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THE SEMI-IMPLICIT EULER–MARUYAMA METHOD FOR NONLINEAR NON-AUTONOMOUS STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY A CLASS OF LÉVY PROCESSES

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Abstract. The strong convergence of the semi-implicit Euler–Maruyama (EM) method for stochastic differential equations with nonlinear coefficients driven by a class of Lévy processes is investigated. The dependence of the convergence order of the numerical scheme on the parameters of the class of Lévy processes is discovered, which is different from existing results. In addition, the existence and uniqueness of the numerical invariant measure for the semi-implicit EM method is studied, and its convergence to the underlying invariant measure is also proved. Numerical examples are provided to confirm our theoretical results.

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1. Introduction

Lévy processes and stochastic differential equations (SDEs) driven by Lévy processes have been widely employed to model uncertain phenomena in various areas [1, 5, 9, 10, 26]. Since different classes of Lévy processes may have distinct properties [2, 32], much attention should be paid for characteristics of different types of Lévy processes, when analyzing numerical methods for SDEs driven by Lévy processes is done.

Numerical methods for SDEs driven different classes of Lévy processes were proposed and studied. Higham and Kloeden [13] studied the semi-implicit Euler–Maruyama (EM) method for solving SDEs with Poisson-driven jumps and obtained its optimal strong convergence order. More recently, the tamed Euler approach [11], the tamed Milstein method [22], general one-step methods [7], and the truncated EM method [36] were proposed for solving SDEs driven by Lévy processes with super-linearly growing coefficients. All the works mentioned above focused on handling the super-linearity of the coefficients but not cared about the effects of the types of Lévy processes on the convergence order.

On the contrary, different types of stable processes do lead to different convergence orders. Pamen and Taguchi [31] investigated the strong convergence order of the EM method for SDEs with Hölder continuous coefficients in both temporal and spatial variables and driven by a truncated symmetric α-stable process.

Keywords and phrases. Lévy process, stochastic differential equation, semi-implicit Euler–Maruyama method, strong convergence, invariant measure.

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Mikulevicius and Xu [30] considered the convergence order of the EM approximation for SDEs driven by an α-stable process, where the drift coefficient is a bounded function of β-Hölder continuity and the diffusion coefficient is a Lipschitz continuous bounded matrix. Huang and Liao [14] and Huang and Yang [15] extended the EM method to stochastic functional differential equations and distribution-dependent SDEs with Hölder drift and α-stable process, respectively. Kühn and Schilling [21] established strong convergence of the EM method for a class of Lévy process-driven SDEs. Although those papers [14, 15, 21, 30, 31] discovered the dependence of the convergence orders on types of stable processes, they focused on the classical EM method, which ruled out the cases of super-linear coefficients [16].

To our best knowledge, few existing works focused on both the super-linearity of the coefficients and the effects of the parameters of the class of Lévy processes on the convergence order.

In this paper, we focus on the class of Lévy processes with the triplet $(0, 0, \nu)$ satisfying $\int_{|z|<1} |z|^{\gamma_0}\nu(\mathrm{d}z) < \infty$ with $\gamma_0 \in [1, 2]$ and $\int_{|z|\geq 1} |z|^{\gamma_\infty}\nu(\mathrm{d}z) < \infty$ with $\gamma_\infty > 1$ (see Sect. 2 for more details). And we derive the strong convergence order of the semi-implicit EM method that is dependent on γ_0. Such setting on the triplet of the Lévy processes covers many interesting stable processes, such as the Lamperti stable process [6] and the tempered stable process [19]. It is worth noting that the class of Lévy processes we are referring to here does not include α-stable. The fundamental reason why this class of Lévy process does not include α-stable is that it does not have second moments, and in this case, only small pth moments ($p \in (0, \alpha)$) can be considered. Compared with existing results [21, 31], the SDEs studied in this work are allowed to have super-linear coefficients. Therefore, the EM method used in [21, 31] may fail to convergence, due to the combined effects of the unbounded noise and super-linear terms in the coefficients [16].

The invariant measure of SDEs with different classes of Lévy processes is also essential for many stochastic models [4,34,35]. However, the explicit expression of the invariant measure is rarely obtained. Therefore, deriving the numerical invariant measure that is convergent to the underlying one is crucial in the applications of those models. In this paper, the existence and uniqueness of the numerical invariant measure of the semi-implicit EM method is proved and the convergence of the numerical invariant measure to the underlying one is verified. We should mention that there are many existing works on the numerical invariant measure of SDEs driven by Brownian motion [3, 17, 23–25, 33], but few works are devoted to the case of the Lévy processes.

The main contributions of this paper are as follows.

- The strong convergence in the finite time of the semi-implicit EM method for SDEs driven by a class of Lévy process is established and the dependence of the convergence order on the Hölder continuity and the Lévy process is revealed.
- The existence and uniqueness of the numerical invariant measure of the semi-implicit EM method is proved. Moreover, the convergence of the numerical invariant measure to the underlying one is obtained.

This paper is organized as follows. In Section 2, some preliminaries are briefly introduced. In Section 3, we establish the finite time strong convergence of the semi-implicit EM method. The convergence of the numerical invariant measure is studied in Section 4. Numerical examples are given in Section 5 to illustrate our theoretical results and some concluding remarks are included in Section 6.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.*, it is right continuous and increasing in t while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d$. Let $B(t)$ be the m-dimensional Brownian motion defined on the probability space.

A stochastic process $L = \{L(t), t \geq 0\}$ is called a m-dimensional Lévy process if

- $L(0) = 0$ (a.s.);
- For any integer $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n \leq T$, the random variables $L(t_0), L(t_1) - L(t_0), \ldots, L(t_n) - L(t_{n-1})$ have independent and stationary increments;

– L is stochastically continuous, *i.e.* for all $\epsilon > 0$ and all $s \geq 0$,

$$\lim_{t\to s} \mathbb{P}(|L(t) - L(s)| > \epsilon) = 0.$$

The Lévy–Khintchine formula [2] for the Lévy process L is

$$\begin{aligned}\varphi_L(\theta)(t) &:= \mathbb{E}\Big[e^{i\theta^{\mathrm{T}} L(t)}\Big]\\ &= \exp\left(t\left(ib^{\mathrm{T}}\theta - \frac{1}{2}\theta^{\mathrm{T}} A\theta + \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\theta^{\mathrm{T}} z} - 1 - i\theta^{\mathrm{T}} z \mathbb{1}_{\{|z|<1\}}\right)\nu(\mathrm{d}z)\right)\right),\end{aligned}$$

where $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ is a positive definite symmetric matrix and the Lévy measure ν is a σ-finite measure such that $\int_{\mathbb{R}^d\setminus\{0\}} \min(1, z^2)\nu(\mathrm{d}z) < \infty$. The Lévy triplet (b, A, ν) is the characteristics of the infinitely divisible random variable $L(1)$, which has a one-to-one correspondence with the Lévy–Khintchine formula.

In this paper, we consider a SDE driven by the multiplicative Brownian motion and the additive Lévy process of the form

$$y(t) = y(0) + \int_0^t f(s, y(s))\,\mathrm{d}s + \int_0^t g(s, y(s))\,\mathrm{d}B(s) + \int_0^t \mathrm{d}L(s), \quad t \in (0, T], \tag{2.1}$$

where $f : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}_+\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $\mathbb{E}|y(0)|^p < +\infty$ for some $2 \leq p < +\infty$. For simplicity, the Lévy triplet is assumed to be $(0, 0, \nu)$ and satisfies $\int_{|z|<1} |z|^{\gamma_0}\nu(\mathrm{d}z) < \infty$ with $\gamma_0 \in [1, 2]$ and $\int_{|z|\geq 1} |z|^{\gamma_\infty}\nu(\mathrm{d}z) < \infty$ with $\gamma_\infty > 1$ (see [21] for more details). In addition, we need some hypotheses on the drift and diffusion coefficients.

Assumption 2.1. *There exist constants $H > 0$ and $\sigma > 0$ such that*

$$|f(t, x) - f(t, y)|^2 \vee |g(t, x) - g(t, y)|^2 \leq H(1 + |x|^\sigma + |y|^\sigma)|x - y|^2$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$.

It can be observed from Assumption 2.1 that for all $t \in [0, T]$ and $x \in \mathbb{R}^d$

$$|f(t, x)|^2 \vee |g(t, y)|^2 \leq \widetilde{H}\big(1 + |x|^{\sigma+2}\big), \tag{2.2}$$

where $\widetilde{H}$ depends on H and $\sup_{0\leq t\leq T}\big(|f(t, 0)|^2 + |g(t, 0)|^2\big)$.

Assumption 2.2. *There exist constants $p_0 \geq 1$ and $K_3 \in \mathbb{R}$ such that*

$$(x - y)^{\mathrm{T}}(f(t, x) - f(t, y)) + p_0|g(t, x) - g(t, y)|^2 \leq K_3|x - y|^2$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$.

Assumption 2.3. *There exist constants $p_1 \geq \left(\frac{1}{2}\sigma + 1\right)^2$, $m > 0$ and $M \in \mathbb{R}$ such that*

$$x^{\mathrm{T}} f(t, x) + p_1|g(t, x)|^2 \leq M|x|^2 + m$$

for all $x \in \mathbb{R}^d$ and $t \in [0, T]$, where σ is given in Assumption 2.1.

Assumption 2.4. *There exist constants $K_1 > 0$, $K_2 > 0$, $\gamma_1 \in (0, 1)$ and $\gamma_2 \in (0, 1)$ such that*

$$|f(s, x) - f(t, x)| \leq K_1\big(1 + |x|^{\sigma+1}\big)|t - s|^{\gamma_1}$$

and

$$|g(s, x) - g(t, x)| \leq K_2\big(1 + |x|^{\sigma+1}\big)|t - s|^{\gamma_2}$$

for all $x \in \mathbb{R}^d$ and $s, t \in [0, T]$, where σ is given in Assumption 2.1.

