ESAIM: M2AN 59 (2025) 2763–2788
https://doi.org/10.1051/m2an/2025074

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

HDG METHODS FOR INCOMPRESSIBLE AND IMMISCIBLE TWO-PHASE FLOW IN POROUS MEDIA: THE EXISTENCE AND CONVERGENCE OF DISCRETE SOLUTIONS

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Abstract. In this paper, we investigate the numerical approximation of incompressible, immiscible two-phase flow in porous media using backward Euler methods for time discretization and hybridized discontinuous Galerkin (HDG) methods for saturation and pressure discretizations. Using the Brouwer fixed-point theorem, we establish the existence of a discrete solution for the proposed scheme. Furthermore, the sequence of discrete solutions solved by the discrete scheme converges, potentially after passing to a subsequence, to a weak solution of the continuous problem. Finally, two numerical examples are presented to validate both the accuracy and performance of the discrete scheme.

Mathematics Subject Classification. 35K55, 65M60, 76S05, 76T10.

Received February 9, 2025. Accepted September 4, 2025.

1. Introduction

The modeling and simulation of two-phase flow in porous media are of significant interest and importance in various real-world applications, including petroleum reservoir engineering and subsurface CO_2 sequestration. The mathematical model of two-phase flow in porous media comprises a coupled system of nonlinear partial differential equations derived from the mass conservation law and Darcy's law for each phase. The nonlinearity of the system primarily arises from the relative permeability and capillary pressure functions, which characterize the interactions between the porous medium and the fluids. Over the past few decades, various numerical strategies have been developed to handle this inherent nonlinearity, including implicit-explicit (IMPES), semi-sequential, semi-implicit, and fully implicit methods [12]. In the standard IMPES method [14], the saturation equation is treated explicitly, making the method computationally efficient and memory-efficient at each time step compared to the fully implicit approach. However, IMPES has several drawbacks. Notably, due to the hyperbolic nature of the saturation equation, the method often requires the use of slope limiters [2, 30, 31, 36] to suppress spurious oscillations, yet formulating these limiters can be complex. In contrast, the fully implicit method does not relax the system's nonlinearity and therefore demands significant computational resources to solve the nonlinear system at each time step, but it avoids the need for slope limiters [4, 24].

Keywords and phrases. HDG methods, two-phase flow, existence of discrete solutions, stability, convergence.

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The selection of spatial discretization also plays a critical role in the solution process of two-phase flow in porous media, since local mass conservation property and efficiency of implementation are all important features. Currently, mixed finite element (MFE) methods have been extensively used in the simulation of two-phase flow in porous media [13,23] for the reason that it can be able to compute the pressure and the velocity simultaneously and maintain the local mass conservation property. In heterogeneous porous media with different distributions of permeability, discontinuous saturation may arise in the entire domain. Thus the discontinuous Galerkin (DG) methods, which possess local mass conservation, are preferable [24, 26] in this case because its approximation space is discontinuous. Recently, the HDG methods, proposed by Cockburn *et al.* [15] for second order elliptic problem, also have been successfully used in the simulation of two-phase flow in porous media [16, 27] without numerical analysis. One advantage of HDG methods is that the resulting algebraic system based on the HDG method is only due to the unknowns on the skeleton of meshes, and this will reduce significantly the computational complexity compared with the classical DG method, especially when high order polynomial approximations are used. Up to now, HDG methods have been widely used in engineering and scientific computing, see [9,35,40,41,45] and references therein for more details. Here we also mention [28,30,44] for the coupled MFE and DG methods, and [8,43,46] for the finite volume method.

Although numerical simulation of two-phase flow in porous media has been investigated for many years, the corresponding numerical analysis, including the existence and convergence of discrete solutions, is rarely involved. By introducing the global and complementary pressures, Chen [10] proved the existence and uniqueness of weak solutions for an appropriate weak formulation of two-phase incompressible flow in porous media, and analyzed *a priori* error estimates for a discrete scheme with MFE methods for the global pressure and velocity and Galerkin methods for the complementary pressure [11]. For a numerical scheme designed in [24] with DG methods for the global pressure and non-wetting saturation, Epshteyn and Rivière [25] proved the existence of discrete solutions by Leray–Schauder theorem and derived *a priori* error estimates. In [37], Kou and Sun proved *a priori* error estimates for a semi-discrete scheme with DG methods for the wetting-phase saturation and pressure. In [38], they proved the existence and uniqueness of discrete solutions and derived *a priori* error estimates for a numerical scheme using MFE methods for the global pressure and velocity and DG methods for the wetting-phase saturation. Here we also mention [7] for *a posteriori* error estimates of finite volume methods, and [8, 32] for the existence and convergence of discrete solutions for finite volume methods of two-phase flow in porous media involving dynamic effects in capillary pressure.

In [25,32,37,38], the convergence results, deduced from the standard *a priori* error analysis, exclude the case of nonsmooth solutions which may be present in physically realistic scenarios. For this reason, here we conduct the numerical analysis that can be used to deduce convergence to weak solutions in the absence of additional regularity for the wetting-phase saturation-pressure formulation of two-phase flow in porous media. The reason for choosing the wetting-phase saturation-pressure formulation is that the complementary pressure introduced in [10,11] has no physical meaning and it is difficult to use it to solve the saturation.

This paper is a continuous of our previous works [40,41]. In this paper, we design a fully discrete scheme with backward Euler methods for the time discretization and HDG methods for the wetting-phase saturation and pressure discretizations. By the Brouwer fixed-point theorem [51], we prove that the proposed discrete scheme has a discrete solution under assumptions (A1)–(A5) (see Sect. 2). Based on the uniform boundedness of discrete solutions in time and space. By the analysis tools developed in [3,18], we also prove that the sequence of discrete solutions converges, possibly after passing to a subsequence, to a weak solution of the continuous problem. It is worth noting that the proof of the convergence is made challenging by the discontinuous nature of our numerical method, since the standard compactness result like Aubin–Lions theorem can not be used directly. Moreover, from numerical experiments provided in Section 5 we can observe that the discrete scheme proposed in this paper is able to be applied to the problem which does not satisfy the assumption (A4) and the problem with the realistic permeability.

Discrete compactness results of HDG methods appear scarce. In fact, we only find two relevant literatures [34, 35]. In [34], Kikuchi proved a discrete version of the Rellich–Kondrachov theorem and applied it to the convergence analysis of second order elliptic equations. In [35], Kirk *et al.* proved a variation of the discrete

Aubin–Lions–Simon theorem [47, 50] by the discrete functional analysis tools [18] and applied it to the convergence analysis of time-dependent incompressible Navier–Stokes equations. Compared with HDG methods, discrete compactness results for DG methods have been well studied. We refer the reader to [6,18] for the discrete version of Rellich–Kondrachov theorem and [42] for the discrete version of Aubin–Lions–Simon theorem. Here we also mention [17, 20] for the discrete version of Rellich–Kondrachov theorem for hybrid high-order methods proposed by Di Pietro and Ern [19].

The rest of this paper is arranged as follows. In Section 2, we describe the model and the corresponding assumptions on the problem data. In Section 3, we introduce a fully discrete scheme and prove that it has a discrete solution by the Brouwer fixed-point theorem. In Section 4, by the compactness arguments we prove that the sequence of discrete solutions converges to a weak solution of the continuous problem. In Section 5, two numerical examples are shown to validate the numerical result and the performance of the discrete scheme. Finally, we conclude the paper in Section 6.

2. The derivation of mathematical model

The mathematical model of incompressible and immiscible two-phase flow in porous media is described by the phase mass conservation law

$$\phi\frac{\partial S_\alpha}{\partial t}+\nabla\cdot\mathbf{u}_\alpha=F_\alpha,\quad \alpha=n,w, \tag{2.1}$$

Darcy's law

$$\mathbf{u}_\alpha=-\frac{k_{r\alpha}}{\mu_\alpha}\mathbf{K}(\nabla p_\alpha+\rho_\alpha g\nabla z),\quad \alpha=n,w, \tag{2.2}$$

and constraint relations

$$S_w+S_n=1, \tag{2.3}$$

$$p_c(S_w)=p_n-p_w, \tag{2.4}$$

where ϕ, g, z, $\mathbf{K}$ and p_c denote the porosity of the medium, the gravity acceleration, the depth, the absolute permeability tensor and the capillary pressure, the subscripts n and w denote the non-wetting and wetting phases, S_α, p_α, $\mathbf{u}_\alpha$, ρ_α, μ_α, $k_{r\alpha}$ and F_α denote the saturation, the pressure, the velocity, the density, the viscosity, the relative permeability and the external volumetric flow rate of the phase α. Note that in this model the capillary pressure and relative permeabilities are the given functions of the wetting-phase saturation S_w.

For ease of presentation, we disregard the effect of gravity in this paper. We set $S=S_w$ and $p=p_w$. Moreover we define the phase mobility $\lambda_\alpha=\frac{k_{r\alpha}}{\mu_\alpha}$, the total mobility $\lambda_t=\lambda_w+\lambda_n$, and $F_t=F_w+F_n$. Then summing the mass conservation equations (2.1) and noting the constraint conditions (2.3) and (2.4), we have

$$-\nabla\cdot(\lambda_t\mathbf{K}\nabla p)-\nabla\cdot(\lambda_n\mathbf{K}\nabla p_c)=F_t. \tag{2.5}$$

Applying the Darcy's law to the mass conservation equation (2.1) of the wetting phase, we obtain that

$$\phi\frac{\partial S}{\partial t}-\nabla\cdot(\lambda_w\mathbf{K}\nabla p)=F_w. \tag{2.6}$$

With (2.5) and (2.6) at hand, we can use (2.5) to calculate the pressure and update the saturation by (2.6), thus we call (2.5) and (2.6) the pressure and saturation equations respectively. In order to close the model, we still need to specify the boundary and initial conditions. To this end, we consider an open and bounded polygonal or polyhedral domain $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ with Lipschitz boundary $\Gamma=\partial\Omega$, and a time interval $J=(0,T)$ with final time T that is a positive constant. Then the boundary conditions are specified by

$$S=S_D,\quad p=p_D,\qquad\qquad \text{on } J\times\Gamma_D, \tag{2.7}$$

$$\lambda_w \mathbf{K}\nabla p \cdot \mathbf{n} = \mathbf{K}(\lambda_t \nabla p + \lambda_n \nabla p_c) \cdot \mathbf{n} = 0, \quad \text{on } J \times \Gamma_N, \tag{2.8}$$

and the initial condition is given by

$$S = S_0, \quad \text{in } \{0\} \times \Omega, \tag{2.9}$$

where Γ_D denotes both inflow and outflow boundaries, Γ_N indicates the no-flow boundary, and $\mathbf{n}$ denotes the unit outward normal vector to Γ_N. In addition, $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

Next we introduce a weak formulation for the coupled system (2.5)–(2.9) and state the existence of weak solutions. Before this, we introduce some notations and assumptions on the problem data. For any nonnegative integer s, open subset $\mathcal{D} \subset \Omega$, and $1 \le q \le \infty$, let $W^{s,q}(\mathcal{D})$ be the Sobolev spaces (see [1]) with norms $\|\cdot\|_{W^{s,q}(\mathcal{D})}$ and seminorms $|\cdot|_{W^{s,q}(\mathcal{D})}$. In particular, if $q = 2$, $W^{s,2}(\mathcal{D})$ is denoted by $H^s(\mathcal{D})$. If $s = 0$, $W^{0,q}(\mathcal{D})$ is replaced by $L^q(\mathcal{D})$. For any Banach space X, $L^q(J;X)$ and $W^{1,q}(J;X)$ denote the Bochner spaces with norms

$$\|v\|^q_{L^q(J;X)} = \int_0^T \|v\|^q_X \,\mathrm{d}x \quad 1 \le q < \infty, \quad \|v\|_{L^\infty(J;X)} = \max_{t\in J} \|v\|_X,$$
$$\|v\|_{W^{1,q}(J;X)} = \|v\|_{L^q(J;X)} + \|\partial_t v\|_{L^q(J;X)}.$$

Throughout the paper, we make the following assumptions on the data of the model.

(A1) The porosity of the medium $\phi \in L^\infty(\Omega)$ satisfies that $0 < \phi_* \le \phi(x) \le \phi^* < \infty$. The absolute permeability tensor $\mathbf{K}(x)$ is bounded, symmetric, and uniformly positive definite, *i.e.*,

$$0 < K_* \le \frac{\boldsymbol{\xi}^T \mathbf{K}(x)\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \le K^* < \infty, \quad x \in \Omega,\ 0 \ne \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2.10}$$

We denote $\mathbf{K} = (\mathbf{K}^{\frac{1}{2}})^T \mathbf{K}^{\frac{1}{2}}$. The external volumetric flow rates F_n and F_w belong to $L^\infty(J; L^2(\Omega))$, and $|\Gamma_D| \ne \emptyset$.

(A2) The phase mobility $\lambda_\alpha(S)$ is a Lipschitz continuous function in S and satisfies that $0 < \lambda^w_* \le \lambda_w(S) \le \lambda^*_w < \infty$ and $0 < \lambda^n_* \le \lambda_n(S) \le \lambda^*_n < \infty$ for any $S \in [0,1]$.

(A3) For any $S \in [0,1]$, $-p_c'$ is a Lipschitz continuous function in S and satisfies that $0 < p^c_* \le -p_c'(S) \le p^*_c < \infty$.

(A4) There exists a positive constant ϵ such that

$$\max\left\{\sqrt{\frac{\lambda_n}{-\lambda_t p_c'}}(1 - p_c')\right\} \le \epsilon < 2.$$

(A5) $S_0 \in W^{1,\infty}(\Omega)$ and $0 \le S_0(x) \le 1$ a.e. in Ω. On the inflow and outflow boundaries, S_D and p_D are two constants independent of the time.

Now we are ready to introduce the weak formulation of the coupled system (2.5)–(2.9): find $S \in L^2(J; H^1_D(\Omega)) + S_D$ with $\partial_t S \in L^2(J; H_D^{-1}(\Omega))$ and $p \in L^2(J; H^1_D(\Omega)) + p_D$ such that

$$\int_0^T \langle \phi \partial_t S, v\rangle \,\mathrm{d}t + \int_0^T (\lambda_w \mathbf{K}\nabla p, \nabla v)\,\mathrm{d}t = \int_0^T (F_w, v)\,\mathrm{d}t, \tag{2.11a}$$
$$\int_0^T (\lambda_t \mathbf{K}\nabla p, \nabla w) + (\lambda_n \mathbf{K}\nabla p_c, \nabla w)\,\mathrm{d}t = \int_0^T (F_t, w)\,\mathrm{d}t, \tag{2.11b}$$

for any $v, w \in L^2(J; H^1_D(\Omega))$ and $S(0,\cdot) = S_0$, where $H^1_D(\Omega) := \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$, $H_D^{-1}(\Omega)$ is the dual space of $H^1_D(\Omega)$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between spaces $H^1_D(\Omega)$ and $H_D^{-1}(\Omega)$, and $(\cdot,\cdot)$ denotes the standard inner product in $L^2(\Omega)$. By Theorem 2.1 of [10], we know that under assumptions (A1)–(A5) the weak formulation (2.11) has a weak solution.

Remark 2.1. If we introduce the global and complementary pressures and view them as unknowns, the boundedness of λ_α ($\alpha \in \{w, n\}$) and $-p_c'$ stated in assumptions (A2) and (A3) is not necessary on the existence of weak solutions, see [10] for more detail. However, in the practice, the complementary pressure has no physical meaning and it is difficult to use it to solve the saturation. Therefore in this paper we consider the pressure-saturation formulation (2.11) under assumptions (A2) and (A3).

