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POINTWISE CONVERGENCE OF THE LOCAL DISCONTINUOUS GALERKIN METHOD ON A SHISHKIN MESH FOR 2D REACTION-DIFFUSION PROBLEMS

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Abstract. A local discontinuous Galerkin (LDG) method, based on a recently-developed "layer-upwind" flux, is applied on a Shishkin mesh to solve a reaction-diffusion problem posed on the 2D unit square. Typical solutions of this problem have an exponential boundary layer along each edge of the square. Energy-norm supercloseness, discrete Green's functions, and local analysis with a cut-off function are combined to establish pointwise convergence for the LDG solution in the entire domain; in this derivation the coarse mesh, the highly anisotropic boundary layer mesh, and the fine corner mesh are each handled with a different technique to gain the sharpest local results. The convergence rates obtained are nearly optimal (provided that one disregards logarithmic factors and error terms that are multiplied by the small singular perturbation parameter) and are the best pointwise error bounds currently known for the LDG method when solving 2D reaction-diffusion problems with boundary layers. Numerical results are also included to test the sharpness of our results.

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1. Introduction

In this paper we shall analyse the error when a local discontinuous Galerkin (LDG) method is used to solve the singularly perturbed reaction-diffusion boundary value problem

$$-\varepsilon\Delta u + bu = f \qquad \text{in } \Omega := (0,1)^2, \tag{1.1a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.1b}$$

where the parameter ε satisfies $0 < \varepsilon \ll 1$ and the functions $b = b(x,y)$ and $f = f(x,y)$ are sufficiently smooth in $\bar{\Omega} = [0,1]^2$. We assume that $b \geq 2\beta^2 > 0$ on $\bar{\Omega}$, where $\beta > 0$ is a constant. Further assumptions are made in Assumption 2.1 below.

Keywords and phrases. Local DG method, singularly perturbed, reaction-diffusion, Shishkin meshes, pointwise error estimate, discrete Green's function.

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Solutions of the problem (1.1) typically display exponential boundary layers along the edges of the domain boundary $\partial\Omega$. A significant portion of [28] is devoted to deriving regularity results for the problem (1.1), showing that its solution contains these layers – in particular, Theorem 7.4.5 of [28] establishes comprehensive bounds on higher-order derivatives for the layer and non-layer components of the solution. To obtain accurate numerical approximations of the solution, a variety of numerical methods have been developed and analyzed; see for instance [2, 13, 20, 23, 28, 31, 35] and their lists of references.

For finite element methods it is natural to carry out error estimates in an energy norm framework, but in the case of reaction-diffusion problems with exponential boundary layers, the standard energy norm is scaled incorrectly in a way that damps out the error in approximating the layer component of the exact solution. This shortcoming has led to the development of *balanced norms* where the gradient component of the energy norm is weighted in such a way that the layer and smooth parts of the exact solution of (1.1) have the same order of magnitude for all values of ε. These balanced norms have generated much interest; see for example [17, 22, 27] and the references in these papers.

Nevertheless, compared with $L^2(\Omega)$, $H^1(\Omega)$ and related integral norms including balanced norms, *pointwise error estimates* provide a more reliable measure of the difference between the true and computed solutions; see for example [13, 19]. But it is well known that the derivation of satisfactory pointwise error bounds for finite element methods is difficult, even when the boundary value problem is not singularly perturbed, as is evident from Chapter 8 of [4].

In the present paper we shall derive pointwise error estimates when the problem (1.1) is solved numerically using our LDG method on a Shishkin mesh. Our aim is to derive convergence rates for these errors that are locally sharp both inside and outside the boundary layers; moreover, these rates should be robust, *i.e.*, they should remain valid as the small parameter ε approaches zero.

The LDG method solves second-order boundary value problems by transforming the differential equation into first-order systems then applying the DG method [29] to approximate each variable. The method has several desirable features: it achieves high-order accuracy in solutions and their gradients, tends to suppress oscillations when solving boundary layer problems on standard meshes [39], offers flexibility in its *hp*-adaptivity, and does not require mesh-dependent artificial parameters to achieve stability, unlike interior penalty DG methods [15]. For further information on the LDG method, see, *e.g.*, [14, 33].

The LDG method has been applied to address a broad spectrum of singularly perturbed problems, such as convection-diffusion problems [9, 43], reaction-diffusion problems [7, 21], third-order problems [40], and fourth-order problems [26]. The error estimates for all these problems are presented in energy or balanced norms. Pointwise error estimates have been proved only for quasi-uniform or shape-regular meshes, *e.g.*, elliptic problems [5] and convection-diffusion problems [8]. In these papers, weighted energy norms and discrete Green's functions are combined to derive pointwise error estimates in subregions that are far from boundary layers; this is also the case in the well-known paper [18] which applies a standard finite element method to solve a convection-diffusion problem.

But in some applications, such as the computation of lift and drag [32], one needs to compute accurately the solution inside the boundary layers. For this purpose it is beneficial to employ layer-adapted meshes that are capable of effectively resolving these layers. Although some uniform-in-ε pointwise error estimates inside boundary layers are known for convection-diffusion problems [24, 34, 36], analogous results for reaction-diffusion problems like (1.1) do not appear in the research literature except as described below.

Remark 1.1. The reaction-diffusion problem (1.1) has a simpler structure than that of related convection-diffusion problems, but this does not imply that a pointwise error estimate for (1.1) is easier to establish. In particular, no stability is available from some type of streamline stabilisation, since (1.1) has no streamlines, nor across interior element interfaces, unlike the convection-diffusion case. Thus, there are significant difficulties in proving optimal pointwise convergence results inside the boundary layers of the true solution.

For the 1D reaction-diffusion problem analogous to (1.1), in [6, 42] suboptimal pointwise error estimates were deduced from L^2 error bounds and an inverse inequality. In [37], an improved 1D pointwise error estimate was obtained for an LDG method based on alternating numerical fluxes, using composite projectors and a discrete Green's function.

In the recent paper [38], which again used an LDG method based on alternating numerical fluxes, we derived pointwise error bounds for the 2D problem (1.1): it was proved that the LDG method on a Shishkin mesh satisfies pointwise error estimates that are $O(N^{-(k+1)})$ outside layers, $O(N^{-k})$ within boundary layer regions, and $O(N^{-(k-1)})$ within corner layer regions. Here N is the number of mesh elements along each coordinate axis and k denotes the degree of the piecewise polynomials within the finite element space. While these results show some progress in obtaining pointwise error estimates for 2D reaction-diffusion problems, they are unsatisfactory inside the layers; for instance, if bilinear elements are used so $k = 1$, then the above $O(N^{-(k-1)})$ result does not give a positive order of convergence in the corner layer.

The objective of the present paper is to obtain optimal pointwise error estimates in the entire domain Ω for the LDG method on a Shishkin mesh applied to the reaction-diffusion problem (1.1). The error analysis has substantial differences from [38]:

– Our LDG method employs a recently-developed "layer-upwind" numerical flux [11], in contrast to the alternating numerical flux of [38]. Where the mesh is fine near the boundary $\partial\Omega$, the layer-upwind flux takes its value in the direction from which the layer originates, while on the coarse mesh portion of Ω the flux simply uses a local average value. For this special formulation of the LDG method, we can prove a significant superclose property: the energy norm error between the computed solution and a composite projection of the true solution includes a factor $\varepsilon^{1/4}$, which greatly facilitates our analysis.
– Applying an inverse inequality and the aforementioned superclose property, in Theorem 3.5 we achieve pointwise convergence with dominant order $O(N^{-(k+1)})$ in the coarse-mesh portion of Ω. (Here N and k have the same meanings as described above for [38].) Although this dominant convergence rate is the same as in Theorem 4.1 from [38], the remaining terms in the error bound, which contain positive powers of the small parameter ε, are slightly sharper than in [38]; see Remark 3.6.
– In the corner layer region Ω_f where the mesh is fine in both coordinate directions, we use a discrete Sobolev inequality (Lem. 4.1) to bound the energy norm of the discrete Green's function $\boldsymbol{G}$ in Lemma 4.3. This result improves the bound of Lemma 3.5 from [38] by a factor $O(N^{-1/2})$. The discrete Green's function analysis leads to a nearly optimal pointwise convergence rate of $O\left(N^{-(k+1)}(\ln N)^{k+3}\right)$ in Ω_f, which is $O\left(N^{-2}(\ln N)^{k+3}\right)$ sharper than Theorem 5.1 of [38] when terms multiplied by positive powers of ε are ignored.
– The boundary layer region Ω_b is significantly more difficult to handle because the Shishkin mesh is highly anisotropic, which gives error bounds that are far from optimal if inverse inequalities or the previous discrete Green's function technique is used. Thus in Section 5 we develop a novel weighted energy-norm analysis with cut-off functions to estimate the discrete Green's function, leading to a pointwise error bound in Ω_b with a nearly optimal $O\left(N^{-(k+1)}(\ln N)^{k+5/2}\right)$ convergence rate that is almost in agreement with our numerical results and is $O\left(N^{-1}(\ln N)^{k+5/2}\right)$ sharper than Theorem 4.1 of [38] when terms multiplied by positive powers of ε are ignored.

The above results can be also extended to other boundary conditions, *e.g.*, Neumann boundary condition, where the boundary layers are typically weaker (see [26,40]). It is also relatively straightforward to extend our results to higher-dimensional problems (*i.e.*, $\Omega = (0,1)^d$ for $d > 2$), provided that the obvious generalisation of the *a priori* decomposition and derivative bounds of Assumption 2.1 is valid for the true solution of the problem. To the best of our knowledge, the pointwise error estimates that we establish in the present paper are the sharpest known uniform-in-ε error bounds for any LDG method when solving a two-dimensional reaction-diffusion problem whose solution exhibits boundary layers.

The paper is organised as follows. Section 2 defines the Shishkin mesh and presents the LDG formulation. In Section 3 we introduce some preliminary analysis including the projectors used in the error analysis, then we derive energy and balanced-norm error estimates, together with pointwise error estimates in the coarse-mesh region. Section 4 presents a pointwise error estimate in the corner layer regions after first investigating the properties of the associated discrete Green's function. In Section 5 we prove pointwise error estimates in the boundary layer region by exploiting various properties of an appropriate cut-off function. Numerical results are provided in Section 6 to test our theoretical error bounds. Finally, we give some concluding remarks in Section 7.

Notation. Throughout the paper, $C > 0$ denotes a generic constant, which may depend on the data b, f of (1.1), the user-chosen parameter σ of (2.2), and the polynomial degree k in the finite element space. Note that C does not depend on ε and N – the number of mesh elements in the x- and y- directions – and the value of C can vary when it is used in different instances.

We will use the standard Sobolev spaces $W^{m,p}(D)$, $H^m(D)$ and $L^p(D)$, where D is a measurable subset of the domain Ω. The $L^2(D)$ and $L^\infty(D)$ norms are denoted by $\|\cdot\|_D$ and $\|\cdot\|_{L^\infty(D)}$ respectively. The $L^2(D)$ inner product is written as $\langle\cdot,\cdot\rangle_D$. If $D = \Omega$, we omit the subscript D from these notations.

2. The problem, mesh and numerical scheme

This section presents the regularity of the solution of (1.1), Shishkin mesh and the LDG scheme.

2.1. Regularity of the true solution

The solution u of (1.1) typically exhibits an exponential boundary layer along each side of $\partial\Omega$. Assuming a sufficient number of smoothness and compatibility conditions at the corners of Ω, one can decompose u into components for which pointwise bounds on the derivatives are explicitly given. Lengthy derivations of such results are given in [1, 13, 16, 28]; see also p.257, Remark III.1.27 from [31] and Remark 4.18 from [35].

Assumption 2.1. *Assume that the solution of* (1.1) *can be decomposed as*

$$u = u^{\mathrm{s}} + \sum_{i=1}^{4} u_{i,\varepsilon}^{\mathrm{b}} + \sum_{i=1}^{4} u_{i,\varepsilon}^{\mathrm{c}},$$

where u^{s} is smooth, each $u_{i,\varepsilon}^{\mathrm{b}}$ corresponds to a boundary layer associated with the edge Γ_i and each $u_{i,\varepsilon}^{\mathrm{c}}$ corresponds to a corner layer associated with the point c_i (see Fig. 1). For all $(x,y) \in \bar{\Omega}$ and $0 \le \ell + m \le \kappa + 2$ (κ is a positive integer), the following pointwise bounds hold true:

$$\left|\frac{\partial^{\ell+m} u^{\mathrm{s}}(x,y)}{\partial x^\ell \partial y^m}\right| \le C, \tag{2.1a}$$

$$\left|\frac{\partial^{\ell+m} u_{1,\varepsilon}^{\mathrm{b}}(x,y)}{\partial x^\ell \partial y^m}\right| \le C\varepsilon^{-\ell/2} e^{-\beta x/\sqrt{\varepsilon}}, \tag{2.1b}$$

$$\left|\frac{\partial^{\ell+m} u_{1,\varepsilon}^{\mathrm{c}}(x,y)}{\partial x^\ell \partial y^m}\right| \le C\varepsilon^{-(\ell+m)/2} e^{-\beta x/\sqrt{\varepsilon}} e^{-\beta y/\sqrt{\varepsilon}} \tag{2.1c}$$

with analogous bounds for the remaining components.

Assumption 2.1 is valid for $\kappa = 2$, if one assumes that f and b belong to the Hölder space $C^{4,\alpha}(\bar{\Omega})$ and satisfy certain compatibility conditions at four corner points, *e.g.*, $f(1,0) = f(0,1) = f(0,0) = f(1,1) = 0$, as discussed in Section 1.2 from [25].

2.2. Shishkin mesh

The Shishkin mesh is well known to yield uniform convergence in ε when it is combined with some suitable numerical method. The 2D Shishkin mesh is simply a tensor product of 1D Shishkin meshes; see [23, 31, 35].

Let N be a multiple of 4. The *mesh transition parameter* is defined by

$$\tau = \min\left\{\frac{1}{4}, \frac{\sigma}{\beta}\sqrt{\varepsilon}\ln N\right\}, \tag{2.2}$$

where the parameter $\sigma > 0$ will affect our error estimates and will be determined later. In Section 2.3, we employ a discrete space that comprises piecewise polynomials of degree $k \geq 0$. Introduce the notation $\tilde{k} = k$ for odd k and $\tilde{k} = k + 1$ for even k. We shall require $\sigma \geq \tilde{k} + 1$ to obtain satisfactory error bounds (cf. [10, 11]). Since we are always interested in the singularly perturbed regime, we assume that $\sigma\sqrt{\varepsilon}\beta^{-1}\ln N \leq 1/4$; if this inequality is not satisfied, one can then carry out the error analysis using conventional methods on uniform meshes. These observations lead us to replace (2.2) by

$$\tau = \sigma\sqrt{\varepsilon}\beta^{-1}\ln N, \quad \text{where } \sigma \geq \tilde{k} + 1.$$

Set

$$H = 2(1 - 2\tau)N^{-1} \quad \text{and} \quad h = 4\tau N^{-1}.$$

The domain Ω is then divided into a mesh Ω^N of rectangular elements $K_{ij} := (x_{i-1}, x_i) \times (y_{j-1}, y_j)$, whose boundaries are parallel to the coordinate axes through the mesh nodes (x_i, y_j) that are defined by $x_0 = y_0 = 0$ and

$$x_i = y_i = \begin{cases} x_{i-1} + h & \text{for } i = 1, 2, \ldots, N/4, \\ x_{i-1} + H & \text{for } i = N/4 + 1, N/4 + 2, \ldots, 3N/4, \\ x_{i-1} + h & \text{for } i = 3N/4 + 1, 3N/4 + 2, \ldots, N. \end{cases} \tag{2.3}$$

Throughout the paper, we shall always write $I_i = (x_{i-1}, x_i)$ and $J_j = (y_{j-1}, y_j)$; thus, the element $K_{ij} = I_i \times J_j$. Furthermore, we set $h_i = x_i - x_{i-1}$; since $x_i = y_i$ for all i, we then have $h_j = y_j - y_{j-1}$.

The horizontal lines $y = \tau$ and $y = 1 - \tau$ and the vertical lines $x = \tau$ and $x = 1 - \tau$ divide $\bar{\Omega}$ into the 9 subdomains Ω_{ij} $(i, j = 1, 2, 3)$ depicted in Figure 1. Three types of subdomain appear: the coarse-mesh domain Ω_{22}, the 4 boundary layer subdomains $\Omega_{12}, \Omega_{21}, \Omega_{23}, \Omega_{32}$ where the mesh is highly anisotropic, and the 4 corner layer subdomains $\Omega_{11}, \Omega_{13}, \Omega_{31}, \Omega_{33}$ where the mesh is fine in both coordinate directions. Furthermore, we define macro-regions as follows:

$$\Omega_i^x := \bigcup_{j=1}^{3} \Omega_{ij} \quad \text{for } i = 1, 2, 3, \qquad \Omega_j^y := \bigcup_{i=1}^{3} \Omega_{ij} \quad \text{for } j = 1, 2, 3,$$
$$\Omega_b := \Omega_{12} \cup \Omega_{21} \cup \Omega_{23} \cup \Omega_{32}, \qquad \Omega_f := \Omega_{11} \cup \Omega_{13} \cup \Omega_{31} \cup \Omega_{33}.$$

In what follows, we will also use D^N to represent the set of rectangular elements $K_{ij} \subset D$, where $D = \Omega_{ij}, \Omega_i^x, \Omega_j^y, \Omega_b, \Omega_f$ for $i, j = 1, 2, 3$.

