ESAIM: M2AN 59 (2025) 1023–1042
https://doi.org/10.1051/m2an/2025016

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

THE SOLUTION METHOD AND ITS ACCURACY FOR A NONLINEAR TIMOSHENKO AXISYMMETRIC SHELL MODEL

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Abstract. The boundary value problem of the axisymmetric displacement of a Timoshenko (cylindrical) shell in static regime was considered. From the given nonlinear system of ordinary differential equations with respect to the displacement functions u, w and ψ, using the Green functions, we obtain an integro-differential equation for the function w. This equation, together with the Dirichlet boundary condition, forms the boundary value problem. To obtain an approximate solution of the problem for w we apply the method that includes the Galerkin method and the Jacobi–Cardano iteration. Obtained values of w and of its error estimate are used for constructing the approximations for other displacement functions u and ψ and for estimation of their errors. The results of the numerical experiment are given.

Mathematics Subject Classification. 34B27, 65L10, 65L60, 65L70, 74B20, 74K25.

Received June 5, 2024. Accepted March 6, 2025.

1. Introduction

1.1. Some facts of the problem

The significance of the nonlinear system of the Timoshenko [26] shell equations lies in their importance from both theoretical and engineering perspectives. This system is employed when the assumption that the normal is perpendicular to the mid-surface after deformation is not met. In other words, the Timoshenko type shell in contrast to the Kirchhoff–Love model takes into account the additional deformations associated with rotational inertia.

Vorovich [33] added two tasks to the list of unsolved problems of the mathematical theory of shells:

(1) Construction of a mathematical theory of boundary value problems for shells of the type considered by Timoshenko in which in addition to geometric nonlinearity, shear stresses are considered.
(2) Justification of approximation methods.

To date, these problems have persisted in various forms. The complexity associated with investigating Timoshenko nonlinear equations becomes more apparent when comparing them to Kirchhoff–Love shell equations. The classical Kirchhoff–Love [28] system involves a biharmonic term $\Delta^2 w$ with respect to the lateral deflection

Keywords and phrases. Timoshenko shell, approximate method, Green function, Galerkin method, Jacobi type iteration method, Cardano formula.

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function, whereas the Timoshenko system exhibits a similar nonlinearity with term Δw. This distinction is crucial. In the foundational studies by Vorovich [31, 32] on static and dynamic Kirchhoff–Love problems, the presence of $\Delta^2 w$ enabled the derivation of *a priori* estimates for the function w in the Sobolev space of functions with square integrable second derivatives. These estimates form the basis for proving the existence of solutions using the Faedo–Galerkin method and justifying certain numerical methods. For Timoshenko shell system, for which the important term in getting estimates is Δw, obtaining *a priori* estimates is possible, but it is insufficient to prove the solvability of the system using the Faedo–Galerkin procedure and justify some numerical methods. The method of *a priori* estimates in Timoshenko's theory requires further exploration. However, using the densifying operator theory and the principle of the compressed mapping in case of sufficiently small initial data, the existence of a solution of some Timoshenko two-dimensional problems are shown in the papers of Kharasova [11], Peradze [18] and Timergaliev [24]. Note that the researchers are also interested in equations of Timoshenko beam, see Feng and Soufiane [7] paper and the publications cited there.

The authors propose that the examination of one-dimensional variants of the Timoshenko system, as presented in this paper for the case of axially symmetric deformation, can enhance our understanding of the inherent nonlinearity in nonlinear models. Furthermore, it may pave the way for investigating two-dimensional models.

The work is organized as follows. After setting the problem, using Green's functions, u and ψ are expressed through the function w. Thus, the system is reduced to an integro-differential equation of Kirchhoff type with respect to function w. The method of approximate solution of the resulting boundary value problem is described. The notion of error is introduced for function w and the errors of the component parts of the method are estimated separately for the Galerkin method and for the Jacobi–Cardano iteration. The error is estimated for function w in general. The obtained result is formulated in form of Theorem 4.4. The truncations and approximations of functions u and ψ are constructed. The errors for functions u and ψ are estimated using the gotten error estimation for the function w. The result obtained is formulated as Theorem 6.1. A test example is solved to verify the effectiveness of the proposed method. At the end, the description of the past solved problems and planned future problem solutions is provided.

1.2. Statement of the problem

If in the Timoshenko system of equations for a shell, see [29, 30], we preserve the terms with cubic nonlinearity and omit the variables t and y, then we obtain a one-dimensional system of differential equations which characterizes the axisymmetric displacement of a (cylindrical) shell in static regime. The system has the form

$$\begin{aligned} N' &= 0, \\ Q' + kN + (Nw')' + f &= 0, \\ M' - Q &= 0, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} N &= \frac{Eh}{1-\nu^2}\left(u' - kw + \frac{1}{2}\, w'^2\right), \\ Q &= k_0^2\, \frac{Eh}{2(1+\nu)}\,(\psi + w'), \\ M &= D\psi'. \end{aligned} \tag{1.2}$$

Applying (1.2) in conjunction with the expression

$$D = \frac{Eh^3}{12(1-\nu^2)},$$

equation (1.1) can be reformulated as a system

$$\begin{aligned}
&u'' - (kw)' + \frac{1}{2}\,(w'^2)' = 0,\\
&k_0^2\,\frac{Eh}{2(1+\nu)}\,(\psi' + w'') + k\,\frac{Eh}{1-\nu^2}\left(u' - kw + \frac{1}{2}\,w'^2\right)\\
&\qquad + \frac{Eh}{1-\nu^2}\left(\left(u' - kw + \frac{1}{2}\,w'^2\right)w'\right)' + f = 0,\\
&\frac{h^2}{6(1-\nu)}\,\psi'' - k_0^2(\psi + w') = 0, \qquad 0 < x < 1.
\end{aligned} \tag{1.3}$$

Assume that the following boundary conditions are fulfilled

$$u(0) = u(1) = 0, \quad w(0) = w(1) = 0, \quad \psi'(0) = \psi'(1) = 0. \tag{1.4}$$

In system (1.3) the longitudinal and transverse displacements, respectively, u, w, of the shell middle surface and the angle of rotation ψ of the normal to the shell middle surface are the functions we want to define, the shell curvature k and the force f are the given functions, ν, E, h and k_0 are the given positive constants, $0 < \nu < 0.5$ and E are respectively Poisson's ratio and Young's modulus, h is the thickness, k_0 is the lateral shear coefficient.

It's noteworthy that the system (1.3) with $k = 0$ can be derived from the Timoshenko plate system outlined in the work of Lagnese and Lions [13].

1.3. Reformulation of the problem

Applying the first and third equations from (1.3) and considering the corresponding boundary conditions outlined in (1.4), the functions u and ψ can be represented in terms of the function w in the following manner

$$\begin{aligned}
u(x) &= \int_0^1 G_u(x,\xi)\big(-2k(\xi)w(\xi) + w'^2(\xi)\big)\,\mathrm{d}\xi,\\
\psi(x) &= \int_0^1 G_\psi(x,\xi)w'(\xi)\,\mathrm{d}\xi,
\end{aligned} \tag{1.5}$$

where

$$\begin{aligned}
G_u(x,\xi) &= \begin{cases} \frac{1}{2}\,(x-1), & x > \xi,\\ \frac{1}{2}\,x, & x < \xi,\end{cases}\\
G_\psi(x,\xi) &= \begin{cases} -\dfrac{\sqrt{\sigma}}{\sinh\sqrt{\sigma}}\,\cosh\sqrt{\sigma}(x-1)\cosh\sqrt{\sigma}\xi, & x > \xi,\\[2mm] -\dfrac{\sqrt{\sigma}}{\sinh\sqrt{\sigma}}\,\cosh\sqrt{\sigma}\,x\cosh\sqrt{\sigma}(\xi-1), & x < \xi,\end{cases}\\
\sigma &= \frac{6(1-\nu)k_0^2}{h^2}.
\end{aligned} \tag{1.6}$$

Applying (1.5) and (1.6), the second equation of system (1.3) yields an integro-differential equation of Kirchhoff [12] static type with respect to w

$$\begin{aligned}
&k_0^2\,\frac{Eh}{2(1+\nu)}\,w''(x) + \frac{Eh}{1-\nu^2}\int_0^1\left(-k(\xi)w(\xi) + \frac{1}{2}\,w'^2(\xi)\right)\mathrm{d}\xi(k(x) + w''(x))\\
&\qquad - \frac{3(1-\nu)Ek_0^4}{(1+\nu)h\sinh\sqrt{\sigma}}\bigg(\sinh\sqrt{\sigma}\,(x-1)\int_0^x \cosh\sqrt{\sigma}\,\xi w'(\xi)\,\mathrm{d}\xi
\end{aligned}$$

$$+\sinh\sqrt{\sigma}\,x\int_x^1\cosh\sqrt{\sigma}\,(\xi-1)w'(\xi)\,\mathrm{d}\xi\Big)+f(x)=0,\qquad 0<x<1. \tag{1.7}$$

Additionally, we augment the equation with the corresponding boundary condition

$$w(0)=w(1)=0. \tag{1.8}$$

Consequently, the problem (1.3), (1.4) comes down to the problem (1.7), (1.8) for the function w. Upon solving the last-mentioned problem, we build the functions u and ψ using explicit formulas (1.5).

