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MEAN FIELD ERROR ESTIMATE OF THE RANDOM BATCH METHOD FOR LARGE INTERACTING PARTICLE SYSTEM

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Abstract. The random batch method (RBM) proposed in Jin *et al.* [*J. Comput. Phys.* **400** (2020) 108877] for large interacting particle systems is an efficient with linear complexity in particle numbers and highly scalable algorithm for N-particle interacting systems and their mean-field limits when N is large. We consider in this work the quantitative error estimate of RBM toward its mean-field limit, the Fokker–Planck equation. Under mild assumptions, we obtain a uniform-in-time $O(\tau^2 + 1/N)$ bound on the scaled relative entropy between the joint law of the random batch particles and the tensorized law at the mean-field limit, where τ is the time step size and N is the number of particles. Therefore, we improve the existing rate in discretization step size from $O(\sqrt{\tau})$ to $O(\tau)$ in terms of the Wasserstein distance.

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1. Introduction

Interacting particle systems arise in a variety of important problems in physical, social, and biological sciences, for example, molecular dynamics [14], swarming [5,6,12,44], chemotaxis [2,19], flocking [1,11,18], synchronization [10,17], consensus [38] and random vortex model [40]. In this paper, we consider the following general first order system of N particles:

$$\mathrm{d}X^i = b\big(X^i\big)\mathrm{d}t + \frac{1}{N-1}\sum_{j:j\neq i} K\big(X^i - X^j\big)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W^i, \quad i = 1, 2, \ldots, N. \tag{1.1}$$

Here $X^i \in \mathbb{R}^d$ are the labels for particles, $b : \mathbb{R}^d \to \mathbb{R}^d$ is the drift force, $K : \mathbb{R}^d \to \mathbb{R}^d$ is the interaction kernel, $\sigma > 0$ is the diffusion coefficient and W^i are N independent d-dimensional Wiener processes (standard Brownian motions). We assume that the initial values $X^i(0) =: X^i_0$ are drawn independently from the same distribution ρ_0.

Keywords and phrases. Relative entropy, random batch method, interacting particle system, propagation of chaos.

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As is well known, under certain conditions, the mean field limit (*i.e.*, $N \to \infty$) of (1.1) is given by the following nonlinear Fokker–Planck equation:

$$\partial_t \rho = -\nabla \cdot ((b + K * \rho)\rho) + \sigma \Delta \rho, \tag{1.2}$$

where $\rho(t,x)$ is the particle density distribution. The convergence of the interacting particle system (1.1) towards the Fokker–Planck equation (1.2) as $N \to \infty$ has been systemically studied in [8, 9, 15, 21, 30, 37].

If one numerically discretizes (1.1) directly, the computational cost per time step is $O(N^2)$, which is prohibitively expensive for large N. The Random Batch Method (RBM) proposed in [26] is a simple and generic random algorithm to reduce the computation cost per time step from $O(N^2)$ to $O(N)$. The idea was to utilize small but random batch, so that interactions only occur inside the small batches at each time step. This random mini batch idea was the key component of the so-called stochastic gradient descent (SGD) [4, 42] in machine learning. Due to the simplicity and scalability, it already has a variety of applications in solving the Poisson–Nernst–Planck, Poisson–Boltzmann and Fokker–Planck–Landau equations [7, 34], efficient sampling [33], molecular simulation [28, 35], and quantum Monte-Carlo method [23]. Readers can refer to the review article [24].

The RBM algorithm corresponding to (1.1) is given in Algorithm 1. Suppose one aims to do the simulation until time $T > 0$. One first chooses a time step $\tau > 0$ and a batch size $p \ll N, p \geqslant 2$ that divides N. Define the discrete time $t_k := k\tau, k \in \mathbb{N}$. For each time subinterval $[t_{k-1}, t_k)$, there are two steps: (1) at time t_{k-1}, one divides the N particles into $n := N/p$ groups (batches) randomly; (2) the particles evolve with interaction inside the batches only.

Algorithm 1. The Random Batch Method (RBM).

1: **for** $k = 1 : [T/\tau]$ **do**
2: Divide $\{1, 2, \ldots, N\}$ into $n = N/p$ batches randomly.
3: **for** each batch ξ_k **do**
4: Update $\tilde{X}^{i,N}$'s $(i \in \xi_k)$ by solving the following stochastic differential equation (SDE) with $t \in [t_{k-1}, t_k)$:

$$\mathrm{d}\tilde{X}^{i,N} = b\left(\tilde{X}^{i,N}\right)\mathrm{d}t + \frac{1}{p-1}\sum_{j \in \xi_k, j \neq i} K\left(\tilde{X}^{i,N} - \tilde{X}^{j,N}\right)\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}W^i. \tag{1.3}$$

5: **end for**
6: **end for**

The RBM is not only an efficient numerical method for approximating the original particle system (1.1), but also, when N is large, a numerical particle method for the Fokker–Planck equation (1.2). The goal of this paper is to provide an explicit bound $O(\tau^2 + 1/N)$ of the rescaled relative entropy between the random batch system (1.3) and the Fokker–Planck equation (1.2), as given in Theorem 2.1. More precisely, denote

$$\tilde{\rho}_t^N = \mathrm{Law}(\tilde{X}_t^{1,N}, \cdots, \tilde{X}_t^{N,N}),$$

here $\tilde{X}_t^{1,N}, \cdots, \tilde{X}_t^{N,N}$ are the continuous version of the random batch system given by the following for $t \in [T_k, T_{k+1})$

$$\tilde{X}_t^{i,N} = \tilde{X}_{T_k}^{i,N} + (t - T_k)b\left(\tilde{X}_{T_k}^{i,N}\right) + \frac{(t - T_k)}{p-1}\sum_{j \in \xi_k} K(\tilde{X}_{T_k}^{i,N} - \tilde{X}_{T_k}^{j,N}) + \sqrt{2\sigma}(W_t^i - W_{T_k}^i), \tag{1.4}$$

which coincide with the Euler–Maruyama (EM) discretization of (1.3) at time T_k: In this paper, we will prove the following main results:

– (Thm. 2.1) The uniform-in-time relative entropy bound is $O(\tau^2 + \frac{1}{N})$:

$$\sup_t \mathcal{H}_N\left(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\right) \le \mathcal{H}_N\left(\tilde{\rho}_0^N \mid \rho_0^{\otimes N}\right) + c_1\tau^2 + \frac{c_2}{N}, \tag{1.5}$$

where the constants c_1 and c_2 are independent of N and τ. Here,

$$\mathcal{H}_N\left(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\right) = \frac{1}{N}\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N\left(\mathrm{x}^N\right) \log \frac{\tilde{\rho}_t^N\left(\mathrm{x}^N\right)}{\rho_t^{\otimes N}(\mathrm{x}^N)} \,\mathrm{d}\mathrm{x}^N,$$

is the rescaled relative entropy and for convenience we use in the article the bond notation $\mathrm{x}^N = (x_1, \cdots, x_N) \in \mathbb{R}^{Nd}$.

– (Cor. 2.1) The strong propagation of chaos is $O(\tau + \frac{1}{\sqrt{N}})$:

$$\|\tilde{\rho}_t^{N,k} - \rho_t^{\otimes k}\|_{L^1} + W_2\left(\tilde{\rho}_t^{N,k}, \rho_t^{\otimes k}\right) \le C_1\tau + \frac{C_2}{\sqrt{N}}. \tag{1.6}$$

Here we define $\tilde{\rho}_t^{N,k}$ to be the density of the law of the k marginals of the random batch N particle system,

$$\tilde{\rho}_t^{N,k}(x_1, \cdots, x_k) = \int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N(x_1, \cdots, x_N)\,\mathrm{d}x_{k+1}\cdots\mathrm{d}x_N.$$

And we define the usual Wasserstein-2 distance by

$$W_2(\mu,\nu) = \left(\inf_{\gamma\in\Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2\,\mathrm{d}\gamma\right)^{1/2}.$$

Our proof consists of the following ingredients. Firstly, we study the time evolution of the relative entropy of the joint law and the tensorized law based on the Markov property of the discrete random batch. In order to do so, like [32], we derive a Liouville equation for the joint law for the fixed batch version of the RBM, where the drift term is the backward conditional expectation at the previous step, conditioned on the current value of N particles. Then one can obtain the desired time evolution equation. Secondly, during the proof, like in [32], techniques including integration by parts (to eliminate the Gaussian noise that would possibly lower the convergence rate) and the Girsanov transform are used. We also obtain the estimate of the Fisher information for the joint law of the fixed batch version of the RBM, as was done in [39]. Thirdly, we use the Law of Large Number at the exponential scale as in [22]. The main idea of the current work is to obtain the error estimates of the random batch interacting particle system by adapting the tools mentioned. Besides these, there are indeed some new estimates needed for the interacting particle system. The most important feature is the careful analysis of the dependence on the particle number. This is reflected, for example, in estimates of the Fisher information and various local error terms.

Several systems will be involved for our analysis. For clarity, in Figure 1, we depict the relationships between these systems.

Recently, there have been some theoretical results on the random batch method. The strong and weak error analysis for the RBM has been conducted in [27] and they showed that the strong error is of $O(\sqrt{\tau})$ while the weak error is of $O(\tau)$ for interacting particles with disparate species and weights. Note that the weak error used in [27] is computed for a fixed test smooth function. In this paper, the estimate under Wasserstein distances can be used for a family of non-smooth test functions. Moreover, the theoretical justification for the sampling accuracy was done in [29] and they give the geometric ergodicity and the long time behavior of the RBM for interacting particle systems. They showed that the Wasserstein distance between the invariant distributions of the interacting particle systems and the random batch interacting particle systems is bounded by $O(\sqrt{\tau})$. Besides, the error estimate and the long-time behavior of the discrete time approximation of RBM was studied

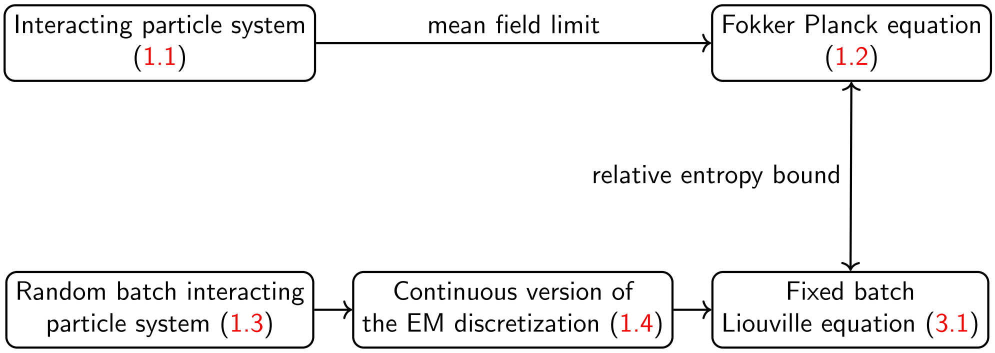

FIGURE 1. Illustration of the various equations.

in [45] and still an $O(\sqrt{\tau})$ bound is obtained. We note that the order of accuracy in the time step τ may not be optimal in the work [29, 45]. This is because they used the strong error estimate of RBM. The novelty of our result is to go beyond the existing strong error of RBM. We analyze the law of RBM at the level of the Liouville equation, enabling us to improve the order of convergence. In this sense, our work indeed represents an improvement in the Wasserstein distance. It is worth mentioning that [25] also investigates the mean field limit of RBM, which gives rise to a mean field limit for p-body particle systems, unlike the classical one-body mean field limit for interacting particle systems. Our work can be seen as an extension and improvement of the work [25]. More precisely, unlike the propagation of chaos in the classical mean field limit for interacting particle systems, their argument of the mean field limit relies on the fact that two particles are unlikely to be related in RBM when $N \to \infty$ for finite iterations. Our results provide a uniform-in-time explicit error estimate between system (1.3) and the mean-field limit (1.2) and directly prove the classical propagation of chaos for RBM. We also clarify that the analogous result in [39] is for the discretization of the original Langevin diffusion where no random approximation is adopted for the drift. So our work provides an improved error estimate for the random batch approximation of the interacting particle systems.

The rest of the paper is organized as follows. We first introduce some preliminaries in Section 2 including the Euler–Maruyama discretization of random batch and the Liouville equation for the joint law for the case of the fixed batch size. In Section 3, we present some auxiliary results which are useful in the proof of the main results. In Section 4, we present the main result: the uniform-in-time estimate of the relative entropy between the joint law of the random batch particles and the tensorized law for the limit equation (the Fokker–Planck equation) and provide the detailed proof. Some technical details that are not so essential are moved to Appendix B.

2. The main results

In the following section, we will briefly introduce the basic setting and the necessary assumptions for the proof in this paper, and then present the main results of this work.