2.3. The LDG formulation

This section presents the LDG method with layer-upwind flux, which was proposed in [11] for the first time. Let Ω^N be the Shishkin mesh with N^2 rectangular elements $K_{ij} = I_i \times J_j$. Define the finite dimensional space as

$$\mathcal{V}_N = \{\chi \in L^2(\Omega) \colon \chi|_{K_{ij}} \in \mathcal{Q}^k(K_{ij}) \ \forall K_{ij} \in \Omega^N\},$$

where $\mathcal{Q}^k(K_{ij})$ denotes the space of functions that are tensor-product polynomials of degree at most $k \geq 0$ defined on K_{ij}. The jumps and averages across the vertical edges of K_{ij} are respectively denoted by $[\![\chi]\!]_{i,y} = \chi_{i,y}^+ - \chi_{i,y}^-$ and $\{\{\chi\}\}_{i,y} = \frac{1}{2}(\chi_{i,y}^+ + v_{i,y}^-)$ for $i = 1, 2, \ldots, N-1$, $[\![\chi]\!]_{0,y} = \chi_{0,y}^+ = \{\{\chi\}\}_{0,y}$ and $\{\{\chi\}\}_{N,y} = \chi_{N,y}^- = -[\![\chi]\!]_{N,y}$, where $\chi_{i,y}^\pm = \chi(x_i^\pm, y)$. One can similarly define $\chi_{x,j}^\pm$, jumps $[\![\chi]\!]_{x,j}$ and averages $\{\{\chi\}\}_{x,j}$ on the horizontal edges of each K_{ij}.

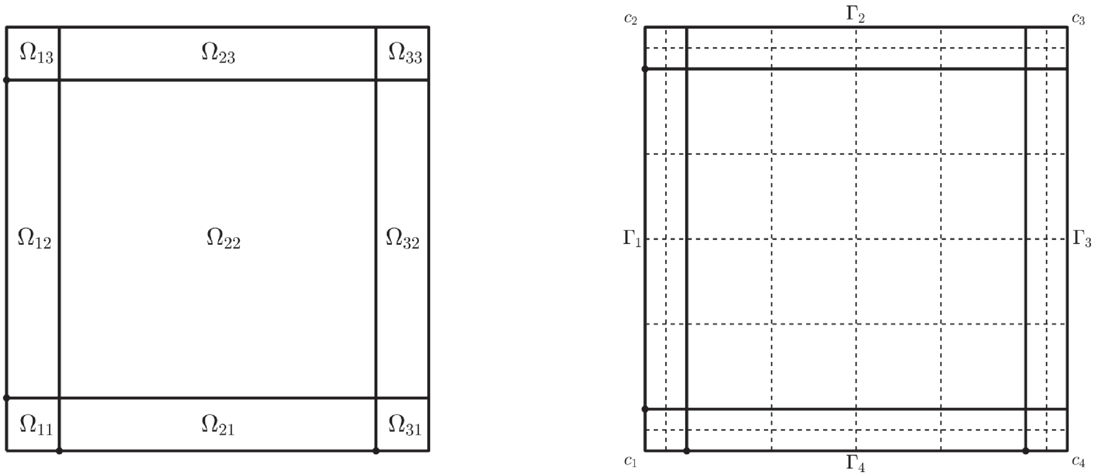

FIGURE 1. Domain division of Ω and Shishkin mesh Ω^8.

Let $\boldsymbol{w} = (u, p, q) := (u, \varepsilon u_x, \varepsilon u_y)$, then reformulate (1.1a) as a system of first-order differential equations:

$$bu - p_x - q_y = f, \qquad \varepsilon^{-1} p - u_x = 0 \quad \text{and} \quad \varepsilon^{-1} q - u_y = 0 \ \text{ in } \Omega \tag{2.4}$$

with the same boundary condition (1.1b). The LDG formulation is then obtained by finding the numerical approximation $\mathbb{W} = (\mathbb{U}, \mathbb{P}, \mathbb{Q}) \in [\mathcal{V}_N]^3 := \mathcal{V}_N \times \mathcal{V}_N \times \mathcal{V}_N$ such that the following variational equations are satisfied:

$$\langle b\mathbb{U}, \mathbb{v} \rangle_{K_{ij}} + \mathcal{D}_{ij}^x(\mathbb{P}, \mathbb{v}; \hat{\mathbb{P}}) + \mathcal{D}_{ij}^y(\mathbb{Q}, \mathbb{v}; \hat{\mathbb{Q}}) = \langle f, \mathbb{v} \rangle_{K_{ij}}, \tag{2.5a}$$

$$\varepsilon^{-1} \langle \mathbb{P}, \mathbb{s} \rangle_{K_{ij}} + \mathcal{D}_{ij}^x(\mathbb{U}, \mathbb{s}; \hat{\mathbb{U}}) = 0, \tag{2.5b}$$

$$\varepsilon^{-1} \langle \mathbb{Q}, \mathbb{r} \rangle_{K_{ij}} + \mathcal{D}_{ij}^y(\mathbb{U}, \mathbb{r}; \hat{\mathbb{U}}) = 0, \tag{2.5c}$$

in each element $K_{ij} \in \Omega^N$, for each test function $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_N]^3$, where

$$\mathcal{D}_{ij}^x(w, \chi; \hat{w}) := \langle w, \chi_x \rangle_{K_{ij}} - \left\langle \hat{w}_{i,y}, \chi_{i,y}^- \right\rangle_{J_j} + \left\langle \hat{w}_{i-1,y}, \chi_{i-1,y}^+ \right\rangle_{J_j}, \tag{2.6a}$$

$$\mathcal{D}_{ij}^y(w, \chi; \hat{w}) := \langle w, \chi_y \rangle_{K_{ij}} - \left\langle \hat{w}_{x,j}, \chi_{x,j}^- \right\rangle_{I_i} + \left\langle \hat{w}_{x,j-1}, \chi_{x,j-1}^+ \right\rangle_{I_i} \tag{2.6b}$$

are two bilinear functionals for DG spatial discretizations, defined on the element K_{ij}, with respect to x- and y-directions respectively. In the definition (2.5), the "hat" terms represent the numerical fluxes, which contribute significantly to the performance of the LDG scheme. We take their values in the "layer-upwind" direction, *i.e.*, on the fine mesh, the flux is chosen from the direction where the layer originates, while on the coarse mesh we use the usual central flux. Precise definitions are given in Table 1; see [11] for more details.

Set $\mathcal{D}^x(w, \chi; \hat{w}) = \sum_{K_{ij} \in \Omega^N} \mathcal{D}_{ij}^x(w, \chi; \hat{w})$ and $\mathcal{D}^y(w, \chi; \hat{w}) = \sum_{K_{ij} \in \Omega^N} \mathcal{D}_{ij}^y(w, \chi; \hat{w})$. Summing (2.5) over all $K_{ij} \in \Omega^N$, we get a compact form of the LDG scheme: Find $\mathbb{W} = (\mathbb{U}, \mathbb{P}, \mathbb{Q}) \in [\mathcal{V}_N]^3$ such that $B(\mathbb{W}; \mathbb{z}) = \langle f, \mathbb{v} \rangle$ for all $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_N]^3$, where

$$\begin{aligned} B(\mathbb{W}; \mathbb{z}) := {} & \langle b\mathbb{U}, \mathbb{v} \rangle + \varepsilon^{-1} \langle \mathbb{P}, \mathbb{s} \rangle + \varepsilon^{-1} \langle \mathbb{Q}, \mathbb{r} \rangle \\ & + \mathcal{D}^x(\mathbb{U}, \mathbb{s}; \hat{\mathbb{U}}) + \mathcal{D}^x(\mathbb{P}, \mathbb{v}; \hat{\mathbb{P}}) + \mathcal{D}^y(\mathbb{U}, \mathbb{r}; \hat{\mathbb{U}}) + \mathcal{D}^y(\mathbb{Q}, \mathbb{v}; \hat{\mathbb{Q}}). \end{aligned} \tag{2.7}$$

Note that the notation (2.7) can be extended to $B(\boldsymbol{w}; \mathbb{z})$ for $\boldsymbol{w} = (u, p, q)$ and $\mathbb{z} \in [\mathcal{V}_N]^3$ in a straightforward way.

TABLE 1. Numerical fluxes in the LDG scheme (2.5).

	$i=0$	$i=1,\dots,N/4$	$i=N/4+1,\dots,3N/4-1$	$i=3N/4,\dots,N-1$	$i=N$
$\hat{\mathbb{U}}_{i,y}$	0	$\mathbb{U}^-_{i,y}$	$\{\{\mathbb{U}\}\}_{i,y}$	$\mathbb{U}^+_{i,y}$	0
$\hat{\mathbb{P}}_{i,y}$	$\mathbb{P}^+_{0,y}$	$\mathbb{P}^+_{i,y}$	$\{\{\mathbb{P}\}\}_{i,y}$	$\mathbb{P}^-_{i,y}$	$\mathbb{P}^-_{N,y}$
$\hat{\mathbb{U}}_{x,i}$	0	$\mathbb{U}^-_{x,i}$	$\{\{\mathbb{U}\}\}_{x,i}$	$\mathbb{U}^+_{x,i}$	0
$\hat{\mathbb{Q}}_{x,i}$	$\mathbb{Q}^+_{x,0}$	$\mathbb{Q}^+_{x,i}$	$\{\{\mathbb{Q}\}\}_{x,i}$	$\mathbb{Q}^-_{x,i}$	$\mathbb{Q}^-_{x,N}$

3. Preliminary results

In this section, we introduce some preliminary results, including several inverse estimates, projectors, existing energy/balanced-norm error estimates and a preliminary pointwise error estimate in each part of Ω.

3.1. Inverse estimates

For any $i \in \{1,\dots,N-1\}$ and any nonnegative integer k, let $\mathcal{P}^k(I_i)$ be the space of polynomials of degree at most k defined on I_i. For each $\chi \in \mathcal{P}^k(I_i)$, one has the 1D inverse inequality:

$$|\chi(x)| \le \|\chi\|_{L^\infty(I_i)} \le C h_i^{-1/2} \|\chi\|_{I_i} \tag{3.1}$$

for any $x \in I_i$, where $C>0$ is a constant that is independent of h_i and χ.

For any $(x,y) \in K_{ij} = I_i \times J_j \in \Omega^N$ and each $\chi \in \mathcal{Q}^k(K_{ij})$, one has the 2D inverse inequalities:

$$\|\chi\|_{L^\infty(K_{ij})} \le C|K_{ij}|^{-1/\ell} \|\chi\|_{L^\ell(K_{ij})}, \qquad \ell > 0 \tag{3.2a}$$

$$\left\|\partial_x^m \partial_y^n \chi\right\|_{L^\ell(K_{ij})} \le C h_i^{-m} h_j^{-n} \|\chi\|_{L^\ell(K_{ij})}, \qquad \ell > 0, \tag{3.2b}$$

$$\|\chi(x,\cdot)\|_{J_j} \le C h_i^{-1/2} \|\chi\|_{K_{ij}} \qquad \forall x \in I_i, \tag{3.2c}$$

$$\|\chi(\cdot,y)\|_{I_i} \le C h_j^{-1/2} \|\chi\|_{K_{ij}} \qquad \forall y \in J_j, \tag{3.2d}$$

where $C>0$ is a constant that is independent of h_i, h_j and χ. For more details see Theorem 3.2.6 from [12].

3.2. Projectors

We shall use several composite projectors in the error analysis:

$$\mathcal{I}\phi := \begin{cases} \Pi^-_{xy}\phi & \text{in } \Omega_{11}, \\ \Pi^-_{y}\phi & \text{in } \Omega_{21}, \\ \Pi^\pm_{xy}\phi & \text{in } \Omega_{31}, \\ \Pi^-_{x}\phi & \text{in } \Omega_{12}, \\ \Pi\phi & \text{in } \Omega_{22}, \\ \Pi^+_{x}\phi & \text{in } \Omega_{32}, \\ \Pi^\mp_{xy}\phi & \text{in } \Omega_{13}, \\ \Pi^+_{y}\phi & \text{in } \Omega_{23}, \\ \Pi^+_{xy}\phi & \text{in } \Omega_{33}, \end{cases} \quad \mathcal{J}\phi := \begin{cases} \Pi^+_{x}\phi & \text{in } \Omega^x_1, \\ \Pi\phi & \text{in } \Omega^x_2, \\ \Pi^-_{x}\phi & \text{in } \Omega^x_3, \end{cases} \quad \text{and} \quad \mathcal{K}\phi := \begin{cases} \Pi^+_{y}\phi & \text{in } \Omega^y_1, \\ \Pi\phi & \text{in } \Omega^y_2, \\ \Pi^-_{y}\phi & \text{in } \Omega^y_3. \end{cases} \tag{3.3}$$

The projectors Π, Π^-_x, Π^-_{xy} and others will be defined piecewise. To be precise, on each rectanguar element K_{ij} they are defined as follows:

– For each $\phi \in L^2(K_{ij})$, define $\Pi\phi \in \mathcal{Q}^k(K_{ij})$ by

$$\langle \Pi\phi, \chi \rangle_{K_{ij}} = \langle \phi, \chi \rangle_{K_{ij}} \quad \forall \chi \in \mathcal{Q}^k(K_{ij}). \tag{3.4}$$

– Define $\mathcal{Q}_{ij}^{m,n} = \mathcal{P}^m(I_i) \otimes \mathcal{P}^n(J_j)$. For each $\phi \in C(\bar{K}_{ij})$, define $\Pi_x^-\phi, \Pi_x^+\phi, \Pi_y^-\phi, \Pi_y^+\phi \in \mathcal{Q}^k(K_{ij})$ by

$$\left\langle \Pi_x^+\phi - \phi, \chi \right\rangle_{K_{ij}} = 0 \quad \forall \chi \in \mathcal{Q}_{ij}^{k-1,k}, \qquad \left\langle (\Pi_x^+\phi - \phi)_{i-1,y}^+, \chi \right\rangle_{J_j} = 0 \quad \forall \chi \in \mathcal{P}^k(J_j); \tag{3.5a}$$

$$\left\langle \Pi_x^-\phi - \phi, \chi \right\rangle_{K_{ij}} = 0 \quad \forall \chi \in \mathcal{Q}_{ij}^{k-1,k}, \qquad \left\langle (\Pi_x^-\phi - \phi)_{i,y}^-, \chi \right\rangle_{J_j} = 0 \quad \forall \chi \in \mathcal{P}^k(J_j); \tag{3.5b}$$

$$\left\langle \Pi_y^+\phi - \phi, \chi \right\rangle_{K_{ij}} = 0 \quad \forall \chi \in \mathcal{Q}_{ij}^{k,k-1}, \qquad \left\langle (\Pi_y^+\phi - \phi)_{x,j-1}^+, \chi \right\rangle_{I_i} = 0 \quad \forall \chi \in \mathcal{P}^k(I_i); \tag{3.5c}$$

$$\left\langle \Pi_y^-\phi - \phi, \chi \right\rangle_{K_{ij}} = 0 \quad \forall \chi \in \mathcal{Q}_{ij}^{k,k-1}, \qquad \left\langle (\Pi_y^-\phi - \phi)_{x,j}^-, \chi \right\rangle_{I_i} = 0 \quad \forall \chi \in \mathcal{P}^k(I_i). \tag{3.5d}$$

– For each $\phi \in C(\bar{K}_{ij})$, define $\Pi_{xy}^+\phi \in \mathcal{Q}^k(K_{ij})$ by requiring the projection error $g(\phi) = \Pi_{xy}^+\phi - \phi$ to satisfy

$$\langle g(\phi), \chi \rangle_{K_{ij}} = 0 \qquad \forall \chi \in \mathcal{Q}^{k-1}(K_{ij}); \qquad \left\langle (g(\phi))_{i-1,y}^+, \chi \right\rangle_{J_j} = 0 \quad \forall \chi \in \mathcal{P}^{k-1}(J_j); \tag{3.6a}$$

$$\left\langle (g(\phi))_{x,j-1}^+, \chi \right\rangle_{I_i} = 0 \quad \forall \chi \in \mathcal{P}^{k-1}(I_i); \qquad g(\phi)(x_{i-1}^+, y_{j-1}^+) = 0. \tag{3.6b}$$

One can likewise define the projectors $\Pi_{xy}^{\pm}\phi$, $\Pi_{xy}^{\mp}\phi$ and $\Pi_{xy}^-\phi$; see Section 3.1 from [11]. These projectors are well-defined and satisfy the following approximation properties.

Lemma 3.1. *Suppose that Assumption 2.1 is valid for $\kappa = k$, then one has*

$$\|u - \mathcal{I}u\|_{\Omega\setminus\Omega_{22}} \le C\varepsilon^{1/4}(N^{-1}\ln N)^{k+1}, \qquad \|u - \mathcal{I}u\|_{\Omega_{22}} \le C\left[N^{-(k+1)} + \varepsilon^{1/4}N^{-\sigma}\right], \tag{3.7a}$$

$$\|u - \mathcal{I}u\|_{L^\infty(\Omega\setminus\Omega_{22})} \le C(N^{-1}\ln N)^{k+1}, \qquad \|u - \mathcal{I}u\|_{L^\infty(\Omega_{22})} \le CN^{-(k+1)}, \tag{3.7b}$$

$$\|p - \mathcal{J}p\|_{\Omega_1^x\cup\Omega_3^x} \le C\varepsilon^{3/4}(N^{-1}\ln N)^{k+1}, \qquad \|p - \mathcal{J}p\|_{\Omega_2^x} \le C\left[\varepsilon N^{-(k+1)} + \varepsilon^{1/2}N^{-\sigma}\right], \tag{3.7c}$$

$$\|p - \mathcal{J}p\|_{L^\infty(\Omega_1^x\cup\Omega_3^x)} \le C\varepsilon^{1/2}(N^{-1}\ln N)^{k+1}, \qquad \|p - \mathcal{J}p\|_{L^\infty(\Omega_2^x)} \le C\varepsilon^{1/2}N^{-(k+1)}, \tag{3.7d}$$

$$\|q - \mathcal{K}q\|_{\Omega_1^y\cup\Omega_3^y} \le C\varepsilon^{3/4}(N^{-1}\ln N)^{k+1}, \qquad \|q - \mathcal{K}q\|_{\Omega_2^y} \le C\left[\varepsilon N^{-(k+1)} + \varepsilon^{1/2}N^{-\sigma}\right], \tag{3.7e}$$

$$\|q - \mathcal{K}q\|_{L^\infty(\Omega_1^y\cup\Omega_3^y)} \le C\varepsilon^{1/2}(N^{-1}\ln N)^{k+1}, \qquad \|q - \mathcal{K}q\|_{L^\infty(\Omega_2^y)} \le C\varepsilon^{1/2}N^{-(k+1)}, \tag{3.7f}$$

where $C > 0$ is a constant independent of ε and N.