2. Assumptions

Suppose that for each $i=1,2,\dots$ the following integrals exist

$$f_i=4\int_0^1 f(x)\sin i\pi x\,\mathrm{d}x,\quad k_i=\int_0^1 k(x)\sin i\pi x\,\mathrm{d}x,$$

the series $\sum_{i=1}^{\infty}\Big(\frac{f_i}{i\pi}\Big)^2$ and $\sum_{i=1}^{\infty}\big(\frac{k_i}{i\pi}\big)^2$ converge and, also, $k\in C^1(0,1)$.

Let us assume that the following inequality is fulfilled

$$\max(\tau_1,\tau_2)\,\tau_2\le\frac{1}{2}\,\varepsilon(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(h\pi)^2}}, \tag{2.1}$$

where ε is some value from the interval $(0,1)$, while τ_1 and τ_2 are defined by the formulas

$$\tau_1=\left(\frac{8(1-\nu^2)}{Eh}\right)^{\frac{1}{3}}\left(\sum_{i=1}^{\infty}\Big(\frac{f_i}{i\pi}\Big)^2\right)^{\frac{1}{6}},\quad \tau_2=\left(\sum_{i=1}^{\infty}\Big(\frac{k_i}{i\pi}\Big)^2\right)^{\frac{1}{2}}. \tag{2.2}$$

Assume that a solution of problem (1.7), (1.8) exists and can be expressed as a series

$$w(x)=\sum_{i=1}^{\infty}w_i\sin i\pi x, \tag{2.3}$$

and the coefficients of this series satisfy the system of equations

$$\begin{aligned}2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi w_i+\sum_{j=1}^{\infty}\big(-4k_jw_j+(j\pi w_j)^2\big)\Big(-2\,\frac{k_i}{i\pi}+i\pi w_i\Big)&\\ -\frac{2(1-\nu^2)}{Ehi\pi}\,f_i=0,\quad i=1,2,\dots.&\end{aligned} \tag{2.4}$$

Notice that the i-th equation in system (2.4) is derived by substituting (2.3) into (1.7), multiplying the resulting equation by $\sin i\pi x$, integrating over x from 0 to 1, and applying the formulas, see [6],

$$\begin{aligned}\int_0^1\sin i\pi x\sin j\pi x\,\mathrm{d}x&=\begin{cases}0, & i\ne j,\\ \frac{1}{2}, & i=j,\end{cases}\\ \int e^{ax}\sin bx\,\mathrm{d}x&=\frac{e^{ax}}{a^2+b^2}\,(a\sin bx-b\cos bx),\\ \int e^{ax}\cos bx\,\mathrm{d}x&=\frac{e^{ax}}{a^2+b^2}\,(a\cos bx+b\sin bx).\end{aligned} \tag{2.5}$$

3. Method description

We write an approximate solution of problem (1.7), (1.8) as follows

$$w_n(x) = \sum_{i=1}^{n} w_{n,i} \sin i\pi x. \tag{3.1}$$

Here, the coefficients $w_{n,i}$ are determined using the Galerkin method, see [15], solving the system of nonlinear equations

$$2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi w_{n,i} + \sum_{j=1}^{n}\big(-4k_j w_{n,j} + (j\pi w_{n,j})^2\big)\left(-2\,\frac{k_i}{i\pi} + i\pi w_{n,i}\right) - \frac{2(1-\nu^2)}{Ehi\pi}\,f_i = 0, \quad i = 1,2,\ldots,n, \tag{3.2}$$

derived through the use of formulas (2.5), and can be considered as a system of nonlinear equations with respect to $i\pi w_{n,i}$, $i = 1,2,\ldots,n$.

We solve system (3.2) using the Jacobi type iteration method, see [16],

$$\begin{aligned} &2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi w_{n,i}^{(m+1)} \\ &\quad+\left(-4k_i w_{n,i}^{(m+1)} + \left(i\pi w_{n,i}^{(m+1)}\right)^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\left(-4k_j w_{n,j}^{(m)} + \left(j\pi w_{n,j}^{(m)}\right)^2\right)\right)\left(-2\,\frac{k_i}{i\pi} + i\pi w_{n,i}^{(m+1)}\right) \\ &\quad- \frac{2(1-\nu^2)}{Ehi\pi}\,f_i = 0, \quad m = 0,1,\ldots, \quad i = 1,2,\ldots,n, \end{aligned} \tag{3.3}$$

where $w_{n,i}^{(m)}$ is the m-th iteration approximation of $w_{n,i}$, $i = 1,2,\ldots,n$, $m = 0,1,\ldots$.

To implement iteration (3.3), it is necessary to solve a cubic equation for $i\pi w_{n,i}^{(m+1)}$ at the $(m+1)$-th iteration step for each i. This issue can be addressed using Cardano's formula, see [8], expressing $i\pi w_{n,i}^{(m+1)}$ explicitly. It's suitable to note that for the cubic equation

$$y^3 + \alpha y^2 + \beta y + \gamma = 0, \tag{3.4}$$

when the discriminant is positive, real root is given by

$$y = -\frac{\alpha}{3} + \left(-\frac{s}{2} + \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} - \left(\frac{s}{2} + \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \tag{3.5}$$

where

$$r = -\frac{\alpha^2}{3} + \beta, \quad s = \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma. \tag{3.6}$$

Let us write (3.3) in form (3.4) as follows

$$\left(i\pi w_{n,i}^{(m+1)}\right)^3 + \alpha_i\left(i\pi w_{n,i}^{(m+1)}\right)^2 + \beta_i\left(i\pi w_{n,i}^{(m+1)}\right) + \gamma_i = 0, \tag{3.7}$$

where

$$\alpha_i = -6\,\frac{k_i}{i\pi}, \quad \beta_i = \delta_i + 2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}} + 8\left(\frac{k_i}{i\pi}\right)^2,$$

$$\gamma_i = -2\,\frac{k_i}{i\pi}\,\delta_i - \frac{2(1-\nu^2)}{Ehi\pi}\,f_i, \quad \delta_i = \sum_{\substack{j=1\\ j\neq i}}^{n}\left(-4k_j w_{n,j}^{(m)} + \left(j\pi w_{n,i}^{(m)}\right)^2\right). \tag{3.8}$$

Here and everywhere below where it is not specified, every formula should be understood to be true for all $i \in \{1, 2, \dots, n\}$.

By analogy with (3.6) we introduce the values

$$r_i = -\frac{\alpha_i^2}{3} + \beta_i, \quad s_i = \frac{2\alpha_i^3}{27} - \frac{\alpha_i\beta_i}{3} + \gamma_i.$$

Applying (3.8) we get

$$\begin{aligned} r_i &= \delta_i + 2(1-\nu)\,\frac{1}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}} - 4\left(\frac{k_i}{i\pi}\right)^2, \\ s_i &= 4(1-\nu)\,\frac{k_i}{i\pi}\,\frac{1}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}} - \frac{2(1-\nu^2)}{Ehi\pi}\,f_i. \end{aligned} \tag{3.9}$$

Bearing in mind (3.5), we can express the solution of equation (3.7) as

$$i\pi w_{n,i}^{(m+1)} = -\frac{\alpha_i}{3} + \sigma_{i,1} - \sigma_{i,2}, \qquad m = 0, 1, \dots, \tag{3.10}$$

where

$$\sigma_{i,p} = \left((-1)^p\,\frac{s_i}{2} + \left(\frac{s_i^2}{4} + \frac{r_i^3}{27}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \quad p = 1, 2. \tag{3.11}$$

The solution method for the considered problem (1.7), (1.8) should be interpreted as the computation performed according to formula (3.10). With the values of $w_{n,i}^{(m)}$ for $i = 1, 2, \dots, n$, we then construct the approximate solution of problem (1.7), (1.8).

$$w_n^{(m)}(x) = \sum_{i=1}^{n} w_{n,i}^{(m)} \sin i\pi x. \tag{3.12}$$

The iteration method here is obtained by combining the Jacobi type iteration and the Cardano formula, so we will call it the Jacobi–Cardano iteration.

