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A CONVERGENT TIME SCHEME FOR A CHEMOTAXIS-FLUIDS MODEL WITH POTENTIAL CONSUMPTION

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Abstract. The present work deals with a Keller–Segel–Navier–Stokes system with potential consumption, under homogeneous Neumann boundary conditions for cell density and chemical signal, and Dirichlet type for the velocity field, over a bounded three-dimensional domain. The paper aims to develop a time discretization scheme converging to weak solutions of the system, which are uniformly bounded at infinite time. While global existence results are already known for simplified cases, either in absence of fluid flow or for linear consumption, the existence of global weak solutions for the fully coupled system with potential consumption has remained as an open problem.

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1. Introduction

In this work, we investigate the dynamics of a chemotaxis-fluid system, a mathematical model with an intricate interplay between cellular behavior and fluid dynamics. The system, described within a bounded three-dimensional domain, encompasses three coupled nonlinear partial differential equations governing the evolution of cell density, chemical concentration, and fluid velocity field.

Originating from biological scenarios such as bacterial motility in response to chemical gradients, chemotaxis modelling was first introduced by Keller and Segel [19]. Chemotaxis is seen in many organic functions, even playing an important role in inflammatory diseases, wound healing, cancer metastasis or disease progressions, [23–25]. In recent years, numerous studies have focused on chemotaxis-related models. For comprehensive reviews of this literature, see, for instance, [1, 3, 4, 21].

In cases where a chemical signal attracts cells that they also consume, a useful energy law emerges due to the interplay between the chemotactic and consumption terms. This effect has been explored in previous studies, allowing *a priori* estimates of possible solutions. Global solutions have been found for three-dimensional domains in [26], and, with the addition of a logistic term on the cell density, the results were extended to higher-dimensional cases in [20]. Some closely related considerations also lead to several other works such as [2, 5, 6, 22].

Keywords and phrases. Chemotaxis, Navier–Stokes equations, time discrete scheme, energy law, convergence, weak solutions.

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Recently, an chemoattractive model with a potential consumption rate, was introduced in [9] as

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\nabla c), \\ c_t = \Delta c - n^s c, \end{cases}$$

where $n = n(t,x) \geq 0$ is the cell density and $c = c(t,x) \geq 0$ denotes the chemical concentration, for any $x \in \Omega$, where Ω is a bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}$), and $t > 0$. The term $\nabla \cdot (n\nabla c)$ corresponds to the chemotactic attraction and n^s is the consumption rate of signal c, with $s \geq 1$. In [9], global weak solutions, uniformly bounded in time, are found in three-dimensional domains with minimal regularity, *via* a continuous regularization of the problem. By the contrary, a convergent time discretization is studied in [15] for the same problem.

In the presence of fluid flow, a buoyancy effect known as bioconvection occurs when bacterial populations exhibit collective behavior in an incompressible fluid, as discussed in [17]. This effect, combined with chemotactic behavior may occur; this is the case, for example, where aerobic bacteria in water droplets search for oxygen [11]. This buoyancy effect couples cell density with a fluid velocity field, where additionally convective terms appear. This macro-scale chemotaxis-fluid model has been extensively studied in various works such as [8, 10].

In the present work, we focus on this kind of attraction-consumption chemotaxis models, in the presence of fluid flow by integrating the Navier–Stokes (NS) equations for an incompressible flow, for the velocity field $u = u(t,x) \in \mathbb{R}^3$ and pressure $P = P(t,x) \in \mathbb{R}$. This is achieved by introducing convective terms for both n and c equations and incorporating a source term into the NS equations, accounting for the gravitational effect of the heavier bacteria on the flow. Moreover, homogeneous Neumann boundary conditions for both the cell density n and the chemical concentration c are considered, jointly with Dirichlet boundary conditions for the velocity field u. Then the system remains as

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - n^s c, \\ u_t + u \cdot \nabla u + \nabla P = \Delta u + n\nabla \Phi, \\ \nabla \cdot u = 0, \\ \partial_\eta n|_\Gamma = \partial_\eta c|_\Gamma = u|_\Gamma = 0, \quad n(0) = n^0, \quad c(0) = c^0, \quad u(0) = u^0. \end{cases} \tag{1}$$

Here, η represents the outward normal vector to the boundary Γ of the domain Ω. The gravitational potential, $\Phi = \Phi(x)$, is assumed to be a given $W^{1,\infty}(\Omega)$ function. Finally, n^0, c^0, u^0 are the initial data.

The case $s = 1$ has been treated first in convex domains, obtaining global classical solutions in two-dimensional domains [27]; and after were extended to non-convex domains in [18]. Global weak solutions have also been obtained in three-dimensional domains in [28].

The objective of this work is twofold: on the one hand to extend the existing results of [28] to the case of potential consumption rate n^s with $s > 1$ and through a convergent time discrete scheme, and on the other hand extending the results of the system without fluid given in [15] to the coupled chemotaxis-fluid system. More specifically, the main contributions of this paper are the following:

(i) Getting the convergence of a time discrete scheme towards weak solutions of system (1), using a new truncation operator for the chemotaxis and consumption terms and also truncating the source term from the Navier–Stokes system.
(ii) Proving the existence of global weak solutions for the chemotaxis-Navier–Stokes model with potential consumption $n^s c$ for $s > 1$ which are uniformly bounded in time.

It is worth mentioning that although the numerical simulation of chemotaxis models is a significant and growing area of research, there are relatively few studies focused on the numerical approximation for chemoattraction-consumption models. Beyond [15], to the best of our knowledge, the case $s = 1$ has been addressed in only two studies: [16] and [12]. The former investigates several finite element schemes, while the latter develops numerical approximations for the system coupled with the Navier–Stokes equations. This shortage may also be linked to the complex interaction between chemotaxis and consumption effects, as well as the difficulty for adapting theoretical frameworks into numerical approaches. Moreover, constructing convergent schemes is particularly demanding, as it requires carefully balancing the competition effects between chemoattraction and consumption in the derivation of energy estimates.

This paper is organized as follows. Section 2 presents the main result and the time discrete scheme. For that, we introduce a new truncation operator and the corresponding time discrete regularized system of (1). Then the existence of global weak solutions of (1) is stated. Section 3 focuses on preliminary results that are either referenced or proved. Keys are Lemma 3.6, which deals with the source term $n\nabla\Phi$ of the fluid system and Lemma 3.7 which clarifies the relationship between the direct upper truncation T_0^m and the new one G_0^m, as given in (4) and (5) below. Section 4 treats the existence of the time discrete regularized problem which is obtained by a fixed point argument. Section 5 presents some *a priori* estimates based on an adequate energy inequality. This is done separately for $s \in (1,2)$ and $s \geq 2$ because the energy becomes singular with respect to cell variable n for the case $s < 2$. Section 6 shows the passage to the limit of the time discrete regularized system towards weak solutions of (1).

2. Main theorem and time discrete scheme

Here and henceforth Ω is assumed to be a bounded domain in $\mathbb{R}^3$ with boundary Γ sufficiently regular as specified below. For simplicity, we will omit the domain in the notation of functional spaces. That is, we will denote, for instance, $L^2(\Omega)$ simply as L^2 from now on.

We start by introducing some usual spaces in fluid problems. Let

$$\mathcal{V} := \big\{ v \in C_0^\infty(\Omega)^3, \nabla \cdot v = 0 \big\}.$$

Denoting by $\overline{\mathcal{V}}^X$ the closure of $\mathcal{V}$ in the X-norm, we consider

$$H := \overline{\mathcal{V}}^{L^2} = \big\{ v \in L^2, \nabla \cdot v = 0, \, v \cdot \eta|_\Gamma = 0 \big\},$$
$$V := \overline{\mathcal{V}}^{H_0^1} = \big\{ v \in H_0^1, \nabla \cdot v = 0 \big\}.$$

Both identifications hold true for bounded Lipschitz domains [7]. Moreover, one has the Gelfand Triple $V \subset H \subset V'$, with compact and dense embeddings. We also define $W_{0,\sigma}^{1,p} := \overline{\mathcal{V}}^{W_0^{1,p}}$ for $1 < p < \infty$.

The main properties of problem (1) are nonnegative variables $n, c \geq 0$, n-conservation, c-pointwise estimates and an energy inequality letting appropriate estimates of all variables n, c and u.

We are going to impose conditions on the regularity of the domain Ω such that the boundary terms on the energy inequality associated to problem (1) are dealt with. Specifically, we impose the following hypotheses:

(H2)-regularity: *For any $f \in L^2$, there exists a unique $z \in H^2$ solution to the Poisson-Neumann problem*

$$-\Delta z + z = f \quad \text{in } \Omega, \quad \partial_\eta z|_\Gamma = 0. \tag{2}$$

Moreover, there exists $C > 0$ such that

$$\|z\|_{H^2} \leq C\| - \Delta z + z\|_{L^2}.$$

(H2)-approx.: *For any $z \in H^2$ with $\partial_\eta z|_\Gamma = 0$, there is a sequence $\{\rho_n\} \subset C^2(\overline{\Omega})$ such that $\partial_\eta \rho_n|_\Gamma = 0$ and $\rho_n \to z$ in H^2.*

Note that, as shown in [9], it suffices require Γ to be at least $C^{2,1}$ to both hypotheses hold.

We now introduce what is meant by a weak solution to problem (1).

Definition 2.1 (Weak solution)**.** A triplet (n, c, u) is a weak solution of system (1) in $(0,\infty)\times\Omega$ if the following features hold:

– (nonnegative) $n, c \geq 0$ a.e. in $(0,\infty)\times\Omega$,
– (n-conservation)

$$\int_\Omega n(t,\cdot) = \int_\Omega n^0 \quad \text{a.e. } t \in (0,\infty),$$

– (c-pointwise estimates)

$$0 \leq c(t,x) \leq \left\|c^0\right\|_{L^\infty} \quad \text{a.e. } (t,x) \in (0,\infty)\times\Omega,$$

– (c-weak estimates)

$$\nabla c \in L^2\big(0,\infty;L^2\big),$$

– (energy regularity)

$$\begin{aligned}
&n \in L^\infty(0,\infty;L^s) \cap L^{5s/3}_{\text{loc}}\Big(0,\infty;L^{5s/3}\Big),\\
&c \in L^\infty\big(0,\infty;H^1\big) \cap L^2_{\text{loc}}\big(0,\infty;H^2\big), \quad \nabla c \in L^4_{\text{loc}}\big(0,\infty;L^4\big),\\
&u \in L^\infty(0,\infty;H) \cap L^2_{\text{loc}}(0,\infty;V),
\end{aligned}$$

– (flux regularity)

$$\begin{aligned}
&n\nabla c,\ \nabla n \in L^{5s/(3+s)}_{\text{loc}}\Big(0,\infty;L^{5s/(3+s)}\Big), \quad \text{for } s \in (1,2),\\
&n\nabla c,\ \nabla n \in L^2_{\text{loc}}\big(0,\infty;L^2\big), \qquad\qquad \text{for } s \in [2,\infty),
\end{aligned}$$

– and (n, c, u) satisfies

$$\begin{aligned}
\int_\Omega n_t\phi - \int_\Omega n\, u\cdot\nabla\phi + \int_\Omega \nabla n\cdot\nabla\phi = \int_\Omega n\nabla c\cdot\nabla\phi, &\quad \text{for all } \ \phi \in W^{1,10}, \quad \text{a.e. } t \in (0,\infty),\\
c_t + u\cdot\nabla c - \Delta c = -n^s c, &\quad \text{a.e. } (0,\infty)\times\Omega,\\
\int_\Omega u_t\cdot\varphi + \int_\Omega (u\otimes u)\cdot\nabla\varphi + \int_\Omega \nabla u\cdot\nabla\varphi = \int_\Omega n\nabla\Phi\cdot\varphi, &\quad \text{for all } \ \varphi \in W^{1,5/2}_{0,\sigma}, \quad \text{a.e. } t \in (0,\infty),
\end{aligned}$$

and the initial conditions $(n, c, u)|_{t=0} = (n^0, c^0, u^0)$. Hereafter, $(u\otimes u)_{ij} = u_i u_j$.

Remark 2.1. With the regularity given in previous definition and looking at the system (1), the following time derivative regularity hold:

$$n_t \in L^{10/9}_{\text{loc}}\Big(0,\infty;\big(W^{1,10}\big)'\Big) \quad \text{for } s<2, \quad n_t \in L^{5/3}_{\text{loc}}\left(0,\infty;\left(W^{1,5/2}\right)'\right) \quad \text{for } s \geq 2,$$

$$c_t \in L^{5/3}_{\text{loc}}\Big(0,\infty;L^{5/3}\Big), \quad \text{and} \quad u_t \in L^{5/3}_{\text{loc}}\left(0,\infty;\left(W^{1,5/2}_{0,\sigma}\right)'\right).$$

In particular, the initial conditions $(n, c, u)|_{t=0} = (n^0, c^0, u^0)$ have sense.

