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TIME DISCRETIZATION IN CONVECTED LINEARIZED THERMO-VISCO-ELASTODYNAMICS AT LARGE DISPLACEMENTS

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Abstract. The fully-implicit time discretization (*i.e.* the backward Euler formula) is applied to compressible nonlinear dynamical models of thermo-viscoelastic solids in the Eulerian description, *i.e.* in the actual deforming configuration, formulated in terms of rates. The Kelvin–Voigt rheology or also, in the deviatoric part, the Jeffreys rheology (which covers creep or plasticity) are considered, using the additive Green–Naghdi decomposition of total strain into the elastic and the inelastic strains formulated in terms of (objective) rates exploiting the Zaremba–Jaumann time derivative. A linearized convective model at large displacements is considered, focusing on the case where the internal energy additively splits the (convex) mechanical and the thermal parts. A fully implicit time-discrete scheme is devised. Considering the multipolar 2nd-grade viscosity, the numerical stability and convergence towards weak solutions are proven by exploiting, in particular, the convexity of the kinetic energy when written in terms of linear momentum instead of velocity and by estimating the temperature gradient from the entropy-like inequality.

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1. Introduction

The models of *thermo-visco-elastodynamics* in continuum mechanics *at finite* (also called *large*) *strains* lead to strongly nonlinear systems of evolution partial differential equations. The basic modelling paradigms are the Lagrangian description (using a referential configuration) *versus* the *Eulerian description* (using the actual deforming configuration). In this paper, we focus on the latter. The Eulerian approach is particularly justified in situations where no referential configuration exists for physical reasons, such as in fluids or in solids undergoing creep as in geophysical models on long timescales of millions of years). The description in the actual configuration reveals the "actual" physics more explicitly than Lagrangian approach.

To avoid truly large-strain Eulerian models, a certain modeling compromise is to use a linearization that leads to a symmetric small-strain tensor while allowing *large displacements* and incorporating suitable convected objective time derivatives. Such models are widely used in engineering and geophysics.

Keywords and phrases. Thermodynamics, visco-elastodynamics, Kelvin–Voigt rheology, anti-Zener rheology, plasticity, large displacements, linearized Euler description, convected model, backward Euler time discretization, Rothe method, entropy inequality, weak solutions.

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Such models can be approximated numerically using the time discretization (*Rothe*'s) *method*. After further space discretization, this method yields computationally implementable schemes. An analytically rigorous treatment of time discretization has been launched for isothermal compressible fluids in [5–7, 11, 17, 39]. and for isothermal compressible solids in [34]. This paper aims to extend this treatment to extend this treatment to the anisothermal compressible solids.

The main attributes of this paper are as follows:

– the (generalized) entropy inequality is exploited for *a-priori* estimation of the temperature gradient,
– the convexity of the kinetic energy of the linear momentum is used for the time discretization as in [34], as well as the convexity of $\varrho \mapsto \sigma := 1/\varrho$ for estimation of the so-called sparsity σ,
– the discrete Gronwall inequality is used for a sufficiently small time step without regularing the discrete kinematics when estimating the strain gradient, which also allows the strong convexity assumption in [34] to be weakened to mere convexity, and
– the time discretization is applied directly to the original, non-regularized problem.

In comparison with the aforementioned paper [34], we account for Maxwellian *creep* in the deviatoric part, which, in combination with the Kelvin–Voigt rheology, yields the *Jeffreys* (also called the anti-Zener) *viscoelastic model*. This creep may be nonlinear and may include plasticity, see Remark 4.6 below. Although we focus on the basic visco-elastodynamics, the coupling with other phenomena such as damage or aging (like [30]) or, *e.g.* also diffusion in poroelasticity or magnetoelasticity, is conceptually well possible and would expand the applicability of this model and approach. Even in the basic scenario presented here, we improve [30] where only a semi-compressible model and a thermally decoupled free energy were considered.

In Section 2 we formulate the linearized convective model involving a small-strain tensor at large displacements. We use the objective strain rate due to the Zaremba–Jaumann time derivative, which is most commonly used in linearized Eulerian models for solid mechanics. It should be emphasized that this model is formulated entirely in terms of rates, so that neither the displacement nor the deformation occurs explicitly in the resulting system. In Section 3, we will devise a fully coupled implicit time discretization. Finally, in Section 4, we perform the stability and convergence analysis in a special (but commonly used) case when the heat capacity is temperature dependent only.

In order to facilitate the rigorous analysis particularly for models of solids (and to model various dispersion of the velocity of propagation of elastic waves, as analyzed in [31]), some higher-order gradients in the dissipative part of the models can be considered. These higher (here 2nd order) gradients lead to the concept of (here 2nd-grade) *nonsimple media*, which has been discussed in literature since the works by Toupin [36] and Mindlin [22]. In the dissipative part as used in this paper, it was also developed by Nečas *et al.* [2, 24, 25] as *multipolar fluids* and later *e.g.* in [10, 27].

For readers' convenience, let us summarize the basic notation used in what follows (Tab. 1).

We will use the standard notation for the Lebesgue and the Sobolev spaces of functions on the Lipschitz bounded domain $\Omega \subset \mathbb{R}^3$, namely $L^p(\Omega;\mathbb{R}^n)$ for Lebesgue measurable $\mathbb{R}^n$-valued functions $\Omega \to \mathbb{R}^n$ whose Euclidean norm is integrable with p-power, and $W^{k,p}(\Omega;\mathbb{R}^n)$ for functions from $L^p(\Omega;\mathbb{R}^n)$ whose all derivatives up to the order k have their Euclidean norm integrable with p-power. For short, we also write $H^k = W^{k,2}$. We have the embedding $H^1(\Omega) \subset L^6(\Omega)$. Furthermore, for a Banach space X and for $I = [0, T]$, we will use the notation $L^p(I;X)$ for the Bochner space of Bochner measurable functions $I \to X$ whose norm is in $L^p(I)$, and $H^1(I;X)$ for functions $I \to X$ whose distributional derivative is in $L^2(I;X)$. Occasionally, we will use $L^p_{\rm w}(I;X)$ for the space of weakly* measurable mappings $I \to X$ if X has a predual, *i.e.* there is X' such that $X = (X')^*$ where $(\cdot)^*$ denotes the dual space. The space of continuous functions on the closure $\overline{\Omega}$ of Ω will be denoted by $C(\overline{\Omega})$.

2. Linearized large-deformation convective model

Most materials typically cannot withstand too much large elastic strains without initiating inelastic processes such as damage, creep, or plastification. Thus, elastic strains always remain rather small (and the elastic

TABLE 1. Summary of the basic notation used.

$\boldsymbol{y}$ deformation,	$\boldsymbol{T}$ Cauchy stress,
$\boldsymbol{v}$ velocity,	$\boldsymbol{D}$ dissipative stress,
ϱ mass density,	$\boldsymbol{\Pi}$ inelastic distortion rate,
$\boldsymbol{p} = \varrho\boldsymbol{v}$ the linear momentum,	$\boldsymbol{g}$ gravity acceleration,
θ (absolute) temperature,	$\mathbb{I}$ the identity matrix,
$\boldsymbol{E}$ small strain,	$\mathrm{tr}(\cdot)$ trace of a matrix,
u thermal part of the internal energy,	$\mathrm{sph}(\cdot)$ spherical part of a matrix, $\mathrm{sph}\,E := (\mathrm{tr}\,E)\mathbb{I}/3$,
$K_{\rm E}, G_{\rm E}$ elastic bulk and shear moduli,	$\mathrm{dev}(\cdot)$ deviatoric part of a matrix, $\mathrm{dev}E := E - \mathrm{sph}E$,
$K_{\rm V}, G_{\rm V}$ viscosity bulk and shear moduli,	$\mathscr{R} : \mathbb{R}^{3\times3}_{\rm sym}\times\mathbb{R} \to \mathbb{R}^{3\times3}_{\rm dev}$ the inelastic-strain rate,
$G_{\rm M}$ Maxwellian viscosity modulus,	$\mathbb{R}^{3\times3}_{\rm sym}$ set of symmetric matrices,
μ the hyper-viscosity coefficient,	$\mathbb{R}^{3\times3}_{\rm dev} = \{A \in \mathbb{R}^{3\times3}_{\rm sym};\ \mathrm{tr}A = 0\}$,
$I = [0, T]$ a time interval, $T > 0$,	"$\cdot$", ":" scalar products of vectors or matrices,
$\psi : \mathbb{R}^{3\times3} \to \mathbb{R}$ free energy,	"$\vdots$" scalar products 3rd-order tensors,
$(\cdot)'$ (partial) derivative of a mapping,	$\tau > 0$ a time step for discretization,
$(\cdot)^{\boldsymbol{\cdot}}$ convective time derivative,	$(\cdot)^{\circ}$ Zaremba–Jaumann corotational time derivative.

distortion tensor is close to the identity tensor $\mathbb{I}$), and a common small-strain linearization is well acceptable and widely used in many applications. However, small strains do not preclude large displacements, which typically occur in fluids but also in solids. In the latter case large displacements can occur especially when the Kelvin–Voigt model is combined with Maxwellian-type rheology in the deviatoric part. This suggests the use of the Eulerian small-strain models combined with a properly designed transport of the strain tensor, in addition to the usual density transport. Ultimately, any rational model must respect thermodynamic consistency in terms of mass, momentum, and energy.

This compromise between small elastic strains and large deformations and displacements requires appropriate formulations in a convected coordinate system. In particular, it requires the proper choice and treatment of *objective rates*. Here, objectivity means that the time derivatives do not depend on the evolving reference frame. Many possibilities are used in the literature for different models. For applications in solid mechanics, it is reasonable to require that the tensor time derivatives, *i.e.* the tensor rates (in particular the stress rate), be identical and *corotational*. This means respectively that the stress rate vanishes for all rigid body motions and, roughly speaking, only takes into account the rotation of the material element. Importantly, practically all corotational rates enjoy a chain-rule property for isotropic tensor functions, which is an attribute not shared by other, merely objective rates, as discussed in [26]. The simplest corotational variant is the *Zaremba–Jaumann time derivative* [15, 38], denoted it by a circle $(\cdot)^{\circ}$, which is well justified when applied to the Cauchy stress tensor, as demonstrated by Biot ([3], p.494), cf. also [4, 9, 23]. This choice is most commonly used especially in geophysics and more generally in engineering, See, for example, the monographs Chapter 12 of [12] or also Section 8.6 of [14] or Section 8.3 [19]. However, in some other applications in which cycling regimes are expected, this choice may exhibit undesirable "ratchetting" effects [16, 21].