Let's make a small digression. The two-step numerical method described here for the solution of problem (1.7), (1.8) is applied by Peradze [20] for the following boundary value problem

$$\begin{aligned} & w''''(x) - \left(\alpha + \beta\int_0^L (w'(\xi))^2 \mathrm{d}\xi\right) w''(x) = f(x), \qquad 0 < x < L, \\ & w(0) = w(L) = 0, \quad w''(0) = w''(L) = 0, \end{aligned} \tag{3.13}$$

where α and β are a given positive constants, f is a given function. Equation (3.13) is somewhat similar to equation (1.7) and belongs to the class of Kirchhoff type equations. For $f(x) = 0$ it is a static variant of the beam equation obtained by Woinowsky-Krieger [34].

Also note, that in the paper, see [35], the finite difference numerical method is used for the problem which is a model of the propagation of axisymmetric longitudinal-bending waves in the cylindrical shell of Timoshenko type. The numerical methods for the solution of equations for shells are discussed in the books authored by Chapelle, Bathe [3] and Ventsel, Krauthammer [27].

4. Approximation error of function w

4.1. Definition of approximation error of function w

We can assess the accuracy of the approximate solution (3.12) by comparing it with the n-th truncation of the exact solution (2.3)

$$p_n w(x) = \sum_{i=1}^{n} w_i \sin i\pi x. \tag{4.1}$$

This implies that the approximation error of the function w is determined by the difference

$$p_n w(x) - w_n^{(m)}(x). \tag{4.2}$$

We express this as a sum

$$p_n w(x) - w_n^{(m)}(x) = \Delta w_n(x) + \Delta w_n^{(m)}(x). \tag{4.3}$$

Here, $\Delta w_n(x)$ represents the Galerkin method error, and $\Delta w_n^{(m)}(x)$ denotes the Jacobi–Cardano iteration error, both of which are given as

$$\Delta w_n(x) = p_n w(x) - w_n(x), \quad \Delta w_n^{(m)}(x) = w_n(x) - w_n^{(m)}(x). \tag{4.4}$$

Our objective is to evaluate the $L^2(0,1)$-norm of $p_n w - w_n^{(m)}$. To achieve this, we need to estimate the errors associated with both the Galerkin method and the Jacobi–Cardano iteration.

4.2. Error estimate of Galerkin method

Expanding Δw_n into a series, considering (4.4), (4.1), and (3.1), we express it as

$$\Delta w_n(x) = \sum_{i=1}^{n} \Delta w_{n,i} \sin i\pi x, \tag{4.5}$$

where

$$\Delta w_{n,i} = w_i - w_{n,i}. \tag{4.6}$$

From (4.5) we have

$$\left\| \frac{\mathrm{d}^l}{\mathrm{d}x^l} \Delta w_n(x) \right\|_{L^2(0,1)} \le \left(\frac{1}{2} \sum_{i=1}^{n} (i\pi)^{2l} \Delta w_{n,i}^2 \right)^{\frac{1}{2}}, \quad l = 0, 1. \tag{4.7}$$

We will revisit (4.7) later; for now, let's rewrite equation (2.4) in the form

$$2(1-\nu)\, \frac{1}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}}\, i\pi w_i + \Bigg(\sum_{j=1}^{n} \big(-4k_j w_j + (j\pi w_j)^2\big) \\ + \sum_{j=n+1}^{\infty} \big(-4k_j w_j + (j\pi w_j)^2\big) \Bigg) \left(-2\,\frac{k_i}{i\pi} + i\pi w_i \right) - \frac{2(1-\nu^2)}{Ehi\pi}\, f_i = 0.$$

Subtracting equation (3.2) from this relation and taking (4.6) into consideration together with the equality

$$ab - cd = \frac{1}{2}\big((a-c)(b+d) + (a+c)(b-d)\big),$$

we obtain the equation for the error

$$
\begin{aligned}
&2(1-\nu)\,\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}\,i\pi\Delta w_{n,i}+\frac{1}{2}\Bigg(\sum_{j=1}^{n}\Big(-4\,\frac{k_j}{j\pi}+j\pi(w_j+w_{n,j})\Big)j\pi\Delta w_{n,j}\\
&\quad\times\Big(-4\,\frac{k_i}{i\pi}+i\pi(w_i+w_{n,i})\Big)+\sum_{j=1}^{n}\big(-4k_j(w_j+w_{n,j})+(j\pi)^2(w_j^2+w_{n,j}^2)\big)i\pi\Delta w_{n,i}\Bigg)\\
&\quad+\sum_{j=n+1}^{\infty}\big(-4k_jw_j+(j\pi w_j)^2\big)\Big(-2\,\frac{k_i}{i\pi}+i\pi w_i\Big)=0.
\end{aligned}
$$

Multiply this equation by $i\pi\Delta w_{n,i}$ and sum the obtained expression over $i=1,2,\dots,n$. The result will be as follows

$$
\begin{aligned}
&2(1-\nu)\sum_{i=1}^{n}\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}(i\pi\Delta w_{n,i})^2\le 2\left|\sum_{j=1}^{n}k_j(w_j+w_{n,j})\right|\sum_{i=1}^{n}(i\pi\Delta w_{n,i})^2\\
&\quad+\left(4\left(\sum_{j=n+1}^{\infty}\left(\frac{k_j}{j\pi}\right)^2\right)^{\frac12}\left(\sum_{j=n+1}^{\infty}(j\pi w_j)^2\right)^{\frac12}+\sum_{j=n+1}^{\infty}(j\pi w_j)^2\right)\\
&\quad\times\left(2\left(\sum_{i=1}^{n}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac12}+\left(\sum_{i=1}^{n}(i\pi w_i)^2\right)^{\frac12}\right)\left(\sum_{i=1}^{n}(i\pi\Delta w_{n,i})^2\right)^{\frac12}.
\end{aligned}\tag{4.8}
$$

Now, let's assess the right-hand side of inequality (4.8). For this, we have to obtain *a priori* estimates for the coefficients from expansions (2.3) and (3.1).

Lemma 4.1. *The estimate*

$$
\left(\sum_{i=1}^{\infty}(i\pi w_i)^2\right)^{\frac12}\le\tau_1 \tag{4.9}
$$

is valid.

Proof. Multiplying equation (2.4) by $i\pi w_i$ and summing the resulting equality over $i=1,2,\dots,$ we obtain

$$
\begin{aligned}
&2(1-\nu)\sum_{i=1}^{\infty}\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}(i\pi w_i)^2+8\left(\sum_{i=1}^{\infty}k_iw_i\right)^2+\left(\sum_{i=1}^{\infty}(i\pi w_i)^2\right)^2\\
&\qquad=6\sum_{i=1}^{\infty}k_iw_i\sum_{i=1}^{\infty}(i\pi w_i)^2+\frac{2(1-\nu^2)}{Eh}\sum_{i=1}^{\infty}f_iw_i.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&2(1-\nu)\sum_{i=1}^{\infty}\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}(i\pi w_i)^2+\frac14\left(\sum_{i=1}^{\infty}(i\pi w_i)^2\right)^2\le 4\left(\sum_{i=1}^{\infty}k_iw_i\right)^2+\frac{2(1-\nu^2)}{Eh}\sum_{i=1}^{\infty}|f_iw_i|\\
&\quad\le 4\sum_{i=1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\sum_{i=1}^{\infty}(i\pi w_i)^2+\frac{2(1-\nu^2)}{Eh}\left(\sum_{i=1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac12}\left(\sum_{i=1}^{\infty}(i\pi w_i)^2\right)^{\frac12}.
\end{aligned}
$$

By virtue of (2.1) and (2.2) we conclude that (4.9) is valid. □

Similar to the above reasoning, using equation (3.2) and also (2.1), (2.2) and manipulating the finite sums, we obtain

Lemma 4.2. *The inequality*

$$\left(\sum_{i=1}^{n}(i\pi w_{n,i})^2\right)^{\frac{1}{2}} \le \tau_{1,n}, \tag{4.10}$$

where

$$\tau_{1,n} = \left(\frac{8(1-\nu^2)}{Eh}\right)^{\frac{1}{3}}\left(\sum_{i=1}^{n}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac{1}{6}} \le \tau_1, \tag{4.11}$$

is satisfied.

Lemma 4.3. *The relation*

$$\left(\sum_{i=n+1}^{\infty}(i\pi w_i)^2\right)^{\frac{1}{2}} \le \left(c_1\left(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}} + c_2\left(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}, \tag{4.12}$$

where

$$c_1 = 2\tau_1(\tau_1 + 4\tau_2), \quad c_2 = \frac{2(1-\nu^2)}{Eh}, \tag{4.13}$$

holds.