Remark 2.2. Assumption (A4) stated as above is similar with H6 of [37]. Moreover, the assumption is seemingly necessary in the stability analysis, since one can find that it is difficult to derive an appropriate bound for $\|S\|_{L^2(J;H^1(\Omega))}$ and $\|p\|_{L^2(J;H^1(\Omega))}$ directly by the weak formulation (2.11) if assumption (A4) is not satisfied. By the way, in order to obtain assumption (A4), we can make $\frac{\lambda_w}{\lambda_n}$ to be sufficiently large, or set $-p_c'$ to be an appropriate positive constant (for example, $-p_c' = 1$). For the first situation, one may require that the wetting-phase saturation S has a lower bound that is far away from zero. For the second situation, one can observe that the capillary pressure p_c is a linear function of the wetting-phase saturation S. This type of dependency holds, however, only if measurements are carried out under equilibrium conditions. In other word, between two successive measurements the fluids have sufficient time to redistribute inside the medium.

3. Numerical discretization

Let $\mathcal{T}_h$ be a conforming and shape-regular triangulation of the domain Ω. Denote by $\mathcal{E}_h^o$ the set of all interior edges/faces of $\mathcal{T}_h$ and $\mathcal{E}_h^\partial$ the set of all boundary edges/faces. We define $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial$, $\mathcal{E}_h^\beta = \{F \in \mathcal{E}_h : F \subset \Gamma_\beta\}$ for $\beta = N, D$, and $\partial\mathcal{T}_h = \{\partial K : K \in \mathcal{T}_h\}$, where ∂K denotes the boundary of the element K. For each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$, let h_K and h_F be the diameters of K and F. Moreover we define $h = \max_{K\in\mathcal{T}_h} h_K$.

Based on the triangulation $\mathcal{T}_h$, we define a mesh-dependent seminorm:

$$|||(v,\mu)|||^2 = \|\nabla v\|^2_{L^2(\mathcal{T}_h)} + \sum_{K\in\mathcal{T}_h} h_K^{-1}\|v-\mu\|^2_{L^2(\partial K)},$$

and discontinuous finite element spaces:

$$V_h := \big\{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}^k(K),\ \forall K \in \mathcal{T}_h\big\},$$
$$\widehat{V}_h := \big\{\widehat{v}_h \in L^2(\mathcal{E}_h) : \widehat{v}_h|_F \in \mathcal{P}^k(F)\ \forall F \in \mathcal{E}_h,\ \widehat{v}_h|_F = 0\ \forall F \in \mathcal{E}_h^D\big\},$$

for $k \geq 1$, where $\mathcal{P}^k(\mathcal{D})$ denotes the set of polynomials of degree no larger than k on the domain $\mathcal{D} \subset \Omega$.

Associated with each mesh $\mathcal{T}_h$, we consider a partition $\{t_h^n\}_{n=0}^{N_h}$ of the time interval $[0,T]$ such that $0 = t_h^0 < t_h^1 < \cdots < t_h^{N_h} = T$ and $\tau_h^n = t_h^n - t_h^{n-1}$. In addition, we assume that the time partition is quasi-uniform, that is, there exists $\vartheta \in (0,1]$ such that

$$\vartheta \max_{1\leq n\leq N_h} \tau_h^n \leq \min_{1\leq n\leq N_h} \tau_h^n.$$

In this paper, we adopt a backward Euler method for the time discretization and an HDG method for the spatial discretization. Then the approximation scheme of the coupled system (2.11) reads as follows: for the given $S_h^{n-1} \in V_h$ and $1 \leq n \leq N_h$, find $(S_h^n, \widehat{S}_h^n) \in V_h \times (\widehat{V}_h + S_D)$ and $(p_h^n, \widehat{p}_h^n) \in V_h \times (\widehat{V}_h + p_D)$ such that

$$\left(\phi\frac{S_h^n - S_h^{n-1}}{\tau_h^n}, v_h\right) + a_w(S_h^n; (p_h^n, \widehat{p}_h^n), (v_h, \widehat{v}_h)) = \left(\overline{F}_w^n, v_h\right), \tag{3.1a}$$

$$a_t(S_h^n; (p_h^n, \widehat{p}_h^n), (w_h, \widehat{w}_h)) + b_n\Big(S_h^n; \Big(S_h^n, \widehat{S}_h^n\Big), (w_h, \widehat{w}_h)\Big) = \Big(\overline{F}_t^n, w_h\Big), \tag{3.1b}$$

for any $(v_h, \widehat{v}_h), (w_h, \widehat{w}_h) \in V_h \times \widehat{V}_h$ and $(S_h^0, \widehat{S}_h^0) = (\Pi_k S_0, \widehat{\Pi}_k S_0)$, where $\Pi_k|_K$ and $\widehat{\Pi}_k|_F$ denote the L^2-projection operators onto $\mathcal{P}^k(K)$ and $\mathcal{P}^k(F)$ for each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h$,

$$\overline{F}_\beta^n = \frac{1}{\tau_h^n}\int_{t_h^{n-1}}^{t_h^n} F_\beta(t,\cdot)\,\mathrm{d}t,$$

and

$$\begin{aligned}
a_\beta(V;(p_h^n,\widehat{p}_h^n),(v_h,\widehat{v}_h)) &= (\lambda_\beta(V)\mathbf{K}\nabla p_h^n,\nabla v_h)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h}\delta_\beta^p h_K^{-1}\langle p_h^n-\widehat{p}_h^n, v_h-\widehat{v}_h\rangle_{\partial K}\\
&\quad - \langle(\lambda_\beta(V)\mathbf{K}\nabla p_h^n)\cdot\mathbf{n}, v_h-\widehat{v}_h\rangle_{\partial\mathcal{T}_h} - \langle(\lambda_\beta(V)\mathbf{K}\nabla v_h)\cdot\mathbf{n}, p_h^n-\widehat{p}_h^n\rangle_{\partial\mathcal{T}_h},\\
b_n\Big(V;\Big(S_h^n,\widehat{S}_h^n\Big),(v_h,\widehat{v}_h)\Big) &= (\lambda_n(V)p_c'(V)\mathbf{K}\nabla S_h^n,\nabla v_h)_{\mathcal{T}_h} - \sum_{K\in\mathcal{T}_h}\delta_n^S h_K^{-1}\Big\langle S_h^n-\widehat{S}_h^n, v_h-\widehat{v}_h\Big\rangle_{\partial K}\\
&\quad - \langle(\lambda_n(V)p_c'(V)\mathbf{K}\nabla S_h^n)\cdot\mathbf{n}, v_h-\widehat{v}_h\rangle_{\partial\mathcal{T}_h} - \Big\langle(\lambda_n(V)p_c'(V)\mathbf{K}\nabla v_h)\cdot\mathbf{n}, S_h^n-\widehat{S}_h^n\Big\rangle_{\partial\mathcal{T}_h},
\end{aligned}$$

for $\beta\in\{w,n,t\}$. Here $\delta_n^S,\delta_\beta^p>0$ are stabilization parameters with $\delta_t^p=\delta_w^p+\delta_n^p$,

$$(\cdot,\cdot)_D=\sum_{K\in D}(\cdot,\cdot)_K \quad\text{and}\quad \langle\cdot,\cdot\rangle_{\partial D}=\sum_{K\in D}\langle\cdot,\cdot\rangle_{\partial K},$$

for any $D\subset\mathcal{T}_h$, where $(\cdot,\cdot)_K$ and $\langle\cdot,\cdot\rangle_{\partial K}$ denote the standard inner products in $L^2(K)$ and $L^2(\partial K)$ respectively.

Next we are going to study the existence of discrete solutions of the coupled system (3.1). Before this, we extend λ_w, λ_n, and $-p_c'$ with the constant outside the interval $[0,1]$. More specifically, we set $\lambda_w(S)=\lambda_*^w$, $\lambda_n(S)=\lambda_*^n$, and $-p_c'(S)=p_*^c$ if $S<0$, and $\lambda_w(S)=\lambda_w^*$, $\lambda_n(S)=\lambda_n^*$, and $-p_c'(S)=p_c^*$ if $S>1$. Furthermore we provide the following well-known properties.

Lemma 3.1 ([5], Lem. 4.5.3, [29], (2.3), and [39], Lem. 3.1)**.** *For each $D\in\mathcal{T}_h$ or $D\in\mathcal{E}_h$,*

$$\|v\|_{W^{s,l}(D)}\le C_v h_D^{m-s+\frac{d_D}{l}-\frac{d_D}{q}}\|v\|_{W^{m,q}(D)},\quad 1\le q\le l\le\infty,\ 0\le m\le s\le k+1, \tag{3.2}$$

for any $v\in\mathcal{P}^k(D)$, where h_D and d_D denote the diameter and dimension of the domain D. Moreover for each $K\in\mathcal{T}_h$ we have

$$\|w\|_{L^s(\partial K)}\le C_t\Big(h_K^{-\frac{1}{s}}\|w\|_{L^s(K)}+h_K^{1-\frac{1}{s}}\|\nabla w\|_{L^s(K)}\Big),\quad 1\le s\le\infty, \tag{3.3}$$

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{H^{-1/2}(\partial K)}\le C_t\big(\|\boldsymbol{v}\|_{L^2(K)}+h_K\|\nabla\cdot\boldsymbol{v}\|_{L^2(K)}\big) \tag{3.4}$$

for any $w\in W^{1,s}(K)$ and $\boldsymbol{v}\in H(div,K)$, where $H(div,\mathcal{D})=\{\boldsymbol{v}\in(L^2(\mathcal{D}))^d:\nabla\cdot\boldsymbol{v}\in L^2(\mathcal{D})\}$ for $\mathcal{D}\subset\Omega$.

Lemma 3.2 ([18], Thm. 2.1)**.** *If $1\le q<\infty$ for $d=2$ and $1\le q\le 6$ for $d=3$, we have*

$$\|v_h\|_{L^q(\Omega)}\le C_e|||(v_h,\widehat{v}_h)|||,\quad \forall(v_h,\widehat{v}_h)\in V_h\times\widehat{V}_h. \tag{3.5}$$

Now we want to use the Brouwer fixed-point theorem ([51], Prop. 2.6) to show the existence of discrete solutions of the coupled system (3.1). To this end, we consider $\mathcal{F}_h$ as a map on $V_h\times(\widehat{V}_h+S_D)$ such that for any $(\Phi_h,\widehat{\Phi}_h)\in V_h\times(\widehat{V}_h+S_D)$, $\mathcal{F}_h(\Phi_h,\widehat{\Phi}_h)=(S_h,\widehat{S}_h)$ is a part of the solution $(S_h,\widehat{S}_h,p_h,\widehat{p}_h)\in V_h\times(\widehat{V}_h+S_D)\times V_h\times(\widehat{V}_h+p_D)$ of

$$\Big(\phi\frac{S_h-S_h^{n-1}}{\tau_h^n},v_h\Big)+a_w(\Phi_h;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))=\Big(\overline{F}_w^n,v_h\Big), \tag{3.6a}$$

$$a_t(\Phi_h;(p_h,\widehat{p}_h),(w_h,\widehat{w}_h))+b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)=\Big(\overline{F}_t^n,w_h\Big), \tag{3.6b}$$

for any $(v_h,\widehat{v}_h),(w_h,\widehat{w}_h)\in V_h\times\widehat{V}_h$.

Lemma 3.3. *For $\beta\in\{w,n,t\}$, let the stabilization parameters δ_β^p and δ_n^S be constants such that δ_n^S is large enough and δ_w^p is far greater than δ_n^S. Then if assumptions (A1)–(A4) hold, the problem (3.6) has a unique solution.*

Proof. For any given $\Phi_h \in V_h$, it is obvious that the problem (3.6) is a linear system. Thus we only need to prove that the problem (3.6) just has a zero solution if $S_D = p_D = \overline{F}_w^n = \overline{F}_t^n = S_h^{n-1} = 0$.

Sum the two equations in (3.6) and set $(v_h, \widehat{v}_h) = (S_h, \widehat{S}_h)$ and $(w_h, \widehat{w}_h) = (p_h - S_h, \widehat{p}_h - \widehat{S}_h)$ to infer that

$$\begin{aligned}\frac{1}{\tau_h^n}\left\|\sqrt{\phi}S_h\right\|^2_{L^2(\Omega)} &+ a_t(\Phi_h;(p_h,\widehat{p}_h),(p_h,\widehat{p}_h)) - b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),\Big(S_h,\widehat{S}_h\Big)\Big)\\ &= a_n\Big(\Phi_h;(p_h,\widehat{p}_h),\Big(S_h,\widehat{S}_h\Big)\Big) - b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),(p_h,\widehat{p}_h)\Big).\end{aligned} \tag{3.7}$$

According to the definitions of a_t and b_n, we obtain by Lemma 3.1 that

$$\begin{aligned}\frac{2+\epsilon}{4}\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac{1}{2}}\nabla p_h\right\|^2_{L^2(\mathcal{T}_h)} &+ \left(\delta_t^p - \frac{4(C_t(1+C_v)\lambda^*K^*)^2}{(2-\epsilon)\lambda_*K_*}\right)\sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_h-\widehat{p}_h\|^2_{L^2(\partial K)}\\ &\le a_t(\cdot;(p_h,\widehat{p}_h),(p_h,\widehat{p}_h)),\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\frac{2+\epsilon}{4}\left\|\sqrt{-\lambda_n p_c'}\mathbf{K}^{\frac{1}{2}}\nabla S_h\right\|^2_{L^2(\mathcal{T}_h)} &+ \left(\delta_n^S - \frac{4(C_t(1+C_v)\lambda_n^*p_c^*K^*)^2}{(2-\epsilon)\lambda_*^n p_*^c K_*}\right)\sum_{K\in\mathcal{T}_h} h_K^{-1}\|S_h-\widehat{S}_h\|^2_{L^2(\partial K)}\\ &\le -b_n\Big(\cdot;\Big(S_h,\widehat{S}_h\Big),\Big(S_h,\widehat{S}_h\Big)\Big),\end{aligned} \tag{3.9}$$

where $\lambda^* = \max \lambda_t$ and $\lambda_* = \min \lambda_t$. Similarly, by Lemma 3.1 we also have

$$\begin{aligned}&a_n\Big(\cdot;(p_h,\widehat{p}_h),\Big(S_h,\widehat{S}_h\Big)\Big) - b_n\Big(\cdot;\Big(S_h,\widehat{S}_h\Big),(p_h,\widehat{p}_h)\Big)\\ &\le \epsilon\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac{1}{2}}\nabla p_h\right\|_{L^2(\mathcal{T}_h)}\left\|\sqrt{-\lambda_n p_c'}\mathbf{K}^{\frac{1}{2}}\nabla S_h\right\|_{L^2(\mathcal{T}_h)} + \left(\delta_n^p+\delta_n^S\right)\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_h-\widehat{p}_h\|^2_{L^2(\partial K)}\right)^{1/2}\\ &\quad\times\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|S_h-\widehat{S}_h\|^2_{L^2(\partial K)}\right)^{1/2} + C_t(1+C_v)\lambda_n^*K^*(1+p_c^*)\|\nabla p_h\|_{L^2(\mathcal{T}_h)}\\ &\quad\times\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|S_h-\widehat{S}_h\right\|^2_{L^2(\partial K)}\right)^{1/2} + C_t(1+C_v)\lambda_n^*K^*(1+p_c^*)\|\nabla S_h\|_{L^2(\Omega)}\\ &\quad\times\left(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_h-\widehat{p}_h\|^2_{L^2(\partial K)}\right)^{1/2}.\end{aligned} \tag{3.10}$$

Thus by (3.7)–(3.10) we obtain that

$$\begin{aligned}&\frac{1}{\tau_h^n}\left\|\sqrt{\phi}S_h\right\|^2_{L^2(\Omega)} + \frac{2-\epsilon}{8}\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac{1}{2}}\nabla p_h\right\|^2_{L^2(\mathcal{T}_h)} + \left(\delta_t^p - \frac{\delta_n^p+\delta_n^S}{2} - \frac{2(C_t(1+C_v)K^*)^2}{(2-\epsilon)K_*}\left(\frac{2(\lambda^*)^2}{\lambda_*}\right.\right.\\ &\quad\left.\left.+\frac{(\lambda_n^*(1+p_c^*))^2}{\lambda_*^n p_*^c}\right)\right)\sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_h-\widehat{p}_h\|^2_{L^2(\partial K)} + \frac{2-\epsilon}{8}\left\|\sqrt{-\lambda_n p_c'}\mathbf{K}^{\frac{1}{2}}\nabla S_h\right\|^2_{L^2(\mathcal{T}_h)}\\ &\quad+\left(\frac{\delta_n^S-\delta_n^p}{2} - \frac{2(C_t(1+C_v)\lambda_n^*K^*)^2}{(2-\epsilon)K_*}\left(\frac{(1+p_c^*)^2}{\lambda_*} + \frac{2(p_c^*)^2}{\lambda_*^n p_*^c}\right)\right)\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|S_h-\widehat{S}_h\right\|^2_{L^2(\partial K)} \le 0.\end{aligned}$$

Then we finish the proof. □

Remark 3.4. From the proof of Lemma 3.3, one can observe that the assumption (A4) is mainly used to ensure that the first term on the right-hand side of (3.10) can be absorbed to the left-hand side of (3.7). In fact, by using the inverse estimate in Lemma 3.1 we have

$$\epsilon\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac{1}{2}}\nabla p_h\right\|_{L^2(\mathcal{T}_h)}\left\|\sqrt{-\lambda_n p_c'}\mathbf{K}^{\frac{1}{2}}\nabla S_h\right\|_{L^2(\mathcal{T}_h)} \le \frac{\epsilon}{4}\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac{1}{2}}\nabla p_h\right\|^2_{L^2(\mathcal{T}_h)} + \frac{\epsilon\lambda_n^* p_c^* K^* C_v^2}{\underline{h}^2\phi_*}\left\|\sqrt{\phi}S_h\right\|^2_{L^2(\Omega)}.$$

Thus if $\frac{\tau_h^n}{\underline{h}^2}$ is sufficiently small, by the above inequality we know that the first term on the right-hand side of (3.10) still can be kicked to the left-hand side of (3.7) without using the assumption (A4), hence the result stated in Lemma 3.3 is still correct. Here $\underline{h} = \min_{K\in\mathcal{T}_h} h_K$.