We may now state our main result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain satisfying* **(H2)-regularity** *and* **(H2)-approx.**. *Let $n^0 \in L^s$, $c^0 \in H^1 \cap L^\infty$ with $n^0 \geq 0, c^0 \geq 0$ a.e. in Ω, and $u^0 \in H$. Then, one has a time discrete problem (see (8) below) which is convergent towards a weak solution (n, c, u) of problem (1) in $(0,\infty)\times\Omega$.*

2.1. Definition of the time discrete scheme

A convenient way to rewrite the equations is by setting an adequate change of variables, changing the chemical variable c by the auxiliary variable $z = \sqrt{c + \alpha^2}$ for some $\alpha > 0$, which simplifies the test functions involved for obtaining an energy law of system (1). In fact, c-equation $(1)_2$ is rewritten as

$$z_t + u \cdot \nabla z - \Delta z - \frac{|\nabla z|^2}{z} = -\frac{1}{2} n^s \left(z - \frac{\alpha^2}{z} \right).$$

A time discretization approach is considered in [15] in the case of the corresponding chemotaxis model, but without the interaction with a fluid flow. A regularization to this system is made with upper and lower truncations in order to have bounded terms and prevent division by zero, respectively. For that reason, we define the lower truncation for z as

$$T_\alpha(z) := \begin{cases} \alpha & \text{if } z \le \alpha, \\ z & \text{if } z \ge \alpha, \end{cases} \tag{3}$$

and the lower-upper truncation for n,

$$T_0^m(n) := \begin{cases} 0 & \text{if } n \le 0, \\ n & \text{if } n \in [0, m], \\ m & \text{if } n \ge m. \end{cases} \tag{4}$$

In this work, a new regularization is proposed. Compared with [15], this regularization fits better concerning the test functions employed in the n and z equations. To do that, we define the new lower-upper truncation operator but for n^s/s,

$$G_0^m(n) := \begin{cases} 0 & \text{if } n \le 0, \\ \frac{1}{s} n^s & \text{if } n \in [0, m], \\ \frac{m}{s-1} n^{s-1} - \frac{m^s}{s(s-1)} & \text{if } n \ge m. \end{cases} \tag{5}$$

In fact, $G_0^m(n)$ is defined such that $G_0^m \in C^1(\mathbb{R})$ and satisfying the important equality

$$(G_0^m)'(n) = T_0^m(n) n^{s-2} 1_{\{n \ge 0\}}. \tag{6}$$

At this point, we consider the truncated (n, z, u)-system as follows

$$\begin{cases} n_t + u \cdot \nabla n - \Delta n + \nabla \cdot (T_0^m(n) 2 T_\alpha(z) \nabla z) = 0, \\ z_t + u \cdot \nabla z - \Delta z - \dfrac{|\nabla z|^2}{T_\alpha(z)} = -\dfrac{s}{2} G_0^m(n) \left(z - \dfrac{\alpha^2}{T_\alpha(z)} \right), \\ u_t + u \cdot \nabla u - \Delta u + \nabla P = T_0^m(n) \nabla \Phi, \\ \nabla \cdot u = 0, \\ \partial_\eta n|_\Gamma = \partial_\eta z|_\Gamma = u|_\Gamma = 0, \quad n(0) = n^0, \quad z(0) = z^0, \quad u(0) = u^0, \end{cases} \tag{7}$$

where $z^0 = \sqrt{c^0 + \alpha^2}$.

As for the time discretization, we will divide the interval $[0, \infty)$ into subintervals denoted by $I_i = [t_{i-1}, t_i)$, with $t_0 = 0$ and $t_i = t_{i-1} + k$, where $k > 0$ is the time step. We use the notation

$$\delta_t n^i := \frac{n^i - n^{i-1}}{k}, \text{ for all } i \ge 1,$$

for the discrete time derivative of a sequence n^i. Then, we define the following Backward Euler time scheme of truncated system (7) with semi-implicit approximation of the convection of the fluid: given $(n^{i-1}, z^{i-1}, u^{i-1})$, to find (n^i, z^i, u^i, P^i) solving

$$\begin{cases} \delta_t n^i + u^i \cdot \nabla n^i - \Delta n^i + \nabla \cdot \left(T_0^m(n^i) 2T_\alpha(z^i) \nabla z^i\right) = 0, \\ \delta_t z^i + u^i \cdot \nabla z^i - \Delta z^i - \dfrac{\left|\nabla z^i\right|^2}{T_\alpha(z^i)} = -\dfrac{s}{2} G_0^m(n^i)\left(z^i - \dfrac{\alpha^2}{T_\alpha(z^i)}\right), \\ \delta_t u^i + u^{i-1} \cdot \nabla u^i - \Delta u^i + \nabla P^i = T_0^m(n^i) \nabla \Phi, \\ \nabla \cdot u^i = 0, \\ \partial_\eta n^i\big|_\Gamma = \partial_\eta z^i\big|_\Gamma = u^i\big|_\Gamma = 0. \end{cases} \tag{8}$$

In fact, (8) is a fully coupled and nonlinear time scheme.

Since $z^0 = \sqrt{c^0 + \alpha^2} \geq \alpha$ and $n^0 \geq 0$ in Ω, we will determine that $z^i \geq \alpha$ and $n^i \geq 0$ in $(0, \infty) \times \Omega$, hence we will drop the lower truncations of z^i by α and n^i by 0 in (8). Thus, to return to the original system (1), we will check what happens to the solution of (8) as $(m, k) \to (\infty, 0)$.

In order to get energy estimates (see Sect. 6.2) we will need to impose more regular initial data for the time discrete problem (8) than $(n^0, z^0, u^0) \in L^s \times (H^1 \cap L^\infty) \times H$ imposed in Theorem 2.1. In fact, we consider a sequence (n_j^0, z_j^0, u_j^0) satisfying the following features:

– (regularity)

$$n_j^0 \in H^1 \cap L^s, \quad z_j^0 \in H^2, \quad u_j^0 \in V, \tag{9}$$

– (nonnegative and conservation)

$$n_j^0 \geq 0, \ z_j^0 \geq \alpha \ \text{ in } \Omega, \quad \int_\Omega n_j^0 = \int_\Omega n^0, \tag{10}$$

– (approximation)

$$\left(n_j^0, z_j^0, u_j^0\right) \to \left(n^0, z^0, u^0\right) \quad \text{in } L^s \times \left(H^1 \cap L^\infty\right) \times H, \text{ as } j \to \infty. \tag{11}$$

Finally, to get that all estimates of this paper be j-independent, it suffices to consider for each j a small enough time step $k(j)$ such that the following estimates hold:

$$\begin{aligned} k(j) \left\|n_j^0\right\|_{W^{1,5s/(s+3)}}^{5s/(s+3)} \leq 1 \ \text{ if } s \leq 2, \qquad & k(j) \left\|n_j^0\right\|_{H^1}^2 \leq 1 \ \text{ if } s \geq 2, \\ k(j) \left\|z_j^0\right\|_{H^2}^2 \leq 1, \qquad & k(j) \left\|u_j^0\right\|_V^2 \leq 1. \end{aligned} \tag{12}$$

Although the solution of (8) depends on j, by simplicity in the notation, from now on we omit the index j.

3. Preliminary results

This section presents several technical results that will be essential tools throughout the paper.

Lemmas 3.1 through 3.3, previously established and proven in [9,15], are restated here for completeness. The following lemma is used to obtain an energy inequality without a boundary term otherwise present.

Lemma 3.1. *Suppose that* **(H2)-regularity** *and* **(H2)-approx** *hold. Let $z \in H^2$ be such that $\partial_\eta z|_\Gamma = 0$ and $z \geq \alpha$ for some $\alpha > 0$. Then there exist positive constants $C_1, C_2 > 0$, independent of α, such that*

$$\int_\Omega |\Delta z|^2 \,\mathrm{d}x + \int_\Omega \frac{|\nabla z|^2}{z} \Delta z \,\mathrm{d}x \geq C_1 \left(\int_\Omega \left|D^2 z\right|^2 \mathrm{d}x + \int_\Omega \frac{|\nabla z|^4}{z^2} \,\mathrm{d}x \right) - C_2 \int_\Omega |\nabla z|^2 \,\mathrm{d}x.$$

As for the time discretization, when dealing with time discrete derivatives the following holds.

Lemma 3.2. *Let $z^n, z^{n-1} \in L^\infty$, and let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function. Then*

$$\int_\Omega \delta_t z^n(x) f'(z^n(x))\,\mathrm{d}x = \delta_t \int_\Omega f(z^n(x))\,\mathrm{d}x + \frac{1}{2k}\int_\Omega f''(b^n(x))\big(z^n(x) - z^{n-1}(x)\big)^2\,\mathrm{d}x,$$

where $b^n(x)$ is an intermediate point between $z^n(x)$ and $z^{n-1}(x)$.

To establish the convergence of the powers of a function, we will utilize the following result.

Lemma 3.3. *Let $p \in (1,\infty)$, and let $\{w_i\}$ be a sequence of nonnegative functions such that $w_i \to w$ in $L^p(0,T;L^p)$ as $i \to \infty$. Then, for every $r \in (1,p)$, $w_i^r \to w^r$ in $L^{p/r}(0,T;L^{p/r})$ as $i \to \infty$.*

We also make use of the classical version of Aubin-Lions theorem for compactness [7].

Lemma 3.4. *Let X, B and Y be three Banach spaces with $X \subset B \subset Y$. Suppose that X is compactly embedded in B and B is continuously embedded in Y. For $1 \le p, q \le \infty$, let*

$$W = \{u \in L^p(0,T;X) \mid \partial_t u \in L^q(0,T;Y)\}.$$

(i) *If $p < \infty$ then the embedding of W into $L^p(0,T;B)$ is compact.*
(ii) *If $p = \infty$ and $q > 1$ then the embedding of W into $C([0,T];B)$ is compact.*

We present a discrete version of uniform in time Gronwall's estimates.

Lemma 3.5. *Consider a sequence of inequalities*

$$\delta_t a^i + \lambda\, a^i \le C, \quad \text{for all } \ i \ge 1,$$

where $\lambda > 0$ and $\{a^i\}_{i\in\mathbb{N}} \subset \mathbb{R}_+$. Then, the following estimate holds

$$a^i \le (1+\lambda k)^{-i} a^0 + \frac{C}{\lambda}\big(1 - (1+\lambda k)^{-i}\big), \quad \text{for all } \ i \ge 1.$$

In particular,

$$a^i \le a^0 + \frac{C}{\lambda}, \quad \text{for all } \ i \ge 1.$$

Proof. We adapt a proof given in [13]. Let $\tilde a^i := (1+\lambda k)^i a^i$. Then, from the recursive inequality

$$\delta_t \tilde a^i \le (1+\lambda k)^{i-1} C, \quad \text{for all } \ i \ge 1,$$

Now, summing over i leads to

$$\tilde a^i \le \tilde a^0 + C\,k \sum_{j=0}^{i-1} (1+\lambda k)^j = a^0 + \frac{C}{\lambda}\big((1+\lambda k)^i - 1\big),$$

and finally

$$a^i \le (1+\lambda k)^{-i} a^0 + \frac{C}{\lambda}\big(1 - (1+\lambda k)^{-i}\big).$$

□

Lemma 3.6. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, with $d \le 3$. Let also f, g be functions defined on Ω such that*

$$\int_\Omega |f| = K, \quad |f|^{s/2} \in H^1 \ (s > 1) \quad \text{and} \quad g \in H^1_0.$$

Then for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there exists $C > 0$ such that

$$\int_\Omega |f||g| \le \epsilon_1 \|\nabla g\|^2_{L^2} + \epsilon_2 \Big\|\nabla |f|^{s/2}\Big\|^2_{L^2} + C,$$

where C depends on ϵ_1, ϵ_2 and $\int_\Omega |f|$.

This lemma will be used to deal with the source term $T_0^m(n^i)\nabla\Phi$ on the fluid velocity equation, when it is multiplied by u^i, by taking $f = T_0^m(n^i)$ and $g = u^i$.