Proof. We multiply equation (2.4) by $i\pi w_i$ and sum the obtained equality over $i = n+1, n+2, \ldots$. As a result of simple transformations we find that

$$\begin{aligned} 2(1-\nu)\sum_{i=n+1}^{\infty}\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(hi\pi)^2}}(i\pi w_i)^2 - 4\sum_{j=1}^{\infty}k_j w_j\sum_{i=n+1}^{\infty}(i\pi w_i)^2 + \left(\sum_{i=n+1}^{\infty}(i\pi w_i)^2\right)^2 \\ \le \left(2\sum_{j=1}^{\infty}(j\pi w_j)^2 - 8\sum_{j=1}^{\infty}k_j w_j\right)\sum_{i=n+1}^{\infty}k_i w_i + \frac{2(1-\nu^2)}{Eh}\sum_{i=n+1}^{\infty}f_i w_i. \end{aligned} \tag{4.14}$$

Since by virtue of (4.9), (2.1) and (2.2)

$$\left|\sum_{j=1}^{\infty}k_j w_j\right| \le \frac{1}{2}(1-\nu)\frac{1}{\frac{1}{k_0^2}+\frac{6(1-\nu)}{(h\pi)^2}},$$

from (4.14) follows the inequality

$$\begin{aligned} \left(\sum_{i=n+1}^{\infty}(i\pi w_i)^2\right)^{\frac{3}{2}} \le \left(2\sum_{j=1}^{\infty}(j\pi w_j)^2 + 8\left(\sum_{j=1}^{\infty}\left(\frac{k_j}{j\pi}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}(j\pi w_j)^2\right)^{\frac{1}{2}}\right) \\ \times\left(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac{1}{2}} + \frac{2(1-\nu^2)}{Eh}\left(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac{1}{2}}, \end{aligned}$$

which together with (4.9) and (2.2) leads to (4.12). □

Taking (4.8)–(4.11) and relations (2.1), (2.2) into account, we get the inequality

$$\sum_{i=1}^{n}(i\pi\Delta w_{n,i})^2 \le \tau\sqrt{2}\Bigg(\bigg(c_0\sum_{j=n+1}^{\infty}\Big(\frac{k_j}{j\pi}\Big)^2\bigg)^{\frac12}\bigg(\sum_{j=n+1}^{\infty}(j\pi w_j)^2\bigg)^{\frac12}+\sum_{j=n+1}^{\infty}(j\pi w_j)^2\Bigg)\bigg(\sum_{i=1}^{n}(i\pi\Delta w_{n,i})^2\bigg)^{\frac12},$$

where

$$c_0=16,\quad \tau=\frac{1}{2\sqrt{2}(1-\varepsilon)(1-\nu)}(\tau_1+2\tau_2)\bigg(\frac{1}{k_0^2}+\frac{6(1-\nu)}{(h\pi)^2}\bigg). \tag{4.15}$$

If this inequality is used in (4.7) together with (4.12), then we obtain the desired estimate

$$\begin{aligned}\bigg\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,\Delta w_n(x)\bigg\|_{L^2(0,1)} &\le \frac{1}{\pi^{1-l}}\,\tau\sum_{p=0}^{1}\bigg(c_0\sum_{i=n+1}^{\infty}\Big(\frac{k_i}{i\pi}\Big)^2\bigg)^{\frac12(1-p)}\\ &\quad\times\Bigg(c_1\bigg(\sum_{i=n+1}^{\infty}\Big(\frac{k_i}{i\pi}\Big)^2\bigg)^{\frac12}+c_2\bigg(\sum_{i=n+1}^{\infty}\Big(\frac{f_i}{i\pi}\Big)^2\bigg)^{\frac12}\Bigg)^{\frac13(1+p)},\quad l=0,1.\end{aligned} \tag{4.16}$$

4.3. Error estimate of Jacobi–Cardano iteration

Using (4.4), (3.1) and (3.12), we represent $\Delta w_n^{(m)}(x)$ as a series

$$\Delta w_n^{(m)}(x)=\sum_{i=1}^{n}\Delta w_{n,i}^{(m)}\sin i\pi x, \tag{4.17}$$

where

$$\Delta w_{n,i}^{(m)}=w_{n,i}-w_{n,i}^{(m)}. \tag{4.18}$$

The series in (4.17) leads to the formula

$$\bigg\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,\Delta w_n^{(m)}(x)\bigg\|_{L^2(0,1)}=\bigg(\frac12\sum_{i=1}^{n}(i\pi)^{2l}(\Delta w_{n,i}^{(m)})^2\bigg)^{\frac12},\quad l=0,1, \tag{4.19}$$

which will be employed later.

Let's express system (3.10) in the following manner

$$i\pi w_{n,i}^{(m+1)}=\varphi_i\Big(1\pi w_{n,1}^{(m)},2\pi w_{n,2}^{(m)},\dots,n\pi w_{n,n}^{(m)}\Big) \tag{4.20}$$

and examine the Jacobi matrix

$$J=\left(\frac{\partial\varphi_i}{\partial\Big(j\pi w_{n,j}^{(m)}\Big)}\right)_{i,j=1}^{n}.$$

Based on (3.8)–(3.11) and (4.20), the diagonal elements of the matrix J are zero, and the non-diagonal elements are given by

$$\frac{\partial\varphi_i}{\partial\Big(j\pi w_{n,j}^{(m)}\Big)}=\frac{1}{27}\Big(j\pi w_{n,j}^{(m)}-2\,\frac{k_j}{j\pi}\Big)r_i^2\Big(\frac{s_i^2}{4}+\frac{r_i^3}{27}\Big)^{-\frac12}\Big(\frac{1}{\sigma_{i,1}^2}-\frac{1}{\sigma_{i,2}^2}\Big).$$

If in this equality we employ the following relations

$$\sigma_{i,1}\sigma_{i,2} = \frac{r_i}{3}, \quad \sigma_{i,2}^3 - \sigma_{i,1}^3 = s_i, \quad \left(\frac{s_i^2}{4} + \frac{r_i^3}{27}\right)^{\frac{1}{2}} = \frac{\sigma_{i,1}^3 + \sigma_{i,2}^3}{2}, \tag{4.21}$$

which are derived from (3.11), we obtain

$$\frac{\partial \varphi_i}{\partial (j\pi w_{n,j}^{(m)})} = \frac{2}{3}\Big(j\pi w_{n,j}^{(m)} - 2\,\frac{k_j}{j\pi}\Big) s_i \Big(\sigma_{i,1}^4 + \Big(\frac{r_i}{3}\Big)^2 + \sigma_{i,2}^4\Big)^{-1}.$$

Let us apply the estimate

$$\sigma_{i,1}^4 + \sigma_{i,2}^4 \geq 2\Big(\frac{r_i}{3}\Big)^2$$

to this equality, which follows from the first relation in (4.21). We get the estimate

$$\left|\frac{\partial \varphi_i}{\partial\Big(j\pi w_{n,j}^{(m)}\Big)}\right| \leq \left(8(1-\nu)\,\frac{|k_i|}{i\pi}\,\frac{1}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}} + \frac{4(1-\nu^2)}{Eh}\,\frac{|f_i|}{i\pi}\right) \times \left(j\pi\Big|w_{n,j}^{(m)}\Big| + 2\,\frac{|k_j|}{j\pi}\right) r_i^{-2}. \tag{4.22}$$

Using the parameter ε from condition (2.1) we introduce the values ρ and δ according to the relations

$$\begin{aligned} &\varepsilon < \rho < 1, \\ &\delta = \frac{1}{2}\Big(1 + \frac{\rho}{\varepsilon}\Big) - \Big(1 + \frac{1}{4}\Big(1 - \frac{\rho}{\varepsilon}\Big)^2\Big)^{\frac{1}{2}}. \end{aligned}$$

By (3.8) and (3.9),

$$|r_i| \geq \left|\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\Big(j\pi w_{n,j}^{(m)}\Big)^2 - 4k_j w_{n,j}^{(m)} + 4\,\frac{\rho}{\varepsilon}\Big(\frac{k_j}{j\pi}\Big)^2\right) + \mu\right| \geq \delta \sum_{\substack{j=1\\ j\neq i}}^{n}\left(\Big(j\pi w_{n,j}^{(m)}\Big)^2 + 4\Big(\frac{k_j}{j\pi}\Big)^2\right) + \mu,$$

where

$$\mu = 4\Big(\frac{\rho}{\varepsilon} - 1\Big)\Big(\frac{k_i}{i\pi}\Big)^2 + (1-\rho)\,\frac{2(1-\nu)}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}}\cdot \tag{4.23}$$

