

and

$$\sup_{0\le n\le N} \|u^n - u_\tau^n\|_{H^1} + \tau^2 \sup_{0\le n\le N} \|u^n - u_\tau^n\|_{H^2} \le C_{E,\tau}\tau^2, \tag{4.30}$$

holds true, where $C_{u,\tau}$, K_0 and $C_{E,\tau}$ are positive constants independent of τ.

Proof. Since $u_\tau^{T,n} = u_\tau^n$, we can prove the existence of the solution u_τ^n directly by the existence of the truncated solution $u_\tau^{T,n}$. It remains to prove the uniqueness of the solution u_τ^n. Let $u_\tau^{\mathrm{I},n+1}$ and $u_\tau^{\mathrm{II},n+1}$ are two solutions to the time-discrete system (2.1), and denote $v^{n+1} = u_\tau^{\mathrm{I},n+1} - u_\tau^{\mathrm{II},n+1}$ and $v^n = 0$. Similar as the error analysis in Section 4.1.3, we can easily obtain

$$\|v^{n+1}\|_{L^2}^2 \le C\tau \|v^{n+1}\|_{L^2}^2, \tag{4.31}$$

with implies that $v^{n+1} = 0$ if τ is selected sufficiently small. Therefore, the unique existence of the solution to the time-discrete system (2.1) has been demonstrated. In addition, the results (4.29) and (4.30) can be directly obtained by the analysis in Section 4.1.3. □

We can also enhance the rate of convergence of $\sup_{0\le n\le N}\|u_\tau^n - u^n\|_{H^2}$ to $O(\tau^2)$. In the next subsection, we will demonstrate the necessity of achieving a higher-order convergence rate.

Lemma 4.7. *Under the conditions of Lemma 4.6, there holds*

$$\sup_{0\le n\le N} \|u^n - u_\tau^n\|_{H^2} \le C_{E,\tau}\tau^2, \tag{4.32}$$

provided $\tau \le \tau_0^*$ with a positive constant τ_0^*, where $C_{E,\tau}$ denotes a positive constant independent of τ.

Proof. Define $e_\tau^n = u^n - u_\tau^n$ for $n \ge 0$. From (2.1) and (1.1), we have

$$iD_\tau e_\tau^{n+\frac{1}{2}} = -\frac{1}{2}\Delta \hat{e}_\tau^{n+\frac{1}{2}} + V\hat{e}_\tau^{n+\frac{1}{2}} - \Omega L_z \hat{e}_\tau^{n+\frac{1}{2}} + \mathcal{N}_\tau^{n+\frac{1}{2}} + \mathcal{R}_\tau^{n+\frac{1}{2}}, \tag{4.33}$$

where $R_\tau^{n+\frac12} = R_\tau^{T,n+\frac12}$ and

$$\mathcal{N}_\tau^{n+\frac12} = \beta\Big[\frac{|u^n|^2+|u^{n+1}|^2}{2}\hat{u}^{n+\frac12} - \frac{|u_\tau^n|^2+|u_\tau^{n+1}|^2}{2}\hat{u}_\tau^{n+\frac12}\Big]. \tag{4.34}$$

Taking the inner product of (4.33) with $D_\tau\Delta e_\tau^{n+\frac12}$, and then selecting the real parts, we get

$$\begin{aligned}\frac{\|\Delta e_\tau^{n+1}\|_{L^2}^2 - \|\Delta e_\tau^n\|_{L^2}^2}{4\tau} &= \mathrm{Re}\Big(V\hat{e}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) - \Omega\mathrm{Re}\Big(L_z\hat{e}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big)\\ &\quad + \mathrm{Re}\Big(\mathcal{N}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) + \mathrm{Re}\Big(\mathcal{R}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big).\end{aligned} \tag{4.35}$$

We next analyze the four terms on the right-hand side of (4.34). Firstly, we have

$$\Big(V\hat{e}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) = \frac{\Big(V\hat{e}_\tau^{n+\frac12}, \Delta e_\tau^{n+1}\Big) - \Big(V\hat{e}_\tau^{n-\frac12}, \Delta e_\tau^{n}\Big)}{\tau} - \frac12\Big(V\Big[D_\tau e_\tau^{n+\frac12} + D_\tau e_\tau^{n-\frac12}\Big], \Delta e_\tau^n\Big), \tag{4.36}$$

$$\Big(\mathcal{N}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) = \frac{\Big(\mathcal{N}_\tau^{n+\frac12}, \Delta e_\tau^{n+1}\Big) - \Big(\mathcal{N}_\tau^{n-\frac12}, \Delta e_\tau^{n}\Big)}{\tau} - \left(\frac{\mathcal{N}_\tau^{n+\frac12} - \mathcal{N}_\tau^{n-\frac12}}{\tau}, \Delta e_\tau^n\right), \tag{4.37}$$

$$\Big(\mathcal{R}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) = \frac{\Big(\mathcal{R}_\tau^{n+\frac12}, \Delta e_\tau^{n+1}\Big) - \Big(\mathcal{R}_\tau^{n-\frac12}, \Delta e_\tau^{n}\Big)}{\tau} - \left(\frac{\mathcal{R}_\tau^{n+\frac12} - \mathcal{R}_\tau^{n-\frac12}}{\tau}, \Delta e_\tau^n\right). \tag{4.38}$$

Then, similar as (2.4), we have

$$\begin{aligned}-\Omega\mathrm{Re}\Big(L_z\hat{e}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) &= \Omega\mathrm{Re}\Big(i[x\partial_y - y\partial_x]\hat{e}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big)\\ &= -\Omega\mathrm{Im}\Big([x\partial_y - y\partial_x]\hat{e}_\tau^{n+\frac12}, \Delta D_\tau e_\tau^{n+\frac12}\Big) = \Omega\mathrm{Im}\Big(\nabla[x\partial_y - y\partial_x]\hat{e}_\tau^{n+\frac12}, \nabla D_\tau e_\tau^{n+\frac12}\Big)\\ &= \Omega\mathrm{Im}\Big([x\partial_y - y\partial_x]\nabla\hat{e}_\tau^{n+\frac12}, \nabla D_\tau e_\tau^{n+\frac12}\Big)\\ &= \frac{\Omega}{2\tau}\mathrm{Im}\big([x\partial_y - y\partial_x](\nabla e^{n+1} + \nabla e^n), \nabla e^{n+1} - \nabla e^n\big)\\ &= \frac{\Omega}{2\tau}\mathrm{Im}\big([x\partial_y - y\partial_x]\nabla e^{n+1}, \nabla e^{n+1}\big) - \frac{\Omega}{2\tau}\mathrm{Im}\big([x\partial_y - y\partial_x]\nabla e^{n+1}, \nabla e^{n}\big)\\ &\quad + \frac{\Omega}{2\tau}\mathrm{Im}\big([x\partial_y - y\partial_x]\nabla e^{n}, \nabla e^{n+1}\big) - \frac{\Omega}{2\tau}\mathrm{Im}([x\partial_y - y\partial_x]\nabla e^{n}, \nabla e^{n})\\ &= \frac{\Omega}{2\tau}\mathrm{Im}\big([x\partial_y - y\partial_x]\nabla e^{n+1}, \nabla e^{n+1}\big) - \frac{\Omega}{2\tau}\mathrm{Im}([x\partial_y - y\partial_x]\nabla e^{n}, \nabla e^{n})\\ &\quad + \frac{\Omega}{2\tau}\mathrm{Im}\Big\{\big(\nabla e^{n+1}, [x\partial_y - y\partial_x]\nabla e^n\big) + \big([x\partial_y - y\partial_x]\nabla e^n, \nabla e^{n+1}\big)\Big\}\\ &= \frac{\Omega}{2\tau}\mathrm{Im}\big([x\partial_y - y\partial_x]\nabla e^{n+1}, \nabla e^{n+1}\big) - \frac{\Omega}{2\tau}\mathrm{Im}([x\partial_y - y\partial_x]\nabla e^{n}, \nabla e^{n}).\end{aligned} \tag{4.39}$$

For convenience, we denote

$$\frac{\mathcal{N}_\tau^{n+\frac12} - \mathcal{N}_\tau^{n-\frac12}}{\tau} = \frac{\beta}{4} * \left[\frac{K_1^n - K_1^{n-1}}{\tau} + \frac{K_2^n - K_2^{n-1}}{\tau} + \frac{K_3^n - K_3^{n-1}}{\tau} + \frac{K_4^n - K_4^{n-1}}{\tau}\right], \tag{4.40}$$

where $K_1^n := |u^n|^2u^n - |u_\tau^n|^2u_\tau^n, K_2^n := |u^n|^2u^{n+1} - |u_\tau^n|^2u_\tau^{n+1}, K_3^n := |u^{n+1}|^2u^n - |u_\tau^{n+1}|^2u_\tau^n$ and $K_4^n := |u^{n+1}|^2u^{n+1} - |u_\tau^{n+1}|^2u_\tau^{n+1}$. We further have

$$\frac{K_1^n - K_1^{n-1}}{\tau} = \frac{\big[|u^n|^2e_\tau^n + (u^n\bar{e}_\tau^n + e_\tau^n\bar{u}_\tau^n)u_\tau^n\big] - \big[|u^{n-1}|^2e_\tau^{n-1} + (u^{n-1}\bar{e}_\tau^{n-1} + e_\tau^{n-1}\bar{u}_\tau^{n-1})u_\tau^{n-1}\big]}{\tau}$$

$$= \frac{|u^n|^2 e_\tau^n - |u^{n-1}|^2 e_\tau^{n-1}}{\tau} + \frac{u^n \bar{e}_\tau^n u_\tau^n - u^{n-1}\bar{e}_\tau^{n-1} u_\tau^{n-1}}{\tau} + \frac{e_\tau^n |u_\tau^n|^2 - e_\tau^{n-1}|u_\tau^{n-1}|^2}{\tau}$$
$$= \left(|u^n|^2 \frac{e_\tau^n - e_\tau^{n-1}}{\tau} + \frac{|u^n|^2 - |u^{n-1}|^2}{\tau} e_\tau^{n-1}\right)$$
$$+ \left(\frac{u^n - u^{n-1}}{\tau}\bar{e}_\tau^n u_\tau^n + u^{n-1}\frac{\bar{e}_\tau^n - \bar{e}_\tau^{n-1}}{\tau} u_\tau^n + u^{n-1}\bar{e}_\tau^{n-1}\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right)$$
$$+ \left(\frac{e_\tau^n - e_\tau^{n-1}}{\tau}(u_\tau^n)^2 + e_\tau^{n-1}\frac{\bar{u}_\tau^n - \bar{u}_\tau^{n-1}}{\tau} u_\tau^n + e_\tau^{n-1}\bar{u}_\tau^{n-1}\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right). \tag{4.41}$$