2.1. General setting

For the convenience of the analysis, we define

$$K^{\xi_k}(\tilde{X}_t^{i,N}) = \frac{1}{p-1} \sum_{j \in \xi_k, j \neq i} K(\tilde{X}_t^{i,N} - \tilde{X}_t^{j,N}), \quad t \in [T_k, T_{k+1}). \tag{2.1}$$

Consider the Euler–Maruyama scheme with constant time step for (1.1):

$$X_{T_{k+1}}^i = X_{T_k}^i + \tau b(X_{T_k}^i) + \frac{\tau}{N-1} \sum_{j: j \neq i} K(X_{T_k}^i - X_{T_k}^j) + \sqrt{2\sigma}\Big(W_{T_{k+1}}^i - W_{T_k}^i\Big). \tag{2.2}$$

Similarly, the Euler–Maruyama scheme for the RBM with time step τ at the k-th iteration for the i-th particle is:

$$\tilde{X}^{i,N}_{T_{k+1}} = \tilde{X}^{i,N}_{T_k} + \tau b\Big(\tilde{X}^{i,N}_{T_k}\Big) + \tau K^{\xi_k}(\tilde{X}^{i,N}_{T_k}) + \sqrt{2\sigma}\Big(W^i_{T_{k+1}} - W^i_{T_k}\Big), \tag{2.3}$$

with $T_k := k\tau$. Here, ξ_k are i.i.d. sampled. Also we consider the continuous version which is given for $t \in [T_k, T_{k+1})$ by the following and coincides with the discrete RBM at grid point T_k:

$$\tilde{X}^{i,N}_t = \tilde{X}^{i,N}_{T_k} + (t - T_k) b\Big(\tilde{X}^{i,N}_{T_k}\Big) + (t - T_k) K^{\xi_k}(\tilde{X}^{i,N}_{T_k}) + \sqrt{2\sigma}(W^i_t - W^i_{T_k}). \tag{2.4}$$

Denote

$$\tilde{\mathrm{X}}^N_{T_k} = (\tilde{X}^{1,N}_{T_k}, \cdots, \tilde{X}^{N,N}_{T_k})^T \in \mathbb{R}^{Nd},$$

then the Euler–Maruyama scheme (2.3) for the RBM for N particles with time step τ at the k-th iteration can be written as:

$$\tilde{\mathrm{X}}^N_{T_{k+1}} = \tilde{\mathrm{X}}^N_{T_k} + \tau \mathrm{b}^N(\tilde{\mathrm{X}}^N_{T_k}) + \tau \mathrm{K}^{N,\xi_k}(\tilde{\mathrm{X}}^N_{T_k}) + \sqrt{2\sigma}\Big(\mathrm{W}^N_{T_{k+1}} - \mathrm{W}^N_{T_k}\Big), \tag{2.5}$$

where

$$\mathrm{b}^N(\tilde{\mathrm{X}}^N_{T_k}) = \Big(b\Big(\tilde{X}^{1,N}_{T_k}\Big), \cdots, b\Big(\tilde{X}^{N,N}_{T_k}\Big)\Big)^T \in \mathbb{R}^{Nd}; \tag{2.6}$$

$$\mathrm{K}^{N,\xi_k}(\tilde{\mathrm{X}}^N_{T_k}) = \Big(K^{\xi_k}(\tilde{X}^{1,N}_{T_k}), \cdots, K^{\xi_k}(\tilde{X}^{N,N}_{T_k})\Big)^T \in \mathbb{R}^{Nd}; \tag{2.7}$$

$$\mathrm{W}^N_{T_k} = \big(W^1_{T_k}, \cdots, W^N_{T_k}\big)^T \in \mathbb{R}^{Nd}.$$

At time t, the continuous version for N particles at grid point T_k (2.4) can be written as:

$$\tilde{\mathrm{X}}^N_t = \tilde{\mathrm{X}}^N_{T_k} + (t - T_k)\mathrm{b}^N(\tilde{\mathrm{X}}^N_{T_k}) + (t - T_k)\mathrm{K}^{N,\xi_k}(\tilde{\mathrm{X}}^N_{T_k}) + \sqrt{2\sigma}\big(\mathrm{W}^N_t - \mathrm{W}^N_{T_k}\big). \tag{2.8}$$

We refer (2.5) as the discrete-RBM (or fixed batch size RBM), abbreviated as d-RBM and (2.8) as the continuous-RBM, abbreviated as c-RBM.

Furthermore, denote the N particle joint law of $\tilde{\mathrm{X}}^N_t$ by $\tilde{\rho}^N_t$. Following the basic approach introduced in [20,22], our main idea is to use the relative entropy methods to compare the joint law $\tilde{\rho}^N_t(x_1, \ldots, x_N)$ of the c-RBM (2.8) to the tensorized law

$$\rho^{\otimes N}_t(x_1, \ldots, x_N) = \Pi^N_{i=1}\rho^i_t(x_i),$$

consisting of N independent copies of a process following the law ρ_t, solution to the mean-field limit equation, the Fokker–Planck equation (1.2).

The proof of our main results is based on the Markov property of the d-RBM. More precisely, let $\tilde{\varrho}^{N,\boldsymbol{\xi}}_t$ be the probability density of c-RBM (2.8) for a given sequence of batches $\boldsymbol{\xi} := (\xi_0, \xi_1, \cdots, \xi_k, \cdots)$. Consequently, we have the following expression of the density:

$$\tilde{\rho}^N_t(\mathrm{x}^N) = \mathbb{E}_{\boldsymbol{\xi}}\Big[\tilde{\varrho}^{N,\boldsymbol{\xi}}_t(\mathrm{x}^N)\Big].$$

Here, $\mathbb{E}_{\boldsymbol{\xi}}[\tilde{\varrho}^{N,\boldsymbol{\xi}}_t(\cdot)]$ means taking expectation for all possible choice of batch $\boldsymbol{\xi}$. Note that $\tilde{\varrho}^{N,\boldsymbol{\xi}}_t$ is consistent with an analogue of the Liouville equation, whose explicit expression will be given in Lemma 3.1.

2.2. Assumptions

Before the main results and proofs, we firstly give the following assumptions we use throughout this paper.

Assumption 2.1. (a) *The field b is Lipschitz:*

$$|b(x_1) - b(x_2)| \le r|x_1 - x_2|.$$

Moreover, the field b is twice differentiable and its Hessian have at most polynomial growth:

$$\big|\nabla^2 b(x)\big| \le C(1 + |x|)^q.$$

(b) *The field b is strongly confining in the sense that there exists two constants α and β such that for any $x_1 \neq x_2$:*

$$(x_1 - x_2) \cdot (b(x_1) - b(x_2)) \leq \alpha - \beta |x_1 - x_2|^2$$

for some constant $\beta > 0$.

(c) *The interaction kernel K is bounded, and Lipschitz:*

$$|K(x) - K(y)| \leq L|x - y|.$$

Moreover, the interaction kernel K is twice differentiable and their Hessians have at most polynomial growth:

$$\left|\nabla^2 K(x)\right| \leq \tilde{C}(1 + |x|)^q.$$

In condition (b), the confining property of b holds if there exists a compact set $S \subset \mathbb{R}^d$ such that for any $x_1, x_2 \notin S$, it holds that:

$$(x_1 - x_2) \cdot (b(x_1) - b(x_2)) \leq -\beta |x_1 - x_2|^2.$$

Utilization of the confining property is essential in the proof of moment control. We mention that if the constant satisfies $\beta > 2L$, one can obtain the uniform-in-time moment control for random batch particles. One can refer [25, 32, 45].

By definition, K^{ξ_k} satisfies the same conditions as those for K for any ξ_k. Our assumptions here are kind of standard in literature. See [26, 27, 29, 45] for examples.

We would like to mention that there should be some straightforward relaxation for our assumptions. For example, the boundedness of the kernel K could be relaxed to be of the form $f(X_i)\tilde{K}(X_i - X_j)$ where f is not necessarily bounded (*e.g.*, with linear growth). Besides, the generalization to non-constant σ is possible for $\sigma = \sigma(X_i)$, namely it depends on X_i itself only. However, there is some intrinsic difficulty if there is more general dependence in the diffusion coefficient like $\sigma(X_i - X_j)$. These generalized assumptions would not change the essential proof but we choose not to involve them for the clarity our proof.

In this paper, we obtain the uniform in time estimate under an additional assumption of a Log-Sobolev inequality (LSI) for ρ_t, uniformly in t. We remark that such assumption is a common ingredient in the proof of uniform-in-time propagation of chaos starting from [36], see also the recent paper [13, 16, 31].

Assumption 2.2. *The Log-Sobolev inequality (LSI): There exists a constant $C_{LS} > 0$ such that for any non-negative smooth functions f, one has*

$$\operatorname{Ent}_{\rho_t}(f) := \int f \log f \, \mathrm{d}\rho_t - \left(\int f \, \mathrm{d}\rho_t\right) \log\left(\int f \, \mathrm{d}\rho_t\right) \leq C_{LS} \int \frac{|\nabla f|^2}{f} \, \mathrm{d}\rho_t. \tag{2.9}$$

One crucial property of the LSI is the tensorization, *i.e.*, if ρ_t satisfies a LSI then $\rho_t^{\otimes N}$ satisfies the same inequality with the same constant (and thus independent of N).

2.3. Main results

In the following, the notation $f \lesssim g$ indicates that there exists a universal constant C such that $f \leq Cg$ holds. We state the main theorem as follows:

Theorem 2.1. *Consider the joint law $\tilde{\rho}_t^N$ for random batch N particle system defined in (2.8) and the tensorized law $\rho_t^{\otimes N}$ of the Fokker–Plank equation (1.2). Suppose $\sigma > 0$ and take $T > 0$. Then, under Assumptions 2.1 and 2.2, there exist positive constants c_1 and Δ independent of N and T, and constant $c_2 = c_2(T)$ independent of N, such that for all $\tau \in (0, \Delta)$ and $t \leq T$,*

$$\mathcal{H}_N\left(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\right) \leq e^{c_1 t} \mathcal{H}_N\left(\tilde{\rho}_0^N \mid \rho_0^{\otimes N}\right) + c_2\left(\tau^2 + \frac{1}{N}\right). \tag{2.10}$$

Moreover, if $\beta > 2L$ and $\|K\|_{L^\infty}^2 \le \frac{\sigma}{8e^2 C_{LS}}$, then $c_1 < 0$ and c_2 can be taken to be independent of T so the above bound is uniform-in-time. Here σ is he diffusion coefficient in (1.1) and C_{LS} is the Log-Sobolev inequality coefficient in (2.9).

It is well-known that one can bound the L^1 norm with square root of the relative entropy by the Csiszár–Kullback–Pinsker inequality, see for instance [3]. Besides, under the Log-Sobolev inequality Assumption 2.2 for ρ_t, one also can bound the Wasserstein-2 distance with square root of the relative entropy by transport inequality, see for instance [41].

Corollary 2.1. *We define $\tilde{\rho}_t^{N,k}$ to be the density of the law of the k marginals of the random batch N particle system, under the same assumptions as in Theorem 4, the following holds:*

$$\|\tilde{\rho}_t^{N,k} - \rho_t^{\otimes k}\|_{L^1} + W_2\left(\tilde{\rho}_t^{N,k}, \rho_t^{\otimes k}\right) \le C_1\tau + \frac{C_2}{\sqrt{N}}. \tag{2.11}$$

Here the constants C_1, C_2 are independent of N and τ. In particular, if $\beta > 2L$ and $\|K\|_{L^\infty}^2 \le \frac{\sigma}{8e^2 C_{LS}}$, then the above bound is uniform-in-time.

3. Some useful estimates

In this section, we make some preparation of the proof by providing some useful estimates, which will be used in the proof of the main result in the next section. In particular, we will first of all provide the dynamics of the law of the particle system, and then give estimates for the moments and Fisher information.

3.1. An analogue of the Liouville equation

Consider the c-RBM (2.8). Our goal is to prove that the probability density of the continuous version satisfies an analogue of the Liouville equation with time-varying drift terms.

Lemma 3.1. *Denote by $\tilde{\varrho}_t^{N,\boldsymbol{\xi}}$ the probability density function of $\tilde{\mathrm{X}}_t^N = \left(\tilde{X}_t^{1,N}, \cdots, \tilde{X}_t^{N,N}\right)$ defined in (2.8) for $t \in [T_k, T_{k+1})$. Then the following Liouville equation holds:*

$$\partial_t \tilde{\varrho}_t^{N,\boldsymbol{\xi}} + \sum_{i=1}^N \mathrm{div}_{x_i}\left(\tilde{\varrho}_t^{N,\boldsymbol{\xi}}\left(\hat{b}_t^{\boldsymbol{\xi},i}(\mathrm{x}^N) + \hat{K}_t^{\boldsymbol{\xi},i}(\mathrm{x}^N)\right)\right) = \sum_{i=1}^N \sigma \Delta_{x_i} \tilde{\varrho}_t^{N,\boldsymbol{\xi}}, \tag{3.1}$$

where

$$\hat{b}_t^{\boldsymbol{\xi},i}(\mathrm{x}^N) = \mathbb{E}\left[b\left(\tilde{X}_{T_k}^{i,N}\right) \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \boldsymbol{\xi}\right], \quad t \in [T_k, T_{k+1}), \tag{3.2}$$

and

$$\hat{K}_t^{\boldsymbol{\xi},i}(\mathrm{x}^N) := \mathbb{E}\left[K^{\xi_k}\left(\tilde{X}_{T_k}^{i,N}\right) \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \boldsymbol{\xi}\right], \quad t \in [T_k, T_{k+1}). \tag{3.3}$$

Here, $\mathrm{x}^N = (x_1, \cdots, x_n) \in \mathbb{R}^{Nd}$.

The derivation is not difficult and has appeared in many previous work [32, 39]. For the convenience of the readers, we also attach a proof in Appendix A.