In this paper, we focus on *isotropic materials*, for which this derivative also affects the symmetric small-strain tensor $\boldsymbol{E}$, which is considered as an independent variable in the models. For an overview of objective corotational strain rates see [20, 37]. Specifically, given the Eulerian velocity $\boldsymbol{v}$, we define

$$\overset{\circ}{\boldsymbol{E}} = \frac{\partial \boldsymbol{E}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{E} - \boldsymbol{W}\boldsymbol{E} + \boldsymbol{E}\boldsymbol{W} \quad \text{with} \quad \boldsymbol{W} = \mathrm{skw}(\nabla\boldsymbol{v}) = \frac{\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^\top}{2}\cdot \tag{2.1}$$

The Eulerian velocity $\boldsymbol{v}$ is also used in the convective time derivative for scalars and, component-wise, for vectors or tensors

$$(\cdot)^{\boldsymbol{\cdot}} = \frac{\partial}{\partial t}(\cdot) + (\boldsymbol{v}\cdot\nabla)(\cdot). \tag{2.2}$$

Then (2.1) can be written shortly as $\mathring{\boldsymbol{E}} = \dot{\boldsymbol{E}} - \mathrm{skw}(\nabla\boldsymbol{v})\boldsymbol{E} + \boldsymbol{E}\mathrm{skw}(\nabla\boldsymbol{v})$.

2.1. The thermodynamical system and its energetics

We consider the visco-elastodynamics in the *Kelvin–Voigt rheology* in the volumetric part and the *Jeffreys* (also called *anti-Zener*) *rheology* in the deviatoric (isochoric) part. This is a fairly general model that allows for an isochoric creep or plasticity. For the displacement $\boldsymbol{u}$, we implement the *Green–Naghdi additive decomposition* [13] of the total small strain $\boldsymbol{\varepsilon}(\boldsymbol{u})$ into the elastic and the inelastic strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{E} + \boldsymbol{P}$, where $\boldsymbol{P}$ is in the position of an internal variable. This decomposition will be expressed in objective rates as $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \mathring{\boldsymbol{E}} + \mathring{\boldsymbol{P}}$, to be further written as

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \mathring{\boldsymbol{E}} + \boldsymbol{\Pi} \quad \text{with} \quad \boldsymbol{\Pi} := \mathring{\boldsymbol{P}}, \tag{2.3}$$

where $\boldsymbol{P}$ denotes the inelastic strain. Thus both $\boldsymbol{u}$ and $\boldsymbol{P}$ can be eliminated in the end, although these variables can be reconstructed *a-posteriori* if the corresponding initial conditions were prescribed. Therefore, the model allows for *large displacements* while still using small elastic strains.

The main ingredient is the Helmholtz' free energy $\psi : \mathbb{R}^{3\times3}_{\mathrm{sym}}\times\mathbb{R}^+ \to \mathbb{R}$ acting on the small-strain tensor $\boldsymbol{E} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and on temperature θ. Here, and in what follows, we will not notationally distinguish the variable as a placeholder and a time-space field. Here, it means both $\boldsymbol{E} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and $\boldsymbol{E} : I\times\Omega \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$. Standardly, we define the (conservative part of the) Cauchy stress tensor $\boldsymbol{T}$ as

$$\boldsymbol{T} = \mathscr{T}(\boldsymbol{E},\theta) \quad \text{with} \quad \mathscr{T}(\boldsymbol{E},\theta) := \psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta) + \psi(\boldsymbol{E},\theta)\mathbb{I}. \tag{2.4}$$

We note that the "negative-pressure" term $\psi(\boldsymbol{E},\theta)\mathbb{I}$ in (2.4) balances the energetics in the convected model, cf. the calculations (2.29)–(2.31) below. From the free energy ψ, we also obtain (by Gibbs' relation) the *internal energy* $\mathscr{E}$ and the *entropy* η, and also the *heat part* $\mathscr{U}$ of $\mathscr{E}$ as

$$\mathscr{E}(\boldsymbol{E},\theta) = \psi(\boldsymbol{E},\theta) + \theta\eta(\boldsymbol{E},\theta) \quad \text{with} \quad \eta(\boldsymbol{E},\theta) = -\psi'_\theta(\boldsymbol{E},\theta) \quad \text{and} \tag{2.5a}$$

$$\mathscr{U}(\boldsymbol{E},\theta) = \mathscr{E}(\boldsymbol{E},\theta) - \mathscr{E}(\boldsymbol{E},0). \tag{2.5b}$$

Another ingredient is the (pseudo)potential of the dissipative forces $\zeta : \mathbb{R}^+\times\mathbb{R}^{3\times3}_{\mathrm{sym}}\times\mathbb{R}^{3\times3}_{\mathrm{dev}} \to \mathbb{R}$ acting on the rates $\boldsymbol{e} := \boldsymbol{\varepsilon}(\boldsymbol{v})$ and $\boldsymbol{\Pi}$, and also depending on θ. This potential governs the dissipative contribution to the Cauchy stress $\boldsymbol{D} = \zeta'_{\boldsymbol{e}}(\theta,\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi})$. Based on the Kelvin–Voigt rheology summing the stresses $\boldsymbol{T}$ and $\boldsymbol{D}$, we can set up the *momentum equation*

$$\varrho\dot{\boldsymbol{v}} = \mathrm{div}\big(\boldsymbol{T}+\boldsymbol{D}\big) + \varrho\boldsymbol{g} \tag{2.6}$$

with ϱ the mass density which is governed by the mass *continuity equation*

$$\dot{\varrho} = -\varrho\,\mathrm{div}\,\boldsymbol{v}. \tag{2.7}$$

From the potential ζ, we also obtain a "creep stress" $\zeta'_{\boldsymbol{\Pi}}$ which balances the deviatoric part of $\boldsymbol{T}$ and which stands here in the position of a "linearized Mandel stress". Thus we compose the flow rule for the creep rate as

$$\zeta'_{\boldsymbol{\Pi}}(\theta;\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi}) = \mathrm{dev}\,\boldsymbol{T}. \tag{2.8}$$

Of course, the pressure contribution $\psi\,\mathbb{I}$ of $\boldsymbol{T}$ in (2.4) is purely volumetric and thus has no effect on (2.8).

The entropy, considered in Pa/K = J/(m^{3}K), is an intensive variable and is governed by the *entropy equation*:

$$\frac{\partial\eta}{\partial t} + \mathrm{div}(\eta\boldsymbol{v}) = \frac{\xi - \mathrm{div}\,\boldsymbol{j}}{\theta} \qquad \text{with the heat flux} \quad \boldsymbol{j} = -\kappa(\theta)\nabla\theta, \tag{2.9}$$

where $\xi = \xi(\theta;\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi})$ denotes the heat production rate (= the specific power in W/m^3) and is considered equal to the dissipation rate of the mechanical energy

$$\xi(\theta;\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi}) = \zeta'_{\boldsymbol{e}}(\theta,\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi}):\boldsymbol{\varepsilon}(\boldsymbol{v}) + \zeta'_{\boldsymbol{\Pi}}(\theta;\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi}):\boldsymbol{\Pi} = \boldsymbol{D}:\boldsymbol{\varepsilon}(\boldsymbol{v}) + \boldsymbol{T}:\boldsymbol{\Pi}. \tag{2.10}$$

The latter equality in (2.9) is the *Fourier law* which phenomenologically determines the heat flux $\boldsymbol{j}$ to be proportional to the negative temperature gradient through the thermal conductivity coefficient $\kappa = \kappa(\theta)$ considered independent of $\boldsymbol{E}$; for a general continuous dependence of κ also on $\boldsymbol{E}$ see Remark 4.5 below. Assuming $\xi \geq 0$ and $\kappa \geq 0$ and integrating (2.9) over the domain Ω while imposing the impenetrability of the boundary in the sense that the normal velocity $\boldsymbol{v}\cdot\boldsymbol{n}$ vanishes over the boundary Γ of Ω, we obtain the *Clausius-Duhem inequality*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_\Omega \eta\,\mathrm{d}\boldsymbol{x}}_{\text{total entropy}} = \int_\Omega \underbrace{\frac{\xi}{\theta} + \kappa(\theta)\frac{|\nabla\theta|^2}{\theta^2}}_{\text{entropy production rate}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \underbrace{\Big(\kappa(\theta)\frac{\nabla\theta}{\theta} - \eta\boldsymbol{v}\Big)}_{\text{entropy flux}}\cdot\boldsymbol{n}\,\mathrm{d}S \geq \int_\Gamma \kappa(\theta)\frac{\nabla\theta}{\theta}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{2.11}$$

If the system is thermally isolated in the sense that the normal heat flux $\boldsymbol{j}\cdot\boldsymbol{n}$ vanishes across the boundary Γ, we recover the *2nd law of thermodynamics*, *i.e.* the total entropy in isolated systems is nondecreasing with time.

Restricting our attention to a case when

$$\boldsymbol{T} \text{ and } \boldsymbol{E} \text{ commute}, \tag{2.12}$$

we perform the calculus for the time derivative of the internal energy $e = \mathscr{E}(\boldsymbol{E},\theta)$ from (2.5b) as an extensive variable:

$$\begin{aligned}
\frac{\partial e}{\partial t} + \mathrm{div}(e\boldsymbol{v}) &= \dot{e} + e\,\mathrm{div}\,\boldsymbol{v} = \overset{\bullet}{\overline{\psi(\boldsymbol{E},\theta) - \theta\psi'_\theta(\boldsymbol{E},\theta)}} + \big(\psi(\boldsymbol{E},\theta) - \theta\psi'_\theta(\boldsymbol{E},\theta)\big)\,\mathrm{div}\,\boldsymbol{v}\\
&= \psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta):\dot{\boldsymbol{E}} + \psi'_\theta(\boldsymbol{E},\theta)\dot{\theta} - \theta\psi''_{\boldsymbol{E}\theta}(\boldsymbol{E},\theta)\dot{\boldsymbol{E}} - \theta\psi''_{\theta\theta}(\boldsymbol{E},\theta)\dot{\theta} - \dot{\theta}\psi'_\theta(\boldsymbol{E},\theta)\\
&\quad + \big(\psi(\boldsymbol{E},\theta) - \theta\psi'_\theta(\boldsymbol{E},\theta)\big)\,\mathrm{div}\,\boldsymbol{v}\\
&= \psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta):\dot{\boldsymbol{E}} + \theta\,\overset{\bullet}{\overline{\eta(\boldsymbol{E},\theta)}} + \big(\psi(\boldsymbol{E},\theta) - \theta\psi'_\theta(\boldsymbol{E},\theta)\big)\,\mathrm{div}\,\boldsymbol{v}\\
&\overset{(2.9)}{=} \psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta):\dot{\boldsymbol{E}} + \xi - \mathrm{div}\,\boldsymbol{j} - \theta\eta(\boldsymbol{E},\theta)\,\mathrm{div}\,\boldsymbol{v} + \big(\psi(\boldsymbol{E},\theta) - \theta\psi'_\theta(\boldsymbol{E},\theta)\big)\,\mathrm{div}\,\boldsymbol{v}\\
&\overset{(2.3)}{=} \psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta):\big(\boldsymbol{\varepsilon}(\boldsymbol{v}) + \mathrm{skw}(\nabla\boldsymbol{v})\boldsymbol{E} + \boldsymbol{E}\mathrm{skw}(\nabla\boldsymbol{v}) - \boldsymbol{\Pi}\big) + \xi - \mathrm{div}\,\boldsymbol{j} + \psi(\boldsymbol{E},\theta)\,\mathrm{div}\,\boldsymbol{v}\\
&\overset{(2.14)}{\underset{(2.4)}{=}} \xi - \mathrm{div}\,\boldsymbol{j} + \boldsymbol{T}:\big(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\Pi}\big),
\end{aligned} \tag{2.13}$$

where we have used the matrix algebra $A:(BC) = (B^\top A):C = (AC^\top):B$ so that, for $W = \mathrm{skw}\,L$, it holds

$$\begin{aligned}
S:(WE - EW) &= \frac{1}{2}S:\big(LE - L^\top E - EL + EL^\top\big) = \frac{1}{2}\big(SE^\top - E^\top S\big):L - \frac{1}{2}\big(LS - SL\big):E\\
&= \frac{1}{2}\big(SE^\top - ES^\top - E^\top S + S^\top E\big):L = \big(\underbrace{SE - ES}_{=0}\big):L = 0.
\end{aligned} \tag{2.14}$$