In the following, we will prove that $\mathcal{F}_h$ is a map on a nonempty, closed, and convex subset. To this end, we consider an auxiliary problem

$$-\Delta\psi = 0 \quad \text{in } \Omega, \tag{3.11a}$$
$$\psi = g \quad \text{on } \Gamma_D, \tag{3.11b}$$
$$\nabla\psi\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_N, \tag{3.11c}$$

for any constant g. It is well-known (see [33], Thm. 1.6 and [49], Thm. 2.7) that the problem (3.11) has a unique solution $\psi \in H^1(\Omega)$ satisfying

$$\|\psi\|_{H^1(\Omega)} \le C\|g\|_{L^\infty(\Gamma_D)}. \tag{3.12}$$

Let $(\psi_h, \widehat{\psi}_h) \in V_h \times (\widehat{V}_h + g)$ be the HDG approximation of ψ such that

$$a\Big(\Big(\psi_h, \widehat{\psi}_h\Big), (v_h, \widehat{v}_h)\Big) = 0, \quad \forall (v_h, \widehat{v}_h) \in V_h \times \widehat{V}_h, \tag{3.13}$$

where

$$\begin{aligned} a\Big(\Big(\psi_h, \widehat{\psi}_h\Big), (v_h, \widehat{v}_h)\Big) = (\nabla\psi_h, \nabla v_h)_{\mathcal{T}_h} &+ \sum_{K\in\mathcal{T}_h} \delta h_K^{-1}\Big\langle \psi_h - \widehat{\psi}_h, v_h - \widehat{v}_h\Big\rangle_{\partial K} \\ &- \langle \nabla\psi_h\cdot\mathbf{n}, v_h - \widehat{v}_h\rangle_{\partial\mathcal{T}_h} - \Big\langle \nabla v_h\cdot\mathbf{n}, \psi_h - \widehat{\psi}_h\Big\rangle_{\partial\mathcal{T}_h} \end{aligned}$$

and δ is a positive constant denoting the stabilization parameter. From the proof of Lemma 3.3, we know that the bilinear form a is coercive in $V_h \times \widehat{V}_h$ with respect to the seminorm $|||(\cdot,\cdot)|||$ if δ is sufficiently large. Thus the problem (3.13) has a unique solution.

Lemma 3.5. *For any given constant g, let $(\psi_h, \widehat{\psi}_h)$ be the solution of problem (3.13). Then we have*

$$\|\psi_h\|_{L^2(\Omega)} + \Big|\Big|\Big|\Big(\psi_h, \widehat{\psi}_h\Big)\Big|\Big|\Big| \le C_s\|g\|_{L^\infty(\Gamma_D)}, \tag{3.14}$$

where C_s is a positive constant independent of the mesh size.

Proof. Due to $\nabla\psi \in H(\mathrm{div}, \Omega)$, by a simple calculation it is easy to verify that the bilinear form a satisfies the following consistency

$$a\Big(\Big(\psi - \psi_h, \psi - \widehat{\psi}_h\Big), (v_h, \widehat{v}_h)\Big) = 0, \quad \forall (v_h, \widehat{v}_h) \in V_h \times \widehat{V}_h.$$

With the coercivity of a, the above consistency, and Lemma 3.1, we have for any $(\chi, \widehat{\chi}) \in V_h \times (\widehat{V}_h + g)$

$$\Big|\Big|\Big|\Big(\chi - \psi_h, \widehat{\chi} - \widehat{\psi}_h\Big)\Big|\Big|\Big|^2 \le Ca\Big(\Big(\chi - \psi_h, \widehat{\chi} - \widehat{\psi}_h\Big), \Big(\chi - \psi_h, \widehat{\chi} - \widehat{\psi}_h\Big)\Big)$$

$$
\begin{aligned}
&= Ca\Big((\chi-\psi,\widehat{\chi}-\psi),\Big(\chi-\psi_h,\widehat{\chi}-\widehat{\psi}_h\Big)\Big)\\
&\le C|||(\psi-\chi,\psi-\widehat{\chi})|||\,\Big|\!\Big|\!\Big|\Big(\chi-\psi_h,\widehat{\chi}-\widehat{\psi}_h\Big)\Big|\!\Big|\!\Big|\\
&\quad - C\Big\langle\nabla(\chi-\psi)\cdot\mathbf{n},(\chi-\psi_h)-\Big(\widehat{\chi}-\widehat{\psi}_h\Big)\Big\rangle_{\partial\mathcal{T}_h}\\
&\le C\Big(|||(\psi-\chi,\psi-\widehat{\chi})||| + h\|\Delta(\psi-\chi)\|_{L^2(\mathcal{T}_h)}\Big)\Big|\!\Big|\!\Big|\Big(\chi-\psi_h,\widehat{\chi}-\widehat{\psi}_h\Big)\Big|\!\Big|\!\Big|,
\end{aligned}
$$

which, together with the triangle inequality, leads to

$$
\Big|\!\Big|\!\Big|\Big(\psi-\psi_h,\psi-\widehat{\psi}_h\Big)\Big|\!\Big|\!\Big| \le C\big(|||(\psi-\chi,\psi-\widehat{\chi})||| + h\|\Delta(\psi-\chi)\|_{L^2(\mathcal{T}_h)}\big). \tag{3.15}
$$

With the fact that $\Delta\psi=0$, we obtain by using the triangle inequality and setting $\chi=\Pi_1\psi$ and $\widehat{\chi}=\widehat{\Pi}_1\psi$ that

$$
\begin{aligned}
\Big|\!\Big|\!\Big|\Big(\psi_h,\widehat{\psi}_h\Big)\Big|\!\Big|\!\Big| &\le C\Big(\|\nabla\psi\|_{L^2(\Omega)} + \Big|\!\Big|\!\Big|\Big(\psi-\psi_h,\psi-\widehat{\psi}_h\Big)\Big|\!\Big|\!\Big|\Big)\\
&\le C\big(\|\nabla\psi\|_{L^2(\Omega)} + \|\nabla(\psi-\chi)\|_{L^2(\mathcal{T}_h)}\big)\\
&\le C\|\nabla\psi\|_{L^2(\Omega)} \le C\|g\|_{L^\infty(\Gamma_D)}.
\end{aligned} \tag{3.16}
$$

Note that in the derivation of (3.16) we have used the stability of Π_k and the fact that

$$
\begin{aligned}
\|\chi-\widehat{\chi}\|^2_{L^2(\partial K)} &= \langle\chi-\widehat{\chi},\chi-\psi\rangle_{\partial K} \le \|\chi-\widehat{\chi}\|_{L^2(\partial K)}\|\psi-\chi\|_{L^2(\partial K)}\\
&\le C\|\chi-\widehat{\chi}\|_{L^2(\partial K)}\Big(h_K^{-1/2}\|\psi-\chi\|_{L^2(K)} + h_K^{1/2}\|\nabla(\psi-\chi)\|_{L^2(K)}\Big)\\
&\le C\|\chi-\widehat{\chi}\|_{L^2(\partial K)}h_K^{1/2}\|\nabla\psi\|_{L^2(K)}.
\end{aligned} \tag{3.17}
$$

Next we are going to bound $\|\psi_h\|_{L^2(\Omega)}$. By Lemma 3.2 and the triangle inequality we have

$$
\begin{aligned}
\|\psi_h\|_{L^2(\Omega)} &\le \|\psi-\chi\|_{L^2(\Omega)} + \|\chi-\psi_h\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\\
&\le C\Big(\|\psi\|_{H^1(\Omega)} + \Big|\!\Big|\!\Big|\Big(\chi-\psi_h,\widehat{\chi}-\widehat{\psi}_h\Big)\Big|\!\Big|\!\Big|\Big)\\
&\le C\|\psi\|_{H^1(\Omega)} \le C\|g\|_{L^\infty(\Gamma_D)},
\end{aligned}
$$

which, together with (3.16), concludes the proof. □

Now we are ready to prove that $\mathcal{F}_h$ is a map on a nonempty, closed and convex subset.

Lemma 3.6. *Let the map $\mathcal{F}_h$ be defined in (3.6). Let the stabilization parameters δ_β^p and δ_n^S be constants for $\beta\in\{w,n,t\}$ such that δ_n^S is large enough and δ_w^p is far greater than δ_n^S. Then if assumptions (A1)–(A5) hold, $\mathcal{F}_h$ is well-defined in*

$$
\begin{aligned}
\mathcal{S}_h := \Big\{\Big(S_h,\widehat{S}_h\Big)\in V_h\times\Big(\widehat{V}_h+S_D\Big) : C^s_{\mathcal{F}}\Big(\|S_h\|^2_{L^2(\Omega)} + \Big|\!\Big|\!\Big|\Big(S_h,\widehat{S}_h\Big)\Big|\!\Big|\!\Big|^2\Big) \le C^b_{\mathcal{F}}\Big(\Big\|\overline{F}_t^n\Big\|^2_{L^2(\Omega)} + \Big\|\overline{F}_n^n\Big\|^2_{L^2(\Omega)}\\
+ \|p_D\|^2_{L^\infty(\Gamma_D)} + \|S_D\|^2_{L^\infty(\Gamma_D)}\Big) + \frac{1}{\tau_h^n}\phi^*\Big(\big\|S_h^{n-1}\big\|^2_{L^2(\Omega)} + C_s^2\|S_D\|^2_{L^\infty(\Gamma_D)}\Big)\Big\},
\end{aligned}
$$

and $\mathcal{F}_h(\mathcal{S}_h)\subset\mathcal{S}_h$, where $C^s_{\mathcal{F}}$ and $C^b_{\mathcal{F}}$ are two positive constants defined in the proof.

Proof. Let ψ^S and $(\psi_h^S,\widehat{\psi}_h^S)$ be the solutions of problems (3.11) and (3.13) with g replaced by S_D. Let ψ^p and $(\psi_h^p,\widehat{\psi}_h^p)$ be the solutions of problems (3.11) and (3.13) with g replaced by p_D. Then we sum the two equations in (3.6) and set $(v_h,\widehat{v}_h)=(S_h-\psi_h^S,\widehat{S}_h-\widehat{\psi}_h^S)$ and $(w_h,\widehat{w}_h)=(p_h-\psi_h^p-v_h,\widehat{p}_h-\widehat{\psi}_h^p-\widehat{v}_h)$ to derive that

$$
\frac{1}{\tau_h^n}\big(\phi(S_h-S_h^{n-1}),(S_h-\psi_h^S)\big) + a_t(\Phi_h;(p_h,\widehat{p}_h),(p_h,\widehat{p}_h)) - b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),\Big(S_h,\widehat{S}_h\Big)\Big)
$$

$$
\begin{aligned}
&= \Big(a_n\Big(\Phi_h;(p_h,\widehat{p}_h),\Big(S_h-\psi_h^S,\widehat{S}_h-\widehat{\psi}_h^S\Big)\Big)-b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),\Big(p_h-\psi_h^p,\widehat{p}_h-\widehat{\psi}_h^p\Big)\Big)\Big) \\
&\quad+\Big(a_t\Big(\Phi_h;(p_h,\widehat{p}_h),\Big(\psi_h^p,\widehat{\psi}_h^p\Big)\Big)-b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),\Big(\psi_h^S,\widehat{\psi}_h^S\Big)\Big)\Big) \\
&\quad+\Big(\Big(\overline{F}_t^n,p_h-\psi_h^p\Big)-\Big(\overline{F}_n^n,S_h-\psi_h^S\Big)\Big)=\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
\tag{3.18}
$$