3.2. Moment control

In this section, we aim to control the moments $\mathbb{E}\left|\tilde{X}_t^{i,N}\right|^p$. We have the following fact. The proof is similar to Lemma 3.6 of [25], so we omit it.

Lemma 3.2. *Under Assumption 2.1, for any $p > 2$, there exists constants C_p and α independent of N and $\boldsymbol{\xi}$ such that for any i*

$$\sup_{t\le T}\mathbb{E}\left[\left|\tilde{X}_t^{i,N}\right|^p \mid \mathcal{F}_{\boldsymbol{\xi}}\right] \le C_p e^{\alpha T}, \tag{3.4}$$

where $\mathcal{F}_{\boldsymbol{\xi}}$ denotes the σ-algebra generated by sequence $\boldsymbol{\xi}$. If $\beta > 2L$, then the above bound is uniform-in-time, namely

$$\sup_{t\ge 0}\mathbb{E}\left[\left|\tilde{X}_t^{i,N}\right|^p \mid \mathcal{F}_{\boldsymbol{\xi}}\right] \le C_p. \tag{3.5}$$

The above result indicates that for a fixed sequence of divisions of random batches $\boldsymbol{\xi} = (\xi_k)_{k\ge 0}$, the p-th moment of $\tilde{X}_t^{i,N}$ is bounded by C_p independent of $\boldsymbol{\xi}$. We remark that using the moment control, we can bound terms like $b(\tilde{X}_t^{i,N})$ and $K^{N,\xi_k}(\tilde{X}_t^{i,N})$ which can be found in the proof Theorem 2.1.

3.3. Estimate of the Fisher information

The Fisher information for a probability density ρ is defined by

$$\mathcal{I}(\rho) = \int |\nabla \log \rho|^2 \rho \, \mathrm{d}x. \tag{3.6}$$

And $\mathcal{I}(\rho) = \infty$ if ρ does not have a density that has a gradient. In our analysis, we require a bound for the Fisher information of $\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}$, which is the law of RBM for a given sequence of batches $\boldsymbol{\xi} = (\xi_0, \cdots, \xi_k, \cdots)$ at grid point T_k. Our proof is based on Stam's convolution inequality for Fisher information ([43], Eq. (2.9)). This inequality guarantees that for any pair of suitably regular probability density functions p and q on $\mathbb{R}^d$ that ensure the existence of the integral of the Fisher information, then the Fisher information satisfies the inequality

$$\frac{1}{\mathcal{I}(p*q)} \ge \frac{1}{\mathcal{I}(p)} + \frac{1}{\mathcal{I}(q)}, \tag{3.7}$$

where $p*q$ denotes the convolution of p and q. The d-RBM (2.5) for N particles at time T_{k+1} can be seen as a combination of applying the deterministic mapping

$$\psi_\tau^{\xi_k}(\mathrm{x}^N) := \mathrm{x}^N + \tau\big(\mathrm{b}^N(\mathrm{x}^N) + \mathrm{K}^{N,\xi_k}(\mathrm{x}^N)\big), \quad \mathrm{x}^N = (x_1, \cdots, x_n) \in \mathbb{R}^{Nd}, \tag{3.8}$$

with a convolution step with a Gaussian kernel, where b^N and K^{N,ξ_k} are defined as (2.6) and (2.7) respectively. More precisely,

$$\mathrm{b}^N(\mathrm{x}^N) = (b(x_1), \cdots, b(x_n))^T \in \mathbb{R}^{Nd},$$

and

$$\mathrm{K}^{N,\xi_k}(\mathrm{x}^N) = \big(K^{\xi_k}(x_1), \cdots, K^{\xi_k}(x_n)\big)^T \in \mathbb{R}^{Nd}.$$

We exploit the inequality (3.7) so as to bound the Fisher information $\mathcal{I}(\tilde{\varrho}_{T_{k+1}}^{N,\boldsymbol{\xi}})$ in terms of $\mathcal{I}(\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}})$. In order to do so, we bound the Fisher information for the intermediate density after the first step.

Lemma 3.3. *For every step size $\tau \in \left(0, \frac{1}{r+L}\right)$, let $p_k(\cdot)$ be the density of the random variable $\mathrm{Z}_k^N = \psi_\tau^{\xi_k}(\tilde{\mathrm{X}}_{T_k}^N)$ obtained by applying the deterministic mapping $\psi_\tau^{\xi_k}$. Then under Assumption 2.1, we have the bound*

$$\mathcal{I}(p_k) \le \frac{1+\tau(r+L)}{1-\tau(r+L)}\left(\mathcal{I}(\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}) + \frac{M_k(r+L)N\tau}{1-\tau(r+L)}\right), \tag{3.9}$$

where $M_k \le C_q e^{\alpha k\tau}$ as in (3.4) for some $q > 0$ and is independent of N.

If $\beta > 2L$, then M_k can be taken to be a constant M that is independent of k.

Proof. Consider the norm of x^N defined as $\|\mathrm{x}^N\|^2 = \sum_i^n \|x_i\|_2^2$. Then, Assumption 2.1 implies that for $\tau < \frac{1}{r+L}$, the mapping $\psi_\tau^{\xi_k}(\mathrm{x}^N) := \mathrm{x}^N + \tau\big(\mathrm{b}^N(\mathrm{x}^N) + \mathrm{K}^{N,\xi_k}(\mathrm{x}^N)\big)$ is a bi-Lipschitz mapping, *i.e.*,

$$(1-\tau(r+L))\|\mathrm{x}^N - \mathrm{y}^N\| \le \|\psi_\tau^{\xi_k}(\mathrm{x}^N) - \psi_\tau^{\xi_k}(\mathrm{y}^N)\| \le (1+\tau(r+L))\|\mathrm{x}^N - \mathrm{y}^N\|,$$

from which one deduces that

$$\|\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)\| \le (1+\tau(r+L)), \quad \|\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)^{-1}\| \le (1-\tau(r+L))^{-1}. \tag{3.10}$$

By the change of variable formula, we have

$$p_k(\mathrm{z}^N) = \frac{\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)}{\left|\det(\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N))\right|}.$$

Consequently, we have the bound on the Fisher information:

$$\begin{aligned}
\mathcal{I}(p_k) &= \int_{\mathbb{R}^{Nd}} p_k(\mathrm{z}^N)\big|\nabla_{\mathrm{z}^N}\log p_k(\mathrm{z}^N)\big|^2\,\mathrm{d}\mathrm{z}^N \\
&= \int_{\mathbb{R}^{Nd}} \tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\Big|\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)^{-1}\Big(\nabla_{\mathrm{x}^N}\log\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N) - \nabla_{\mathrm{x}^N}\log\big|\det\big(\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)\big)\big|\Big)\Big|^2\,\mathrm{d}\mathrm{x}^N \\
&\le (1+\tau(r+L))\int_{\mathbb{R}^{Nd}} \tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\Big|\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)^{-1}\nabla_{\mathrm{x}^N}\log\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\Big|^2\,\mathrm{d}\mathrm{x}^N \\
&\quad + \left(1+\frac{1}{\tau(r+L)}\right)\int_{\mathbb{R}^{Nd}} \tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\big|\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)^{-1}\nabla_{\mathrm{x}^N}\log\big|\det\big(\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)\big)\big|\big|^2\,\mathrm{d}\mathrm{x}^N \\
&=: I_1 + I_2.
\end{aligned}$$

The inequality follows from the simple fact $(a+b)^2 \le (1+\lambda)a^2 + (1+\frac{1}{\lambda})b^2$. It is clear that

$$I_1 \le \frac{1+\tau(r+L)}{1-\tau(r+L)}\int_{\mathbb{R}^{Nd}} \tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\Big|\nabla\log\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\Big|^2\,\mathrm{d}\mathrm{x}^N = \frac{1+\tau(r+L)}{1-\tau(r+L)}\mathcal{I}(\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}).$$

The second term can be estimated by

$$\begin{aligned}
I_2 &= \frac{1+\tau(r+L)}{\tau(r+L)}\int_{\mathbb{R}^{Nd}} \tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\big|\nabla\psi_\tau^{\xi_k}(\mathrm{x}^N)^{-2}\nabla\cdot\big(I_{Nd} + \tau\nabla\mathrm{b}^N(\mathrm{x}^N) + \tau\nabla\mathrm{K}^{N,\xi_k}(\mathrm{x}^N)\big)\big|^2\,\mathrm{d}\mathrm{x}^N \\
&\le \frac{(1+\tau(r+L))\tau}{(1-\tau(r+L))^2(r+L)}\int_{\mathbb{R}^{Nd}} \tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\big|\nabla\cdot\big(\nabla\mathrm{b}^N(\mathrm{x}^N) + \nabla\mathrm{K}^{N,\xi_k}(\mathrm{x}^N)\big)\big|^2\,\mathrm{d}\mathrm{x}^N \\
&\le \frac{(1+\tau(r+L))N\tau}{(1-\tau(r+L))^2}\frac{M_k}{r+L},
\end{aligned}$$

where the last inequality above is due to the polynomial growth of the Hessians and the boundness of the moments in Lemma 3.2. The moment bound M_k is independent of N.

If $\beta > 2L$, we have the uniform-in-time bound (3.5) for the moments, thus M_k can be taken to be independent of k. ☐

Let q_τ denote the Nd-dimensional Gaussian distribution $\mathcal{N}(0, 2\sigma\tau\mathrm{I}_{Nd})$. Clearly we have the identity $\mathcal{I}(q_\tau) = \frac{Nd}{2\sigma\tau}$. By the d-RBM (2.5), we have that $\tilde{\varrho}_{T_{k+1}}^{N,\boldsymbol{\xi}} = p_k(\mathrm{x}^N) * q_\tau$. Let us take $k_0 > 0$. Then, $M_k \le M_{k_0}$ for all $k \le k_0$. We have for $\tau < \frac{1}{2(r+L)}$ by invoking the convolution inequality (3.7) the following the bound

$$\frac{1}{\mathcal{I}\Big(\tilde{\varrho}_{T_{k+1}}^{N,\boldsymbol{\xi}}\Big)} \ge \frac{1}{\mathcal{I}(p_k)} + \frac{1}{\mathcal{I}(q_\tau)} \ge \frac{(1-\tau(r+L))^2}{1+\tau(r+L)}\frac{1}{\max(\mathcal{I}\Big(\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}\Big), M_{k_0}N)} + \frac{2\sigma\tau}{Nd}. \tag{3.11}$$

Let $u_k = \frac{1}{\mathcal{I}\left(\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}\right)}$. Then, one has a relation like

$$u_{k+1} \geq \min(\gamma u_k + \delta, \gamma B + \delta)$$

where $\gamma = (1 - \tau(r + L))^2/(1 + \tau(r + L))$, $B = \frac{1}{M_{k_0} N}$ and $\delta = \frac{2\sigma\tau}{Nd}$. For this recursion, one easily obtains

$$u_k \geq \min\left(u_0, \gamma B + \delta, \frac{\delta}{1-\gamma}\right),$$

because $u < \gamma u + \delta$ for $u < \delta/(1-\gamma)$ and $u > \gamma u + \delta$ for $u > \delta/(1-\gamma)$.

Consequently, one can obtain the following control for the Fisher information (by taking $k = k_0$ and then rename k_0 to be k). The upper bound clearly scales linearly with N.

Lemma 3.4. *Under Assumption 2.1, we have the following bound of the Fisher information independent of the batch* $\boldsymbol{\xi} = (\xi_0, \cdots, \xi_k, \cdots)$*:*

$$\mathcal{I}\left(\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}\right) \leq \max\left(\mathcal{I}\left(\rho_0^N\right), \frac{1 + \tau(r+L)}{(1 - \tau(r+L))^2} M_k N, \frac{Nd(r+L)(3 + \tau(r+L))}{2\sigma}\right), \tag{3.12}$$

where M_k *has the same bound as in Lemma 3.3. If* $\beta > 2L$*, then* M_k *can be taken to be a constant* M*, independent of* k*.*

4. Proof of the main results

Equipped with the preparation work before, we now establish an $O(\tau^2)$ error estimate for RBM in terms of the scaled relative entropy.