Note that in (2.13), we have used (2.14) for $\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta)$, $\boldsymbol{E}$, $\nabla\boldsymbol{v}$, and $\mathrm{skw}(\nabla\boldsymbol{v})$ in place of S, E, L, and W, respectively, so that, in particular, the last equality in (2.14) exploited (2.12). Moreover, it should be emphasized that in (2.14), we have to assume that the initial condition for $\boldsymbol{E}$ is symmetric and to exploit that the Zaremba–Jaumann corotational derivative in (2.3) maintains the symmetry of $\boldsymbol{E}$ throughout the entire evolution.

In fact, (2.13) with $\boldsymbol{T} = \mathscr{T}(\boldsymbol{E},\theta)$ forms the *internal-energy equation* for $e = \mathscr{E}(\boldsymbol{E},\theta)$ with $\mathscr{E}$ from (2.5a). Similarly, by making an obvious modification of (2.13), we can obtain the equation for the heat part of the internal energy $u = \mathscr{U}(\boldsymbol{E},\theta)$:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \mathrm{div}\big(u\boldsymbol{v}+\boldsymbol{j}\big) = \xi + \big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big):\big(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\Pi}\big) \quad \text{with} \quad u = \mathscr{U}(\boldsymbol{E},\theta) \quad \text{and}\\
&\qquad\qquad \text{with} \quad \boldsymbol{j} = -\kappa(\theta)\nabla\theta, \quad \text{where } \mathscr{T} \text{ is from (2.4) and } \mathscr{U} \text{ from (2.5)}.
\end{aligned} \tag{2.15}$$

2.2. The thermomechanical system with a multipolar viscosity

The momentum equation (2.6), the mass continuity equation (2.7), the kinematic equation (2.3) reflecting the Green–Naghdi decomposition, the creep flow rule (2.8), and the heat internal energy equation (2.15) altogether form a closed system for $(\varrho, \boldsymbol{v}, \boldsymbol{E}, \boldsymbol{\Pi}, \theta)$.

We restrict our focus to the thermal coupling only in the volumetric part and on the additive split of the Stokes and the Maxwellian viscosities, specifically

$$\operatorname{dev}\big(\psi''_{\boldsymbol{E}\theta}(\boldsymbol{E},\theta)\big) \equiv 0 \quad \text{and} \quad \zeta(\theta,\boldsymbol{e},\boldsymbol{\Pi}) = \tfrac{1}{2}\mathbb{D}\boldsymbol{e}:\boldsymbol{e} + \zeta_{\rm p}(\theta,\boldsymbol{\Pi}). \tag{2.16}$$

The former means that temperature may influence only the volumetric part of the Cauchy stress but does not influence the deviatoric part. Indeed, realizing that

$$\begin{aligned}\mathscr{T}(\boldsymbol{E},\theta) = \psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta) + \psi(\boldsymbol{E},\theta)\mathbb{I} &= \psi'_{\boldsymbol{E}}(\boldsymbol{E},0) + \psi(\boldsymbol{E},0)\mathbb{I} + \int_0^\theta \psi''_{\boldsymbol{E}\theta}(\boldsymbol{E},\vartheta) + \psi'_\theta(\boldsymbol{E},\vartheta)\mathbb{I}\,\mathrm{d}\vartheta \\ &= \mathscr{T}(\boldsymbol{E},0) + \int_0^\theta \psi''_{\boldsymbol{E}\theta}(\boldsymbol{E},\vartheta) + \psi'_\theta(\boldsymbol{E},\vartheta)\mathbb{I}\,\mathrm{d}\vartheta,\end{aligned} \tag{2.17}$$

the former condition in (2.16) implies that

$$\operatorname{dev}\mathscr{T}(\boldsymbol{E},\theta) = \operatorname{dev}\mathscr{T}(\boldsymbol{E},0). \tag{2.18}$$

Notably, (2.16) simplifies the adiabatic heat source in (2.15) as

$$\begin{aligned}\big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big):\big(\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\Pi}\big) &= \operatorname{sph}\big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big):\boldsymbol{\varepsilon}(\boldsymbol{v}) \\ &= \tfrac{1}{3}\operatorname{tr}\big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big)\operatorname{div}\boldsymbol{v}\end{aligned} \tag{2.19}$$

because the inelastic strain (and its rate) naturally does not affect the volumetric part of the model and is thus modelled as isochoric, *i.e.* $\operatorname{tr}\boldsymbol{\Pi} = 0$. Furthermore, for the latter in (2.16), we assume (as physically natural) the convexity of $\zeta_{\rm p}(\theta,\cdot)$ and use the standard convex-analysis construction of the convex conjugate $\zeta^*_{\rm p}(\theta,\cdot) := [\zeta_{\rm p}(\theta,\cdot)]^*$. Realizing that $\zeta^*_{\rm p}(\theta,\cdot)' = [\zeta_{\rm p}(\theta,\cdot)']^{-1}$, we will use the abbreviation for the inelastic-strain rate $\mathscr{R} : \mathbb{R}^{3\times3}_{\rm sym}\times\mathbb{R} \to \mathbb{R}^{3\times3}_{\rm dev}$ defined by

$$\mathscr{R} : (\boldsymbol{E},\theta) \mapsto [\zeta_{\rm p}(\theta,\cdot)^*]'(\operatorname{dev}\mathscr{T}(\boldsymbol{E},\theta)). \tag{2.20}$$

In addition, as mentioned in the Introduction, we incorporate a nonlinear dissipative higher-order enhancement of the Stokes-type viscosity, also known as *hyper-viscosity*. For simplicity, we adopt this concept of *multipolar continua* equally in the volumetric and in the shear parts using just one coefficient $\mu > 0$ and one exponent p, leading to the extended dissipation potential and the extended dissipation rate

$$\begin{aligned}\zeta_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi}) &= \zeta(\theta,\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi}) + \tfrac{\mu}{p}|\nabla^2\boldsymbol{v}|^p \overset{(2.16)}{=} \tfrac{1}{2}\mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v}) + \zeta_{\rm p}(\theta,\boldsymbol{\Pi}) + \tfrac{\mu}{p}|\nabla^2\boldsymbol{v}|^p \quad \text{and} \\ \xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi}) &= \xi(\theta,\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\Pi}) + \mu|\nabla^2\boldsymbol{v}|^p \overset{(2.16)}{=} \mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v}) + [\zeta_{\rm p}]'_{\boldsymbol{\Pi}}(\theta,\boldsymbol{\Pi}):\boldsymbol{\Pi} + \mu|\nabla^2\boldsymbol{v}|^p,\end{aligned} \tag{2.21}$$

respectively. Mechanically, such higher gradient only amplifies the effect of normal dispersion and attenuation of elastic waves as exhibited already for simple Stokes viscosity in the Kelvin–Voigt rheology. In other words, waves with ultra-high frequencies are more attenuated and do not propagate at all, while waves with very low frequencies can propagate without substantial attenuation, cf. Section 3.1 of [31]. When $\zeta_{\rm p} \not\equiv 0$, the contribution from the Maxwellian-type viscosity combines with the anomalous dispersion, so that also ultra low frequency waves are also attenuated and eventually cannot propagate at all, cf. again [31].

We express the system in terms of the linear momentum $\boldsymbol{p} = \varrho\boldsymbol{v}$ and, later, exploit the convexity of the kinetic energy $(\boldsymbol{p},\varrho) \mapsto \frac{1}{2}|\boldsymbol{p}|^2/\varrho$, in contrast to the nonconvex (although equivalent) form $(\boldsymbol{v},\varrho) \mapsto \frac{1}{2}\varrho|\boldsymbol{v}|^2$, cf.

[34]. The explicit use of the convexity of the kinetic energy expressed in terms of the momentum $\boldsymbol{p}$ is also in [8]. Exploiting that $\varrho\dot{\boldsymbol{v}} = \frac{\partial}{\partial t}(\varrho\boldsymbol{v}) + \mathrm{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v})$ granted by (2.7), we will deal with the system for $(\varrho, \boldsymbol{v}, \boldsymbol{E}, \theta)$ and thus also for $\boldsymbol{p}$ and u:

$$\frac{\partial\varrho}{\partial t} = -\mathrm{div}\,\boldsymbol{p} \quad \text{with} \quad \boldsymbol{p} = \varrho\boldsymbol{v}, \tag{2.22a}$$

$$\frac{\partial\boldsymbol{p}}{\partial t} = \mathrm{div}\big(\mathscr{T}(\boldsymbol{E},\theta)+\boldsymbol{D}-\boldsymbol{p}\otimes\boldsymbol{v}\big) + \varrho\boldsymbol{g} \quad \text{with} \quad \mathscr{T} \ \text{ from (2.4)}$$
$$\text{and} \quad \boldsymbol{D} = \mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}) - \mathrm{div}\mathfrak{H} \ \text{ with } \ \mathfrak{H} = \mu\big|\nabla^2\boldsymbol{v}\big|^{p-2}\nabla^2\boldsymbol{v}, \tag{2.22b}$$

$$\frac{\partial\boldsymbol{E}}{\partial t} = \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\Pi} - \boldsymbol{B}_{\rm ZJ}(\boldsymbol{v},\boldsymbol{E}) \ \text{ with } \ \boldsymbol{\Pi} = \mathscr{R}(\boldsymbol{E},\theta) \ \text{ where } \mathscr{R} \text{ is from (2.20)}, \tag{2.22c}$$

$$\frac{\partial u}{\partial t} = \mathrm{div}\big(\kappa(\theta)\nabla\theta - u\boldsymbol{v}\big) + \xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi}) + \tfrac{1}{3}\mathrm{tr}\big(\mathscr{T}(\boldsymbol{E},\theta)-\mathscr{T}(\boldsymbol{E},0)\big)\,\mathrm{div}\,\boldsymbol{v} \ \text{ with}$$
$$\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi}) \ \text{ from (2.21) and } \ u = \mathscr{U}(\boldsymbol{E},\theta) := \psi(\boldsymbol{E},\theta) - \theta\psi'_\theta(\boldsymbol{E},\theta) - \psi(\boldsymbol{E},0), \tag{2.22d}$$

where, in (2.22c), we have used a shorthand notation for the bi-linear operator involved in the Zaremba–Jaumann derivative:

$$\boldsymbol{B}_{\rm ZJ}(\boldsymbol{v},\boldsymbol{E}) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{E} - \mathrm{skew}(\nabla\boldsymbol{v})\boldsymbol{E} + \boldsymbol{E}\,\mathrm{skew}(\nabla\boldsymbol{v}). \tag{2.23}$$