Proof. By using the embedding $H^1 \hookrightarrow L^6$, the equivalence between $\|g\|_{H^1}$ and $\|\nabla g\|_{L^2}$ for any $g \in H_0^1$ and Young's inequality, we obtain

$$\int_\Omega |f||g| \le \|f\|_{L^{6/5}}\|g\|_{L^6} \le C\Big\||f|^{s/2}\Big\|_{L^{12/(5s)}}^{2/s}\|\nabla g\|_{L^2} \le C\Big\||f|^{s/2}\Big\|_{L^{12/(5s)}}^{4/s} + \epsilon_1\|\nabla g\|_{L^2}^2.$$

Since $2/s < 12/5s < 6$, by interpolating

$$\Big\||f|^{s/2}\Big\|_{L^{12/(5s)}}^{4/s} \le \left(\Big\||f|^{s/2}\Big\|_{L^6}^{a}\right)^{4/s}\left(\Big\||f|^{s/2}\Big\|_{L^{2/s}}^{(1-a)}\right)^{4/s} = \Big\||f|^{s/2}\Big\|_{L^6}^{4a/s}\left(\int_\Omega |f|\right)^{2(1-a)}$$

where $a = s/2(3s-1)$. Then, by using again the embedding $H^1 \hookrightarrow L^6$, and as $4a/s = 2/(3s-1) < 1$ for any $s > 1$, applying Young's inequality we arrive at

$$\int_\Omega |f||g| \le \epsilon\Big\|\nabla|f|^{s/2}\Big\|_{L^2}^2 + \epsilon\Big\||f|^{s/2}\Big\|_{L^2}^2 + C(K) + \epsilon_1\|\nabla g\|_{L^2}^2. \tag{13}$$

By using Poincaré's inequality we have

$$\Big\||f|^{s/2}\Big\|_{L^2}^2 \le C_1\Big\|\nabla|f|^{s/2}\Big\|_{L^2}^2 + C_1\Big\||f|^{s/2}\Big\|_{L^1}^2. \tag{14}$$

The remaining term $\big\||f|^{s/2}\big\|_{L^1}^2$ is dealt separately in the cases $s \le 2$ and $s > 2$. Indeed, if $s \le 2$, then

$$\Big\||f|^{s/2}\Big\|_{L^1}^2 = \left(\int_\Omega |f|^{s/2}\right)^2 \le \|f\|_{L^1}^s|\Omega|^{(2-s)} \le C(K).$$

If $s > 2$, by interpolation and Young inequalities, for $a = \frac{s-2}{s-1} \in (0,1)$ we have

$$\Big\||f|^{s/2}\Big\|_{L^1}^2 = \|f\|_{L^{s/2}}^s \le \|f\|_{L^s}^{sa}\|f\|_{L^1}^{s(1-a)} \le \frac{1}{2C_1}\|f\|_{L^s}^s + C(K) = \frac{1}{2C_1}\Big\||f|^{s/2}\Big\|_{L^2}^2 + C(K).$$

Therefore, in both cases, from (14), we arrive at

$$\Big\||f|^{s/2}\Big\|_{L^2}^2 \le C\Big\|\nabla|f|^{s/2}\Big\|_{L^2}^2 + C(K),$$

hence using (13) the proof is finished. □

We also make use of a comparison inequality between the truncation operators.

Lemma 3.7. *Let T_0^m and G_0^m be the truncations defined in (4) and (5). Then for every $s > 1$ and $q \ge 1$, it holds that*

$$n^{s-q}\,T_0^m(n)^q \le s\,G_0^m(n), \quad \text{for all } n \in \mathbb{R}.$$

Proof. If $0 \le n \le m$, the equality becomes $n^s = s\,G_0^m(n)$ which is true looking at (4) and (5), and if $n < 0$, the equality reduces to $0 = 0$. Let $n > m$ and define the auxiliary function,

$$f(n) := \frac{1}{s}n^{s-q}\,T_0^m(n)^q - G_0^m(n) = \frac{1}{s}n^{s-q}m^q - \frac{m}{s-1}n^{s-1} + \frac{m^s}{s(s-1)}.$$

It suffices to prove that $f(n) \leq 0$ for all $n > m$. Note that

$$\lim_{n\to m^+} f(n) = \frac{1}{s}m^s - \frac{m^s}{s-1} + \frac{m^s}{s(s-1)} = 0.$$

On the other hand, for any $n > m$

$$\begin{aligned} f'(n) &= \frac{s-q}{s} n^{s-q-1} m^q - m n^{s-2} = \frac{s-q}{s} n^{s-q-1} m^q - n^{s-q-1} n^{q-1} m \\ &\leq \frac{s-q}{s} n^{s-q-1} m^q - n^{s-q-1} m^q = n^{s-q-1} m^q \left(\frac{s-q}{s} - 1 \right) \leq 0, \end{aligned}$$

because as $q \geq 1$ it holds $(s-q)/s \leq 1$. In summary, $f(n)$ is a non-increasing negative function for all $n > m$. □

4. Existence of solutions for the time discrete scheme

Proposition 4.1. *Let Ω be as in Theorem 2.1. Let $(n^{i-1}, z^{i-1}, u^{i-1}) \in (L^2 \cap L^s) \times L^\infty \times (H \cap L^5)$, for $s > 1$, with $n^{i-1} \geq 0$, $z^{i-1} \geq \alpha$ where $z^{i-1} = \sqrt{c^{i-1} + \alpha^2}$. Then for any $\alpha < \min\{\frac{2}{s}, 1\}$, there exists $(n^i, z^i, u^i, P^i) \in H^2 \times H^2 \times (H^2 \cap V) \times H^1$ solving the time discrete scheme (8).*

Proof. The existence of solutions is proven by a fixed-point argument. Given $(\overline{n}, \overline{z}, \overline{u}) \in (W^{1,4})^2 \times V$, we consider the (decoupled) auxiliary problem to define (n, z, u, P) solving

$$\begin{cases} \dfrac{u}{k} + (u^{i-1} \cdot \nabla) u - \Delta u + \nabla P = T_0^m(\overline{n}) \nabla \Phi + \dfrac{u^{i-1}}{k}, \quad \nabla \cdot u = 0, \\ \dfrac{z}{k} - \Delta z + \dfrac{s}{2} G_0^m(\overline{n}) z = \alpha^2 \dfrac{s}{2} \dfrac{G_0^m(\overline{n})}{T_\alpha(\overline{z})} + \dfrac{|\nabla \overline{z}|^2}{T_\alpha(\overline{z})} + \dfrac{z^{i-1}}{k} - \overline{u} \cdot \nabla \overline{z}, \\ \dfrac{n}{k} + \overline{u} \cdot \nabla n - \Delta n = -2 \nabla \cdot (T_0^m(\overline{n}) T_\alpha(\overline{z}) \nabla z) + \dfrac{n^{i-1}}{k}, \\ \partial_\eta n|_\Gamma = \partial_\eta z|_\Gamma = u|_\Gamma = 0. \end{cases}$$

We first ensure, through the Lax–Milgram theorem, the existence of u and z, and then the existence of n (depending on z), in such a way that a compact map

$$S : (\overline{n}, \overline{z}, \overline{u}) \to (n, z, u)$$

is defined from $(W^{1,4})^2 \times V$ to itself. Finally, by Leray-Schauder fixed-point theorem [14], we will obtain the desired solution of the discrete scheme (8).

Step 1. Operator S is well-defined.

Given $\overline{n} \in W^{1,4}$, the existence and uniqueness of the pair (u, P) follow a standard argument, by first applying Lax–Milgram theorem to get only $u \in V$, and after *via* De Rham's Lemma and the regularity of the Stokes problem, in a bootstrapping argument, getting that there exists a unique $(u, P) \in H^2 \times (H^1 \cap L^2_0)$, where $L^2_0 = \{f \in L^2 : \int_\Omega f = 0\}$.

The existence and uniqueness of $z \in H^1$ follow from the Lax–Milgram theorem, obtaining that in fact $z \in H^2$ *via* regularity of the Poisson-Neumann problem (2). From that, looking at n equation and applying Lax–Milgram theorem, we again obtain, through a bootstrapping argument, necessary due to the presence of the convective term $\bar{u} \cdot \nabla n$, that there exists a unique $n \in H^2$.

The bounds obtained throughout the bootstrapping argument are sufficient to conclude that S is a well-defined compact mapping, defined by the composition of a sequentially continuous map from $(W^{1,4})^2 \times V$ into $(H^2)^2 \times (H^2 \cap V)$ and the compact embedding from $(H^2)^2 \times (H^2 \cap V)$ into $(W^{1,4})^2 \times V$.

Step 2. A priori estimates of possible fixed-points.

To apply the fixed point theorem, we now need λ-independent estimates for all possible triplets (n, z, u) in $(W^{1,4})^2 \times V$ satisfying $(n, z, u) = \lambda\, S(n, z, u)$ for some $\lambda \in [0, 1]$.

The case $\lambda = 0$ is trivially satisfied because $(n, z, u) \equiv (0, 0, 0)$. Let $\lambda \in (0, 1]$, therefore if $(n, z, u) = \lambda\, S(n, z, u)$, then (n, z, u) satisfies the equations

$$\frac{u}{k} + u^{i-1} \cdot \nabla u - \Delta u + \nabla P = \lambda\, T_0^m(n) \nabla \Phi + \lambda \frac{u^{i-1}}{k}, \tag{1_λ}$$

$$\frac{n}{k} + u \cdot \nabla n - \Delta n = -\nabla \cdot (T_0^m(n) 2 T_\alpha(z) \nabla z) + \lambda \frac{n^{i-1}}{k}, \tag{2_λ}$$

$$\frac{z}{k\lambda} + u \cdot \nabla z - \Delta \frac{z}{\lambda} + \frac{s}{2\lambda} G_0^m(n) z = \alpha^2 \frac{s}{2} \frac{G_0^m(n)}{T_\alpha(z)} + \frac{|\nabla z|^2}{T_\alpha(z)} + \frac{z^{i-1}}{k}\cdot \tag{3_λ}$$

Firstly, as $u \in V$, we can test (1_λ) by u, obtaining

$$\begin{aligned}\frac{1}{k} \int |u|^2 + \int |\nabla u|^2 &= \lambda \int T_0^m(n) \nabla \Phi \cdot u + \int \lambda \frac{u^{i-1}}{k} \cdot u \\ &\leq \delta \int |u|^2 + C_\delta \left(\frac{1}{k^2} \left\| u^{i-1} \right\|_{L^2}^2 + m^2 \|\nabla \Phi\|_{L^\infty}^2 \right).\end{aligned}$$

By taking $\delta < 1/k$ there follows that u is bounded in V (independently of λ).

We now deal separately with n and z. The proof is analogous to that presented in Theorem 3.1 of [15], and here we only point out the main differences.

The first step is to show that n is a nonnegative function. For that, n equation (2_λ) is tested by the negative part of n, defined as $n_- \coloneqq \min\{0, n\}$. Noting that for the convective term

$$\int (u \cdot \nabla n) n_- = \frac{1}{2} \int u \cdot \nabla (n_-^2) = 0,$$

then it can be shown that $n_- = 0$ a.e. in Ω by the proof in Theorem 3.1 from [15], and therefore $n \geq 0$ a.e. in Ω.

Now, we will show the following pointwise bounds, not only on z but also on z/λ:

$$\alpha \leq \frac{z}{\lambda} \leq \left\| z^{i-1} \right\|_{L^\infty} \quad \text{a.e. in } \Omega. \tag{15}$$

To prove (15), the calculations are similar to the ones presented in Theorem 3.1 from [15]. The only difference is given by the convective term $u \cdot \nabla z$. However, rewriting (3_λ) in terms of $\tilde{z} \coloneqq z/\lambda - \left\| z^{i-1} \right\|_{L^\infty}$ and testing by $\tilde{z}_+$ where $\tilde{z}_+ \coloneqq \max\{0, \tilde{z}\}$, the convective term becomes

$$\lambda \int (u \cdot \nabla z) \tilde{z}_+ = \lambda \int (u \cdot \nabla \tilde{z}) \tilde{z}_+ = -\frac{\lambda}{2} \int u \cdot \nabla (\tilde{z}_+^2) = 0.$$

Similarly, when dealing with the lower bound, the equation is rewritten in terms of $\tilde{z} \coloneqq z/\lambda - \alpha$ and testing by $\tilde{z}_-$, the convective term also vanishes.

Note that, with pointwise estimates (15) for z/λ we also gain a pointwise estimate for z in L^∞ (independent of λ).