Then, by using (3.1) and the results given in Lemma 4.6, we derive

$$\left(\frac{K_1^n - K_1^{n-1}}{\tau}, \Delta e_\tau^n\right) \le C\Bigg(\left\|\frac{e_\tau^n - e_\tau^{n-1}}{\tau}\right\|_{L^2}\|\Delta e_\tau^n\|_{L^2} + \|e_\tau^n\|_{L^2}\|\Delta e_\tau^n\|_{L^2} + \left\|e_\tau^{n-1}\right\|_{L^2}\|\Delta e_\tau^n\|_{L^2}$$
$$+ \left\|e_\tau^{n-1}\right\|_{L^4}\left\|\frac{u_\tau^n - u_\tau^{n-1}}{\tau}\right\|_{L^4}\|\Delta e_\tau^n\|_{L^2}\Bigg)$$
$$\le C\left(\left\|\frac{e_\tau^n - e_\tau^{n-1}}{\tau}\right\|_{L^2}^2 + \|\Delta e_\tau^n\|_{L^2}^2 + \tau^4\right). \tag{4.42}$$

Similarly, we have

$$\left(\frac{K_2^n - K_2^{n-1}}{\tau} + \frac{K_3^n - K_3^{n-1}}{\tau} + \frac{K_4^n - K_4^{n-1}}{\tau}, \Delta e_\tau^n\right)$$
$$\le C\left(\left\|\frac{e_\tau^{n+1} - e_\tau^n}{\tau}\right\|_{L^2}^2 + \left\|\frac{e_\tau^n - e_\tau^{n-1}}{\tau}\right\|_{L^2}^2 + \|\Delta e_\tau^n\|_{L^2}^2 + \tau^4\right). \tag{4.43}$$

Hence, from (4.42) and (4.43), we get

$$\left(\frac{\mathcal{N}_\tau^{n+\frac12} - \mathcal{N}_\tau^{n-\frac12}}{\tau}, \Delta e_\tau^n\right) \le C_{\mathcal{N}}\left(\left\|\frac{e_\tau^{n+1} - e_\tau^n}{\tau}\right\|_{L^2}^2 + \left\|\frac{e_\tau^n - e_\tau^{n-1}}{\tau}\right\|_{L^2}^2 + \|\Delta e_\tau^n\|_{L^2}^2 + \tau^4\right). \tag{4.44}$$

In addition, using Taylor formulation, we have

$$\left\|\frac{\mathcal{R}_\tau^{n+\frac12} - \mathcal{R}_\tau^{n-\frac12}}{\tau}\right\|_{L^2} \le C_R\tau^2,$$

which implies that

$$\left(\frac{\mathcal{R}_\tau^{n+\frac12} - \mathcal{R}_\tau^{n-\frac12}}{\tau}, \Delta e_\tau^n\right) \le C\left(\|\Delta e_\tau^n\|_{L^2}^2 + \tau^4\right). \tag{4.45}$$

Therefore, using (4.36)–(4.45) in (4.35) gives

$$\frac{\|\Delta e_\tau^{n+1}\|_{L^2}^2 - \|\Delta e_\tau^n\|_{L^2}^2}{4\tau} \le \frac{J^{n+1} - J^n}{\tau} + C\left(\left\|\frac{e_\tau^{n+1} - e_\tau^n}{\tau}\right\|_{L^2}^2 + \left\|\frac{e_\tau^n - e_\tau^{n-1}}{\tau}\right\|_{L^2}^2 + \|\Delta e_\tau^n\|_{L^2}^2 + \tau^4\right), \tag{4.46}$$

where

$$J^n := \left(V\hat{e}_\tau^{n-\frac12}, \Delta e_\tau^n\right) + \frac{\Omega}{2}([x\partial y - y\partial x]\nabla e_\tau^n, \nabla e_\tau^n) + \left(\mathcal{N}_\tau^{n-\frac12}, \Delta e_\tau^n\right) + \left(\mathcal{R}_\tau^{n-\frac12}, \Delta e_\tau^n\right).$$

Replacing n by m in (4.46), and then summing the inequality from 2 to n, we get

$$\|\Delta e_\tau^{n+1}\|_{L^2}^2 \le \|\Delta e_\tau^0\|_{L^2}^2 + J^{n+1} + C\tau\sum_{m=0}^{n}\left\|\frac{e_\tau^{m+1}-e_\tau^m}{\tau}\right\|_{L^2}^2 + C\tau\sum_{m=0}^{n}\|\Delta e_\tau^m\|_{L^2}^2 + C\tau^4. \tag{4.47}$$

By virtue of Young's inequality, we have

$$J^{n+1} \le \varepsilon\|\Delta e_\tau^n\|_{L^2}^2 + C_\varepsilon\|\mathcal{N}_\tau^{n-\frac{1}{2}}\|_{L^2}^2 + C_\varepsilon\|\mathcal{R}_\tau^{n-\frac{1}{2}}\|_{L^2}^2 + C_\varepsilon\tau^4. \tag{4.48}$$

Since $\mathcal{N}_\tau^{n-\frac{1}{2}} = \mathcal{N}_\tau^{T,n-\frac{1}{2}}$ that satisfies (4.22), and $\mathcal{R}_\tau^{n-\frac{1}{2}} = \mathcal{R}_\tau^{T,n-\frac{1}{2}}$ that satisfies (4.20), we obtain from (4.48) and the known results given in Lemma 4.6 that

$$J^{n+1} \le \varepsilon\|\Delta e_\tau^n\|_{L^2}^2 + C_\varepsilon + C_\varepsilon\tau^4. \tag{4.49}$$

Selecting sufficient small ε, then combining (4.47) and (4.49) shows

$$\|\Delta e_\tau^{n+1}\|_{L^2}^2 \le C\tau\sum_{m=0}^{n}\left\|\frac{e_\tau^{m+1}-e_\tau^m}{\tau}\right\|_{L^2}^2 + C\tau\sum_{m=0}^{n}\|\Delta e_\tau^m\|_{L^2}^2 + C\tau^4. \tag{4.50}$$

Taking the difference of (4.33) between two consecutive steps, we obtain

$$\begin{aligned} i\frac{e_\tau^{n+1}-2e_\tau^n+e_\tau^{n-1}}{\tau^2} &= -\frac{\Delta e_\tau^{n+1}-\Delta e_\tau^{n-1}}{4\tau} + \frac{Ve_\tau^{n+1}-Ve_\tau^{n-1}}{2\tau} - \Omega\frac{L_z e_\tau^{n+1}-L_z e_\tau^{n-1}}{2\tau} \\ &\quad + \frac{\mathcal{N}_\tau^{n+\frac{1}{2}}-\mathcal{N}_\tau^{n-\frac{1}{2}}}{\tau} + \frac{\mathcal{R}_\tau^{T,n+\frac{1}{2}}-\mathcal{R}_\tau^{T,n-\frac{1}{2}}}{\tau}. \end{aligned} \tag{4.51}$$

For simplicity, we denote $e_\tau^{n+1} - e_\tau^n = \tau H_\tau^{n+1}$. Then we rewrite (4.51) as

$$iD_\tau H_\tau^{n+\frac{1}{2}} = -\frac{1}{2}\Delta\hat{H}_\tau^{n+\frac{1}{2}} + V\hat{H}_\tau^{n+\frac{1}{2}} - \Omega L_z\hat{H}_\tau^{n+\frac{1}{2}} + \frac{\mathcal{N}_\tau^{n+\frac{1}{2}}-\mathcal{N}_\tau^{n-\frac{1}{2}}}{\tau} + \frac{\mathcal{R}_\tau^{T,n+\frac{1}{2}}-\mathcal{R}_\tau^{T,n-\frac{1}{2}}}{\tau}. \tag{4.52}$$

Taking the inner product of (4.52) with $\hat{H}_\tau^{T,n+\frac{1}{2}}$, and selecting the imaginary part of the resulting, we have

$$\frac{\|H_\tau^{n+1}\|_{L^2}^2 - \|H_\tau^n\|_{L^2}^2}{2\tau} = \left(\frac{\mathcal{N}_\tau^{n+\frac{1}{2}}-\mathcal{N}_\tau^{n-\frac{1}{2}}}{\tau}, \hat{H}_\tau^{T,n+\frac{1}{2}}\right) + \left(\frac{\mathcal{R}_\tau^{T,n+\frac{1}{2}}-\mathcal{R}_\tau^{T,n-\frac{1}{2}}}{\tau}, \hat{H}_\tau^{T,n+\frac{1}{2}}\right). \tag{4.53}$$

Similar as (4.44) and (4.45), we can obtain

$$\left(\frac{\mathcal{N}_\tau^{n+\frac{1}{2}}-\mathcal{N}_\tau^{n-\frac{1}{2}}}{\tau}, \hat{H}_\tau^{T,n+\frac{1}{2}}\right) + \left(\frac{\mathcal{R}_\tau^{T,n+\frac{1}{2}}-\mathcal{R}_\tau^{T,n-\frac{1}{2}}}{\tau}, \hat{H}_\tau^{T,n+\frac{1}{2}}\right) \le C\Big(\left\|H_\tau^{n+1}\right\|_{L^2}^2 + \|H_\tau^n\|_{L^2}^2\Big) + C\tau^4. \tag{4.54}$$

From (4.53) and (4.54), and by using discrete Gronwall's inequality, there exists a positive constant τ_2 such that when $\tau \le \tau_2$,

$$\|H_\tau^{n+1}\|_{L^2} = \left\|\frac{e_\tau^{n+1}-e_\tau^n}{\tau}\right\|_{L^2} \le C_H\tau^2, \tag{4.55}$$

where $C_H > 0$ is a constant independent of τ.

Substituting (4.55) into (4.50), it is sufficient to prove that

$$\|\Delta e_\tau^{n+1}\|_{L^2} \le C_{E,\tau}\tau^2, \tag{4.56}$$

provided that $\tau < \tau_3$ with a positive constant τ_3. Setting $\tau_0^* = \min\{\tau_0, \tau_2, \tau_3\}$, then if $\tau \le \tau_0^*$, and thanks to the given results in Lemma 4.6, we finally obtain the high-order convergence result (4.32). $\square$

Remark 4.8. The idea of the proof is to first establish the L^∞-norm boundedness of the solution to the truncated time-discrete method. Subsequently, we discard the truncated function and exclusively investigate the H^2-norm convergence of the time-discrete system. In addition, although in the proof it is also existing a mild condition that τ should be sufficiently small, we can also remove this condition by using the conservation properties given in Theorem 2.1. But For simplicity, we do not intend to list its proof in the paper.

4.2. Spatial error analysis

In this subsection, we delve into the optimal and high-order error analysis of the fully discrete method outlined in Definition 2.2. As for the time-discrete method, we also take a detour over its auxiliary system. For simplicity, our primary focus remains on the conforming FEM, while it's noteworthy that a similar convergence analysis can be carried out for the nonconforming FEM, and a concise proof of the nonconforming case is provided in the appendix.