Therefore, keeping in mind the fact that $\max_\xi \frac{\xi}{(\xi^2+\alpha^2)^2}$, where $0 < \xi < \infty$, $\alpha = const > 0$, is equal to $\frac{3\sqrt{3}}{16\alpha^3}$, we get

$$\Big(j\pi|w_{n,j}^{(m)}| + 2\,\frac{|k_j|}{j\pi}\Big) r_i^{-2} \leq \frac{3}{8\mu}\Big(\frac{3}{2\delta\mu}\Big)^{\frac{1}{2}}. \tag{4.24}$$

We will employ vector and matrix norms, defined as $\sum_{i=1}^n |\nu_i|$ and $\max_{1\leq j\leq n}\sum_{i=1}^n |m_{ij}|$ for $\nu = (\nu_i)_{i=1}^n$ and $M = (m_{ij})_{i,j=1}^n$. For any set of values $w_{n,j}^{(m)}$, $j = 1, 2, \ldots, n$, it is necessary that the elements of the matrix J meet the condition $\max_{1\leq j\leq n}\ \sum_{i=1}^n \left|\frac{\partial\varphi_i}{\partial(j\pi w_{n,j}^{(m)})}\right| \leq q < 1$. Applying (4.22)–(4.24) we come to the conclusion that it is sufficient

$$\frac{1-\nu}{\sqrt{\delta}} \sum_{i=1}^{n}\left(\frac{1}{4}\,\frac{|k_i|}{i\pi}\,\frac{1}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}} + \frac{1+\nu}{8Eh}\,\frac{|f_i|}{i\pi}\right)\left(\frac{2}{3}\Big(\frac{\rho}{\varepsilon} - 1\Big)\Big(\frac{k_i}{i\pi}\Big)^2 + \frac{1-\rho}{3}\,\frac{1-\nu}{\frac{1}{k_0^2} + \frac{6(1-\nu)}{(hi\pi)^2}}\right)^{-\frac{3}{2}} \leq q < 1 \tag{4.25}$$

to hold.

In accordance with Banach's compression principle, see [14], system (3.2) possesses a unique solution $w_{n,i}$, $i=1,2,\dots,n$. The iteration process (3.10) converges, $\lim_{m\to\infty} w_{n,i}^{(m)} = w_{n,i}$, $i=1,2,\dots,n$, with the convergence rate determined by the inequality

$$\sum_{i=1}^{n} i\Big|w_{n,i}-w_{n,i}^{(m)}\Big| \le \frac{q^m}{1-q}\sum_{i=1}^{n} i\Big|w_{n,i}^{(1)}-w_{n,i}^{(0)}\Big|, \qquad m=1,2,\dots$$

Considering this, along with (4.18) and (4.19), we get the estimate for the Jacobi–Cardano iteration error

$$\left\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,\Delta w_n^{(m)}(x)\right\|_{L^2(0,1)} \le \frac{\pi^l}{\sqrt{2}}\,\frac{q^m}{1-q}\sum_{i=1}^{n} i\Big|w_{n,i}^{(1)}-w_{n,i}^{(0)}\Big|, \qquad l=0,1, \quad m=1,2,\dots. \tag{4.26}$$

4.4. Result

Let's assess the error (4.2). From (4.3), we have

$$\left\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\Big(p_n w(x)-w_n^{(m)}(x)\Big)\right\|_{L^2(0,1)} \le \left\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,\Delta w_n(x)\right\|_{L^2(0,1)} + \left\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,\Delta w_n^{(m)}(x)\right\|_{L^2(0,1)}.$$

Thus, applying (4.16) and (4.26), we obtain the inequality

$$\begin{aligned}\left\|\frac{\mathrm{d}^l}{\mathrm{d}x^l}\Big(p_n w(x)-w_n^{(m)}(x)\Big)\right\|_{L^2(0,1)} &\le \frac{1}{\pi^{1-l}}\,\tau\sum_{p=0}^{1}\left(c_0\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac12(1-p)} \\ &\quad\times\left(c_1\left(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\right)^{\frac12} + c_2\left(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\right)^{\frac12}\right)^{\frac13(1+p)} \\ &\quad+\frac{\pi^l}{\sqrt{2}}\,\frac{q^m}{1-q}\sum_{i=1}^{n} i\Big|w_{n,i}^{(1)}-w_{n,i}^{(0)}\Big|, \quad l=0,1, \quad m=1,2,\dots.\end{aligned} \tag{4.27}$$

In summary, the obtained result can be expressed as follows.

Theorem 4.4. *Let q be some number from the interval $(0,1)$. Suppose the conditions of Section 2 and the restriction (4.25) are satisfied. In that case, the approximation error of the function w is bounded by inequality (4.27), where the constants c_0, c_1, c_2 and τ are computed using formulas (4.15) and (4.13).*

5. Formulas for truncations of functions u, ψ and its approximations

Let's refer to formulas (1.5). Applying $p_n w(x)$ and $w_n^{(m)}(x)$, we build the n-th truncation of the functions u and ψ

$$\begin{aligned} p_n u(x) &= \int_0^1 G_u(x,\xi)\Big(-2k(\xi)p_n w(\xi) + \big((p_n w(\xi))'\big)^2\Big)\,\mathrm{d}\xi, \\ p_n\psi(x) &= \int_0^1 G_\psi(x,\xi)(p_n w(\xi))'\,\mathrm{d}\xi, \end{aligned} \tag{5.1}$$

and the approximation of these functions

$$\begin{aligned} u_n^{(m)}(x) &= \int_0^1 G_u(x,\xi)\left(-2k(\xi)w_n^{(m)}(\xi) + \left(\Big(w_n^{(m)}(\xi)\Big)'\right)^2\right)\mathrm{d}\xi, \\ \psi_n^{(m)}(x) &= \int_0^1 G_\psi(x,\xi)\Big(w_n^{(m)}(\xi)\Big)'\,\mathrm{d}\xi. \end{aligned} \tag{5.2}$$

6. Approximation errors of functions u and ψ

6.1. Definition of approximation errors of functions u and ψ

Similar to (4.2), we determine the approximation errors of the functions u and ψ as the differences $p_n u(x) - u_n^{(m)}(x)$ and $p_n\psi(x) - \psi_n^{(m)}(x)$, respectively. We then estimate the $L^2(0,1)$-norm of each of them. Applying (5.1) and (5.2), we get

$$p_n u(x) - u_n^{(m)}(x) = \int_0^1 G_u(x,\xi)\left(-2k(\xi)\left(p_n w(\xi) - w_n^{(m)}(\xi)\right) + \left((p_n w(\xi))'\right)^2 - \left(\left(w_n^{(m)}(\xi)\right)'\right)^2\right) \mathrm{d}\xi \tag{6.1}$$

and

$$p_n\psi(x) - \psi_n^{(m)}(x) = \int_0^1 G_\psi(x,\xi)\left(p_n w(\xi) - w_n^{(m)}(\xi)\right)' \mathrm{d}\xi. \tag{6.2}$$

6.2. Result

From (6.1) and (1.6) we obtain

$$\left(p_n u(x) - u_n^{(m)}(x)\right)^2 = \frac{1}{4}\left((x-1)^2\left(\int_0^x H(\xi)\,\mathrm{d}\xi\right)^2 + 2x(x-1)\int_0^x H(\xi)\,\mathrm{d}\xi \int_x^1 H(\xi)\,\mathrm{d}\xi + x^2\left(\int_x^1 H(\xi)\,\mathrm{d}\xi\right)^2\right),$$

where

$$H(\xi) = -2k(\xi)\left(p_n w(\xi) - w_n^{(m)}(\xi)\right) + \left((p_n w(\xi))'\right)^2 - \left(\left(w_n^{(m)}(\xi)\right)'\right)^2.$$

Consequently,

$$\left(p_n u(x) - u_n^{(m)}(x)\right)^2 \le \frac{1}{4}\left(\int_0^1 |H(\xi)|\,\mathrm{d}\xi\right)^2. \tag{6.3}$$

Let's estimate $\int_0^1 |H(\xi)|\,\mathrm{d}\xi$. Because of

$$w_n^{(m)}(\xi) = p_n w(\xi) + \left(w_n^{(m)}(\xi) - p_n w(\xi)\right)$$

we write

$$\begin{aligned}\int_0^1 |H(\xi)|\,\mathrm{d}\xi \le{}& 2\|k(x)\|_{L^2(0,1)}\left\|p_n w(x) - w_n^{(m)}(x)\right\|_{L^2(0,1)} \\ &+ \left\|\left(p_n w(x) - w_n^{(m)}(x)\right)'\right\|_{L^2(0,1)}\left(2\|(p_n w(x))'\|_{L^2(0,1)} + \left\|\left(p_n w(x) - w_n^{(m)}(x)\right)'\right\|_{L^2(0,1)}\right).\end{aligned}$$