Proof. See Lemma 3.2 from [11]. □

3.3. Energy/balanced-norm error estimates

Let the energy norm $|||\cdot|||_E$ be defined on $[\mathcal{V}_N]^3$ through $|||\mathbb{z}|||_E^2 = B(\mathbb{z};\mathbb{z})$ for $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_N]^3$. Integrating by parts and straightforward calculations yield

$$|||\mathbb{z}|||_E^2 = \varepsilon^{-1}\|\mathbb{s}\|^2 + \varepsilon^{-1}\|\mathbb{r}\|^2 + \left\|b^{1/2}\mathbb{v}\right\|^2. \tag{3.8}$$

To recognise the contribution of the boundary layers, we also define the balanced norm

$$|||\mathbb{z}|||_B^2 = \varepsilon^{-3/2}\|\mathbb{s}\|^2 + \varepsilon^{-3/2}\|\mathbb{r}\|^2 + \left\|b^{1/2}\mathbb{v}\right\|^2,$$

which treats the regular and layer components of the true solution equally [10,17,22,30].

For the true solution $\boldsymbol{w} = (u, p, q)$ of (2.4), set $\mathcal{P}\boldsymbol{w} = (\mathcal{I}u, \mathcal{J}p, \mathcal{K}q) \in [\mathcal{V}_N]^3$. Write the error as $\boldsymbol{e} = \boldsymbol{w} - \mathbb{W}$, and split it into two parts, $\boldsymbol{e} = (\boldsymbol{w} - \mathcal{P}\boldsymbol{w}) - (\mathbb{W} - \mathcal{P}\boldsymbol{w}) := \boldsymbol{\eta} - \boldsymbol{\xi}$. For $\boldsymbol{\phi} \in \{\boldsymbol{e}, \boldsymbol{\eta}, \boldsymbol{\xi}\}$, we write $\boldsymbol{\phi} = (\phi_u, \phi_p, \phi_q)$. Then $\boldsymbol{\eta} = (\eta_u, \eta_p, \eta_q) = (u - \mathcal{I}u, p - \mathcal{J}p, q - \mathcal{K}q)$ and $\boldsymbol{\xi} = (\xi_u, \xi_p, \xi_q) = (\mathbb{U} - \mathcal{I}u, \mathbb{P} - \mathcal{J}p, \mathbb{Q} - \mathcal{K}q) \in [\mathcal{V}_N]^3$.

Recall that $\tilde{k} = k$ if k is odd and $\tilde{k} = k + 1$ if k is even. Define

$$\theta(\varepsilon, N, k) := (N^{-1} \ln N)^{k+1} + \varepsilon^{1/4} N^{-\tilde{k}} + \varepsilon^{1/4} N^{-(k+1/2)} + \varepsilon^{1/2} N^{-k} (\ln N)^{k+1}. \tag{3.9}$$

The quantity θ reflects the optimal convergence rate on the Shishkin mesh; see Remark 5.3 from [11]. In our convergence analysis, it is frequently multiplied by a positive power of ε. Such terms are considered less significant as ε is very small, *i.e.*, they play a subordinate role in the overall estimates.

For the energy and balanced-norm errors, we have the following estimates.

Lemma 3.2 ([11], Thm. 5.1)**.** *There exists a constant $C > 0$, which is independent of ε and N, such that*

$$|||\boldsymbol{\xi}|||_E \le C\varepsilon^{1/4}\theta(\varepsilon, N, k). \tag{3.10}$$

Furthermore, the following energy and balanced-norm error estimates hold true:

$$|||\boldsymbol{e}|||_E \le C\big[N^{-(k+1)} + \varepsilon^{1/4}\theta(\varepsilon, N, k)\big] \quad \text{and} \quad |||\, \boldsymbol{e}|||_B \le C\theta(\varepsilon, N, k).$$

For $\xi_p = p - \mathcal{J}p$, we have estimates at the interfaces where the mesh size changes abruptly.

Lemma 3.3. *We have*

$$\left(\sum_{j=1}^{N/4} \left\|(\xi_p)^+_{N/4,y}\right\|^2_{J_j}\right)^{1/2} \le C\varepsilon \left[N(\ln N)^{1/2} + \varepsilon^{1/4}N^{3/2}\right] \theta(\varepsilon, N, k); \tag{3.11a}$$

$$\left(\sum_{j=N/4+1}^{3N/4} \left\|(\xi_p)^+_{N/4,y}\right\|^2_{J_j}\right)^{1/2} \le C\varepsilon \left[N(\ln N)^{1/2} + \varepsilon^{1/4}N^{3/2}\right] \theta(\varepsilon, N, k). \tag{3.11b}$$

Proof. Since u and p satisfy the variational equation (2.5b), one has the Galerkin orthogonality property

$$\varepsilon^{-1} \langle e_p, \mathbb{s}\rangle_{K_{N/4+1,j}} + \mathcal{D}^x_{N/4+1,j}(e_u, \mathbb{s}; \hat{e_u}) = 0$$

for each $\mathbb{s} \in \mathcal{Q}^k(K_{N/4+1,j})$. It is easy to see that $\langle \eta_p, \mathbb{s}\rangle_{K_{N/4+1,j}} = \langle p - \Pi p, \mathbb{s}\rangle_{K_{N/4+1,j}} = 0$ for $j = 1, 2, \ldots, N/4$, so the decomposition $e_p = \eta_p - \xi_p$ gives

$$\varepsilon^{-1} \langle \xi_p, \mathbb{s}\rangle_{K_{N/4+1,j}} = \mathcal{D}^x_{N/4+1,j}(e_u, \mathbb{s}; \hat{e_u}) \tag{3.12}$$

for any $\mathbb{s} \in \mathcal{Q}^k(K_{N/4+1,j})$ and $j = 1, 2, \ldots, N/4$. Invoking the inequalities (3.2b) and (3.2c) yields

$$\begin{aligned} \left|\mathcal{D}^x_{N/4+1,j}(\xi_u, \mathbb{s}; \hat{\xi_u})\right| &= \left|\langle \xi_u, \mathbb{s}_x\rangle_{K_{N/4+1,j}} - \left\langle \{\{\xi_u\}\}_{N/4+1,y}, \mathbb{s}^-_{N/4+1,y}\right\rangle_{J_j} + \left\langle (\xi_u)^-_{N/4,y}, \mathbb{s}^+_{N/4,y}\right\rangle_{J_j}\right| \\ &\le C\left(N\left\|\xi_u\right\|_{K_{N/4+1,j}\cup K_{N/4+2,j}} + N^{1/2}\left\|(\xi_u)^-_{N/4,y}\right\|_{J_j}\right)\|\mathbb{s}\|_{K_{N/4+1,j}} \end{aligned} \tag{3.13}$$

and

$$\left|\mathcal{D}^x_{N/4+1,j}\left(\eta_u, \mathbb{s}; \hat{\eta_u}\right)\right| \le C\left(N\left\|\eta_u\right\|_{K_{N/4+1,j}} + N^{1/2}h_j^{1/2}\left\|\eta_u\right\|_{L^\infty(\Omega)}\right)\|\mathbb{s}\|_{K_{N/4+1,j}}. \tag{3.14}$$

Take $\mathbb{s} = \xi_p$ in (3.12); then (3.13) and (3.14) give

$$\|\xi_p\|_{K_{N/4+1,j}} \le C\varepsilon\Big[N\,\|\xi_u\|_{K_{N/4+1,j}\cup K_{N/4+2,j}} + N^{1/2}\left\|(\xi_u)^-_{N/4,y}\right\|_{J_j} + N\,\|\eta_u\|_{K_{N/4+1,j}} + N^{1/2}h_j^{1/2}\,\|\eta_u\|_{L^\infty(\Omega)}\Big]. \tag{3.15}$$

It follows from inequality (4.6a) of [11] that

$$\left(\sum_{j=1}^{N/4}\left\|(\xi_u)^-_{N/4,y}\right\|^2_{J_j}\right)^{1/2} \le C\varepsilon^{-1/2}\tau^{1/2}\Big[\,|||\,\boldsymbol{\xi}\,|||_E + \varepsilon^{1/4}(N^{-1}\ln N)^{k+1}\Big]. \tag{3.16}$$

Applying (3.2c), (3.15), $\|\xi_u\| \le C\,|||\,\boldsymbol{\xi}|||_E$, (3.16) and Lemma 3.1, we obtain

$$\begin{aligned}\sum_{j=1}^{N/4}\left\|(\xi_p)^+_{N/4,y}\right\|^2_{J_j} &\le C\sum_{j=1}^{N/4}N\,\|\xi_p\|^2_{K_{N/4+1,j}}\\ &\le C\varepsilon^2N^3\,|||\,\boldsymbol{\xi}\,|||_E^2 + C\varepsilon N^2\tau\Big[\,|||\,\boldsymbol{\xi}\,|||_E^2 + \varepsilon^{1/2}(N^{-1}\ln N)^{2(k+1)}\Big]\\ &\quad + C\varepsilon^2N^3\,\|\eta_u\|^2_{\Omega_{21}} + C\varepsilon^2N^2\tau\,\|\eta_u\|^2_{L^\infty(\Omega)}\\ &\le C\varepsilon^2\left[N^2\ln N + \varepsilon^{1/2}N^3\right]\theta^2(\varepsilon,N,k).\end{aligned}$$

The bound (3.11a) follows immediately. One can prove (3.11b) in a similar way. □

Remark 3.4. The inverse estimate (3.2c) and the superclose inequality (3.10) quickly yield

$$\left(\sum_{j=1}^{N/4}\left\|(\xi_p)^+_{N/4,y}\right\|^2_{J_j}\right)^{1/2} \le CN^{1/2}\,\|\xi_p\| \le C\varepsilon^{1/2}N^{1/2}\,|||\,\boldsymbol{\xi}|||_E \le C\varepsilon^{3/4}N^{1/2}\theta(\varepsilon,N,k),$$

but the bound (3.11a) is sharper than this simpler bound by a factor $\varepsilon^{1/4}$.

3.4. Pointwise error estimates in the coarse-mesh region Ω_{22}

Energy-norm and balanced-norm error estimates for our LDG method (2.7) were derived in [11]; the present paper extends this work by establishing local pointwise error bounds. We now prove our first result of this type.

Theorem 3.5. *There exists a constant C such that for all $(x^\star,y^\star)\in\Omega$ one has*

$$|(u-\mathbb{U})(x^\star,y^\star)| \le \begin{cases} C\left[N^{-(k+1)} + \left(\varepsilon^{1/4}N\right)\,\theta(\varepsilon,N,k)\right] & \text{if } (x^\star,y^\star)\in\Omega_{22},\\ CN(\ln N)^{-1/2}\theta(\varepsilon,N,k) & \text{if } (x^\star,y^\star)\in\Omega_b,\\ C\varepsilon^{-1/4}N(\ln N)^{-1}\theta(\varepsilon,N,k) & \text{if } (x^\star,y^\star)\in\Omega_f.\end{cases}$$

Proof. Suppose that $(x^\star,y^\star)\in K\in\Omega^N$. Then inequality (3.7b) of Lemma 3.1 yields

$$|\eta_u(x^\star,y^\star)| \le \|\eta_u\|_{L^\infty(K)} \le \begin{cases} CN^{-(k+1)} & \text{if } K\subset\Omega_{22},\\ C(N^{-1}\ln N)^{k+1} & \text{if } K\subset\Omega\backslash\Omega_{22}.\end{cases}$$

Using (3.2a), Lemma 3.2 and $\|\xi_u\| \le C\,|||\,\boldsymbol{\xi}|||_E$, one gets

$$|\xi_u(x^\star,y^\star)| \le C|K|^{-1/2}\,|||\,\boldsymbol{\xi}|||_E \le \begin{cases} C(\varepsilon^{1/4}N)\theta(\varepsilon,N,k) & \text{if } K\subset\Omega_{22},\\ CN(\ln N)^{-1/2}\theta(\varepsilon,N,k) & \text{if } K\subset\Omega_b,\\ C\varepsilon^{-1/4}N(\ln N)^{-1}\theta(\varepsilon,N,k) & \text{if } K\subset\Omega_f.\end{cases}$$

As $u-\mathbb{U} = \eta_u+\xi_u$, the desired result follows from the above bounds. □

Remark 3.6. In the coarse-mesh domain Ω_{22}, Theorem 3.5 and the definition (3.9) of $\theta(\varepsilon, N, k)$ yield the pointwise error estimate

$$|(u-\mathbb{U})(x^\star, y^\star)| \le C\Big[N^{-(k+1)} + \varepsilon^{1/4}N^{-k}(\ln N)^{k+1} + \varepsilon^{1/2}N^{-(k-1/2)} + \varepsilon^{1/2}N^{-(\tilde{k}-1)} + \varepsilon^{3/4}N^{-(k-1)}(\ln N)^{k+1}\Big].$$

When ε is very small, the dominant rate $O(N^{-(k+1)})$ here is the same as in Theorem 4.1 from [38], but the other terms are slightly sharper. This dominant rate agrees exactly with the numerical results of Section 6.

The convergence rates of Theorem 3.5 in the boundary and corner layer parts of the mesh (*i.e.*, when $K \subset \Omega_b \cup \Omega_f$) are significantly lower than the rates observed numerically in Section 6. Much deeper analyses in Sections 4 and 5 will improve these bounds to a satisfactory standard.

4. Pointwise convergence in the corner layer region Ω_f

Theorem 3.5 gives a pointwise convergence rate that is almost $O(\varepsilon^{-1/4}N^{-k}(\ln N)^k)$ in the corner layer region Ω_f, but this rate is significantly inferior to the accuracy that is demonstrated in the numerical tests of Section 6. Furthermore, the presence of the factor $\varepsilon^{-1/4}$ in the bound is undesirable because it makes the error bound non-uniform with respect to ε. To derive a sharper error bound in Ω_f that is uniform in ε and resembles the numerical results, we shall use a discrete Green's function.

4.1. The compact LDG scheme in Ω_f

We consider the pointwise error estimate for an arbitrary point $(x^\star, y^\star) \in \Omega_{11}$. All constants C that appear in Section 4 are independent of the particular choice of $(x^\star, y^\star)$. Define on Ω_{11} the discrete space

$$\mathcal{V}_C := \left\{\chi \in L^2(\Omega_{11})\colon \chi|_{K_{ij}} \in \mathcal{Q}^k(K_{ij})\ \forall K_{ij} \in \Omega_{11}^N\right\}.$$

Observe that $\mathcal{V}_C \subset W^{1,\ell}(\Omega_{11}^N) := \{\phi \in L^2(\Omega_{11})\colon \phi|_{K_{ij}} \in W^{1,\ell}(K_{ij})\ \forall K_{ij} \in \Omega_{11}^N\}$ for all $\ell \in (0, +\infty]$. For any $\phi \in W^{1,\ell}(\Omega_{11}^N)$, we use the notation

$$\|\phi\|_{L^\ell(\Omega_{11}^N)} := \left(\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}} \phi^\ell \,\mathrm{d}x\,\mathrm{d}y\right)^{1/\ell}, \quad 0 < \ell < +\infty.$$

Define the H^1 semi-norm by $|\phi|_{H^1(\Omega_{11}^N)} := (\|\phi_x\|^2_{L^2(\Omega_{11}^N)} + \|\phi_y\|^2_{L^2(\Omega_{11}^N)})^{1/2}$, and set $|[\![\phi]\!]|_{\Gamma_{11}^N} = (|[\![\phi]\!]|^2_{\Gamma^N_{11,x}} + |[\![\phi]\!]|^2_{\Gamma^N_{11,y}})^{1/2}$, where

$$|[\![\phi]\!]|_{\Gamma^N_{11,x}} := \left(\sum_{i,j=1}^{N/4}\|[\![\phi]\!]_{i-1,y}\|^2_{J_j}\right)^{1/2} \quad\text{and}\quad |[\![\phi]\!]|_{\Gamma^N_{11,y}} := \left(\sum_{i,j=1}^{N/4}\|[\![\phi]\!]_{x,j-1}\|^2_{I_i}\right)^{1/2}.$$

We sum (2.5) over $K_{ij} \in \Omega_{11}^N$ to get a compact form of the LDG scheme in Ω_{11}: Find the numerical solution $\mathbb{W} = (\mathbb{U}, \mathbb{P}, \mathbb{Q}) \in [\mathcal{V}_C]^3 := \mathcal{V}_C \times \mathcal{V}_C \times \mathcal{V}_C$ such that

$$B_C(\mathbb{W}; \mathbb{z}) = \langle f, \mathbb{v}\rangle_{\Omega_{11}} - F_C(\mathbb{W}, \mathbb{v}) \qquad \forall \mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_C]^3, \tag{4.1}$$

where

$$\begin{aligned} B_C(\mathbb{W}; \mathbb{z}) :=\ & \langle b\mathbb{U}, \mathbb{v}\rangle_{\Omega_{11}} + \varepsilon^{-1}\langle \mathbb{P}, \mathbb{s}\rangle_{\Omega_{11}} + \varepsilon^{-1}\langle \mathbb{Q}, \mathbb{r}\rangle_{\Omega_{11}} \\ & + \sum_{K_{ij}\subset\Omega_{11}} \mathcal{D}^x_{ij}\left(\mathbb{U}, \mathbb{s}; \mathbb{U}^-\right) + \sum_{K_{ij}\subset\Omega_{11}} \mathcal{D}^y_{ij}\left(\mathbb{U}, \mathbb{r}; \mathbb{U}^-\right) \\ & + \langle \mathbb{P}, \mathbb{v}_x\rangle_{\Omega_{11}} + \sum_{j=1}^{N/4}\sum_{i=0}^{N/4-1}\left\langle \mathbb{P}^+_{i,y}, [\![\mathbb{v}]\!]_{i,y}\right\rangle_{J_j} + \langle \mathbb{Q}, \mathbb{v}_y\rangle_{\Omega_{11}} + \sum_{i=1}^{N/4}\sum_{j=0}^{N/4-1}\left\langle \mathbb{Q}^+_{x,j}, [\![\mathbb{v}]\!]_{x,j}\right\rangle_{I_i}; \end{aligned} \tag{4.2}$$

with $\mathbb{U}^-_{0,y} = \mathbb{U}^-_{x,0} = 0$ and

$$F_C(\mathbb{W}, \mathbb{v}) := \sum_{j=1}^{N/4} \left\langle \mathbb{P}^+_{N/4,y}, \mathbb{v}^-_{N/4,y} \right\rangle_{J_j} + \sum_{i=1}^{N/4} \left\langle \mathbb{Q}^+_{x,N/4}, \mathbb{v}^-_{x,N/4} \right\rangle_{I_i}. \tag{4.3}$$

Set $|||\mathbb{z}|||^2_{E,\Omega_{11}} := B_C(\mathbb{z}; \mathbb{z})$. One can verify that

$$|||\mathbb{z}|||^2_{E,\Omega_{11}} = \varepsilon^{-1}\left(\|\mathbb{s}\|^2_{\Omega_{11}} + \|\mathbb{r}\|^2_{\Omega_{11}}\right) + \left\|b^{1/2}\mathbb{v}\right\|^2_{\Omega_{11}}. \tag{4.4}$$

4.2. A discrete Sobolev inequality

We shall present a discrete Sobolev inequality that bounds the maximum norm of discrete space functions by the L^2-norm of their gradients and the jumps across the element boundary interfaces. An inequality of this type was established in equation (28) of [3], but our inequality is sharper in two essential ways: (i) the analysis of equation (28) from [3] does not take account of the area of the domain but here we work on the corner layer region Ω_{11}, whose small area affects the bound significantly; (ii) in equation (28) of [3] the H^1 and L^2 components of the H^1 norm are treated equally but we show that they should be scaled differently from each other.