4.2.1. Fully discrete method with truncation

Definition 4.3 (Fully discrete method with truncation)**.** Let $u_{h,\tau}^{T,0}$ be a suitable interpolation of u^0. Then for $n \geq 1$, we define the following truncated fully discrete system, which is to find $u_{h,\tau}^{T,n+1} \in V_h^C$, $0 \leq n \leq N-1$ such that

$$i\Big(D_\tau u_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) = \frac{1}{2}\Big(\nabla \hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \nabla\omega_h\Big) + \Big(V\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Omega\Big(L_z \hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) + \beta\left(\frac{\mu_A\Big(|u_{h,\tau}^{T,n}|^2\Big) + \mu_A\Big(|u_{h,\tau}^{T,n+1}|^2\Big)}{2}\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\right), \tag{4.57}$$

for any $\omega_h \in V_h^C$.

The existence of the solution to the truncated system (4.57) is covered similar as the proof of truncated time-discrete method. Before obtaining the L^∞-norm boundedness of $u_{h,\tau}^n$ for $n \geq 1$, we should first consider the truncated system (4.57).

Lemma 4.9. *Suppose the regularity condition* (3.1) *holds, and denote* $u_{h,\tau}^{T,n} \in V_h$ *as a solution of the truncated fully discrete system* (4.57)*. Then there exists a positive constant* τ_0^{**}*, such that when* $\tau \leq \tau_0^{**}$*, it holds*

$$\sup_{0\leq n\leq N}\Big\|R_h u_\tau^n - u_{h,\tau}^{T,n}\Big\|_{L^2} \leq C_{E,h}(h^2 + h\tau^2 + h^2\tau), \tag{4.58}$$

where $C_{E,h}$ *is a positive constant independent of* h *and* τ*.*

Proof. For convenience, we denote

$$e_{h,\tau}^{T,n} = u_\tau^n - u_{h,\tau}^{T,n} = (u_\tau^n - R_h u_\tau^n) + \Big(R_h u_\tau^n - u_{h,\tau}^{T,n}\Big) =: \rho_{h,\tau}^{T,n} + \theta_{h,\tau}^{T,n}.$$

Subtracting (4.57) from the variational formulation of the time-discrete method (2.1) gives

$$i\Big(D_\tau e_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) = \frac{1}{2}\Big(\nabla \hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \nabla\omega_h\Big) + \Big(V\hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Omega\Big(L_z \hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) + \Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big), \quad \forall \omega_h \in V_h^C, \tag{4.59}$$

where

$$\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}} = \beta\left[\frac{|u_\tau^n|^2 + |u_\tau^{n+1}|^2}{2}\hat{u}_\tau^{n+\frac{1}{2}} - \frac{\mu_A(|u_{h,\tau}^{T,n}|^2) + \mu_A(|u_{h,\tau}^{T,n+1}|^2)}{2}\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}\right]. \tag{4.60}$$

Then, by using the definition of the projection R_h, (4.59) is equivalent to the following one

$$i\Big(D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big) = -i\Big(D_\tau\rho_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big) + \frac{1}{2}\Big(\nabla\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}},\nabla\omega_h\Big) + \Big(V\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big) + \Big(V\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big)$$
$$-\Omega\Big(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big) - \Omega\Big(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big) + \Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}},\omega_h\Big), \quad \forall\omega_h\in V_h^C. \tag{4.61}$$

Denoting $\omega_h = \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}$ in (4.61), and taking the imaginary part of the resulting, we have

$$\frac{\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|\theta_{h,\tau}^{T,n}\|_{L^2}^2}{2\tau} = -\mathrm{Re}\Big(D_\tau\rho_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) + \mathrm{Im}\Big(V\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) - \Omega\mathrm{Im}\Big(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)$$
$$-\Omega\mathrm{Im}\Big(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) + \mathrm{Im}\Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big). \tag{4.62}$$

From Lemma 4.7 and using (2.11), we obtain

$$\mathrm{Re}\Big(D_\tau\rho_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) \le \Big\|D_\tau\rho_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} \le C_{R_h}h^2\Big\|D_\tau u_\tau^{n+\frac{1}{2}}\Big\|_{H^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$= C_{R_h}h^2\Big\|D_\tau u^{n+\frac{1}{2}} - D_\tau e_\tau^{n+\frac{1}{2}}\Big\|_{H^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} \le C_{R_h}h^2\Big(\Big\|D_\tau u^{n+\frac{1}{2}}\Big\|_{H^2} + \Big\|D_\tau e_\tau^{n+\frac{1}{2}}\Big\|_{H^2}\Big)\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$\le C\Big(\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|\theta_{h,\tau}^{T,n}\|_{L^2}^2\Big) + C\big(h^4 + h^4\tau^2\big). \tag{4.63}$$

For the second term on the right-hand side of (4.62), it follows from Lemma 4.6 that

$$\mathrm{Im}\Big(V\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) \le \|V\|_{L^\infty}\Big\|\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$\le Ch^2\Big\|\hat{u}_\tau^{n+\frac{1}{2}}\Big\|_{H^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} \le C\Big(\|\theta_{h,\tau}^{n+1}\|_{L^2}^2 + \|\theta_{h,\tau}^{n}\|_{L^2}^2\Big) + Ch^4. \tag{4.64}$$

In addition, by virtue of (2.11) and Lemma 4.7, we get

$$-\Omega\mathrm{Im}\Big(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}},\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) = -\Omega\mathrm{Im}\Big(L_z\Big[\hat{u}_\tau^{n+\frac{1}{2}} - R_h\hat{u}_\tau^{n+\frac{1}{2}}\Big],\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)$$
$$= \Omega\mathrm{Im}\Big(L_z\Big[\hat{e}_\tau^{n+\frac{1}{2}} - R_h\hat{e}_\tau^{n+\frac{1}{2}}\Big],\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) - \Omega\mathrm{Im}\Big(L_z\Big[\hat{u}^{n+\frac{1}{2}} - R_h\hat{u}^{n+\frac{1}{2}}\Big],\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)$$
$$= \Omega\mathrm{Im}\Big(L_z\Big[\hat{e}_\tau^{n+\frac{1}{2}} - R_h\hat{e}_\tau^{n+\frac{1}{2}}\Big],\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) - \Omega\mathrm{Im}\Big(L_z\Big[\hat{u}^{n+\frac{1}{2}} - I_h\hat{u}^{n+\frac{1}{2}}\Big],\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)$$
$$-\Omega\mathrm{Im}\Big(L_z\Big[I_h\hat{u}^{n+\frac{1}{2}} - R_h\hat{u}^{n+\frac{1}{2}}\Big],\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)$$
$$\le |\Omega|\Big\|L_z\Big[\hat{e}_\tau^{n+\frac{1}{2}} - R_h\hat{e}_\tau^{n+\frac{1}{2}}\Big]\Big\|_{L^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} + Ch^2\Big\|\hat{u}^{n+\frac{1}{2}}\Big\|_{H^3}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$+ C\Big\|I_h\hat{u}^{n+\frac{1}{2}} - R_h\hat{u}^{n+\frac{1}{2}}\Big\|_{H^1}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$\le C\Big\|\hat{e}_\tau^{n+\frac{1}{2}} - R_h\hat{e}_\tau^{n+\frac{1}{2}}\Big\|_{H^1}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} + Ch^2\Big\|\hat{u}^{n+\frac{1}{2}}\Big\|_{H^3}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$\le Ch\Big\|\hat{e}_\tau^{n+\frac{1}{2}}\Big\|_{H^2}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} + Ch^2\Big\|\hat{u}^{n+\frac{1}{2}}\Big\|_{H^3}\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2}$$
$$\le C\big(h\tau^2 + h^2\big)\Big\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\|_{L^2} \le C\Big(\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|\theta_{h,\tau}^{T,n}\|_{L^2}^2\Big) + C\big(h^4 + h^2\tau^4\big), \tag{4.65}$$

where we have used the following results [44, 45]

$$(L_z(u - I_h u), v) = O(h^2)\,\|u\|_{H^3}\|v\|_{L^2}, \quad \|I_h u - R_h u\|_{H^1} = O(h^2)\|u\|_{H^3}. \tag{4.66}$$

Moreover, it is obvious that

$$-\Omega \mathrm{Im}\Big(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) = 0. \tag{4.67}$$

Thanks to the definition of the truncated function μ_A, we have

$$\begin{aligned}
\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}} &= \beta\left[\frac{|u_\tau^n|^2 + \left|u_\tau^{n+1}\right|^2}{2}\hat{u}_\tau^{n+\frac{1}{2}} - \frac{\mu_A\left(\left|u_{h,\tau}^{T,n}\right|^2\right) + \mu_A\left(\left|u_{h,\tau}^{T,n+1}\right|^2\right)}{2}\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}\right] \\
&= \beta\left[\frac{\mu_A\left(|u_\tau^n|^2\right) + \mu_A\left(\left|u_\tau^{n+1}\right|^2\right)}{2}\hat{u}_\tau^{n+\frac{1}{2}} - \frac{\mu_A\left(\left|u_{h,\tau}^{T,n}\right|^2\right) + \mu_A\left(\left|u_{h,\tau}^{T,n+1}\right|^2\right)}{2}\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}\right] \\
&= \beta\left[\frac{\mu_A\left(|u_\tau^n|^2\right) + \mu_A\left(\left|u_\tau^{n+1}\right|^2\right)}{2} - \frac{\mu_A\left(\left|u_{h,\tau}^{T,n}\right|^2\right) + \mu_A\left(\left|u_{h,\tau}^{T,n+1}\right|^2\right)}{2}\right]\hat{u}_\tau^{n+\frac{1}{2}} \\
&\quad + \beta\frac{\mu_A\left(\left|u_{h,\tau}^{T,n}\right|^2\right) + \mu_A\left(\left|u_{h,\tau}^{T,n+1}\right|^2\right)}{2}\hat{e}_{h,\tau}^{T,n+\frac{1}{2}}.
\end{aligned} \tag{4.68}$$

Then, following the properties of the truncated function μ_A, it is easy to obtain that

$$\left\|\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}\right\|_{L^2} \le C_{\mathcal{N}}\left(\left\|e_{h,\tau}^{T,n}\right\|_{L^2} + \left\|e_{h,\tau}^{T,n+1}\right\|_{L^2}\right). \tag{4.69}$$

Therefore, for the last term on the right-hand side of (4.62), it follows from (4.69) and (2.11) that

$$\mathrm{Im}\Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) \le C\left(\left\|\theta_{h,\tau}^{T,n+1}\right\|_{L^2}^2 + \left\|\theta_{h,\tau}^{T,n}\right\|_{L^2}^2\right) + Ch^4. \tag{4.70}$$

Now, using (4.63)–(4.65), (4.67) and (4.70) in (4.62) obtains that

$$\frac{\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|\theta_{h,\tau}^{T,n}\|_{L^2}^2}{2\tau} \le C\Big(\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|\theta_{h,\tau}^{T,n}\|_{L^2}^2\Big) + C\big(h^4 + h^2\tau^4 + h^4\tau^2\big). \tag{4.71}$$

By using the discrete Gronwall's inequality in (4.71), there exists a positive constant τ_4, such that when $\tau \le \tau_4$, it holds

$$\|\theta_{h,\tau}^{T,n+1}\|_{L^2} \le C_{E,h}\big(h^2 + h\tau^2 + h^2\tau\big). \tag{4.72}$$