To estimate z in H^2 we use the energy structure of the z equation. We first multiply (3_λ) by $\lambda T_\alpha(z)/z$, recalling that $z > 0$, and arrive at

$$\frac{T_\alpha(z)}{k} - T_\alpha(z) \frac{\Delta z}{z} - \lambda \frac{|\nabla z|^2}{z} = -\frac{s}{2} G_0^m(n) T_\alpha(z) + \lambda \frac{s\alpha^2}{2} G_0^m(n) \frac{1}{z} + \lambda u \cdot \nabla z \frac{T_\alpha(z)}{z} + \lambda \frac{z^{i-1}}{k} \frac{T_\alpha(z)}{z}\cdot$$

By testing by $-\Delta z$, and integrating by parts, we have

$$-\frac{1}{k}\int T_\alpha(z)\Delta z+\lambda\int\frac{|\nabla z|^2}{z}\Delta z+\int T_\alpha(z)\frac{|\Delta z|^2}{z}$$
$$=-\frac{s}{2}\int T_\alpha(z)\nabla G_0^m(n)\cdot\nabla z-\frac{s}{2}\int G_0^m(n)(T_\alpha(z))'|\nabla z|^2-\lambda\frac{s\alpha^2}{2}\int\frac{G_0^m(n)}{z^2}|\nabla z|^2$$
$$+\lambda\frac{s\alpha^2}{2}\int\frac{1}{z}\nabla G_0^m(n)\cdot\nabla z-\lambda\int u\cdot\nabla z\frac{T_\alpha(z)}{z}\Delta z-\lambda\int\frac{z^{i-1}}{k}\frac{T_\alpha(z)}{z}\Delta z. \tag{16}$$

For the LHS of (16), using the fact that for $T_\alpha(z)/z\ge 1$ and integrating by parts, we obtain

$$-\frac{1}{k}\int T_\alpha(z)\Delta z+\lambda\int\frac{|\nabla z|^2}{z}\Delta z+\int T_\alpha(z)\frac{|\Delta z|^2}{z}\ge\frac{1}{k}\int T'_\alpha(z)|\nabla z|^2+\lambda\int\frac{|\nabla z|^2}{z}\Delta z+\int|\Delta z|^2$$
$$\ge\lambda\int\frac{|\nabla z|^2}{z}\Delta z+\lambda\int|\Delta z|^2+(1-\lambda)\int|\Delta z|^2.$$

Then by Lemma 3.1, and noting that $1+\lambda(C_1-1)\ge\min\{1,C_1\}=C_3$ for all $\lambda\in[0,1]$, we get

$$-\frac{1}{k}\int T_\alpha(z)\Delta z+\lambda\int\frac{|\nabla z|^2}{z}\Delta z+\int T_\alpha(z)\frac{|\Delta z|^2}{z}\ge(\lambda C_1+1-\lambda)\int|\Delta z|^2-C_2\lambda\int|\nabla z|^2+C_1\lambda\int\frac{|\nabla z|^4}{z^2}$$
$$\ge C_3\int|\Delta z|^2-C_2\lambda\int|\nabla z|^2+C_1\lambda\int\frac{|\nabla z|^4}{z^2}.$$

Now, we deal with the RHS of (16). Note that using the already obtained pointwise estimates (15) gives us

$$\lambda\frac{T_\alpha(z)}{z}=\frac{T_\alpha(z)}{z/\lambda}\le\frac{T_\alpha(z)}{\alpha}\le\frac{\|z^{i-1}\|_{L^\infty}}{\alpha}. \tag{17}$$

Therefore, by first applying Hölder inequality and using (17) yields

$$\lambda\int u\cdot\nabla z\frac{T_\alpha(z)}{z}\Delta z\le C\|u\|_{H^1}\|\nabla z\|_{L^4}\|\Delta z\|_{L^2}.$$

Now by applying Young's inequality twice, multiplying and dividing the gradient term by z^2/λ^2, and using (15), we have

$$\|u\|_{H^1}\|\nabla z\|_{L^4}\|\Delta z\|_{L^2}\le C\big(\epsilon,\|z^{i-1}\|_{L^\infty}\big)\|u\|_{H^1}^4+\lambda^2\|z^{i-1}\|_{L^\infty}^2\epsilon\int\frac{|\nabla z|^4}{z^2}+\frac{C_3}{4}\|\Delta z\|_{L^2}^2,$$

hence

$$\lambda\int u\cdot\nabla z\frac{T_\alpha(z)}{z}\Delta z\le C\big(\epsilon,\|z^{i-1}\|_{L^\infty},\|u\|_{H^1}\big)+\lambda\epsilon C_4\int\frac{|\nabla z|^4}{z^2}+\frac{C_3}{4}\|\Delta z\|_{L^2}^2.$$

It also holds true that

$$\lambda\int\frac{z^{i-1}}{k}\frac{T_\alpha(z)}{z}\Delta z\le C\big(\epsilon,\|z^{i-1}\|_{L^\infty},k\big)+\frac{C_3}{4}\|\Delta z\|_{L^2}^2.$$

Moreover, using (6) one has $\nabla G_0^m(n)=T_0^m(n)n^{s-2}\nabla n=(2/s)T_0^m(n)n^{s/2-1}\nabla(n^{s/2})$, hence we have

$$\frac{\lambda s\alpha^2}{2}\int\frac{1}{z}\nabla G_0^m(n)\cdot\nabla z=\lambda\alpha^2\int\frac{n^{s/2-1}}{z}T_0^m(n)\nabla z\cdot\nabla n^{s/2}$$

$$\leq \frac{\lambda\alpha^3}{2}\int \frac{n^{s-2}}{z^2}T_0^m(n)^2|\nabla z|^2 + \frac{\lambda\alpha}{2}\int\left|\nabla n^{s/2}\right|^2.$$

By Lemma 3.7, one has $n^{s-2}T_0^m(n)^2 \leq s\, G_0^m(n)$, so

$$\frac{\lambda s\alpha^2}{2}\int \frac{1}{z}\nabla G_0^m(n)\cdot\nabla z \leq \frac{\lambda s\alpha^3}{2}\int\frac{G_0^m(n)}{z^2}|\nabla z|^2 + \frac{\lambda\alpha}{2}\int\left|\nabla n^{s/2}\right|^2.$$

Then by setting $\alpha < 1$ the first term will be absorbed in (16). We also note that $\nabla n^{s/2}$ might be singular for $s < 2$. For an adaptation of this argument on the case $s < 2$, the use of a translation on n is necessary. This will be made more accurate in Lemma 5.2 below.

Altogether, we arrive at the inequality

$$\begin{aligned}\frac{C_3}{2}\int|\Delta z|^2 - C_2\lambda\int|\nabla z|^2 + (C_1-\epsilon C_4)\lambda\int\frac{|\nabla z|^4}{z^2} + \frac{s}{2}\int G_0^m(n)(T_\alpha(z))'|\nabla z|^2\\ \leq \frac{\lambda\alpha}{2}\int\left|\nabla n^{s/2}\right|^2 - \frac{s}{2}\int T_\alpha(z)\nabla G_0^m(n)\cdot\nabla z + C\big(\epsilon, \|u\|_{H^1}, \left\|z^{i-1}\right\|_{L^\infty}\big).\end{aligned} \tag{18}$$

On the other hand, we multiply (2_λ) by n^{s-1} and use (6) to obtain

$$\begin{aligned}\int\frac{n^s}{k} + \frac{4(s-1)}{s^2}\int\left|\nabla n^{s/2}\right|^2 &= (s-1)\int T_0^m(n)n^{s-2}2T_\alpha(z)\nabla z\cdot\nabla n + \frac{\lambda}{k}\int n^{i-1}n^{s-1}\\ &\leq (s-1)\int 2T_\alpha(z)\nabla z\cdot\nabla G_0^m(n) + C\big(k, \left\|n^{i-1}\right\|_{L^s}\big) + \frac{1}{2k}\int n^s,\end{aligned}$$

and then

$$\frac{s}{8(s-1)}\int\frac{n^s}{k} + \frac{1}{2s}\int\left|\nabla n^{s/2}\right|^2 \leq \frac{s}{2}\int T_\alpha(z)\nabla z\cdot\nabla G_0^m(n) + C\big(k, \left\|n^{i-1}\right\|_{L^s}\big). \tag{19}$$

Summing up (18) and (19), and by taking ϵ small enough, we get

$$\begin{aligned}\frac{s}{8(s-1)}\int\frac{n^s}{k} + \frac{1}{s}\int\left|\nabla n^{s/2}\right|^2 + \frac{C_3}{2}\int|\Delta z|^2 + C_5\lambda\int\frac{|\nabla z|^4}{z^2} + \frac{s}{2}\int G_0^m(n)(T_\alpha(z))'|\nabla z|^2\\ \leq \frac{\alpha}{2}\int\left|\nabla n^{s/2}\right|^2 + C_2\lambda\int|\nabla z|^2 + C.\end{aligned}$$

Thus, if we set $\alpha < 2/s$, we may absorb the first term of the RHS. As for the second one, just note that

$$C_2\lambda\int|\nabla z|^2 \leq \frac{C_5}{2}\lambda\int\frac{|\nabla z|^4}{z^2} + C\big(\left\|z^{i-1}\right\|_{L^\infty}\big).$$

Therefore Δz is bounded in L^2 independently of λ, hence z is bounded in H^2 independently of λ through the Poisson-Neumann regularity.

Now to check that n is bounded in H^2 (independently of λ), we start by testing (2_λ) by n, getting that n is bounded independently of λ in H^1. Indeed,

$$\begin{aligned}\frac{1}{k}\|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 &= \int T_0^m(n)\nabla(z^2)\cdot\nabla n + \frac{\lambda}{k}\int n^{i-1}n\\ &\leq \epsilon\left(\frac{1}{k}\|n\|_{L^2}^2 + \|\nabla n\|_{L^2}^2\right) + C\big(\epsilon, m, k, \|z\|_{H^1}, \left\|z^{i-1}\right\|_{L^\infty}, \left\|n^{i-1}\right\|_{L^2}\big),\end{aligned}$$

so taking $\epsilon < 1$ ensures $n \in H^1$.

On the other hand, since $z \in H^2$ we can use the identity

$$\nabla \cdot (T_0^m(n)T_\alpha(z)\nabla z) = \nabla n \cdot \nabla z (T_0^m)'(n)T_\alpha(z) + T_0^m(n)(T_\alpha(z))'|\nabla z|^2 + T_0^m(n)T_\alpha(z)\Delta z.$$

Then $\nabla \cdot (T_0^m(n)T_\alpha(z)\nabla z)$ is bounded in L^2 and we may then test (2_λ) by $-\Delta n$. We observe that by interpolation

$$\begin{aligned}\int u\cdot\nabla n\,\Delta n &= -\int(\nabla u\,\nabla n)\cdot\nabla n \le \int|\nabla n|^2|\nabla u| \\ &\le C(\epsilon, \|u\|_{H^1}, \|n\|_{H^1}) + \epsilon\|\Delta n\|^2_{L^2},\end{aligned}$$

and therefore

$$\begin{aligned}\frac{1}{k}\int|\nabla n|^2 + \int|\Delta n|^2 &= -2\int\nabla\cdot(T_0^m(n)T_\alpha(z)\nabla z)\Delta n - \frac{\lambda}{k}\int n^{i-1}\Delta n + \int u\cdot\nabla n\Delta n \\ &\le C_1\big(\epsilon, m, k, \|z\|_{H^2}, \|z^{i-1}\|_{L^\infty}, \|n^{i-1}\|_{L^2}, \|u\|_{H^1}, \|n\|_{H^1}\big) + 3\epsilon\int|\Delta n|^2.\end{aligned}$$

By taking ϵ small enough it follows that n is bounded in H^2 (and in particular in $W^{1,4}$) independently of λ.

Step 3. Conclusion.

In summary, the fixed-point gives a triplet $(u^i, n^i, z^i) \in V \times (H^2)^2$, with an associated pressure $P^i \in H^1$, solution of the discrete scheme (8). □

5. First uniform (m, k)-estimates and energy inequalities

By using the pointwise estimates made in previous section, estimate (15) (for $\lambda = 1$) readily implies that

$$n^i \ge 0, \quad \alpha \le z^i \le \|z^{i-1}\|_{L^\infty} \quad \text{a.e. in } \Omega. \tag{20}$$

This guarantees the lower bound for n^i and z^i and also provides a L^∞ bound for z^i. Moreover, from now on, the lower truncations given by (3)–(5) in the time discrete scheme (8) may be simplified by making the following changes:

$$T_\alpha(z^i) = z^i, \quad T_0^m(n^i) = T^m(n^i) \quad \text{and} \quad G_0^m(n^i) = G^m(n^i),$$

where

$$T^m(n) := \begin{cases} n & \text{if } n \le m, \\ m & \text{if } n \ge m, \end{cases} \qquad G^m(n) := \begin{cases} \frac{1}{s}n^s & \text{if } n \le m, \\ \frac{m}{s-1}n^{s-1} - \frac{m^s}{s(s-1)} & \text{if } n \ge m. \end{cases}$$

Specifically, time scheme (8) reduces to find $(n^i, z^i, u^i, P^i) \in (H^2)^2 \times (H^2 \times V) \times H^1$ satisfying pointwise bounds (20) and problem:

$$\begin{cases} \delta_t n^i + u^i\cdot\nabla n^i - \Delta n^i + \nabla\cdot(T^m(n^i)2\,z^i\nabla z^i) = 0, \\ \delta_t z^i + u^i\cdot\nabla z^i - \Delta z^i - \dfrac{|\nabla z^i|^2}{z^i} = -\dfrac{s}{2}G^m(n^i)\Big(z^i - \dfrac{\alpha^2}{z^i}\Big), \\ \delta_t u^i + u^{i-1}\cdot\nabla u^i - \Delta u^i + \nabla P^i = T^m(n^i)\nabla\Phi, \\ \nabla\cdot u^i = 0, \\ \partial_\eta n^i\big|_\Gamma = \partial_\eta z^i\big|_\Gamma = u^i\big|_\Gamma = 0. \end{cases} \tag{21}$$

The next step is to obtain (m, k)-independent estimates. The first result is obtained directly through testing the equations of the discrete scheme (21), and reads as follows.