Next we estimate I, II, III and the left-hand side of (3.18) separately. Similar with (3.8), (3.9) and (3.10), we have by Lemma 3.1

$$
\begin{aligned}
&\frac{1}{\tau_h^n}\big(\phi\big(S_h-S_h^{n-1}\big),\big(S_h-\psi_h^S\big)\big)+a_t\big(\Phi_h;(p_h,\widehat{p}_h),(p_h,\widehat{p}_h)\big)-b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),\Big(S_h,\widehat{S}_h\Big)\Big) \\
&\quad\ge\frac{1}{2\tau_h^n}\Big(\Big\|\sqrt{\phi}\big(S_h-S_h^{n-1}\big)\Big\|_{L^2(\Omega)}^2+\Big\|\sqrt{\phi}\big(S_h-\psi_h^S\big)\Big\|_{L^2(\Omega)}^2-\Big\|\sqrt{\phi}\big(S_h^{n-1}-\psi_h^S\big)\Big\|_{L^2(\Omega)}^2\Big) \\
&\qquad+\left(\frac{2+\epsilon}{4}\Big\|\sqrt{\lambda_t}\mathbf{K}^{\frac12}\nabla p_h\Big\|_{L^2(\mathcal{T}_h)}^2+\left(\delta_t^p-\frac{4(\lambda^*K^*C_t(1+C_v))^2}{(2-\epsilon)\lambda_*K_*}\right)\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2\right) \\
&\qquad+\left(\frac{2+\epsilon}{4}\Big\|\sqrt{-\lambda_np_c'}\mathbf{K}^{\frac12}\nabla S_h\Big\|_{L^2(\mathcal{T}_h)}^2+\left(\delta_n^S-\frac{4(\lambda_n^*p_c^*K^*C_t(1+C_v))^2}{(2-\epsilon)\lambda_*^np_*^cK_*}\right)\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\|S_h-\widehat{S}_h\Big\|_{L^2(\partial K)}^2\right),
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\mathrm{I}&\le\epsilon\Big\|\sqrt{\lambda_t}\mathbf{K}^{\frac12}\nabla p_h\Big\|_{L^2(\mathcal{T}_h)}\Big\|\sqrt{-\lambda_np_c'}\mathbf{K}^{\frac12}\nabla S_h\Big\|_{L^2(\Omega)}+\big(\delta_n^p+\delta_n^S\big)\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\langle S_h-\widehat{S}_h,p_h-\widehat{p}_h\Big\rangle_{\partial K} \\
&\quad+\lambda_n^*(1+p_c^*)K^*C_t(1+C_v)\Bigg(\|\nabla p_h\|_{L^2(\mathcal{T}_h)}\Bigg(\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\|S_h-\widehat{S}_h\Big\|_{L^2(\partial K)}^2\Bigg)^{1/2} \\
&\quad+\|\nabla S_h\|_{L^2(\mathcal{T}_h)}\Bigg(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2\Bigg)^{1/2}\Bigg)+\Big(\|\lambda_n\mathbf{K}\nabla p_h\|_{L^2(\mathcal{T}_h)}\big\|\nabla\psi_h^S\big\|_{L^2(\mathcal{T}_h)} \\
&\quad+\|(-\lambda_np_c'\mathbf{K})\nabla S_h\|_{L^2(\mathcal{T}_h)}\|\nabla\psi_h^p\|_{L^2(\mathcal{T}_h)}\Big)-\delta_n^p\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\langle p_h-\widehat{p}_h,\psi_h^S-\widehat{\psi}_h^S\Big\rangle_{\partial K} \\
&\quad-\delta_n^S\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\langle S_h-\widehat{S}_h,\psi_h^p-\widehat{\psi}_h^p\Big\rangle_{\partial K}+\lambda_n^*K^*C_t(1+C_v)\Bigg(\|\nabla p_h\|_{L^2(\mathcal{T}_h)} \\
&\quad\times\Bigg(\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\|\psi_h^S-\widehat{\psi}_h^S\Big\|_{L^2(\partial K)}^2\Bigg)^{1/2}+\big\|\nabla\psi_h^S\big\|_{L^2(\mathcal{T}_h)}\Bigg(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2\Bigg)^{1/2}\Bigg) \\
&\quad+\lambda_n^*p_c^*K^*C_t(1+C_v)\Bigg(\|\nabla S_h\|_{L^2(\mathcal{T}_h)}\Bigg(\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\|\psi_h^p-\widehat{\psi}_h^p\Big\|_{L^2(\partial K)}^2\Bigg)^{1/2} \\
&\quad+\|\nabla\psi_h^p\|_{L^2(\mathcal{T}_h)}\Bigg(\sum_{K\in\mathcal{T}_h}h_K^{-1}\Big\|S_h-\widehat{S}_h\Big\|_{L^2(\partial K)}^2\Bigg)^{1/2}\Bigg) \\
&\le\left(\frac{\epsilon}{2}+\frac{3(2-\epsilon)}{40}\right)\Big\|\sqrt{\lambda_t}\mathbf{K}^{\frac12}\nabla p_h\Big\|_{L^2(\mathcal{T}_h)}+\left(\frac{3\delta_n^p+2\delta_n^S+1}{4}+\frac{10(\lambda_n(1+p_c^*)K^*C_t(1+C_v))^2}{(2-\epsilon)\lambda_*^np_*^cK_*}\right) \\
&\quad\times\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2+\left(\frac{\epsilon}{2}+\frac{3(2-\epsilon)}{40}\right)\Big\|\sqrt{-\lambda_np_c'}\mathbf{K}^{\frac12}\nabla S_h\Big\|_{L^2(\mathcal{T}_h)}^2
\end{aligned}
\tag{3.20}
$$

$$+\left(\frac{4\delta_n^p+5\delta_n^S+2}{8}+\frac{10(\lambda_n^*(1+p_c^*)K^*C_t(1+C_v))^2}{(2-\epsilon)\lambda_*K_*}\right)\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|S_h-\widehat{S}_h\right\|_{L^2(\partial K)}^2$$
$$+C_1\left(\left|\!\left|\!\left|(\psi_h^S,\widehat{\psi}_h^S)\right|\!\right|\!\right|^2+\left|\!\left|\!\left|\left(\psi_h^p,\widehat{\psi}_h^p\right)\right|\!\right|\!\right|^2\right),$$

and

$$\begin{aligned}
\mathrm{II}\le{}&\|\lambda_t\mathbf{K}\nabla p_h\|_{L^2(\mathcal{T}_h)}\|\nabla\psi_h^p\|_{L^2(\mathcal{T}_h)}+\delta_t^p\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\langle p_h-\widehat{p}_h,\psi_h^p-\widehat{\psi}_h^p\right\rangle_{\partial K}\\
&+\lambda^*K^*C_t(1+C_v)\left(\|\nabla p_h\|_{L^2(\mathcal{T}_h)}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|\psi_h^p-\widehat{\psi}_h^p\right\|_{L^2(\partial K)}^2\right)^{1/2}\right.\\
&\left.+\|\nabla\psi_h^p\|_{L^2(\mathcal{T}_h)}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2\right)^{1/2}\right)\\
&+\|(-\lambda_np_c'\mathbf{K})\nabla S_h\|_{L^2(\mathcal{T}_h)}\left\|\nabla\psi_h^S\right\|_{L^2(\mathcal{T}_h)}+\delta_n^S\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\langle S_h-\widehat{S}_h,\psi_h^S-\widehat{\psi}_h^S\right\rangle_{\partial K}\qquad(3.21)\\
&+\lambda_n^*p_c^*K^*C_t(1+C_v)\left(\|\nabla S_h\|_{L^2(\mathcal{T}_h)}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|\psi_h^S-\widehat{\psi}_h^S\right\|_{L^2(\partial K)}^2\right)^{1/2}\right.\\
&\left.+\left\|\nabla\psi_h^S\right\|_{L^2(\mathcal{T}_h)}\left(\sum_{K\in\mathcal{T}_h}h_K^{-1}\|S_h-\widehat{S}_h\|_{L^2(\partial K)}^2\right)^{1/2}\right)\\
\le{}&\frac{2-\epsilon}{20}\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac12}\nabla p_h\right\|_{L^2(\mathcal{T}_h)}^2+\frac{\delta_t^p+1}{4}\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2+\frac{2-\epsilon}{20}\left\|\sqrt{-\lambda_np_c'}\mathbf{K}^{\frac12}\nabla S_h\right\|_{L^2(\mathcal{T}_h)}^2\\
&+\frac{\delta_n^S+2}{8}\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|S_h-\widehat{S}_h\right\|_{L^2(\partial K)}^2+C_2\left(\left|\!\left|\!\left|\left(\psi_h^S,\widehat{\psi}_h^S\right)\right|\!\right|\!\right|^2+\left|\!\left|\!\left|\left(\psi_h^p,\widehat{\psi}_h^p\right)\right|\!\right|\!\right|^2\right),
\end{aligned}$$

where C_1 and C_2 are two positive constants depending on λ_n^*, λ_*^n, K^*, K_*, λ^*, λ_*, p_c^*, p_*^c, C_t, C_v and the stabilization parameters. Then by (3.20) and (3.21) and Young's inequality, we obtain that

$$\begin{aligned}
\mathrm{I}+\mathrm{II}\le{}&\frac{2+3\epsilon}{8}\left(\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac12}\nabla p_h\right\|_{L^2(\mathcal{T}_h)}^2+\left\|\sqrt{-\lambda_np_c'}\mathbf{K}^{\frac12}\nabla S_h\right\|_{L^2(\mathcal{T}_h)}^2\right)+\left(\frac{\delta_t^p+3\delta_n^p+2\delta_n^S+2}{4}\right.\\
&\left.+\frac{10(\lambda_n^*(1+p_c^*)K^*C_t(1+C_v))^2}{(2-\epsilon)\lambda_*^np_*^cK_*}\right)\sum_{K\in\mathcal{T}_h}h_K^{-1}\|p_h-\widehat{p}_h\|_{L^2(\partial K)}^2+\left(\frac{2\delta_n^p+3\delta_n^S+2}{4}\right.\\
&\left.+\frac{10(\lambda_n^*(1+p_c^*)K^*C_t(1+C_v))^2}{(2-\epsilon)\lambda_*K_*}\right)\sum_{K\in\mathcal{T}_h}h_K^{-1}\left\|S_h-\widehat{S}_h\right\|_{L^2(\partial K)}^2\\
&+(C_1+C_2)\left(\left|\!\left|\!\left|\left(\psi_h^S,\widehat{\psi}_h^S\right)\right|\!\right|\!\right|^2+\left|\!\left|\!\left|\left(\psi_h^p,\widehat{\psi}_h^p\right)\right|\!\right|\!\right|^2\right),
\end{aligned}$$

which, together with (3.18), (3.19) and Lemma 3.5, yields

$$\begin{aligned}
&\frac{1}{2\tau_h^n}\left(\left\|\sqrt{\phi}(S_h-S_h^{n-1})\right\|_{L^2(\Omega)}^2+\left\|\sqrt{\phi}(S_h-\psi_h^S)\right\|_{L^2(\Omega)}^2-\left\|\sqrt{\phi}(S_h^{n-1}-\psi_h^S)\right\|_{L^2(\Omega)}^2\right)\\
&+\frac{2-\epsilon}{8}\left(\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac12}\nabla p_h\right\|_{L^2(\mathcal{T}_h)}^2+\left\|\sqrt{-\lambda_np_c'}\mathbf{K}^{\frac12}\nabla S_h\right\|_{L^2(\mathcal{T}_h)}^2\right)+\left(\frac{3\delta_w^p-2\delta_n^S-2}{4}\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{2(C_t(1+C_v)K^*)^2}{(2-\epsilon)K_*}\left(\frac{2(\lambda^*)^2}{\lambda_*} + \frac{5(\lambda_n^*(1+p_c^*))^2}{\lambda_*^n p_*^c}\right)\Bigg) \sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_h - \widehat{p}_h\|^2_{L^2(\partial K)} \\
&+ \left(\frac{\delta_n^S - 2\delta_n^p - 2}{4} - \frac{2(\lambda_n^* K^* C_t(1+C_v))^2}{(2-\epsilon)K_*}\left(\frac{2(p_c^*)^2}{\lambda_*^n p_*^c} + \frac{5(1+p_c^*)^2}{\lambda_*}\right)\right) \sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|S_h - \widehat{S}_h\right\|^2_{L^2(\partial K)} \\
&\le (C_1+C_2)C_s^2\left(\|S_D\|^2_{L^\infty(\Gamma_D)} + \|p_D\|^2_{L^\infty(\Gamma_D)}\right) + \mathrm{III}.
\end{aligned} \tag{3.22}
$$

Now we turn to estimate the term III. By Cauchy–Schwarz inequality, Lemmas 3.2 and 3.5, we have

$$
\begin{aligned}
\mathrm{III} &\le \left\|\overline{F}_t^n\right\|_{L^2(\Omega)}\left(\|p_h\|_{L^2(\Omega)} + \|\psi_h^p\|_{L^2(\Omega)}\right) + \left\|\overline{F}_n^n\right\|_{L^2(\Omega)}\left(\|S_h\|_{L^2(\Omega)} + \|\psi_h^S\|_{L^2(\Omega)}\right) \\
&\le \left\|\overline{F}_t^n\right\|_{L^2(\Omega)}\left(\|p_h - \psi_h^p\|_{L^2(\Omega)} + 2\|\psi_h^p\|_{L^2(\Omega)}\right) + \left\|\overline{F}_n^n\right\|_{L^2(\Omega)}\left(\|S_h - \psi_h^S\|_{L^2(\Omega)} + 2\|\psi_h^S\|_{L^2(\Omega)}\right) \\
&\le \left\|\overline{F}_t^n\right\|_{L^2(\Omega)}\left(C_e\left|\!\left|\!\left|\left(p_h - \psi_h^p, \widehat{p}_h - \widehat{\psi}_h^p\right)\right|\!\right|\!\right| + 2\|\psi_h^p\|_{L^2(\Omega)}\right) \\
&\quad + \left\|\overline{F}_n^n\right\|_{L^2(\Omega)}\left(C_e\left|\!\left|\!\left|\left(S_h - \psi_h^S, \widehat{S}_h - \widehat{\psi}_h^S\right)\right|\!\right|\!\right| + 2\|\psi_h^S\|_{L^2(\Omega)}\right) \\
&\le \left\|\overline{F}_t^n\right\|_{L^2(\Omega)}\left(C_e|\!|\!|(p_h, \widehat{p}_h)|\!|\!| + (2+C_e)C_s\|p_D\|_{L^\infty(\Gamma_D)}\right) \\
&\quad + \left\|\overline{F}_n^n\right\|_{L^2(\Omega)}\left(C_e\left|\!\left|\!\left|\left(S_h, \widehat{S}_h\right)\right|\!\right|\!\right| + (2+C_e)C_s\|S_D\|_{L^\infty(\Gamma_D)}\right) \\
&\le \frac{2-\epsilon}{16}\left(\left\|\sqrt{\lambda_t}\mathbf{K}^{\frac{1}{2}}\nabla p_h\right\|^2_{L^2(\mathcal{T}_h)} + \left\|\sqrt{-\lambda_n p_c'}\mathbf{K}^{\frac{1}{2}}\nabla S_h\right\|^2_{L^2(\mathcal{T}_h)}\right) + \frac{1}{4}\Bigg(\sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_h - \widehat{p}_h\|^2_{L^2(\partial K)} \\
&\quad + \sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|S_h - \widehat{S}_h\right\|^2_{L^2(\partial K)}\Bigg) + C_3\left(\left\|\overline{F}_t^n\right\|^2_{L^2(\Omega)} + \left\|\overline{F}_n^n\right\|^2_{L^2(\Omega)} + \|S_D\|^2_{L^\infty(\Gamma_D)} + \|p_D\|^2_{L^\infty(\Gamma_D)}\right),
\end{aligned} \tag{3.23}
$$

where C_3 is a positive constant depending on λ_*^n, λ_*, p_*^c, K_*, C_e and C_s, which, together with (3.22), leads to

$$
\begin{aligned}
&\frac{1}{2\tau_h^n}\left(\left\|\sqrt{\phi}(S_h - S_h^{n-1})\right\|^2_{L^2(\Omega)} + \left\|\sqrt{\phi}(S_h - \psi_h^S)\right\|^2_{L^2(\Omega)} - \left\|\sqrt{\phi}(S_h^{n-1} - \psi_h^S)\right\|^2_{L^2(\Omega)}\right) \\
&\quad + C_4\left(|\!|\!|(p_h, \widehat{p}_h)|\!|\!|^2 + \left|\!\left|\!\left|\left(S_h, \widehat{S}_h\right)\right|\!\right|\!\right|^2\right) \le \left(C_3 + (C_1+C_2)C_s^2\right)\left(\left\|\overline{F}_t^n\right\|^2_{L^2(\Omega)} + \left\|\overline{F}_n^n\right\|^2_{L^2(\Omega)}\right. \\
&\quad \left. + \|p_D\|^2_{L^\infty(\Gamma_D)} + \|S_D\|^2_{L^\infty(\Gamma_D)}\right),
\end{aligned} \tag{3.24}
$$

where

$$
\begin{aligned}
C_4 := \min\Bigg\{&\frac{(2-\epsilon)\lambda_* K_*}{16}, \frac{3\delta_w^p - 2\delta_n^S - 3}{4} - \frac{2(C_t(1+C_v)K^*)^2}{(2-\epsilon)K_*}\left(\frac{2(\lambda^*)^2}{\lambda_*} + \frac{5(\lambda_n^*(1+p_c^*))^2}{\lambda_*^n p_*^c}\right), \\
&\frac{(2-\epsilon)\lambda_*^n p_*^c K_*}{16}, \frac{\delta_n^S - 2\delta_n^p - 3}{4} - \frac{2(\lambda_n^* K^* C_t(1+C_v))^2}{(2-\epsilon)K_*}\left(\frac{2(p_c^*)^2}{\lambda_*^n p_*^c} + \frac{5(1+p_c^*)^2}{\lambda_*}\right)\Bigg\}.
\end{aligned}
$$

Thus by (3.24) and the triangle inequality, we have

$$
C_{\mathcal{F}}^s\left(\|S_h\|^2_{L^2(\Omega)} + \left|\!\left|\!\left|\left(S_h, \widehat{S}_h\right)\right|\!\right|\!\right|^2\right) \le C_{\mathcal{F}}^b\left(\left\|\overline{F}_t^n\right\|^2_{L^2(\Omega)} + \left\|\overline{F}_n^n\right\|^2_{L^2(\Omega)} + \|p_D\|^2_{L^\infty(\Gamma_D)} + \|S_D\|^2_{L^\infty(\Gamma_D)}\right)
$$

$$+\frac{1}{\tau_h^n}\phi^*\Big(\big\|S_h^{n-1}\big\|^2_{L^2(\Omega)}+C_s^2\|S_D\|^2_{L^\infty(\Gamma_D)}\Big),$$

where

$$C_{\mathcal{F}}^s=\min\{C_4,\phi_*\}\quad\text{and}\quad C_{\mathcal{F}}^b=C_3+(C_1+C_2+\phi^*)C_s^2.$$

Then we conclude the proof. □

With Lemmas 3.3 and 3.6 at hand, now we prove the existence of discrete solutions of the coupled system (3.1) by the Brouwer fixed-point theorem in the following.