The existence and uniqueness of the solution $y(t)$ to SDE (2.1) is guaranteed [7, 12] under Assumptions 2.2 and 2.3.

3. Main results on the finite time strong convergence

Given $\Delta t \in (0,1)$ and time T, let $\mathbf{N} = \lfloor T/\Delta t \rfloor$ and $t_i = i\Delta t, i = 0,1,\dots,\mathbf{N}$. The semi-implicit EM method of SDE (2.1) is defined by

$$Y_{i+1} = Y_i + f(t_{i+1}, Y_{i+1})\Delta t + g(t_i, Y_i)\Delta B_{i+1} + \Delta L_{i+1}, \qquad i = 0,1,2,\dots,\mathbf{N}, \tag{3.1}$$

where Y_i approximates $y(t_i)$, $\Delta B_{i+1} = B(t_{i+1}) - B(t_i)$, and $\Delta L_{i+1} = L(t_{i+1}) - L(t_i)$.

The continuous-time semi-implicit EM solution of SDE (2.1) is constructed in the following manner

$$Y(t) = Y_i, \quad t \in [t_i, t_{i+1}), \quad i = 0,1,2,\dots,\mathbf{N}. \tag{3.2}$$

The implementation of (3.1) requires solving a nonlinear equation at each iteration. The well-posedness of the difference equation (3.1) is proved in the next lemma. The following lemma is very classic, as seen in [25, 28]. For the completeness of the article, we present it here.

Lemma 3.1. *Let Assumption 2.2 holds. If $K_3 < 0$, the semi-implicit EM method is well defined. If $K_3 > 0$, the method is well defined when $\Delta t < \frac{1}{K_3}$.*

Proof. For any $i \in \mathbb{N}$, we have

$$Y_{i+1} - f(t_{i+1}, Y_{i+1})\Delta t = Y_i + g(t_i, Y_i)\Delta B_{i+1} + \Delta L_{i+1}.$$

Define $F(u) = u - f(t,u)\Delta t$ for $u \in \mathbb{R}^d$. By Assumption 2.2, it is straightforward to see

$$(u_1 - u_2)^{\mathrm{T}}(F(u_1) - F(u_2)) \geq (1 - K_3\Delta t)|u_1 - u_2|^2, \quad \text{for all } u_1, u_2 \in \mathbb{R}^d,\ u_1 \neq u_2.$$

When $K_3 < 0$, $1 - K_3\Delta t > 0$ holds for all Δt, which means that $F(\cdot)$ is monotonic. When $K_3 > 0$, we need to assume $\Delta t < \frac{1}{K_3}$ to ensure that $1 - K_3\Delta t > 0$ holds. Therefore, $F(\cdot)$ has its inverse function $F^{-1}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ such that for any $i \in \mathbb{N}$,

$$Y_{i+1} = F^{-1}(Y_i + g(t_i, Y_i)\Delta B_{i+1} + \Delta L_{i+1}).$$

That is to say, for any $i \in \mathbb{N}$ the unique Y_{i+1} can always be found for the given $Y_i + g(t_i, Y_i)\Delta B_{i+1} + \Delta L_{i+1}$, which completes the proof. □

Our first main result is on the finite time strong convergence.

Theorem 3.2. *Suppose that Assumptions 2.1 and 2.4 hold, and let Assumption 2.2 holds for some $K_3 > 0$. For any $\Delta t \in (0, 1 \wedge \frac{1}{3K_3})$, then the semi-implicit EM method (3.1) and (3.2) for solving (2.1) is convergent and satisfies*

$$\mathbb{E}|y(t) - Y(t)|^2 \leq C\big(\Delta t^{2\gamma_1} + \Delta t^{2\gamma_2} + \Delta t\big), \quad \forall\ t \in [0,T], \tag{3.3}$$

where the positive constant C is independent of Δt.

The proof of Theorem 3.2 will be given in Section 3.2.

Now we look at a special interesting case that $g(\cdot,\cdot) \equiv 0$, *i.e.* the Brownian motion is removed

$$u(t) = u(0) + \int_0^t f(s, u(s))\,\mathrm{d}s + \int_0^t \mathrm{d}L(s). \tag{3.4}$$

Then the semi-implicit EM method becomes

$$U_{i+1} = U_i + f(t_{i+1}, U_{i+1})\Delta t + \Delta L_{i+1} \tag{3.5}$$

and its continuous-time version is

$$U(t) = U_i, \quad t \in [t_i, t_{i+1}), \quad i = 0,1,2,\dots,\mathbf{N}. \tag{3.6}$$

The second main result of this paper, which we think is more interesting, states that the convergence order depends on the parameter γ_0 of the Lévy process and the Hölder index γ_1.

Corollary 3.3. *Suppose that Assumptions 2.1 and 2.4 hold, and let Assumption 2.2 hold for some $K_3 > 0$. For any $\Delta t \in (0, 1 \wedge \frac{1}{K_3})$, then the semi-implicit EM method (3.5) and (3.6) for solving (3.4) is convergent and satisfies*

$$\mathbb{E}|u(t) - U(t)|^2 \leq C\Big(\Delta t^{2\gamma_1} + \Delta t^{\frac{2}{\gamma_0}}\Big), \quad \forall\, t \in [0, T], \tag{3.7}$$

where the positive constant C is independent of Δt.

When the Brownian motion term is removed ($i.e.$, $g(\cdot,\cdot) \equiv 0$), the main modification of the proof is that the exponent η in Lemma 3.6 will be taken as $\eta = \frac{2}{\gamma_0}$ in the proof of Theorem 3.2.

Remark 3.4. The above conclusions prove that the convergence order under non–global Lipschitz conditions is related to the Hölder continuity index and the parameters of the calss of Lévy processes. It is worth noting that when the noise is a specific stable process, including Lamperti stable, tempered stable, isotropic stable, and relativistic stable, the conclusions are applicable.

3.1. Some necessary lemmas

In this section, we show the boundedness of the pth moment and the continuity of the solution.

Lemma 3.5. *Suppose that Assumptions 2.1 and 2.3 hold with $M > 0$. Then, for fixed $p \in [2, (2p_1 + 1) \wedge \gamma_\infty)$,*

$$\mathbb{E}|y(t)|^p \leq C_p, \quad \forall\, t \in [0, T],$$

where $C_p := C(p, T, M, \mathbb{E}|y(0)|^p, \int_{|z|<1} |z|^2 \nu(\mathrm{d}z), \int_{|z|\geq 1} |z|^p \nu(\mathrm{d}z))$.

Proof. Let N be the Poisson measure on $[0,\infty) \times (\mathbb{R}\backslash\{0\})$ with $\mathbb{E}N(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t\nu(\mathrm{d}z)$. Define the compensated Poisson random measure $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\,\mathrm{d}t$. From the Itô formula [2], it follows

$$\begin{aligned}
|y(t)|^p &= |y(0)|^p + \int_0^t p|y(s)|^{p-2} y^{\mathrm{T}}(s) f(s, y(s))\,\mathrm{d}s + \frac{p(p-2)}{2}\int_0^t |y(s)|^{p-4} |g(s, y(s)) y(s)|^2\,\mathrm{d}s \\
&\quad + \frac{p}{2}\int_0^t |y(s)|^{p-2} |g(s, y(s))|^2\,\mathrm{d}s + \int_0^t p|y(s)|^{p-2} y^{\mathrm{T}}(s) g(s, y(s))\,\mathrm{d}B(s) \\
&\quad + \int_0^t \int_{|z|<1} (|y(s) + z|^p - |y(s)|^p)\widetilde{N}(\mathrm{d}z,\,\mathrm{d}s) + \int_0^t \int_{|z|\geq 1} (|y(s) + z|^p - |y(s)|^p) N(\mathrm{d}z,\,\mathrm{d}s) \\
&\quad + \int_0^t \int_{\mathbb{R}^d\backslash\{0\}} (|y(s) + z|^p - |y(s)|^p - p|y(s)|^{p-2} y^{\mathrm{T}}(s) z \mathbb{1}_{(0,1)}(|z|))\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leq |y(0)|^p + \int_0^t p|y(s)|^{p-2} y^{\mathrm{T}}(s) f(s, y(s)) + \frac{1}{2} p(p-1)|y(s)|^{p-2} |g(s, y(s))|^2\,\mathrm{d}s \\
&\quad + \int_0^t p|y(s)|^{p-2} y^{\mathrm{T}}(s) g(s, y(s))\,\mathrm{d}B(s) + \int_0^t \int_{|z|<1} (|y(s) + z|^p - |y(s)|^p)\widetilde{N}(\mathrm{d}z,\,\mathrm{d}s) \\
&\quad + \int_0^t \int_{|z|\geq 1} (|y(s) + z|^p - |y(s)|^p)\widetilde{N}(\mathrm{d}z,\,\mathrm{d}s) + 2\int_0^t \int_{|z|\geq 1} (|y(s) + z|^p - |y(s)|^p)\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\quad + \int_0^t \int_{|z|<1} \big(|y(s) + z|^p - |y(s)|^p - p|y(s)|^{p-2} y^{\mathrm{T}}(s) z\big)\nu(\mathrm{d}z)\,\mathrm{d}s.
\end{aligned}$$