Lemma 5.1. *Let (n^i, z^i, u^i) be any solution of (21). Then the following estimates hold*

(i) $\int n^i \,\mathrm{d}x = \int n^0 \,\mathrm{d}x$, *for all* $i \in \mathbb{N}$;
(ii) $\|z^i\|_{L^2}^2 + \sum_{j=1}^i \|z^j - z^{j-1}\|_{L^2}^2 \le \|z^0\|_{L^2}^2$ *for all* $i \in \mathbb{N}$;
(iii) $k\sum_{j=1}^i \|\nabla z^j\|_{L^2}^2 \le \frac{1}{4\alpha^2}\|c^0 + \alpha^2\|_{L^2}^2$ *for all* $i \in \mathbb{N}$.

Proof. All items will follow exactly from Lemma 3.2 of [15]. The first item is obtained by integrating n^i equation, in which the convective term vanishes. The second and third items are obtained by testing z^i equation, by z^i and $k(z^i)^3$ respectively, and, in both cases, the convective term also vanishes. In the latter case, it holds the intermediate inequality $k\sum_{j=1}^i \|\nabla (z^j)^2\|_{L^2}^2 \le \|(z^0)^2\|_{L^2}^2$, hence one arrives at the third item using that $z^i \ge \alpha$. □

We now make use of the structure of the equations, and the cancellation effects between chemotactic attraction and consumption terms, which is formally achieved by testing n^i equation by $(n^i)^{s-1}$ and z^i equation by $-\Delta z^i$ and balancing out the results. On the other hand, we use Lemma 3.6 to bound the source term of the u^i system.

When the diffusion term $-\Delta n^i$ is tested by $(n^i)^{s-1}$, a term of the form $\nabla (n^i)^{s/2}$ appears, which may be singular in the case $s < 2$. For this reason, the energy inequality is obtained separately for $s \in (1,2)$ and $s \ge 2$, avoiding this singular term $\nabla (n^i)^{s/2}$ in the case $s \in (1,2)$.

Lemma 5.2 (Energy inequality for $s \in (1,2)$)**.** *Let* (n^i, z^i, u^i) *be a solution of* (21) *with* $s \in (1,2)$. *Then, if* α *is small enough, there exist constants* $C_1, C_2, C_3, C_4 > 0$ *such that,*

$$\frac{1}{2}\delta_t\left[\frac{1}{2(s-1)}\|n^i\|_{L^s}^s + \|\nabla z^i\|_{L^2}^2 + C_1\|u^i\|_{L^2}^2\right] + \frac{1}{2k}\left[\|\nabla z^i - \nabla z^{i-1}\|_{L^2}^2 + C_1\|u^i - u^{i-1}\|_{L^2}^2\right]$$
$$+ C_2\|\nabla u^i\|_{L^2}^2 + C_3\left(\|D^2 z^i\|_{L^2}^2 + \int \frac{|\nabla z^i|^4}{(z^i)^2}\right) + \frac{s}{4}\int G^m(n^i)|\nabla z^i|^2 \le C_4.$$

Proof. We start by testing z^i equation by $-\Delta z^i$, obtaining

$$\int \frac{z^{i-1} - z^i}{k}\Delta z^i - \int u^i \cdot \nabla z^i \Delta z^i + \|\Delta z^i\|_{L^2}^2 + \int \frac{|\nabla z^i|^2}{z^i}\Delta z^i$$
$$= \frac{s}{2}\int \left(z^i - \frac{\alpha^2}{z^i}\right) G^m(n^i)\Delta z^i$$
$$= -\frac{s}{2}\int \left(z^i - \frac{\alpha^2}{z^i}\right)\nabla G^m(n^i)\cdot \nabla z^i - \frac{s}{2}\int \left(1 + \frac{\alpha^2}{(z^i)^2}\right) G^m(n^i)|\nabla z^i|^2.$$

Then, making use of Lemma 3.1 to bound the diffusive terms, bounding the convective term as

$$\int u^i \cdot \nabla z^i \Delta z^i = -\int (\nabla u^i \,\nabla z^i)\cdot \nabla z^i \le \int |\nabla z^i|^2 |\nabla u^i|$$
$$\le \epsilon \int \frac{|\nabla z^i|^4}{(z^i)^2} + C_\epsilon \int |\nabla u^i|^2,$$

one has, taking ϵ small enough and using (6),

$$\frac{1}{2}\delta_t\|\nabla z^i\|_{L^2}^2 + \frac{1}{2k}\|\nabla z^i - \nabla z^{i-1}\|_{L^2}^2 + \frac{C_1}{2}\left(\|D^2 z^i\|_{L^2}^2 + \int \frac{|\nabla z^i|^4}{(z^i)^2}\right) + \frac{s}{2}\int G^m(n^i)|\nabla z^i|^2$$
$$\le -\frac{s}{2}\int z^i\left(1 - \frac{\alpha^2}{(z^i)^2}\right) T^m(n^i)(n^i)^{s-2}\nabla n^i \cdot \nabla z^i + C_2\int |\nabla z^i|^2 + C_3\int |\nabla u^i|^2. \tag{22}$$

Due to the presence of the term $\int |\nabla u^i|^2$ in the RHS of (22), we look at the usual u^i energy equation, testing the u^i system by u^i (accounting that $\int (u^{i-1}\cdot\nabla)u^i\cdot u^i = 0$),

$$\frac{1}{2}\delta_t\|u^i\|_{L^2}^2 + \frac{1}{2k}\|u^i-u^{i-1}\|_{L^2}^2 + \|\nabla u^i\|_{L^2}^2 = \int T^m(n^i)\,u^i\cdot\nabla\Phi \le C_4\int n^i|u^i|. \tag{23}$$

Finally, we can then add up $2C_3$ times (23) and (22), and get

$$\begin{aligned}\frac{1}{2}\delta_t\Big[\|\nabla z^i\|_{L^2}^2 + 2C_3\|u^i\|_{L^2}^2\Big] &+ \frac{1}{2k}\Big[\|\nabla z^i-\nabla z^{i-1}\|_{L^2}^2 + 2C_3\|u^i-u^{i-1}\|_{L^2}^2\Big]\\ &+ C_3\|\nabla u^i\|_{L^2}^2 + \frac{C_1}{2}\left(\|D^2z^i\|_{L^2}^2 + \int\frac{|\nabla z^i|^4}{(z^i)^2}\right) + \frac{s}{2}\int G^m(n^i)|\nabla z^i|^2\\ &\le -\frac{s}{2}\int z^i\left(1-\frac{\alpha^2}{(z^i)^2}\right)T^m(n^i)(n^i)^{s-2}\nabla n^i\cdot\nabla z^i + C_2\int|\nabla z^i|^2 + C_5\int n^i|u^i|.\end{aligned}\tag{24}$$

Now, when testing n^i equation by $(n^i)^{s-1}$ the Laplacian term becomes $\nabla(n^i)^{s/2}$ which is singular for $s<2$, and for that reason we translate the test function by $1/j$ and then pass the limit as $j\to+\infty$. Thus, testing n^i equation by $(n^i+1/j)^{s-1}/(s-1)$, and using Lemma 3.2 there follows

$$\begin{aligned}\frac{1}{s(s-1)}\delta_t\int\left(n^i+1/j\right)^s + \frac{4}{s^2}\left\|\nabla\left(n^i+1/j\right)^{s/2}\right\|_{L^2}^2 + \frac{1}{2k}\int\left(b^i+1/j\right)^{s-2}\left(n^i-n^{i-1}\right)^2\\ = \int T^m\left(n^i\right)\left(n^i+1/j\right)^{s-2}\nabla\left(z^i\right)^2\cdot\nabla n^i.\end{aligned}$$

Since $(b^i+1/j)^{s-2}\ge 0$, we drop this term and arrive at

$$\frac{1}{s(s-1)}\delta_t\int\left(n^i+1/j\right)^s + \frac{4}{s^2}\left\|\nabla\left(n^i+1/j\right)^{s/2}\right\|_{L^2}^2 \le \int T^m\left(n^i\right)\left(n^i+1/j\right)^{s-2}\nabla(z^i)^2\cdot\nabla n^i. \tag{25}$$

So adding $s/4$ times (25) to (24) we obtain

$$\begin{aligned}\frac{1}{2}\delta_t\Big[\frac{1}{2(s-1)}\left\|\left(n^i+1/j\right)\right\|_{L^s}^s + \|\nabla z^i\|_{L^2}^2 + 2C_3\|u^i\|_{L^2}^2\Big] + \frac{1}{2k}\Big[\|\nabla z^i-\nabla z^{i-1}\|_{L^2}^2 + 2C_3\|u^i-u^{i-1}\|_{L^2}^2\Big]\\ + \frac{1}{s}\left\|\nabla\left(n^i+1/j\right)^{s/2}\right\|_{L^2}^2 + C_3\|\nabla u^i\|_{L^2}^2 + \frac{C_1}{2}\left(\|D^2z^i\|_{L^2}^2 + \int\frac{|\nabla z^i|^4}{(z^i)^2}\right) + \frac{s}{2}\int G^m(n^i)|\nabla z^i|^2\\ \le \frac{s}{2}\int\frac{\alpha^2}{z^i}T^m(n^i)(n^i)^{s-2}\nabla n^i\cdot\nabla z^i + C_2\int|\nabla z^i|^2 + C_5\int n^i|u^i|\\ - \frac{s}{4}\int T^m(n^i)\nabla(z^i)^2\cdot\nabla n^i\left((n^i)^{s-2} - \left(n^i+1/j\right)^{s-2}\right).\end{aligned}$$

For the first term on the RHS we note that

$$\nabla n^i = \nabla\left(n^i+1/j\right) = \nabla\left(\left(n^i+1/j\right)^{s/2}\right)^{2/s} = \frac{2}{s}\left(n^i+1/j\right)^{1-s/2}\nabla\left(n^i+1/j\right)^{s/2},$$

hence by Young's inequality, the first term remains

$$\begin{aligned}&\int\frac{\alpha^2}{z^i}T^m\left(n^i\right)\left(n^i\right)^{s-2}\left(n^i+1/j\right)^{1-s/2}\nabla\left(n^i+1/j\right)^{s/2}\cdot\nabla z^i\\ &\quad\le C(\epsilon)\alpha^2\int\left(n^i\right)^{2s-4}T^m\left(n^i\right)^2\left(n^i+1/j\right)^{2-s}|\nabla z^i|^2 + \epsilon\int\left|\nabla\left(n^i+1/j\right)^{s/2}\right|^2\\ &\quad\le C(\epsilon)\alpha^2 s\int G^m(n^i)|\nabla z^i|^2(n^i)^{s-2}\left(n^i+1/j\right)^{2-s} + \epsilon\int\left|\nabla\left(n^i+1/j\right)^{s/2}\right|^2.\end{aligned}$$

In the last line, we have used Lemma 3.7. First, we choose ϵ small enough to absorb the second term and then set a small α accordingly, depending on ϵ. We also note that even with the presence of the singular term $(n^i)^{s-2}$ the whole term is still not singular because close to zero $G^m(n^i)$ is of the order of $(n^i)^s$ making the product $(n^i)^{s-2}G^m(n^i)$ well defined. Overall, for α small enough we get

$$\frac{s}{2}\int\frac{\alpha^2}{z^i}T^m\left(n^i\right)\left(n^i\right)^{s-2}\nabla n^i\cdot\nabla z^i\le\frac{s}{4}\int G^m\left(n^i\right)\left|\nabla z^i\right|^2\left(n^i\right)^{s-2}\left(n^i+1/j\right)^{2-s}+\frac{1}{4s}\int\left|\nabla\left(n^i+1/j\right)^{s/2}\right|^2. \quad (26)$$

For the source term of the u^i system, applying Lemma 3.6 with $f=n^i$ and $g=u^i$ and choosing $\epsilon_1=\frac{C_3}{2C_5}$ and $\epsilon_2=\frac{1}{4sC_5}$ we have that

$$C_5\int_\Omega n^i|u^i|\le C_5\int_\Omega(n^i+1/j)|u^i|\le\frac{C_3}{2}\|\nabla u^i\|_{L^2}^2+\frac{1}{4s}\left\|\nabla(n^i+1/j)^{s/2}\right\|_{L^2}^2+C_6.$$

We can also estimate

$$C_2\int\left|\nabla z^i\right|^2=C_2\int\frac{\left|\nabla z^i\right|^2}{z^i}z^i\le\frac{C_1}{4}\int\frac{\left|\nabla z^i\right|^4}{(z^i)^2}+C. \quad (27)$$