Theorem 3.7. *For $\beta\in\{w,n,t\}$, let the stabilization parameters δ_β^p and δ_n^S be constants such that δ_n^S is large enough and δ_w^p is far greater than δ_n^S. Then if assumptions (A1)–(A5) hold, the coupled system (3.1) has a solution $(S_h^n,\widehat{S}_h^n,p_h^n,\widehat{p}_h^n)\in\mathcal{S}_h\times V_h\times(\widehat{V}_h+p_D)$ for the given $S_h^{n-1}\in V_h$ and each $1\le n\le N_h$.*

Proof. Obviously, if we want to prove the existence of discrete solutions of the coupled system (3.1), we only need to prove that the map $\mathcal{F}_h$ defined in (3.6) has a fixed point in $\mathcal{S}_h$. According to the Brouwer fixed-point theorem ([51], Prop. 2.6), we know that the remaining task is to prove the continuity of the map $\mathcal{F}_h$ in $\mathcal{S}_h$.

To this end, let $(\Phi_h^m,\widehat{\Phi}_h^m)\in\mathcal{S}_h$ be a sequence converging to $(\Phi_h,\widehat{\Phi}_h)\in\mathcal{S}_h$ as m tends to infinity. In the following, we prove that $(S_h^m,\widehat{S}_h^m)\to(S_h,\widehat{S}_h)$ in $\mathcal{S}_h$ as $m\to\infty$, where $(S_h^m,\widehat{S}_h^m)=\mathcal{F}_h(\Phi_h^m,\widehat{\Phi}_h^m)$ and $(S_h,\widehat{S}_h)=\mathcal{F}_h(\Phi_h,\widehat{\Phi}_h)$. By a simple calculation, we obtain that

$$\begin{aligned}&\frac{1}{\tau_h^n}(\phi(S_h-S_h^m),v_h)+a_w(\Phi_h^m;(p_h-p_h^m,\widehat{p}_h-\widehat{p}_h^m),(v_h,\widehat{v}_h))\\&\quad=a_w(\Phi_h^m;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))-a_w(\Phi_h;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))\quad\forall(v_h,\widehat{v}_h)\in V_h\times\widehat{V}_h,\end{aligned}\tag{3.25}$$

and

$$\begin{aligned}&a_t(\Phi_h^m;(p_h-p_h^m,\widehat{p}_h-\widehat{p}_h^m),(w_h,\widehat{w}_h))+b_n\Big(\Phi_h^m;\Big(S_h-S_h^m,\widehat{S}_h-\widehat{S}_h^m\Big),(w_h,\widehat{w}_h)\Big)\\&\quad=a_t(\Phi_h^m;(p_h,\widehat{p}_h),(w_h,\widehat{w}_h))-a_t(\Phi_h;(p_h,\widehat{p}_h),(w_h,\widehat{w}_h))\\&\qquad+b_n\Big(\Phi_h^m;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)-b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)\quad\forall(w_h,\widehat{w}_h)\in V_h\times\widehat{V}_h.\end{aligned}\tag{3.26}$$

In (3.25) and (3.26), we set $(v_h,\widehat{v}_h)=(S_h-S_h^m,\widehat{S}_h-\widehat{S}_h^m)$ and $(w_h,\widehat{w}_h)=(p_h-p_h^m-v_h,\widehat{p}_h-\widehat{p}_h^m-\widehat{v}_h)$. Then we sum the two equations and know with (3.8)–(3.10) that there exists a positive constant C such that

$$\begin{aligned}&C\Big(\|S_h-S_h^m\|^2_{L^2(\Omega)}+\Big|\Big|\Big|\Big(S_h-S_h^m,\widehat{S}_h-\widehat{S}_h^m\Big)\Big|\Big|\Big|^2+|||(p_h-p_h^m,\widehat{p}_h-\widehat{p}_h^m)|||^2\Big)\\&\quad\le a_w(\Phi_h^m;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))-a_w(\Phi_h;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))\\&\qquad+a_t(\Phi_h^m;(p_h,\widehat{p}_h),(w_h,\widehat{w}_h))-a_t(\Phi_h;(p_h,\widehat{p}_h),(w_h,\widehat{w}_h))\\&\qquad+b_n\Big(\Phi_h^m;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)-b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)=J_1+J_2+J_3.\end{aligned}\tag{3.27}$$

According to the definitions of a_β and b_n for $\beta\in\{w,n,t\}$, we have by Lemma 3.1

$$\begin{aligned}&a_\beta(\Phi_h^m;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))-a_\beta(\Phi_h;(p_h,\widehat{p}_h),(v_h,\widehat{v}_h))\\&\quad\le C\|\Phi_h-\Phi_h^m\|_{L^2(\Omega)}|||(v_h,\widehat{v}_h)|||\Big(\|\nabla p_h\|_{L^\infty(\mathcal{T}_h)}+\underline{h}^{-1}\|p_h-\widehat{p}_h\|_{L^\infty(\partial\mathcal{T}_h)}\Big),\end{aligned}\tag{3.28}$$

and

$$b_n\Big(\Phi_h^m;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)-b_n\Big(\Phi_h;\Big(S_h,\widehat{S}_h\Big),(w_h,\widehat{w}_h)\Big)\tag{3.29}$$

$$\leq C\|\Phi_h - \Phi_h^m\|_{L^2(\Omega)}|||(w_h, \widehat{w}_h)|||\left(\|\nabla S_h\|_{L^\infty(\mathcal{T}_h)} + \underline{h}^{-1}\left\|S_h - \widehat{S}_h\right\|_{L^\infty(\partial\mathcal{T}_h)}\right),$$

where $\underline{h} = \min_{K\in\mathcal{T}_h} h_K$, $\|\cdot\|_{L^\infty(\mathcal{T}_h)} = \max_{K\in\mathcal{T}_h}\|\cdot\|_{L^\infty(K)}$, and $\|\cdot\|_{L^\infty(\partial\mathcal{T}_h)} = \max_{K\in\mathcal{T}_h}\|\cdot\|_{L^\infty(\partial K)}$. Thus

$$J_1 \to 0, \quad J_2 \to 0, \quad J_3 \to 0, \quad \text{as} \quad m \to 0.$$

and this leads to

$$\|S_h - S_h^m\|^2_{L^2(\Omega)} + \left|\left|\left|\left(S_h - S_h^m, \widehat{S}_h - \widehat{S}_h^m\right)\right|\right|\right|^2 \to 0, \quad \text{as} \quad m \to 0.$$

We conclude the proof. □

Remark 3.8. From the proof of Theorem 3.7, one can find that the quasi-uniformity of the time partition $\{t_h^n\}_{n=0}^{N_h}$ has not used. Thus the result presented in Theorem 3.7 are valid for any time partitions if assumptions (A1)–(A5) hold.

4. Convergence

In this section, we are devoted to prove that there exists a subsequence of numerical solutions solved by (3.1) converging to a weak solution of (2.11). Before this, we define piecewise constant functions $(S_{h\tau}, \widehat{S}_{h\tau})$ and $(p_{h\tau}, \widehat{p}_{h\tau})$ on the time partition such that

$$\left(S_{h\tau}, \widehat{S}_{h\tau}\right)|_{[t_h^{n-1}, t_h^n]} = \left(S_h^n, \widehat{S}_h^n\right) \quad \text{and} \quad (p_{h\tau}, \widehat{p}_{h\tau})|_{[t_h^{n-1}, t_h^n]} = (p_h^n, \widehat{p}_h^n), \quad n = 1, 2, \cdots, N_h,$$

and piecewise linear functions $\widetilde{S}_{h\tau}$ and $\widetilde{\widehat{S}}_{h\tau}$ satisfying

$$\widetilde{S}_{h\tau}(t,\cdot)|_{[t_h^{n-1}, t_h^n]} = \frac{t - t_h^{n-1}}{\tau_h^n} S_h^n + \frac{t_h^n - t}{\tau_h^n} S_h^{n-1} \quad \text{and} \quad \widetilde{\widehat{S}}_{h\tau}(t,\cdot)|_{[t_h^{n-1}, t_h^n]} = \frac{t - t_h^{n-1}}{\tau_h^n} \widehat{S}_h^n + \frac{t_h^n - t}{\tau_h^n} \widehat{S}_h^{n-1}.$$

Note that in the subsequent analysis we will use $A \lesssim B$ to denote $A \leq CB$, where C is a generic positive constant independent of the time and spatial discretization parameters. Moreover the value of C might change at each occurrence.

Lemma 4.1. *Let $(S_h^n, \widehat{S}_h^n, p_h^n, \widehat{p}_h^n)$ be the solution of problem (3.1) for $1 \leq n \leq N_h$. Let $(S_{h\tau}, \widehat{S}_{h\tau})$, $(p_{h\tau}, \widehat{p}_{h\tau})$, and $\widetilde{S}_{h\tau}$ be defined as above. Then for any $1 \leq m \leq N_h$, we have*

$$\begin{aligned}
&\sum_{n=1}^{m} \int_{t_h^{n-1}}^{t_h^n} \tau_h^n \left\|\sqrt{\phi}\partial_t \widetilde{S}_{h\tau}\right\|^2_{L^2(\Omega)} \mathrm{d}t + \left\|\sqrt{\phi} S_h^m\right\|^2_{L^2(\Omega)} + \int_0^{t_h^m} \left(\left|\left|\left|\left(S_{h\tau}, \widehat{S}_{h\tau}\right)\right|\right|\right|^2 + |||(p_{h\tau}, \widehat{p}_{h\tau})|||^2\right) \mathrm{d}t \\
&\quad \lesssim \left(\|S_0\|^2_{L^2(\Omega)} + \|S_D\|^2_{L^\infty(\Gamma_D)}\right) + t_h^m \left(\|F_t\|^2_{L^\infty(J;L^2(\Omega))} + \|F_n\|^2_{L^\infty(J;L^2(\Omega))} + \|p_D\|^2_{L^\infty(\Gamma_D)} + \|S_D\|^2_{L^\infty(\Gamma_D)}\right).
\end{aligned}$$

Proof. From the proof of Lemma 3.6, we know that for the given $S_h^{n-1} \in V_h$, $(S_h^n, \widehat{S}_h^n, p_h^n, \widehat{p}_h^n)$ solved by (3.1) satisfies (3.24) (there $(S_h, \widehat{S}_h, p_h, \widehat{p}_h)$ is replaced by $(S_h^n, \widehat{S}_h^n, p_h^n, \widehat{p}_h^n)$). Therefore, by summing (3.24) over the time partition we have for any integer $1 \leq m \leq N_h$

$$\begin{aligned}
&\sum_{n=1}^{m} \left\|\sqrt{\phi}\left(S_h^n - S_h^{n-1}\right)\right\|^2_{L^2(\Omega)} + \left\|\sqrt{\phi}\left(S_h^m - \psi_h^S\right)\right\|^2_{L^2(\Omega)} + 2C_4 \sum_{n=1}^{m} \tau_h^n \left(|||(p_h^n, \widehat{p}_h^n)|||^2 + \left|\left|\left|\left(S_h^n, \widehat{S}_h^n\right)\right|\right|\right|^2\right) \\
&\quad \leq \left\|\sqrt{\phi}\left(S_h^0 - \psi_h^S\right)\right\|^2_{L^2(\Omega)} + 2C_{\mathcal{F}}^b \sum_{n=1}^{m} \tau_h^n \left(\left\|\overline{F}_t^n\right\|^2_{L^2(\Omega)} + \left\|\overline{F}_n^n\right\|^2_{L^2(\Omega)} + \|p_D\|^2_{L^\infty(\Gamma_D)} + \|S_D\|^2_{L^\infty(\Gamma_D)}\right),
\end{aligned}$$

which, together with Lemma 3.5, the triangle inequality and the definitions of $\widetilde{S}_{h\tau}, S_{h\tau}, \widehat{S}_{h\tau}, p_{h\tau}, \widehat{p}_{h\tau}$, leads to

$$\begin{aligned}
&\sum_{n=1}^{m}\int_{t_h^{n-1}}^{t_h^n}\tau_h^n\left\|\sqrt{\phi}\partial_t\widetilde{S}_{h\tau}\right\|_{L^2(\Omega)}^2\mathrm{d}t+\left\|\sqrt{\phi}S_h^m\right\|_{L^2(\Omega)}^2+\int_0^{t_h^m}\left(\left|\!\left|\!\left|\left(S_{h\tau},\widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2+|\!|\!|(p_{h\tau},\widehat{p}_{h\tau})|\!|\!|^2\right)\mathrm{d}t\\
&\quad\lesssim\left\|S_h^0\right\|_{L^2(\Omega)}^2+\left\|\psi_h^S\right\|_{L^2(\Omega)}^2+\sum_{n=1}^{m}t_h^n\left(\|F_t\|_{L^\infty(J;L^2(\Omega))}^2+\|F_n\|_{L^\infty(J;L^2(\Omega))}^2+\|p_D\|_{L^\infty(\Gamma_D)}^2+\|S_D\|_{L^\infty(\Gamma_D)}^2\right)\\
&\quad\lesssim\left\|S_h^0\right\|_{L^2(\Omega)}^2+t_h^m\left(\|F_t\|_{L^\infty(J;L^2(\Omega))}^2+\|F_n\|_{L^\infty(J;L^2(\Omega))}^2+\|p_D\|_{L^\infty(\Gamma_D)}^2+\|S_D\|_{L^\infty(\Gamma_D)}^2\right).\quad\text{(by Lemma 3.5)}
\end{aligned}$$

Thus we conclude the proof. □

Lemma 4.2. *Let $(S_h^n,\widehat{S}_h^n,p_h^n,\widehat{p}_h^n)$ be the solution of problem (3.1) for $1\le n\le N_h$. Let $(S_{h\tau},\widehat{S}_{h\tau})$, $(p_{h\tau},\widehat{p}_{h\tau})$, $\widetilde{S}_{h\tau}$, and $\widetilde{\widehat{S}}_{h\tau}$ be defined as above. Then we have for any $1\le m\le N_h$*

$$\begin{aligned}
\int_0^{t_h^m}\left(\|S_{h\tau}\|_{L^2(\Omega)}^2+\|p_{h\tau}\|_{L^2(\Omega)}^2\right)\mathrm{d}t&\lesssim\int_0^{t_h^m}\left(\left|\!\left|\!\left|\left(S_{h\tau},\widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2+|\!|\!|(p_{h\tau},\widehat{p}_{h\tau})|\!|\!|^2\right)\mathrm{d}t\\
&\quad+t_h^m\left(\|p_D\|_{L^\infty(\Gamma_D)}^2+\|S_D\|_{L^\infty(\Gamma_D)}^2\right),\qquad(4.1)\\
\int_0^{t_h^m}\left(\left\|\widetilde{S}_{h\tau}\right\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(\widetilde{S}_{h\tau},\widetilde{\widehat{S}}_{h\tau}\right)\right|\!\right|\!\right|^2\right)\mathrm{d}t&\lesssim\int_0^{t_h^m}\left(\|S_{h\tau}\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(S_{h\tau},\widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2\right)\mathrm{d}t+t_h^1\|S_0\|_{H^1(\Omega)}^2\qquad(4.2)
\end{aligned}$$

and

$$\int_0^T\left\|\partial_t\widetilde{S}_{h\tau}\right\|_{H_D^{-1}(\Omega)}^2\mathrm{d}t\lesssim\left\|\overline{F}_w\right\|_{L^2(J;L^2(\Omega))}^2+\int_0^T|\!|\!|(p_{h\tau},\widehat{p}_{h\tau})|\!|\!|^2\,\mathrm{d}t,\qquad(4.3)$$

where $\overline{F}_\beta|_{[t_h^{n-1},t_h^n]}=\overline{F}_\beta^n$ for $\beta\in\{w,n,t\}$ and $1\le n\le N_h$.