Lemma 4.1. *For each $\chi \in \mathcal{V}_C$, we have*

$$\|\chi\|_{L^\infty(\Omega_{11})} \le C(\ln N)\left[|\chi|^2_{H^1(\Omega^N_{11})} + h^{-1}\|[\![\chi]\!]\|^2_{\Gamma^N_{11}}\right]^{1/2}. \tag{4.5}$$

Proof. Our proof follows the approach of Proposition 3.3 from [18] but the discontinuous nature of the function $\chi \in \mathcal{V}_C$ introduces significant difficulties.

For any $(x, y) \in K_{ij} = I_i \times J_j \in \Omega^N_{11}$ and each $\phi \in W^{1,\infty}(\Omega^N_{11})$, one has

$$\phi(x, y) = \left(\int_{x_{i-1}}^{x} + \int_{x_{i-2}}^{x_{i-1}} + \ldots + \int_{x_0}^{x_1}\right) \phi_{x'}(x', y)\,\mathrm{d}x' + \sum_{m=1}^{i} [\![\phi]\!]_{m-1,y}.$$

Hence

$$|\phi(x, y)| \le \sum_{m=1}^{N/4} \int_{I_m} |\phi_{x'}(x', y)|\,\mathrm{d}x' + \sum_{m=1}^{N/4} |[\![\phi]\!]_{m-1,y}|. \tag{4.6}$$

Similarly,

$$|\phi(x, y)| \le \sum_{n=1}^{N/4} \int_{J_n} |\phi_{y'}(x, y')|\,\mathrm{d}y' + \sum_{n=1}^{N/4} |[\![\phi]\!]_{x,n-1}|. \tag{4.7}$$

Multiplying (4.6) and (4.7), integrating over K_{ij} and removing the primes then summing over all $K_{ij} \in \Omega^N_{11}$, one obtains

$$\int_{\Omega^N_{11}} |\phi(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y \le A_1 + A_2 + A_3 + A_4, \tag{4.8}$$

where

$$A_1=\sum_{j=1}^{N/4}\sum_{m=1}^{N/4}\int_{K_{mj}}|\phi_x(x,y)|\,\mathrm{d}x\,\mathrm{d}y\sum_{i=1}^{N/4}\sum_{n=1}^{N/4}\int_{K_{in}}|\phi_y(x,y)|\,\mathrm{d}x\,\mathrm{d}y;$$
$$A_2=\sum_{j=1}^{N/4}\sum_{m=1}^{N/4}\int_{K_{mj}}|\phi_x(x,y)|\,\mathrm{d}x\,\mathrm{d}y\sum_{i=1}^{N/4}\sum_{n=1}^{N/4}\int_{I_i}|[\![\phi]\!]_{x,n-1}|\,\mathrm{d}x;$$
$$A_3=\sum_{j=1}^{N/4}\sum_{m=1}^{N/4}\int_{J_j}|[\![\phi]\!]_{m-1,y}|\,\mathrm{d}y\sum_{i=1}^{N/4}\sum_{n=1}^{N/4}\int_{K_{in}}|\phi_y(x,y)|\,\mathrm{d}x\,\mathrm{d}y;$$
$$A_4=\sum_{j=1}^{N/4}\sum_{m=1}^{N/4}\int_{J_j}|[\![\phi]\!]_{m-1,y}|\,\mathrm{d}y\sum_{i=1}^{N/4}\sum_{n=1}^{N/4}\int_{I_i}|[\![\phi]\!]_{x,n-1}|\,\mathrm{d}x.$$

Let $\phi=\chi^\ell$, where $\chi\in\mathcal{V}_C$ and $\ell>0$ is an integer yet to be determined. Then

$$\int_{\Omega_{11}^N}|\phi(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y=\|\chi\|^{2\ell}_{L^{2\ell}(\Omega_{11}^N)}\,. \tag{4.9}$$

Using Hölder's inequality in its integral and discrete forms, one has

$$\begin{aligned}\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{2(\ell-1)}\,\mathrm{d}x\,\mathrm{d}y&\le\sum_{K_{ij}\in\Omega_{11}^N}\left[|K_{ij}|^{1/\ell}\left(\int_{K_{ij}}|\chi|^{2\ell}\,\mathrm{d}x\,\mathrm{d}y\right)^{(\ell-1)/\ell}\right]\\&\le\left(\sum_{K_{ij}\in\Omega_{11}^N}|K_{ij}|\right)^{1/\ell}\left(\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{2\ell}\,\mathrm{d}x\,\mathrm{d}y\right)^{(\ell-1)/\ell}\\&=|\Omega_{11}|^{1/\ell}\,\|\chi\|^{2(\ell-1)}_{L^{2\ell}(\Omega_{11}^N)}\,.\end{aligned}\tag{4.10}$$

A Cauchy–Schwarz inequality and (4.10) give

$$\begin{aligned}\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{\ell-1}|\chi_x|\,\mathrm{d}x\,\mathrm{d}y&\le\left(\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi_x|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\left(\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{2(\ell-1)}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}\\&\le|\Omega_{11}|^{1/(2\ell)}\,\|\chi_x\|_{L^2(\Omega_{11}^N)}\,\|\chi\|^{\ell-1}_{L^{2\ell}(\Omega_{11}^N)}\,.\end{aligned}\tag{4.11}$$

Since $\phi_x=\ell\chi^{\ell-1}\chi_x$ and $\phi_y=\ell\chi^{\ell-1}\chi_y$, we use (4.11) to get

$$\begin{aligned}A_1&=\ell^2\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{\ell-1}|\chi_x|\,\mathrm{d}x\,\mathrm{d}y\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{\ell-1}|\chi_y|\,\mathrm{d}x\,\mathrm{d}y\\&\le\ell^2|\Omega_{11}|^{1/\ell}\,\|\chi_x\|_{L^2(\Omega_{11}^N)}\,\|\chi_y\|_{L^2(\Omega_{11}^N)}\,\|\chi\|^{2(\ell-1)}_{L^{2\ell}(\Omega_{11}^N)}\,.\end{aligned}\tag{4.12}$$

Next, we consider A_2. For $x\in I_i$ and $j=1,2,\dots,N/4-1$, one has

$$\begin{aligned}|[\![\phi]\!]_{x,j}|&=|(\chi^+_{x,j})^\ell-(\chi^-_{x,j})^\ell|\le|[\![\chi]\!]_{x,j}|\sum_{m=0}^{\ell-1}|\chi^+_{x,j}|^m|\chi^-_{x,j}|^{\ell-1-m}\\&\le\ell|[\![\chi]\!]_{x,j}|\,\|\chi\|^{\ell-1}_{L^\infty(K_{ij}\cup K_{i,j+1})}\le C\ell|K_{ij}\cup K_{i,j+1}|^{-(\ell-1)/(2\ell)}|[\![\chi]\!]_{x,j}|\,\|\chi\|^{\ell-1}_{L^{2\ell}(K_{ij}\cup K_{i,j+1})}\,,\end{aligned}\tag{4.13a}$$

where we applied the inverse inequality (3.2a) to χ in the final inequality. Furthermore, for $x \in I_i$ one has

$$|[\![\phi]\!]_{x,0}| = |(\chi_{x,0}^+)^\ell| \le |[\![\chi]\!]_{x,0}|\,\|\chi\|_{L^\infty(K_{i1})}^{\ell-1} \le C|[\![\chi]\!]_{x,0}||K_{i1}|^{-(\ell-1)/(2\ell)}\,\|\chi\|_{L^{2\ell}(K_{i1})}^{\ell-1}. \tag{4.13b}$$

Now $K_{ij} \in \Omega_{11}^N$ implies $|K_{ij}| = h^2$. We use (4.13), Cauchy–Schwarz and Hölder inequalities to get

$$\begin{aligned}
\sum_{i=1}^{N/4}\sum_{j=1}^{N/4}\int_{I_i} |[\![\phi]\!]_{x,j-1}|\,\mathrm{d}x &\le \sum_{i=1}^{N/4}\left(\int_{I_i}|[\![\chi]\!]_{x,0}|\,\mathrm{d}x\right)\left(|K_{i1}|^{-(\ell-1)/(2\ell)}\,\|\chi\|_{L^{2\ell}(K_{i1})}^{\ell-1}\right)\\
&\quad + \ell\sum_{i=1}^{N/4}\sum_{j=2}^{N/4}\left(\int_{I_i}|[\![\chi]\!]_{x,j-1}|\,\mathrm{d}x\right)\left(|K_{i,j-1}\cup K_{ij}|^{-(\ell-1)/(2\ell)}\,\|\chi\|_{L^{2\ell}(K_{i,j-1}\cup K_{ij})}^{\ell-1}\right)\\
&\le C\ell\left(\sum_{i=1}^{N/4}\sum_{j=1}^{N/4} h\,\|[\![\chi]\!]_{x,j-1}\|_{I_i}^2\right)^{1/2}\left(\sum_{i=1}^{N/4}\sum_{j=1}^{N/4} h^{-2}|K_{ij}|^{1/\ell}\,\|\chi\|_{L^{2\ell}(K_{ij})}^{2(\ell-1)}\right)^{1/2}\\
&\le C\ell h^{-1/2}|[\![\chi]\!]|_{\Gamma_{11,y}^N}\left(\sum_{K_{ij}\in\Omega_{11}^N}|K_{ij}|\right)^{1/(2\ell)}\left(\sum_{K_{ij}\in\Omega_{11}^N}\|\chi\|_{L^{2\ell}(K_{ij})}^{2\ell}\right)^{(\ell-1)/(2\ell)}\\
&= C\ell|\Omega_{11}|^{1/(2\ell)}\left(h^{-1/2}|[\![\chi]\!]|_{\Gamma_{11,y}^N}\right)\|\chi\|_{L^{2\ell}(\Omega_{11}^N)}^{\ell-1}.
\end{aligned} \tag{4.14}$$

Combining (4.11) and (4.14), one gets

$$\begin{aligned}
A_2 &= \ell\sum_{K_{ij}\in\Omega_{11}^N}\int_{K_{ij}}|\chi|^{\ell-1}|\chi_x|\,\mathrm{d}x\,\mathrm{d}y\sum_{i=1}^{N/4}\sum_{j=1}^{N/4}\int_{I_i}|[\![\phi]\!]_{x,j-1}|\,\mathrm{d}x\\
&\le \ell^2|\Omega_{11}|^{1/\ell}\,\|\chi_x\|_{L^2(\Omega_{11}^N)}\left(h^{-1/2}|[\![\chi]\!]|_{\Gamma_{11,y}^N}\right)\|\chi\|_{L^{2\ell}(\Omega_{11}^N)}^{2(\ell-1)}.
\end{aligned} \tag{4.15}$$

Similarly,

$$A_3 \le \ell^2|\Omega_{11}|^{1/\ell}\,\|\chi_y\|_{L^2(\Omega_{11}^N)}\left(h^{-1/2}|[\![\chi]\!]|_{\Gamma_{11,x}^N}\right)\|\chi\|_{L^{2\ell}(\Omega_{11}^N)}^{2(\ell-1)}, \tag{4.16}$$

$$A_4 \le \ell^2|\Omega_{11}|^{1/\ell}h^{-1}|[\![\chi]\!]|_{\Gamma_{11,x}^N}|[\![\chi]\!]|_{\Gamma_{11,y}^N}\,\|\chi\|_{L^{2\ell}(\Omega_{11}^N)}^{2(\ell-1)}. \tag{4.17}$$

Substituting (4.9), (4.12) and (4.15)–(4.17) into (4.8), cancelling the factor $\|\chi\|_{L^{2\ell}(\Omega_{11})}^{2(\ell-1)}$, then using $ab \le (a^2+b^2)/2$, one has

$$\begin{aligned}
\|\chi\|_{L^{2\ell}(\Omega_{11}^N)} &\le C\ell|\Omega_{11}|^{1/(2\ell)}\left[|\chi|_{H^1(\Omega_{11}^N)}^2 + h^{-1}|[\![\chi]\!]|_{\Gamma_{11}^N}^2\right]^{1/2}\\
&= C\ell\tau^{1/\ell}\left[|\chi|_{H^1(\Omega_{11}^N)}^2 + h^{-1}|[\![\chi]\!]|_{\Gamma_{11}^N}^2\right]^{1/2},
\end{aligned} \tag{4.18}$$

since $|\Omega_{11}| = \tau^2$. For any $(x,y) \in K_{ij} \subset \Omega_{11}$, the inverse inequality (3.2a) and (4.18) with $\ell = \lceil \ln N\rceil$ give

$$\begin{aligned}
|\chi(x,y)| &\le C|K_{ij}|^{-1/(2\ell)}\,\|\chi\|_{L^{2\ell}(K_{ij})} \le Ch^{-1/\ell}\,\|\chi\|_{L^{2\ell}(\Omega_{11}^N)}\\
&\le Ch^{-1/\ell}\ell\tau^{1/\ell}\left[|\chi|_{H^1(\Omega_{11}^N)}^2 + h^{-1}|[\![\chi]\!]|_{\Gamma_{11}^N}^2\right]^{1/2}\\
&\le C(\ln N)\left[|\chi|_{H^1(\Omega_{11}^N)}^2 + h^{-1}|[\![\chi]\!]|_{\Gamma_{11}^N}^2\right]^{1/2}.
\end{aligned}$$

The conclusion (4.5) follows. $\square$

4.3. The discrete Green's function

Recall that $(x^\star, y^\star) \in \Omega_{11}$. Associate with $(x^\star, y^\star)$ the discrete Green's function $\boldsymbol{G} \in [\mathcal{V}_C]^3$ that is defined by

$$B_C(\mathbb{z}; \boldsymbol{G}) = \mathbb{v}(x^\star, y^\star) \qquad \forall \mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_C]^3, \tag{4.19}$$

where $B_C(\cdot;\cdot)$ is given by (4.2). We shall write

$$\boldsymbol{G} := \boldsymbol{G}(x^\star, y^\star; \cdot) = (G_u(x^\star, y^\star; \cdot), G_p(x^\star, y^\star; \cdot), G_q(x^\star, y^\star; \cdot)) = (G_u, G_p, G_q).$$

The identity (4.4) implies that $\boldsymbol{G}$ is well defined.

Here $(G_u)_x \approx \varepsilon^{-1} G_p$ and $(G_u)_y \approx \varepsilon^{-1} G_q$ because the LDG scheme (2.5) is derived from the first-order system (2.4).

Lemma 4.2. *For any $K_{ij} = I_i \times J_j \in \Omega_{11}^N$, one has*

$$\|(G_u)_x\|_{K_{ij}} + h_i^{-1/2} \|[\![G_u]\!]_{i-1,y}\|_{J_j} \le C\varepsilon^{-1} \|G_p\|_{K_{ij}} \quad \text{for } i,j = 1,2,\ldots,N/4, \tag{4.20a}$$

$$\|(G_u)_y\|_{K_{ij}} + h_j^{-1/2} \|[\![G_u]\!]_{x,j-1}\|_{I_i} \le C\varepsilon^{-1} \|G_q\|_{K_{ij}} \quad \text{for } i,j = 1,2,\ldots,N/4. \tag{4.20b}$$

Proof. Taking $\mathbb{v} = \mathbb{r} = 0$ in (4.19), we get

$$\varepsilon^{-1} \langle \mathbb{s}, G_p \rangle_{K_{ij}} + \langle \mathbb{s}, (G_u)_x \rangle_{K_{ij}} + \left\langle \mathbb{s}_{i-1,y}^{+}, [\![G_u]\!]_{i-1,y} \right\rangle_{J_j} = 0 \tag{4.21}$$

for each $\mathbb{s} \in \mathcal{Q}^k(K_{ij})$.