Dropping the gradient term of $(n^i+1/j)^{s/2}$, avoiding any division by zero, we get that

$$\begin{aligned}&\frac{1}{2}\delta_t\left[\frac{1}{2(s-1)}\|n^i+1/j\|_{L^s}^s+\|\nabla z^i\|_{L^2}^2+2C_3\|u^i\|_{L^2}^2\right]+\frac{1}{2k}\left[\|\nabla z^i-\nabla z^{i-1}\|_{L^2}^2+2C_3\|u^i-u^{i-1}\|_{L^2}^2\right]\\&\quad+\frac{C_3}{2}\|\nabla u^i\|_{L^2}^2+\frac{C_1}{4}\left(\|D^2z^i\|_{L^2}^2+\int\frac{|\nabla z^i|^4}{(z^i)^2}\right)+\frac{s}{2}\int G^m(n^i)|\nabla z^i|^2\left(1-\frac{(n^i)^{s-2}(n^i+1/j)^{2-s}}{2}\right)\\&\quad\le C_6-\frac{s}{4}\int T^m(n^i)\nabla(z^i)^2\cdot\nabla n^i\left((n^i)^{s-2}-(n^i+1/j)^{s-2}\right).\end{aligned}$$

Finally, we can pass to the limit as $j\to\infty$, *via* the Dominated Convergence Theorem, to finish the proof, remarking only that

$$\begin{aligned}G^m(n^i)|\nabla z^i|^2(n^i)^{s-2}(n^i+1/j)^{2-s}&\le\frac{1}{s}|\nabla z^i|^2(n^i)^s(n^i)^{s-2}(n^i+1/j)^{2-s}\\&\le\frac{1}{s}|\nabla z^i|^2(n^i+1)^s,\end{aligned}$$

which is a L^1 function, and then

$$\frac{s}{2}\int G^m(n^i)|\nabla z^i|^2\left(1-\frac{(n^i)^{s-2}(n^i+1/j)^{2-s}}{2}\right)\to\frac{s}{4}\int G^m(n^i)|\nabla z^i|^2.$$

The remaining terms can be treated similarly. □

Lemma 5.3 (Energy inequality for $s\ge2$)**.** *Let (n^i,z^i,u^i) be a solution of (21) with $s\ge2$. Then, if α is small enough, there exist constants $C_1,C_2,C_3,C_4>0$ such that*

$$\begin{aligned}&\frac{1}{2}\delta_t\left[\frac{1}{2(s-1)}\|n^i\|_{L^s}^s+\|\nabla z^i\|_{L^2}^2+C_1\|u^i\|_{L^2}^2\right]+\frac{1}{2k}\left[\|\nabla z^i-\nabla z^{i-1}\|_{L^2}^2+C_1\|u^i-u^{i-1}\|_{L^2}^2\right]\\&\quad+\frac{1}{4s}\left\|\nabla(n^i)^{s/2}\right\|_{L^2}^2+C_2\|\nabla u^i\|_{L^2}^2+C_3\left(\|D^2z^i\|_{L^2}^2+\int\frac{|\nabla z^i|^4}{(z^i)^2}\right)+\frac{s}{4}\int G^m(n^i)|\nabla z^i|^2\le C_4.\end{aligned}$$

Proof. Up until (24) there is no distinction whether $s \in (1,2)$ or $s \geq 2$. Now as there is no singularity, testing n^i equation by $(n^i)^{s-1}/(s-1)$ and using Lemma 3.2 yields

$$\frac{1}{s(s-1)}\delta_t \int (n^i)^s + \frac{4}{s^2}\left\|\nabla (n^i)^{s/2}\right\|_{L^2}^2 \leq \int T^m(n^i)(n^i)^{s-2}\nabla(z^i)^2 \cdot \nabla n^i. \tag{28}$$

Adding $s/4$ times (28) to (24), we obtain

$$\begin{aligned}
&\frac{1}{2}\delta_t\left[\frac{1}{2(s-1)}\left\|n^i\right\|_{L^s}^s + \left\|\nabla z^i\right\|_{L^2}^2 + 2C_3\left\|u^i\right\|_{L^2}^2\right] + \frac{1}{2k}\left[\left\|\nabla z^i - \nabla z^{i-1}\right\|_{L^2}^2 + 2C_3\left\|u^i - u^{i-1}\right\|_{L^2}^2\right]\\
&\quad + \frac{1}{s}\left\|\nabla (n^i)^{s/2}\right\|_{L^2}^2 + C_3\left\|\nabla u^i\right\|_{L^2}^2 + \frac{C_1}{2}\left(\left\|D^2 z^i\right\|_{L^2}^2 + \int \frac{\left|\nabla z^i\right|^4}{(z^i)^2}\right) + \frac{s}{2}\int G^m(n^i)\left|\nabla z^i\right|^2\\
&\quad \leq \frac{s}{2}\int \frac{\alpha^2}{z^i} T^m(n^i)(n^i)^{s-2}\nabla n^i \cdot \nabla z^i + C_2\int \left|\nabla z^i\right|^2 + C_5\int n^i\left|u^i\right|.
\end{aligned}$$

Following the same calculations to get (26) changing $(n^i + 1/j)$ by n^i, we have, for α small enough, that

$$\frac{s}{2}\int \frac{\alpha^2}{z^i}T^m(n^i)(n^i)^{s-2}\nabla n^i \cdot \nabla z^i \leq \frac{s}{4}\int G^m(n^i)\left|\nabla z^i\right|^2 + \frac{1}{4s}\int\left|\nabla (n^i)^{s/2}\right|^2.$$

For the source term of u^i system, applying Lemma 3.6 with $f = n^i$ and $g = u^i$ and choosing $\epsilon_1 = \frac{C_3}{2C_5}$ and $\epsilon_2 = \frac{1}{4sC_5}$ we have that

$$C_5\int_\Omega n^i\left|u^i\right| \leq \frac{C_3}{2}\left\|\nabla u^i\right\|_{L^2}^2 + \frac{1}{4s}\left\|\nabla (n^i)^{s/2}\right\|_{L^2}^2 + C_6.$$

By plugging (27), we obtain the result. ☐

6. Energy (m,k)-estimates and passage to the limit as $(m,k) \to (\infty, 0)$

With the (m,k)-estimates which will provide Lemmas 5.1–5.3 at hand, we are now prepared to prove our main result, Theorem 2.1. We start rewriting the scheme (21) as a continuous in-time system, by considering the piecewise constant function n_m^k and the locally linear and globally continuous function $\tilde{n}_m^k$ defined by

$$n_m^k(t,x) := n^i(x) \quad \text{ if } t \in I_i$$

and

$$\tilde{n}_m^k(t,x) := n^i(x) + \frac{(t-t_i)}{k}\left(n^i(x) - n^{i-1}(x)\right) \quad \text{ if } t \in I_i.$$

Analogously, we define the functions $z_m^k, \tilde{z}_m^k, u_m^k$, $\tilde{u}_m^k$ and P_m^k. We also define

$$\hat{u}_m^k(t,x) := u^{i-1}(x) \quad \text{if } t \in I_i.$$

In particular, one has $\partial_t \tilde{n}_m^k = \delta_t n^i$ for $t \in I_i$. Then, the discrete scheme (21) can be rewritten as

$$\begin{cases}
\partial_t \tilde{n}_m^k + u_m^k \cdot \nabla n_m^k - \Delta n_m^k + \nabla \cdot \left(T^m(n_m^k)\nabla (z_m^k)^2\right) = 0,\\
\partial_t \tilde{z}_m^k + u_m^k \cdot \nabla z_m^k - \Delta z_m^k - \dfrac{\left|\nabla z_m^k\right|^2}{z_m^k} = -\dfrac{s}{2}G^m(n_m^k)\left(z_m^k - \dfrac{\alpha^2}{z_m^k}\right),\\
\partial_t \tilde{u}_m^k + (\hat{u}_m^k \cdot \nabla)u_m^k - \Delta u_m^k + \nabla P_m^k = T^m(n_m^k)\nabla\Phi,\\
\nabla \cdot u_m^k = 0,\\
\partial_\eta \tilde{n}_m^k\big|_\Gamma = \partial_\eta \tilde{z}_m^k\big|_\Gamma = \tilde{u}_m^k\big|_\Gamma = 0, \quad \tilde{n}(0) = n^0, \quad \tilde{z}(0) = z^0, \quad \tilde{u}(0) = u^0.
\end{cases} \tag{29}$$

As a direct consequence of those definitions, from estimate (20) and Lemma 5.1, one has

$$\int n_m^k = \int n^0 \quad \text{and} \quad \|z_m^k\|_{L^\infty} \le \|z^0\|_{L^\infty}, \tag{30}$$

and the following bound (independent of m, k) holds

$$\nabla z_m^k \text{in } L^2(0,\infty; L^2). \tag{31}$$

6.1. First energy estimates

We sum up the energy inequalities obtained in Lemmas 5.2 and 5.3, from $j=1$ to i, for any i, and multiply it by k to obtain

$$\begin{aligned}\frac{1}{4(s-1)}\int (n^i)^s + \frac{1}{2}\int |\nabla z^i|^2 + \frac{C_1}{2}\int |u^i|^2 + \frac{1}{2}\sum_{j=1}^{i}\Big[\|\nabla z^j - \nabla z^{j-1}\|_{L^2}^2 + C_1\|u^j - u^{j-1}\|_{L^2}^2\Big] \\ + k\sum_{j=1}^{i}\left(C_2\|\nabla u^j\|_{L^2}^2 + C_3\left(\|D^2 z^j\|_{L^2}^2 + \int \frac{|\nabla z^j|^4}{(z^j)^2}\right) + \frac{s}{4}\int G^m(n^j)|\nabla z^j|^2\right) \\ \le \frac{1}{4(s-1)}\int (n^0)^s + \frac{1}{2}\int |\nabla z^0|^2 + \frac{C_1}{2}\int |u^0|^2 + C_4\, t_i.\end{aligned} \tag{32}$$

Note that, for each regularized initial data (n_j^0, z_j^0, u_j^0) satisfying (9), (10) and (11), the estimates obtained from the energy inequality only depend on $\|n_j^0\|_{L^s}^s + \|\nabla z_j^0\|_{L^2}^2 + \|u_j^0\|_{L^2}^2$ which is bounded independently of j. Then we may conclude that the RHS of (32) is bounded at finite time, therefore for any fixed $T>0$, the following bounds hold

$$\nabla z_m^k \text{ in } L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \cap L^4(0,T;L^4), \tag{33}$$

$$u_m^k \text{ in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{34}$$

$$n_m^k \text{ in } L^\infty(0,T;L^s), \tag{35}$$

$$G^m\big(n_m^k\big)^{1/2}\nabla z_m^k \text{ in } L^2\big(0,T;L^2\big). \tag{36}$$

Note that by Lemma 3.7 one has $T^m(n^i)^s \le s\, G^m(n^i)$, hence from (36)

$$T^m(n_m^k)^{s/2}\nabla z_m^k \text{ is bounded in } L^2\big(0,T;L^2\big). \tag{37}$$

On the other hand, estimates (30) and (33) imply that

$$z_m^k \text{ is bounded in } L^\infty\big(0,T;H^1\cap L^\infty\big) \cap L^2\big(0,T;H^2\big). \tag{38}$$

By starting from (34) and using $3D$ interpolation, one also has

$$u_m^k \text{ is bounded in } L^{10/3}\Big(0,T;L^{10/3}\Big). \tag{39}$$

Lemma 5.1 and energy estimate (32) also imply

$$\sum_{j=1}^{i}\|z^j - z^{j-1}\|_{H^1}^2 + \sum_{j=1}^{i}\|u^j - u^{j-1}\|_{L^2}^2 \le C\,T, \quad \text{for all } i \ge 1,$$

hence we may infer,

$$\|z_m^k - \tilde z_m^k\|_{L^2(0,T;H^1)}^2 + \|u_m^k - \tilde u_m^k\|_{L^2(0,T;L^2)}^2 + \|\hat u_m^k - \tilde u_m^k\|_{L^2(0,T;L^2)}^2 \le C\,T\,k. \tag{40}$$

6.2. Additional estimates for $s \in (1,2)$

We next obtain an estimate for the difference $n_m^k - \tilde{n}_m^k$ and for n_m^k and ∇n_m^k.