Taking expectations on both sides, we obtain

$$\mathbb{E}|y(t)|^p \leq \mathbb{E}|y(0)|^p + p\mathbb{E}\int_0^t |y(s)|^{p-2}\left[y^{\mathrm{T}}(s) f(s, y(s)) + \frac{p-1}{2}|g(s, y(s))|^2\right]\mathrm{d}s$$

$$
\begin{aligned}
&+ 2\mathbb{E}\int_0^t \int_{|z|\geq 1} (|y(s)+z|^p - |y(s)|^p)\nu(\mathrm{d}z)\,\mathrm{d}s \\
&+ \mathbb{E}\int_0^t \int_{|z|<1} \big(|y(s)+z|^p - |y(s)|^p - p|y(s)|^{p-2}y^{\mathrm{T}}(s)z\big)\nu(\mathrm{d}z)\,\mathrm{d}s \\
=:\ & \mathbb{E}|y(0)|^p + I_1 + I_2 + I_3.
\end{aligned}
$$

It follows from Assumption 2.3 that

$$
\begin{aligned}
I_1 &= p\mathbb{E}\int_0^t |y(s)|^{p-2}\left[y^{\mathrm{T}}(s)f(s,y(s)) + \frac{p-1}{2}|g(s,y(s))|^2\right]\mathrm{d}s \\
&\leq p\mathbb{E}\int_0^t |y(s)|^{p-2}(M|y(s)|^2 + m)\,\mathrm{d}s \\
&\leq pmT + p(M+m)\mathbb{E}\int_0^t |y(s)|^p\,\mathrm{d}s.
\end{aligned}
$$

In order to estimate I_2, we construct a function $F(\tau) = |y(s)+\tau z|^p$ for $\tau \geq 0$. By the mean value theorem, there exists a constant $\xi \in (0,1)$ such that

$$
F(1) - F(0) = |y(s)+z|^p - |y|^p = \int_0^1 p|y(s)+\xi z|^{p-2}(y(s)+\xi z)^{\mathrm{T}} z\,\mathrm{d}\xi.
$$

It follows from the elementary inequality and the Young inequality,

$$
\begin{aligned}
|y(s)+z|^p - |y(s)|^p &\leq \int_0^1 p2^{p-2}|y(s)|^{p-2}(y(s)+\xi z)^{\mathrm{T}} z\,\mathrm{d}\xi + \int_0^1 p2^{p-2}\xi^{p-2}|z|^{p-2}(y(s)+\xi z)^{\mathrm{T}} z\,\mathrm{d}\xi \\
&\leq \int_0^1 p2^{p-2}|y(s)|^{p-2}y^{\mathrm{T}}(s)z\,\mathrm{d}\xi + \int_0^1 p2^{p-2}|y(s)|^{p-2}\xi|z|^2\,\mathrm{d}\xi \\
&\quad + \int_0^1 p2^{p-2}\xi^{p-2}|z|^{p-2}y^{\mathrm{T}}(s)z\,\mathrm{d}\xi + \int_0^1 p2^{p-2}\xi^{p-1}|z|^p\,\mathrm{d}\xi \\
&= p2^{p-2}|y(s)|^{p-2}y^{\mathrm{T}}(s)z + p2^{p-3}|y(s)|^{p-2}|z|^2 + \frac{p2^{p-2}}{p-1}|z|^{p-2}y^{\mathrm{T}}(s)z + 2^{p-2}|z|^p \\
&\leq \left(p2^{p-3} + (p-2)2^{p-2} + \frac{2^{p-2}}{p-1}\right)|y(s)|^p + 2^p|z|^p.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
I_2 &= 2\mathbb{E}\int_0^t \int_{|z|\geq 1} (|y(s)+z|^p - |y(s)|^p)\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\leq C_p + C_p\mathbb{E}\int_0^t |y(s)|^p\,\mathrm{d}s.
\end{aligned}
$$

Note that for any $y_1, y_2 \in \mathbb{R}^d$ [11],

$$
\begin{aligned}
|y_1+y_2|^p - |y_1|^p - p|y_1|^{p-2}y_1^{\mathrm{T}}y_2 &\leq C_p\int_0^1 |y_1+\theta y_2|^{p-2}|y_2|^2 d\theta \\
&\leq C_p\big(|y_1|^{p-2}|y_2|^2 + |y_2|^p\big).
\end{aligned}
$$

So,

$$
I_3 = \mathbb{E}\int_0^t \int_{|z|<1} \big(|y(s)+z|^p - |y(s)|^p - p|y(s)|^{p-2}y^{\mathrm{T}}(s)z\big)\nu(\mathrm{d}z)\,\mathrm{d}s
$$

$$
\begin{aligned}
&\le C_p \mathbb{E} \int_0^t \int_{|z|<1} \big(|y(s)|^{p-2}|z|^2 + |z|^p\big)\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\le C_p \mathbb{E} \int_0^t \int_{|z|<1} \big[2|z|^2 + |z|^2|y(s)|^p\big]\nu(\mathrm{d}z)\,\mathrm{d}s \\
&\le C_p + C_p \mathbb{E} \int_0^t |y(s)|^p \,\mathrm{d}s.
\end{aligned}
$$

Combining the estimates of I_1, I_2 and I_3, we arrive at

$$
\mathbb{E}|y(t)|^p \le C_p + C_p \mathbb{E} \int_0^t |y(s)|^p \,\mathrm{d}s.
$$

The desired result follows from the Gronwall inequality. $\square$

Lemma 3.6. *Suppose that Assumption 2.1 holds. Then, for any $2 \le \kappa < 2p/(\sigma+2)$, $0 \le s < t \le T$ and $|t-s|<1$, we have*

$$
\mathbb{E}|y(t)-y(s)|^\kappa \le C|t-s|^\eta, \tag{3.8}
$$

where the constant C only depends on $\widetilde{H}$, C_p, M_2, m_2 and κ, and $\eta = \begin{cases} \frac{\kappa}{\gamma_0}, & g(\cdot,\cdot)\equiv 0, \\ \frac{\kappa}{2}, & g(\cdot,\cdot)\neq 0. \end{cases}$

Proof. For any $0 \le s < t \le T$, the solution y satisfies

$$
y(t)-y(s) = \int_s^t f(r,y(r))\,\mathrm{d}r + \int_s^t g(r,y(r))\,\mathrm{d}B(r) + (L(t)-L(s)).
$$

By the Hölder inequality, the Burkholder–Davis–Gundy (BDG) inequality [27], the fractional moment estimate for the Lévy process [20], and the fact that

$$
L(t)-L(s) \overset{d}{=} L(t-s),
$$

we conclude that

$$
\begin{aligned}
\mathbb{E}|y(t)-y(s)|^\kappa &= \mathbb{E}\bigg|\int_s^t f(r,y(r))\,\mathrm{d}r + \int_s^t g(r,y(r))\,\mathrm{d}B(r) + (L(t)-L(s))\bigg|^\kappa \\
&\le 3^{\kappa-1}\bigg(\mathbb{E}\bigg|\int_s^t f(r,y(r))\,\mathrm{d}r\bigg|^\kappa + \mathbb{E}\bigg|\int_s^t g(r,y(r))\,\mathrm{d}B(r)\bigg|^\kappa + \mathbb{E}|L(t)-L(s)|^\kappa\bigg) \\
&\le 3^{\kappa-1}\bigg(|t-s|^{\kappa-1}\mathbb{E}\int_s^t |f(r,y(r))|^\kappa\,\mathrm{d}r + c_\kappa \mathbb{E}\bigg(\int_s^t |g(r,y(r))|^2\,\mathrm{d}r\bigg)^{\frac{\kappa}{2}} + \mathbb{E}|L(t)-L(s)|^\kappa\bigg) \\
&\le 3^{\kappa-1}\bigg(|t-s|^{\kappa-1}\mathbb{E}\int_s^t |f(r,y(r))|^\kappa\,\mathrm{d}r + c_\kappa |t-s|^{\frac{\kappa-2}{2}}\mathbb{E}\int_s^t |g(r,y(r))|^\kappa\,\mathrm{d}r + \mathbb{E}|L(t)-L(s)|^\kappa\bigg) \\
&\le 3^{\kappa-1}\bigg(|t-s|^{\kappa-1}\widetilde{H}^{\frac{\kappa}{2}}\int_s^t \mathbb{E}\big(1+|y(r)|^{\sigma+2}\big)^{\frac{\kappa}{2}}\,\mathrm{d}r + |t-s|^{\frac{\kappa-2}{2}} c_\kappa \widetilde{H}^{\frac{\kappa}{2}}\int_s^t \mathbb{E}\big(1+|y(r)|^{\sigma+2}\big)^{\frac{\kappa}{2}}\,\mathrm{d}r\bigg) \\
&\quad + 3^{\kappa-1}|t-s|^{\frac{\kappa}{\gamma_0}} \\
&\le C\Big(|t-s|^\kappa + |t-s|^{\frac{\kappa}{2}} + |t-s|^{\frac{\kappa}{\gamma_0}}\Big) \\
&\le C|t-s|^\eta,
\end{aligned}
$$

where the constant $C>0$ depends on $\widetilde{H}$, C_p, M_2, m_2 and κ. This completes the proof. $\square$

3.2. Proof of Theorem 3.2

Proof. It follows from (2.1) and (3.1) that

$$\begin{aligned}
y(t_{i+1}) &= y(t_i) + \int_{t_i}^{t_{i+1}} f(s, y(s))\,\mathrm{d}s + \int_{t_i}^{t_{i+1}} g(s, y(s))\,\mathrm{d}B(s) + \int_{t_i}^{t_{i+1}} \mathrm{d}L(s) \\
&= y(t_i) + \int_{t_i}^{t_{i+1}} f(s, y(s)) - f(t_{i+1}, y(t_{i+1}))\,\mathrm{d}s + f(t_{i+1}, y(t_{i+1}))\Delta t \\
&\quad + \int_{t_i}^{t_{i+1}} g(s, y(s)) - g(t_i, y(t_i))\,\mathrm{d}B(s) + g(t_i, y(t_i))\Delta B_{i+1} + \int_{t_i}^{t_{i+1}} \mathrm{d}L(s)
\end{aligned}$$

and

$$Y_{i+1} = Y_i + f(t_{i+1}, Y_{i+1})\Delta t + g(t_i, Y_i)\Delta B_{i+1} + \Delta L_{i+1}, \quad i = 0, 1, \ldots, \mathbf{N}.$$