Setting $\tau_0^{**} = \min\{\tau_0^*, \tau_4\}$, we have completed the proof of this lemma. □

4.3. Proof of L^2-norm convergence

Thanks to the inverse inequality, and using the results given in Lemmas 4.6 and 4.9, it holds

$$\begin{aligned}
\left\|u_{h,\tau}^{T,n}\right\|_{L^\infty} &\le \|u^n\|_{L^\infty} + C\|u^n - u_\tau^n\|_{H^2} + C\|u_\tau^n - R_hu_\tau^n\|_{H^2} + \left\|R_hu_\tau^n - u_{h,\tau}^{T,n}\right\|_{L^\infty} \\
&\le \|u^n\|_{L^\infty} + CC_{E,\tau}\tau^2 + CC_{E,h}h^2\|u_\tau^n\|_{H^2} + CC_{E,h}h^{-1}(h^2 + h\tau^2 + h^2\tau) \le K_0,
\end{aligned} \tag{4.73}$$

provided $h \leq h_0$ and $\tau \leq \tau_0^{**}$ with positive constants h_0 and τ_0^{**}. The L^∞-norm boundedness of $u_{h,\tau}^{T,n}$ can guarantee $\mu_A(|u_{h,\tau}^{T,n}|^2) = |u_{h,\tau}^{T,n}|^2$, which further implies the equivalence of the schemes (2.13) and (4.57). Hence, $u_{h,\tau}^{T,n} = u_{h,\tau}^n$ holds. Then we can prove the unique existence of the fully discrete system (2.13) similar as Lemma 4.6. From Lemma 4.9, we get

$$\sup_{0\leq n\leq N} \left\| R_h u_\tau^n - u_{h,\tau}^n \right\|_{L^2} \leq C_{E,h}(h^2 + h\tau^2 + h^2\tau). \tag{4.74}$$

Then, using (2.11) and (4.74), we obtain

$$\begin{aligned} \left\| u^n - u_{h,\tau}^n \right\|_{L^2} &\leq \|u^n - u_\tau^n\|_{L^2} + \|u_\tau^n - R_h u_\tau^n\|_{L^2} + \left\| R_h u_\tau^n - u_{h,\tau}^n \right\|_{L^2} \\ &\leq C_{E,\tau}\tau^2 + C_{R_h} C_{u,\tau} h^2 + C_{E,h}(h^2 + h\tau^2 + h^2\tau) \leq C_E(h^2 + \tau^2). \end{aligned} \tag{4.75}$$

We have completed the proof of the optimal convergence in the sense of L^2-norm.

Remark 4.10. It is noticed that the L^2-norm convergence rate is optimal without any time-space ratio restriction, and also without any requirements on the parameters. Furthermore, observed from the proof, the results provided in Lemma 4.6 are sufficient to yield the convergence result (4.75). Indeed, under the results of Lemma 4.6, (4.63) can be modified by

$$\operatorname{Re}\Big(D_\tau \rho_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}} \Big) \leq C\Big(\|\theta_{h,\tau}^{n+1}\|_{L^2}^2 + \|\theta_{h,\tau}^n\|_{L^2}^2 \Big) + Ch^2. \tag{4.76}$$

Thus, (4.72) will be updated by

$$\|\theta_{h,\tau}^{n+1}\|_{L^2} \leq C_{E,h} h. \tag{4.77}$$

Then the L^∞-boundedness of $u_{h,\tau}^{T,n}$ can be derived similarly to (4.73), but the upper bound will not be K_0. Additionally, if we consider the FE space in 3D, the L^∞-boundedness of $u_{h,\tau}^{T,n}$ cannot be obtained by using the inverse inequality. Therefore, it is evident that the improved result presented in Lemma 4.7 is a preferable choice. Indeed, in 3D, using the inverse inequality,

$$\begin{aligned} \|u_{h,\tau}^{n+1}\|_{L^\infty} &\leq \|R_h u^{n+1}\|_{L^\infty} + \|\theta_{h,\tau}^{n+1}\|_{L^\infty} \leq \|R_h u^{n+1}\|_{L^\infty} + Ch^{-\frac{3}{2}} \|\theta_{h,\tau}^{n+1}\|_{L^2} \\ &\leq \|u^{n+1}\|_{H^2} + Ch^{-\frac{3}{2}}\big(h^2 + h\tau^2 + h^2\tau\big), \end{aligned} \tag{4.78}$$

which means we only need a relatively mild condition: $h^{-1/2}\tau^2 = O(1)$. We can also consider removing this condition since we do not actually require the L^∞-norm of the numerical solution. To obtain the convergence results, we may only need the H^1-norm boundedness of the numerical solution, which can be derived by

$$\|u_{h,\tau}^{n+1}\|_{H^1} \leq \|R_h u^{n+1}\|_{H^1} + \|\theta_{h,\tau}^{n+1}\|_{H^1} \leq \|R_h u^{n+1}\|_{H^1} + Ch^{-1}\big(h^2 + h\tau^2 + h^2\tau\big) \leq K_0. \tag{4.79}$$

We will carefully consider these issues in the near future.

4.4. Proof of H^1-norm high-order convergence

In [39], the authors employed the finite difference method to tackle the model described by equation (1.1). They conducted an analysis that yielded an error estimate in the H^1-norm, demonstrating a convergence rate of $O(h^{3/2} + \tau^{3/2})$. Henning and Wärnegård [36] studied the H^1-error estimates for the Crank–Nicolson FEM for the GPE without the angular momentum rotation term. In Lemma 4.9, we established an error estimate in the L^2-norm with an optimal convergence rate of $O(h^2)$. Consequently, this allowed us to achieve the optimal H^1-norm error estimate by using the inverse inequality, which exhibits a convergence rate of $O(h)$. In what follows, we will conduct a comprehensive analysis to establish high-order convergence in the H^1-norm.

Denoting $\omega_h = D_\tau \theta_{h,\tau}^{T,n+\frac{1}{2}}$ in (4.61), and selecting the real parts of the resulting, we get

$$\frac{\|\nabla\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|\nabla\theta_{h,\tau}^{T,n}\|_{L^2}^2}{4\tau} + \frac{\left(V\theta_{h,\tau}^{T,n+1}, \theta_{h,\tau}^{T,n+1}\right) - \left(V\theta_{h,\tau}^{T,n}, \theta_{h,\tau}^{T,n}\right)}{2\tau} = \mathrm{Im}\left(D_\tau\rho_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right)$$
$$- \mathrm{Re}\left(V\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right) + \Omega\mathrm{Re}\left(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right)$$
$$+ \Omega\mathrm{Re}\left(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right) - \mathrm{Re}\left(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right). \tag{4.80}$$

Similar as (4.63)–(4.65), we have

$$\mathrm{Im}\left(D_\tau\rho_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right) - \mathrm{Re}\left(V\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right) + \Omega\mathrm{Re}\left(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right)$$
$$\le C\left\|D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right\|_{L^2}^2 + C\left(h^4 + h^4\tau^2 + h^2\tau^4\right). \tag{4.81}$$

Meanwhile, similar as (4.70), it follows that

$$\mathrm{Re}\left(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right) \le C\left\|D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\right\|_{L^2}^2 + Ch^4. \tag{4.82}$$

Substituting (4.81) and (4.82) into (4.80) obtains

$$\frac{\|\nabla\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|\nabla\theta_{h,\tau}^{T,n}\|_{L^2}^2}{4\tau} + \frac{\left(V\theta_{h,\tau}^{T,n+1}, \theta_{h,\tau}^{T,n+1}\right) - \left(V\theta_{h,\tau}^{T,n}, \theta_{h,\tau}^{T,n}\right)}{2\tau}$$
$$\le C\left(\|\nabla\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|\nabla\theta_{h,\tau}^{T,n}\|_{L^2}^2 + \|D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\|_{L^2}^2\right) + C\left(h^4 + h^4\tau^2 + h^2\tau^4\right). \tag{4.83}$$

On the other hand, we take the difference of (4.61) between two consecutive steps to arrive at

$$i\left(\frac{\theta_{h,\tau}^{T,n+1} - 2\theta_{h,\tau}^{T,n} + \theta_{h,\tau}^{T,n-1}}{\tau^2}, \omega_h\right) = -i\left(\frac{\rho_{h,\tau}^{T,n+1} - 2\rho_{h,\tau}^{T,n} + \rho_{h,\tau}^{T,n-1}}{\tau^2}, \omega_h\right) + \left(\frac{\nabla\theta_{h,\tau}^{T,n+1} - \nabla\theta_{h,\tau}^{T,n-1}}{4\tau}, \nabla\omega_h\right)$$
$$+ \left(\frac{V\rho_{h,\tau}^{T,n+1} - V\rho_{h,\tau}^{T,n-1}}{2\tau}, \omega_h\right) + \left(\frac{V\theta_{h,\tau}^{T,n+1} - V\theta_{h,\tau}^{T,n-1}}{2\tau}, \omega_h\right)$$
$$- \Omega\left(\frac{L_z\rho_{h,\tau}^{T,n+1} - L_z\rho_{h,\tau}^{T,n-1}}{2\tau}, \omega_h\right) - \Omega\left(\frac{L_z\theta_{h,\tau}^{T,n+1} - L_z\theta_{h,\tau}^{T,n-1}}{2\tau}, \omega_h\right)$$
$$+ \left(\frac{\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}} - \mathcal{N}_{h,\tau}^{T,n-\frac{1}{2}}}{\tau}, \omega_h\right). \tag{4.84}$$

For simplicity, we denote $\theta_{h,\tau}^{T,n+1} - \theta_{h,\tau}^{T,n} = \tau H_{h,\tau}^{T,n+1}$. Then we rewrite (4.84) as

$$i\left(D_\tau H_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\right) = -i\left(\frac{\rho_{h,\tau}^{T,n+1} - 2\rho_{h,\tau}^{T,n} + \rho_{h,\tau}^{T,n-1}}{\tau^2}, \omega_h\right) + \frac{1}{2}\left(\nabla\hat{H}_{h,\tau}^{T,n+\frac{1}{2}}, \nabla\omega_h\right) + \left(\frac{V\rho_{h,\tau}^{T,n+1} - V\rho_{h,\tau}^{T,n-1}}{2\tau}, \omega_h\right)$$
$$+ \left(V\hat{H}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\right) - \Omega\left(\frac{L_z\rho_{h,\tau}^{T,n+1} - L_z\rho_{h,\tau}^{T,n-1}}{2\tau}, \omega_h\right) - \Omega\left(L_z\hat{H}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\right)$$
$$+ \left(\frac{\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}} - \mathcal{N}_{h,\tau}^{T,n-\frac{1}{2}}}{\tau}, \omega_h\right). \tag{4.85}$$