Lemma 6.1. *For $s \in (1,2)$, there exists $C > 0$, independent of m and k, such that*

$$\left\| n_m^k - \tilde{n}_m^k \right\|_{L^2(0,T;L^s)}^2 \leq C\, k. \tag{41}$$

Moreover, the following estimates hold

$$n_m^k \text{ in } L^{5s/3}\Big(0,T;L^{5s/3}\Big), \tag{42}$$

and

$$\nabla n_m^k \text{ in } L^{5s/(s+3)}\Big(0,T;L^{5s/(s+3)}\Big). \tag{43}$$

Proof. Testing n^i equation of (21) by $(n^i+1)^{s-1}/(s-1)$, and using Lemma 3.2, we have

$$\frac{1}{s(s-1)}\delta_t \int \big(n^i+1\big)^s + \frac{4}{s^2}\left\|\nabla\big(n^i+1\big)^{s/2}\right\|_{L^2}^2 + \frac{1}{2k}\int \big(b^i+1\big)^{s-2}\big(n^i - n^{i-1}\big)^2$$
$$= \int T^m(n^i)\big(n^i+1\big)^{s-2}\nabla(z^i)^2\cdot\nabla n^i,$$

where $b^i(x)$ is an intermediate function between $n^i(x)$ and $n^{i-1}(x)$. Then by using that $1 - s/2 > 0$ (recall that $s < 2$), the equality

$$(n^i+1)^{s/2-1}\nabla n^i = \frac{2}{s}\nabla(n^i+1)^{s/2},$$

and the inequality $T^m(n^i) \leq n^i$, one has

$$\frac{1}{s(s-1)}\delta_t \int \big(n^i+1\big)^s + \frac{4}{s^2}\left\|\nabla\big(n^i+1\big)^{s/2}\right\|_{L^2}^2 \leq 2\int T^m(n^i)(n^i+1)^{s/2-1}z^i\nabla z^i\cdot\nabla n^i(n^i+1)^{s/2-1}$$
$$= \frac{4}{s}\int \frac{T^m\big(n^i\big)^{1-s/2}}{\big(n^i+1\big)^{1-s/2}}T^m(n^i)^{s/2}z^i\nabla z^i\cdot\nabla(n^i+1)^{s/2}$$
$$\leq C\left\|z^0\right\|_{L^\infty}^2\int T^m(n^i)^s\left|\nabla z^i\right|^2 + \frac{2}{s^2}\left\|\nabla(n^i+1)^{s/2}\right\|_{L^2}^2.$$

Thus, we arrive at

$$\frac{1}{s(s-1)}\delta_t \int (n^i+1)^s + \frac{2}{s^2}\left\|\nabla(n^i+1)^{s/2}\right\|_{L^2}^2 \leq C\int T^m(n^i)^s\left|\nabla z^i\right|^2. \tag{44}$$

Now multiplying (44) by k and summing up, from 1 to i, for any $i \in \mathbb{N}$, we obtain

$$\frac{1}{s(s-1)}\int \big(n^i+1\big)^s + \frac{2}{s^2}k\sum_{j=1}^{i}\left\|\nabla\big(n^j+1\big)^{s/2}\right\|_{L^2}^2 \leq Ck\sum_{j=1}^{i}\int T^m(n^j)^s\left|\nabla z^j\right|^2 + \frac{1}{s(s-1)}\int \big(n^0+1\big)^s$$
$$\leq C_1(T) + C_2\bigg(\int (n^0)^s + |\Omega|\bigg),$$

where estimate (37) has been applied in the last inequality. Therefore, we infer

$$\big(n_m^k+1\big)^{s/2} \text{ is bounded in } L^\infty\big(0,T;L^2\big)\cap L^2\big(0,T;H^1\big). \tag{45}$$

Note that (45) implies, by interpolation, that

$$\left(n_m^k+1\right)^{s/2} \text{ is bounded in } L^{10/3}\left(0,T;L^{10/3}\right),$$

hence (42) is proved.

Since $s<2$, from (42) we get that $(n_m^k+1)^{1-s/2}$ is bounded in $L^{10s/3(2-s)}(0,T;L^{10s/3(2-s)})$, which jointly with (45) imply (43) due to

$$\nabla n_m^k=\frac{2}{s}\left(n_m^k+1\right)^{1-s/2}\nabla\left(n_m^k+1\right)^{s/2}.$$

Finally, using previous bounds and following line by line ([15], Lem. 4.2), we can obtain (41). □

The next lemma provides uniform in time energy estimates.

Lemma 6.2. *For $s\in(1,2)$, we have*

$$\left(n_m^k,\nabla z_m^k,u_m^k\right) \text{ is bounded in } L^\infty\left(0,\infty;L^s\times L^2\times H\right). \tag{46}$$

Proof. By Lemma 3.7 we have that $T^m(n^i)^s\le s\,G^m(n^i)$, so we can add $1/(8C)$ times (44) to the energy inequality given in Lemma 5.2, and obtain the recursive inequality

$$\delta_t a_i+d_i\le C_4, \tag{47}$$

where

$$a_i:=\frac{1}{4(s-1)}\left\|n^i\right\|_{L^s}^s+\frac{1}{8Cs(s-1)}\left\|n^i+1\right\|_{L^s}^s+\frac{1}{2}\left\|\nabla z^i\right\|_{L^2}^2+\frac{C_1}{2}\left\|u^i\right\|_{L^2}^2,$$

and

$$d_i:=\frac{1}{4Cs^2}\left\|\nabla\left(n^i+1\right)^{s/2}\right\|_{L^2}^2+C_2\left\|\nabla u^i\right\|_{L^2}^2+C_3\left(\left\|D^2z^i\right\|_{L^2}^2+\int\frac{\left|\nabla z^i\right|^4}{\left(z^i\right)^2}\right)+\frac{s}{8}\int G^m\left(n^i\right)\left|\nabla z^i\right|^2.$$

Now we show that there exist $K_1,K_2>0$ such that

$$a_i\le K_1d_i+K_2,\qquad \text{for all } i\ge 1. \tag{48}$$

We estimate each term of a_i separately. For n^i, we note that by applying Poincaré's inequality in $(n^i+1)^{s/2}$ one has

$$\left\|n^i+1\right\|_{L^s}^s=\left\|\left(n^i+1\right)^{s/2}\right\|_{L^2}^2\le C\left\|\nabla\left(n^i+1\right)^{s/2}\right\|_{L^2}^2+C\left\|\left(n^i+1\right)^{s/2}\right\|_{L^1}^2. \tag{49}$$

For the second term on the RHS, as $s<2$, by using Hölder inequality, we have that

$$\left\|(n^i+1)^{s/2}\right\|_{L^1}^2\le|\Omega|^{2-s}\left\|n^i+1\right\|_{L^1}^s\le C\left(\left\|n^0\right\|_{L^1}\right), \tag{50}$$

where in the last estimate we have used Lemma 5.1. Therefore,

$$\frac{1}{4(s-1)}\left\|n^i\right\|_{L^s}^s+\frac{1}{8C(s-1)}\left\|n^i+1\right\|_{L^s}^s\le C\left\|\nabla(n^i+1)^{s/2}\right\|_{L^2}^2+K_2\le K_1d_i+K_2.$$

For z^i, we use its L^∞ bound

$$\left\|\nabla z^i\right\|_{L^2}^2\le\left\|z^i\right\|_{L^\infty}\int\frac{\left|\nabla z^i\right|^2}{z^i}\le C_3\int\frac{\left|\nabla z^i\right|^4}{(z^i)^2}+K_2\le d_i+K_2.$$

Finally, for u^i we just note that by Poincaré's inequality, there exists $C > 0$ such that

$$\left\|u^i\right\|_{L^2}^2 \leq C\left\|\nabla u^i\right\|_{L^2}^2 \leq K_1 d_i.$$

Plugging the previous bounds, we find (48). Hence, the recursive inequality (47) can be written as

$$\delta_t a_i + C_6 a_i \leq C_4 + C_5.$$

Then, by applying Lemma 3.5 with constants $\lambda = C_6$ and $C = C_4 + C_5$ yields to

$$a_i \leq a_0 + C/C_6, \quad \text{for all } i \geq 1.$$

By taking into account the expression of a_i, one has the uniform in time energy estimates (46). □

We also have the following estimates.

Lemma 6.3. *For $s \in (1,2)$, it holds*

$$\partial_t \tilde{n}_m^k \text{ is bounded in } L^{10/9}\Big(0,T;\big(W^{1,10}\big)'\Big), \tag{51}$$

$$\partial_t \tilde{z}_m^k \text{ is bounded in } L^{5/3}\Big(0,T;L^{5/3}\Big), \tag{52}$$

$$\tilde{n}_m^k \text{ is bounded in } L^\infty(0,\infty;L^s) \cap L^{5s/3}\Big(0,T;L^{5s/3}\Big) \cap L^{5s/(s+3)}\Big(0,T;W^{1,5s/(s+3)}\Big), \tag{53}$$

$$\tilde{z}_m^k \text{ is bounded in } L^\infty\big(0,\infty;L^\infty \cap H^1\big) \cap L^2\big(0,T;H^2\big), \tag{54}$$

$$\tilde{u}_m^k,\ \hat{u}_m^k \text{ are bounded in } L^\infty(0,\infty;H) \cap L^2(0,T;V), \tag{55}$$

$$\partial_t \tilde{u}_m^k \text{ is bounded in } L^{5/3}\left(0,T;\left(W_{0,\sigma}^{1,5/2}\right)'\right). \tag{56}$$

Proof. To estimate $\partial_t \tilde{n}_m^k$, considering the chemotaxis flux, as $s < 2$ we may write $T^m(n_m^k)\nabla(z_m^k)^2$ as

$$T^m\big(n_m^k\big)\nabla\big(z_m^k\big)^2 = 2\,T^m\big(n_m^k\big)^{1-s/2} T^m\big(n_m^k\big)^{s/2} z_m^k \nabla z_m^k.$$

As from (42) we have that $(n_m^k)^{1-s/2}$ (and $T^m(n_m^k)^{1-s/2}$) is bounded in $L^{10s/3(2-s)}(0,T;L^{10s/3(2-s)})$, and using also estimates (30) and (37), we conclude

$$T^m\big(n_m^k\big)\nabla\big(z_m^k\big)^2 \text{ is bounded in } L^{5s/(s+3)}\Big(0,T;L^{5s/(s+3)}\Big). \tag{57}$$

On the other hand, we rewrite the coupled term $u_m^k \cdot \nabla n_m^k = \nabla \cdot (u_m^k n_m^k)$, and from (39) and (42)

$$u_m^k n_m^k \text{ is bounded in } L^{10s/(3s+6)}\Big(0,T;L^{10s/(3s+6)}\Big). \tag{58}$$

From $(29)_1$, by any $\phi \in W^{1,10}$, we have that

$$\left\langle \partial_t \tilde{n}_m^k, \phi \right\rangle = \int \Big[u_m^k n_m^k + \nabla n_m^k + T^m\big(n_m^k\big)\nabla\big(z_m^k\big)^2\Big] \cdot \nabla\phi.$$

From (58), (43), and (57), it holds that the term in brackets is bounded in $L^{10/9}(0,T;L^{10/9})$, we conclude (51).

Next, we deduce an estimate for $\partial_t z_m^k$. Note that by (33) and (20),

$$\frac{\left|\nabla z_m^k\right|^2}{z_m^k} \text{ is bounded in } L^2\big(0,T;L^2\big).$$

Moreover as $s\,G^m(n_m^k) \le (n_m^k)^s$, which is bounded in $L^{5/3}(0,T;L^{5/3})$ owing to (42), then

$$G^m\left(n_m^k\right),\ \frac{G^m\left(n_m^k\right)}{z_m^k} \text{ are bounded in } L^{5/3}\left(0,T;L^{5/3}\right).$$

For the convective term $u_m^k \cdot \nabla z_m^k$, since u_m^k and ∇z_m^k are bounded in $L^{10/3}(0,T;L^{10/3})$, it holds

$$u_m^k \cdot \nabla z_m^k \text{ is bounded in } L^{5/3}\left(0,T;L^{5/3}\right).$$

Finally, from equation $(29)_2$ we may conclude (52).

In order to obtain estimates for the global continuous and local linear in-time sequence $\tilde{n}_m^k$ (and $\tilde{z}_m^k, \tilde{u}_m^k$), we use the triangular inequality

$$\left\|\tilde{n}_m^k(t)\right\| \le \left\|n_m^k(t)\right\| + \left\|n^{j-1}\right\|, \quad \text{for all } t \in I_j.$$

Since there is a dependence on the previous time step, more regular initial data is required, as stated in (9)–(12).

Then, from (35), (42) and (43), and the initial estimate (12), we infer (53). On the other hand, from (38) and (34), and the initial estimate (12), one has that (54) and (55) hold.

To estimate $\partial_t \tilde{u}_m^k$ from equation $(29)_3$, we observe that $\hat{u}_m^k \otimes u_m^k$ and n_m^k are bounded in $L^{5/3}(0,T;L^{5/3})$, thus (56) follows. □

6.3. Passing the limit as $(m,k) \to (\infty, 0)$ for the case $s \in (1,2)$

Having all the necessary estimates, we can start looking for strong convergence to pass to the limit in nonlinear terms.