Define $e_i = y(t_i) - Y_i$. So,

$$\begin{aligned}
e_{i+1} - e_i &= \int_{t_i}^{t_{i+1}} f(s, y(s)) - f(t_{i+1}, y(t_{i+1}))\,\mathrm{d}s + (f(t_{i+1}, y(t_{i+1})) - f(t_{i+1}, Y_{i+1}))\Delta t \\
&\quad + \int_{t_i}^{t_{i+1}} g(s, y(s)) - g(t_i, y(t_i))\,\mathrm{d}B(s) + (g(t_i, y(t_i)) - g(t_i, Y_i))\Delta B_{i+1}. \qquad (3.9)
\end{aligned}$$

First note that

$$|b|^2 - |a|^2 + |b - a|^2 = 2b^{\mathrm{T}}(b - a), \quad \text{for any } a, b \in \mathbb{R}^d. \qquad (3.10)$$

Then, we get

$$|e_{i+1}|^2 - |e_i|^2 + |e_{i+1} - e_i|^2 = 2e_{i+1}^{\mathrm{T}}(e_{i+1} - e_i).$$

Combining (3.9), we have

$$\begin{aligned}
2e_{i+1}^{\mathrm{T}}(e_{i+1} - e_i) &= 2e_{i+1}^{\mathrm{T}}\Bigg(\int_{t_i}^{t_{i+1}} f(s, y(s)) - f(t_{i+1}, y(t_{i+1}))\,\mathrm{d}s + (f(t_{i+1}, y(t_{i+1})) - f(t_{i+1}, Y_{i+1}))\Delta t \\
&\quad + \int_{t_i}^{t_{i+1}} g(s, y(s)) - g(t_i, y(t_i))\,\mathrm{d}B(s) + (g(t_i, y(t_i)) - g(t_i, Y_i))\Delta B_{i+1}\Bigg) \\
&= 2e_{i+1}^{\mathrm{T}}\left(\int_{t_i}^{t_{i+1}} f(s, y(s)) - f(t_{i+1}, y(t_{i+1}))\,\mathrm{d}s\right) + 2\Delta t e_{i+1}^{\mathrm{T}}(f(t_{i+1}, y(t_{i+1})) - f(t_{i+1}, Y_{i+1})) \\
&\quad + 2e_{i+1}^{\mathrm{T}}\left(\int_{t_i}^{t_{i+1}} g(s, y(s)) - g(t_i, y(t_i))\,\mathrm{d}B(s)\right) + 2e_{i+1}^{\mathrm{T}}[(g(t_i, y(t_i)) - g(t_i, Y_i))\Delta B_{i+1}] \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Note that

$$f(s, y(s)) - f(t_{i+1}, y(t_{i+1})) = f(s, y(t_{i+1})) - f(t_{i+1}, y(t_{i+1})) + f(s, y(s)) - f(s, y(t_{i+1})).$$

By the Young inequality $2ab \le \epsilon^2 a^2 + \frac{b^2}{\epsilon^2}$ and Assumptions 2.1 and 2.4, we choose $\epsilon^2 = K_3\Delta t$ such that

$$\begin{aligned}
J_1 &= 2e_{i+1}^{\mathrm{T}}\left(\int_{t_i}^{t_{i+1}} f(s, y(s)) - f(t_{i+1}, y(t_{i+1}))\,\mathrm{d}s\right) \\
&\le \epsilon^2|e_{i+1}|^2 + \frac{\Delta t}{\epsilon^2}\int_{t_i}^{t_{i+1}} |f(s, y(s)) - f(t_{i+1}, y(t_{i+1}))|^2\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le \epsilon^2|e_{i+1}|^2 + \frac{2\Delta t}{\epsilon^2}\int_{t_i}^{t_{i+1}} |f(s,y(t_{i+1})) - f(t_{i+1},y(t_{i+1}))|^2\,\mathrm{d}s \\
&\quad + \frac{2\Delta t}{\epsilon^2}\int_{t_i}^{t_{i+1}} |f(s,y(s)) - f(s,y(t_{i+1}))|^2\,\mathrm{d}s \\
&\le \epsilon^2|e_{i+1}|^2 + \frac{2\Delta t}{\epsilon^2}\int_{t_i}^{t_{i+1}} 2K_1^2(1+|y(t_{i+1})|^{2\sigma+2})|t_{i+1}-s|^{2\gamma_1}\,\mathrm{d}s \\
&\quad + \frac{2\Delta t}{\epsilon^2}\int_{t_i}^{t_{i+1}} H(1+|y(s)|^\sigma + |y(t_{i+1})|^\sigma)|y(s)-y(t_{i+1})|^2\,\mathrm{d}s.
\end{aligned}
$$

Applying the Hölder inequality and Lemmas 3.5 and 3.6 (for the case $g(\cdot,\cdot)\neq 0$), we have

$$
\begin{aligned}
&\mathbb{E}\big((1+|y(s)|^\sigma + |y(t_{i+1})|^\sigma)|y(s)-y(t_{i+1})|^2\big) \\
&\quad\le 3^{\frac{2}{\sigma+2}}\big(\mathbb{E}\big(1+|y(s)|^{\sigma+2}+|y(t_{i+1})|^{\sigma+2}\big)\big)^{\frac{\sigma}{\sigma+2}}\big(\mathbb{E}|y(s)-y(t_{i+1})|^{\sigma+2}\big)^{\frac{2}{\sigma+2}} \\
&\quad\le 3^{\frac{2}{\sigma+2}}\big(1+\mathbb{E}|y(s)|^{\sigma+2}+\mathbb{E}|y(t_{i+1})|^{\sigma+2}\big)^{\frac{\sigma}{\sigma+2}}\big(\mathbb{E}|y(s)-y(t_{i+1})|^{\sigma+2}\big)^{\frac{2}{\sigma+2}} \\
&\quad\le 2\times 3^{\frac{2}{\sigma+2}}(1+C_p)^{\frac{\sigma}{\sigma+2}}\Big(C|t_{i+1}-s|^{\frac{\sigma+2}{2}}\Big)^{\frac{2}{\sigma+2}} \\
&\quad\le \bar{C}|t_{i+1}-s|.
\end{aligned} \tag{3.11}
$$

From the above estimates, it yields

$$
\begin{aligned}
\mathbb{E}[J_1] &\le \epsilon^2\mathbb{E}|e_{i+1}|^2 + \frac{2\Delta t^{2\gamma_1+2}}{\epsilon^2}2K_1^2\Big(1+\mathbb{E}|y(t_{i+1})|^{2\sigma+2}\Big) + \frac{2\bar{C}H}{\epsilon^2}\Delta t^3 \\
&\le \epsilon^2\mathbb{E}|e_{i+1}|^2 + \frac{4K_1^2(1+C_p)}{\epsilon^2}\Delta t^{2\gamma_1+2} + \frac{2\bar{C}H}{\epsilon^2}\Delta t^3.
\end{aligned} \tag{3.12}
$$

Due to Assumption 2.2, we have

$$
\begin{aligned}
J_2 &= 2\Delta t e_{i+1}^{\mathrm{T}}(f(t_{i+1},y(t_{i+1})) - f(t_{i+1},Y_{i+1})) \\
&\le 2K_3\Delta t|e_{i+1}|^2 - 2p_0\Delta t|g(t_{i+1},y(t_{i+1})) - g(t_{i+1},Y_{i+1})|^2.
\end{aligned}
$$

This leads to

$$
\mathbb{E}[J_2] \le 2K_3\Delta t\mathbb{E}|e_{i+1}|^2 - 2p_0\Delta t\mathbb{E}|g(t_{i+1},y(t_{i+1})) - g(t_{i+1},Y_{i+1})|^2. \tag{3.13}
$$

The term J_3 can be estimated as

$$
\begin{aligned}
J_3 &= 2e_{i+1}^{\mathrm{T}}\bigg(\int_{t_i}^{t_{i+1}} g(s,y(s)) - g(t_i,y(t_i))\,\mathrm{d}B(s)\bigg) \\
&= 2(e_{i+1}-e_i)^{\mathrm{T}}\bigg(\int_{t_i}^{t_{i+1}} g(s,y(s)) - g(t_i,y(t_i))\,\mathrm{d}B(s)\bigg) + 2e_i^{\mathrm{T}}\bigg(\int_{t_i}^{t_{i+1}} g(s,y(s)) - g(t_i,y(t_i))\,\mathrm{d}B(s)\bigg).
\end{aligned}
$$

For $t\in[t_i,t_{i+1}]$, the term $\int_{t_i}^{t_{i+1}} e_i^{\mathrm{T}}(g(s,y(s)) - g(t_i,y(t_i)))\,\mathrm{d}B(s)$ is a martingale [25]. Utilizing the martingale property, the isometric property of stochastic integral, and Young's inequality, we obtain

$$
\mathbb{E}[J_3] \le \frac{1}{2}\mathbb{E}|e_{i+1}-e_i|^2 + 2\int_{t_i}^{t_{i+1}} \mathbb{E}|g(s,y(s)) - g(t_i,y(t_i))|^2\,\mathrm{d}s.
$$