The equations (2.22b) and (2.22d) are to be completed by the boundary conditions. For velocity, we allow for free slip but, as is common in the Eulerian formulation, we prescribe zero normal velocity (so that the shape of Ω does not evolve). This is here adapted for the multipolar 4th-order enhancement. For the temperature, we consider the Fourier-type boundary condition. Altogether,

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0, \quad [(\mathscr{T}(\boldsymbol{E},\theta)+\boldsymbol{D})\boldsymbol{n} + \mathrm{div}_{\rm S}(\mathfrak{H}\boldsymbol{n})]_{\rm T} = \boldsymbol{0}, \quad \mathfrak{H}\,{:}\,(\boldsymbol{n}\otimes\boldsymbol{n}) = \boldsymbol{0}, \quad \text{and}$$
$$\kappa(\theta)\nabla\theta\cdot\boldsymbol{n} + h(\theta) = h_{\rm ext} \quad \text{on} \quad \Gamma \tag{2.24}$$

with $h_{\rm ext}$ a prescribed external heat flux and with $h(\cdot)$ a temperature-dependent boundary heat out-flux. We consider the initial-value problem for (2.22) with the initial conditions

$$\varrho|_{t=0} = \varrho_0, \qquad \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0, \qquad \boldsymbol{E}|_{t=0} = \boldsymbol{E}_0, \quad \text{and} \quad \theta|_{t=0} = \theta_0. \tag{2.25}$$

Now, let us reveal the energy balance behind this system. To this aim, we test (2.22a) by $|\boldsymbol{p}|^2/(2\varrho^2)$ and (2.22b) by $\boldsymbol{p}/\varrho$, which yields

$$\frac{\partial}{\partial t}\Big(\frac{|\boldsymbol{p}|^2}{2\varrho}\Big) = \frac{\boldsymbol{p}}{\varrho}\cdot\frac{\partial\boldsymbol{p}}{\partial t} - \frac{|\boldsymbol{p}|^2}{2\varrho^2}\frac{\partial\varrho}{\partial t} = \frac{\boldsymbol{p}}{\varrho}\cdot\Big(\mathrm{div}\big(\boldsymbol{T}+\boldsymbol{D} - \boldsymbol{v}\otimes\boldsymbol{p}\big) + \varrho\boldsymbol{g}\Big) + \frac{|\boldsymbol{p}|^2}{2\varrho^2}\mathrm{div}\,\boldsymbol{p}. \tag{2.26}$$

Then, we use the calculus

$$\int_\Omega \frac{\boldsymbol{p}}{\varrho}\cdot\mathrm{div}\big(\boldsymbol{p}\otimes\boldsymbol{v}\big) - \frac{|\boldsymbol{p}|^2}{2\varrho^2}\mathrm{div}\,\boldsymbol{p}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{v}\cdot\mathrm{div}\big(\varrho\boldsymbol{v}\otimes\boldsymbol{v}\big) - \frac{|\boldsymbol{v}|^2}{2}\,\mathrm{div}(\varrho\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} = 0, \tag{2.27}$$

based on the Green formula with $\boldsymbol{v}\cdot\boldsymbol{n} = 0$ for the calculus

$$\int_\Omega \varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Gamma \varrho|\boldsymbol{v}|^2\,\underbrace{\boldsymbol{v}\cdot\boldsymbol{n}}_{=0}\,\mathrm{d}S - \int_\Omega \boldsymbol{v}\cdot\mathrm{div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}$$
$$= -\int_\Omega \varrho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{v} + \mathrm{div}(\varrho\boldsymbol{v})|\boldsymbol{v}|^2\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \mathrm{div}(\varrho\boldsymbol{v})\frac{|\boldsymbol{v}|^2}{2}\,\mathrm{d}\boldsymbol{x}. \tag{2.28}$$

Let us denote the mere *stored energy* part of ψ as $\varphi(\boldsymbol{E}) := \psi(\boldsymbol{E},0)$ and the corresponding part of the Cauchy stress tensor $\boldsymbol{T}$ as $\boldsymbol{T}_0 = \mathscr{T}(\boldsymbol{E},0) = \varphi'(\boldsymbol{E}) + \varphi(\boldsymbol{E})\mathbb{I}$. For this part of the Cauchy stress, we use the calculus

$$\begin{aligned}\int_\Omega(\mathrm{div}\,\boldsymbol{T}_0)\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} &= \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{T}_0\boldsymbol{n}\,\mathrm{d}S - \int_\Omega \boldsymbol{T}_0:\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \overset{(2.6)}{=} \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{T}_0\boldsymbol{n}\,\mathrm{d}S - \int_\Omega(\varphi'(\boldsymbol{E})+\varphi(\boldsymbol{E})\mathbb{I}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\\ &\overset{(2.1)}{=} \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{T}_0\boldsymbol{n}\,\mathrm{d}S - \int_\Omega \varphi'(\boldsymbol{E}):\Big(\frac{\partial\boldsymbol{E}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{E} - \boldsymbol{W}\boldsymbol{E} + \boldsymbol{E}\boldsymbol{W}\Big) - \mathrm{dev}\boldsymbol{T}_0:\mathscr{R}(\boldsymbol{E},\theta) + \varphi(\boldsymbol{E})\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}\\ &\overset{(2.31)}{\underset{(2.14)}{=}} \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{T}_0\boldsymbol{n}\,\mathrm{d}S - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi(\boldsymbol{E})\,\mathrm{d}\boldsymbol{x} - \int_\Omega [\zeta_\mathrm{p}]'_{\boldsymbol{\Pi}}(\theta,\boldsymbol{\Pi}):\boldsymbol{\Pi}\,\mathrm{d}\boldsymbol{x}. \qquad (2.29)\end{aligned}$$

Here, in addition to (2.19), we have also used that

$$\begin{aligned}\mathrm{dev}\,\boldsymbol{T}_0:\mathscr{R}(\boldsymbol{E},\theta) &\overset{(2.18)}{=} \mathrm{dev}\mathscr{T}(\boldsymbol{E},\theta):\mathscr{R}(\boldsymbol{E},\theta)\\ &= \mathrm{dev}\mathscr{T}(\boldsymbol{E},\theta):[\zeta_\mathrm{p}(\theta,\cdot)^*]'(\mathrm{dev}\mathscr{T}(\boldsymbol{E},\theta)) = [\zeta_\mathrm{p}]'_{\boldsymbol{\Pi}}(\theta,\boldsymbol{\Pi}):\boldsymbol{\Pi} \qquad (2.30)\end{aligned}$$

and the calculus

$$\int_\Omega \varphi'(\boldsymbol{E}):(\boldsymbol{v}\cdot\nabla)\boldsymbol{E} + \varphi(\boldsymbol{E})\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\varphi(\boldsymbol{E})\cdot\boldsymbol{v} + \varphi(\boldsymbol{E})\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \int_\Gamma \varphi(\boldsymbol{E})\,\underbrace{(\boldsymbol{v}\cdot\boldsymbol{n})}_{=0}\,\mathrm{d}S. \qquad (2.31)$$

Also, taking into account the form of the spin $\boldsymbol{W} = \frac{1}{2}\nabla\boldsymbol{v} - \frac{1}{2}(\nabla\boldsymbol{v})^\top$, we have used $\varphi'(\boldsymbol{E}):(\boldsymbol{W}\boldsymbol{E} - \boldsymbol{E}\boldsymbol{W}) = \boldsymbol{0}$, cf. (2.14).

Relying on the boundary conditions (2.24), we finally obtain the *energy-dissipation balance*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{\varphi(\boldsymbol{E})}_{\substack{\text{stored}\\ \text{energy}}}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \underbrace{\mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v})}_{\substack{\text{disipation rate due to}\\ \text{the Stokes viscosity}}} + \underbrace{\mu|\nabla^2\boldsymbol{v}|^p}_{\substack{\text{disipation rate due}\\ \text{to the hyper-viscosity}}} + \underbrace{[\zeta_\mathrm{p}]'_{\boldsymbol{\Pi}}(\theta,\boldsymbol{\Pi}):\boldsymbol{\Pi}}_{\substack{\text{disipation rate due to}\\ \text{Maxwellian viscosity}}}\mathrm{d}\boldsymbol{x}\\ = \int_\Omega \underbrace{\varrho\,\boldsymbol{g}\cdot\boldsymbol{v}}_{\substack{\text{power of}\\ \text{gravity field}}} - \underbrace{\tfrac{1}{3}\mathrm{tr}\big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big)\,\mathrm{div}\,\boldsymbol{v}}_{\text{power of adiabatic effects}}\,\mathrm{d}\boldsymbol{x}; \qquad (2.32)\end{aligned}$$

cf. also Formulas (2.10)–(2.13) of [34] for detailed handling of the boundary conditions by using Green's formula over Ω twice and the Green surface formula over Γ.

By adding (2.22d) tested by 1, we obtain the *total-energy balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{\mathscr{E}(\boldsymbol{E},\theta)}_{\substack{\text{internal}\\ \text{energy}}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \underbrace{h(\theta)}_{\substack{\text{power of}\\ \text{boundary}\\ \text{flux}}}\,\mathrm{d}S = \int_\Omega \underbrace{\varrho\,\boldsymbol{g}\cdot\boldsymbol{v}}_{\substack{\text{power of}\\ \text{gravity field}}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \underbrace{h_\mathrm{ext}}_{\substack{\text{external}\\ \text{heat}\\ \text{flux}}}\,\mathrm{d}S, \qquad (2.33)$$

expressing the *1st law of thermodynamics*: in particular, in isolated systems, the total energy (here as a sum of the kinetic and the internal energies) is conserved.