With respect to $\{\tilde{n}_m^k\}$ sequence, from estimates (53), (51), and the compactness Lemma 3.4, there exist a subsequence of $\{\tilde{n}_m^k\}$, relabeled the same, and a limit function n such that, as $(m,k) \to (\infty,0)$,

$$\tilde{n}_m^k \to n \ \text{ in } C\left([0,T];\left(W^{1,10}\right)'\right) \cap L^{5s/(s+3)}\left(0,T;L^{5s/(s+3)}\right).$$

By accounting (41), we also have the same convergences for n_m^k towards the same limit n. Moreover, from (53),

$$n_m^k \to n \ \text{ weakly in } L^{5s/3}\left(0,T;L^{5s/3}\right) \cap L^{5s/(s+3)}\left(0,T;W^{1,5s/(s+3)}\right), \text{ weakly}* \text{ in } L^\infty(0,\infty;L^s),$$

and

$$n_m^k \to n \ \text{ in } L^p(0,T;L^p), \quad \text{for all } p \in [1, 5s/3), \tag{59}$$

and from (51)

$$\partial_t \tilde{n}_m^k \to \partial_t n \ \text{ weakly in } L^{10/9}\left(0,T;\left(W^{1,10}\right)'\right).$$

By applying the Dominated Convergence Theorem jointly to (59),

$$T^m\left(n_m^k\right) \to n \ \text{ in } L^p(0,T;L^p), \quad \text{for all } p \in [1,5s/3).$$

Similarly, again by Dominated Convergence Theorem, (59) and Lemma 3.3, it follows that

$$G^m\left(n_m^k\right) \to \frac{n^s}{s} \ \text{ in } L^p(0,T;L^p), \quad \text{for all } p \in [1,5/3). \tag{60}$$

Arguing now for $\{\tilde{z}_m^k\}$, using estimates (54) and (52), from the compactness of Lemma 3.4 we get the strong convergence, up to subsequences, and a limit function z such that

$$\tilde{z}_m^k \to z \ \text{ in } C\left([0,T];L^2\right) \cap L^2\left(0,T;H^1\right) \cap L^p(0,T;L^p), \ \text{ for all } p \in [1,\infty). \tag{61}$$

Due to (40), the same convergences will hold true to z_m^k towards the same limit function z. Also, from estimate (38) we have, up to a subsequence,

$$z_m^k \xrightarrow{*} z \quad \text{weakly}* \text{ in } L^\infty\big(0,\infty; H^1 \cap L^\infty\big) \text{ and weakly in } L^2\big(0,T;H^2\big),$$

by (33)

$$\nabla z_m^k \to \nabla z \quad \text{weakly in } L^4\big(0,T;L^4\big), \tag{62}$$

and by (52)

$$\partial_t \tilde{z}_m^k \to \partial_t z \quad \text{weakly in } L^{5/3}\Big(0,T;L^{5/3}\Big).$$

With respect to velocity sequence $\{\tilde{u}_m^k\}$, we use estimates (55) and (56), and that V is compactly embedded in H and H is continuously embedded in $(W_{0,\sigma}^{1,5/2})'$. Therefore, by Lemma 3.4, there exist a subsequence of $\{\tilde{u}_m^k\}$, again relabeled the same, and a limit function u such that

$$\tilde{u}_m^k \to u \quad \text{in } C\bigg([0,T];\Big(W_{0,\sigma}^{1,5/2}\Big)'\bigg) \cap L^2(0,T;H) \cap L^p(0,T;L^p), \qquad \text{for all } p < 10/3. \tag{63}$$

Again, by (40), those convergences will also hold for u_m^k. Moreover, from (34) and (39)

$$u_m^k \to u \quad \text{weakly}* \text{ in } L^\infty(0,\infty;H), \text{ weakly in } L^2(0,T;V) \cap L^{10/3}\Big(0,T;L^{10/3}\Big), \tag{64}$$

and from (56)

$$\partial_t \tilde{u}_m^k \to \partial_t u \quad \text{weakly in } L^{5/3}\bigg(0,T;\Big(W_{0,\sigma}^{1,5/2}\Big)'\bigg).$$

We now handle the nonlinear terms. Since $g(x) = x^{-1}$ is a Lipschitz function on $[\alpha,\infty)$, from (61), one has

$$\begin{aligned} \frac{1}{z_m^k} &\to \frac{1}{z} && \text{in } L^p(0,T;L^p), && \text{for all } \ p < \infty, \\ \nabla\big(z_m^k\big)^2 &\to \nabla(z^2) && \text{in } L^p(0,T;L^p), && \text{for all } \ p < 4, \end{aligned} \tag{65}$$

and from (57)

$$T^m\big(n_m^k\big)\nabla\big(z_m^k\big)^2 \to n\nabla\big(z^2\big) \quad \text{weakly in } L^{5s/(s+3)}\Big(0,T;L^{5s/(s+3)}\Big).$$

Arguing as above, from (61) and (62)

$$\frac{\big|\nabla z_m^k\big|^2}{z_m^k} \to \frac{|\nabla z|^2}{z} \quad \text{weakly in } L^2\big(0,T;L^2\big),$$

from (60) and (61)

$$G^m\big(n_m^k\big)z_m^k \to \frac{1}{s}n^s z \quad \text{weakly in } L^{5/3}\Big(0,T;L^{5/3}\Big),$$

and from (60) and (65)

$$\frac{s\,G^m\big(n_m^k\big)}{z_m^k} \to \frac{n^s}{z} \quad \text{weakly in } L^{5/3}\Big(0,T;L^{5/3}\Big).$$

Moreover, from (62) and (63),

$$u_m^k \cdot \nabla z_m^k \to u \cdot \nabla z \quad \text{weakly in } L^{20/11}\Big(0,T;L^{20/11}\Big).$$

As for the convective term on n_m^k, note that due to (59), (64) and (58), we get that

$$u_m^k n_m^k \to u\,n \ \text{ weakly in } L^{10s/(3s+6)}\Big(0,T;L^{10s/(3s+6)}\Big).$$

For the fluid convective term, from (63) and (64),

$$\hat{u}_m^k \otimes u_m^k \to u \otimes u \ \text{ weakly in } L^{5/3}\Big(0,T;L^{5/3}\Big).$$

By accounting all previous convergences, it is possible to pass the limit in each term of problem (29) as $(m,k) \to (\infty, 0)$ and obtain that the limit (n,z,u) is a weak solution of system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla(z^2)), \\ z_t + u\cdot\nabla z = \Delta z - \dfrac{1}{2}n^s\left(z - \dfrac{\alpha^2}{z}\right) + \dfrac{|\nabla z|^2}{z}, \\ u_t + u\cdot\nabla u + \nabla P = \Delta u + n\nabla\Phi, \\ \nabla\cdot u = 0, \\ \partial_\eta n|_\Gamma = \partial_\eta z|_\Gamma = u|_\Gamma = 0, \quad n(0)=n^0, \quad z(0)=z^0, \quad u(0)=u^0. \end{cases} \tag{66}$$

Note that the initial conditions are also well-defined in $L^s \times (H^1\cap L^\infty)\times H$ due to the weak continuity in time of the solution (n,z,u) in this space.

As systems (1) and (66) are equivalent in the sense that (n,z,u) is a weak solution of (66) if and only if (n,c,u) is a weak solution of (1), with $c = z^2 - \alpha^2$, we can deduce the existence of a weak solution (n,c,u) to system (1).

6.4. Passage to the limit as $(m,k) \to (\infty,0)$ for the case $s \geq 2$.

We first consider the energy estimates, and then proceed to get flux estimates.

Energy estimates

From Lemma 5.3, we obtain all energy estimates from the previous case $s<2$, and the additional estimate

$$\nabla\big(n_m^k\big)^{s/2} \text{ is bounded in } L^2\big(0,T;L^2\big),$$

which allows us to simplify the argument to gain the uniform in time energy estimates given in Lemma 6.2. Indeed, now it is enough to consider

$$a_i := \frac{1}{4(s-1)}\|n^i\|_{L^s}^s + \frac{1}{2}\|\nabla z^i\|_{L^2}^2 + \frac{C_1}{2}\|u^i\|_{L^2}^2,$$

and

$$d_i := \frac{1}{4s}\Big\|\nabla\big(n^i\big)^{s/2}\Big\|_{L^2}^2 + C_2\|\nabla u^i\|_{L^2}^2 + C_3\left(\|D^2 z^i\|_{L^2}^2 + \int\frac{|\nabla z^i|^4}{(z^i)^2}\right) + \frac{s}{4}\int G^m\big(n^i\big)|\nabla z^i|^2.$$

We just notice that instead of (49) and (50), now the argument is

$$\|n^i\|_{L^s}^s = \Big\|\big(n^i\big)^{s/2}\Big\|_{L^2}^2 \leq C\Big\|\nabla\big(n^i\big)^{s/2}\Big\|_{L^2}^2 + C\Big\|\big(n^i\big)^{s/2}\Big\|_{L^1}^2,$$

and for the second term on the RHS, using interpolation, it holds for $a = \frac{s-2}{s-1} \in (0,1)$, that

$$\Big\|\big(n^i\big)^{s/2}\Big\|_{L^1}^2 = \|n^i\|_{L^{s/2}}^s \leq \|n^i\|_{L^s}^{sa}\|n^i\|_{L^1}^{s(1-a)} \leq \epsilon\|n^i\|_{L^s}^s + C\big(\|n^0\|_{L^1},\epsilon\big),$$

for any $\epsilon>0$ small enough. In conclusion, for the case $s\geq 2$, we also get the uniform in time energy estimates given in (46).

Flux estimates

Following a similar idea stated in [9], an estimate of the chemotactic flux $T^m(n_m^k)\nabla z_m^k$ can be obtained by splitting the domain in terms of $n_m^k(t,\cdot)$ by defining the sets

$$\{0 \le n_m^k(t,\cdot) \le 1\} = \{x \in \Omega \mid 0 \le n_m^k(t,x) \le 1\}$$

and

$$\{n_m^k(t,\cdot) > 1\} = \{x \in \Omega \mid n_m^k(t,x) > 1\}.$$

It holds that, a.e. in $(0,T)$

$$\begin{aligned}\int T^m(n_m^k)^2|\nabla z_m^k|^2 &= \int_{\{0\le n_m^k\le 1\}} T^m(n_m^k)^2|\nabla z_m^k|^2 + \int_{\{n_m^k\ge 1\}} T^m(n_m^k)^2|\nabla z_m^k|^2 \\ &\le \int_{\{0\le n_m^k\le 1\}} |\nabla z_m^k|^2 + \int_{\{n_m^k\ge 1\}} T^m(n_m^k)^s|\nabla z_m^k|^2 \\ &\le \int |\nabla z_m^k|^2 + \int T^m(n_m^k)^s|\nabla z_m^k|^2.\end{aligned}$$

By integrating in time, we can conclude that

$$\int_0^T \int T^m(n_m^k)^2|\nabla z_m^k|^2 \le \int_0^T \int |\nabla z_m^k|^2 + \int_0^T \int T^m(n_m^k)^s|\nabla z_m^k|^2,$$

which is bounded by (31) and (37). Therefore,

$$T^m(n_m^k)\nabla z_m^k \text{ is bounded in } L^2(0,T;L^2), \tag{67}$$

hence passing to the limit one arrives at $n\nabla z \in L^2(0,T;L^2)$.

Estimate (67) will help us to obtain the $L^2(0,T;L^2)$ estimate for the diffusion flux ∇n_m^k. Indeed, by testing n^i equation by n^i, we get

$$\begin{aligned}\frac{1}{2}\delta_t \int |n^i|^2 + \frac{1}{2k}\int |n^i - n^{i-1}|^2 + \int |\nabla n^i|^2 &= 2\int z^i\, T^m(n^i)\nabla z^i \cdot \nabla n^i \\ \le 2\|z^i\|_{L^\infty}\|T^m(n^i)\nabla z^i\|_{L^2}\|\nabla n^i\|_{L^2} &\le C\int T^m(n^i)^2|\nabla z^i|^2 + \frac{1}{2}\int |\nabla n^i|^2.\end{aligned}$$

Then multiplying by $2k$ and summing in i, and using (67), we end up with

$$\nabla n_m^k \text{ is bounded in } L^2(0,T;L^2). \tag{68}$$

At this point, we can repeat the process for passing to the limit as $(m,k) \to (\infty,0)$ as in the case $s < 2$. In fact the main differences in convergences compared with the case $s < 2$ arise from the flux estimates (67) and (68), hence in particular one arrives at $n\nabla z$ and $\nabla n \in L^2(0,T;L^2)$ which also imply that n_t lies in a different space than in case $s < 2$ (cf. Rem. 2.1).

As in the case $s < 2$, we can conclude that the limit functions n, z, and u are a weak solution to the system (66), completing the proof of Theorem 2.1.

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The research data associated with this article are included in the article.

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