By Assumptions 2.1 and 2.4, we can easily get

$$
|g(s,y(s)) - g(t_i,y(t_i))|^2 = |g(s,y(s)) - g(s,y(t_i)) + g(s,y(t_i)) - g(t_i,y(t_i))|^2
$$

$$\leq 2|g(s,y(s))-g(s,y(t_i))|^2+2|g(s,y(t_i))-g(t_i,y(t_i))|^2$$
$$\leq 2H(1+|y(s)|^\sigma+|y(t_i)|^\sigma)|y(t_i)-y(s)|^2+4K_2^2(1+|y(t_i)|^{2\sigma+2})|t_i-s|^{2\gamma_2}.$$

So, by the Hölder inequality and Lemmas 3.5 and 3.6, we have

$$\begin{aligned}\int_{t_i}^{t_{i+1}}\mathbb{E}|g(s,y(s))-g(t_i,y(t_i))|^2\,\mathrm{d}s &\leq 2H3^{\frac{2}{\sigma+2}}(1+C_p)^{\frac{\sigma}{\sigma+2}}\int_{t_i}^{t_{i+1}}\left(\mathbb{E}|y(s)-y(t_i)|^{\sigma+2}\right)^{\frac{2}{\sigma+2}}\mathrm{d}s\\ &\quad+4K_2^2\Delta t^{2\gamma_2}\int_{t_i}^{t_{i+1}}(1+\mathbb{E}|y(t_i)|^{2\sigma+2})\,\mathrm{d}s\\ &\leq 2H\bar{C}\Delta t^2+4K_2^2(1+C_p)\Delta t^{2\gamma_2+1}.\end{aligned}$$

Therefore,

$$\mathbb{E}[J_3]\leq\frac{1}{2}\mathbb{E}|e_{i+1}-e_i|^2+4H\bar{C}\Delta t^2+8K_2^2(1+C_p)\Delta t^{2\gamma_2+1}. \tag{3.14}$$

For J_4, we have

$$J_4=2(e_{i+1}-e_i)^{\mathrm{T}}(g(t_i,y(t_i))-g(t_i,Y_i))\Delta B_{i+1}+2e_i^{\mathrm{T}}(g(t_i,y(t_i))-g(t_i,Y_i))\Delta B_{i+1}.$$

Note that

$$\mathbb{E}\left[e_i^{\mathrm{T}}(g(t_i,y(t_i))-g(t_i,Y_i))\Delta B_{i+1}\right]=0.$$

By the martingale property, the isometric property and Young's inequality

$$\mathbb{E}[J_4]\leq\frac{1}{2}\mathbb{E}|e_{i+1}-e_i|^2+2\Delta t\mathbb{E}|g(t_i,y(t_i))-g(t_i,Y_i)|^2. \tag{3.15}$$

Combining (3.12)–(3.15), it can be concluded that

$$\begin{aligned}\mathbb{E}|e_{i+1}|^2-\mathbb{E}|e_i|^2+\mathbb{E}|e_{i+1}-e_i|^2&=2\mathbb{E}\left(e_{i+1}^{\mathrm{T}}(e_{i+1}-e_i)\right)\\ &\leq\epsilon^2\mathbb{E}|e_{i+1}|^2+\frac{4K_1^2(1+C_p)}{\epsilon^2}\Delta t^{2\gamma_1+2}+\frac{2\bar{C}H}{\epsilon^2}\Delta t^3\\ &\quad+2K_3\Delta t\mathbb{E}|e_{i+1}|^2-2p_0\Delta t\mathbb{E}|g(t_{i+1},y(t_{i+1}))-g(t_{i+1},Y_{i+1})|^2\\ &\quad+\mathbb{E}|e_{i+1}-e_i|^2+4H\bar{C}\Delta t^2+8K_2^2(1+C_p)\Delta t^{2\gamma_2+1}\\ &\quad+2\Delta t\mathbb{E}|g(t_i,y(t_i))-g(t_i,Y_i)|^2.\end{aligned}$$

Then the inequality above can be rearranged to

$$\begin{aligned}&(1-3K_3\Delta t)\mathbb{E}|e_{i+1}|^2+2p_0\Delta t\mathbb{E}|g(t_{i+1},y(t_{i+1}))-g(t_{i+1},Y_{i+1})|^2\\ &\quad\leq\mathbb{E}|e_i|^2+2\Delta t\mathbb{E}|g(t_i,y(t_i))-g(t_i,Y_i)|^2+\frac{4K_1^2(1+C_p)}{K_3}\Delta t^{2\gamma_1+1}\\ &\quad+\left(\frac{2\bar{C}H}{K_3}+4H\bar{C}\right)\Delta t^2+8K_2^2(1+C_p)\Delta t^{2\gamma_2+1}.\end{aligned}$$

Since $1-3K_3\Delta t\leq p_0$, and define

$$\zeta=\frac{4K_1^2(1+C_p)}{3K_3^2}\Delta t^{2\gamma_1}+\frac{8K_2^2(1+C_p)}{3K_3}\Delta t^{2\gamma_2}+\left(\frac{2\bar{C}H}{3K_3^2}+\frac{4H\bar{C}}{3K_3}\right)\Delta t,$$

we have

$$(1-3K_3\Delta t)\left(\mathbb{E}|e_{i+1}|^2+2\Delta t\mathbb{E}|g(t_{i+1},y(t_{i+1}))-g(t_{i+1},Y_{i+1})|^2+\zeta\right)$$

$$\leq \mathbb{E}|e_i|^2 + 2\Delta t \mathbb{E}|g(t_i, y(t_i)) - g(t_i, Y_i)|^2 + \zeta.$$

Since $\Delta t < \frac{1}{3K_3}$, by iteration we get

$$\begin{aligned}
&\mathbb{E}|e_{i+1}|^2 + 2\Delta t \mathbb{E}|g(t_{i+1}, y(t_{i+1})) - g(t_{i+1}, Y_{i+1})|^2 + \zeta \\
&\quad \leq \frac{1}{1 - 3K_3\Delta t}\big(\mathbb{E}|e_i|^2 + 2\Delta t \mathbb{E}|g(t_i, y(t_i)) - g(t_i, Y_i)|^2 + \zeta\big) \\
&\quad \leq \left(\frac{1}{1 - 3K_3\Delta t}\right)^{i+1}\left(\frac{4K_1^2(1+C_p)}{3K_3^2}\Delta t^{2\gamma_1} + \frac{8K_2^2(1+C_p)}{3K_3}\Delta t^{2\gamma_2} + \left(\frac{2\bar{C}H}{3K_3^2} + \frac{4H\bar{C}}{3K_3}\right)\Delta t\right).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\mathbb{E}|y(t_i) - Y_i|^2 &\leq \left(\frac{1}{1 - 3K_3\Delta t}\right)^{i}\left(\frac{4K_1^2(1+C_p)}{3K_3^2}\Delta t^{2\gamma_1} + \frac{8K_2^2(1+C_p)}{3K_3}\Delta t^{2\gamma_2} + \left(\frac{2\bar{C}H}{3K_3^2} + \frac{4H\bar{C}}{3K_3}\right)\Delta t\right) \\
&\leq e^{Ct_i}\left(\frac{4K_1^2(1+C_p)}{3K_3^2}\Delta t^{2\gamma_1} + \frac{8K_2^2(1+C_p)}{3K_3}\Delta t^{2\gamma_2} + \left(\frac{2\bar{C}H}{3K_3^2} + \frac{4H\bar{C}}{3K_3}\right)\Delta t\right) \\
&\leq C(\Delta t^{2\gamma_1} + \Delta t^{2\gamma_2} + \Delta t)e^{CT}, \qquad (3.16)
\end{aligned}$$

where we used the fact that $(1 + C\Delta t)^i \leq e^{Ct_i}$ for any $C > 0$.

For any $t \in [i\Delta t, (i+1)\Delta t)$, it follows from (3.2), (3.8) and (3.16) that

$$\begin{aligned}
\mathbb{E}|y(t) - Y(t)|^2 &\leq 2\mathbb{E}|y(t) - y(t_i)|^2 + 2\mathbb{E}|y(t_i) - Y_i|^2 \\
&\leq 2\Delta t + 2C\big(\Delta t^{2\gamma_1} + \Delta t^{2\gamma_2} + \Delta t\big)e^{3K_3T} \\
&\leq C\big(\Delta t^{2\gamma_1} + \Delta t^{2\gamma_2} + \Delta t\big).
\end{aligned}$$

This completes the proof. □

4. Numerical invariant measure

In this section, we discuss the invariant measure of the numerical solution generated by the semi-implicit EM method. In order to simplify the analysis, we consider the following autonomous SDEs

$$x(t) = x(0) + \int_0^t f(x(s))\,\mathrm{d}s + \int_0^t g(x(s))\,\mathrm{d}B(s) + L(t), \quad t > 0, \tag{4.1}$$

where the Lévy process $L(t)$ is defined as that in Section 2 with an additional requirement that $\mathbb{E}(L(t)) = 0$. The semi-implicit EM method reduces to

$$X_{i+1} = X_i + f(X_{i+1})\Delta t + g(X_i)\Delta B_{i+1} + \Delta L_{i+1}, \quad i = 0, 1, 2, \ldots \tag{4.2}$$

To explore the invariant measure of the numerical method, we need more notations. Let $\mathcal{P}(\mathbb{R}^d)$ denote the family of all probability measures on $\mathbb{R}^d$. Define the corresponding Wasserstein-1 distance between $\omega \in \mathcal{P}(\mathbb{R}^d)$ and $\omega' \in \mathcal{P}(\mathbb{R}^d)$ by

$$W_1(\omega, \omega') = \inf \mathbb{E}|u - v|,$$

where the infimum is taken over all pairs of random variables u and v on $\mathbb{R}^d$ with respect to the laws ω and ω'.