Let $\omega_h = \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}$ in (4.85), and take the imaginary part of the resulting, then we get

$$\frac{\|H_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|H_{h,\tau}^{T,n}\|_{L^2}^2}{2\tau} = -\mathrm{Re}\left(\frac{\rho_{h,\tau}^{T,n+1} - 2\rho_{h,\tau}^{T,n} + \rho_{h,\tau}^{T,n-1}}{\tau^2}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right) + \mathrm{Im}\left(\frac{V\rho_{h,\tau}^{T,n+1} - V\rho_{h,\tau}^{T,n-1}}{2\tau}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right)$$
$$- \Omega\left(\frac{L_z\rho_{h,\tau}^{T,n+1} - L_z\rho_{h,\tau}^{T,n-1}}{2\tau}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right) + \left(\frac{\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}} - \mathcal{N}_{h,\tau}^{T,n-\frac{1}{2}}}{\tau}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right). \quad (4.86)$$

By using Lemma 4.7, we can easily obtain

$$\mathrm{Re}\left(\frac{\rho_{h,\tau}^{T,n+1} - 2\rho_{h,\tau}^{T,n} + \rho_{h,\tau}^{T,n-1}}{\tau^2}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right) + \mathrm{Im}\left(\frac{V\rho_{h,\tau}^{T,n+1} - V\rho_{h,\tau}^{T,n-1}}{2\tau}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right)$$
$$\leq C\left(\|H_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|H_{h,\tau}^{T,n}\|_{L^2}^2\right) + C\left(h^4 + h^4\tau^2\right). \quad (4.87)$$

Thanks to

$$\frac{L_z\rho_{h,\tau}^{T,n+1} - L_z\rho_{h,\tau}^{T,n-1}}{2\tau} = \left[L_z\left(\frac{u^{n+1} - u^{n-1}}{2\tau}\right) - L_zR_h\left(\frac{u^{n+1} - u^{n-1}}{2\tau}\right)\right]$$
$$- \left[L_z\left(\frac{e_\tau^{n+1} - e_\tau^{n-1}}{2\tau}\right) - L_zR_h\left(\frac{e_\tau^{n+1} - e_\tau^{n-1}}{2\tau}\right)\right]. \quad (4.88)$$

Then, similar as the deduction of (4.65), we conclude

$$-\Omega\left(\frac{L_z\rho_{h,\tau}^{T,n+1} - L_z\rho_{h,\tau}^{T,n-1}}{2\tau}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right) \leq C\left(\|H_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|H_{h,\tau}^{T,n}\|_{L^2}^2\right) + C\left(h^4 + h^2\tau^2\right). \quad (4.89)$$

Moreover, using similar methods as (4.44), we have

$$\left(\frac{\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}} - \mathcal{N}_{h,\tau}^{T,n-\frac{1}{2}}}{\tau}, \hat{H}_{h,\tau}^{T,n+\frac{1}{2}}\right) \leq C\left(\|H_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|H_{h,\tau}^{T,n}\|_{L^2}^2\right) + C\left(h^4 + h^2\tau^4 + h^4\tau^2\right). \quad (4.90)$$

Hence, substituting (4.87), (4.89) and (4.90) into (4.86), we have

$$\frac{\|H_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|H_{h,\tau}^{T,n}\|_{L^2}^2}{2\tau} \leq C\left(\|H_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|H_{h,\tau}^{T,n}\|_{L^2}^2\right) + C\left(h^4 + \tau^4\right). \quad (4.91)$$

By using discrete Gronwall's inequality, (4.91) further implies that there exists a positive constant τ_5, such that when $\tau \leq \tau_5$, there holds

$$\|H_{h,\tau}^{T,n+1}\|_{L^2} = \|D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\|_{L^2} \leq C(h^2 + \tau^2). \quad (4.92)$$

Therefore, from (4.92) and (4.83), we have

$$\frac{\|\nabla\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|\nabla\theta_{h,\tau}^{T,n}\|_{L^2}^2}{4\tau} + \frac{\left(V\theta_{h,\tau}^{T,n+1}, \theta_{h,\tau}^{T,n+1}\right) - \left(V\theta_{h,\tau}^{T,n}, \theta_{h,\tau}^{T,n}\right)}{2\tau}$$
$$\leq C\left(\|\nabla\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|\nabla\theta_{h,\tau}^{T,n}\|_{L^2}^2\right) + C\left(h^4 + \tau^4\right). \quad (4.93)$$

Then by using discrete Gronwall's inequality again, we can get

$$\|\nabla\theta_{h,\tau}^{n+1}\|_{L^2} = \|\nabla\theta_{h,\tau}^{T,n+1}\|_{L^2} \leq C_E(h^2+\tau^2), \tag{4.94}$$

provided that $\tau \leq \tau_6$ with a positive constant τ_6 independent of h and τ.

Finally, we have

$$\begin{aligned}\left\|\nabla(I_h u^{n+1} - u_{\tau,h}^n)\right\|_{L^2} &\leq \left\|\nabla(I_h u^{n+1} - R_h u^{n+1})\right\|_{L^2} + \left\|\nabla(R_h u^{n+1} - R_h u_\tau^{n+1})\right\|_{L^2} \\ &+ \left\|\nabla(R_h u_\tau^{n+1} - u_{\tau,h}^{n+1})\right\|_{L^2} \leq C_E(h^2+\tau^2),\end{aligned} \tag{4.95}$$

and based on interpolated postprocessing technique, we can obtain

$$\left\|u^{n+1} - I_{2h}U_h^{n+1}\right\|_{H^1} \leq C_E(h^2+\tau^2), \tag{4.96}$$

where I_{2h} is the interpolated postprocessing operator [44].

Setting $\tau_0^{***} = \min\{\tau_0^{**}, \tau_5, \tau_6\}$, we have completed the proof.

Remark 4.11. When in the nonconforming FE space, we do not need to provide a complete error analysis, as most of it is similar to the conforming case. The brief proof is shown in appendix.

5. Numerical results

In this section, we first verify the accuracy of the conforming and nonconforming FEMs shown in Theorem 3.1. Then, we check the discrete mass and energy conservations of the both schemes. Finally, we show some experiments about the dynamics of vortex lattice in rotating BEC.

5.1. Accuracy test

To test the accuracy of our numerical methods in 2D, we add a source term at the right-hand side of the model (1.1) with the exact solution $u(x,y,t) = (t+1)^2 \sin\pi x \sin\pi y$, and take $U = [0,1]\times[0,1]$, $\Omega = 0.8$, $\beta = 1$, $\gamma_x = 1$ and $\gamma_y = 2$. For convenience, we select $h_x = h_y = h = \tau$, with h_x and h_y denoting the spatial step sizes in x and y directions, respectively. We numerically verify the optimal error estimates in the L^2-norm and H^1-norm, as well as the high-order error estimates in H^1-norm, using both the conforming and nonconforming FEMs proposed in Definition 2.2. The corresponding results are illustrated in Figure 2, and they align with the theoretical analysis presented in Theorem 3.1. Because we add a source term in this example, we clarify that this experiment is not for the model studied in this paper. Our aim is solely to test the convergence of the numerical method using this example. Consequently, the solution to the numerical scheme does not satisfy mass and energy conservation.

Furthermore, we present numerical tests to compare the errors using conforming and nonconforming FEMs for the GPE model (1.1) with a relatively non-smooth solution, given by $u(x,y,t) = (t+1)^2(1-|2x-1|^2)^s(1-|2y-1|^2)^s$, where $s \in [1/2,1]$. By selecting different values of the parameter s, we plot the errors in Figure 3. We can clearly observe that when the solution is not sufficiently smooth, the nonconforming FEM exhibits better convergence compared to the conforming FEM. Additionally, when the solution is smooth, the errors of both methods are nearly identical.

5.2. Structure-preservation test

In this subsection, we test the structure-preserving properties proposed in Theorem 2.2, including the mass conservation and energy conservation in the discrete senses. We define the relative errors of the discrete mass and discrete energy by

$$ER(N(u_{h,\tau}^n)) = \frac{N(u_{h,\tau}^n) - N(u_{h,\tau}^0)}{N(u_{h,\tau}^0)}, \quad ER(E_h(u_{h,\tau}^n)) = \frac{E_h(u_{h,\tau}^n) - E_h(u_{h,\tau}^0)}{E_h(u_{h,\tau}^0)}\cdot \tag{5.1}$$

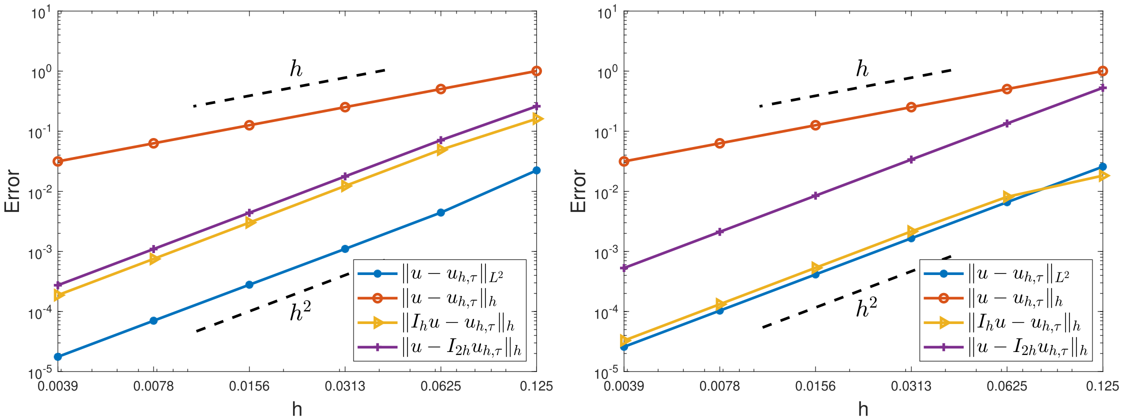

FIGURE 2. The error estimates and convergence rates for the conforming (*left*) and nonconforming (*right*) FEMs.

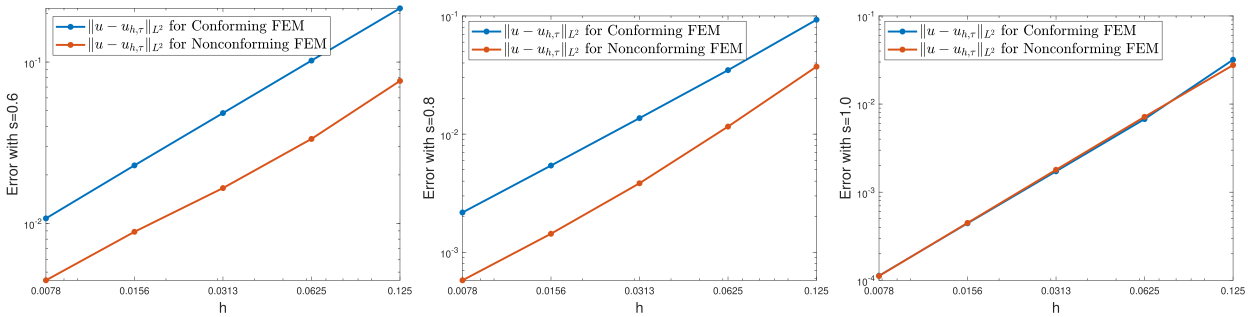

FIGURE 3. The errors for the conforming and nonconforming FEMs with different values of the parameter s.