Furthermore, for the (here rather formal) test of (2.15) by the so-called *coldness* $1/\theta$, we need to assume positivity of θ and then use the calculus

$$\begin{aligned}\frac{1}{\theta}\Big(\frac{\partial e}{\partial t} + \mathrm{div}(e\boldsymbol{v})\Big) &= \frac{1}{\theta}\,\dot{\overline{\mathscr{E}(\boldsymbol{E},\theta)}} + \frac{\mathscr{E}(\boldsymbol{E},\theta)}{\theta}\,\mathrm{div}\,\boldsymbol{v}\\ &= \frac{\mathscr{E}'_\theta(\boldsymbol{E},\theta)}{\theta}\dot{\theta} + \frac{\mathscr{E}'_{\boldsymbol{E}}(\boldsymbol{E},\theta)}{\theta}:\dot{\boldsymbol{E}} + \frac{\mathscr{E}(\boldsymbol{E},\theta)}{\theta}\,\mathrm{div}\,\boldsymbol{v}\end{aligned}$$

$$
\begin{aligned}
&= \overline{\eta(\boldsymbol{E},\theta)}^{\boldsymbol{\cdot}} + \Big(\frac{\mathscr{E}'_{\boldsymbol{E}}(\boldsymbol{E},\theta)}{\theta} - \eta'_{\boldsymbol{E}}(\boldsymbol{E},\theta)\Big) : \dot{\boldsymbol{E}} + \frac{\mathscr{E}(\boldsymbol{E},\theta)}{\theta}\,\mathrm{div}\,\boldsymbol{v} \\
&= \overline{\eta(\boldsymbol{E},\theta)}^{\boldsymbol{\cdot}} + \eta(\boldsymbol{E},\theta)\mathrm{div}\,\boldsymbol{v} + \frac{\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta)}{\theta} : \dot{\boldsymbol{E}} + \frac{\psi(\boldsymbol{E},\theta)}{\theta}\,\mathrm{div}\,\boldsymbol{v} \\
&= \frac{\partial}{\partial t}\eta(\boldsymbol{E},\theta) + \mathrm{div}(\eta(\boldsymbol{E},\theta)\boldsymbol{v}) + \underbrace{\frac{\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta)}{\theta} : \dot{\boldsymbol{E}} + \frac{\psi(\boldsymbol{E},\theta)}{\theta}\,\mathrm{div}\,\boldsymbol{v}}_{=\,\boldsymbol{T}(\boldsymbol{E},\theta):(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\Pi})/\theta}. \qquad (2.34)
\end{aligned}
$$

Here, in addition to (2.14), we also used

$$
\eta'_\theta(\boldsymbol{E},\theta) = \frac{\mathscr{E}'_\theta(\boldsymbol{E},\theta)}{\theta} \quad \text{and} \quad \frac{\psi(\boldsymbol{E},\theta)}{\theta} - \frac{\mathscr{E}(\boldsymbol{E},\theta)}{\theta} = \psi'_\theta(\boldsymbol{E},\theta) = -\eta(\boldsymbol{E},\theta). \qquad (2.35)
$$

The positivity of θ mentioned above does not seem easy to prove, cf. also Remark 3.4 of [33], and we leave it open here. In any case, by the mentioned (formal) test of (2.13), we obtain the *entropy balance*:

$$
\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\boldsymbol{E},\theta)\,\mathrm{d}\boldsymbol{x}}_{\substack{\text{total}\\\text{entropy}}} = \int_\Omega \underbrace{\frac{\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})}{\theta}}_{\substack{\text{entropy}\\\text{production due}\\\text{to viscosity}}} + \underbrace{\kappa(\theta)\frac{|\nabla\theta|^2}{\theta^2}}_{\substack{\text{entropy produ-}\\\text{ction due to}\\\text{heat transfer}}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \underbrace{\frac{h_{\rm ext} - h(\theta)}{\theta}}_{\substack{\text{entropy flux}\\\text{through}\\\text{boundary}}}\,\mathrm{d}S \qquad (2.36)
$$

with the dissipation rate $\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})$ from (2.22d). The identity (2.36) expresses the *2nd law of thermodynamics*: in isolated systems, the total entropy is not decaying.

Using (2.19), the modification of (2.34) for $u = \mathscr{U}(\boldsymbol{E},\theta)$ instead of $e = \mathscr{E}(\boldsymbol{E},\theta)$ gives

$$
\frac{1}{\theta}\Big(\frac{\partial u}{\partial t} + \mathrm{div}(u\boldsymbol{v})\Big) = \frac{\partial}{\partial t}\eta(\boldsymbol{E},\theta) + \mathrm{div}(\eta(\boldsymbol{E},\theta)\boldsymbol{v}) + \frac{1}{3\theta}\mathrm{tr}(\boldsymbol{T}(\boldsymbol{E},\theta) - \boldsymbol{T}(\boldsymbol{E},0))\,\mathrm{div}\,\boldsymbol{v}. \qquad (2.37)
$$

The weak formulation of (2.22b) and (2.22d) uses the by-part integration in time and the Green formula in Ω, in the former equation even twice and still combined with a surface Green formula over Γ:

Definition 2.1 (Weak formulation of (2.22))**.** The quadruple $(\varrho,\boldsymbol{v},\boldsymbol{E},\theta)$ with $\varrho \in L^\infty(I\times\Omega)\cap W^{1,1}(I\times\Omega)$, $\boldsymbol{v} \in L^p(I;W^{2,p}(\Omega;\mathbb{R}^3))$, $\boldsymbol{E} \in W^{1,1}(I\times\Omega;\mathbb{R}^{3\times3}_{\rm sym})$, and $\theta \in L^1(I\times\Omega)$ such that $\mathrm{tr}(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)) \in L^{p'}(I;L^1(\Omega))$, $\mathscr{U}(\boldsymbol{E},\theta) \in L^{p'}(I;L^1(\Omega))$, and $\varkappa(\theta) \in L^1(I\times\Omega)$ is called a weak solution to the system (2.22) with the boundary conditions (2.24) and the initial conditions (2.25) if

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \bigg(\Big(\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta) + \mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}) - \varrho\boldsymbol{v}\otimes\boldsymbol{v}\Big):\boldsymbol{\varepsilon}(\widetilde{\boldsymbol{v}}) + \psi(\boldsymbol{E},\theta)\,\mathrm{div}\,\widetilde{\boldsymbol{v}} - \varrho\boldsymbol{v}\cdot\frac{\partial\widetilde{\boldsymbol{v}}}{\partial t} \\
&\qquad + \mu|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v}\,\vdots\,\nabla^2\widetilde{\boldsymbol{v}}\bigg)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \varrho\boldsymbol{g}\cdot\widetilde{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega \varrho_0\boldsymbol{v}_0\cdot\widetilde{\boldsymbol{v}}(0)\,\mathrm{d}\boldsymbol{x} \qquad (2.38a)
\end{aligned}
$$

holds for any $\widetilde{\boldsymbol{v}}$ with $\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n} = \boldsymbol{0}$ on $I\times\Gamma$ and $\widetilde{\boldsymbol{v}}(T) = 0$, and

$$
\begin{aligned}
&\int_0^T\!\!\int_\Omega \bigg(\mathscr{U}(\boldsymbol{E},\theta)\frac{\partial\widetilde{\theta}}{\partial t} + \mathscr{U}(\boldsymbol{E},\theta)\boldsymbol{v}\cdot\nabla\widetilde{\theta} - \varkappa(\theta)\Delta\widetilde{\theta} \\
&\qquad + \Big(\tfrac{1}{3}\mathrm{tr}\big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big)\,\mathrm{div}\,\boldsymbol{v} + \xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})\Big)\,\widetilde{\theta}\bigg)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_0^T\!\!\int_\Gamma h(\theta)\widetilde{\theta}\,\mathrm{d}S\,\mathrm{d}t + \int_\Omega \mathscr{U}(\boldsymbol{E}_0,\theta_0)\,\widetilde{\theta}(0)\,\mathrm{d}\boldsymbol{x} = \int_0^T\!\!\int_\Gamma h_{\rm ext}\widetilde{\theta}\,\mathrm{d}S\,\mathrm{d}t \qquad (2.38b)
\end{aligned}
$$

with $\varkappa(\theta) := \int_0^\theta \kappa(\vartheta)\,\mathrm{d}\vartheta$ and with $\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})$ from (2.22d) and $\boldsymbol{\Pi} = \mathscr{R}(\boldsymbol{E},\theta)$ holds for any $\tilde{\theta}$ smooth with $\widetilde{\theta}(T) = 0$ and $\boldsymbol{n}\cdot\nabla\widetilde{\theta} = 0$ on $I\times\Gamma$, and if (2.22a) and (2.22c) hold a.e. on $I\times\Omega$ together with the respective initial conditions for ϱ and $\boldsymbol{E}$ in (2.25).

Notably, due to the general embedding $W^{1,1}(I;X)\subset C(I;X)$, the initial conditions mentioned in the above definition for ϱ and $\boldsymbol{E}$ have a good sense in $L^1(\Omega)$ and $L^1(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$, respectively. In fact, it will be improved further in $W^{1,r}(\Omega)$ and $W^{1,s}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$ later on, relying on the estimates (4.21) and (4.49) below.

Remark 2.2 (The condition $\boldsymbol{v}\cdot\boldsymbol{n}=0$ in (2.24))**.** A common disadvantage of the Eulerian approach is that we have assumed the shape of the considered bounded domain Ω fixed by assuming zero normal velocity on the boundary. This simplification is commonly used in both theoretical and computational studies. In applications where the shape of the domain intended to evolve, this disadvantage can be overcome by embedding Ω into an artificial soft medium whose boundary is fixed, which is known as the sticky-air approach (in geophysics) or the fictitious-domain approach or the immersed-boundary method (in engineering).

3. Fully implicit time discretization of (2.22)

For simplicity, the viscosity coefficients $\mathbb{D}$ and μ in (2.22) are considered constant here, although their continuous dependence on θ and on $\boldsymbol{E}$ could easily be considered, too. Similarly, a continuous dependence of κ also on $\boldsymbol{E}$ could be considered for a stable scheme (*i.e.* for *a-priori* estimates), although convergence would then be analytically troublesome, cf. Remark 4.5.