Let $\overline{\mathbb{P}}_t(\cdot,\cdot)$ be the transition probability kernel of the underlying solution $x(t)$, with the notation $\delta_x\overline{\mathbb{P}}_t$ emphasizing the initial value x. The probability measure $\pi(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ is called an invariant measure of $x(t)$ if

$$\pi(B) = \int_{\mathbb{R}^d} \overline{\mathbb{P}}_t(x, B)\pi(\mathrm{d}x)$$

holds for any $t > 0$ and Borel set $B \in \mathcal{B}(\mathbb{R}^d)$.

Let $\mathbb{P}_i(\cdot,\cdot)$ be the transition probability kernel of the numerical solution $\{X_i\}_{i\geq 0}$ with the notation $\delta_x\mathbb{P}_i$ emphasizing the initial value x. The probability measure $\Pi_{\Delta t}(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ is called an invariant measure of the numerical solution $\{X_i\}_{i\geq 0}$ if

$$\Pi_{\Delta t}(B) = \int_{\mathbb{R}^d} \mathbb{P}_i(x, B)\Pi_{\Delta t}(\mathrm{d}x), \quad \forall i = 0, 1, 2, ...,$$

for any $B \in \mathcal{B}(\mathbb{R}^d)$.

Lemma 4.1. *Suppose that Assumption 2.3 holds with $M < 0$. Then for any $\Delta t \in (0, 1 \wedge \frac{2p_1-1}{-2M})$, the numerical solution is uniformly bounded, i.e.,*

$$\mathbb{E}|X_i|^2 < |X_0|^2 + |g(X_0)|^2 - \frac{2m+1}{M}\cdot$$

Proof. Recall equations (3.10) and (4.2), we have

$$\begin{aligned}
|X_{i+1}|^2 - |X_i|^2 + |X_{i+1} - X_i|^2 &= 2X_{i+1}^{\mathrm{T}}(X_{i+1} - X_i)\\
&= 2X_{i+1}^{\mathrm{T}}(f(X_{i+1})\Delta t + g(X_i)\Delta B_{i+1} + \Delta L_{i+1})\\
&= 2\Delta t X_{i+1}^{\mathrm{T}} f(X_{i+1}) + 2X_{i+1}^{\mathrm{T}}(g(X_i)\Delta B_{i+1} + \Delta L_{i+1})\\
&= 2\Delta t X_{i+1}^{\mathrm{T}} f(X_{i+1}) + 2(X_{i+1} - X_i)^{\mathrm{T}}(g(X_i)\Delta B_{i+1} + \Delta L_{i+1})\\
&\quad + 2X_i^{\mathrm{T}}(g(X_i)\Delta B_{i+1} + \Delta L_{i+1}).
\end{aligned} \tag{4.3}$$

Note that

$$\mathbb{E}\big(X_i^{\mathrm{T}}(g(X_i)\Delta B_{i+1})\big) = 0 \quad\text{and}\quad \mathbb{E}\big(X_i^{\mathrm{T}}\Delta L_{i+1}\big) = 0.$$

Taking expectations of both sides of (4.3) and making use of Assumption 2.3 give

$$\begin{aligned}
\mathbb{E}|X_{i+1}|^2 - \mathbb{E}|X_i|^2 + \mathbb{E}|X_{i+1} - X_i|^2 &= 2\mathbb{E}\big(X_{i+1}^{\mathrm{T}}(X_{i+1} - X_i)\big)\\
&\leq 2M\Delta t\mathbb{E}|X_{i+1}|^2 + 2m\Delta t - 2p_1\Delta t\mathbb{E}|g(X_{i+1})|^2 + \mathbb{E}|X_{i+1} - X_i|^2\\
&\quad + \Delta t\mathbb{E}|g(X_i)|^2 + \Delta t^{2/\gamma_0}.
\end{aligned}$$

Then cancelling the same term on both sides and using the fact $\Delta t^{2/\gamma_0} < \Delta t$ yield

$$\begin{aligned}
(1 - 2M\Delta t)\mathbb{E}|X_{i+1}|^2 + 2p_1\Delta t\mathbb{E}|g(X_{i+1})|^2 &\leq \mathbb{E}|X_i|^2 + \Delta t\mathbb{E}|g(X_i)|^2 + 2m\Delta t + \Delta t^{2/\gamma_0}\\
&\leq \mathbb{E}|X_i|^2 + \Delta t\mathbb{E}|g(X_i)|^2 + (2m+1)\Delta t.
\end{aligned}$$

Choose Δt such that $1 - 2M\Delta t \leq 2p_1$ and let $c = -\frac{2m+1}{M}$, we can obtain

$$(1 - 2M\Delta t)\big(\mathbb{E}|X_{i+1}|^2 + \Delta t\mathbb{E}|g(X_{i+1})|^2 - c\big) \leq \mathbb{E}|X_i|^2 + \Delta t\mathbb{E}|g(X_i)|^2 - c.$$

By iteration, this leads to

$$\begin{aligned}
\mathbb{E}|X_{i+1}|^2 + \Delta t\mathbb{E}|g(X_{i+1})|^2 - c &\leq \frac{1}{1 - 2M\Delta t}\big(\mathbb{E}|X_i|^2 + \Delta t\mathbb{E}|g(X_i)|^2 - c\big)\\
&\leq \frac{1}{(1 - 2M\Delta t)^{i+1}}\big(|X_0|^2 + \Delta t|g(X_0)|^2 - c\big),
\end{aligned}$$

where $1 - 2M\Delta t > 1$. Finally, we have

$$\mathbb{E}|X_i|^2 < |X_0|^2 + \Delta t|g(X_0)|^2 + c,$$

which completes the proof. ∎

Let $\{X_i^x\}_{i\geq 0}$ and $\{X_i^y\}_{i\geq 0}$ be the numerical solutions with respect to two different initial values x and y.

Lemma 4.2. *Suppose that Assumption 2.2 holds with $K_3 < 0$. For any $\Delta t \in (0, 1 \wedge \frac{2p_0-1}{-2K_3})$, the numerical solutions satisfy*

$$\lim_{i\to\infty} \mathbb{E}|X_i^x - X_i^y|^2 = 0.$$

Proof. Define $E_i = X_i^x - X_i^y$, we obtain that

$$E_{i+1} = E_i + \big(f(X_{i+1}^x) - f(X_{i+1}^y)\big)\Delta t + (g(X_i^x) - g(X_i^y))\Delta B_{i+1}.$$

Using the same technique as in the proof of Lemma 4.1, it can be concluded that,

$$\begin{aligned} |E_{i+1}|^2 - |E_i|^2 + |E_{i+1} - E_i|^2 &= 2E_{i+1}^{\mathrm{T}}(E_{i+1} - E_i) \\ &= 2E_{i+1}^{\mathrm{T}}\big(f(X_{i+1}^x) - f(X_{i+1}^y)\big)\Delta t + 2(E_{i+1} - E_i)^{\mathrm{T}}(g(X_i^x) - g(X_i^y))\Delta B_{i+1} \\ &\quad + 2E_i^{\mathrm{T}}(g(X_i^x) - g(X_i^y))\Delta B_{i+1}. \end{aligned} \tag{4.4}$$

Due to

$$\mathbb{E}\big(E_i^{\mathrm{T}}(g(X_i^x) - g(X_i^y))\Delta B_{i+1}\big) = 0.$$

Taking expectations on both sides of (4.4) and using Assumption 2.2 result in

$$\begin{aligned} \mathbb{E}|E_{i+1}|^2 - \mathbb{E}|E_i|^2 + \mathbb{E}|E_{i+1} - E_i|^2 &= 2\mathbb{E}\big(E_{i+1}^{\mathrm{T}}(E_{i+1} - E_i)\big) \\ &\leq 2K_3\Delta t\mathbb{E}|E_{i+1}|^2 - 2p_0\Delta t\mathbb{E}\big|g(X_{i+1}^x) - g(X_{i+1}^y)\big|^2 \\ &\quad + \mathbb{E}|E_{i+1} - E_i|^2 + \Delta t\mathbb{E}|g(X_i^x) - g(X_i^y)|^2. \end{aligned}$$

Choose Δt such that $1 - 2K_3\Delta t \leq 2p_0$,

$$(1 - 2K_3\Delta t)\Big(\mathbb{E}|E_{i+1}|^2 + \Delta t\mathbb{E}\big|g(X_{i+1}^x) - g(X_{i+1}^y)\big|^2\Big) \leq \mathbb{E}|E_i|^2 + \Delta t\mathbb{E}|g(X_i^x) - g(X_i^y)|^2.$$

Since $K_3 < 0$, $1 - 2K_3\Delta t > 1$ holds. By iteration we have

$$\begin{aligned} \mathbb{E}|E_{i+1}|^2 + \Delta t\mathbb{E}\big|g(X_{i+1}^x) - g(X_{i+1}^y)\big|^2 &\leq \frac{1}{1 - 2K_3\Delta t}\Big(\mathbb{E}|E_i|^2 + \Delta t\mathbb{E}|g(X_i^x) - g(X_i^y)|^2\Big) \\ &\leq \frac{1}{(1 - 2K_3\Delta t)^{i+1}}\Big(\mathbb{E}|E_0|^2 + \Delta t\mathbb{E}|g(X_0^x) - g(X_0^y)|^2\Big) \\ &\leq \frac{1 + C}{(1 - 2K_3\Delta t)^{i+1}}|x - y|^2. \end{aligned}$$

This completes the proof. $\square$

We now present the existence and uniqueness of the invariant measure of the numerical solution $\{X_i\}_{i\geq 0}$.