4.1. Time evolution of the relative entropy

Firstly, we notice that the d-RBM (2.5) at discrete time points is a time homogeneous Markov chain, and $\tilde{\rho}_{T_k}^N$ is the law at T_k. Recall that $\tilde{\varrho}_t^{N,\boldsymbol{\xi}}$ is the probability density of the c-RBM (2.8) for a given sequence of batches $\boldsymbol{\xi} := (\xi_0, \xi_1, \cdots, \xi_k, \cdots)$, so that $\tilde{\rho}_{T_k}^N = \mathbb{E}_{\boldsymbol{\xi}}\left[\tilde{\varrho}_{T_k}^{N,\boldsymbol{\xi}}\right]$. Moreover, by the Markov property, we are able to define the conditional expectation given batch ξ_k:

$$\tilde{\rho}_t^{N,\xi_k} := \mathbb{E}\left[\tilde{\varrho}_t^{N,\boldsymbol{\xi}} \mid \xi_k\right] = \mathcal{S}_{T_k,t}^{N,\xi_k} \tilde{\rho}_{T_k}^N, \quad t \in [T_k, T_{k+1}), \tag{4.1}$$

where the operator $\mathcal{S}_{T_k,t}^{N,\xi_k}$ is the evolution operator from T_k to t for the Liouville equation of the c-RBM (2.8) derived in Lemma 3.1, for some given ξ_k:

$$\partial_t \tilde{\rho}_t^{N,\xi_k} + \sum_{i=1}^N \operatorname{div}_{x_i}\left(\tilde{\rho}_t^{N,\xi_k}\left(\tilde{b}_t^{\xi_k,i}\left(\mathrm{x}^N\right) + \tilde{K}_t^{\xi_k,i}\left(\mathrm{x}^N\right)\right)\right) = \sum_{i=1}^N \sigma \Delta_{x_i} \tilde{\rho}_t^{N,\xi_k}, \quad \tilde{\rho}_{T_k}^{N,\xi_k} = \tilde{\rho}_{T_k}^N, \tag{4.2}$$

where

$$\tilde{b}_t^{\xi_k,i}\left(\mathrm{x}^N\right) := \mathbb{E}\left[b\left(\tilde{X}_{T_k}^{i,N}\right) \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\right], \quad t \in [T_k, T_{k+1}),$$

and

$$\tilde{K}_t^{\xi_k,i}\left(\mathrm{x}^N\right) := \mathbb{E}\left[K^{\xi_k}\left(\tilde{X}_{T_k}^{i,N}\right) \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\right], \quad t \in [T_k, T_{k+1}).$$

Here, $\tilde{b}_t^{\xi_k,i}\left(\mathrm{x}^N\right)$ and $\tilde{K}_t^{\xi_k,i}\left(\mathrm{x}^N\right)$, the expectation conditional on the k-th batch ξ_k, are a little bit different from (3.2) and (3.3) which are the expectation conditional on all fixed batches $\boldsymbol{\xi} = (\xi_0, \cdots, \xi_k, \cdots)$.

Since for $t \in [T_k, T_{k+1})$,

$$\tilde{\rho}_t^N = \mathbb{E}_{\xi_k}\left[\tilde{\rho}_t^{N,\xi_k}\right],$$

then by (4.2) one gets

$$\partial_t \tilde{\rho}_t^N = -\sum_{i=1}^N \mathbb{E}_{\xi_k}\left[\mathrm{div}_{x_i}\left(\tilde{\rho}_t^{N,\xi_k}\left(\tilde{b}_t^{\xi_k,i}(\mathrm{x}^N) + \tilde{K}_t^{\xi_k,i}(\mathrm{x}^N)\right)\right) + \sigma \Delta_{x_i}\tilde{\rho}_t^{N,\xi_k}\right]. \tag{4.3}$$

Recall the definition of the tensorized law $\rho_t^{\otimes N}$, one can readily check that $\rho_t^{\otimes N}$ solves

$$\partial_t \rho_t^{\otimes N} + \sum_{i=1}^N \mathrm{div}_{x_i}\left(\rho_t^{\otimes N}(b(x_i) + K * \rho_t(x_i))\right) = \sum_{i=1}^N \sigma \Delta_{x_i}\rho_t^{\otimes N}. \tag{4.4}$$

By using the equations (4.4) and (4.3) for $\rho_t^{\otimes N}$ and $\tilde{\rho}_t^N$, respectively, we calculate the time derivative of the relative entropy in the time interval $[T_k, T_{k+1})$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N\left(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\right) &= \frac{1}{N}\int_{\mathbb{R}^{Nd}}\left(\partial_t \tilde{\rho}_t^N\right)\left(\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}} + 1\right)\mathrm{d}\mathrm{x}^N + \frac{1}{N}\int_{\mathbb{R}^{Nd}}\left(\partial_t \rho_t^{\otimes N}\right)\left(-\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right)\mathrm{d}\mathrm{x}^N \\
&= \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\left(\mathbb{E}_{\xi_k}\left(\tilde{\rho}_t^{N,\xi_k}\left(\tilde{b}_t^{\xi_k,i}(\mathrm{x}^N) + \tilde{K}_t^{\xi_k,i}(\mathrm{x}^N)\right)\right) - \sigma\nabla_{x_i}\tilde{\rho}_t^N\right)\cdot\left(\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right)\mathrm{d}\mathrm{x}^N \\
&\quad + \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\left(\rho_t^{\otimes N}(b(x_i) + K * \rho_t(x_i)) - \sigma\nabla_{x_i}\rho_t^{\otimes N}\right)\cdot\left(-\nabla_{x_i}\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right)\mathrm{d}\mathrm{x}^N \\
&= \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\left(\mathbb{E}_{\xi_k}\left(\tilde{\rho}_t^{N,\xi_k}\tilde{b}_t^{\xi_k,i}(\mathrm{x}^N)\right) - \tilde{\rho}_t^N b(x_i)\right)\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N \\
&\quad + \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\mathbb{E}_{\xi_k}\left(\tilde{\rho}_t^{N,\xi_k}\tilde{K}_t^{\xi_k,i}(\mathrm{x}^N)\right)\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N \\
&\quad - \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\rho_t^{\otimes N}(K * \rho_t(x_i))\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N \\
&\quad - \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\sigma\nabla_{x_i}\tilde{\rho}_t^N\cdot\nabla\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N + \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\sigma\nabla_{x_i}\rho_t^{\otimes N}\cdot\nabla_{x_i}\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N.
\end{aligned} \tag{4.5}$$

Note that

$$\begin{aligned}
&-\frac{1}{N}\int \sigma\nabla_{x_i}\tilde{\rho}_t^N\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N + \frac{1}{N}\int \sigma\nabla_{x_i}\rho_t^{\otimes N}\cdot\nabla_{x_i}\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N \\
&\qquad = -\frac{\sigma}{N}\int \tilde{\rho}_t^N\left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2\mathrm{d}\mathrm{x}^N.
\end{aligned}$$

Introduce

$$F^N(x_i) = \frac{1}{N-1}\sum_{j:j\neq i} K(x_i - x_j),$$

and rearrange the terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N\left(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\right) = \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\mathbb{E}_{\xi_k}\left(\rho_t^{N,\xi_k}\left(\tilde{b}_t^{\xi_k,i}(\mathrm{x}^N) - b(x_i)\right)\right)\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\mathrm{d}\mathrm{x}^N$$

$$
\begin{aligned}
&+\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{Nd}}\mathbb{E}_{\xi_k}\Big(\tilde{\rho}_t^{N,\xi_k}\tilde{K}_t^{\xi_k,i}(\mathrm{x}^N)-\tilde{\rho}_t^{N,\xi_k}K^{\xi_k}(x_i)\Big)\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\,\mathrm{dx}^N\\
&+\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{Nd}}\mathbb{E}_{\xi_k}\Big(\tilde{\rho}_t^{N,\xi_k}K^{\xi_k}(x_i)-\tilde{\rho}_t^{N}F^N(x_i)\Big)\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\,\mathrm{dx}^N\\
&+\frac{1}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{Nd}}\big(F^N(x_i)-K*\rho_t(x_i)\big)\tilde{\rho}_t^N\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\,\mathrm{dx}^N\\
&-\frac{\sigma}{N}\sum_{i=1}^{N}\int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^N\left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2\mathrm{dx}^N\\
:=&\ \frac{1}{N}\sum_{i=1}^{N}(J_1^i+J_2^i+J_3^i+J_4^i+J_5^i).
\end{aligned}
\tag{4.6}
$$

As a remark, the term $\left|\tilde{\rho}_t^{N,\xi_k}K^{\xi_k}(x_i)-\tilde{\rho}_t^NF^N(x_i)\right|$ in J_3^i is of $O(1)$, since $|K^{\xi_k}-F^N(x_i)|=O(1)$, which is not small. This is the difference of random approximation from the usual deterministic approximation. Direct bounding this term by $O(1)$ bound would not give convergence. We will use the averaging effect in $\mathbb{E}_{\xi_k}$ largely.

As done in [32], our approach to use the the averaging effect in $\mathbb{E}_{\xi_k}$ is to introduce another copy of RBM $\hat{\mathrm{X}}^N$ that depends on another batch $\tilde{\xi}_k$ such that

- $\hat{\mathrm{X}}_{T_k}^N=\tilde{\mathrm{X}}_{T_k}^N$;
- the Brownian motions are the same in $[T_k,T_{k+1})$;
- the batch $\tilde{\xi}_k$ on $[T_k,T_{k+1})$ is independent of ξ_k.

Consequently, density of the law $\tilde{\rho}_t^{N,\tilde{\xi}_k}$ for $\hat{\mathrm{X}}^N$ satisfies both (4.1) and (4.2). Then

$$
\begin{aligned}
J_3^i&=\int_{\mathbb{R}^{Nd}}\mathbb{E}_{\xi_k}\Big[\big(K^{\xi_k}(x_i)-F^N(x_i)\big)\Big(\tilde{\rho}_t^{N,\xi_k}-\tilde{\rho}_t^N\Big)\Big]\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\,\mathrm{dx}^N\\
&=\int_{\mathbb{R}^{Nd}}\mathbb{E}_{\xi_k,\tilde{\xi}_k}\Big[\big(K^{\xi_k}(x_i)-F^N(x_i)\big)\Big(\tilde{\rho}_t^{N,\xi_k}-\tilde{\rho}_t^{N,\tilde{\xi}_k}\Big)\Big]\cdot\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\,\mathrm{dx}^N\\
&\le\frac{2}{\sigma}\mathbb{E}_{\xi_k,\tilde{\xi}_k}\left[\int_{\mathbb{R}^{Nd}}\big|K^{\xi_k}(x_i)-F^N(x_i)\big|^2\frac{\left|\tilde{\rho}_t^{N,\xi_k}-\tilde{\rho}_t^{N,\tilde{\xi}_k}\right|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\,\mathrm{dx}^N\right]+\frac{\sigma}{8}\int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^N\left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2\mathrm{dx}^N.
\end{aligned}
\tag{4.7}
$$

Here the first equality is due to the fact

$$\mathbb{E}_{\xi_k}\left[\tilde{\rho}_t^{N,\xi_k}\right]=\tilde{\rho}_t^N,$$

and the consistency of the random batch, that is

$$\mathbb{E}_{\xi_k}\left[K^{\xi_k}(x_i)\right]=F^N(x_i).$$

The second equality is due to $\mathbb{E}_{\tilde{\xi}_k}\left[\tilde{\rho}_t^{N,\tilde{\xi}_k}\right]=\tilde{\rho}_t^N$. In the last inequality, we applied Young's inequality and the fact $\mathbb{E}_{\xi_k,\tilde{\xi}_k}\left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2\tilde{\rho}_t^{N,\tilde{\xi}_k}=\left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2\tilde{\rho}_t^N$. The introduction of the independent copy of $\tilde{\xi}_k$ is useful since we may apply the Girsanov transform later to estimate this quantitatively.

Then apply Young's inequality for J_1^i, J_2^i, J_4^i, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N\big(\tilde{\rho}_t^N\mid\rho_t^{\otimes N}\big)\le\mathcal{A}_N(t)+\mathcal{B}_N(t)+\mathcal{C}_N(t)+\mathcal{D}_N(t)-\frac{\sigma}{2}\mathcal{I}_N(t),\tag{4.8}$$

where

$$
\begin{aligned}
\mathcal{A}_N(t) &= \frac{2}{\sigma N}\sum_{i=1}^N \mathbb{E}_{\xi_k}\int_{\mathbb{R}^{Nd}}\Big|\tilde{b}_t^{\xi_k,i}(\mathrm{x}^N)-b(x_i)\Big|^2\tilde{\rho}_t^{N,\xi_k}\,\mathrm{dx}^N;\\
\mathcal{B}_N(t) &= \frac{2}{\sigma N}\sum_{i=1}^N \mathbb{E}_{\xi_k}\int_{\mathbb{R}^{Nd}}\Big|\tilde{K}_t^{\xi_k,i}(\mathrm{x}^N)-K^{\xi_k}(x_i)\Big|^2\tilde{\rho}_t^{N,\xi_k}\,\mathrm{dx}^N;\\
\mathcal{C}_N(t) &= \frac{2}{\sigma N}\sum_{i=1}^N \mathbb{E}_{\xi_k,\tilde{\xi}_k}\left[\int_{\mathbb{R}^{Nd}}\big|K^{\xi_k}(x_i)-F^N(x_i)\big|^2\frac{\Big|\tilde{\rho}_t^{N,\xi_k}-\tilde{\rho}_t^{N,\tilde{\xi}_k}\Big|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\right]\mathrm{dx}^N;\\
\mathcal{D}_N(t) &= \frac{2}{\sigma N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^N\big|F^N(x_i)-K*\rho_t(x_i)\big|^2\,\mathrm{dx}^N;\\
\mathcal{I}_N(t) &= \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^N\left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2\mathrm{dx}^N.
\end{aligned}
\tag{4.9}
$$

4.2. Bounding the terms $\mathcal{A}_N(t)$ and $\mathcal{B}_N(t)$

Before bounding these two terms, we first prove the following useful lemma.