We use the so-called *Rothe method*, *i.e.* the fully implicit time discretization with an equidistant partition of the time interval I with the time step $\tau>0$ such that T/τ is an integer. We denote by ϱ_τ^k, $\boldsymbol{v}_\tau^k$, $\boldsymbol{E}_\tau^k$, θ_τ^k, ... the approximate values of ϱ, $\boldsymbol{v}$, $\boldsymbol{E}$, θ, ... at time instants $t=k\tau$ with $k=1,2,\ldots,T/\tau$. Using $\mathscr{T}$ from (2.4) and $\mathscr{R}$ from (2.20), we will then use the following recursive regularized time-discrete scheme

$$\frac{\varrho_\tau^k-\varrho_\tau^{k-1}}{\tau}=-{\rm div}\,\boldsymbol{p}_\tau^k\qquad\text{with}\quad \boldsymbol{p}_\tau^k=\varrho_\tau^k\boldsymbol{v}_\tau^k,\tag{3.1a}$$

$$\begin{aligned}\frac{\boldsymbol{p}_\tau^k-\boldsymbol{p}_\tau^{k-1}}{\tau}&={\rm div}\big(\mathscr{T}\big(\boldsymbol{E}_\tau^k,\theta_\tau^k\big)+\boldsymbol{D}_\tau^k-\boldsymbol{p}_\tau^k\otimes\boldsymbol{v}_\tau^k\big)+\varrho_\tau^k\boldsymbol{g}_\tau^k,\\ &\quad\text{where}\ \ \boldsymbol{D}_\tau^k=\mathbb{D}\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big)-{\rm div}\mathfrak{H}_\tau^k\ \ \text{with}\ \ \mathfrak{H}_\tau^k=\mu\big|\nabla^2\boldsymbol{v}_\tau^k\big|^{p-2}\nabla^2\boldsymbol{v}_\tau^k,\end{aligned}\tag{3.1b}$$

$$\frac{\boldsymbol{E}_\tau^k-\boldsymbol{E}_\tau^{k-1}}{\tau}=\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big)-\boldsymbol{\Pi}_\tau^k-\boldsymbol{B}_{\rm ZJ}\big(\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k\big)\quad\text{with}\ \ \boldsymbol{\Pi}_\tau^k=\mathscr{R}\big(\boldsymbol{E}_\tau^k,\theta_\tau^k\big),\tag{3.1c}$$

$$\begin{aligned}\frac{u_\tau^k-u_\tau^{k-1}}{\tau}&={\rm div}\big(\kappa\big(\theta_\tau^k\big)\nabla\theta_\tau^k+u_\tau^k\boldsymbol{v}_\tau^k\big)+\xi_{\rm ext}\big(\theta_\tau^k;\boldsymbol{v}_\tau^k,\boldsymbol{\Pi}_\tau^k\big)\\ &\quad+\tfrac{1}{3}{\rm tr}\big(\mathscr{T}\big(\boldsymbol{E}_\tau^k,\theta_\tau^k\big)-\mathscr{T}\big(\boldsymbol{E}_\tau^k,0\big)\big)\,{\rm div}\,\boldsymbol{v}_\tau^k\quad\text{with}\quad u_\tau^k=\mathscr{U}\big(\boldsymbol{E}_\tau^k,\theta_\tau^k\big),\end{aligned}\tag{3.1d}$$

where $\xi_{\rm ext}$ is from (2.22d) and $\mathscr{U}(E,\theta)=\psi(E,\theta)-\theta\psi'_\theta(E,\theta)-\psi(E,0)$ is from (2.5).

The corresponding boundary conditions (2.24) lead to

$$\boldsymbol{v}_\tau^k\cdot\boldsymbol{n}=0,\quad\big[\big(\mathscr{T}\big(\boldsymbol{E}_\tau^k,\theta_\tau^k\big)+\boldsymbol{D}_\tau^k\big)\boldsymbol{n}-{\rm div}_{\rm S}\big(\mathfrak{H}_\tau^k\cdot\boldsymbol{n}\big)\big]_{\rm T}=\boldsymbol{0},\tag{3.2a}$$

$$\nabla^2\boldsymbol{v}_\tau^k:(\boldsymbol{n}\otimes\boldsymbol{n})=0,\quad\text{and}\quad\kappa\big(\theta_\tau^k\big)\nabla\theta_\tau^k\cdot\boldsymbol{n}+h\big(\theta_\tau^k\big)=h_{{\rm ext},\tau}^k,\tag{3.2b}$$

where $h_{{\rm ext},\tau}^k(\boldsymbol{x}):=\int_{(k-1)\tau}^{k\tau}h_{\rm ext}(t,\boldsymbol{x})\,{\rm d}t$. Similarly, in (3.1b), we used $\boldsymbol{g}_\tau^k(\boldsymbol{x}):=\int_{(k-1)\tau}^{k\tau}\boldsymbol{g}(t,\boldsymbol{x})\,{\rm d}t$.

This system of boundary-value problems (3.1) and (3.2) for the quadruple $(\varrho_\tau^k,\boldsymbol{p}_\tau^k,\boldsymbol{E}_\tau^k,\theta_\tau^k)$ and thus also for u_τ^k is to be solved recursively for $k=1,2,\ldots,T/\tau$, starting for $k=1$ with the initial conditions analogous to (2.25), *i.e.*

$$\varrho_\tau^0=\varrho_0,\qquad\boldsymbol{p}_\tau^0=\varrho_0\boldsymbol{v}_0,\qquad\boldsymbol{E}_\tau^0=\boldsymbol{E}_0,\quad\text{and}\quad u_\tau^0=\mathscr{U}(\boldsymbol{E}_0,\theta_0).\tag{3.3}$$

Thus, from (3.1a), we obtain also $\boldsymbol{v}_\tau^k=\boldsymbol{p}_\tau^k/\varrho_\tau^k$ provided that $\varrho_\tau^k>0$, as will indeed be proved in Step 2 below in the next section 4.2. In that section, we will also prove the existence of a solution to (3.1) with (3.2) in Step 5.

We will use the "compact" notation that exploits the interpolants. Specifically, using the values $(\boldsymbol{v}_\tau^k)_{k=0}^{T/\tau}$, we define the piecewise constant and the piecewise affine interpolants respectively as

$$\bar{\boldsymbol{v}}_\tau(t) := \boldsymbol{v}_\tau^k \quad \text{and} \quad \boldsymbol{v}_\tau(t) := \Big(\frac{t}{\tau} - k+1\Big)\boldsymbol{v}_\tau^k + \Big(k - \frac{t}{\tau}\Big)\boldsymbol{v}_\tau^{k-1} \quad \text{for} \quad (k-1)\tau < t \le k\tau \tag{3.4}$$

for $k = 0, 1, \ldots, T/\tau$. Analogously, we define also $\bar\varrho_\tau$, ϱ_τ, $\bar{\boldsymbol{p}}_\tau$, $\boldsymbol{p}_\tau$, $\bar{\boldsymbol{E}}_\tau$, $\boldsymbol{E}_\tau$, etc. Written "compactly" in terms of these interpolants, the recursive system (3.1) writes as

$$\frac{\partial\varrho_\tau}{\partial t} = -\mathrm{div}\,\bar{\boldsymbol{p}}_{\varepsilon\delta\tau} \quad \text{with} \quad \bar{\boldsymbol{p}}_\tau = \bar\varrho_\tau\bar{\boldsymbol{v}}_\tau, \tag{3.5a}$$

$$\frac{\partial\boldsymbol{p}_\tau}{\partial t} = \mathrm{div}\big(\mathscr{T}\big(\bar{\boldsymbol{E}}_\tau,\bar\theta_\tau\big)+\bar{\boldsymbol{D}}_\tau-\bar{\boldsymbol{p}}_\tau\otimes\bar{\boldsymbol{v}}_\tau\big) + \bar\varrho_\tau\bar{\boldsymbol{g}}_\tau,$$
$$\text{where} \quad \bar{\boldsymbol{D}}_{\varepsilon\delta\tau} = \mathbb{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau) - \mathrm{div}\bar{\mathfrak{H}}_\tau \quad \text{with} \quad \bar{\mathfrak{H}}_\tau = \mu\big|\nabla^2\bar{\boldsymbol{v}}_\tau\big|^{p-2}\nabla^2\bar{\boldsymbol{v}}_\tau, \tag{3.5b}$$

$$\frac{\partial\boldsymbol{E}_\tau}{\partial t} = \boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau) - \bar{\boldsymbol{\Pi}}_\tau - \boldsymbol{B}_{\rm ZJ}\big(\bar\theta_\tau,\bar{\boldsymbol{E}}_\tau\big) \quad \text{with} \quad \bar{\boldsymbol{\Pi}}_\tau = \mathscr{R}\big(\bar{\boldsymbol{E}}_\tau,\bar\theta_\tau\big), \tag{3.5c}$$

$$\frac{\partial u_\tau}{\partial t} = \mathrm{div}\big(\kappa\big(\bar\theta_\tau\big)\nabla\bar\theta_\tau-\bar u_\tau\bar{\boldsymbol{v}}_\tau\big) + \xi_{\rm ext}\big(\bar\theta_\tau;\bar{\boldsymbol{v}}_\tau,\bar{\boldsymbol{\Pi}}_\tau\big) \tag{3.5d}$$

$$+ \tfrac13\mathrm{tr}\big(\mathscr{T}\big(\bar{\boldsymbol{E}}_{\varepsilon\delta\tau},\bar\theta_\tau\big) - \mathscr{T}\big(\bar{\boldsymbol{E}}_\tau,0\big)\big)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau \quad \text{with} \quad \bar u_\tau = \mathscr{U}\big(\bar{\boldsymbol{E}}_\tau,\bar\theta_\tau\big), \tag{3.5e}$$

where the dissipation rate $\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})$ is from (2.22d). Of course, the boundary conditions (3.2) are "translated" accordingly, *i.e.*

$$\bar{\boldsymbol{v}}_\tau\cdot\boldsymbol{n} = 0, \quad \big[(\mathscr{T}(\bar{\boldsymbol{E}}_\tau,\bar\theta_\tau)+\bar{\boldsymbol{D}}_\tau)\boldsymbol{n} + \mathrm{div}_{\rm S}(\bar{\mathfrak{H}}_\tau\boldsymbol{n})\big]_{\rm T} = \boldsymbol{0}, \tag{3.6a}$$

$$\bar{\mathfrak{H}}_\tau : (\boldsymbol{n}\otimes\boldsymbol{n}) = \boldsymbol{0}, \quad \text{and} \quad \kappa(\bar\theta_\tau)\nabla\bar\theta_\tau\cdot\boldsymbol{n} + h(\bar\theta_\tau) = \bar h_{{\rm ext},\tau} \quad \text{on} \quad \varGamma, \tag{3.6b}$$

and with the corresponding initial conditions for $(\varrho_\tau, \boldsymbol{p}_\tau, \boldsymbol{E}_\tau, u_\tau)$ arising for $t = 0$ from (3.3).