Theorem 4.3. *Suppose that Assumptions 2.2 and 2.3 hold. If Δt satisfies*

$$\Delta t \in \left(0, 1 \wedge \frac{2p_1 - 1}{-2M} \wedge \frac{2p_0 - 1}{-2K_3}\right),$$

then the numerical solution $\{X_i\}_{i\geq 0}$ has a unique invariant measure $\Pi_{\Delta t}$.

Proof. For each integer $n \geq 1$ and any Borel set $B \subset \mathbb{R}^d$, define the measure

$$\omega_n(B) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(X_i \in B).$$

It follows from Lemma 4.1 and the Chebyshev inequality that the measure sequence $\{\omega_n(B)\}_{n\geq 1}$ is tight such that there exists a subsequence which converges to an invariant measure [3,25]. This proves the existence of the invariant measure of the numerical solution. In the following, we show the uniqueness of the invariant measure. Let $\Pi^x_{\Delta t}$ and $\Pi^y_{\Delta t}$ be invariant measures of $\{X^x_i\}_{i\geq 0}$ and $\{X^y_i\}_{i\geq 0}$, respectively. Then

$$\begin{aligned} W_1(\Pi^x_{\Delta t}, \Pi^y_{\Delta t}) &= W_1(\Pi^x_{\Delta t}\mathbb{P}_i, \Pi^y_{\Delta t}\mathbb{P}_i) \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Pi^x_{\Delta t}(\mathrm{d}x)\Pi^y_{\Delta t}(\mathrm{d}y) W_1(\delta_x\mathbb{P}_i, \delta_y\mathbb{P}_i). \end{aligned}$$

From Lemma 4.2, we get

$$W_1(\delta_x\mathbb{P}_i, \delta_y\mathbb{P}_i) \leq \left(\frac{1+C}{(1-2K_3\Delta t)^i}|x-y|^2\right)^{\frac{1}{2}} \to 0, \quad \text{as} \quad i\to\infty.$$

Then, we have

$$W_1(\Pi^x_{\Delta t}, \Pi^y_{\Delta t}) = 0,$$

which completes the proof. □

The following theorem states the numerical invariant measure $\Pi_{\Delta t}$ converges to the underlying one π in the Wassertein distance.

Theorem 4.4. *Suppose that Assumptions 2.2 and 2.3 hold. Then*

$$\lim_{\Delta t\to 0+} W_1(\pi, \Pi_{\Delta t}) = 0.$$

Proof. It is clear to see that

$$W_1(\delta_x\overline{\mathbb{P}}_{i\Delta t}, \pi) \leq \int_{\mathbb{R}^d} \pi(\mathrm{d}y) W_1(\delta_x\overline{\mathbb{P}}_{i\Delta t}, \delta_y\overline{\mathbb{P}}_{i\Delta t})$$

and

$$W_1(\delta_x\mathbb{P}_i, \Pi_{\Delta t}) \leq \int_{\mathbb{R}^d} \Pi_{\Delta t}(\mathrm{d}y) W_1(\delta_x\mathbb{P}_i, \delta_y\mathbb{P}_i).$$

Because of the existence and uniqueness of the invariant measure for the numerical method (4.2) and Theorem 4.3, for any $\Delta t \in (0, 1\wedge \frac{2p_1-1}{-2M} \wedge \frac{2p_0-1}{-2K_3})$ and $\varepsilon > 0$, there exists a $i > 0$ sufficiently large such that

$$W_1\big(\delta_x\overline{\mathbb{P}}_{i\Delta t}, \pi\big) \leq \frac{\varepsilon}{3} \quad \text{and} \quad W_1(\delta_x\mathbb{P}_i, \Pi_{\Delta t}) \leq \frac{\varepsilon}{3}.$$

Then, for the chosen i, by Theorem 3.2, we get

$$W_1\big(\delta_x\overline{\mathbb{P}}_{i\Delta t}, \delta_x\mathbb{P}_i\big) \leq \frac{\varepsilon}{3}.$$

Therefore,

$$W_1(\pi, \Pi_{\Delta t}) \leq W_1(\delta_x\overline{\mathbb{P}}_{i\Delta t}, \pi) + W_1(\delta_x\mathbb{P}_i, \Pi_{\Delta t}) + W_1(\delta_x\overline{\mathbb{P}}_{i\Delta t}, \delta_x\mathbb{P}_i) \leq \varepsilon,$$

which completes the proof. □

Remark 4.5. Our conclusions reveal the existence and uniqueness of numerical invariant measure in the sense of Wasserstein-1 distance and the convergence of the numerical invariant measure to the underlying counterpart.

5. Numerical examples

In this section, we conduct some numerical experiments to verify our theoretical results. Examples 5.1 and 5.2 are used to illustrate the results of Theorem 3.2 and Corollary 3.3, respectively. Examples 5.3 and 5.4 are given to demonstrate the results of Theorems 4.3 and 4.4.

It is should be noted that the Lévy process we used in our numerical experiments is the tempered stable process, whose generating algorithm is borrowed from [18].

Example 5.1. *Consider the following SDE*

$$\mathrm{d}y(t) = \Big([(t-1)(2-t)]^{\frac{1}{5}} y^2(t) - 2y^5(t)\Big)\,\mathrm{d}t + \Big(2[(t-1)(2-t)]^{\frac{2}{5}} y^2(t)\Big)\,\mathrm{d}B(t) + \mathrm{d}L(t),$$

where $t \in [1,2]$, $y(0) = 1$.

For any $x, y \in \mathbb{R}$, we have

$$\begin{aligned} |f(t,x) - f(t,y)|^2 &= \Big|[(t-1)(2-t)]^{\frac{1}{5}}\big(x^2 - y^2\big) - 2\big(x^5 - y^5\big)\Big|^2 \\ &\le \Big|[(t-1)(2-t)]^{\frac{1}{5}}(x+y) - \big(x^4 + y^4\big)\Big|^2 |x-y|^2 \\ &\le H\big(1 + |x|^8 + |y|^8\big)|x-y|^2 \end{aligned}$$

and

$$\begin{aligned} |g(t,x) - g(t,y)|^2 &= \Big|2[(t-1)(2-t)]^{\frac{2}{5}}\big(x^2 - y^2\big)\Big|^2 \\ &\le H\big(1 + |x|^2 + |y|^2\big)|x-y|^2, \end{aligned}$$

which shows that Assumption 2.1 is satisfied with $\sigma = 8$.

Using the same approach, for any $t \in [1,2]$, we can prove that Assumptions 2.2 and 2.3 are satisfied. Assumption 2.4 also holds with $\gamma_1 = 1/5$ and $\gamma_2 = 2/5$. Thus, according to the Theorem 3.2, we obtain

$$\mathbb{E}|y(t) - Y(t)|^2 \le C\big(\Delta t^{2\gamma_1} + \Delta t^{2\gamma_2} + \Delta t\big).$$

Note that the parameter γ_0 does not affect the convergence order.

We conduct 1000 independent trajectories with different step sizes 2^{-9}, 2^{-10}, 2^{-11}, 2^{-12} and 2^{-15}, respectively. Since the true solution of the SDE is difficult to obtain, we take the numerical solution solved with the minimum step size 2^{-15} as a reference solution. We choose different γ_0 and record the errors *versus* the step sizes. Figures 1a and 1b show that the convergence order is about 0.2 when $\gamma_0 = 1.3, 1.5$. For $\gamma_0 = 1.3$, we take $f(t,y) = [(t-1)(2-t)]^{4/5} y^2(t) - 2y^5(t)$, and thus $\gamma_1 = 0.8$. Figure 1c shows the convergence order is about 0.5. Numerical results confirm a close agreement between theoretical and numerical convergence order.

Example 5.2. *Consider the following SDE*

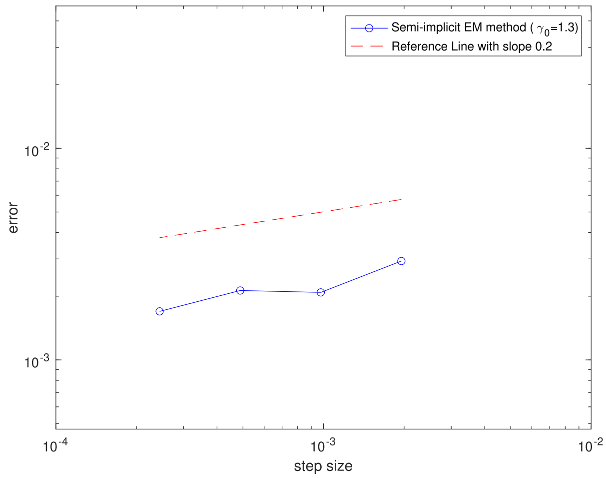
$$\mathrm{d}u(t) = \big([(t-1)(2-t)]^{\gamma_1} u^2(t) - 2u^5(t)\big)\,\mathrm{d}t + \mathrm{d}L(t), \quad t \in [1,2],$$

where $u(0) = 1$.

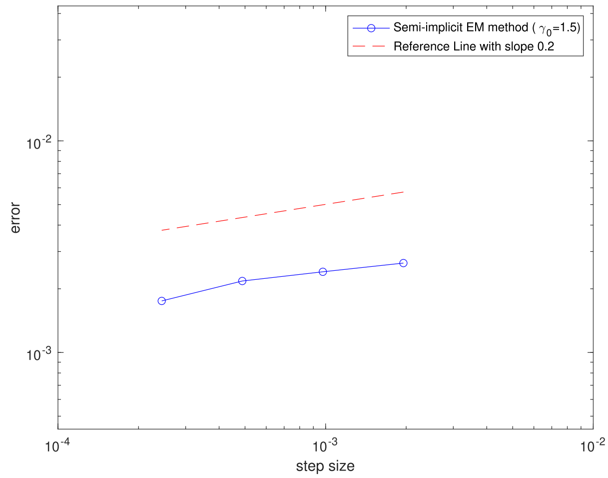
The Brownian motion doesn't appear in this example. According to Corollary 3.3, the convergence order of the semi-implicit EM method is $\min\{\gamma_1, 1/\gamma_0\}$. For $\gamma_1 = 0.9$ and $\gamma_0 = 1.3$, we observe from Figure 2 that the convergence order is close to $1/\gamma_0 \approx 0.77$, which agrees with Corollary 3.3.