(i) Take $\mathbb{s}|_{K_{ij}} = \frac{x - x_{i-1}}{h_i}(G_u)_x$. Then $\mathbb{s}_{i-1,y}^{+} = 0$. From (4.21), a scaling argument using norm equivalence on a reference element and the Cauchy–Schwarz inequality give

$$\|(G_u)_x\|_{K_{ij}}^2 \le C \left| \langle \mathbb{s}, (G_u)_x \rangle_{K_{ij}} \right| \le C \left| \varepsilon^{-1} \langle \mathbb{s}, G_p \rangle_{K_{ij}} \right| \le C\varepsilon^{-1} \|G_p\|_{K_{ij}} \|(G_u)_x\|_{K_{ij}}.$$

That is,

$$\|(G_u)_x\|_{K_{ij}} \le C\varepsilon^{-1} \|G_p\|_{K_{ij}}. \tag{4.22}$$

(ii) Take $\mathbb{s}|_{K_{ij}} = G_u - (G_u)_{i-1,y}^{-}$. Then $\mathbb{s}_{i-1,y}^{+} = [\![G_u]\!]_{i-1,y}$. For any $(x,y) \in K_{ij}$, it is easy to see that

$$|\mathbb{s}(x,y)|^2 = \left(\int_{x_{i-1}}^{x} (G_u)_x(x,y)\,\mathrm{d}x + [\![G_u]\!]_{i-1,y} \right)^2 \le 2h_i \int_{x_{i-1}}^{x_i} \left[(G_u)_x(x,y) \right]^2 \mathrm{d}x + 2[\![G_u]\!]_{i-1,y}^2.$$

Consequently,

$$\|\mathbb{s}\|_{K_{ij}}^2 \le 2h_i^2 \|(G_u)_x\|_{K_{ij}}^2 + 2h_i \|[\![G_u]\!]_{i-1,y}\|_{J_j}^2. \tag{4.23}$$

From (4.21), one uses the Cauchy–Schwarz and Young's inequalities and (4.23) to get

$$\begin{aligned}
\left\|[\![G_u]\!]_{i-1,y}\right\|_{J_j}^2 &= \left| -\varepsilon^{-1} \langle \mathbb{s}, G_p \rangle_{K_{ij}} - \langle \mathbb{s}, (G_u)_x \rangle_{K_{ij}} \right| \\
&\le \left(\varepsilon^{-1} \|G_p\|_{K_{ij}} + \|(G_u)_x\|_{K_{ij}} \right) \|\mathbb{s}\|_{K_{ij}} \\
&\le \frac{1}{4h_i} \|\mathbb{s}\|_{K_{ij}}^2 + 2h_i \left(\varepsilon^{-2} \|G_p\|_{K_{ij}}^2 + \|(G_u)_x\|_{K_{ij}}^2 \right) \\
&\le \frac{1}{2} \left\|[\![G_u]\!]_{i-1,y}\right\|_{J_j}^2 + Ch_i \left(\varepsilon^{-2} \|G_p\|_{K_{ij}}^2 + \|(G_u)_x\|_{K_{ij}}^2 \right).
\end{aligned}$$

Rearranging this inequality and invoking (4.22), we find that

$$h_i^{-1/2}\left\|[\![G_u]\!]_{i-1,y}\right\|_{J_j} \le C\varepsilon^{-1}\left\|G_p\right\|_{K_{ij}}. \tag{4.24}$$

The two inequalities (4.22) and (4.24) give the bound (4.20a). The inequality (4.20b) can be proved in a similar manner.

□

Lemma 4.3. *Let $\boldsymbol{G} \in [\mathcal{V}_C]^3$ be the solution of (4.19). Then one has*

$$|||\boldsymbol{G}|||_{E,\Omega_{11}} \le C\varepsilon^{-1/2}\ln N. \tag{4.25}$$

Proof. Taking $\mathbb{z} = \boldsymbol{G}$ in (4.19) gives

$$|||\boldsymbol{G}|||^2_{E,\Omega_{11}} = B_C(\boldsymbol{G};\boldsymbol{G}) = G_u(x^\star,y^\star). \tag{4.26}$$

But Lemmas 4.1 and 4.2 yield

$$\begin{aligned}|G_u(x^\star,y^\star)| &\le \|G_u\|_{L^\infty(\Omega_{11})} \le C(\ln N)\left[|G_u|^2_{H^1(\Omega^N_{11})} + h^{-1}|[\![G_u]\!]|^2_{\Gamma^N_{11}}\right]^{1/2}\\ &\le C\varepsilon^{-1}(\ln N)\left\|G_p\right\|_{\Omega_{11}} \le C\varepsilon^{-1/2}(\ln N)\,|||\,\boldsymbol{G}\,|||_{E,\Omega_{11}}.\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) gives (4.25). □

Remark 4.4. It follows from (4.25) and (4.27) that

$$\|G_u\|_{L^1(\Omega_{11})} \le C\tau^2\left\|G_u\right\|_{L^\infty(\Omega_{11})} \le C\varepsilon^{1/2}(\ln N)^3\,|||\,\boldsymbol{G}|||_{E,\Omega_{11}} \le C(\ln N)^4 \tag{4.28}$$

for any $(x^\star,y^\star) \in \Omega_{11}$.

Let us compare this bound with the analogous bound for the corresponding continuous Green's function $\overline{G}(x^\star,y^\star;x,y)$. Assume that $b>0$ is constant for simplicity. Following the analysis in Section 5.1 of [19], one has

$$\overline{G}(x^\star,y^\star;x,y) = g(x^\star,y^\star;x,y) - g(-x^\star,y^\star;x,y) - g(x^\star,-y^\star;x,y) + g(-x^\star,-y^\star;x,y),$$

where the function $g(x^\star,y^\star;x,y) = (2\pi\varepsilon)^{-1}K_0(r\sqrt{b/\varepsilon})$, with $r = \sqrt{(x-x^\star)^2+(y-y^\star)^2}$ and K_0 is the modified Bessel function of the second kind of order zero, which satisfies $0 < K_0(s) < Cs^{-1}e^{-s}$ for all $s>0$. Let $s = r\sqrt{b/\varepsilon}$. By a transformation into polar coordinates, one has $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta = C\varepsilon s\;\mathrm{d}s\,\mathrm{d}\theta$. This yields

$$\|g(x^\star,y^\star;\cdot)\|_{L^1(\Omega_{11})} \le C\varepsilon^{-1}\int_{\Omega_{11}} K_0\left(r\sqrt{\frac{b}{\varepsilon}}\right)\mathrm{d}x\,\mathrm{d}y \le C\int_0^{+\infty} sK_0(s)\,\mathrm{d}s \le C.$$

Hence $\left\|\overline{G}(x^\star,y^\star;\cdot)\right\|_{L^1(\Omega_{11})} \le C$. Thus our bound (4.28) is sharp up to the factor $(\ln N)^4$.

The next lemma bounds G_u along the fine/coarse mesh interfaces.

Lemma 4.5. *We have*

$$\left(\sum_{j=1}^{N/4}\left\|(G_u)^-_{N/4,y}\right\|^2_{J_j}\right)^{1/2} \le C\sqrt{\frac{\tau}{\varepsilon}}\,|||\,\boldsymbol{G}\,|||_{E,\Omega_{11}}. \tag{4.29}$$

Proof. It is easy to see that $(G_u)^-_{N/4,y} = \sum_{i=1}^{N/4} \int_{I_i} (G_u)_x \,\mathrm{d}x + \sum_{i=1}^{N/4} [\![G_u]\!]_{i-1,y}$. Now a Cauchy–Schwarz inequality gives

$$|(G_u)^-_{N/4,y}| \le \tau^{1/2} \left(\sum_{i=1}^{N/4} \left[\|(G_u)_x\|^2_{I_i} + h_i^{-1} |[\![G_u]\!]_{i-1,y}|^2 \right] \right)^{1/2}.$$

Hence, recalling (4.20a), we get

$$\begin{aligned}\sum_{j=1}^{N/4} \left\| (G_u)^-_{N/4,y} \right\|^2_{J_j} &\le \tau \sum_{j=1}^{N/4} \sum_{i=1}^{N/4} \left(\|(G_u)_x\|^2_{K_{ij}} + h_i^{-1} \|[\![G_u]\!]_{i-1,y}\|^2_{J_j} \right) \\ &\le C\tau\varepsilon^{-2} \|G_p\|^2_{\Omega_{11}} \le C\tau\varepsilon^{-1} \interleave \boldsymbol{G} \interleave^2_{E,\Omega_{11}}.\end{aligned}$$

The proof is complete. □

Remark 4.6. The Cauchy–Schwarz and inverse inequalities give

$$\left(\sum_{j=1}^{N/4} \left\| (G_u)^-_{N/4,y} \right\|^2_{J_j} \right)^{1/2} \le Ch^{-1/2} \|G_u\|_{\Omega_{11}} \le CN^{1/2} \sqrt{\frac{\tau}{\varepsilon}} \interleave \boldsymbol{G} \interleave_{E,\Omega_{11}}.$$

The bound established in Lemma 4.5 is $O(N^{-1/2})$ sharper than this direct estimate.

4.4. Pointwise error estimate in the corner layer region Ω_f

In Theorem 4.7 we present the main result of Section 4: a pointwise error bound in the corner layer region Ω_f. We prove that the dominant convergence rate (*i.e.*, neglecting terms multiplied by positive powers of ε) is $O\left(N^{-(k+1)}(\ln N)^{k+3}\right)$, which (modulo a logarithmic factor) fits the computational results of Section 6.

Theorem 4.7. *There exists a constant C such that*

$$|(u - \mathbb{U})(x^\star, y^\star)| \le C \left\{ N^{-(k+1)} (\ln N)^{k+3} + \varepsilon^{1/4} \left[N(\ln N)^2 + \varepsilon^{1/4} (N \ln N)^{3/2} \right] \theta(\varepsilon, N, k) \right\} \tag{4.30}$$

for all $(x^\star, y^\star) \in \Omega_f$.

Proof. Without loss of generality, suppose that $(x^\star, y^\star) \in \Omega_{11}$. One has the Galerkin orthogonality property

$$B_C(\boldsymbol{e}; \mathbb{z}) = -F_C(\boldsymbol{e}, \mathbb{v}) \qquad \forall \mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_C]^3, \tag{4.31}$$

where B_C and F_C were defined in (4.2) and (4.3). Recall the decomposition $\boldsymbol{e} = \boldsymbol{\eta} - \boldsymbol{\xi}$. Take $\mathbb{z} = \boldsymbol{\xi}$ in (4.19) to get

$$\xi_u(x^\star, y^\star) = B_C(\boldsymbol{\xi}; \boldsymbol{G}) = B_C(\boldsymbol{\eta}; \boldsymbol{G}) + F_C(\boldsymbol{e}, G_u) := \sum_{i=1}^{5} W_i + F_C(\boldsymbol{\eta}, G_u) - F_C(\boldsymbol{\xi}, G_u) \tag{4.32}$$

using (4.31), where

$$\begin{aligned} W_1 &:= \langle b\eta_u, G_u \rangle_{\Omega_{11}} + \varepsilon^{-1} \langle \eta_p, G_p \rangle_{\Omega_{11}} + \varepsilon^{-1} \langle \eta_q, G_q \rangle_{\Omega_{11}}, \\ W_2 &:= \sum_{K_{ij} \subset \Omega_{11}} \mathcal{D}^x_{ij}(\eta_u, G_p; \eta_u^-), \qquad W_3 := \sum_{K_{ij} \subset \Omega_{11}} \mathcal{D}^y_{ij}(\eta_u, G_q; \eta_u^-), \\ W_4 &:= \langle \eta_p, (G_u)_x \rangle_{\Omega_{11}} + \sum_{j=1}^{N/4} \sum_{i=0}^{N/4-1} \left\langle (\eta_p)^+_{i,y}, [\![G_u]\!]_{i,y} \right\rangle_{J_j}, \end{aligned}$$

and

$$W_5 := \langle \eta_q, (G_u)_y \rangle_{\Omega_{11}} + \sum_{i=1}^{N/4} \sum_{j=0}^{N/4-1} \left\langle (\eta_q)^+_{x,j}, [\![G_u]\!]_{x,j} \right\rangle_{I_i}.$$

We bound each of the W_i separately. First, Cauchy–Schwarz inequalities and Lemma 3.1 yield

$$\begin{aligned} |W_1| &\le C|\Omega_{11}|^{1/2} \left(\|\eta_u\|_{L^\infty(\Omega_{11})} + \varepsilon^{-1/2} \|\eta_p\|_{L^\infty(\Omega_{11})} + \varepsilon^{-1/2} \|\eta_q\|_{L^\infty(\Omega_{11})} \right) ||| \boldsymbol{G} |||_{E,\Omega_{11}} \\ &\le C\varepsilon^{1/2} N^{-(k+1)} (\ln N)^{k+2} ||| \boldsymbol{G} |||_{E,\Omega_{11}}. \end{aligned}$$

Next, for any $K_{ij} \subset \Omega_{11}$, one has $\eta_p = p - \Pi_x^+ p$. Then the definition (3.5a) gives $\langle \eta_p, \mathbb{v}_x \rangle_{K_{ij}} = \left\langle (\eta_p)^+_{i-1,y}, \mathbb{v}_{i,y} \right\rangle_{J_j} = 0$ for $i, j = 1, \dots, N/4$ and $\mathbb{v} \in \mathcal{Q}^k(K_{ij})$, so $W_4 = 0$ and similarly $W_5 = 0$.

Turning our attention to W_2, observe that $\eta_u = u - \Pi_{xy}^- u$ on each $K_{ij} \subset \Omega_{11}$. Then by equation (3.5a) of Lemma 3.3 from [9] and the decomposition of Assumption 2.1, one has

$$\left| \mathcal{D}^x_{ij}(\eta_\phi, z; \eta_\phi^-) \right| \le \begin{cases} C h_i^{-1} N^{-(k+2)} |K_{ij}|^{1/2} \|z\|_{K_{ij}} & \text{if } \phi = \bar{u}, \\ C h_i^{-1} (N^{-1}\ln N)^{k+2} \left\| e^{-\beta x/\sqrt{\varepsilon}} \right\|_{K_{ij}} \|z\|_{K_{ij}} & \text{if } \phi = u^{\mathrm{b}}_{1,\varepsilon}, \\ C h_i^{-1} (N^{-1}\ln N)^{k+2} \left\| e^{-\beta x/\sqrt{\varepsilon}} e^{-\beta y/\sqrt{\varepsilon}} \right\|_{K_{ij}} \|z\|_{K_{ij}} & \text{if } \phi = u^{\mathrm{c}}_{1,\varepsilon}, \end{cases}$$

where $\eta_\phi = \phi - \Pi_{xy}^- \phi$. But $\left\| e^{-\beta x/\sqrt{\varepsilon}} \right\|_{\Omega_{11}} \le C(\varepsilon \ln N)^{1/2}$ and $\left\| e^{-\beta x/\sqrt{\varepsilon}} e^{-\beta y/\sqrt{\varepsilon}} \right\|_{\Omega_{11}} \le C\varepsilon^{1/2}$; thus, a Cauchy–Schwarz inequality yields

$$\sum_{K_{ij} \subset \Omega_{11}} |\mathcal{D}^x_{ij}(\eta_\phi, z; \eta_\phi^-)| \le \begin{cases} C N^{-(k+1)} \|z\|_{\Omega_{11}} & \text{if } \phi = \bar{u}, \\ C N^{-(k+1)} (\ln N)^{k+3/2} \|z\|_{\Omega_{11}} & \text{if } \phi = u^{\mathrm{b}}_{1,\varepsilon}, \\ C (N^{-1}\ln N)^{k+1} \|z\|_{\Omega_{11}} & \text{if } \phi = u^{\mathrm{c}}_{1,\varepsilon}. \end{cases}$$

The other components $u^{\mathrm{b}}_{i,\varepsilon}$ and $u^{\mathrm{c}}_{i,\varepsilon}$ (for $i = 2, 3, 4$) can be handled similarly, where we recall that $u = u^{\mathrm{s}} + \sum_{i=1}^4 u^{\mathrm{b}}_{i,\varepsilon} + \sum_{i=1}^4 u^{\mathrm{c}}_{i,\varepsilon}$. Hence

$$|W_2| \le C N^{-(k+1)} (\ln N)^{k+3/2} \|G_p\|_{\Omega_{11}} \le C\varepsilon^{1/2} N^{-(k+1)} (\ln N)^{k+3/2} ||| \boldsymbol{G} |||_{E,\Omega_{11}}.$$

In a similar manner, one can prove $|W_3| \le C\varepsilon^{1/2} N^{-(k+1)} (\ln N)^{k+3/2} ||| \boldsymbol{G} |||_{E,\Omega_{11}}$. These bounds on the W_i for $i = 1, 2, \dots, 5$ and Lemma 4.3 give

$$\left| \sum_{i=1}^5 W_i \right| \le C\varepsilon^{1/2} N^{-(k+1)} (\ln N)^{k+2} ||| \boldsymbol{G} |||_{E,\Omega_{11}} \le C N^{-(k+1)} (\ln N)^{k+3}. \tag{4.33}$$

We move on to the term $F_C(\boldsymbol{\eta}, G_u)$ in (4.32). A Cauchy–Schwarz inequality, (3.7d), (4.29) and Lemma 4.3 yield

$$\begin{aligned} \left| \sum_{j=1}^{N/4} \left\langle (\eta_p)^+_{N/4,y}, (G_u)^-_{N/4,y} \right\rangle_{J_j} \right| &\le \left(\sum_{j=1}^{N/4} h_j \right)^{1/2} \max_{1 \le j \le N/4} \|\eta_p\|_{L^\infty(K_{N/4+1,j})} \left(\sum_{j=1}^{N/4} \left\| (G_u)^-_{N/4,y} \right\|^2_{J_j} \right)^{1/2} \\ &\le C\tau^{1/2} \|\eta_p\|_{L^\infty(\Omega_2^x)} \sqrt{\frac{\tau}{\varepsilon}} ||| \boldsymbol{G} |||_{E,\Omega_{11}} \le C N^{-(k+1)} (\ln N)^2. \end{aligned}$$

One can bound similarly the second term of $F_C(\boldsymbol{\eta}, G_u)$ and finally obtain

$$|F_C(\boldsymbol{\eta}, G_u)| \le CN^{-(k+1)}(\ln N)^2. \tag{4.34}$$

For the final term $|F_C(\boldsymbol{\xi}, G_u)|$ in (4.32), a Cauchy–Schwarz inequality and Lemmas 3.3, 4.5 and 4.3 give us

$$\begin{aligned}\left|\sum_{j=1}^{N/4}\left\langle (\xi_p)^+_{N/4,y}, (G_u)^-_{N/4,y}\right\rangle_{J_j}\right| &\le \left(\sum_{j=1}^{N/4}\left\|(\xi_p)^+_{N/4,y}\right\|^2_{J_j}\right)^{1/2}\left(\sum_{j=1}^{N/4}\left\|(G_u)^-_{N/4,y}\right\|^2_{J_j}\right)^{1/2}\\ &\le C\varepsilon\left[N(\ln N)^{1/2}+\varepsilon^{1/4}N^{3/2}\right]\theta(\varepsilon,N,k)\sqrt{\frac{\tau}{\varepsilon}}\,|||\,\boldsymbol{G}|||_{E,\Omega_{11}}\\ &\le C\varepsilon^{1/4}\left[N(\ln N)^2+\varepsilon^{1/4}(N\ln N)^{3/2}\right]\theta(\varepsilon,N,k).\end{aligned}$$

Now the definition (4.3) shows immediately that

$$|F_C(\boldsymbol{\xi}, G_u)| \le C\varepsilon^{1/4}\left[N(\ln N)^2+\varepsilon^{1/4}(N\ln N)^{3/2}\right]\theta(\varepsilon,N,k). \tag{4.35}$$

Substituting (4.33)–(4.35) into (4.32), we obtain

$$|\xi_u(x^\star,y^\star)| \le C\Big\{N^{-(k+1)}(\ln N)^{k+3}+\varepsilon^{1/4}\Big[N(\ln N)^2+\varepsilon^{1/4}(N\ln N)^{3/2}\Big]\theta(\varepsilon,N,k)\Big\}. \tag{4.36}$$

Then (4.30) follows from (3.7b) and a triangle inequality. □

5. Pointwise convergence in the boundary layer region Ω_b

This section proves a pointwise convergence estimate in the boundary layer region Ω_b. We shall use a discrete Green's function like Section 4, but the analysis of this function here is much more complicated since it involves cut-off functions.