For the mechanical part (2.22a)–(2.22c) alone, such a time discretization was used for the compressible (isothermal) Navier–Stokes equations in [17, 39] or Chapter 7 of [5]. The explicit use of the convexity of the kinetic energy expressed in terms of the momentum $\boldsymbol{p}$ is in [8].

4. Analysis of the time-discrete scheme (3.1) in a special case

Proving the stability of the discrete scheme (3.1) is quite technical and it is worth simplifying by making the heat capacity independent of the strain variable $\boldsymbol{E}$.

4.1. A special ansatz of partly linearized thermomechanical coupling

Without significant loss of applicability, we consider a commonly used special form

$$\psi(\boldsymbol{E},\theta) = \varphi(\boldsymbol{E}) + \theta\,\phi(\mathrm{tr}\,\boldsymbol{E}) + \gamma(\theta) \tag{4.1}$$

with $\gamma(0) = 0$, which will allow for a relatively simple estimation strategy by exploting the entropy balance. It is physically natural to consider also $\gamma'(0) = 0$. The ansatz (4.1) implies that $\mathrm{dev}(\psi''_{\boldsymbol{E}\theta}(\boldsymbol{E},\theta)) = \mathrm{dev}(\phi'(\mathrm{tr}\,\boldsymbol{E})\mathbb{I}) \equiv 0$ so, in particular, it complies with (2.16). Also, the adiabatic power (2.19) takes a specific form as

$$\tfrac13\mathrm{tr}\big(\mathscr{T}(\boldsymbol{E},\theta) - \mathscr{T}(\boldsymbol{E},0)\big)\,\mathrm{div}\,\boldsymbol{v} = \big(\theta\,\phi'(\mathrm{tr}\,\boldsymbol{E}) + \theta\,\phi(\mathrm{tr}\,\boldsymbol{E}) + \gamma(\theta)\big)\,\mathrm{div}\,\boldsymbol{v}. \tag{4.2}$$

The ansatz (4.1) has the effect of separating the mechanical and the thermal variables additively in the internal energy $\mathscr{E}$ and the entropy η defined in (2.5). Here it results in:

$$\mathscr{E}(\boldsymbol{E},\theta) = \varphi(\boldsymbol{E}) + \underbrace{\gamma(\theta) - \theta\gamma'(\theta)}_{=:\,\mathscr{U}(\theta)} \quad \text{and} \quad \eta(\boldsymbol{E},\theta) = -\,\phi(\mathrm{tr}\,\boldsymbol{E})\underbrace{-\,\gamma'(\theta)}_{=:\,\eta_1(\theta)}; \tag{4.3}$$

note that the thermal part of the internal energy $\mathscr{U}(\theta)=\mathscr{E}(\boldsymbol{E},\theta)-\mathscr{E}(\boldsymbol{E},0)=\mathscr{E}(\boldsymbol{E},\theta)-\varphi(\boldsymbol{E})$ here depends only on temperature, *i.e.* $\mathscr{U}_{\boldsymbol{E}}'=0$. Likewise, the heat capacity $\mathscr{E}_\theta'(\boldsymbol{E},\theta)=\mathscr{U}'(\theta)=-\theta\gamma''(\theta)=:c(\theta)$ depends only on temperature. Also, the thermal part of the entropy $\eta_1(\theta)=\eta(\boldsymbol{E},\theta)-\eta(\boldsymbol{E},0)$ depends only on temperature. This simplifies many calculations but it slightly corrupts the thermodynamical consistency (cf. Rem. 4.1 below) no matter how often such models are used in the literature.

Given this special ansatz (4.1), we have the following calculus

$$\begin{aligned}\frac{1}{\theta}\bigg(\frac{\partial u}{\partial t}+\mathrm{div}(u\boldsymbol{v})\bigg)&=\frac{1}{\theta}\bigg(\frac{\partial}{\partial t}\mathscr{U}(\theta)+\mathrm{div}(\mathscr{U}(\theta)\boldsymbol{v})\bigg)\\&=\frac{1}{\theta}\frac{\partial}{\partial t}(\gamma(\theta)-\theta\gamma'(\theta))+\frac{1}{\theta}\mathrm{div}((\gamma(\theta)-\theta\gamma'(\theta))\boldsymbol{v})\\&=-\frac{\partial}{\partial t}\gamma'(\theta)-\mathrm{div}(\gamma'(\theta)\boldsymbol{v})+\frac{\gamma(\theta)}{\theta}\mathrm{div}\,\boldsymbol{v}.\end{aligned}\tag{4.4}$$

Reminding the thermal part of the entropy $\eta_1=\eta_1(\theta)=-\gamma'(\theta)$ from (4.3) and testing (2.22d) by the coldness $1/\theta$, we obtain the balance of η_1 as

$$\frac{\partial}{\partial t}\eta_1(\theta)=\frac{\mathrm{div}(\kappa(\theta)\nabla\theta)}{\theta}-\mathrm{div}(\eta_1(\theta)\boldsymbol{v})+\frac{\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})}{\theta}+\underbrace{\bigg(\mathrm{tr}\frac{\mathscr{T}(\boldsymbol{E},\theta)-\mathscr{T}(\boldsymbol{E},0)}{3\theta}-\frac{\gamma(\theta)}{\theta}\bigg)}_{=\,\phi'(\mathrm{tr}\,\boldsymbol{E})+\phi(\mathrm{tr}\,\boldsymbol{E})}\mathrm{div}\,\boldsymbol{v}\tag{4.5}$$

with the heat power $\xi_{\rm ext}=\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})$ from (2.22d).

Remark 4.1 (3rd law of thermodynamics)**.** A physically relevant requirement is that the entropy at zero temperature is bounded from below and independent of the mechanical state. This is called the 3rd law of thermodynamics. By default, the entropy at zero temperature $\eta(\cdot,0)$ is thus calibrated to zero. It should be openly pointed out that the ansatz (4.1) satisfies this 3rd law of thermodynamic only partially. Namely, the entropy $\eta(\boldsymbol{E},\theta)=\eta_1(\theta)-\phi(\mathrm{tr}\,\boldsymbol{E})$ is bounded from below at zero absolute temperature when proving that the small-strain field $\boldsymbol{E}$ and hence also ϕ is bounded, as we will indeed proved later; recall that $\gamma'(0)$ is assumed to be bounded (even zero). However, if $\phi(\cdot)$ is not constant, then $\eta(\cdot,0)$ is not constant either, and thus does not fully comply with the third law.

4.2. Stability and convergence of the time-discrete scheme (3.1)

We will now perform the analysis for the special ansatz (4.1).

Recalling the notation $\mathscr{E}$ and $\mathscr{U}$ from (2.5) and $\mathscr{R}$ from (2.20), let us summarize the assumptions used in what follows. For suitable exponents α and β, which will be specified later in (4.7) and which refer, respectively, to the polynomial growth of the heat capacity and heat conductivity, we assume

$$\psi\in C^2(\mathbb{R}^{3\times3}_{\rm sym}\times\mathbb{R}),\qquad \kappa\in C(\mathbb{R}),\qquad h\in C(\mathbb{R}),\qquad \phi(\cdot,0)\ \text{ convex},\tag{4.6a}$$

$$\begin{aligned}&\exists\,0<c_0\le C_1<+\infty\quad\forall(\boldsymbol{E},\theta)\in\mathbb{R}^{3\times3}_{\rm sym}\times\mathbb{R}:\\&\qquad\mathscr{E}(\boldsymbol{E},\theta)\ge c_0(|\boldsymbol{E}|^2+(\theta^+)^{1+\alpha})\ \text{ and }\ \mathscr{E}_\theta'(\boldsymbol{E},\cdot)>0\ \text{ on }(0,+\infty),\\&\qquad|\mathscr{E}_{\boldsymbol{E}}'(\boldsymbol{E},\theta)|\le C_1\big(1{+}|\boldsymbol{E}|\big)\ \text{ and }\ \theta^+\mathscr{E}_\theta'(\boldsymbol{E},\theta)\le C_1\big(1{+}(\theta^+)^{1+\alpha}\big),\end{aligned}\tag{4.6b}$$

$$\max\big(|\eta(\boldsymbol{E},0)|,\,|\eta_{\boldsymbol{E}}'(\boldsymbol{E},0)|\big)\le C_1\big(1{+}|\boldsymbol{E}|^{C_1}\big),\tag{4.6c}$$

$$\big|\mathscr{T}(\boldsymbol{E},\theta)\big|\le C_1\big(1+\mathscr{E}(\boldsymbol{E},\theta)\big),\tag{4.6d}$$

$$\max\big(|\mathscr{R}(\boldsymbol{E},\theta)|,\,|\mathscr{R}_\theta'(\boldsymbol{E},\theta)|\big)\le C_1(1{+}|\boldsymbol{E}|)\ \text{ and }\mathscr{R}_{\boldsymbol{E}}'\ \text{ positive semi-definite},\tag{4.6e}$$

$$\mathbb{D}:\mathbb{R}^{3\times3}_{\rm sym}\to\mathbb{R}^{3\times3}_{\rm sym}\ \text{ linear symmetric positive semi-definite},\qquad \mu>0,\tag{4.6f}$$

$$\exists\,\kappa_{\max}\ge\kappa_{\min}>0\ \ \forall\,\theta\in\mathbb{R}:\quad \kappa_{\min}(\theta^+)^{\beta^+}\le\kappa(\theta)\le\kappa_{\max}\big(1{+}(\theta^+)^{\beta}\big),\tag{4.6g}$$

$$\zeta_{\rm p}\in C(\mathbb{R}\times\mathbb{R}^{3\times3}_{\rm dev}),\quad \theta\in\mathbb{R}:\quad \zeta_{\rm p}(\theta,\cdot):\mathbb{R}^{3\times3}_{\rm dev}\to\mathbb{R}\ \text{ convex},$$

$$0 < \inf_{\theta\in\mathbb{R}^+,\ \boldsymbol{\Pi}\in\mathbb{R}^{3\times3}_{\rm dev}\setminus\{0\}} \frac{\zeta_{\rm p}(\theta,\boldsymbol{\Pi})}{|\boldsymbol{\Pi}|^2} \quad\text{ and }\quad \sup_{\theta\in\mathbb{R}^+,\ \boldsymbol{\Pi}\in\mathbb{R}^{3\times3}_{\rm dev}\setminus\{0\}} \frac{\zeta_{\rm p}(\theta,\boldsymbol{\Pi})}{1+|\boldsymbol{\Pi}|^2} < +\infty, \tag{4.6h}$$

$$h:\mathbb{R}^+\to\mathbb{R}^+ \text{ increasing},\quad h(0)=0,\quad h_{\rm ext}\in L^1(I{\times}\Gamma),\quad h_{\rm ext}\ge 0 \text{ a.e. on } I{\times}\Gamma, \tag{4.6i}$$

$$\varrho_0\in W^{1,r}(\Omega),\quad \min\nolimits_{\overline\Omega}\varrho_0>0,\quad \boldsymbol{v}_0\in L^2(\Omega;\mathbb{R}^3),\quad \boldsymbol{g}\in L^1(I;L^\infty(\Omega;\mathbb{R}^3)), \tag{4.6j}$$

$$\boldsymbol{E}_0\in W^{1,\infty}(\Omega;\mathbb{R}^{3\times3}_{\rm sym}),\quad \theta_0\in L^\alpha(\Omega),\quad \theta_0>0 \text{ on } \Omega,\quad \mathscr{E}(\boldsymbol{E}_0,\theta_0)\in L^1(\Omega). \tag{4.6k}$$

An example for h complying with (4.6i), occurred in (2.24), is $h(\theta)=\alpha_1\theta+\alpha_1\theta^4$ with $\alpha_1\ge 0$, $\alpha_2\ge 0$, and $\alpha_1+\alpha_2>0$ should hold for h to be increasing. Yet, in fact, the an analysis in (4.36) allows for $h\equiv 0$, too.