Applying (3.12) and (4.1) with (4.9), we obtain

$$\int_0^1 |H(\xi)|\,\mathrm{d}\xi \le \sum_{l=1}^2 c_{l+2}\left\|\left(p_n w(x) - w_n^{(m)}(x)\right)'\right\|_{L^2(0,1)}^l,$$

where

$$c_3 = \tau_1\sqrt{2} + \frac{2}{\pi}\left(\int_0^1 k^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}, \quad c_4 = 1. \tag{6.4}$$

Therefore by (6.3) and (4.27) we have

$$\begin{aligned}\left\|p_n u(x) - u_n^{(m)}(x)\right\|_{L^2(0,1)} \le \frac{1}{2}\sum_{l=1}^{2} c_{l+2}\Bigg(\tau \sum_{p=0}^{1}\Bigg(c_0 \sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}(1-p)} \\ \times\Bigg(c_1\Bigg(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}} + c_2\Bigg(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}}\Bigg)^{\frac{1}{3}(1+p)} \\ + \frac{\pi}{\sqrt{2}}\frac{q^m}{1-q}\sum_{i=1}^{n} i\left|w_{n,i}^{(1)} - w_{n,i}^{(0)}\right|\Bigg)^l, \qquad m = 1, 2, \ldots .\end{aligned} \tag{6.5}$$

Continuing, (6.2) and (1.6) imply

$$\begin{aligned}\left(p_n\psi(x) - \psi_n^{(m)}(x)\right)^2 = \frac{\sigma}{\sinh^2\sqrt{\sigma}}\Bigg(\cosh\sqrt{\sigma}\,(x-1)\int_0^x \cosh\sqrt{\sigma}\,\xi\left(p_n w(\xi) - w_n^{(m)}(\xi)\right)'\mathrm{d}\xi \\ + \cosh\sqrt{\sigma}\,x\int_x^1 \cosh\sqrt{\sigma}\,(\xi-1)\left(p_n w(\xi) - w_n^{(m)}(\xi)\right)'\mathrm{d}\xi\Bigg)^2.\end{aligned}$$

So,

$$\left\|p_n\psi(x) - \psi_n^{(m)}(x)\right\|_{L^2(0,1)} \le c_5\left\|\left(p_n w(x) - w_n^{(m)}(x)\right)'\right\|_{L^2(0,1)}, \tag{6.6}$$

where

$$c_5 = \left(\int_0^1 F(x)\,\mathrm{d}x\right)^{\frac{1}{2}} \tag{6.7}$$

and

$$F(x) = \frac{2\sigma}{\sinh^2\sqrt{\sigma}}\left(\cosh^2\sqrt{\sigma}\,(x-1)\int_0^x \cosh^2\sqrt{\sigma}\,\xi\,\mathrm{d}\xi + \cosh^2\sqrt{\sigma}\,x\int_x^1 \cosh^2\sqrt{\sigma}\,(\xi-1)\,\mathrm{d}\xi\right).$$

Applying the formula

$$\int \cosh^2 x\,\mathrm{d}x = \frac{\sinh 2x}{4} + \frac{x}{2},$$

after evaluating the integrals, we obtain

$$\begin{aligned}F(x) = \frac{\sigma}{2\sinh^2\sqrt{\sigma}}\Bigg(1 + \frac{1}{2\sqrt{\sigma}}\sinh 2\sqrt{\sigma} + \cosh 2\sqrt{\sigma}x \\ + \sinh\sqrt{\sigma}\left(\frac{1}{\sqrt{\sigma}}\cosh\sqrt{\sigma}\,(2x-1) - 2x\sinh\sqrt{\sigma}\,(2x-1)\right)\Bigg).\end{aligned}$$

We substitute this expression into (6.7), performing additional integration, and applying formula

$$\int x\sinh x\,\mathrm{d}x = x\cosh x - \sinh x.$$

As a result we get

$$c_5 = \frac{1}{\sinh\sqrt{\sigma}}\left(\frac{\sigma}{2} + \frac{\sqrt{\sigma}}{4}\sinh 2\sqrt{\sigma} + \sinh^2\sqrt{\sigma}\right)^{\frac{1}{2}}. \tag{6.8}$$

By (6.6) and (4.27) we obtain

$$\begin{aligned}\left\|p_n\psi(x)-\psi_n^{(m)}(x)\right\|_{L^2(0,1)} \le c_5\Bigg(\tau\sum_{p=0}^{1}\Bigg(c_0\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}(1-p)}\\ \times\Bigg(c_1\Bigg(\sum_{i=n+1}^{\infty}\left(\frac{k_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}}+c_2\Bigg(\sum_{i=n+1}^{\infty}\left(\frac{f_i}{i\pi}\right)^2\Bigg)^{\frac{1}{2}}\Bigg)^{\frac{1}{3}(1+p)}\\ +\frac{\pi}{\sqrt{2}}\frac{q^m}{1-q}\sum_{i=1}^{n} i|w_{n,i}^{(1)}-w_{n,i}^{(0)}|\Bigg),\qquad m=1,2,\dots. \qquad (6.9)\end{aligned}$$

The received result can be formulated as follows.

Theorem 6.1. *Suppose the conditions of Theorem 4.4 are satisfied. In that case, the approximation errors of the functions u and ψ are estimated by inequalities (6.5) and (6.9), respectively. Here, the coefficients c_i for $i=0,1,\dots,5$, and τ are determined by formulas (4.15), (4.13), (6.4) and (6.8).*

7. Numerical example

To check the effectiveness of the method, we solved a test example. Let us consider the shell with $k(x)=\exp(-\pi x)$, $\nu=0.3$, $E=2.1\cdot10^{11}$, $h=0.005$ and as in paper of Timoshenko,[25] suppose $k_0^2=\frac{2}{3}$. Assume that

$$\begin{aligned}f(x)=\frac{0.1Eh}{1-\nu^2}\left(\frac{0.1\pi}{8}(1-\exp(2\pi))+\frac{2}{\pi}\right)(\exp(-\pi x)+0.2\pi^2\exp(\pi x)\cos\pi x)\\ -k_0^2\frac{0.1Eh}{2(1+\nu)}(2\pi^2\exp(\pi x)\cos\pi x+\alpha),\end{aligned}$$

where

$$\begin{aligned}\alpha=\frac{2\pi^2\sigma}{((\pi+\sqrt{\sigma})^2+\pi^2)((\pi-\sqrt{\sigma})^2+\pi^2)}\Big(\exp(\pi x)\big(2\pi^2\sin\pi x-\sigma\cos\pi x\big)\\ -\frac{\sigma}{\sinh\sqrt{\sigma}}\big(\exp(\pi x)\sinh\sqrt{\sigma}x+\sinh\sqrt{\sigma}(x-1)\big)\Big).\end{aligned}$$

Then the exact solution of problem (1.3), (1.4) is

$$\begin{aligned}u(x)&=\left(\frac{0.01\pi}{8}(\exp(2\pi)-1)-\frac{0.2}{\pi}\right)x+\frac{0.1}{\pi}(1-\cos\pi x)-\frac{0.01\pi}{8}\exp(2\pi x)(2+\sin2\pi x-\cos2\pi x)+\frac{0.01\pi}{8},\\ w(x)&=0.1\exp(\pi x)\sin\pi x,\\ \psi(x)&=\frac{0.1\pi\sigma}{\big((\pi+\sqrt{\sigma})^2+\pi^2\big)\big((\pi-\sqrt{\sigma})^2+\pi^2\big)}\Big(\exp(\pi x)\big(\big(2\pi^2-\sigma\big)\sin\pi x-\big(2\pi^2+\sigma\big)\cos\pi x\big)\\ &\quad-\frac{2\pi\sqrt{\sigma}}{\sinh\sqrt{\sigma}}\big(\exp(\pi x)\cosh\sqrt{\sigma}x+\cosh\sqrt{\sigma}(x-1)\big)\Big).\end{aligned}$$

The method is applied with $n=3,5,10,\dots 50$. The initial iteration approximation was $w_n^{(0)}(x)=0$. The method error for $g_i(x),\ i=1,2,3$ is calculated using the formula

$$e_{n,i}^{(m)}=\left\|p_ng_i(x)-g_{n,i}^{(m)}(x)\right\|_{L^2(0,1)}, \qquad (7.1)$$

TABLE 1. Values of the method error.