Proof. By Lemmas 3.2 and 3.5, we have

$$\begin{aligned}
&\int_0^{t_h^m}\left(\|S_{h\tau}\|_{L^2(\Omega)}^2+\|p_{h\tau}\|_{L^2(\Omega)}^2\right)\mathrm{d}t\\
&\quad\lesssim\sum_{n=1}^{m}\int_{t_h^{n-1}}^{t_h^n}\left(\|S_h^n-\psi_h^S\|_{L^2(\Omega)}^2+\|\psi_h^S\|_{L^2(\Omega)}^2+\|p_h^n-\psi_h^p\|_{L^2(\Omega)}^2+\|\psi_h^p\|_{L^2(\Omega)}^2\right)\mathrm{d}t\\
&\quad\lesssim\sum_{n=1}^{m}\int_{t_h^{n-1}}^{t_h^n}\left(\left|\!\left|\!\left|\left(S_h^n-\psi_h^S,\widehat{S}_h^n-\widehat{\psi}_h^S\right)\right|\!\right|\!\right|^2+\|\psi_h^S\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(p_h^n-\psi_h^p,\widehat{p}_h^n-\widehat{\psi}_h^p\right)\right|\!\right|\!\right|^2+\|\psi_h^p\|_{L^2(\Omega)}^2\right)\mathrm{d}t\\
&\quad\lesssim\int_0^{t_h^m}\left(\left|\!\left|\!\left|\left(S_{h\tau},\widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2+|\!|\!|(p_{h\tau},\widehat{p}_{h\tau})|\!|\!|^2\right)\mathrm{d}t+t_h^m\left(\|p_D\|_{L^\infty(\Gamma_D)}^2+\|S_D\|_{L^\infty(\Gamma_D)}^2\right),
\end{aligned}$$

and this is the desired result (4.1). Next we are going to prove (4.2). According to the definitions of $(\widetilde{S}_{h\tau},\widetilde{\widehat{S}}_{h\tau})$ and $(S_h^0,\widehat{S}_h^0)$, we have by the quasi-uniformity of the time partition, the stability of Π_k, and (3.17)

$$\begin{aligned}
&\int_0^{t_h^m}\left(\left\|\widetilde{S}_{h\tau}\right\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(\widetilde{S}_{h\tau},\widetilde{\widehat{S}}_{h\tau}\right)\right|\!\right|\!\right|^2\right)\mathrm{d}t\\
&\quad\lesssim\sum_{n=1}^{m}\int_{t_h^{n-1}}^{t_h^n}\left(\frac{t-t_h^{n-1}}{\tau_h^n}\right)^2\left(\|S_h^n\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(S_h^n,\widehat{S}_h^n\right)\right|\!\right|\!\right|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{t_h^n-t}{\tau_h^n}\right)^2\left(\|S_h^{n-1}\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(S_h^{n-1},\widehat{S}_h^{n-1}\right)\right|\!\right|\!\right|^2\right)\mathrm{d}t\\
&\lesssim\int_0^{t_h^m}\left(\|S_{h\tau}\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(S_{h\tau},\widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2\right)\mathrm{d}t+\tau_h^1\left(\|S_h^0\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(S_h^0,\widehat{S}_h^0\right)\right|\!\right|\!\right|^2\right)\\
&\lesssim\int_0^{t_h^m}\left(\|S_{h\tau}\|_{L^2(\Omega)}^2+\left|\!\left|\!\left|\left(S_{h\tau},\widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2\right)\mathrm{d}t+t_h^1\|S_0\|_{H^1(\Omega)}^2.
\end{aligned}
$$

Now we turn to prove (4.3). For any $w\in L^2(J;H_D^1(\Omega))$, let $w_h(t,\cdot)=\Pi_k w$ and $\widehat{w}_h(t,\cdot)=\widehat{\Pi}_k w$. Then we have by (3.1a), the stability of Π_k, and (3.17)

$$
\begin{aligned}
\int_0^T\left(\partial_t\widetilde{S}_{h\tau},w\right)\mathrm{d}t&\lesssim\sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\left(\phi\partial_t\widetilde{S}_{h\tau},w_h\right)\mathrm{d}t\\
&=\sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\left(\overline{F}_w^n,w_h\right)-a_w(S_h^n;(p_h^n,\widehat{p}_h^n),(w_h,\widehat{w}_h))\,\mathrm{d}t\\
&\lesssim\left\|\overline{F}_w\right\|_{L^2(J;L^2(\Omega))}\|w\|_{L^2(J;L^2(\Omega))}+\left(\int_0^T|\!|\!|(p_{h\tau},\widehat{p}_{h\tau})|\!|\!|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T|\!|\!|(w_h,\widehat{w}_h)|\!|\!|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\lesssim\left\|\overline{F}_w\right\|_{L^2(J;L^2(\Omega))}\|w\|_{L^2(J;L^2(\Omega))}+\left(\int_0^T|\!|\!|(p_{h\tau},\widehat{p}_{h\tau})|\!|\!|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\|\nabla w\|_{L^2(J;L^2(\Omega))}.
\end{aligned}
$$

Thus we conclude the proof. □

With Lemmas 4.1 and 4.2 at hand, we are going to prove the first convergence result.

Theorem 4.3. *Let $\tau_h\to 0$ as $h\to 0$ and $(S_h^n,\widehat{S}_h^n,p_h^n,\widehat{p}_h^n)$ $(1\le n\le N_h)$ be a sequence of numerical solutions solved by (3.1), where $\tau_h=\max_{1\le n\le N_h}\tau_h^n$. Let $(S_{h\tau},\widehat{S}_{h\tau})$ and $(p_{h\tau},\widehat{p}_{h\tau})$ be defined as above. Then there exist $S\in L^2(J;H^1(\Omega))$ with $S|_{\Gamma_D}=S_D$ and $p\in L^2(J;H^1(\Omega))$ with $p|_{\Gamma_D}=p_D$ such that, possibly after passing to a subsequence,*

$$
\begin{aligned}
S_{h\tau}&\rightharpoonup S && \text{weakly in } L^2(J;L^2(\Omega)), && (4.4)\\
\nabla_h S_{h\tau}-G_h(S_{h\tau}-\widehat{S}_{h\tau})&\rightharpoonup\nabla S && \text{weakly in } L^2(J;(L^2(\Omega))^d), && (4.5)\\
S_{h\tau}&\to S_D && \text{strongly in } L^2(J;L^2(\Gamma_D)), && (4.6)\\
p_{h\tau}&\rightharpoonup p && \text{weakly in } L^2(J;L^2(\Omega)), && (4.7)\\
\nabla_h p_{h\tau}-G_h(p_{h\tau}-\widehat{p}_{h\tau})&\rightharpoonup\nabla p && \text{weakly in } L^2\big(J;(L^2(\Omega))^d\big), && (4.8)\\
p_{h\tau}&\to p_D && \text{strongly in } L^2\big(J;L^2(\Gamma_D)\big), && (4.9)
\end{aligned}
$$

where ∇_h is the broken gradient operator and G_h is a global lifting operator such that for each $K\in\mathcal{T}_h$ and $\mu\in L^2(\partial K)$, $G_h|_{\partial K}:L^2(\partial K)\to(\mathcal{P}^k(K))^d$ satisfies

$$
(G_h(\mu),\boldsymbol{w}_h)_K=\langle\mu,\boldsymbol{w}_h\cdot\boldsymbol{n}\rangle_{\partial K},\quad\forall\boldsymbol{w}_h\in(\mathcal{P}^k(K))^d.
$$

Proof. In the proof, we only prove the results corresponding to $(S_{h\tau},\widehat{S}_{h\tau})$, since the results associated with $(p_{h\tau},\widehat{p}_{h\tau})$ can be proved similarly.

According to the definition of the lifting operator G_h, we have

$$
\int_0^T\left\|\nabla_h S_{h\tau}-G_h\left(S_{h\tau}-\widehat{S}_{h\tau}\right)\right\|_{L^2(\Omega)}^2\mathrm{d}t\lesssim\int_0^T\|\nabla S_{h\tau}\|_{L^2(\mathcal{T}_h)}^2\,\mathrm{d}t+\sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\sum_{K\in\mathcal{T}_h}\left\|G_h\left(S_h^n-\widehat{S}_h^n\right)\right\|_{L^2(K)}^2\mathrm{d}t
$$

$$\lesssim \int_0^T \|\nabla S_{h\tau}\|^2_{L^2(\mathcal{T}_h)}\,\mathrm{d}t + \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|S_h^n - \widehat{S}_h^n\right\|^2_{L^2(\partial K)}\,\mathrm{d}t$$
$$= \int_0^T \left|\!\left|\!\left|\left(S_{h\tau}, \widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2 \mathrm{d}t.$$

From Lemmas 4.1, 4.2, and the above result, we know that $S_{h\tau}$ and $\nabla_h S_{h\tau} - G_h(S_{h\tau} - \widehat{S}_{h\tau})$ are uniformly bounded in $L^2(J;L^2(\Omega))$ and $L^2(J;(L^2(\Omega))^d)$. Thus there exist $S \in L^2(J;L^2(\Omega))$ and $\mathbf{G}_S \in L^2(J;(L^2(\Omega))^d)$ such that, possibly after passing to a subsequence,

$$S_{h\tau} \rightharpoonup S \qquad \text{weakly in } L^2(J;L^2(\Omega)),$$
$$\nabla_h S_{h\tau} - G_h\left(S_{h\tau} - \widehat{S}_{h\tau}\right) \rightharpoonup \mathbf{G}_S \quad \text{weakly in } L^2\left(J;(L^2(\Omega))^d\right).$$

Let $\varphi \in L^2(J;(\mathcal{C}_0^\infty(\Omega))^d)$, then by the Green's formula and the definition of G_h we obtain that

$$\begin{aligned}\int_0^T (S_{h\tau}, \mathrm{div}\varphi)\,\mathrm{d}t &= \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n} -(\nabla S_h^n, \varphi)_{\mathcal{T}_h} + \left\langle S_h^n - \widehat{S}_h^n, \varphi\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h}\mathrm{d}t \\ &= \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n} -(\nabla S_h^n, \varphi - \Pi_k\varphi)_{\mathcal{T}_h} + \left\langle S_h^n - \widehat{S}_h^n, (\varphi-\Pi_k\varphi)\cdot\mathbf{n}\right\rangle_{\partial\mathcal{T}_h} \\ &\quad + \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n} -\left(\nabla S_h^n - G_h\left(S_h^n - \widehat{S}_h^n\right), \Pi_k\varphi - \varphi\right)_{\mathcal{T}_h}\mathrm{d}t \\ &\quad + \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n} -\left(\nabla S_h^n - G_h(S_h^n - \widehat{S}_h^n), \varphi\right)_{\mathcal{T}_h}\mathrm{d}t,\end{aligned} \tag{4.10}$$

and this leads to

$$\int_0^T (S, \mathrm{div}\varphi)\,\mathrm{d}t = -\int_0^T (\mathbf{G}_S, \varphi)\,\mathrm{d}t, \quad \text{as } h\to 0.$$

Thus we have $\nabla S = \mathbf{G}_S \in L^2(J;(L^2(\Omega))^d)$, and hence $S \in L^2(J;H^1(\Omega))$. If we let $\varphi \in L^2(J;(\mathcal{C}^\infty(\overline{\Omega}))^d)$ with $\varphi = 0$ on Γ_N in (4.10), we have

$$\int_0^T (S, \mathrm{div}\varphi)\,\mathrm{d}t = -\int_0^T (\nabla S, \varphi)\,\mathrm{d}t + \int_0^T \langle S_D, \varphi\cdot\mathbf{n}\rangle_{\Gamma_D}\,\mathrm{d}t, \quad \text{as } h\to 0,$$

which, together with the arbitrariness of φ, yields $S = S_D$ on Γ_D.

Next we are going to prove (4.6). By Lemma 4.1, we derive that

$$\begin{aligned}\int_0^T \|S_{h\tau} - S_D\|^2_{L^2(\Gamma_D)}\,\mathrm{d}t &= \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\sum_{F\in\mathcal{E}_h^D}\left\|S_h^n - \widehat{S}_h^n\right\|^2_{L^2(F)}\mathrm{d}t \\ &\le h\sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n}\sum_{K\in\mathcal{T}_h} h_K^{-1}\left\|S_h^n - \widehat{S}_h^n\right\|^2_{L^2(\partial K\cap\mathcal{E}_h^D)}\mathrm{d}t \\ &\le h\int_0^T \left|\!\left|\!\left|\left(S_{h\tau}, \widehat{S}_{h\tau}\right)\right|\!\right|\!\right|^2 \mathrm{d}t \to 0, \quad \text{as } h\to 0.\end{aligned}$$

Therefore we conclude the proof. □

Before proving the next convergence result, we pause to state the Aubin–Lions lemma.