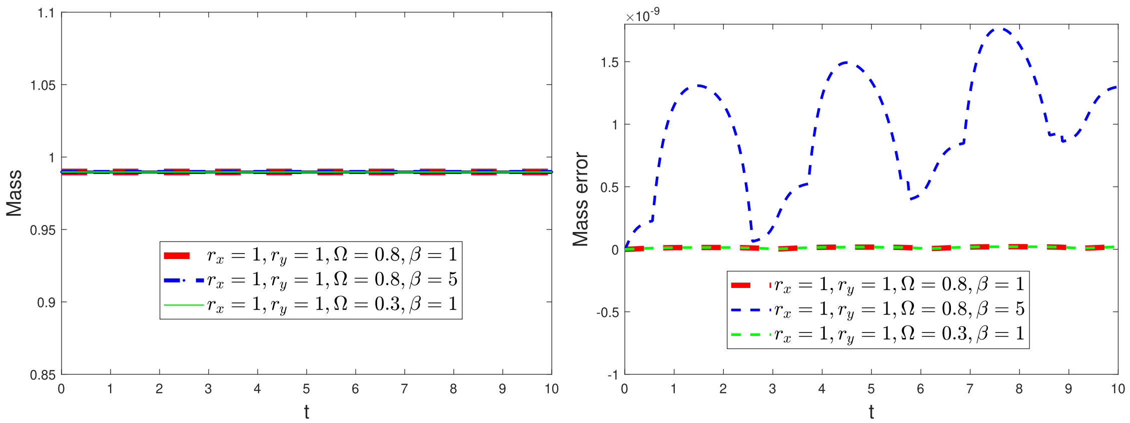

FIGURE 4. The discrete mass and its relative error for the conforming FEM.

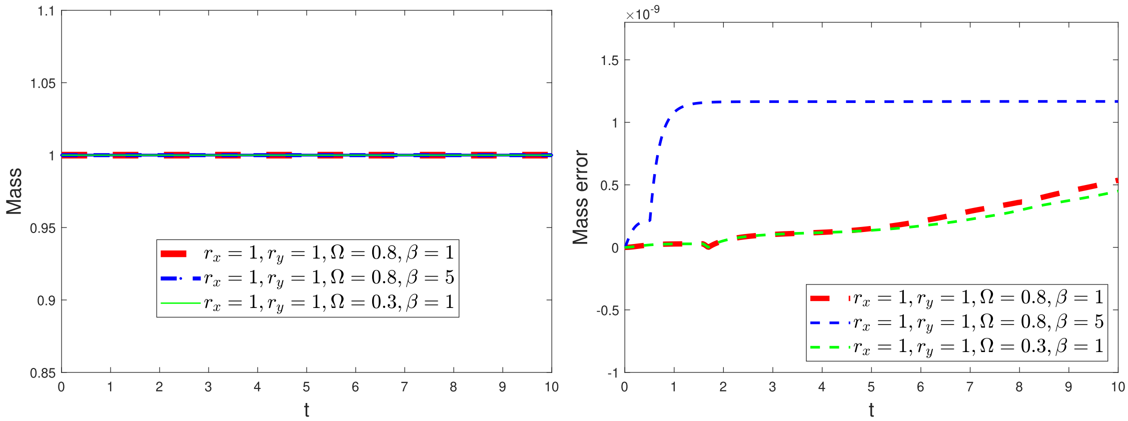

FIGURE 5. The discrete mass and its relative error for the nonconforming FEM.

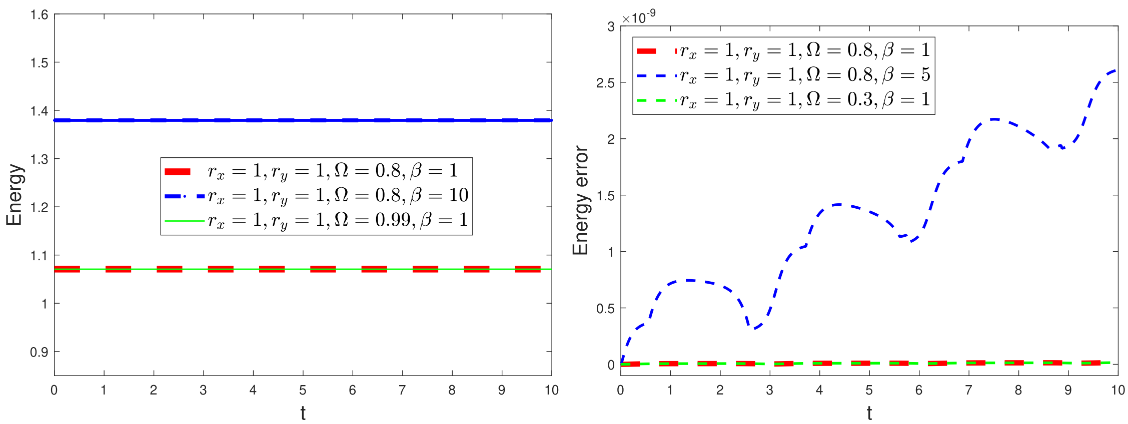

FIGURE 6. The discrete energy and its relative error for the conforming FEM.

The numerical results, as showcased in Figures 4–7, are consistent with the theoretical conclusions that we have proved.

5.3. Dynamics of a vortex lattice in rotating BEC

We in this subsection numerically study the dynamics of a vortex lattice in rotating BEC influenced by the trap frequencies. We consider the model (1.1) with the parameters $\beta = 100$ and $\Omega = 0.99$, and the the domain $U = [-16, 16] \times [-16, 16]$. The initial data is selected as the ground state of the system with $\gamma_x = \gamma_y = 1$, numerically computed by using the normalized gradient flow proposed in [10, 46]. The energy of the ground state is 1.5600. In the subsequent experiments, we will exclusively employ the conforming FEM. We mainly do the following tests:

(I) The first test is to check the free expansion of the quantized vortex lattice when the trap is removed, *i.e.*, $\gamma_x = \gamma_y = 0$. From Figure 8, it is noticeable that the vortex lattice undergoes expansion over time in the absence of any trapping influence during the numerical test, but the vortex structure still keeps the

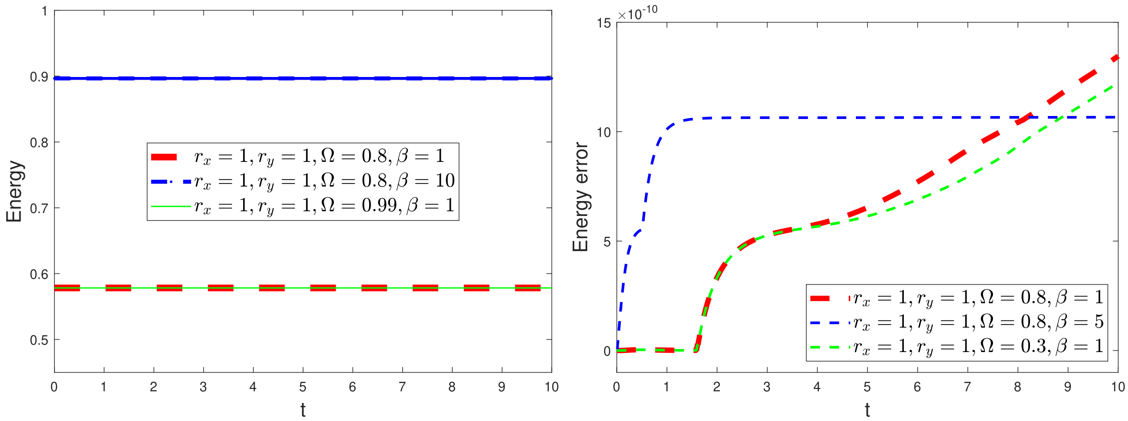

FIGURE 7. The discrete energy and its relative error for the nonconforming FEM.

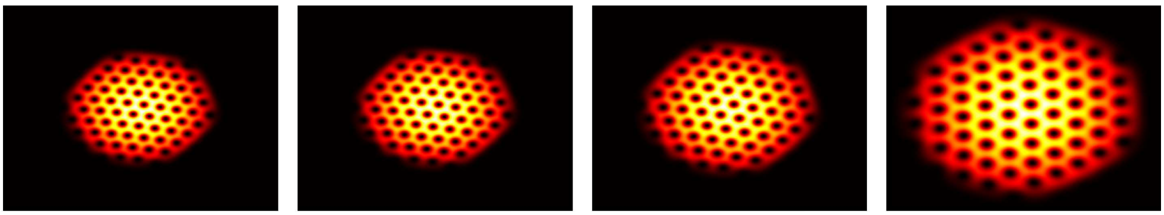

FIGURE 8. Image plots of the density $|u|^2$ at the times $t = 0, 0.3, 0.6$ and 1.2. Here, we select $\gamma_x = \gamma_y = 0$, $h = 1/16$ and $\tau = 0.01$.

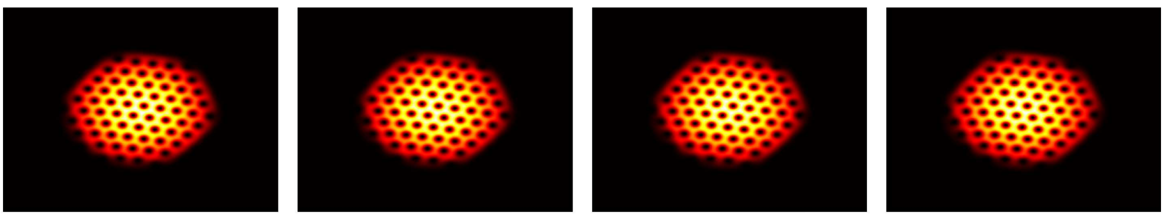

FIGURE 9. Image plots of the density $|u|^2$ at the times $t = 0, 0.75, 1.5$ and 3. Here, we select $\gamma_x = \gamma_y = 1$, $h = 1/16$ and $\tau = 0.01$.

rotational symmetry during the expansion. As a comparison, we also present the dynamics of vortex lattices for the case of $\gamma_x = \gamma_y = 1$, see Figure 9.

(II) Another test focuses on investigating the dynamics of the quantized vortex lattice as the trap frequencies undergo variations. We change the frequencies in y-direction only, in x-direction only, or in both x- and y-directions, see the image plots of the density $|u|^2$ with different cases in Figure 10.

Similar numerical tests were carried out in [7, 47] by different methods, with much less number of vortices in the lattice.