Lemma 4.1. *Under the setting of Theorem 2.1, for any $T>0$, there exists a positive constant $c(T)$ independent of ξ_k, N and τ such that for $t\in[T_k,T_{k+1})$ with $T_{k+1}\le T$,*

$$
\int_{\mathbb{R}^{Nd}}\Big|\mathbb{E}\Big[\tilde{\mathrm{X}}_{T_k}^N-\tilde{\mathrm{X}}_t^N \mid \tilde{\mathrm{X}}_t^N=\mathrm{x}^N,\xi_k\Big]\Big|^2\tilde{\rho}_t^{N,\xi_k}\,\mathrm{dx}^N \le c(T)\tau^2\big(3N+\mathcal{I}\big(\tilde{\rho}_{T_k}^N\big)\big),\tag{4.10}
$$

where $\mathcal{I}\big(\tilde{\rho}_{T_k}^N\big)$ is the Fisher information. If $\beta>2L$, then the constant $c(T)$ can be taken to be independent of T.

Proof. By Bayes's law,

$$
\begin{aligned}
\mathbb{E}\Big[\tilde{\mathrm{X}}_{T_k}^N-\tilde{\mathrm{X}}_t^N \mid \tilde{\mathrm{X}}_t^N=\mathrm{x}^N,\xi_k\Big] &= \int_{\mathbb{R}^{Nd}}(\mathrm{y}^N-\mathrm{x}^N)p\Big(\tilde{\mathrm{X}}_{T_k}^N=\mathrm{y}^N \mid \tilde{\mathrm{X}}_t^N=\mathrm{x}^N,\xi_k\Big)\,\mathrm{dy}^N\\
&= \int_{\mathbb{R}^{Nd}}(\mathrm{y}^N-\mathrm{x}^N)\frac{\tilde{\rho}_{T_k}^N(\mathrm{y}^N)p\Big(\tilde{\mathrm{X}}_t^N=\mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N=\mathrm{y}^N,\xi_k\Big)}{\tilde{\rho}_t^{N,\xi_k}(\mathrm{x}^N)}\,\mathrm{dy}^N.
\end{aligned}
\tag{4.11}
$$

Here, $p\Big(\tilde{\mathrm{X}}_t^N=\mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N=\mathrm{y}^N,\xi_k\Big)$ is the transition density of particle $\tilde{\mathrm{X}}_{T_k}^N$ and $\tilde{\mathrm{X}}_t^N$ from y^N to x^N for given batch ξ_k. Clearly, this conditional density is a Gaussian centered at $\mathrm{y}^N+(t-T_k)\big(\mathrm{b}^N(\mathrm{y}^N)+\mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)$ with fixed covariance, namely,

$$
\begin{aligned}
&p\Big(\tilde{\mathrm{X}}_t^N=\mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N=\mathrm{y}^N,\xi_k\Big)\\
&\quad= (4\pi\sigma(t-T_k))^{-\frac{Nd}{2}}\exp\left(-\frac{\big|\mathrm{x}^N-\mathrm{y}^N-\mathrm{b}^N(\mathrm{y}^N)(t-T_k)-\mathrm{K}^{N,\xi_k}(\mathrm{y}^N)(t-T_k)\big|^2}{4\sigma(t-T_k)}\right).
\end{aligned}
$$

Therefore, the gradient with respect to y^N is the density itself times a linear factor $\mathrm{y}^N-\mathrm{x}^N+(t-T_k)\big(\mathrm{b}^N(\mathrm{y}^N)+\mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)$, with an additional factor depending on the Jacobian of b^N and K^{N,ξ_k}. Using this elementary fact and following the similar strategy in [39], one can make a decomposition whose goal is to express $\mathbb{E}\Big[\tilde{\mathrm{X}}_{T_k}^N-\tilde{\mathrm{X}}_t^N \mid \tilde{\mathrm{X}}_t^N=\mathrm{x}^N,\xi_k\Big]$ using the conditional expectation of $\nabla\log\tilde{\rho}_{T_k}^N$ by integration by parts and some other terms which are easy to control. More precisely, in order to expose a gradient of the Gaussian density, one can split (4.11) into three parts. Let

$$
\begin{aligned}
\mathrm{y}^N - \mathrm{x}^N = &\big(\mathrm{I}_{Nd} + (t-T_k)\big(\nabla \mathrm{b}^N(\mathrm{y}^N) + \nabla \mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)\big)\cdot\big(\mathrm{y}^N - \mathrm{x}^N + (t-T_k)\big(\mathrm{b}^N(\mathrm{y}^N) + \mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)\big)\\
&- (t-T_k)\big(\nabla \mathrm{b}^N(\mathrm{y}^N) + \nabla \mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)\cdot\big(\mathrm{y}^N - \mathrm{x}^N + (t-T_k)\big(\mathrm{b}^N(\mathrm{y}^N) + \mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)\big)\\
&- (t-T_k)\big(\mathrm{b}^N(\mathrm{y}^N) + \mathrm{K}^{N,\xi_k}(\mathrm{y}^N)\big)\\
:= &\, a_1(\mathrm{x}^N,\mathrm{y}^N) - a_2(\mathrm{x}^N,\mathrm{y}^N) - a_3(\mathrm{x}^N,\mathrm{y}^N),
\end{aligned}
$$

and define

$$
I_i(\mathrm{x}^N) := \mathbb{E}\Big[a_i\Big(\tilde{\mathrm{X}}_t^N, \tilde{\mathrm{X}}_{T_k}^N\Big) \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big], \quad i = 1,2,3.
$$

(a) For the term I_1, since the distribution $p\Big(\tilde{\mathrm{X}}_t^N = \mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N, \xi_k\Big)$ is Gaussian, then after integration by parts we obtain:

$$
\begin{aligned}
I_1(\mathrm{x}^N) &= \int_{\mathbb{R}^{Nd}} a_1(\mathrm{x}^N,\mathrm{y}^N)\frac{\tilde{\rho}_{T_k}^N(\mathrm{y}^N)p\Big(\tilde{\mathrm{X}}_t^N = \mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N, \xi_k\Big)}{\tilde{\rho}_t^{N,\xi_k}(\mathrm{x}^N)}\,\mathrm{dy}^N\\
&= 2\sigma(t-T_k)\int_{\mathbb{R}^{Nd}} \frac{\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}{\tilde{\rho}_t^{N,\xi_k}(\mathrm{x}^N)}\nabla p\Big(\tilde{\mathrm{X}}_t^N = \mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N, \xi_k\Big)\,\mathrm{dy}^N\\
&= 2\sigma(t-T_k)\int_{\mathbb{R}^{Nd}} \frac{\nabla\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}{\tilde{\rho}_t^{N,\xi_k}(\mathrm{x}^N)} p\Big(\tilde{\mathrm{X}}_t^N = \mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N, \xi_k\Big)\,\mathrm{dy}^N.
\end{aligned}
$$

Using Bayes's law again, one has

$$
I_1(\mathrm{x}^N) = 2\sigma(t-T_k)\int_{\mathbb{R}^{Nd}} \frac{\nabla\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}{\tilde{\rho}_{T_k}^N(\mathrm{y}^N)} p\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big)\,\mathrm{dy}^N.
$$

Hence, by Bayes's law and Jensen's inequality,

$$
\begin{aligned}
\mathbb{E}\bigg[\Big|I_1\Big(\tilde{\mathrm{X}}_t^N\Big)\Big|^2 \mid \xi_k\bigg] &\le 4\sigma^2(t-T_k)^2\int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^{N,\xi_k}(\mathrm{x}^N)\int_{\mathbb{R}^{Nd}}\left|\frac{\nabla\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}{\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}\right|^2 p\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big)\,\mathrm{dy}^N\,\mathrm{dx}^N\\
&= 4\sigma^2(t-T_k)^2\int_{\mathbb{R}^{Nd}}\left|\frac{\nabla\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}{\tilde{\rho}_{T_k}^N(\mathrm{y}^N)}\right|^2\tilde{\rho}_{T_k}^N(\mathrm{y}^N)\,\mathrm{dy}^N\int_{\mathbb{R}^{Nd}} p\Big(\tilde{\mathrm{X}}_t^N = \mathrm{x}^N \mid \tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N, \xi_k\Big)\,\mathrm{dx}^N\\
&:= c(t-T_k)^2\mathcal{I}\big(\tilde{\rho}_{T_k}^N\big).
\end{aligned}
$$

(b) For the term I_2, using the Lipschitz condition in Assumption 2.1 and Jensen's inequality,

$$
\begin{aligned}
\mathbb{E}\bigg[\Big|I_2\Big(\tilde{\mathrm{X}}_t^N\Big)\Big|^2 \mid \xi_k\bigg] &\le c'(t-T_k)^2\mathbb{E}\bigg[\Big|\mathbb{E}\Big[\tilde{\mathrm{X}}_{T_k}^N - \tilde{\mathrm{X}}_t^N + (t-T_k)\Big(\mathrm{b}^N\Big(\tilde{\mathrm{X}}_{T_k}^N\Big) + \mathrm{K}^{N,\xi_k}\Big(\tilde{\mathrm{X}}_{T_k}^N\Big)\Big) \mid \tilde{\mathrm{X}}_t^N\Big]\Big|^2 \mid \xi_k\bigg]\\
&\le c'(t-T_k)^2\mathbb{E}\bigg[\Big|\tilde{\mathrm{X}}_{T_k}^N - \tilde{\mathrm{X}}_t^N + (t-T_k)\Big(\mathrm{b}^N\Big(\tilde{\mathrm{X}}_{T_k}^N\Big) + \mathrm{K}^{N,\xi_k}\Big(\tilde{\mathrm{X}}_{T_k}^N\Big)\Big)\Big|^2 \mid \xi_k\bigg]\\
&\le c'(t-T_k)^2\mathbb{E}\bigg[\Big|\int_{T_t}^t\sqrt{2\sigma}\,\mathrm{dW}_s^N\Big|^2 \mid \xi_k\bigg] := c'N(t-T_k)^3.
\end{aligned}
$$

(c) For the term I_3, using Jensen's inequality and the polynomial growth assumption for b and boundedness assumption for K^{ξ_k}, combining with the moment control Lemma 3.2, it is clear that

$$
\begin{aligned}
\mathbb{E}\bigg[\Big|I_3\Big(\tilde{\mathrm{X}}_t^N\Big)\Big|^2 \mid \xi_k\bigg] &\le (t-T_k)^2\mathbb{E}\bigg[\Big|\mathrm{b}^N\Big(\tilde{\mathrm{X}}_{T_k}^N\Big) + \mathrm{K}^{N,\xi_k}\Big(\tilde{\mathrm{X}}_{T_k}^N\Big)\Big|^2 \mid \xi_k\bigg]\\
&\le c''N(t-T_k)^2,
\end{aligned}
$$

where c'' depends on the length of the time horizon T considered. Hence, combining the above bounds one gets

$$\begin{aligned}\int_{\mathbb{R}^{Nd}}\Big|\mathbb{E}\Big[\tilde{\mathrm{X}}^N_{T_k}-\tilde{\mathrm{X}}^N_t \mid \tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\Big]\Big|^2\tilde{\rho}^{N,\xi_k}_t\,\mathrm{d}\mathrm{x}^N &\le 3\sum_{k=1}^{3}\mathbb{E}\Big[\Big|I_k\Big(\tilde{\mathrm{X}}^N_t\Big)\Big|^2\mid \xi_k\Big]\\ &\le c(T)\tau^2\big(3N+\mathcal{I}\big(\tilde{\rho}^N_{T_k}\big)\big),\end{aligned}\tag{4.12}$$

where the positive constant $c(T)$ is independent of ξ_k, N and τ, and this is what we desire.

Note that if $\beta > 2L$, then we have the uniform-in-time moment control (3.5), and $c(T)$ can be taken to be independent of T. □

Lemma 4.2. *Under the same assumptions as in Theorem 2.1, for any $T>0$, there exists a positive constant $c(T)$ independent of ξ_k, N and τ such that for $t\in[T_k,T_{k+1})$ with $T_{k+1}\le T$,*

$$\mathcal{A}_N(t)+\mathcal{B}_N(t)\le c(T)\bigg(\tau^2+\frac{1}{N}\mathcal{I}\big(\tilde{\rho}^N_{T_k}\big)\bigg).\tag{4.13}$$

If $\beta>2L$, then the constant $c(T)$ can be taken to be independent of T.

Proof. In the proof here, the constant $c=c(T)$ is a generic constant independent of ξ_k, N and τ, whose concrete meaning can change in different places. The procedure for estimating these two terms is similar so we only show the first one as the example.