5.1. The compact LDG scheme in $\boldsymbol{\Omega_b}$

Consider the pointwise error estimate for an arbitrary point $(x^\star, y^\star) \in \Omega_b$. Without loss of generality we assume that $(x^\star, y^\star) \in \Omega_{12}$. All constants C that appear in Section 5 are independent of the particular choice of $(x^\star, y^\star)$.

Set $\mathcal{V}_L := \{\chi \in L^2(\Omega_{12})\colon \chi|_{K_{ij}} \in \mathcal{Q}^k(K_{ij})\ \forall K_{ij} \in \Omega_{12}^N\}$. Sum (2.5) over all $K_{ij} \in \Omega_{12}^N$ to get a compact form of the LDG method in Ω_{12}: Find $\mathbb{W} = (\mathbb{U}, \mathbb{P}, \mathbb{Q}) \in [\mathcal{V}_L]^3 := \mathcal{V}_L \times \mathcal{V}_L \times \mathcal{V}_L$ such that

$$B_L(\mathbb{W};\mathbb{z}) = \langle f, \mathbb{v}\rangle_{\Omega_{12}} - F_L(\mathbb{W},\mathbb{z}) \quad \forall \mathbb{z} = (\mathbb{v},\mathbb{s},\mathbb{r}) \in [\mathcal{V}_L]^3,$$

where

$$\begin{aligned}B_L(\mathbb{W};\mathbb{z}) :=&\ \langle b\mathbb{U},\mathbb{v}\rangle_{\Omega_{12}} + \varepsilon^{-1}\langle \mathbb{P},\mathbb{s}\rangle_{\Omega_{12}} + \varepsilon^{-1}\langle \mathbb{Q},\mathbb{r}\rangle_{\Omega_{12}}\\ &+ \sum_{K_{ij}\subset\Omega_{12}} \mathcal{D}^x_{ij}(\mathbb{U},\mathbb{s};\hat{\mathbb{U}}) + \langle \mathbb{U},\mathbb{r}_y\rangle_{\Omega_{12}} + \sum_{i=1}^{N/4}\sum_{j=N/4+1}^{3N/4-1}\langle \{\{\mathbb{U}\}\}_{x,j}, [\![\mathbb{r}]\!]_{x,j}\rangle_{I_i}\\ &+ \langle \mathbb{P},\mathbb{v}_x\rangle_{\Omega_{12}} + \sum_{j=N/4+1}^{3N/4}\sum_{i=0}^{N/4-1}\left\langle \mathbb{P}^+_{i,y}, [\![\mathbb{v}]\!]_{i,y}\right\rangle_{J_j} + \sum_{K_{ij}\subset\Omega_{12}} \mathcal{D}^y_{ij}(\mathbb{Q},\mathbb{v};\hat{\mathbb{Q}})\end{aligned} \tag{5.1}$$

with the $\hat{\mathbb{U}}_{i,y}$ $(i = 0, 1, \ldots, N/4)$ and $\hat{\mathbb{Q}}_{x,j}$ $(j = N/4, N/4+1, \ldots, 3N/4)$ defined in Table 1, and

$$F_L(\mathbb{W}, \mathbb{v}) := \sum_{i=1}^{N/4} \left\langle \mathbb{U}^+_{x,3N/4}, \mathbb{r}^-_{x,3N/4} \right\rangle_{I_i} - \sum_{i=1}^{N/4} \left\langle \mathbb{U}^-_{x,N/4}, \mathbb{r}^+_{x,N/4} \right\rangle_{I_i} + \sum_{j=N/4+1}^{3N/4} \left\langle \mathbb{P}^+_{N/4,y}, \mathbb{v}^-_{N/4,y} \right\rangle_{J_j}.$$

Set $|||\mathbb{z}|||^2_{E,\Omega_{12}} := B_L(\mathbb{z}; \mathbb{z})$. One can show easily that

$$|||\mathbb{z}|||^2_{E,\Omega_{12}} = \varepsilon^{-1} \left(\|\mathbb{s}\|^2_{\Omega_{12}} + \|\mathbb{r}\|^2_{\Omega_{12}} \right) + \left\| b^{1/2} \mathbb{v} \right\|^2_{\Omega_{12}}. \tag{5.2}$$

5.2. Cut-off and weight functions

We have $y^\star \in J_j = (y_{j-1}, y_j)$ for some fixed $j \in \{N/4+1, N/4+2, \ldots, 3N/4\}$. Set $\mu = \max\{\varepsilon^{1/2}, N^{-1}\}$. Then

$$\varepsilon^{1/2}\mu^{-1} \leq 1 \quad \text{and} \quad h_k \mu^{-1} \leq 1 \quad \text{for } k = N/4+1, N/4+2, \ldots, 3N/4. \tag{5.3}$$

Define a 1D interval $J^+_{j,s} \supset J_j$ by

$$J^+_{j,s} := (y_{j-1} - s\gamma\mu \ln N, y_j + s\gamma\mu \ln N), \tag{5.4}$$

where $\gamma > 0$ is a sufficiently large constant and $s > 0$ is a constant independent of ε, N and γ.

The interval $J^+_{j,s}$ has length $|J^+_{j,s}| = O(\mu \ln N)$. Let $\mathcal{N}(J^+_{j,s})$ denote the number of the intervals $J_k = (y_{k-1}, y_k)$ that are contained in $J^+_{j,s}$. Then

$$\mathcal{N}(J^+_{j,s}) = O(\mu N \ln N). \tag{5.5}$$

Define the continuous cut-off function $\psi : \mathbb{R} \to (0, 2) \subset \mathbb{R}$ by

$$\psi(\rho) = \begin{cases} e^{-\rho} & \text{for } \rho > 0, \\ 2 - e^{\rho} & \text{for } \rho \leq 0. \end{cases}$$

Then define the weight function $\omega : [0, 1] \to (0, 4)$ by

$$\omega(y) := \psi\left(\frac{y - y_j}{\gamma\mu}\right) \psi\left(\frac{y_{j-1} - y}{\gamma\mu}\right).$$

It is straightforward to verify the properties of ω that are listed in the next lemma.

Lemma 5.1. *There exists a constant C such that*

(i) $0 < \omega(y) < 4$ *for* $y \in (0, 1)$;
(ii) $1 < \omega(y) < 4$ *for* $y \in J_j$;
(iii) $0 < \omega(y) < CN^{-s}$ *for* $y \in (0, 1) \backslash J^+_{j,s}$;
(iv) *for* $J_k = (y_{k-1}, y_k)$ *with* $k = N/4+1, \ldots, 3N/4$, *one has*

$$RO(J_k, \omega) + RO(J_k, \omega') \leq C, \tag{5.6}$$

where $RO(J_k, v) := \max_{y \in J_k} |v(y)| / \min_{y \in J_k} |v(y)|$;
(v) $|\omega'(y)| \leq C(\gamma\mu)^{-1}\omega(y)$ *for* $y \in (0, 1)$.

By virtue of the property (5.6), one can prove the following weighted inverse estimate.

Lemma 5.2. *There exists a constant C such that*

$$\left\|\omega\chi_y\right\|_{K_{ij}} + h_j^{1/2}\left\|(\omega\chi)_{x,j}^-\right\|_{I_i} \le Ch_j^{-1}\left\|\omega\chi\right\|_{K_{ij}} \text{ for all } \chi \in \mathcal{Q}^k(K_{ij}).$$

Let $\mathbb{P}$ denote Π, Π_x^- or Π_x^+. For any given function ϕ, set $\mathbb{P}^\perp\phi := \phi - \mathbb{P}\phi$. The following superapproximation property is crucial.

Lemma 5.3. *There exists a constant C such that*

$$\left\|\omega\mathbb{P}^\perp(\omega^{-2}\chi)\right\|_{K_{ij}} + (h_ih_j)^{1/2}\left\|\omega\mathbb{P}^\perp(\omega^{-2}\chi)\right\|_{L^\infty(K_{ij})} \le Ch_j\mu^{-1}\gamma^{-1}\left\|\omega^{-1}\chi\right\|_{K_{ij}} \tag{5.7a}$$

for all $\chi \in \mathcal{Q}^k(K_{ij})$. Consequently

$$\left\|\omega\mathbb{P}(\omega^{-2}\chi)\right\|_{K_{ij}} + (h_ih_j)^{1/2}\left\|\omega\mathbb{P}(\omega^{-2}\chi)\right\|_{L^\infty(K_{ij})} \le C\left(1 + h_j\mu^{-1}\gamma^{-1}\right)\left\|\omega^{-1}\chi\right\|_{K_{ij}}. \tag{5.7b}$$

Proof. One uses Lemma 5.2 to prove the result, similarly to the proof of Lemma 3.3 from [41]. ☐

For each $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_L]^3$, we define the weighted energy norm

$$|||\mathbb{z}|||_{\omega,\Omega_{12}}^2 := \varepsilon^{-1}\left(\left\|\omega^{-1}\mathbb{s}\right\|_{\Omega_{12}}^2 + \left\|\omega^{-1}\mathbb{r}\right\|_{\Omega_{12}}^2\right) + \left\|\omega^{-1}b^{1/2}\mathbb{v}\right\|_{\Omega_{12}}^2.$$

One can easily prove the coercivity and boundedness properties of the next two lemmas.

Lemma 5.4. *If $\gamma > 0$ is sufficiently large, then for any $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_L]^3$ one has*

$$B_L(\omega^{-2}\mathbb{z}; \mathbb{z}) \ge \frac{1}{2}|||\,\mathbb{z}\,|||_{\omega,\Omega_{12}}^2.$$

Proof. Integration by parts and some simple manipulations yield

$$B_L(\omega^{-2}\mathbb{z}; \mathbb{z}) = |||\mathbb{z}\,|||_{\omega,\Omega_{12}}^2 - \left\langle\left(\omega^{-2}\right)'\mathbb{v}, \mathbb{r}\right\rangle_{\Omega_{12}}.$$

Using the Cauchy–Schwarz inequality, Lemma 5.1(v) and the first inequality of (5.3), one has

$$\begin{aligned}\left|\left\langle\left(\omega^{-2}\right)'\mathbb{v}, \mathbb{r}\right\rangle_{\Omega_{12}}\right| &\le \left\|\omega\left(\omega^{-2}\right)'\mathbb{v}\right\|_{\Omega_{12}}\left\|\omega^{-1}\mathbb{r}\right\|_{\Omega_{12}} \le C\mu^{-1}\gamma^{-1}\left\|\omega^{-1}\mathbb{v}\right\|_{\Omega_{12}}\left\|\omega^{-1}\mathbb{r}\right\|_{\Omega_{12}}\\ &\le C\varepsilon^{1/2}\mu^{-1}\gamma^{-1}|||\,\mathbb{z}|||_{\omega,\Omega_{12}}^2 \le C_1\gamma^{-1}|||\,\mathbb{z}|||_{\omega,\Omega_{12}}^2,\end{aligned}$$

where C_1 is some fixed constant. If $\gamma > 0$ is sufficiently large so that $C_1\gamma^{-1} \le 1/2$, then one gets

$$B_L(\omega^{-2}\mathbb{z}; \mathbb{z}) \ge \left(1 - C_1\gamma^{-1}\right)|||\,\mathbb{z}|||_{\omega,\Omega_{12}}^2 \ge \frac{1}{2}|||\,\mathbb{z}|||_{\omega,\Omega_{12}}^2,$$

which completes the proof. ☐

Set $\mathbb{H}(\omega^{-2}\mathbb{z}) := (\Pi_x^-(\omega^{-2}\mathbb{v}), \Pi_x^+(\omega^{-2}\mathbb{s}), \Pi(\omega^{-2}\mathbb{r}))$ and $\mathbb{H}^\perp(\omega^{-2}\mathbb{z}) := \omega^{-2}\mathbb{z} - \mathbb{H}(\omega^{-2}\mathbb{z})$ for any $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_L]^3$.

Lemma 5.5. *If $\gamma > 0$ is sufficiently large, then for any $\mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_L]^3$,*

$$B_L\left(\mathbb{H}^\perp(\omega^{-2}\mathbb{z}); \mathbb{z}\right) \le \frac{1}{4}|||\,\mathbb{z}\,|||_{\omega,\Omega_{12}}^2.$$

Proof. For notational simplicity we set

$$E := \mathbb{H}^{\perp}\left(\omega^{-2}\mathbb{z}\right) = \left(\omega^{-2}\mathbb{v} - \Pi_x^-\left(\omega^{-2}\mathbb{v}\right), \omega^{-2}\mathbb{s} - \Pi_x^+\left(\omega^{-2}\mathbb{s}\right), \omega^{-2}\mathbb{r} - \Pi\left(\omega^{-2}\mathbb{r}\right)\right) =: (E_v, E_s, E_r).$$

It follows from the definition (5.1) that $B_L(E,\mathbb{z}) := \sum_{i=1}^5 Z_i$, where

$$Z_1 = \langle bE_v, \mathbb{v}\rangle_{\Omega_{12}} + \varepsilon^{-1}\langle E_s, \mathbb{s}\rangle_{\Omega_{12}} + \varepsilon^{-1}\langle E_r, \mathbb{r}\rangle_{\Omega_{12}};$$

$$Z_2 = \sum_{K_{ij}\subset\Omega_{12}} \mathcal{D}_{ij}^x(E_v, \mathbb{s}; \hat{E}_v);\ Z_3 = \langle E_v, \mathbb{r}_y\rangle_{\Omega_{12}} + \sum_{i=1}^{N/4}\sum_{j=N/4+1}^{3N/4-1}\langle \{\{E_v\}\}_{x,j}, [\![\mathbb{r}]\!]_{x,j}\rangle_{I_i};$$

$$Z_4 = \langle E_s, \mathbb{v}_x\rangle_{\Omega_{12}} + \sum_{j=N/4+1}^{3N/4}\sum_{i=0}^{N/4-1}\left\langle (E_s)_{i,y}^+, [\![\mathbb{v}]\!]_{i,y}\right\rangle_{J_j};\ Z_5 = \sum_{K_{ij}\subset\Omega_{12}} \mathcal{D}_{ij}^y(E_r, \mathbb{v}; \hat{E}_r).$$

The Cauchy–Schwarz inequality, Lemma 5.3 and the second inequality of (5.3) give

$$|Z_1| \le C\Big(\|\omega E_v\|_{\Omega_{12}} + \varepsilon^{-1/2}\|\omega E_s\|_{\Omega_{12}} + \varepsilon^{-1/2}\|\omega E_r\|_{\Omega_{12}}\Big)|||\,\mathbb{z}|||_{\omega,\Omega_{12}} \le C\gamma^{-1}|||\,\mathbb{z}\,|||^2_{\omega,\Omega_{12}}.$$

The definitions (3.5a) and (3.5b) of Π_x^- and Π_x^+ imply that $Z_2 = Z_4 = 0$.

The inverse inequality (3.2), Lemma 5.3 and the first inequality of (5.3) yield

$$\begin{aligned}|Z_3| &\le C\left[H^{-1}\|\omega E_v\|_{\Omega_{12}} + H^{-1/2}\left(\sum_{i=1}^{N/4}\sum_{j=N/4+1}^{3N/4-1}\|\{\{\omega E_v\}\}_{x,j}\|^2_{I_i}\right)^{1/2}\right]\left\|\omega^{-1}\mathbb{r}\right\|_{\Omega_{12}}\\ &\le C\mu^{-1}\gamma^{-1}\left\|\omega^{-1}\mathbb{v}\right\|_{\Omega_{12}}\left\|\omega^{-1}\mathbb{r}\right\|_{\Omega_{12}} \le C\gamma^{-1}|||\,\mathbb{z}\,|||^2_{\omega,\Omega_{12}}.\end{aligned}$$

Similarly, $|Z_5| \le C\gamma^{-1}|||\,\mathbb{z}|||^2_{\omega,\Omega_{12}}$.

Combining the estimates for the Z_i $(i = 1, 2, \dots, 5)$ yields $|B_L(E,\mathbb{z})| \le C_2\gamma^{-1}|||\,\mathbb{z}|||^2_{\omega,\Omega_{12}}$ for some fixed constant C_2. Now choose $\gamma > 0$ so large that $C_2\gamma^{-1} \le 1/4$, and we are done. □

5.3. Weighted approximation properties

The weighted estimates of the next lemma are essentially a combination of Lemmas 3.1 and 5.1.