Lemma 4.4 ([48]). *Consider Banach spaces B_0, B_1, and B_2 such that $B_0 \hookrightarrow B_1$ is compact and $B_1 \hookrightarrow B_2$ is continuous. Assume that B_0 is separable and reflexive. Then $W := \{v \in L^2(J;B_0) : \partial_t v \in L^2(J;B_2)\}$ is compactly embedded into $L^2(J;B_1)$.*

Theorem 4.5. *Let $\tau_h \to 0$ as $h \to 0$ and $(S_h^n, \widehat{S}_h^n, p_h^n, \widehat{p}_h^n)$ $(1 \le n \le N_h)$ be a sequence of numerical solutions solved by (3.1), where τ_h is defined in Theorem 4.3. Let $(S_{h\tau}, \widehat{S}_{h\tau})$ and $\widetilde{S}_{h\tau}$ be defined as above. Then for the limit S of $S_{h\tau}$ in the sense of Theorem 4.3, we have, possibly after passing to a subsequence,*

$$\widetilde{S}_{h\tau} \to S \quad \text{strongly in } L^2(J;L^2(\Omega)), \tag{4.11}$$

$$\partial_t \widetilde{S}_{h\tau} \rightharpoonup \partial_t S \quad \text{weakly in } L^2(J;H_D^{-1}(\Omega)), \tag{4.12}$$

$$S_{h\tau} \to S \quad \text{strongly in } L^2\big(J;L^2(\Omega)\big). \tag{4.13}$$

Proof. Associated with the discrete space V_h, we define a DG norm

$$\|v_h\|_{\mathrm{DG}}^2 = \|v_h\|_{L^2(\Omega)}^2 + \|\nabla_h v_h\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{E}_h^o} h_F^{-1}\|[v_h]\|_{L^2(F)}^2, \quad \forall v_h \in V_h,$$

where $[\cdot]|_F$ denotes the jump on the edge/face F. Moreover by Theorem 2.1 of [18] and (6.6) of [3] we have

$$\|v_h\|_{BV(\Omega)} + \|v_h\|_{L^4(\Omega)} \lesssim \|v_h\|_{\mathrm{DG}}, \quad \forall v_h \in V_h, \tag{4.14}$$

where $BV(\Omega)$ is the space of function of bounded variation. Let $\widetilde{M} = [BV(\Omega)\cap L^4(\Omega), L^4(\Omega)]_{1/2}$ and M be the closure of the span of V_h in $\widetilde{M}$, we know by the analysis in the below of Theorem 6.4 from [3] that M is a separable and reflexive space and it is compactly embedded into $L^2(\Omega)$. Thus by the Aubin–Lions lemma stated in Lemma 4.4, we obtain that the space $W = \{v \in L^2(J;M) : \partial_t v \in L^2(J;H_D^{-1})\}$ is compactly embedded into $L^2(J;L^2(\Omega))$ by setting $B_0 = M$, $B_1 = L^2(\Omega)$, and $B_2 = H_D^{-1}(\Omega)$. On the other hand, from Lemma 4.2 and (4.14), we observe that $\widetilde{S}_{h\tau}$ is uniformly bounded in W, hence there exist $\widetilde{S} \in L^2(J;L^2(\Omega))$ and $\mathbb{S} \in L^2(J;H_D^{-1}(\Omega))$ such that, possibly after passing to a subsequence,

$$\widetilde{S}_{h\tau} \to \widetilde{S} \quad \text{strongly in } L^2\big(J;L^2(\Omega)\big),$$

$$\partial_t \widetilde{S}_{h\tau} \rightharpoonup \mathbb{S} \quad \text{weakly in } L^2(J;H_D^{-1}(\Omega)).$$

Next, we are going to prove (4.13). By the definitions of $S_{h\tau}$ and $\widetilde{S}_{h\tau}$ and Lemma 4.1, we have

$$\begin{aligned}\int_0^T \Big\|S_{h\tau} - \widetilde{S}\Big\|_{L^2(\Omega)}^2 \mathrm{d}t &= \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n} \Big\|\widetilde{S}_{h\tau} + (t_h^n - t)\partial_t\widetilde{S}_{h\tau} - \widetilde{S}\Big\|_{L^2(\Omega)}^2 \mathrm{d}t \\ &\lesssim \Big\|\widetilde{S}_{h\tau} - \widetilde{S}\Big\|_{L^2(J;L^2(\Omega))}^2 + \sum_{n=1}^{N_h}\int_{t_h^{n-1}}^{t_h^n} (\tau_h^n)^2 \Big\|\partial_t\widetilde{S}_{h\tau}\Big\|_{L^2(\Omega)}^2 \mathrm{d}t \to 0, \quad \text{as } h \to 0.\end{aligned}$$

Thus $S_{h\tau} \to \widetilde{S}$ strongly in $L^2(J;L^2(\Omega))$, which together with the uniqueness of limits yields $\widetilde{S} = S$.

For arbitrary $v \in H_D^1(\Omega)$ and $\varphi \in \mathcal{C}_0^\infty(J)$, a simple calculation implies that

$$\int_0^T \Big\langle \partial_t \widetilde{S}_{h\tau}, v\varphi \Big\rangle \mathrm{d}t = -\int_0^T \partial_t\varphi\Big(\widetilde{S}_{h\tau}, v\Big)\mathrm{d}t, \tag{4.15}$$

By taking $h \to 0$ in (4.15), we find that for any $\varphi \in \mathcal{C}_0^\infty(J)$

$$\int_0^T \langle \mathbb{S}, v\rangle\varphi \,\mathrm{d}t = -\int_0^T (S, v)\partial_t\varphi \,\mathrm{d}t.$$

Therefore $\mathbb{S} = \partial_t S$ and we conclude the proof. □

Theorem 4.6. *Let $\tau_h \to 0$ as $h \to 0$ and $(S_h^n, \widehat{S}_h^n, p_h^n, \widehat{p}_h^n)$ $(1 \le n \le N_h)$ be a sequence of numerical solutions solved by (3.1), where τ_h is defined in Theorem 4.3. Let $(S_{h\tau}, \widehat{S}_{h\tau})$, $(p_{h\tau}, \widehat{p}_{h\tau})$, and $\widetilde{S}_{h\tau}$ be defined as above, and (S,p) the limit of $(S_{h\tau}, p_{h\tau})$ in the sense of Theorems 4.3 and 4.5. Then (S,p) is a weak solution of (2.11) if $S_h^0 \to S_0$ strongly in $L^2(\Omega)$ and $\boldsymbol{K} \in (W^{1,\infty}(\Omega))^{d\times d}$.*

Proof. From Theorem 1 of [21], we know that $\mathcal{C}^\infty(\overline{\Omega}) \cap H_D^1(\Omega)$ is dense in $H_D^1(\Omega)$. Therefore we sum (3.1) over the time partition and take the test functions as $(v_h, \widehat{v}_h) = (\Pi_k v, \widehat{\Pi}_k v)$ and $(w_h, \widehat{w}_h) = (\Pi_k w, \widehat{\Pi}_k w)$ for $v, w \in \mathcal{C}_0^\infty(J; \mathcal{C}^\infty(\overline{\Omega}) \cap H_D^1(\Omega))$ to derive that

$$\int_0^T \Big(\phi \partial_t \widetilde{S}_{h\tau}, v_h\Big)\,\mathrm{d}t + \int_0^T a_w(S_{h\tau}; (p_{h\tau}, \widehat{p}_{h\tau}), (v_h, \widehat{v}_h))\,\mathrm{d}t = \int_0^T \big(\overline{F}_w, v_h\big)\,\mathrm{d}t, \tag{4.16a}$$

$$\int_0^T a_t(S_{h\tau}; (p_{h\tau}, \widehat{p}_{h\tau}), (w_h, \widehat{w}_h))\,\mathrm{d}t + \int_0^T b_n\Big(S_{h\tau}; \Big(S_{h\tau}, \widehat{S}_{h\tau}\Big), (w_h, \widehat{w}_h)\Big)\,\mathrm{d}t = \int_0^T \big(\overline{F}_t, w_h\big)\,\mathrm{d}t, \tag{4.16b}$$

where $\overline{F}_\beta$ for $\beta \in \{w, n, t\}$ is defined in Lemma 4.2. By Lemma 4.2, Theorem 4.5, and the definition of $\overline{F}_w$, we observe that

$$\begin{aligned}
\int_0^T \big(\overline{F}_w, v_h\big) - (F_w, v)\,\mathrm{d}t &= \sum_{n=1}^{N_h} \int_{t_h^{n-1}}^{t_h^n} \left(F_w, \frac{1}{\tau_h^n}\int_{t_h^{n-1}}^{t_h^n} v_h\,\mathrm{d}t - v\right)\mathrm{d}t \\
&= \sum_{n=1}^{N_h} \int_{t_h^{n-1}}^{t_h^n} \left(F_w, \frac{1}{\tau_h^n}\int_{t_h^{n-1}}^{t_h^n} v_h - v\,\mathrm{d}t\right)\mathrm{d}t + \sum_{n=1}^{N_h} \int_{t_h^{n-1}}^{t_h^n} \left(F_w, \frac{1}{\tau_h^n}\int_{t_h^{n-1}}^{t_h^n} v\,\mathrm{d}t - v\right)\mathrm{d}t \\
&\le \big\|\overline{F}_w\big\|_{L^2(J;L^2(\Omega))} \|v_h - v\|_{L^2(J;L^2(\Omega))} + \|F_w\|_{L^2(J;L^2(\Omega))} \|\overline{v} - v\|_{L^2(J;L^2(\Omega))},
\end{aligned}$$

where $\overline{v}|_{[t_h^{n-1}, t_h^n]} = \frac{1}{\tau_h^n}\int_{t_h^{n-1}}^{t_h^n} v\,\mathrm{d}t$, and

$$\int_0^T \Big(\phi \partial_t \widetilde{S}_{h\tau}, v_h\Big) - \langle \phi \partial_t S, v\rangle\,\mathrm{d}t = \int_0^T \Big\langle \phi \partial_t \widetilde{S}_{h\tau}, v_h - v\Big\rangle + \Big\langle \phi\Big(\partial_t \widetilde{S}_{h\tau} - \partial_t S\Big), v\Big\rangle\,\mathrm{d}t.$$

Thus

$$\int_0^T \big(\overline{F}_w, v_h\big)\,\mathrm{d}t \to \int_0^T (F_w, v)\,\mathrm{d}t, \quad \int_0^T \Big(\phi \partial_t \widetilde{S}_{h\tau}, v_h\Big)\,\mathrm{d}t \to \int_0^T \langle \phi \partial_t S, v\rangle\,\mathrm{d}t, \quad \text{as } h \to 0. \tag{4.17}$$

Next we consider the limit of the term $\int_0^T a_w(S_{h\tau}; (p_{h\tau}, \widehat{p}_{h\tau}), (v_h, \widehat{v}_h))\,\mathrm{d}t$ as $h \to 0$. By a simple calculation and the definition of a_w, we can derive that

$$\begin{aligned}
&\int_0^T a_w(S_{h\tau}; (p_{h\tau}, \widehat{p}_{h\tau}), (v_h, \widehat{v}_h))\,\mathrm{d}t \\
&\quad= \int_0^T ((\lambda_w(S_{h\tau}) - \lambda_w(S))\mathbf{K}\nabla_h p_{h\tau}, \nabla_h v_h)\,\mathrm{d}t - \int_0^T \langle (\lambda_w(S_{h\tau}) - \lambda_w(S))\mathbf{K}\nabla v_h \cdot \mathbf{n}, p_{h\tau} - \widehat{p}_{h\tau}\rangle_{\partial\mathcal{T}_h}\,\mathrm{d}t \\
&\qquad+ \int_0^T (\lambda_w(S)\mathbf{K}(\nabla_h p_{h\tau} - G_h(p_{h\tau} - \widehat{p}_{h\tau})), \nabla_h v_h)\,\mathrm{d}t \\
&\qquad- \int_0^T \langle (\lambda_w(S)\mathbf{K} - \lambda_w(\Pi_0 S)\Pi_0\mathbf{K})\nabla v_h \cdot \mathbf{n}, p_{h\tau} - \widehat{p}_{h\tau}\rangle_{\partial\mathcal{T}_h}\,\mathrm{d}t \\
&\qquad- \int_0^T ((\lambda_w(\Pi_0 S)\Pi_0\mathbf{K} - \lambda_w(S)\mathbf{K})G_h(p_{h\tau} - \widehat{p}_{h\tau}), \nabla_h v_h)\,\mathrm{d}t
\end{aligned} \tag{4.18}$$

$$+\int_0^T \sum_{K\in\mathcal{T}_h} \delta_w^p h_K^{-1}\langle p_{h\tau}-\widehat{p}_{h\tau}, v_h-\widehat{v}_h\rangle_{\partial K} - \langle \lambda_w(S_{h\tau})\mathbf{K}\nabla p_{h\tau}\cdot\mathbf{n}, v_h-\widehat{v}_h\rangle_{\partial\mathcal{T}_h}\,\mathrm{d}t$$
$$= I_1+I_2+I_3+I_4+I_5.$$

Note that in the derivation of (4.18), we have used the fact that $\lambda_w(\Pi_0 S)$ is a piecewise constant function on the domain partition $\mathcal{T}_h$ since in this paper the case we considered is that λ_α $(\alpha = w, n)$ is a function only depending on S. By interpolation error estimates, Lemma 4.1 and Theorem 4.3, we obtain that

$$I_2 \to \int_0^T (\lambda_w(S)\mathbf{K}\nabla p, \nabla v)\,\mathrm{d}t, \quad I_5 \to 0, \quad \text{as } h\to 0. \tag{4.19}$$

Since λ_w is Lipschitz continuous and $S_{h\tau} \to S$ in $L^2(J;L^2(\Omega))$, the interpolation error estimates, Lemmas 3.1 and 4.1 yield

$$\begin{aligned} I_1 \lesssim{}& \|\nabla p_h\|_{L^2(J;L^2(\mathcal{T}_h))}\Big(\|\nabla(v-v_h)\|_{L^2(J;L^2(\mathcal{T}_h))} + \|S_{h\tau}-S\|_{L^2(J;L^2(\Omega))}\|\nabla v\|_{L^\infty(J;L^\infty(\Omega))}\Big)\\ &+\left(\int_0^T \sum_{K\in\mathcal{T}_h} h_K^{-1}\|p_{h\tau}-\widehat{p}_{h\tau}\|_{L^2(\partial K)}^2\,\mathrm{d}t\right)^{1/2}\Big(\|\nabla(v-v_h)\|_{L^2(J;L^2(\mathcal{T}_h))} + h\big\|\nabla^2(v-v_h)\big\|_{L^2(J;L^2(\mathcal{T}_h))}\\ &+\|\nabla v\|_{L^\infty(J;L^\infty(\Omega))}\Big(\|S_{h\tau}-S\|_{L^2(J;L^2(\Omega))} + h\|\nabla(S_{h\tau}-S)\|_{L^2(J;L^2(\mathcal{T}_h))}\Big)\Big) \to 0, \quad \text{as } h\to 0. \end{aligned} \tag{4.20}$$

Similar to (4.20), we can also prove that

$$I_3 + I_4 \to 0, \quad \text{as } h\to 0. \tag{4.21}$$

Then combining (4.18)–(4.21) we derive that

$$\int_0^T a_w(S_{h\tau};(p_{h\tau},\widehat{p}_{h\tau}),(v_h,\widehat{v}_h))\,\mathrm{d}t \to \int_0^T (\lambda_w(S)\mathbf{K}\nabla p,\nabla v)\,\mathrm{d}t, \quad \text{as } h\to 0. \tag{4.22}$$

As for the terms appeared in (4.16b), we can adopt a similar method that has been used to obtain (4.17) and (4.22) to prove that

$$\int_0^T a_t(S_{h\tau};(p_{h\tau},\widehat{p}_{h\tau}),(w_h,\widehat{w}_h))\,\mathrm{d}t \to \int_0^T (\lambda_t(S)\mathbf{K}\nabla p,\nabla w)\,\mathrm{d}t, \tag{4.23}$$

$$\int_0^T b_n\Big(S_{h\tau};\Big(S_{h\tau},\widehat{S}_{h\tau}\Big),(w_h,\widehat{w}_h)\Big)\,\mathrm{d}t \to \int_0^T (\lambda_n(S)\mathbf{K}\nabla p_c,\nabla w)\,\mathrm{d}t, \tag{4.24}$$

$$\int_0^T (\overline{F}_t, w_h)\,\mathrm{d}t \to \int_0^T (F_t, w)\,\mathrm{d}t. \tag{4.25}$$

Therefore, by (4.22)–(4.25) and the fact that $\mathcal{C}_0^\infty(J;\mathcal{C}^\infty(\Omega)\cap H_D^1(\Omega))$ is dense in $L^2(J;H_D^1(\Omega))$, we conclude the proof. □

5. Numerical experiments

In this section, two numerical examples are provided. In the first example, we confirm the convergence of the HDG scheme (3.1). In the second example, we explore the potential of the HDG scheme (3.1) to the numerical simulation of the problem with the realistic permeability. Throughout this section, we choose the stabilization parameters as: $\delta_t^p = 32$, $\delta_w^p = 30$, and $\delta_n^S = 16$. Moreover, we assume that the absolute permeability tensor is given by $\mathbf{K} = K\mathbf{I}$, where K is a positive constant and $\mathbf{I}$ is the identity matrix.