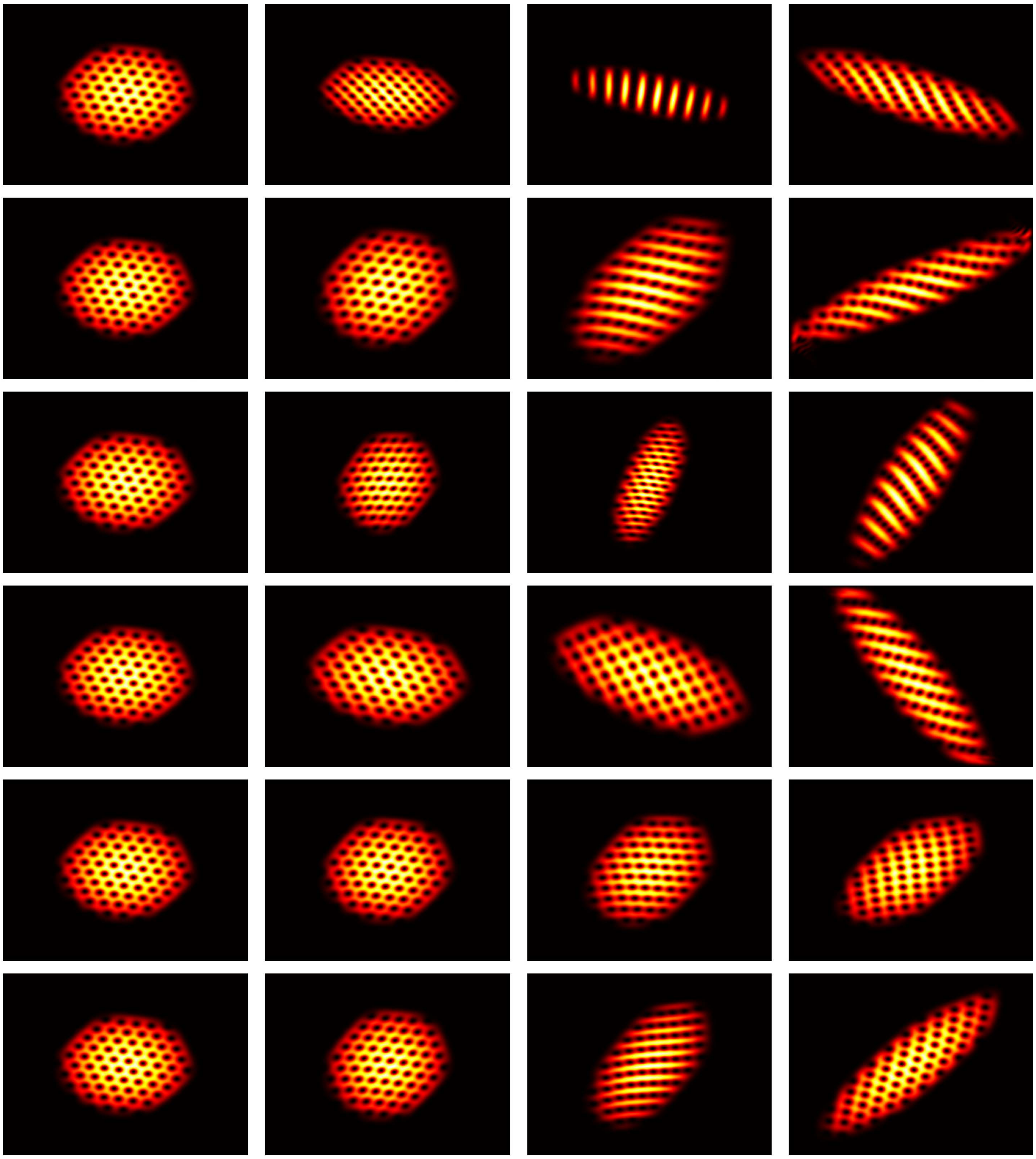

FIGURE 10. Image plots of the density $|u|^2$ at the times $t = 0, 0.75, 1.5$ and 3. Here, we select $h = 1/16$, $\tau = 0.01$, and $\gamma_x = 1, \gamma_y = 1.5$ (*first row*); $\gamma_x = 1, \gamma_y = 0.5$ (*second row*); $\gamma_x = 1.5, \gamma_y = 1$ (*third row*); $\gamma_x = 0.5, \gamma_y = 1$ (*fourth row*); $\gamma_x = \sqrt{1.2}, \gamma_y = \sqrt{0.8}$ (*fifth row*); $\gamma_x = \sqrt{1.4}, \gamma_y = \sqrt{0.6}$ (*sixth row*).

6. Conclusion

This paper focuses on developing and analyzing structure-preserving Galerkin methods for simulating the dynamics of rotating BEC based on the GPE with angular momentum rotation. The challenge lies in constructing FEMs that preserve both mass and energy, particularly in the context of nonconforming FEMs, due to the presence of the rotation term. Furthermore, we provide a comprehensive unconditional error analysis for the structure-preserving FEMs, offering various important generalizations of existing references, such as Bao and Cai [39], Henning and Peterseim [24], Henning and Målqvist [16], and Henning and Wärnegård [48]. To validate the theoretical analysis of the structure-preserving numerical method for rotating BEC, extensive numerical results are presented. The behavior of the quantized vortex lattice is thoroughly examined through a series of numerical tests.

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References

[1] M.R. Matthews, B.P. Anderson, P. Haljan, D. Hall, C. Wieman and E.A. Cornell, Vortices in a Bose–Einstein condensate. *Phys. Rev. Lett.* **83** (1999) 2498.
[2] K.W. Madison, F. Chevy, W. Wohlleben and J. Dalibard, Vortex formation in a stirred Bose–Einstein condensate. *Phys. Rev. Lett.* **84** (2000) 806.
[3] J.R. Abo-Shaeer, C. Raman, J.M. Vogels and W. Ketterle, Observation of vortex lattices in Bose–Einstein condensates. *Science* **292** (2001) 476–479.
[4] C. Raman, J. Abo-Shaeer, J. Vogels, K. Xu and W. Ketterle, Vortex nucleation in a stirred Bose–Einstein condensate. *Phys. Rev. Lett.* **87** (2001) 210402.
[5] A. Aftalion and Q. Du, Vortices in a rotating Bose–Einstein condensate: critical angular velocities and energy diagrams in the Thomas–Fermi regime. *Phys. Rev. A* **64** (2001) 063603.
[6] A. Penckwitt, R. Ballagh and C. Gardiner, Nucleation, growth, and stabilization of Bose–Einstein condensate vortex lattices. *Phys. Rev. Lett.* **89** (2002) 260402.
[7] S. K. Adhikari and P. Muruganandam, Effect of an impulsive force on vortices in a rotating Bose–Einstein condensate. *Phys. Lett. A* **301** (2002) 333–339.
[8] I. Coddington, P. Haljan, P. Engels, V. Schweikhard, S. Tung and E.A. Cornell, Experimental studies of equilibrium vortex properties in a Bose-condensed gas. *Phys. Rev. A* **70** (2004) 063607.
[9] W. Bao and Y. Zhang, Dynamics of the ground state and central vortex states in Bose–Einstein condensation. *Math. Models Methods Appl. Sci.* **15** (2005) 1863–1896.
[10] W. Bao, P.A. Markowich and H. Wang, Ground, symmetric and central vortex states in rotating Bose–Einstein condensates. *Commun. Math. Sci.* **3** (2005) 57–88.
[11] W. Bao, Q. Du and Y. Zhang, Dynamics of rotating Bose–Einstein condensates and its efficient and accurate numerical computation. *SIAM J. Appl. Math.* **66** (2006) 758–786.
[12] Y. Zhang, W. Bao and H. Li, Dynamics of rotating two-component Bose–Einstein condensates and its efficient computation. *Phys. D Nonlinear Phenom.* **234** (2007) 49–69.
[13] W. Bao, H. Li and J. Shen, A generalized-Laguerre–Fourier–Hermite pseudospectral method for computing the dynamics of rotating Bose–Einstein condensates. *SIAM J. Sci. Comput.* **31** (2009) 3685–3711.
[14] W. Bao and Y. Cai, Mathematical theory and numerical methods for Bose–Einstein condensation. *Kinet. Relat. Models* **6** (2013) 1–135.
[15] C. Besse, G. Dujardin and I. Lacroix-Violet, High order exponential integrators for nonlinear Schrödinger equations with application to rotating Bose–Einstein condensates. *SIAM J. Numer. Anal.* **55** (2017) 1387–1411.
[16] P. Henning and A. Målqvist, The finite element method for the time-dependent Gross–Pitaevskii equation with angular momentum rotation. *SIAM J. Numer. Anal.* **55** (2017) 923–952.
[17] W. Bao, Y. Cai and X. Ruan, Ground states of Bose–Einstein condensates with higher order interaction. *Phys. D Nonlinear Phenom.* **386** (2019) 38–48.

[18] A.N. da Silva, R.K. Kumar, A.S. Bradley and L. Tomio, Vortex generation in stirred binary Bose–Einstein condensates. *Phys. Rev. A* **107** (2023) 033314.

[19] H. Chen, G. Dong, W. Liu and Z. Xie, Second-order flows for computing the ground states of rotating Bose–Einstein condensates. *J. Comput. Phys.* **475** (2023) 111872.

[20] R. Seiringer, Gross–Pitaevskii theory of the rotating Bose gas. *Commun. Math. Phys.* **229** (2002) 491–509.

[21] A. Aftalion, R.L. Jerrard and J. Royo-Letelier, Non-existence of vortices in the small density region of a condensate. *J. Funct. Anal.* **260** (2011) 2387–2406.

[22] M. Correggi and N. Rougerie, Inhomogeneous vortex patterns in rotating Bose–Einstein condensates. *Commun. Math. Phys.* **321** (2013) 817–860.

[23] L. Pitaevskii and S. Stringari, Bose–Einstein Condensation and Superfluidity. Vol. 164. Oxford University Press (2016).

[24] P. Henning and D. Peterseim, Crank–Nicolson Galerkin approximations to nonlinear Schrödinger equations with rough potentials. *Math. Models Methods Appl. Sci.* **27** (2017) 2147–2184.

[25] T. Cazenave, Semilinear Schrödinger Equations. Vol. 10. American Mathematical Society (2003).

[26] C. Hao, L. Hsiao and H.-L. Li, Global well posedness for the Gross–Pitaevskii equation with an angular momentum rotational term in three dimensions. *J. Math. Phys.* **48** (2007) 102105.

[27] E.H. Lieb and R. Seiringer, Derivation of the Gross–Pitaevskii equation for rotating Bose gases. *Commun. Math. Phys.* **264** (2006) 505–537.

[28] J. Williams, R. Walser, C. Wieman, J. Cooper and M. Holland, Achieving steady-state Bose–Einstein condensation. *Phys. Rev. A* **57** (1998) 2030.

[29] I. Zapata, F. Sols and A.J. Leggett, Josephson effect between trapped Bose–Einstein condensates. *Phys. Rev. A* **57** (1998) R28.

[30] B. Nikolić, A. Balaž and A. Pelster, Dipolar Bose–Einstein condensates in weak anisotropic disorder. *Phys. Rev. A* **88** (2013) 013624.

[31] J. Sanz-Serna, Methods for the numerical solution of the nonlinear Schrödinger equation. *Math. Comput.* **43** (1984) 21–27.

[32] G.D. Akrivis, V.A. Dougalis and O.A. Karakashian, On fully discrete Galerkin methods of second-order temporal accuracy for the nonlinear Schrödinger equation. *Numer. Math.* **59** (1991) 31–53.