For any $t\in[T_k,T_{k+1})$, by Taylor's expansion, we have

$$\begin{aligned}\tilde{b}^{\xi_k,i}_t(\mathrm{x}^N)-b(x_i)&=\mathbb{E}\Big[b\Big(\tilde{X}^{i,N}_{T_k}\Big)-b\Big(\tilde{X}^{i,N}_t\Big)\mid \tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\Big]\\&=\mathbb{E}\Big[\tilde{X}^{i,N}_{T_k}-\tilde{X}^{i,N}_t\mid \tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\Big]\cdot\nabla_{x_i}b(x_i)+\tilde{r}_t(x_i),\end{aligned}$$

where the remainder takes the form

$$\tilde{r}_t(x_i):=\frac{1}{2}\mathbb{E}\bigg[\Big(\tilde{X}^{i,N}_{T_k}-\tilde{X}^{i,N}_t\Big)^{\otimes 2}:\int_0^1\nabla^2_{x_i}b\Big((1-s)\tilde{X}^{i,N}_t+s\tilde{X}^{i,N}_{T_k}\Big)\,\mathrm{d}s\mid \tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\bigg].\tag{4.14}$$

By Lemma B.1 in Appendix B, we show that for $t\in[T_k,T_{k+1})$,

$$\mathbb{E}\bigg[\Big|\tilde{r}_t\Big(\tilde{X}^{i,N}_t\Big)\Big|^2\mid\xi_k\bigg]\le c(t-T_k)^2.$$

Hence, by Lemma 4.1, we have

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^N\mathbb{E}_{\xi_k}\int_{\mathbb{R}^{Nd}}\Big|\tilde{b}^{\xi_k,i}_t(\mathrm{x}^N)-b(x_i)\Big|^2\tilde{\rho}^{N,\xi_k}_t\,\mathrm{d}\mathrm{x}^N\\&\quad\le 2\mathbb{E}_{\xi_k}\left[\frac{1}{N}\sum_{i=1}^N\mathbb{E}\bigg[\Big|\tilde{r}_t\Big(\tilde{X}^{i,N}_t\Big)\Big|^2\mid\xi_k\bigg]+c\frac{1}{N}\sum_{i=1}^N\int_{\mathbb{R}^{Nd}}\Big|\mathbb{E}\Big[\tilde{X}^{i,N}_{T_k}-\tilde{X}^{i,N}_t\mid\tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\Big]\Big|^2\tilde{\rho}^{N,\xi_k}_t\,\mathrm{d}\mathrm{x}^N\right]\\&\quad\le c(T)\tau^2\bigg(3+\frac{1}{N}\mathcal{I}\big(\tilde{\rho}^N_{T_k}\big)\bigg).\end{aligned}\tag{4.15}$$

Here, we used the simple fact that

$$\Big|\mathbb{E}\Big[\tilde{\mathrm{X}}^N_{T_k}-\tilde{\mathrm{X}}^N_t\mid\tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\Big]\Big|^2=\sum_{i=1}^N\Big|\mathbb{E}\Big[\tilde{X}^{i,N}_{T_k}-\tilde{X}^{i,N}_t\mid\tilde{\mathrm{X}}^N_t=\mathrm{x}^N,\xi_k\Big]\Big|^2.$$

The estimate for $\tilde{K}^{\xi_k,i}_t(\mathrm{x}^N)-K^{\xi_k}(x_i)$ is the same. We skip the details.

Lastly, if $\beta > 2L$ holds, then the moments are bounded uniformly in time. The constants in Lemmas B.1 and 4.1 can be taken independent of T, so the claim about the constant here also holds. □

4.3. Bounding the term $\mathcal{C}_N(t)$

Lemma 4.3. *Under the same assumptions as in Theorem 2.1, for any $T > 0$, there exists a positive constant $c(T)$ independent of ξ_k, N and τ such that for $t \in [T_k, T_{k+1})$ with $T_{k+1} \le T$, the following holds*

$$\mathcal{C}_N(t) \le c(T)\left(\tau^2 + \frac{1}{N}\mathcal{I}\big(\tilde{\rho}^N_{T_k}\big)\right). \tag{4.16}$$

If $\beta > 2L$, then the constant $c(T)$ can be taken to be independent of T.

Proof. As before, the constant $c = c(T)$ is a generic constant independent of ξ_k, $\tilde{\xi_k}$, N and τ, whose concrete meaning can change from line to line.

By the boundedness of $|K^{\xi_k}(x_i) - F^N(x_i)|$ in Assumption 2.1, we only need to estimate

$$\int_{\mathbb{R}^{Nd}} \frac{\left|\tilde{\rho}_t^{N,\tilde{\xi}_k} - \tilde{\rho}_t^{N,\xi_k}\right|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\,\mathrm{dx}^N.$$

First note that

$$\int_{\mathbb{R}^{Nd}} \frac{\left|\tilde{\rho}_t^{N,\tilde{\xi}_k} - \tilde{\rho}_t^{N,\xi_k}\right|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\,\mathrm{dx}^N = \int_{\mathbb{R}^{Nd}} \left|\frac{\tilde{\rho}_t^{N,\xi_k}}{\tilde{\rho}_t^{N,\tilde{\xi}_k}} - 1\right|^2 \tilde{\rho}_t^{N,\tilde{\xi}_k}\,\mathrm{dx}^N.$$

We will make use of the Girsanov transform in the path space to estimate this term. Recall that the two copies of the RBM process, $\tilde{\mathrm{X}}^N_{T_k}$ and $\hat{\mathrm{X}}^N_{T_k}$ share the same initial value $\mathrm{y}^N \sim \tilde{\rho}^N_{T_k}$. Let their laws in the path space $C\big([T_k, T_{k+1}]; \mathbb{R}^{Nd}\big)$ be $P_{\tilde{\mathrm{X}}^N}$ and $P_{\hat{\mathrm{X}}^N}$, respectively. Taking $\phi : \mathrm{X} \to \mathrm{X}_t$, the time projection mapping, then by the Lemma C.1 and the Girsanov transform we can conclude

$$\begin{aligned}
\frac{\tilde{\rho}_t^{N,\xi_k}}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}(\mathrm{x}^N) &= \mathbb{E}\left[\frac{\mathrm{d}P_{\tilde{\mathrm{X}}^N}}{\mathrm{d}P_{\hat{\mathrm{X}}^N}} \mid \hat{\mathrm{X}}^N_t = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\right]\\
&= \mathbb{E}\left[\exp\left(\sqrt{\frac{1}{2\sigma}}\int_{T_k}^t \big(\delta\mathrm{K}^N\big)(\mathrm{y}^N)\,\mathrm{dW}^N_s - \frac{1}{4\sigma}\int_{T_k}^t \big|\big(\delta\mathrm{K}^N\big)(\mathrm{y}^N)\big|^2\,\mathrm{d}s\right) \mid \hat{\mathrm{X}}^N_t = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\right].
\end{aligned}$$

Here, we denote

$$\delta\mathrm{K}^N(\mathrm{y}^N) := \frac{1}{\sqrt{2\sigma}}\Big(\mathrm{K}^{N,\tilde{\xi}_k} - \mathrm{K}^{N,\xi_k}\Big)(\mathrm{y}^N).$$

Notice the fact that $\hat{\mathrm{X}}^N_t = \mathrm{y}^N + \Big(\mathrm{b}^N(\mathrm{y}^N) + \mathrm{K}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big)(t - T_k) + \sqrt{2\sigma}\big(\mathrm{W}^N_t - \mathrm{W}^N_{T_k}\big)$, resulting in that

$$\begin{aligned}
&\exp\left(\int_{T_k}^t \delta\mathrm{K}^N\big(\mathrm{y}^N\big)\,\mathrm{dW}^N_s - \frac{1}{2}\int_{T_k}^t \big|\delta\mathrm{K}^N\big(\mathrm{y}^N\big)\big|^2\,\mathrm{d}s\right) - 1\\
&\quad= \exp\left(\sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\mathrm{y}^N)\Big(\mathrm{x}^N - \mathrm{y}^N - (t - T_k)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big) - \frac{1}{2}|\delta\mathrm{K}^N(\mathrm{y}^N)|^2(t - T_k)\right) - 1,
\end{aligned} \tag{4.17}$$

where $\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N) = \mathrm{b}^N(\mathrm{y}^N) + \mathrm{K}^{N,\tilde{\xi}_k}(\mathrm{y}^N)$. Now we split (4.17) into three terms

$$\begin{aligned}
&\exp\left(\sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\mathrm{y}^N)\Big(\mathrm{x}^N - \mathrm{y}^N - (t - T_k)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big) - \frac{1}{2}|\delta\mathrm{K}^N(\mathrm{y}^N)|^2(t - T_k)\right) - 1\\
&\quad= \sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\mathrm{y}^N)\big(\mathrm{x}^N - \mathrm{y}^N\big) + \left(-\sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\mathrm{y}^N)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N) - \frac{1}{2}|\delta\mathrm{K}^N(\mathrm{y}^N)|^2\right)(t - T_k)\\
&\qquad+ \big(e^{\mathrm{z}} - \mathrm{z} - 1\big)\\
&\quad:= K_1 + K_2 + K_3,
\end{aligned} \tag{4.18}$$

where $\mathrm{z}^N = \sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\mathrm{y}^N)\Big(\mathrm{x}^N - \mathrm{y}^N - (t-T_k)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big) - \frac{1}{2}|\delta\mathrm{K}^N(\mathrm{y}^N)|^2(t-T_k)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^{Nd}}\left|\frac{\tilde{\rho}_t^{N,\xi_k}}{\tilde{\rho}_t^{N,\tilde{\xi}_k}} - 1\right|^2 \tilde{\rho}_t^{N,\tilde{\xi}_k}\,\mathrm{dx}^N \\
&\qquad = \int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\left|\int_{\mathbb{R}^{Nd}}(K_1+K_2+K_3)P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\right|^2 \mathrm{dx}^N.
\end{aligned}$$

For K_2, by Jensen's inequality,

$$\begin{aligned}
&\int \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\left|\int K_2 P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\right|^2 \mathrm{dx}^N \\
&\quad \le \int \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\int |K_2|^2 P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\,\mathrm{dx}^N \\
&\quad = (t-T_k)^2\mathbb{E}\left[\left|\left(\sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\mathrm{y}^N)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N) + \frac{1}{2}|\delta\mathrm{K}^N(\mathrm{y}^N)|^2\right)\right|^2 \mid \xi_k, \tilde{\xi}_k\right].
\end{aligned}$$

By Assumption 2.1, since $\delta\mathrm{K}^N$ is bounded and $\mathrm{F}^{N,\tilde{\xi}_k}$ has polynomial growth, then by moment control Lemma 3.2, we can obtain

$$\int \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\left|\int K_2 P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\right|^2 \mathrm{dx}^N \le c(T)N\tau^2. \tag{4.19}$$

For K_3, by Jensen's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\left|\int_{\mathbb{R}^{Nd}} K_3 P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\right|^2 \mathrm{dx}^N \\
&\quad \le \int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\int_{\mathbb{R}^{Nd}} |K_3|^2 P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\,\mathrm{dx}^N \\
&\quad = \mathbb{E}\left[\left|e^{\mathrm{Z}_t^N} - 1 - \mathrm{Z}_t^N\right|^2 \mid \xi_k, \tilde{\xi}_k\right],
\end{aligned}$$

where we denote the process

$$\begin{aligned}
\mathrm{Z}_t^N &:= \sqrt{\frac{1}{2\sigma}}\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\cdot\Big(\hat{\mathrm{X}}_t^N - \tilde{\mathrm{X}}_{T_k}^N - (t-T_k)\mathrm{F}^{N,\tilde{\xi}_k}(\tilde{\mathrm{X}}_{T_k}^N)\Big) - \frac{1}{2}|\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)|^2(t-T_k) \\
&= -\frac{1}{2}|\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)|^2(t-T_k) + \delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\cdot\int_{T_k}^t \mathrm{dW}_s^N.
\end{aligned}$$

Denote $\tilde{\mathrm{Z}}_t^N = e^{\mathrm{Z}_t^N} - 1 - \mathrm{Z}_t^N$, then by Itô's formula,

$$\begin{aligned}
\tilde{\mathrm{Z}}_t^N &= \frac{1}{2}\left|\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\right|^2(t-T_k) + \delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\cdot\int_{T_k}^t\Big(e^{\mathrm{Z}_s^N} - 1\Big)\,\mathrm{dW}_s^N \\
&= \frac{1}{2}\left|\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\right|^2(t-T_k) + \delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\cdot\int_{T_k}^t\Big(\tilde{\mathrm{Z}}_t^N + \mathrm{Z}_s^N\Big)\,\mathrm{dW}_s^N.
\end{aligned}$$

So

$$\mathbb{E}\left[\left|\tilde{\mathrm{Z}}_t^N\right|^2 \mid \xi_k, \tilde{\xi}_k\right] = \frac{(t-T_k)^2}{4}\mathbb{E}\left[\left|\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\right|^2 \mid \xi_k, \tilde{\xi}_k\right] + \int_{T_k}^t \mathbb{E}\left[\left|\delta\mathrm{K}^N(\tilde{\mathrm{X}}_{T_k}^N)\right|^2\Big(\tilde{\mathrm{Z}}_t^N + \mathrm{Z}_s^N\Big)^2 \mid \xi_k, \tilde{\xi}_k\right]\mathrm{d}s.$$

By Assumption 2.1, since $\delta \mathrm{K}^N(\tilde{\mathrm{X}}^N_{T_k})$ is uniformly bounded, then $\mathbb{E}\Big[\big|\mathrm{Z}^N_t\big|^2 \mid \xi_k, \tilde{\xi}_k\Big] \le cN(t-T_k)^2$. Thus

$$\mathbb{E}\left[\left|\tilde{\mathrm{Z}}^N_t\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le c\int_{T_k}^t \mathbb{E}\left[\left|\tilde{\mathrm{Z}}_s\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \mathrm{d}s + c(T)N(t-T_k)^2.$$

By Grönwall's inequality,

$$\begin{aligned}\int_{\mathbb{R}^{Nd}} \tilde{\rho}^{N,\tilde{\xi}_k}_t(\mathrm{x}^N)\left|\int_{\mathbb{R}^{Nd}} K_3 P\Big(\tilde{\mathrm{X}}^N_{T_k} = \mathrm{y}^N \mid \hat{\mathrm{X}}^N_t = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{d}\mathrm{y}^N\right|^2 \mathrm{d}\mathrm{x}^N \\ \le \mathbb{E}\left[\left|\tilde{\mathrm{Z}}^N_t\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le c(T)N\tau^2.\end{aligned} \tag{4.20}$$