Example 5.3. *Consider the following Ornstein–Uhlenbeck (OU) process*

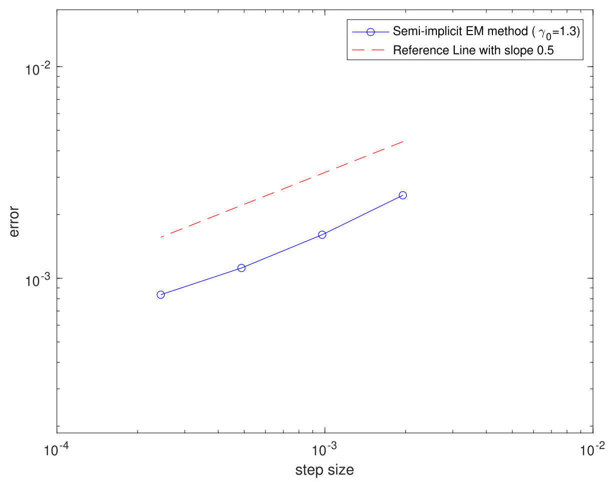
$$\mathrm{d}x(t) = -2x(t)\,\mathrm{d}t + 2L(t), \quad x(0) = 10.$$



(a)



(b)



(c)

FIGURE 1. Errors *versus* step sizes. (a) $\gamma_0 = 1.3$, $\gamma_1 = 0.2$, $\gamma_2 = 0.4$. (b) $\gamma_0 = 1.5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.4$. (c) $\gamma_0 = 1.3$, $\gamma_1 = 0.8$, $\gamma_2 = 0.6$.

The Kolmogorov–Smirnov test [29] helps us to measure the difference in distribution between the numerical and true solutions more clearly. When $L(t)$ is an α-stable process ($\alpha \in (0,2)$), the SDE has an invariant measure, which is $\mathbb{S}(2(\frac{1}{2\alpha})^{1/\alpha}, 0, 0)$ [8]. We choose $\alpha = 1.5$ as the true distribution. We simulate 10 000 paths generated by the semi-implicit EM method. The empirical distributions at $t = 0.1$, $t = 0.3$, $t = 0.7$ and $t = 2$ are plotted in Figure 3a. It can be seen that the empirical distributions get closer to the true distribution as time t increases. Figure 3b shows that the distribution difference between the numerical solution and the true solution decreases as time t increases.

Example 5.4. *Considering the following SDE*

$$\mathrm{d}x(t) = \left(-x^3(t) - 5x(t) + 5\right)\mathrm{d}t + x^2(t)\,\mathrm{d}B(t) + 2\,\mathrm{d}L(t), \quad x(0) = 10.$$

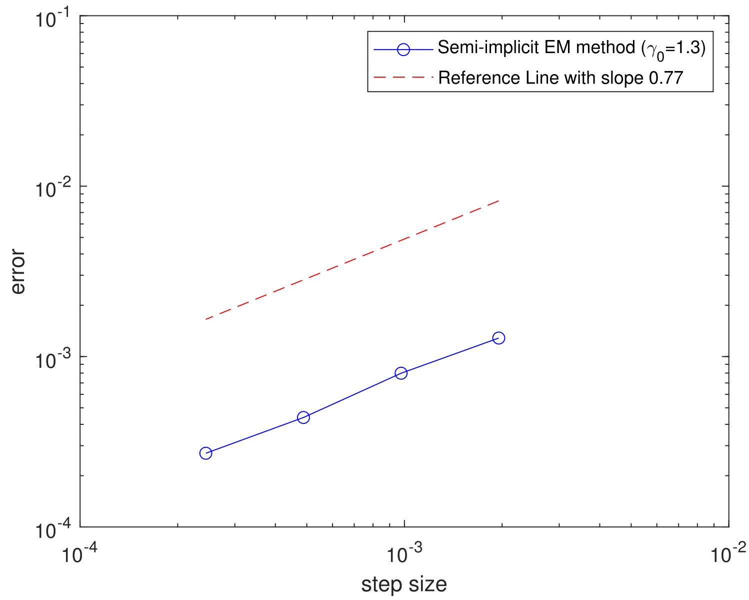
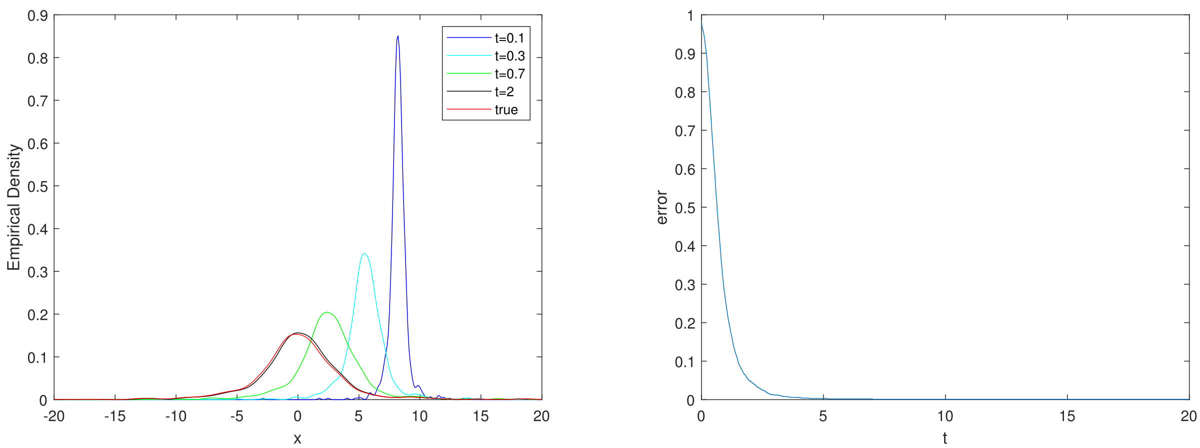


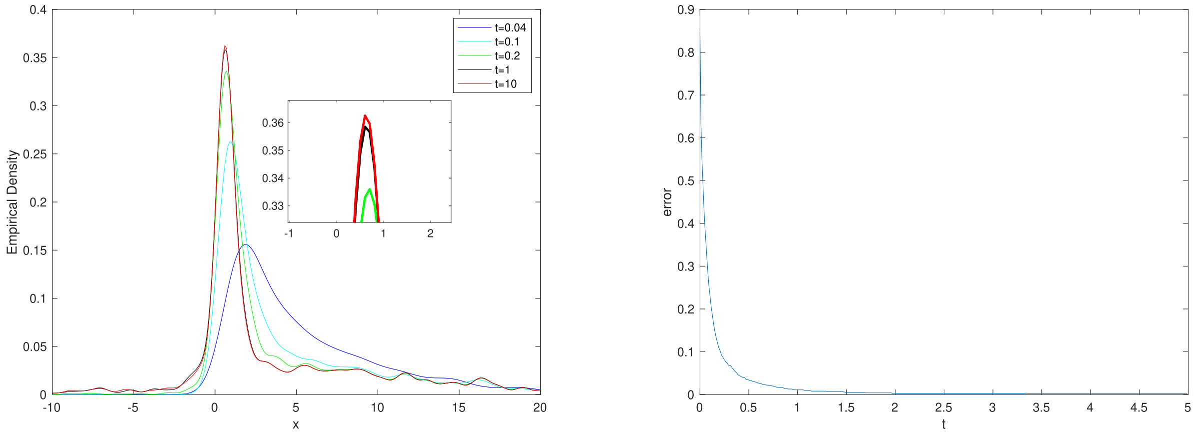
FIGURE 2. Errors *versus* step sizes.



(a) (b)

FIGURE 3. Empirical distributions and long time stability. (a) Empirical density functions at different time points. (b) Differences between empirical distributions and the true stationary distribution.

We assume that step size $h = 0.01$, $T = 10$, and simulate 10 000 paths. The empirical distributions at $t = 0.04$, $t = 0.1$, $t = 0.2$, $t = 1$ and $t = 10$ are plotted in Figure 4a. It can be clearly seen that the empirical density functions with $t = 0.04$, $t = 0.1$ and $t = 0.2$ have quite different shapes, but the shapes with $t = 2$ and $t = 10$ are very close. Figure 4b shows that the distribution difference between the numerical solution and the reference solution decreases as time t increases, which supports our theoretical results as well.



(a) (b)

FIGURE 4. Empirical distributions and long time stability. (a) Empirical density functions at different time points. (b) Differences between empirical distributions and the true stationary distribution.

6. Conclusion and future research

We investigate the finite time strong convergence of the semi-implicit EM method for SDEs with super-linear coefficients driven by a class of Lévy processes. One of the key findings is that the convergence order is related to the parameter of the class of Lévy process, which has not been observed in literatures. In addition, the semi-implicit EM method is capable of providing a good approximation of the invariant measure of the underlying SDEs.

There are still some technical difficulties to overcome in the multiplicative case of the class of Lévy processes. Furthermore, other stable processes not covered by the settings of the class of Lévy processes in Section 2 are worth to try in the future.

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Data availability statement

The research data associated with this article are included in the article.

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