Lemma 5.6. *Suppose that $s \ge 1/2$ in (5.4). Then there exists a constant C such that*

$$\|\omega\eta_u\|_{\Omega_{12}} + \varepsilon^{-1/2}\Big(\|\omega\eta_p\|_{\Omega_{12}} + \|\omega\eta_q\|_{\Omega_{12}}\Big) \le C\Big(1 + \varepsilon^{1/4}N^{1/2}\Big)\varepsilon^{1/4}N^{-(k+3/2)}(\ln N)^{k+2}, \tag{5.8a}$$

$$\left(\sum_{i=1}^{N/4}\sum_{j=N/4+1}^{3N/4-1}\|\{\{\omega\eta_u\}\}_{x,j}\|^2_{I_i}\right)^{1/2} \le C\Big(1 + \varepsilon^{1/4}N^{1/2}\Big)\varepsilon^{1/4}N^{-(k+1)}(\ln N)^{k+2}, \tag{5.8b}$$

$$\left(\sum_{j=N/4+1}^{3N/4}\left\|(\omega\eta_p)^+_{N/4,y}\right\|^2_{J_j}\right) \le C\Big(1 + \varepsilon^{1/4}N^{1/2}\Big)\varepsilon^{1/2}N^{-(k+3/2)}(\ln N)^{1/2}, \tag{5.8c}$$

and

$$\begin{aligned}&\left(\sum_{i=1}^{N/4}\Big[\sum_{j=N/4+1}^{3N/4-1}\|\{\{\omega\eta_q\}\}_{x,j}\|^2_{I_i} + \left\|(\omega\eta_q)^-_{x,3N/4}\right\|^2_{I_i} + \left\|(\omega\eta_q)^+_{x,N/4}\right\|^2_{I_i}\Big]\right)^{1/2}\\ &\qquad\le C\Big(1 + \varepsilon^{1/4}N^{1/2}\Big)\varepsilon^{3/4}N^{-(k+1)}(\ln N)^{k+2}.\end{aligned} \tag{5.8d}$$

Proof of (5.8a). Set $\Omega_{j,s} := (0,\tau) \times J^+_{j,s}$ and $\Omega^c_{j,s} := \Omega_{12} \backslash \Omega_{j,s}$. The 2D areas of these regions can be estimated as $|\Omega_{j,s}| = O(\tau\mu \ln N)$ and $|\Omega^c_{j,s}| = O(\tau)$.

Recall that $\eta_u = u - \mathcal{I}u$. Since $s \geq 1/2$, Lemma 5.1 and (3.7b) yield

$$\begin{aligned}
\|\omega\eta_u\|_{\Omega_{12}} &\leq C|\Omega_{j,s}|^{1/2} \|\eta_u\|_{L^\infty(\Omega_{j,s})} + C|\Omega^c_{j,s}|^{1/2} N^{-s} \|\eta_u\|_{L^\infty(\Omega^c_{j,s})} \\
&\leq C\left[(\tau\mu \ln N)^{1/2} + \tau^{1/2} N^{-s}\right] \|\eta_u\|_{L^\infty(\Omega_{12})} \\
&\leq C\left[\varepsilon^{1/4} N^{-1/2} \ln N + \varepsilon^{1/2} \ln N + \varepsilon^{1/4} N^{-s} (\ln N)^{1/2}\right] (N^{-1} \ln N)^{k+1} \\
&\leq C\left(1 + \varepsilon^{1/4} N^{1/2}\right) \varepsilon^{1/4} N^{-(k+3/2)} (\ln N)^{k+2}.
\end{aligned}$$

One can likewise bound $\varepsilon^{-1/2} \|\omega\eta_p\|_{\Omega_{12}}$ and $\varepsilon^{-1/2} \|\omega\eta_q\|_{\Omega_{12}}$. ☐

Proof of (5.8b). One uses (3.7b), (5.5) and Lemma 5.1 to obtain

$$\begin{aligned}
\sum_{i=1}^{N/4} \sum_{j=N/4+1}^{3N/4-1} \|\{\{\omega\eta_u\}\}_{x,j}\|^2_{I_i} &\leq C \sum_{i=1}^{N/4} \sum_{j=N/4+1}^{3N/4} h_i \|\omega\eta_u\|^2_{L^\infty(K_{ij})} \\
&\leq C \sum_{i=1}^{N/4} h_i \sum_{J_n \in J^+_{j,s}} \|\eta_u\|^2_{L^\infty(K_{in})} + C \sum_{i=1}^{N/4} h_i \sum_{J_n \in (\tau, 1-\tau) \backslash J^+_{j,s}} N^{-2s} \|\eta_u\|^2_{L^\infty(K_{in})} \\
&\leq C\left[\tau \mathcal{N}(J^+_{j,s}) + \tau N^{-2s+1}\right] \|\eta_u\|^2_{L^\infty(\Omega_{12})} \\
&\leq C\left[\varepsilon^{1/2} (\ln N)^2 + \varepsilon N (\ln N)^2 + \varepsilon^{1/2} N^{-2s+1} \ln N\right] (N^{-1} \ln N)^{2(k+1)} \\
&\leq C\left(1 + \varepsilon^{1/2} N\right) \varepsilon^{1/2} N^{-2(k+1)} (\ln N)^{2(k+2)},
\end{aligned}$$

where $s \geq 1/2$ was used to bound N^{-2s+1}. ☐

Proofs of (5.8c) *and* (5.8d). These bounds are proved in a similar way so we omit the details. ☐

5.4. Discrete Green's function

Recall that $(x^\star, y^\star) \in \Omega_{12}$. Associate with this point the discrete Green's function $\boldsymbol{G} \in [\mathcal{V}_L]^3$, defined by

$$B_L(\mathbb{z}; \boldsymbol{G}) = \mathbb{v}(x^\star, y^\star) \quad \forall \mathbb{z} = (\mathbb{v}, \mathbb{s}, \mathbb{r}) \in [\mathcal{V}_L]^3, \tag{5.9}$$

where $B_L(\cdot\,;\cdot)$ is given by (5.1). The identity (5.2) implies that $\boldsymbol{G}$ is uniquely defined. (Here we use the same notation as the discrete Green's function of Section 4.3, since there should be no confusion.)

Lemma 5.7. *There exists a constant C such that*

$$|||\boldsymbol{G}|||_{E,\Omega_{12}} \leq 4\, |||\, \boldsymbol{G}|||_{\omega,\Omega_{12}} \leq C\varepsilon^{-1/4} (N \ln N)^{1/2}. \tag{5.10}$$

Proof. For any $(x, y) \in K_{ij} \subset \Omega_{12}$, one has

$$G_u(x,y) = \left(\int_{x_{i-1}}^{x} + \int_{x_{i-2}}^{x_{i-1}} + \cdots + \int_{x_0}^{x_1}\right) (G_u)_x(x,y)\,\mathrm{d}x + \sum_{m=1}^{i} [\![G_u]\!]_{m-1,y}.$$

Thus, a Cauchy–Schwarz inequality gives

$$|G_u(x,y)| \leq \sum_{m=1}^{i} h_m^{1/2} \Big(\|(G_u)_x(\cdot, y)\|_{I_m} + h_m^{-1/2} \big|[\![G_u]\!]_{m-1,y}\big| \Big). \tag{5.11}$$

Here the inverse inequalities (3.2d) and (3.1) yield

$$\|(G_u)_x(\cdot,y)\|_{I_m} \le Ch_j^{-1/2}\|(G_u)_x\|_{K_{mj}} \text{ and } \left|[\![G_u]\!]_{m-1,y}\right| \le Ch_j^{-1/2}\left\|[\![G_u]\!]_{m-1,y}\right\|_{J_j}. \tag{5.12}$$

Combining (5.11) and (5.12), one obtains

$$|G_u(x,y)| \le C\sum_{m=1}^{i} h_m^{1/2}h_j^{-1/2}\left[\|(G_u)_x\|_{K_{mj}} + h_m^{-1/2}\left\|[\![G_u]\!]_{m-1,y}\right\|_{J_j}\right]. \tag{5.13}$$

But one can verify that Lemma 4.2 remains valid when $K_{mj} \subset \Omega_{12}$, *i.e.*,

$$\|(G_u)_x\|_{K_{mj}} + h_m^{-1/2}\left\|[\![G_u]\!]_{m-1,y}\right\|_{J_j} \le C\varepsilon^{-1}\|G_p\|_{K_{mj}}.$$

Hence (5.13) and a Cauchy–Schwarz inequality imply that

$$|G_u(x,y)| \le C\sum_{m=1}^{i} h_m^{1/2}N^{1/2}\varepsilon^{-1}\|G_p\|_{K_{mj}} \le C\varepsilon^{-1}\tau^{1/2}N^{1/2}\left(\sum_{m=1}^{N/4}\|G_p\|_{K_{mj}}^2\right)^{1/2}.$$

Now Lemma 5.1(iv) yields

$$\left\|\omega^{-1}G_u\right\|_{L^\infty(K_{ij})} \le C\varepsilon^{-1}\tau^{1/2}N^{1/2}\left(\sum_{i=1}^{N/4}\left\|\omega^{-1}G_p\right\|_{K_{ij}}^2\right)^{1/2} \le C\varepsilon^{-1/4}(N\ln N)^{1/2}\,|||\,\boldsymbol{G}\,|||_{\omega,\Omega_{12}}. \tag{5.14}$$

Appealing to Lemmas 5.4 and 5.5, the definitions of $\mathbb{H}^{\perp}$ and $\boldsymbol{G}$, Lemma 5.1(i), Lemma 5.3 and the definition of μ, we get

$$\begin{aligned}\frac{1}{4}\,|||\,\boldsymbol{G}|||_{\omega,\Omega_{12}}^2 &\le B_L(\omega^{-2}\boldsymbol{G};\boldsymbol{G}) - B_L(\mathbb{H}^{\perp}(\omega^{-2}\boldsymbol{G});\boldsymbol{G}) = B_L(\mathbb{H}(\omega^{-2}\boldsymbol{G});\boldsymbol{G})\\ &= [\Pi_x^-(\omega^{-2}G_u)](x^\star,y^\star) \le C\left\|\omega\Pi_x^-(\omega^{-2}G_u)\right\|_{L^\infty(K_{ij})}\\ &\le C(h_ih_j)^{-1/2}(1+h_j\mu^{-1}\gamma^{-1})\left\|\omega^{-1}G_u\right\|_{K_{ij}} \le C\left\|\omega^{-1}G_u\right\|_{L^\infty(K_{ij})}.\end{aligned} \tag{5.15}$$

The desired bound (5.10) now follows easily from $0<\omega<4$ and the inequalities (5.14) and (5.15). ☐

Lemma 5.8. *One has*

$$\left(\sum_{j=N/4+1}^{3N/4}\left\|(\omega^{-1}G_u)^-_{N/4,y}\right\|_{J_j}^2\right)^{1/2} \le C\sqrt{\frac{\tau}{\varepsilon}}\,|||\,\boldsymbol{G}|||_{E,\Omega_{12}} \tag{5.16}$$

for some constant C.

Proof. This result can be proved similarly to Lemma 4.5, using Lemma 5.1(iv). ☐

5.5. Pointwise error estimate in the boundary layer region Ω_b

In Theorem 5.9 we present the main result of Section 5: a pointwise error bound in Ω_b which improves significantly the convergence result in Ω_b that was proved in Theorem 3.5. This new bound differs from Theorem 4.7 by only a factor $(\ln N)^{1/2}$, but the result is obtained *via* a different argument.

Theorem 5.9. *There exists a constant C such that*

$$|(u-\mathbb{U})(x^\star,y^\star)| \le \tilde{C}_3 N^{-(k+1)}(\ln N)^{k+5/2} + C\varepsilon^{1/2}\Big\{ N^{-k}(\ln N)^{k+4} + \Big[N^{3/2}(\ln N)^2 + \varepsilon^{1/4}N^2(\ln N)^3 + \varepsilon^{1/2}(N\ln N)^{5/2}\Big]\theta(\varepsilon,N,k)\Big\} \tag{5.17}$$

for all $(x^\star,y^\star)\in\Omega_b$, where $\tilde{C}_3 = C(1+\varepsilon^{1/4}N^{1/2}+\varepsilon^{1/2}N+\varepsilon^{3/4}N^{3/2})$.

Proof. Without loss of generality, suppose that $(x^\star,y^\star)\in\Omega_{12}$. Define a discrete Green's function $\boldsymbol{G}\in[\mathcal{V}_L]^3$ associated with $(x^\star,y^\star)$ as in (5.9).

Recall the decomposition $\boldsymbol{e}=\boldsymbol{\eta}-\boldsymbol{\xi}$. Take $\mathbb{z}=\boldsymbol{\xi}$ in (5.9); then using the Galerkin orthogonality property $B_L(\boldsymbol{e};\mathbb{z}) = -F_L(\boldsymbol{e},\mathbb{z})$ for all $\mathbb{z}=(\mathbb{v},\mathbb{s},\mathbb{r})\in[\mathcal{V}_L]^3$, one gets

$$\xi_u(x^\star,y^\star) = B_L(\boldsymbol{\xi};\boldsymbol{G}) = B_L(\boldsymbol{\eta};\boldsymbol{G}) + F_L(\boldsymbol{e},\boldsymbol{G}) := \sum_{i=1}^{5} S_i + F_L(\boldsymbol{\eta},\boldsymbol{G}) - F_L(\boldsymbol{\xi},\boldsymbol{G}), \tag{5.18}$$

where

$$\begin{aligned}
S_1 &= \langle b\eta_u, G_u\rangle_{\Omega_{12}} + \varepsilon^{-1}\langle \eta_p, G_p\rangle_{\Omega_{12}} + \varepsilon^{-1}\langle \eta_q, G_q\rangle_{\Omega_{12}},\\
S_2 &= \sum_{K_{ij}\subset\Omega_{12}} \mathcal{D}^x_{ij}(\eta_u, G_p; \hat{\eta_u}), \quad S_3 = \langle \eta_u, (G_q)_y\rangle_{\Omega_{12}} + \sum_{i=1}^{N/4}\sum_{j=N/4+1}^{3N/4-1} \langle \{\{\eta_u\}\}_{x,j}, [\![G_q]\!]_{x,j}\rangle_{I_i},\\
S_4 &= \langle \eta_p, (G_u)_x\rangle_{\Omega_{12}} + \sum_{j=N/4+1}^{3N/4}\sum_{i=0}^{N/4-1} \left\langle (\eta_p)^+_{i,y}, [\![G_u]\!]_{i,y}\right\rangle_{J_j}, \quad S_5 = \sum_{K_{ij}\subset\Omega_{12}} \mathcal{D}^x_{ij}(\eta_q, G_u; \hat{\eta_q}).
\end{aligned}$$

Cauchy–Schwarz inequalities and Lemmas 5.6 and 5.7 yield

$$\begin{aligned}
|S_1| &\le C\big(\|\omega\eta_u\|_{\Omega_{12}} + \varepsilon^{-1/2}\|\omega\eta_p\|_{\Omega_{12}} + \varepsilon^{-1/2}\|\omega\eta_q\|_{\Omega_{12}}\big) |\!|\!| \boldsymbol{G} |\!|\!|_{\omega,\Omega_{12}}\\
&\le C(1+\varepsilon^{1/4}N^{1/2})N^{-(k+1)}(\ln N)^{k+5/2}.
\end{aligned}$$

For any $K_{ij}\subset\Omega_{12}$, one has $\eta_u = u-\Pi_x^- u$ and $\eta_p = p-\Pi_x^+ p$, so (3.5a) and (3.5b) imply that $S_2=S_4=0$. A Cauchy–Schwarz inequality, the inverse inequalities (3.2), and Lemmas 5.6 and 5.7 show that

$$\begin{aligned}
|S_3| &\le C\varepsilon^{1/2}\left[N\|\omega\eta_u\|_{\Omega_{12}} + N^{1/2}\left(\sum_{i=1}^{N/4}\sum_{j=N/4+1}^{3N/4-1} \|\{\{\omega\eta_u\}\}_{x,j}\|^2_{I_i}\right)^{1/2}\right] |\!|\!| \boldsymbol{G} |\!|\!|_{\omega,\Omega_{12}}\\
&\le C\left(\varepsilon^{1/2}N+\varepsilon^{3/4}N^{3/2}\right)N^{-(k+1)}(\ln N)^{k+5/2}.
\end{aligned}$$

Similarly, $|S_5|\le C\left(\varepsilon^{1/2}N+\varepsilon^{3/4}N^{3/2}\right)N^{-(k+1)}(\ln N)^{k+5/2}$.

These bounds on the S_i for $i=1,2,\ldots,5$ give

$$\left|\sum_{i=1}^{5} S_i\right| \le C\left(1+\varepsilon^{1/4}N^{1/2}+\varepsilon^{1/2}N+\varepsilon^{3/4}N^{3/2}\right)N^{-(k+1)}(\ln N)^{k+5/2}. \tag{5.19}$$

Next, we bound $F_L(\boldsymbol{\eta}, \boldsymbol{G}) = \sum_{i=1}^{3} X_i$ where

$$X_1 := \sum_{i=1}^{N/4} \left\langle (\eta_u)^+_{x,3N/4}, (G_q)^-_{x,3N/4} \right\rangle_{I_i}, \quad X_2 := -\sum_{i=1}^{N/4} \left\langle (\eta_u)^-_{x,N/4}, (G_q)^+_{x,N/4} \right\rangle_{I_i},$$
$$X_3 := \sum_{j=N/4+1}^{3N/4} \left\langle (\eta_p)^+_{N/4,y}, (G_u)^-_{N/4,y} \right\rangle_{J_j}.$$

A Cauchy–Schwarz inequality, the inverse inequality (3.2b), then the first inequality of (3.7b) and Lemma 5.7 give

$$\begin{aligned}|X_1| &\le \sum_{i=1}^{N/4} \left\|(\eta_u)^+_{x,3N/4}\right\|_{I_i} \left\|(G_q)^-_{x,3N/4}\right\|_{I_i} \le C \sum_{i=1}^{N/4} h_i \left\|\eta_u\right\|_{L^\infty(K_{i,3N/4+1})} H^{-1/2} \left\|G_q\right\|_{K_{i,3N/4}} \\ &\le C N^{1/2} \tau^{1/2} \left\|\eta_u\right\|_{L^\infty(\Omega_f)} \left\|G_q\right\|_{\Omega_{12}} \le C \varepsilon^{3/4} (N \ln N)^{1/2} \left\|\eta_u\right\|_{L^\infty(\Omega_f)} \|\!|\!| \boldsymbol{G} |\!|\!|_{E,\Omega_{12}} \\ &\le C \varepsilon^{1/2} N^{-k} (\ln N)^{k+2}. \end{aligned} \tag{5.20}$$

Similarly, $|X_2| \le C\varepsilon^{1/2} N^{-k} (\ln N)^{k+2}$.