For K_1, the proof strategy is similar to Lemma 4.1. We split the term K_1 into three parts:

$$\begin{aligned}\delta \mathrm{K}^N(\mathrm{y}^N)\big(\mathrm{x}^N - \mathrm{y}^N\big) &= \delta \mathrm{K}^N(\mathrm{y}^N)\cdot\Big(\mathrm{I}_{Nd} + (t-T_k)\nabla \mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big)\cdot\Big(\mathrm{y}^N - \mathrm{x}^N + (t-T_k)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big) \\ &\quad - (t-T_k)\delta \mathrm{K}^N(\mathrm{y}^N)\cdot\nabla \mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\cdot\Big(\mathrm{y}^N - \mathrm{x}^N + (t-T_k)\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N)\Big) \\ &\quad - (t-T_k)\delta \mathrm{K}^N(\mathrm{y}^N)\cdot \mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N) \\ &:= \tilde{a}_1(\mathrm{x}^N,\mathrm{y}^N) - \tilde{a}_2(\mathrm{x}^N,\mathrm{y}^N) - \tilde{a}_3(\mathrm{x}^N,\mathrm{y}^N),\end{aligned}$$

where $\mathrm{F}^{N,\tilde{\xi}_k}(\mathrm{y}^N) = \mathrm{b}^N(\mathrm{y}^N) + \mathrm{K}^{N,\tilde{\xi}_k}(\mathrm{y}^N)$. Define

$$\tilde{I}_j(\mathrm{x}^N) := \mathbb{E}\Big[\tilde{a}_j(\tilde{\mathrm{X}}^N_{T_k}, \hat{\mathrm{X}}^N_t) \mid \hat{\mathrm{X}}^N_t = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big], \quad j = 1,2,3.$$

For the first term $\tilde{I}_1$, following the same step in the proof of Lemma 4.1, we have

$$\begin{aligned}\tilde{I}_1(\mathrm{x}^N) &= 2\sigma(t-T_k)\int_{\mathbb{R}^{Nd}} \frac{\nabla_{\mathrm{y}^N}\Big(\delta \mathrm{K}^N(\mathrm{y}^N)\tilde{\rho}^{N,\xi_k}_{T_k}(\mathrm{y}^N)\Big)}{\tilde{\rho}^{N,\tilde{\xi}_k}_t(\mathrm{x}^N)} p\Big(\hat{\mathrm{X}}^N_t = \mathrm{x}^N \mid \tilde{\mathrm{X}}^N_{T_k} = \mathrm{y}^N\Big)\,\mathrm{d}\mathrm{y}^N \\ &= 2\sigma(t-T_k)\int_{\mathbb{R}^{Nd}} \left(\nabla\delta \mathrm{K}^N(\mathrm{y}^N) + \delta \mathrm{K}^N(\mathrm{y}^N)\frac{\nabla\tilde{\rho}^{N,\xi_k}_{T_k}(\mathrm{y}^N)}{\tilde{\rho}^{N,\xi_k}_{T_k}(\mathrm{y}^N)}\right) p\Big(\hat{\mathrm{X}}^N_t = \mathrm{x}^N \mid \tilde{\mathrm{X}}^N_{T_k} = \mathrm{y}^N\Big)\,\mathrm{d}\mathrm{y}^N.\end{aligned}$$

Since K^{ξ_k} is uniformly bounded by Assumption 2.1, then $\delta \mathrm{K}^N(\mathrm{y}^N)$ and $\nabla\delta \mathrm{K}^N(\mathrm{y}^N)$ are also uniformly bounded, hence

$$\mathbb{E}\left[\left|\tilde{I}_1(\hat{\mathrm{X}}^N_t)\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le c(T)(t-T_k)^2\big(1 + \mathcal{I}\big(\tilde{\rho}^N_{T_k}\big)\big). \tag{4.21}$$

For the second term $\tilde{I}_2$, by Jensen's inequality, it holds that

$$\mathbb{E}\left[\left|\tilde{I}_2(\hat{\mathrm{X}}^N_t)\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le (t-T_k)^2\mathbb{E}\left[\left|\delta \mathrm{K}^N(\tilde{\mathrm{X}}^N_{T_k})\cdot\nabla \mathrm{F}^{N,\tilde{\xi}_k}(\tilde{\mathrm{X}}^N_{T_k})\cdot\int_{T_k}^t \mathrm{d}\mathrm{W}^N_s\right|^2 \mid \xi_k, \tilde{\xi}_k\right].$$

By Assumption 2.1, since $\delta \mathrm{K}^N$ is bounded and $\nabla \mathrm{F}^{N,\tilde{\xi}_k}$ has polynomial growth, then by the moment control Lemma 3.2, we can obtain

$$\mathbb{E}\left[\left|\tilde{I}_2(\hat{\mathrm{X}}^N_t)\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le c(T)N(t-T_k)^2. \tag{4.22}$$

For the third term $\tilde{I}_3$, by Jensen's inequality, it holds that

$$\mathbb{E}\left[\left|\tilde{I}_3(\hat{\mathrm{X}}^N_t)\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le (t-T_k)^2\mathbb{E}\left[\left|\delta \mathrm{K}^N(\tilde{\mathrm{X}}^N_{T_k})\cdot \mathrm{F}^{N,\tilde{\xi}_k}(\tilde{\mathrm{X}}^N_{T_k})\right|^2 \mid \xi_k, \tilde{\xi}_k\right].$$

By Assumption 2.1, since $\delta\mathrm{K}^N$ is bounded and $\mathrm{F}^{N,\tilde{\xi}_k}$ has polynomial growth, then by moment control Lemma 3.2, we can obtain

$$\mathbb{E}\left[\left|\tilde{I}_3(\hat{\mathrm{X}}_t^N)\right|^2 \mid \xi_k, \tilde{\xi}_k\right] \le c(T)N(t-T_k)^2. \tag{4.23}$$

Finally, combining (4.21), (4.22) and (4.23), we get

$$\begin{aligned}\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^{N,\tilde{\xi}_k}(\mathrm{x}^N)\left|\int_{\mathbb{R}^{Nd}} K_1 P\Big(\tilde{\mathrm{X}}_{T_k}^N = \mathrm{y}^N \mid \hat{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k, \tilde{\xi}_k\Big)\,\mathrm{dy}^N\right|^2 \mathrm{dx}^N \\ \le\ c(T)\tau^2\big(N + \mathcal{I}\big(\tilde{\rho}_{T_k}^N\big)\big).\end{aligned} \tag{4.24}$$

Finally, equipped with the estimation for integration of K_1, K_2 and K_3 from (4.24), (4.19) and (4.20), we can get the estimate

$$\int_{\mathbb{R}^{Nd}} \frac{\left|\tilde{\rho}_t^{N,\tilde{\xi}_k} - \tilde{\rho}_t^{N,\xi_k}\right|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\,\mathrm{dx}^N \le c(T)\tau^2\big(N + \mathcal{I}\big(\tilde{\rho}_{T_k}^N\big)\big).$$

Hence,

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^N \mathbb{E}_{\xi_k,\tilde{\xi}_k}\left[\int_{\mathbb{R}^{Nd}} \left|K^{\xi_k}(x_i) - F^N(x_i)\right|^2 \frac{\left|\tilde{\rho}_t^{N,\xi_k} - \tilde{\rho}_t^{N,\tilde{\xi}_k}\right|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\,\mathrm{dx}^N\right] \\ &\le \frac{1}{N}\mathbb{E}_{\xi_k,\tilde{\xi}_k}\left[\int_{\mathbb{R}^{Nd}} \left|\mathrm{K}^{N,\xi_k}(\mathrm{x}^N) - \mathrm{F}(\mathrm{x}^N)\right|^2 \frac{\left|\tilde{\rho}_t^{N,\xi_k} - \tilde{\rho}_t^{N,\tilde{\xi}_k}\right|^2}{\tilde{\rho}_t^{N,\tilde{\xi}_k}}\,\mathrm{dx}^N\right] \\ &\le c(T)\tau^2\left(1 + \frac{1}{N}\mathcal{I}\big(\tilde{\rho}_{T_k}^N\big)\right).\end{aligned} \tag{4.25}$$

Again, if $\beta > 2L$ holds, then the moments are bounded uniformly in time so the constant c can be taken to be independent of T. □

4.4. Bounding the term $\mathcal{D}_N(t)$

The following lemma describes the Law of Large Numbers at exponential scale. This statement is a crucial estimate proposed by Theorem 3 of [22] which will be useful in order to control the error term $\mathcal{D}_N(t)$.

Lemma 4.4. *Consider any $\rho \in L^1(\mathbb{R}^d)$ with $\rho \ge 0$ and $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1$. Assume that a scalar function $\psi \in L^\infty$ with $\|\psi\|_{L^\infty} < \frac{1}{2e}$, and that for any fixed z, $\int_{\mathbb{R}^d} \psi(z,x)\rho(x)\,\mathrm{d}x = 0$, then*

$$\int_{\mathbb{R}^{Nd}} \rho^N \exp\left(\frac{\eta}{N}\sum_{j_1,j_2=1}^N \psi(x_1,x_{j_1})\psi(x_1,x_{j_2})\right)\mathrm{dx}^N \le C,$$

where $\rho^N(t,\mathrm{x}^N) = \Pi_{i=1}^N \rho(t,x_i)$, η is an arbitrary positive number and constant C only depends on η and $\|\psi\|_{L^\infty}$.

Lemma 4.5. *Under the same assumptions as in Theorem 2.1, the following holds*

$$\mathcal{D}_N(t) \le 4e^2\|K\|_{L^\infty}^2\left(\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) + \frac{C}{N}\right), \tag{4.26}$$

where the constant C is independent of N.

Proof. Firstly, one can perform a change of measure from $\tilde{\rho}_t^N$ to $\rho_t^{\otimes N}$. Let

$$\Phi(x_i) = \big(F^N(x_i) - K * \rho_t(x_i)\big)^2$$

with $F^N(x_i)$ given in (4.1). Define

$$f^N = \frac{1}{\lambda}\exp(N\eta\Phi)\rho_t^{\otimes N}, \quad \lambda^N = \int_{\mathbb{R}^{Nd}} \exp(N\eta\Phi)\rho_t^{\otimes N}\,\mathrm{dx}^N,$$

where η is an arbitrary positive number. Thanks to the function $h(x) = x\log x + 1 - x \geq 0$ for any $x > 0$, we can obtain

$$\begin{aligned}\frac{1}{N}\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N \log\tilde{\rho}_t^N\,\mathrm{dx}^N &= \frac{1}{N}\int_{\mathbb{R}^{Nd}} f^N\left(\frac{\tilde{\rho}_t^N}{f^N}\log\frac{\tilde{\rho}_t^N}{f^N} - \frac{\tilde{\rho}_t^N}{f^N} + 1\right)\mathrm{dx}^N + \frac{1}{N}\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N \log f^N\,\mathrm{dx}^N \\ &\geq \frac{1}{N}\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N \log f^N\,\mathrm{dx}^N.\end{aligned}$$

On the other hand, one can easily check that

$$\frac{1}{N}\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N\log f^N\,\mathrm{dx}^N = \eta\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N\Phi\,\mathrm{dx}^N + \frac{1}{N}\int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^N\log\rho_t^{\otimes N}\,\mathrm{dx}^N - \frac{\log\lambda^N}{N}.$$

Then we can conclude that

$$\int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N\Phi\,\mathrm{dx}^N \leq \frac{1}{\eta}\left(\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) + \frac{1}{N}\log\int_{\mathbb{R}^{Nd}}\rho_t^{\otimes N}\exp(N\eta\Phi)\,\mathrm{dx}^N\right). \tag{4.27}$$

In the following we choose $0 < \eta < \frac{1}{4e^2\|K\|_{L^\infty}^2}$. Applying Lemma 4.4 to

$$\phi(x,z) = \frac{1}{2e\|K\|_{L^\infty}}(K(x-z) - K*\rho_t(x)),$$

which satisfies $\|\phi\|_{L^\infty} \leq \frac{1}{2e}$ and is such that

$$\int \phi(x,z)\rho_t(x)\,\mathrm{d}x = 0.$$

Then by Lemma 4.4, we deduce that

$$\int_{\mathbb{R}^{Nd}}\rho_t^{\otimes N}e^{N\eta\Phi}\,\mathrm{dx}^N = \int_{\mathbb{R}^{Nd}}\rho_t^{\otimes N}\exp\left(\frac{1}{N}\sum_{j_1,j_2=1}^{N}\phi(x_i,x_{j_1})\phi(x_i,x_{j_2})\right)\mathrm{dx}^N \leq C, \tag{4.28}$$

where the constant C is independent of N. Therefore

$$\int_{\mathbb{R}^{Nd}}\tilde{\rho}_t^N\big|F^N(x_i) - K*\rho_t(x_i)\big|^2\,\mathrm{dx}^N \leq 4e^2\|K\|_{L^\infty}^2\left(\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) + \frac{C}{N}\right), \tag{4.29}$$

which concludes the proof. □

4.5. Proof of Theorem 2.1

Proof of Theorem 2.1. Gathering the previous results, by the Log-Sobolev inequality in Assumption 2.2, taking $f = \frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}$, we have

$$\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) = \frac{1}{N}\operatorname{Ent}_{\rho_t^{\otimes N}}\left(\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right) \le \frac{C_{LS}}{N}\sum_{i=1}^N \int_{\mathbb{R}^{Nd}} \tilde{\rho}_t^N \left|\nabla_{x_i}\log\frac{\tilde{\rho}_t^N}{\rho_t^{\otimes N}}\right|^2 \mathrm{d}x^N = C_{LS}\mathcal{I}_N(t).$$

Then by Lemmas 2.9, 4.2, 4.3 and 4.5, we yield the following desired estimate for $t \in [T_k, T_{k+1})$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) &\le \left(4e^2\|K\|_{L^\infty}^2 - \frac{\sigma}{2C_{LS}}\right)\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) + c_1\tau^2\left(1 + \frac{1}{N}\mathcal{I}\big(\tilde{\rho}_{T_k}^N\big)\right) + \frac{c_2}{N}\\ &\le C_0\mathcal{H}_N\big(\tilde{\rho}_t^N \mid \rho_t^{\otimes N}\big) + C_1\tau^2 + \frac{C_2}{N},\end{aligned}$$

where we have applied the Jensen's inequality by the convexity of the functional $\rho \mapsto \mathcal{I}(\rho)$ and Lemma 3.4:

$$\mathcal{I}\big(\tilde{\rho}_{T_k}^N\big) \le \mathbb{E}_{\xi_k}\Big[\mathcal{I}\Big(\tilde{\rho}_{T_k}^{N,\xi_k}\Big)\Big] \le CN.$$

Note that the constants C_0, C_1 and C_2 are independent of N, τ and ξ_k, then by Gronwall's inequality, we end the proof of the first part.