[33] Y. Tourigny, Optimal H^1 estimates for two time-discrete Galerkin approximations of a nonlinear Schrödinger equation. *IMA J. Numer. Anal.* **11** (1991) 509–523.

[34] J. Wang, A new error analysis of Crank–Nicolson Galerkin FEMs for a generalized nonlinear Schrödinger equation. *J. Sci. Comput.* **60** (2014) 390–407.

[35] C. Döding and P. Henning, Uniform L^∞-bounds for energy-conserving higher-order time integrators for the Gross–Pitaevskii equation with rotation. Preprint arXiv:2210.01553 (2022).

[36] P. Henning and J. Wärnegård, A note on optimal H^1-error estimates for Crank–Nicolson approximations to the nonlinear Schrödinger equation. *BIT Numer. Math.* **61** (2021) 37–59.

[37] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods. Springer (2008).

[38] D. Shi, J. Wang and F. Yan, Unconditional superconvergence analysis for nonlinear parabolic equation with EQ_1^{rot} nonconforming finite element. *J. Sci. Comput.* **70** (2017) 85–111.

[39] W. Bao and Y. Cai, Optimal error estimates of finite difference methods for the Gross–Pitaevskii equation with angular momentum rotation. *Math. Comput.* **82** (2013) 99–128.

[40] E. Gagliardo, Proprieta di alcune classi di funzioni in piu variabili. *Matematika* **5** (1961) 87–116.

[41] M. Fila and M. Winkler, A Gagliardo–Nirenberg-type inequality and its applications to decay estimates for solutions of a degenerate parabolic equation. *Adv. Math.* **357** (2019) 106823.

[42] C. Carstensen, R. Khot and A.K. Pani, A priori and a posteriori error analysis of the lowest-order NCVEM for second-order linear indefinite elliptic problems. *Numer. Math.* **151** (2022) 551–600.

[43] G. Leoni, A First Course in Sobolev Spaces, 2nd edition. American Mathematical Society (2017).

[44] Q. Lin and J. Lin, Finite Element Methods: Accuracy and Improvement. Science Press, Beijing (2006).

[45] D. Shi, F. Wang, M. Fan and Y. Zhao, A new approach of the lowest order anisotropic mixed finite element high accuracy analysis for nonlinear sine-Gordon equations. *Math. Numer. Sin.* **37** (2015) 148.

[46] W. Bao and Q. Du, Computing the ground state solution of Bose–Einstein condensates by a normalized gradient flow. *SIAM J. Sci. Comput.* **25** (2004) 1674–1697.

[47] W. Bao and H. Wang, An efficient and spectrally accurate numerical method for computing dynamics of rotating Bose–Einstein condensates. *J. Comput. Phys.* **217** (2006) 612–626.

[48] P. Henning and J. Wärnegård, Superconvergence of time invariants for the Gross–Pitaevskii equation. *Math. Comput.* **91** (2022) 509–555.

Appendix A Convergence analysis for the nonconforming FEM

The fully discrete method with truncation for the nonconforming case is given as follows.

Definition 1.1 (Fully discrete method with truncation)**.** Let $u_{h,\tau}^{T,0}$ be a suitable interpolation of u^0. Then for $n \geq 1$, we define the following truncated fully discrete system, which is to find $u_{h,\tau}^{T,n+1} \in V_h^{NC}$, $0 \leq n \leq N-1$ such that

$$i\Big(D_\tau u_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) = \frac{1}{2}\Big(\nabla \hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \nabla \omega_h\Big)_h + \Big(V\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Omega\Big(L_z\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big)_h + \beta\Bigg(\frac{\mu_A(|u_{h,\tau}^{T,n}|^2) + \mu_A(|u_{h,\tau}^{T,n+1}|^2)}{2}\hat{u}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Bigg) + \Big\langle S^{T,n+1}, \omega_h\Big\rangle, \tag{A.1}$$

for any $\omega_h \in V_h^{NC}$, where

$$\Big\langle S^{T,n+1}, \omega_h\big\rangle := -i\frac{\Omega}{2}\mathrm{Re}\Big\langle (x-y)\hat{u}_{h,\tau}^{n+\frac{1}{2}}, \omega_h\Big\rangle + \frac{\Omega}{2}\mathrm{Im}\big\langle (x-y)u_{h,\tau}^{n+1}, \omega_h\big\rangle.$$

We will use the interpolation operator to split the error. For convenience, we still adopt the same denotations:

$$e_{h,\tau}^{T,n} = u_\tau^n - u_{h,\tau}^{T,n} = (u_\tau^n - I_h u_\tau^n) + \Big(I_h u_\tau^n - u_{h,\tau}^{T,n}\Big) =: \rho_{h,\tau}^{T,n} + \theta_{h,\tau}^{T,n}.$$

Subtracting (A.1) from the variational formulation of the time-discrete method (2.1) gives

$$i\Big(D_\tau e_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) = \frac{1}{2}\Big(\nabla \hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \nabla \omega_h\Big)_h + \Big(V\hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Omega\Big(L_z\hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big)_h + \Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Big\langle S^{T,n+1}, \omega_h\Big\rangle - \frac{1}{2}\sum_K \int_{\partial K}\Big(\nabla \hat{u}_\tau^{n+\frac{1}{2}} \cdot \mathbf{n}\Big)\omega_h \,\mathrm{d}s, \quad \forall \omega_h \in V_h^{NC}. \tag{A.2}$$

Noting the property (2.12), (A.2) is indeed equivalent to

$$i\Big(D_\tau \theta_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) = -i\Big(D_\tau \rho_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) + \frac{1}{2}\Big(\nabla \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}, \nabla \omega_h\Big)_h + \Big(V\hat{e}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Omega\Big(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big)_h + \Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big) - \Omega\Big(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}, \omega_h\Big)_h - \Big\langle S^{T,n+1}, \omega_h\Big\rangle - \frac{1}{2}\sum_K \int_{\partial K}\Big(\nabla \hat{u}_\tau^{n+\frac{1}{2}} \cdot \mathbf{n}\Big)\omega_h \,\mathrm{d}s. \tag{A.3}$$

Denoting $\omega_h = \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}$ in (A.3), and taking the imaginary part of the equation, we obtain

$$\begin{aligned}\frac{\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 - \|\theta_{h,\tau}^{T,n}\|_{L^2}^2}{2\tau} &= -\mathrm{Re}\Big(D_\tau \rho_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) + \mathrm{Im}\Big(V\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) - \Omega\mathrm{Im}\Big(L_z\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)_h \\ &+ \mathrm{Im}\Big(\mathcal{N}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big) - \Omega\mathrm{Im}\Big(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)_h - \mathrm{Im}\Big\langle S^{T,n+1}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\rangle \\ &- \mathrm{Im}\Big\{\frac{1}{2}\sum_K \int_{\partial K}\Big(\nabla \hat{u}_\tau^{n+\frac{1}{2}} \cdot \mathbf{n}\Big)\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\,\mathrm{d}s\Big\}. \end{aligned} \tag{A.4}$$

Compared with the conforming case, we only need to consider the last three terms of (A.4). For the last term of (A.6), we have

$$\begin{aligned} &-\mathrm{Im}\Big\{\frac{1}{2}\sum_K \int_{\partial K}\Big(\nabla \hat{u}_\tau^{n+\frac{1}{2}} \cdot \mathbf{n}\Big)\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\,\mathrm{d}s\Big\} = \mathrm{Im}\Big\{\frac{1}{2}\sum_K \int_{\partial K}\Big(\nabla \hat{e}_\tau^{n+\frac{1}{2}} \cdot \mathbf{n}\Big)\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\,\mathrm{d}s\Big\} \\ &\quad -\mathrm{Im}\Big\{\frac{1}{2}\sum_K \int_{\partial K}\Big(\nabla \hat{u}_\tau^{n+\frac{1}{2}} \cdot \mathbf{n}\Big)\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\,\mathrm{d}s\Big\} \le Ch\|\hat{e}_\tau^{n+\frac{1}{2}}\|_{H^2}\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\|_{1,h} + Ch^2\|\hat{u}^{n+\frac{1}{2}}\|_{H^3}\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\|_{1,h} \\ &\quad \le C\big(h\tau^2 + h^2\big)\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\|_{1,h}. \end{aligned} \tag{A.5}$$

By the definition of $\langle S^{T,n+1}, \cdot\rangle$, using integration by parts, by virtue of $\langle \cdot, \omega_h\rangle = O(h^2)\|\cdot\|_{H^2}\|\omega_h\|_{1,h}$, thanks to the boundedness of $\hat{\rho}_{h,\tau}^{T,n+\frac{1}{2}}$ in H^2-norm, and similar as (A.5), we can get

$$-\Omega\mathrm{Im}\Big(L_z\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big)_h - \mathrm{Im}\Big\langle S^{T,n+1}, \hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\Big\rangle \le C\big(h\tau^2 + h^2\big)\|\hat{\theta}_{h,\tau}^{T,n+\frac{1}{2}}\|_{1,h}. \tag{A.6}$$

Hence, from (A.4) and similar as the conforming case, we can obtain

$$\|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 \le C_{E,h}\big(h^2\tau^4 + h^4\big) + C\tau\sum_{k=1}^n \|\theta_{h,\tau}^{T,k}\|_{1,h}^2. \tag{A.7}$$

In addition, similar as the conforming case (4.83) and using above analytical approach, we can further obtain

$$\|\theta_{h,\tau}^{T,n+1}\|_{1,h}^2 \le C_{E,h}\big(h^2\tau^4 + h^4\big) + C\tau\sum_{k=1}^n \Big(\|\theta_{h,\tau}^{T,k}\|_{1,h}^2 + \|D_\tau\theta_{h,\tau}^{T,k+\frac{1}{2}}\|_{L^2}^2\Big). \tag{A.8}$$

Moreover, like (4.84), we can also take the difference between two consecutive steps, and then adopt similar steps as conforming case to finally obtain that

$$\|D_\tau\theta_{h,\tau}^{T,k+\frac{1}{2}}\|_{L^2}^2 \le C_{E,h}\big(h^2\tau^4 + h^4\big) + C\tau\sum_{k=1}^n \|\theta_{h,\tau}^{T,k}\|_{1,h}^2. \tag{A.9}$$

From (A.7)–(A.9), we get

$$\begin{aligned} \|\theta_{h,\tau}^{T,n+1}\|_{L^2}^2 + \|\theta_{h,\tau}^{T,n+1}\|_{1,h}^2 + \|D_\tau\theta_{h,\tau}^{T,n+\frac{1}{2}}\|_{L^2}^2 &\le C_{E,h}\big(h^2\tau^4 + h^4\big) \\ &+ C\tau\sum_{k=1}^n\Big[\|\theta_{h,\tau}^{T,k}\|_{L^2}^2 + \|\theta_{h,\tau}^{T,k}\|_{1,h}^2 + \|D_\tau\theta_{h,\tau}^{T,k+\frac{1}{2}}\|_{L^2}^2\Big]. \end{aligned} \tag{A.10}$$

In conclusion, by applying the discrete Gronwall's inequality, we derive the desired convergence results that are similar to those in the conforming case.