TABLE 1. (Example 5.1). The convergence histories and orders of $\|S - S_h\|_{L^2(\Omega)}$ and $\|p - p_h\|_{L^2(\Omega)}$ at $t = 0.5$ for $p_c = -S$.

	h	$\|S - S_h\|_{L^2(\Omega)}$	Order	$\|p - p_h\|_{L^2(\Omega)}$	Order
$k = 1$	$\frac{1}{4}$	9.4636e−03	–	1.5019e−03	–
	$\frac{1}{8}$	2.3781e−03	1.99	3.6570e−04	2.04
	$\frac{1}{16}$	5.9409e−04	2.00	9.0807e−05	2.01
	$\frac{1}{32}$	1.4849e−04	2.00	2.2664e−05	2.00
	$\frac{1}{64}$	3.7119e−05	2.00	5.6650e−06	2.00
$k = 2$	$\frac{1}{4}$	3.9401e−04	–	5.5822e−05	–
	$\frac{1}{8}$	4.9975e−05	2.98	6.8523e−06	3.03
	$\frac{1}{16}$	6.2913e−06	2.99	8.5163e−07	3.01
	$\frac{1}{32}$	7.8757e−07	3.00	1.0629e−07	3.00

TABLE 2. (Example 5.1). The convergence histories and orders of $\|S - S_h\|_{L^2(\Omega)}$ and $\|p - p_h\|_{L^2(\Omega)}$ at $t = 0.5$ for $p_c = 10/\sqrt{S}$.

	h	$\|S - S_h\|_{L^2(\Omega)}$	Order	$\|p - p_h\|_{L^2(\Omega)}$	Order
$k = 1$	$\frac{1}{4}$	8.0493e−03	–	1.0250e−03	–
	$\frac{1}{8}$	1.9529e−03	2.04	2.5850e−04	1.99
	$\frac{1}{16}$	4.8562e−04	2.01	6.5036e−05	1.99
	$\frac{1}{32}$	1.2125e−04	2.00	1.6290e−05	2.00
	$\frac{1}{64}$	3.0298e−05	2.00	4.0778e−06	2.00
$k = 2$	$\frac{1}{4}$	3.8856e−04	–	5.7572e−05	–
	$\frac{1}{8}$	4.9660e−05	2.97	6.9007e−06	3.06
	$\frac{1}{16}$	6.2441e−06	2.99	8.5349e−07	3.02
	$\frac{1}{32}$	7.8131e−07	3.00	1.0635e−07	3.00

Example 5.1. We take $\Omega = (0, 1)^2$ and consider a problem with the exact solutions given by

$$S = 0.5(t + 1) - 7xy(1 - x)(1 - y)e^{-x^2-y^2}, \quad p = (t + 1) + xy \tanh(1 - x) \tanh(1 - y).$$

Dirichlet boundary conditions are imposed for both pressure and saturation. Moreover, we invoke Brooks-Corey relative permeabilities

$$\lambda_w = \frac{S^4}{\mu_w}, \quad \lambda_n = \frac{(1 - S)^2(1 - S^2)}{\mu_n},$$

and fix the following parameters as: porosity $\phi = 0.3$, viscosities $\mu_n = 10^{-2}$ and $\mu_w = 10^{-3}$, $K = 10^{-4}$, and $T = 1$.

We set $\tau_h = h^2$ and test this example for two different capillary pressures $p_c = -S$ and $p_c = \frac{10}{\sqrt{S}}$. For the first choice, the assumption (A4) is satisfied. For the second choice, the assumption (A4) is not satisfied. In Tables 1 and 2, we present the convergence histories and orders of $\|S - S_h\|_{L^2(\Omega)}$ and $\|p - p_h\|_{L^2(\Omega)}$ at $t = 0.5$

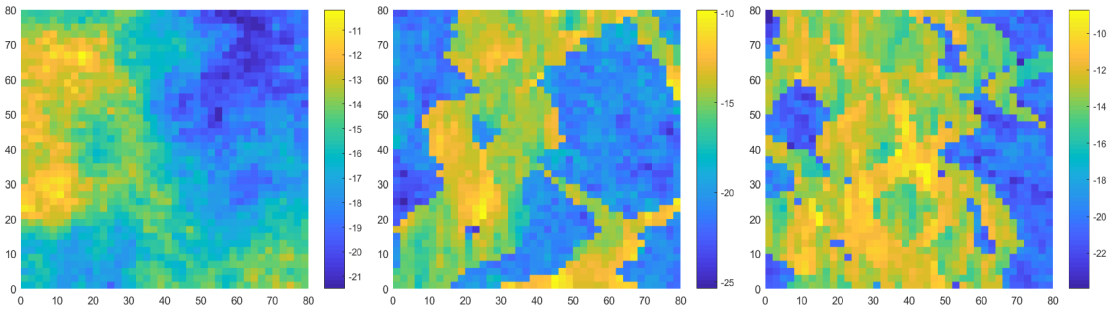


FIGURE 1. (Example 5.2). The permeabilities K.

for these two different capillary pressures. We observe that the errors $\|S - S_h\|_{L^2(\Omega)}$ and $\|p - p_h\|_{L^2(\Omega)}$ reduce with a convergence rate $O(h^{k+1})$ as the mesh size is decreased. This demonstrates that the discrete solution solved by the HDG scheme (3.1) converges to the continuous solution as the mesh size tends to zero.

Example 5.2. In this example, the domain is $\Omega = 80\,\mathrm{m} \times 80\,\mathrm{m}$. The relative permeabilities and the capillary pressure are chosen as

$$\lambda_w = \frac{S_e}{\mu_w}, \quad \lambda_n = \frac{(1 - S_e)^2}{\mu_n}, \quad p_c = -\frac{0.1}{\sqrt{K}} \log S_e,$$

where S_e is the effective saturation defined as $(S - S_{rw})/(1 - S_{rw} - S_{rn})$, $S_{rw} = S_{rn} = 10^{-4}$ are residual wetting and non-wetting phase saturations, and K is the data extracted from horizontal slices of the SPE Comparative Solution Project of model 2 (https://www.spe.org/web/csp/datasets/set02.htm). Moreover we fix the porosity and viscosities as $\phi = 0.2$, $\mu_n = 5 \times 10^{-2}\,\mathrm{Pa.s}$ and $\mu_w = 10^{-3}\,\mathrm{Pa.s}$. At the left boundary $x = 0$, Dirichlet boundaries conditions are imposed as $p_D = 6\,\mathrm{Pa}$ and $S_D = 0.99$. At the right boundary $x = 80$, we set $p_D = 1\,\mathrm{Pa}$ and $S_D = 0.01$. The remaining boundaries are set as no flow.

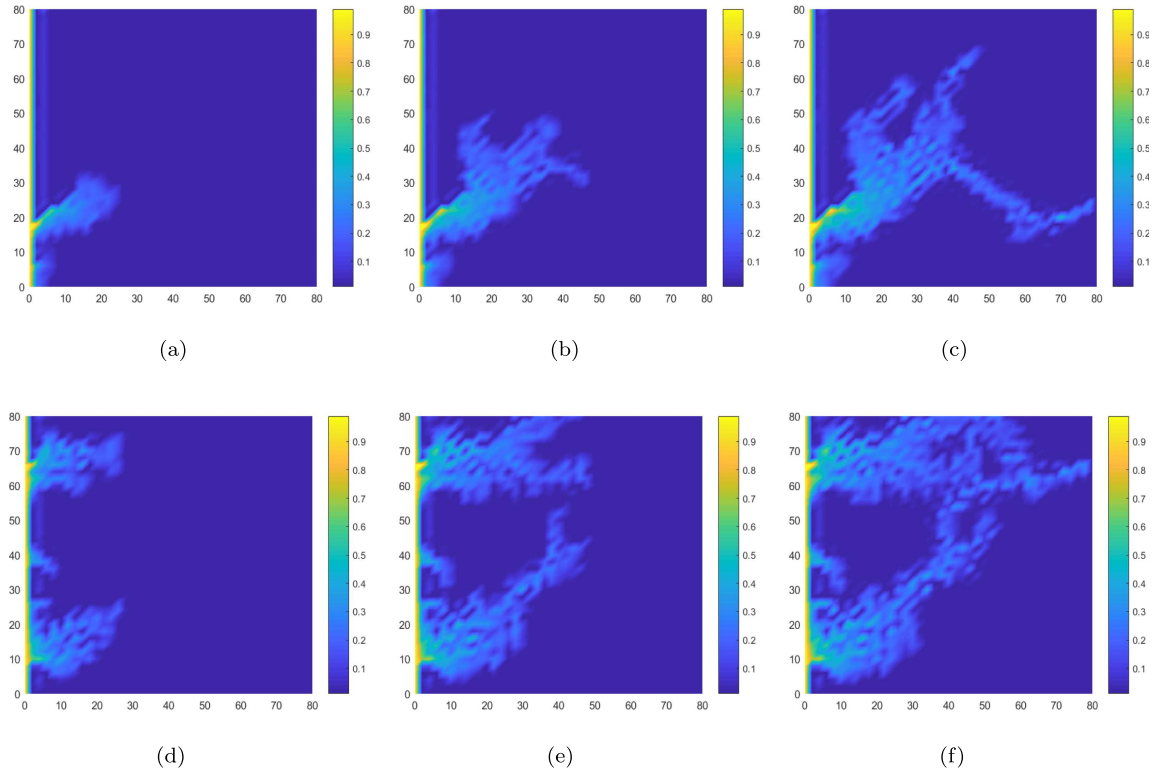
The initial distribution of the wetting-phase saturation is set as $S = 0.01$ in entire domain. As the permeability value is relatively small, its logarithm is presented in Figure 1 to make the illustration more clear. In Figure 2, the evolution of the wetting-phase saturation on a 40×40 grid is shown for the permeabilities provided in the middle and right-hand sides of Figure 1. Moreover, the evolution of the wetting-phase saturation on the 40×40 and 80×80 grids is also shown in Figure 3 for the permeability in the left-hand side of Figure 1. We can see that the wetting-phase saturation avoids to flow to the region with low permeability which acts as a barrier.

Next, we test the convergence of the HDG scheme (3.1) for the example with permeability presented in Figure 4. Since there are no exact solutions, we take the discrete solutions on a 160×160 grid as the reference solutions. In Table 3, we show the convergence histories of $\|S - S_h\|_{L^2(\Omega)}$ and $\|p - p_h\|_{L^2(\Omega)}$ at $t = 3$ day. We can observe that these errors reduce as the mesh size decreases which verifies the convergence analysis.

We should point out that the above numerical results demonstrate that the HDG scheme (3.1) can be used to simulate the problem with the realistic permeability.

6. Conclusion

In this paper, we design a fully discrete scheme for the wetting-phase saturation-pressure formulation of two-phase flow in porous media with a backward Euler method for the time discretization and an HDG method for the saturation and pressure discretizations. We prove by the Brouwer fixed-point theorem that the discrete scheme has a discrete solution under assumptions (A1)–(A5). Moreover by the analysis tools developed in [3,18], we derive that the sequence of discrete solutions converges, possibly after passing to a subsequence, to a weak



(a) (b) (c)

(d) (e) (f)

FIGURE 2. (Example 5.2). From *top* to *bottom*: The evolution of the wetting-phase saturation on a 40×40 grid for the permeabilities presented in the middle and right-hand side of Figure 1. Here $k = 1$ and the time step is set as $\tau = 0.01$ day. (a) $T = 14$ day. (b) $T = 34$ day. (c) $T = 48$ day. (d) $T = 44$ day. (e) $T = 88$ day. (f) $T = 108$ day.

TABLE 3. (Example 5.2). The convergence histories of $\|S - S_h\|_{L^2(\Omega)}$ and $\|p - p_h\|_{L^2(\Omega)}$ at $t = 3$ day. Here $k = 1$ and the time step is chosen as $\tau = h$.

h	$\|S - S_h\|_{L^2(\Omega)}$	$\|p - p_h\|_{L^2(\Omega)}$
$\frac{1}{10}$	2.6365	8.1997
$\frac{1}{20}$	2.1881	3.8479
$\frac{1}{40}$	1.6875	1.5007
$\frac{1}{80}$	1.2066	0.4563

solution of the continuous problem. Finally, two numerical examples are provided to validate the numerical results and explore the potential of the discrete scheme to the numerical simulation of the problem that does not satisfy the assumption (A4) and the problem with the realistic permeability.

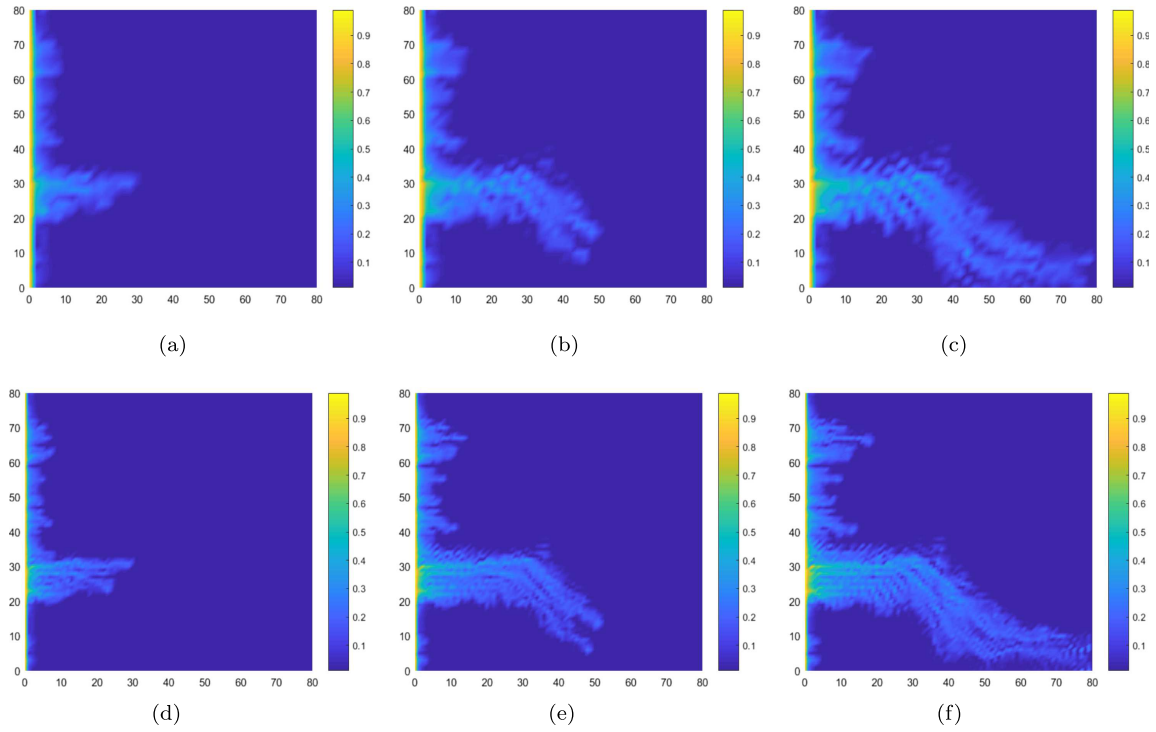


FIGURE 3. (Example 5.2). From *top* to *bottom*: The evolution of the wetting-phase saturation on the 40×40 and 80×80 grids for the permeability presented in the left-hand side of Figure 1. Here $k = 1$ and the time step is set as $\tau = 0.01$ day. (a) $T = 20$ day. (b) $T = 40$ day. (c) $T = 54$ day. (d) $T = 20$ day. (e) $T = 40$ day. (f) $T = 54$ day.

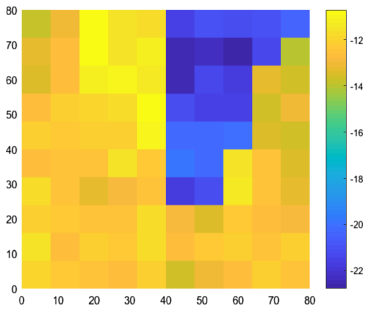


FIGURE 4. (Example 5.2). The permeability K.

FUNDING

This work was supported by National Key Research and Development Project of China (Grant No. 2023YFA1011702), National Natural Science Foundation of China (Grant Nos. 12471345, 12122115), and Guangdong Basic and Applied Basic Research Foundation (Grant Nos. 2024A1515011163, 2025A1515010428).

DATA AVAILABILITY STATEMENT

The research data associated with this article are included in the article.

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