If $\beta > 2L$, by Lemmas 3.4, 4.2, and 4.3, the constants C_1 and C_2 can be made independent of t. Moreover, if $\|K\|_{L^\infty}^2 \le \frac{\sigma}{8e^2 C_{LS}}$, then the constant C_0 becomes negative. Therefore, we have a uniform-in-time bound for the relative entropy. □

5. Conclusion and perspective

In this paper we prove the error estimate of propagation of chaos form for random batch method for $N-$particle systems toward its mean-field limit, the Fokker–Planck equation, based on the relative entropy. We show that the convergence rate is $O(\tau^2 + 1/N)$, where τ is the small time steps. Our result can be seen as an improvement over the previous works about the random batch, and we fill the gap to understand the approximation error of the RBM as a numerical method for its mean-field limit. Our main challenge is dealing with the random batch, where we follow the approach in [32] by using path measures and Girsanov transformations. Our results need some regularity assumptions on the force terms which used in various literature. It will be an interesting topic to investigate the error estimate of random batch method for singular interacting kernel such as the Biot-Savart Law in incompressible flows, among other applications. However, for singular kernels, the application of Girsanov transform seems to be quite challenging as it requires the square integrability of the kernel. In fact, we are working on the random batch approximation of the vortex model and it seems there is some intrinsic difficulty.

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Appendix A. Proof of Lemma 3.1

Proof of Lemma 3.1. Indeed, for $t \in [T_k, T_{k+1})$, consider the random variable $\tilde{\varrho}_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}$, where $\mathcal{F}_{T_k} = \sigma\left(\tilde{\mathrm{X}}_s^N, s \le T_k\right)$. Then, by definition of $\tilde{\mathrm{X}}_t^N$ for $t \in [T_k, T_{k+1})$, $\tilde{\varrho}_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}$ satisfies the Liouville equation:

$$\partial_t\left(\tilde{\varrho}_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}\right) = -\sum_{i=1}^N \mathrm{div}_{x_i}\left(\left(b\left(\tilde{X}_{T_k}^{i,N}\right) + K^{\xi_k}\left(\tilde{X}_{T_k}^{i,N}\right)\right)\left(\hat{\rho}_t^N \mid \mathcal{F}_{T_k}\right)\right) + \sum_{i=1}^N \sigma \Delta_{x_i}\left(\tilde{\varrho}_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}\right).$$

Taking expectation, one has

$$\mathbb{E}\Big[\partial_t\Big(\tilde\varrho_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}\Big)\Big] = \partial_t\tilde\varrho_t^{N,\boldsymbol{\xi}}, \quad \mathbb{E}\Big[\Delta_{x_i}\partial_t\Big(\tilde\varrho_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}\Big)\Big] = \Delta_{x_i}\tilde\varrho_t^{N,\boldsymbol{\xi}},$$

and for the drift term,

$$\begin{aligned}
&\mathbb{E}\Big[\mathrm{div}_{x_i}\Big(\Big(b\Big(\tilde X_{T_k}^{i,N}\Big) + K^{\xi_k}\Big(\tilde X_{T_k}^{i,N}\Big)\Big)\Big(\tilde\varrho_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}\Big)\Big)\Big] \\
&\quad = \mathrm{div}_{x_i}\int\Big(\tilde\varrho_t^{N,\boldsymbol{\xi}} \mid \mathcal{F}_{T_k}\Big)(\mathrm{x}^N \mid \mathrm{y}^N)(b(y_i) + K^{\xi_k}(y_i))\tilde\varrho_t^{N,\boldsymbol{\xi}}(\mathrm{y}^N)\,\mathrm{dy}^N \\
&\quad = \mathrm{div}_{x_i}\int(b(y_i) + K^{\xi_k}(y_i))\tilde\varrho_{t,T_k}^{N,\boldsymbol{\xi}}(\mathrm{x}^N,\mathrm{y}^N)\,\mathrm{dy}^N \\
&\quad = \mathrm{div}_{x_i}\Big(\tilde\varrho_t^{N,\boldsymbol{\xi}}(\mathrm{x}^N)\mathbb{E}\Big[b\Big(\tilde X_{T_k}^{i,N}\Big) + K^{\xi_k}\Big(\tilde X_{T_k}^{i,N}\Big) \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \boldsymbol{\xi}\Big]\Big),
\end{aligned}$$

where $\tilde\varrho_{t,T_k}^{N,\boldsymbol{\xi}}$ denotes the joint distribution of $\tilde{\mathrm{X}}_t^N$ and $\tilde{\mathrm{X}}_{T_k}^N$. Note that we used Bayes' law in the third equality. Combining all the above, we obtain the desired result (3.1). □

Appendix B. Omitted details in Section 4

In this appendix, we prove the details omitted in the proof of Theorem 2.1.

Lemma B.1. *There exist positive constants c_1 and c_2 independent of N and the batch ξ_k but dependent on t such that for all $t \in [T_k, T_{k+1})$, it holds that*

$$\mathbb{E}\Big[\Big|\tilde r_t\Big(\tilde X_t^{i,N}\Big)\Big|^2 \mid \xi_k\Big] \le c_1(t-T_k)^2; \tag{B.1}$$

$$\mathbb{E}\Big[\Big|\hat r_t\Big(\tilde X_t^{i,N}\Big)\Big|^2 |\xi_k\Big] \le c_2(t-T_k)^2. \tag{B.2}$$

Moreover, if $\beta > 2L$ holds, then the constants c_1 and c_2 are independent of t and the bound is uniform-in-time.

Proof. Since the Hessians of b and K^{ξ_k} have polynomial growth and Lipschitz, combining with the moment control yields

$$\begin{aligned}
|\tilde r_t(x_i)| &\le \int_0^1 \mathbb{E}\Big[\Big|\nabla_{x_i}^2 b\Big((1-s)\tilde X_t^{i,N} + s\tilde X_{T_k}^{i,N}\Big)\Big|\cdot\Big|\tilde X_{T_k}^{i,N} - \tilde X_t^{i,N}\Big|^2 \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big] \\
&\le \frac{1}{q+1}\mathbb{E}\Big[(1 + \Big|\tilde X_t^{i,N}\Big|^q + \Big|\tilde X_{T_k}^{i,N}\Big|^q)\cdot\Big|\tilde X_{T_k}^{i,N} - \tilde X_t^{i,N}\Big|^2 \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big] \\
&\le C\mathbb{E}\Big[\Big|\tilde X_{T_k}^{i,N} - \tilde X_t^{i,N}\Big|^2 \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big]; \\
|\hat r_t(x_i)| &\le \int_0^1 \mathbb{E}\Big[\Big|\nabla_{x_i}^2 K^{\xi_k}\Big((1-s)\tilde X_t^{i,N} + s\tilde X_{T_k}^{i,N}\Big)\Big|\cdot\Big|\tilde X_{T_k}^{i,N} - \tilde X_t^{i,N}\Big|^2 \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big] \\
&\le \tilde C\mathbb{E}\Big[\Big|\tilde X_{T_k}^{i,N} - \tilde X_t^{i,N}\Big|^2 \mid \tilde{\mathrm{X}}_t^N = \mathrm{x}^N, \xi_k\Big].
\end{aligned}$$

Hence by Jensen's inequality, taking the expectation and using polynomial growth assumption for b and Lipschitz assumption for K^{ξ_k}, we have

$$\begin{aligned}
\mathbb{E}\Big[\Big|\tilde r_t\Big(\tilde X_t^{i,N}\Big)\Big|^2 \mid \xi_k\Big] &\le C^2\mathbb{E}\Big[\Big|\mathbb{E}\Big[\Big|\tilde X_{T_k}^{i,N} - \tilde X_t^{i,N}\Big|^2 \mid \tilde{\mathrm{X}}_t^N\Big]\Big|^2 \mid \xi_k\Big] \\
&\le C^2\mathbb{E}\Big[\Big|b\Big(\tilde X_{T_k}^{i,N}\Big)(t-T_k) + K^{\xi_k}\Big(\tilde X_{T_k}^{i,N}\Big)(t-T_k) + \int_{T_k}^t \mathrm{d}W_s\Big|^4 \mid \xi_k\Big] \\
&\le C^2\Big((t-T_k)^4\Big(1 + \mathbb{E}\Big[\Big|\tilde X_{T_k}^{i,N}\Big|^q \mid \xi_k\Big]\Big)^4 + (t-T_k)^4\Big(\Big|K^{\xi_k}(0)\Big| + L\mathbb{E}\Big[\Big|\tilde X_{T_k}^{i,N}\Big| \mid \xi_k\Big]\Big)^4 \\
&\qquad + 3(t-T_k)^2\Big);
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\left|\hat{r}_t\left(\tilde{X}_t^{i,N}\right)\right|^2 \mid \xi_k\right] &\le \tilde{C}^2\mathbb{E}\left[\left|\mathbb{E}\left[\left|\tilde{X}_{T_k}^{i,N} - \tilde{X}_t^{i,N}\right|^2 \mid \tilde{\mathrm{X}}_t^N\right]\right|^2 \mid \xi_k\right] \\
&\le \tilde{C}^2\mathbb{E}\left[\left|b\left(\tilde{X}_{T_k}^{i,N}\right)(t-T_k) + K^{\xi_k}\left(\tilde{X}_{T_k}^{i,N}\right)(t-T_k) + \int_{T_k}^t \mathrm{d}W_s\right|^4 \mid \xi_k\right] \\
&\le \tilde{C}^2\bigg((t-T_k)^4\left(1+\mathbb{E}\left[\left|\tilde{X}_{T_k}^{i,N}\right|^q \mid \xi_k\right]\right)^4 + (t-T_k)^4\left(\left|K^{\xi_k}(0)\right| + L\mathbb{E}\left[\left|\tilde{X}_{T_k}^{i,N}\right| \mid \xi_k\right]\right)^4 \\
&\quad + 3(t-T_k)^2\bigg).
\end{aligned}$$

Finally, by Lemma 3.2, we arrive at the desired conclusion. Note that if $\beta > 2L$, we have the uniform-in-time bound for the moment (3.5), and thus we can obtain the uniform-in-time bound for the remainder terms. □

Appendix C. The Radon–Nikodym derivatives of path measures

The following lemma describes the relationship between the two Radon–Nikodym derivatives (of path measures and of push forward measures):

Lemma C.1 ([32], Lem. A.1)**.** *Let Q_1, Q_2 be two probability distributions on $\mathcal{X}$, and Q_2 is absolutely continuous with respect to Q_1. Let $\phi : \mathcal{X} \to \mathbb{R}^d$ be a measurable mapping, and consider the push forward measure $\phi_\# Q_1$ and $\phi_\# Q_2$, denoted by $Q_{1,\phi}$ and $Q_{2,\phi}$, respectively. Then the Randon-Nikodym derivatives $\frac{\mathrm{d}Q_{1,\phi}}{\mathrm{d}Q_{2,\phi}} \in L^1\left(\mathrm{d}Q_{2,\phi}, \mathbb{R}^d\right)$, $\frac{\mathrm{d}Q_1}{\mathrm{d}Q_2} \in L^1(\mathrm{d}Q_2, \mathcal{X})$ are well-defined, and*

$$\frac{\mathrm{d}Q_{1,\phi}}{\mathrm{d}Q_{2,\phi}}(x) = \mathbb{E}_{X\sim Q_2}\left[\frac{\mathrm{d}Q_1}{\mathrm{d}Q_2} \mid \phi(X) = x\right].$$