A Cauchy–Schwarz inequality, the bound (5.8c) of Lemma 5.6, and Lemma 5.8 yield

$$\begin{aligned}|X_3| &\le \left(\sum_{j=N/4+1}^{3N/4} \left\|(\omega \eta_p)^+_{N/4,y}\right\|_{J_j} \right)^{1/2} \left(\sum_{j=N/4+1}^{3N/4} \left\|(\omega^{-1} G_u)^-_{N/4,y}\right\|_{J_j} \right)^{1/2} \\ &\le C\left(1 + \varepsilon^{1/4} N^{1/2}\right) \varepsilon^{1/2} N^{-(k+3/2)} (\ln N)^{1/2} \sqrt{\frac{\tau}{\varepsilon}} \|\!|\!| \boldsymbol{G} |\!|\!|_{E,\Omega_{12}} \\ &\le C(1 + \varepsilon^{1/4} N^{1/2}) N^{-(k+1)} (\ln N)^{3/2}. \end{aligned}$$

Collecting the bounds for the $X_i (i = 1, 2, 3)$, we obtain

$$|F_L(\boldsymbol{\eta}, \boldsymbol{G})| \le C \left[\varepsilon^{1/2} N^{-k} (\ln N)^{k+2} + (1 + \varepsilon^{1/4} N^{1/2}) N^{-(k+1)} (\ln N)^{3/2} \right]. \tag{5.21}$$

We now bound the final term in (5.18): $F_L(\boldsymbol{\xi}, \boldsymbol{G}) = \sum_{i=1}^{3} Y_i$ with

$$Y_1 := \sum_{i=1}^{N/4} \left\langle (\xi_u)^+_{x,3N/4}, (G_q)^-_{x,3N/4} \right\rangle_{I_i}, \quad Y_2 := -\sum_{i=1}^{N/4} \left\langle (\xi_u)^-_{x,N/4}, (G_q)^+_{x,N/4} \right\rangle_{I_i},$$
$$Y_3 := \sum_{j=N/4+1}^{3N/4} \left\langle (\xi_p)^+_{N/4,y}, (G_u)^-_{N/4,y} \right\rangle_{J_j}.$$

Similarly to (5.20), one uses (4.36) and (5.10) to get

$$\begin{aligned}|Y_1| &\le C\varepsilon^{3/4} (N \ln N)^{1/2} \left\|\xi_u\right\|_{L^\infty(\Omega_f)} \|\!|\!| \boldsymbol{G} |\!|\!|_{E,\Omega_{12}} \\ &\le C \left\{ \varepsilon^{1/2} N^{-k} (\ln N)^{k+4} + \left[\varepsilon^{3/4} N^2 (\ln N)^3 + \varepsilon (N \ln N)^{5/2} \right] \theta(\varepsilon, N, k) \right\}. \end{aligned}$$

One can likewise bound Y_2. Lemma 5.1(iv), (3.11b) and (5.16) give

$$\begin{aligned}|Y_3| &\le \left(\sum_{j=N/4+1}^{3N/4}\left\|(\omega\xi_p)^+_{N/4,y}\right\|^2_{J_j}\right)^{1/2}\left(\sum_{j=N/4+1}^{3N/4}\left\|(\omega^{-1}G_u)^-_{N/4,y}\right\|^2_{J_j}\right)^{1/2}\\ &\le C\varepsilon\Big[N(\ln N)^{1/2}+\varepsilon^{1/4}N^{3/2}\Big]\theta(\varepsilon,N,k)\sqrt{\frac{\tau}{\varepsilon}}\,|||\,\boldsymbol{G}|||_{E,\Omega_{12}}\\ &\le C\Big[\varepsilon^{1/2}(N\ln N)^{3/2}+\varepsilon^{3/4}N^2\ln N\Big]\theta(\varepsilon,N,k).\end{aligned}$$

Collecting the bounds for the $Y_i(i=1,2,3)$ yields

$$|F_L(\boldsymbol{\xi},\boldsymbol{G})|\le C\Big\{\varepsilon^{1/2}N^{-k}(\ln N)^{k+4}+[\varepsilon^{1/2}(N\ln N)^{3/2}+\varepsilon^{3/4}N^2(\ln N)^3+\varepsilon(N\ln N)^{5/2}]\theta(\varepsilon,N,k)\Big\}. \tag{5.22}$$

Finally, (5.17) follows from (5.18), (5.19), (5.21), (5.22), (3.7b) and a triangle inequality. □

Remark 5.10. In the singularly perturbed regime when $\varepsilon^{1/2}\le N^{-1}$, clearly $\tilde{C}_3$ can be bounded by a constant C that is independent of ε and N, and Theorem 5.9 then gives an $O(N^{-(k+1)}(\ln N)^{k+5/2})$ pointwise error bound in Ω_b when terms multiplied by positive powers of ε are ignored.

6. Numerical experiments

Example 6.1. Consider the reaction-diffusion problem

$$\begin{aligned}-\varepsilon\Delta u+\big(1+x^2y^2e^{xy/2}\big)u&=f\quad \text{in } \Omega=(0,1)^2,\\ u&=0\quad \text{on } \partial\Omega\end{aligned}$$

where f is chosen such that

$$u(x,y)=\left(1-e^{-x/\sqrt{\varepsilon}}\right)\left(1-e^{-(1-x)/\sqrt{\varepsilon}}\right)\left(1-e^{-y/\sqrt{\varepsilon}}\right)\left(1-e^{-(1-y)/\sqrt{\varepsilon}}\right)\cos(\pi xy).$$

This solution has precisely the layer behaviour that is described in Assumption 2.1 .

We implement the LDG method (2.5), testing various values of k for the finite element space $\mathcal{Q}^k$, using the Shishkin mesh (2.3) with $\sigma=k+1$. For N elements in each coordinate direction of Ω, we set

$$e_N:=\|u-\mathbb{U}\|_{L^\infty(D^N)}=\max_{K\in D^N}\|u-\mathbb{U}\|_{L^\infty(K)},$$

the observed local maximum error in any subregion $D^N\subset\Omega$, where $\|u-\mathbb{U}\|_{L^\infty(K)}$ is computed for each rectangular element K by taking the maximum pointwise error of $u-\mathbb{U}$ over 100 uniformly distributed points within K. Then the numerical convergence rate is calculated from the formulae

$$r_2:=\frac{\ln e_N-\ln e_{2N}}{\ln 2}\quad\text{ and }\quad r_S:=\frac{\ln e_N-\ln e_{2N}}{\ln(2\ln N/\ln(2N))},$$

so r_2 estimates the rate r when $e_N\approx CN^{-r}$, while r_S estimates r when $e_N\approx C(N^{-1}\ln N)^r$.

In our experiments we take $\beta=1/\sqrt{2}$. For $\varepsilon=10^{-4}$ and $\varepsilon=10^{-8}$, Figure 2 displays numerical solutions $\mathbb{U}$ and their pointwise errors $|u-\mathbb{U}|$, which are generated by the LDG method on a Shishkin mesh with $N=32$ and $k=2$. One sees that the LDG method yields an accurate solution, highlighting the LDG method's ability to capture sharp boundary layers without any oscillations in their vicinity.

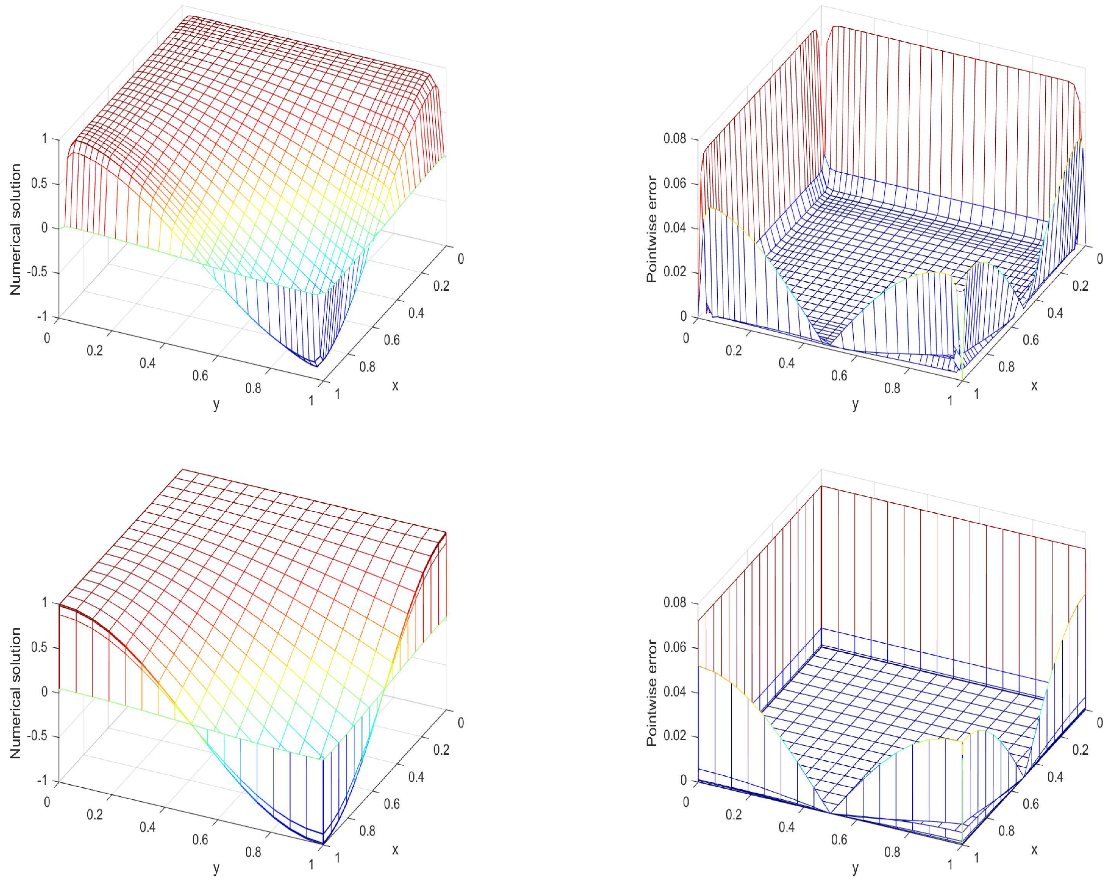

FIGURE 2. The graphics of numerical solutions and pointwise errors, when $\varepsilon = 10^{-4}$ (*top*) and $\varepsilon = 10^{-8}$ (*bottom*).

For various values of k and N, with $\varepsilon = 10^{-8}$, Table 2 presents the maximum errors in the subregions $D = \Omega_{22}, \Omega_b$ and Ω_f. We observe a pointwise convergence rate $O(N^{-(k+1)})$ in the coarse region Ω_{22}, which agrees with Theorem 3.5. The rate of convergence in the boundary layer region Ω_b and corner layer region Ω_f appears to be $O\left((N^{-1}\ln N)^{k+1}\right)$, implying that the convergence rate established in Theorem 4.7 and 5.9 are almost sharp if the logarithmic factors are neglected.

For $k = 1$ and $N = 256$, and various values of ε, Table 3 shows that the local maximum errors in each subregion remain robust as ε approaches zero, demonstrating the reliability of the LDG method in the entire singularly perturbed regime.

7. Concluding remarks

This paper analyses pointwise convergence of an LDG method on a Shishkin mesh for a two-dimensional singularly perturbed reaction-diffusion problem. By means of special projectors, discrete Green's functions and weighted energy norms, we prove that our method achieves pointwise accuracy of order $O(N^{-(k+1)})$ in the coarse-mesh domain, $O((N^{-1}\ln N)^{k+1}(\ln N)^{3/2})$ in the boundary layer region, and $O((N^{-1}\ln N)^{k+1}(\ln N)^2)$ in the corner layer region. These results are consistent with our numerical results and are the sharpest known pointwise error estimates in the layer regions for any DG method applied to two-dimensional reaction-diffusion problems

TABLE 2. Convergence results in local maximum norm.

		$\lVert u-\mathbb{U}\rVert_{L^\infty(\Omega_{22})}$		$\lVert u-\mathbb{U}\rVert_{L^\infty(\Omega_b)}$			$\lVert u-\mathbb{U}\rVert_{L^\infty(\Omega_f)}$		
	N	Error	r_2-Rate	Error	r_2-Rate	r_S-Rate	Error	r_2-Rate	r_S-Rate
$\mathcal{Q}^0$	8	5.4598E−01	–	7.3998E−01	–	-	6.4090E−01	–	-
	16	2.9432E−01	0.8915	6.1077E−01	0.2768	0.4733	5.8200E−01	0.1391	0.2378
	32	1.4915E−01	0.9806	4.5304E−01	0.4310	0.6356	4.4596E−01	0.3841	0.5665
	64	7.4878E−02	0.9941	3.0619E−01	0.5652	0.7669	3.0456E−01	0.5502	0.7466
	128	3.7497E−02	0.9978	1.9267E−01	0.6683	0.8594	1.9231E−01	0.6633	0.8530
	256	1.8759E−02	0.9992	1.1525E−01	0.7413	0.9183	1.1518E−01	0.7396	0.9160
	512	9.3810E−03	0.9998	6.6590E−02	0.7915	0.9534	6.6574E−02	0.7908	0.9528
$\mathcal{Q}^1$	8	9.4720E−02	–	5.2584E−01	–	-	5.0003E−01	–	-
	16	2.5055E−02	1.9186	3.2397E−01	0.6988	1.3180	3.1968E−01	0.6454	1.1033
	32	6.3544E−03	1.9793	1.6444E−01	0.9783	1.4428	1.6390E−01	0.9638	1.4214
	64	1.5938E−03	1.9953	7.0288E−02	1.2262	1.6639	7.0231E−02	1.2227	1.6590
	128	3.9852E−04	1.9997	2.6553E−02	1.4044	1.8061	2.6548E−02	1.4035	1.8049
	256	9.9604E−05	2.0004	9.2231E−03	1.5255	1.8896	9.2227E−03	1.5253	1.8893
	512	2.4941E−05	1.9977	3.0247E−03	1.6084	1.9377	3.0247E−03	1.6084	1.9377
$\mathcal{Q}^2$	8	4.8212E−03	–	2.8653E−01	–	-	2.8582E−01	–	-
	16	5.8824E−04	3.0349	1.4027E−01	1.0305	1.7616	1.4026E−01	1.0271	1.7557
	32	7.2750E−05	3.0154	5.1054E−02	1.4581	2.1504	5.1054E−02	1.4580	2.1502
	64	9.0324E−06	3.0098	1.4536E−02	1.8124	2.4593	1.4536E−02	1.8124	2.4593
	128	1.1245E−06	3.0059	3.4434E−03	2.0777	2.6719	3.4434E−03	2.0777	2.6719
	256	1.4009E−07	3.0048	7.1559E−04	2.2666	2.8075	7.1559E−04	2.2666	2.8075
	512	1.7463E−08	3.0040	1.3570E−04	2.3987	2.8898	1.3570E−04	2.3987	2.8898
$\mathcal{Q}^3$	8	3.6716E−04	–	1.5777E−01	–	-	1.5958E−01	–	-
	16	2.5825E−05	3.8296	6.0612E−02	1.3801	2.3594	6.0612E−02	1.3966	2.3594
	32	1.6767E−06	3.9450	1.5913E−02	1.9294	2.8454	1.5913E−02	1.9294	2.8454
	64	1.0611E−07	3.9821	3.0265E−03	2.3945	3.2491	3.0265E−03	2.3945	3.2491
	128	6.6580E−09	3.9943	4.4940E−04	2.7516	3.5385	4.4940E−04	2.7516	3.5385
	256	4.1802E−10	3.9934	5.5819E−05	3.0092	3.7272	5.5819E−05	3.0092	3.7272

TABLE 3. Robustness of local maximum errors in ε.

ε	$\lVert u-\mathbb{U}\rVert_{L^\infty(\Omega_{22})}$	$\lVert u-\mathbb{U}\rVert_{L^\infty(\Omega_b)}$	$\lVert u-\mathbb{U}\rVert_{L^\infty(\Omega_f)}$
10^{-4}	1.6882E−04	9.2242E−03	9.2243E−03
10^{-5}	1.3523E−04	9.2229E−03	9.2229E−03
10^{-6}	9.9873E−05	9.2227E−03	9.2227E−03
10^{-7}	9.8384E−05	9.2227E−03	9.2227E−03
10^{-8}	9.9604E−05	9.2231E−03	9.2227E−03
10^{-9}	1.0017E−04	9.2232E−03	9.2227E−03
10^{-10}	1.0037E−04	9.2232E−03	9.2227E−03
10^{-11}	1.0043E−04	9.2232E−03	9.2227E−03
10^{-12}	1.0045E−04	9.2232E−03	9.2227E−03
10^{-13}	1.0046E−04	9.2232E−03	9.2227E−03
10^{-14}	1.0046E−04	9.2232E−03	9.2227E−03

with boundary layers. In future work, we aim to derive analogous results for singularly perturbed *convection-diffusion* problems, but for these problems no energy-norm supercloseness result analogous to Lemma 3.2 (*i.e.*, with a factor ε^γ for some $\gamma > 0$) is known; furthermore, the derivation of suitable discrete stability bounds in the crosswind direction appears to be difficult. These are two significant challenges that require further investigation.

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