n	m	$e_{n,1}^{(m)}$	$e_{n,2}^{(m)}$	$e_{n,3}^{(m)}$
3	53	0.0361641791	0.0125063153	0.0556577568
5	101	0.0108661138	0.0031821873	0.0146294241
10	138	0.0015581772	0.0004359962	0.0020326522

TABLE 2. Values of the exact and approximate solutions.

x	0.0	0.25	0.5	0.75	1.0
$p_3u(x)$	0.0	0.4276791	0.7097537	0.9295171	0.0
$u_3^{(53)}(x)$	0.0	0.4526044	0.7502573	0.9816446	0.0
$p_5u(x)$	0.0	0.4635897	0.7483907	1.1102759	0.0
$u_5^{(101)}(x)$	0.0	0.4703572	0.7591218	1.1259984	0.0
$p_3w(x)$	0.0	0.1616577	0.4643126	0.7763954	0.0
$w_3^{(53)}(x)$	0.0	0.1653666	0.4766530	0.7979788	0.0
$p_5w(x)$	0.0	0.1458122	0.4867215	0.7605499	0.0
$w_5^{(101)}(x)$	0.0	0.1466407	0.4900084	0.7659102	0.0
$p_3\psi(x)$	-0.7775047	-0.5898112	-1.9305749	0.5898112	4.6386549
$\psi_3^{(53)}(x)$	-0.7949475	-0.6049209	-1.9867096	0.6049209	4.7683668
$p_5\psi(x)$	-0.5353335	-0.9348537	-1.3371791	-0.2519378	5.5832749
$\psi_5^{(101)}(x)$	-0.5378648	-0.9413017	-1.3470411	-0.2542302	5.6230107

where $g_1(x) = u(x)$, $g_2(x) = w(x)$ and $g_3(x) = \psi(x)$.

The computations were executed using the Julia programming language, and the outcomes are shown in the following tables and graphs.

Table 1 contains the method error values for $n = 3, 5$ and 10 calculated using formula (7.1).

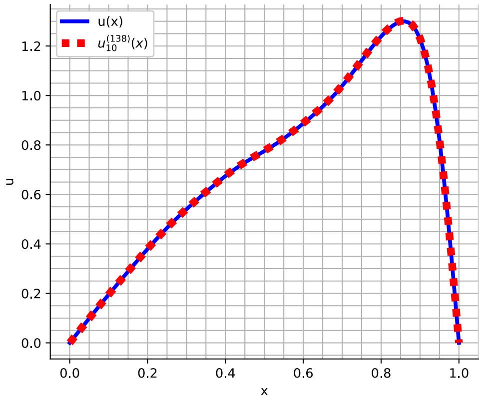
Table 2 contains the values of functions $u_n^{(m)}, w_n^{(m)}, \psi_n^{(m)}$ at the points $x_i = 0.25i,\ i = 0, 1, ..., 4$, for $n = 3$, $m = 53; n = 5$, $m = 101$ and also the n-th truncations of exact solution $p_n u, p_n w, p_n \psi$ values at the same nodes.

Figure 1a presents the graphs of approximate solution $u_{10}^{(138)}$ and exact solution u. Figure 1b presents the graphs of approximate solution $w_{10}^{(138)}$ and exact solution w. Figure 1c presents the graphs of approximate solution $\psi_{10}^{(138)}$ and exact solution ψ.

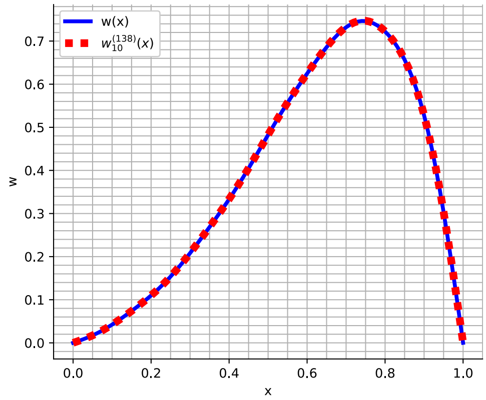
Figure 2a presents the scatter plot of method error for u and $n = 5, 10, \dots, 50$. Figure 2b presents the scatter plot of method error for w and $n = 5, 10, \dots, 50$. Figure 2c presents the scatter plot of method error for ψ and $n = 5, 10, \dots, 50$.

8. CONCLUSIONS, PAST AND FUTURE WORKS

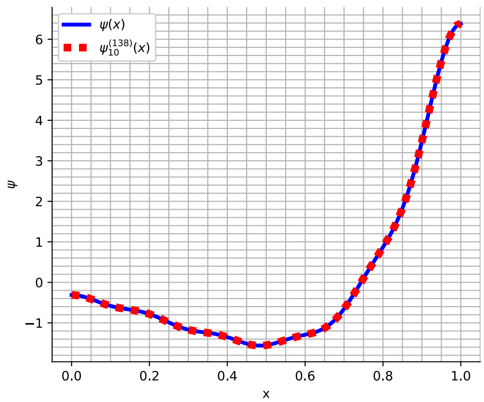
Our goal was to develop a method for approximate solution of a simplified one-dimensional model that can probably be extended to a generalized model. This model consists of a system with respect to five functions of two spatial variables. We believe that the system can be reduced to a single equation with respect to function



(A)



(B)



(C)

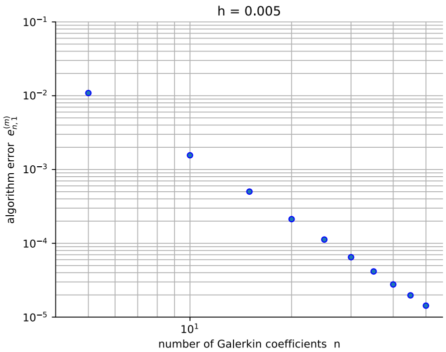
FIGURE 1. Graphs of exact and approximate solutions u, w and ψ. (a) Graphs of $u_{10}^{(138)}$ and u. (b) Graphs of $w_{10}^{(138)}$ and w. (c) Graphs of $\psi_{10}^{(138)}$ and ψ.

w using Green's matrices or the Fourier series method. If we are not able to achieve the result, we hope that the other researchers will be more successful.

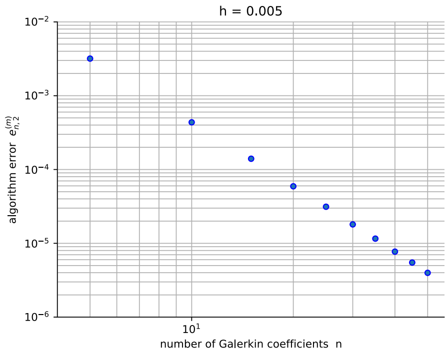
For the solution of a problem on static symmetric state of shell in the Timoshenko theory an approximate method is used, which consists of Green's function method, Galerkin projection method, nonlinear iteration method of Jacobi type together with Cardano formula. The accuracy of the approximate method is estimated. The results of a numerical experiments are given, which indicate the effectiveness of the method.

Briefly about testing the method. The provided table and graphs illustrating algorithm errors indicate that increasing the parameter n leads to an improvement in errors. From the obtained results, one can infer that the numerical algorithm offered for solving problem (1.3), (1.4) proves to be effective.

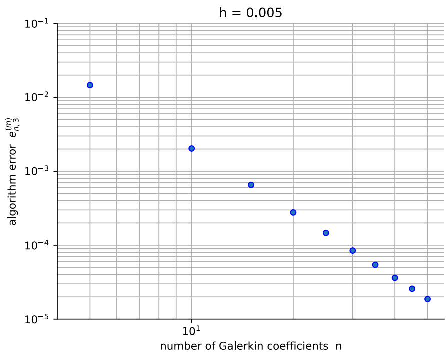
This work is a generalization of the papers, see [21, 23] and [22], in which the problem for the plate is considered. In case of a plate the analog of problem (1.7), (1.8) has a more simpler form, than in case of a shell, because the analog of equation (1.7) does not contain the curvature function, since for plate $k(x) = 0$. More about other differences. In papers, see [21] and [22], the error of only the iteration part of the method is



(A)



(B)



(C)

FIGURE 2. Scatter plots of algorithm error for functions u, w and ψ. (a) Scatter plot of method error for u. (b) Scatter plot of method error for w. (c) Scatter plot of method error for ψ.

estimated. Further, in contrast to the papers, see [21,23] and [22], this work provides the results of computational testing of the applied method.

In this work, the main attention was paid to construction of a numerical method and to estimation of its accuracy for equation (1.7). An important part of the nonlinearity of equation (1.7) also arises in other equations for which the numerical methods have been proposed and studied. In some cases, this enables the use of other methods and investigate its accuracy for equation (1.7). In particular we have in mind one such possibility based on Chipot's method, see [4,5], with which the Kirchhoff generalized equation for string in static regime

$$\varphi\left(\int_0^1 w'^2(x)\,\mathrm{d}x\right)w''(x) = f(x), \qquad 0 < x < 1, \tag{8.1}$$

is solved, where $\varphi : [0,+\infty) \to (\alpha,+\infty)$, $\alpha > 0$, and $f : (0,1) \to \mathbb{R}$, are given functions. In Kachakhidze *et al.* [10] paper the practical aspects of accuracy method is discussed and the numerical examples are given for equation (8.1).

Now regarding the dynamic case of the problem, which describes the axisymmetric oscillation of the Timoshenko shell. It would be natural to conclude the system of equations (1.3) is such that equation (1.7) has become a hyperbolic type partial differential equation with respect to the function w of spatial and time variables x and t. In this case an important part of the nonlinearity of equation (1.7) can be found in other hyperbolic equations as well. Such equations are, for example, the Kirchhoff generalized equation for string in dynamic regime presented in Bernstein's paper, see [2], as

$$w_{tt}(x,t) = \varphi\left(\int_0^l w_x^2(x,t)\,\mathrm{d}x\right) w_{xx}(x,t), \qquad 0 < x < l,\ 0 < t \leq T, \tag{8.2}$$

where $\varphi : [0,+\infty) \to (\alpha,+\infty),\ \alpha > 0$, is a given function and Ball beam equation, see [1],

$$\begin{aligned} w_{tt}(x,t) + \alpha w_{xxxx}(x,t) - \left(\beta + \gamma\int_0^l w_x^2(x,t)\,\mathrm{d}x\right) w_{xx}(x,t) + \delta w_{xxxxt}(x,t) \\ - \xi\left(\int_0^l w_x(x,t) w_{xt}(x,t)\,\mathrm{d}x\right) w_{xx}(x,t) + \eta w_t(x,t) = 0, \qquad 0 < x < l,\ 0 < t \leq T, \end{aligned} \tag{8.3}$$

where the constants α, γ, δ and ξ are positive, and where the constants β and η are unrestricted in sign.

To solve equations (8.2) and (8.3) Peradze [19] and Papukashvili *et al.* [17] used the three-step methods. Namely, the Galerkin method and implicit difference schemes are used for discretization of this equations with respect to the spatial and time variables. And to solve the resulting systems of discrete equations, in one case, the fixed-point iteration method, and in the other Jacobi type iteration method with Cardano formula are used. In the given case, the similarity noted above allows us to solve the hyperbolic analog of equation (1.7) by the methods, which were used in the papers, see [17, 19], for the solution of equations (8.2) and (8.3). Separately, the question then is raised about the accuracy of the methods.

Data availability statement

The code used in this paper is available online in a Github repository: https://github.com/nika3966/timoshenko_shell [9].

References

[1] J. Ball, Stability theory for an extensible beam. *J. Diff. Equ.* **14** (1973) 399–418.

[2] S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles. *Bull. Acad. Sci. URSS. Sér. Math. [Izv. Akad. Nauk SSSR Ser. Mat.]* **14** 1 (1940) 17–26.

[3] D. Chapelle and K.J. Bathe, The Finite Element Analysis of Shells – Fundamentals, 2nd edition. Springer-Verlag, Berlin, Heidelberg (2011).

[4] M. Chipot, Elements of Nonlinear Analysis. Birkhäuser Verlag, Basel (2000).

[5] M. Chipot, Remarks on some class of non-local elliptic problems, in Recent Advances of Elliptic and Parabolic Issues. World Scientific (2006) 79–102.

[6] H.B. Dwight, Tables of Integrals and Other Mathematical Data, 4th edition. The Macmillan Co., New York (1961).

[7] B. Feng and A. Soufyane, Memory-type boundary control of a laminated Timoshenko beam. *Math. Mech. Solids* **25** (2020) 1568–1588.

[8] N. Jacobson, Basic Algebra. I, 2nd edition. W.H. Freeman and Company, New York (1985).

[9] N. Kachakhidze, Julia code for "The solution method and its accuracy for a nonlinear Timoshenko axisymmetric shell model" (2024). https://github.com/nika3966/timoshenko_shell.

[10] N. Kachakhidze, N. Khomeriki, I. Peradze and Z. Tsiklauri, Chipot's method for Kirchhoff's one-dimensional static equation. *Numer. Algor.* **73** (2016) 1091–1106.

[11] L.S. Kharasova, Solvability of one problem for differential equations of shell theory of Timoshenko type. *Int. J. Eng. Res. Tech. (IJET)* **13** (2020) 5328–5334.

[12] G. Kirchhoff, Vorlesungen über Mathematische Physik: Mechanik. Teubner, Leipzig (1876).

[13] J.E. Lagnese and J.-L. Lions, Modelling Analysis and Control of Thin Plates. Springer-Verlag, Berlin, Heidelberg (1990).

[14] L.P. Lebedev, I.I. Vorovich and M.J. Cloud, Functional Analysis in Mechanics. Springer, New York, NY (2013).

[15] S.G. Mikhlin, The Numerical Performance of Variational Methods. Wolters-Noordhoff Publishing, Groningen (1971).

[16] J.M. Ortega and W.C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables. Reprint of the 1970 original, *Classics in Applied Mathematics*. Vol. 30. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2000).
[17] G. Papukashvili, I. Peradze and Z. Tsiklauri, On a stage of a numerical algorithm for Timoshenko type nonlinear equation. *Proc. A. Razmadze Math. Inst.* **158** (2012) 67–77.
[18] J. Peradze, Densifying operator for one system of equations and its solvability. *Proc. Tbilisi State Univ.* **300** (1990) 88–100.
[19] J. Peradze, An approximate algorithm for a Kirchhoff wave equation. *SIAM J. Numer. Anal.* **47** (2009) 2243–2268.
[20] J. Peradze, A numerical algorithm for a Kirchhoff-type nonlinear static beam. *J. Appl. Math.* **2009** (2009). DOI: 10.1155/2009/818269.
[21] J. Peradze, On the convergence of an iteration method in Timoshenko's theory of plates, in Shell-like Structures Nonclassical Theories and Applications. Springer-Verlag, Berlin, Heidelberg (2011) 83–90.
[22] J. Peradze, On an iteration method of finding a solution of a nonlinear equilibrium problem for Timoshenko's plate. *Z. Angew. Math. Mech.* **91** (2011) 993–1001.
[23] J. Peradze and V. Odisharia, A numerical algorithm for a one-dimensional nonlinear Timoshenko system. *Int. J. Appl. Math. Inf.* **2** (2008) 67–75.
[24] S.N. Timergaliev, Method of integral equations for studying the solvability of boundary value problems for the system of nonlinear differential equations of the theory of Timoshenko type shallow inhomogeneous shells. *Part. Diff. Equ.* **55** (2019) 243–259.
[25] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibration of prismatic bar. *Philos. Mag.* **41** (1921) 744–746.
[26] S.P. Timoshenko and S. Woinowsky-Kriger, Theory of Plates and Shells, 2nd edition. McGraw-Hill Book Company, New York (1969).
[27] E. Ventsel and T. Krauthammer, Thin Plates and Shells. Marcel Dekker, Inc., New York (2001).
[28] V.Z. Vlasov, General Theory of Shells and its Application in Engineering. NASA TTF-99, Washington (1964).
[29] A.S. Volmir, The Nonlinear Dynamics of Plates and Shells. Foreign Technology Division, Wright-Patterson AFB, Dayton (1974).
[30] A.S. Volmir, The Shells on the Flow of Fluid and Gas, Problems of Hydroelasticity. Nauka, Moscow (1979).
[31] I.I. Vorovich, On the existence of solutions in the nonlinear theory of shells. *Izv. Akad. Nauk SSSR. Ser. Mat.* **19** (1955) 173–186.
[32] I.I. Vorovich, On some direct methods in the nonlinear theory of vibrations of curved shells. *Izv. Akad. Nauk SSSR. Ser. Mat.* **21** (1957) 747–784.
[33] I.I. Vorovich, Nonlinear Theory of Shallow Shells. *Applied Mathematical Sciences*. Vol. 133. Springer-Verlag, New York (1999).
[34] S. Woinowsky-Krieger, The effect of an axial force on the vibration of hinged bars. *J. Appl. Mech.* **17** (1950) 35–36.
[35] A.I. Zemlyanukhin, A.V. Bochkarev, L.I. Mogilevich and E.G. Tindova, Axisymmetric longitudinal-bending waves in a cylindrical shell interacting with a non-linear elastic medium. *Model. Simul. Eng.* **2016** (2016). DOI: 10.1155/2016/6596231.