Proposition 4.2 (Stability of the discrete scheme and solutions to (3.1))**.** *For the ansatz (4.1) leading, for some $0<\lambda<2$, to the convex function $1/\mathscr{U}(\cdot)^\lambda$, let the assumptions (2.12), (2.16), and (4.6) hold with $p>r>3$ and the exponents α and β in (4.6b) and (4.6g) satisfy*

$$1+\lambda>\beta^+\ge\frac{2}{3}\alpha+\lambda-\frac{1}{3}\quad\text{ and }\quad \alpha\ge\Big(\frac{3}{2}\lambda-1\Big)^+. \tag{4.7}$$

Moreover, let $h_{\rm ext}/h^{-1}(h_{\rm ext})^\lambda\in L^1(I{\times}\Gamma)$. Then:

- (i) *The time-discrete scheme (3.5) possesses a solution $(\varrho_\tau,\boldsymbol{v}_\tau,\boldsymbol{E}_\tau,\theta_\tau)$ and is stable (in the spaces specified later within the proof) with respect to $\tau>0$ provided τ is sufficiently small.*
- (ii) *For $\tau\to0$, $(\varrho_\tau,\boldsymbol{v}_\tau,\boldsymbol{E}_\tau,\theta_\tau)$ converges weakly* (in terms of subsequences) in the topologies specified in (4.58) and (4.61a) below and every such a limit $(\varrho,\boldsymbol{v},\boldsymbol{E},\theta)$ solves (in a weak sense of Definition 2.1) the initial-boundary-value problem in the sense.*
- (iii) *In particular, (2.22) has at least one weak solution in the sense of Definition 2.1 such that also $\varrho\in L^\infty(I;W^{1,r}(\Omega))\cap C(I{\times}\overline\Omega)$ with $\min_{I\times\overline\Omega}\varrho>0$, $\boldsymbol{v}\in L^\infty(I;L^2(\Omega;\mathbb{R}^3))$, $\boldsymbol{E}\in L^\infty(I;W^{1,s}(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$ with any $1\le s<\mu$, and $\theta\in L^\infty(I;L^{1+\alpha}(\Omega))\cap L^\mu(I;W^{1,\mu}(\Omega))$ with*

$$\mu\le\frac{5+2\alpha+3\beta^+-3\lambda}{4+\alpha}. \tag{4.8}$$

- (vi) *Moreover, the energy-dissipation balance (2.32) and the total-energy balance (2.33) integrated over the time interval $[0,t]$ holds for any $t\in I$.*

Proof. We expand and modify the arguments from Section 2 of [34], where the isothermal variant with $\zeta_{\rm p}'=\zeta_{\rm p}'(\boldsymbol{\Pi})$ linear was handled. Compared to [34], the discrete scheme is substantially simpler here without any gradient regularization of the equation (3.1c) for $\boldsymbol{E}$. Here we will need A careful usage of the discrete Gronwall inequality, for which we refer *e.g.* to [28,29,35], is to be used here. Moreover, unlike [34], we avoid any regularization of (3.1a) and (3.1b). For the heat part, which is completely absent in [34], we use the test of a generalized-entropy type, similarly as in [1] in the incompressible case with $\alpha=0=\beta$.

For the sake of clarity, we divide the following argumentation into seven steps, and partly use some calculations for the mechanical part in Section 2 of [34].

Step 1: Basic stability of (3.1) and first a-priori estimates from the total energy. Using the convexity of the kinetic energy as the functional $(\varrho,\boldsymbol{p})\mapsto\int_\Omega\frac12|\boldsymbol{p}|^2/\varrho\,{\rm d}\boldsymbol{x}$, cf. Section 2.4 of [34], and the assumed convexity (4.6a) of the stored energy φ, we now test (3.1b) by $\boldsymbol{v}_\tau^k=\boldsymbol{p}_\tau^k/\varrho_\tau^k$ while using (3.1a) tested by $\frac12|\boldsymbol{p}_\tau^k|/(\varrho_\tau^k)^2$ and (3.1c) tested by $\varphi'(\boldsymbol{E}_\tau^k)$, and derive the discrete analog of (2.32) an inequality:

$$\int_\Omega\frac{\big|\boldsymbol{p}_\tau^k\big|^2}{2\varrho_\tau^k}+\varphi\big(\boldsymbol{E}_\tau^k\big)\,{\rm d}\boldsymbol{x}+\tau\sum_{m=1}^k\int_\Omega\xi_{\rm ext}(\theta_\tau^m;\boldsymbol{v}_\tau^m,\boldsymbol{\Pi}_\tau^m)\,{\rm d}\boldsymbol{x}\le\int_\Omega\frac{\big|\boldsymbol{p}_\tau^0\big|^2}{2\varrho_\tau^0}+\varphi\big(\boldsymbol{E}_\tau^0\big)\,{\rm d}\boldsymbol{x}$$
$$+\tau\sum_{m=1}^k\int_\Omega\big(\varrho_\tau^m\boldsymbol{g}_{\varepsilon\delta}^m{\cdot}\boldsymbol{v}_\tau^m-(\theta_\tau^m\phi'({\rm tr}\,\boldsymbol{E}_\tau^m)+\theta_\tau^m\phi({\rm tr}\,\boldsymbol{E}_\tau^m)+\gamma(\theta_\tau^m))\,{\rm div}\,\boldsymbol{v}_\tau^m\big)\,{\rm d}\boldsymbol{x}. \tag{4.9}$$

In addition, we also perform the test of (3.1d) by 1. This leads to the discrete analog of the total-energy balance (2.33), namely

$$\int_\Omega \frac{\big|\boldsymbol{p}_\tau^k\big|^2}{2\varrho_\tau^k} + \underbrace{\varphi\big(\boldsymbol{E}_\tau^k\big) + \mathscr{U}\big(\theta_\tau^k\big)}_{=\mathscr{E}(\boldsymbol{E}_\tau^k,\theta_\tau^k)}\mathrm{d}\boldsymbol{x} + \tau\sum_{m=1}^k\int_\Gamma h\big(\theta_\tau^k\big)\,\mathrm{d}S$$
$$\leq \int_\Omega \frac{|\boldsymbol{p}_0|^2}{2\varrho_0} + \mathscr{E}(\boldsymbol{E}_0,\theta_0)\,\mathrm{d}\boldsymbol{x} + \tau\sum_{m=1}^k\bigg(\int_\Omega \varrho_\tau^m\boldsymbol{g}_\tau^m\cdot\boldsymbol{v}_\tau^m\,\mathrm{d}\boldsymbol{x} + \int_\Gamma h_{\mathrm{ext},\tau}^k\,\mathrm{d}S\bigg), \tag{4.10}$$

where $\mathscr{E}(\boldsymbol{E},\theta) := \varphi(\boldsymbol{E}) + \mathscr{U}(\theta)$ as in (4.3). To obtain some *a-priori* estimates, we handle the gravity-loading term $\varrho_\tau^m\boldsymbol{g}_\tau^m$ by using the continuity equation (2.22a) and the impenetrability of the boundary, which guarantees

$$\int_\Omega \varrho_\tau^k\,\mathrm{d}\boldsymbol{x} = \int_\Omega \varrho_0\,\mathrm{d}\boldsymbol{x} =: M \qquad \text{for any } k = 1,\dots,T/\tau. \tag{4.11}$$

Let us imagine, for a moment, that (3.1a) is replaced by $\varrho_\tau^k - \varrho_\tau^{k-1} = \tau\mathrm{div}((\varrho_\tau^k)^+\boldsymbol{v}_\tau^k)$ with $(\cdot)^+$ denoting the non-negative part. By testing it by the non-positive part $(\varrho_\tau^k)^-$ of ϱ_τ^k and realizing that $\mathrm{div}((\varrho_\tau^k)^+\boldsymbol{v}_\tau^k)(\varrho_\tau^k)^- = 0$ a.e., we can see that $\varrho_\tau^k \geq 0$ if also $\varrho_\tau^{k-1} \geq 0$. Therefore, $(\varrho_\tau^k)^+ = \varrho_\tau^k$ so that, in fact, we arrive at (3.1a). For another argumentation more directly for the original equation (3.1a), we refer to (4.22) below. Then, by exploiting that $\varrho_\tau^k \geq 0$, we can estimate

$$\begin{aligned}\int_\Omega \varrho_\tau^m\boldsymbol{g}_\tau^m\cdot\boldsymbol{v}_\tau^m\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \boldsymbol{g}_\tau^m\cdot\boldsymbol{p}_\tau^m\,\mathrm{d}\boldsymbol{x} \leq \big\|\sqrt{\varrho_\tau^m}\boldsymbol{g}_\tau^m\big\|_{L^2(\Omega;\mathbb{R}^3)}\bigg\|\frac{\boldsymbol{p}_\tau^m}{\sqrt{\varrho_\tau^m}}\bigg\|_{L^2(\Omega;\mathbb{R}^3)}\\
&\leq \|\varrho_\tau^m\|_{L^1(\Omega)}^{1/2}\|\boldsymbol{g}_\tau^m\|_{L^\infty(\Omega;\mathbb{R}^3)}\bigg\|\frac{\boldsymbol{p}_\tau^m}{\sqrt{\varrho_\tau^m}}\bigg\|_{L^2(\Omega;\mathbb{R}^3)}\\
&\leq M^{1/2}\|\boldsymbol{g}_\tau^m\|_{L^\infty(\Omega;\mathbb{R}^3)}\bigg(1 + \frac14\bigg\|\frac{\boldsymbol{p}_\tau^m}{\sqrt{\varrho_\tau^m}}\bigg\|_{L^2(\Omega;\mathbb{R}^3)}^2\bigg)\\
&= M^{1/2}\|\boldsymbol{g}_\tau^m\|_{L^\infty(\Omega;\mathbb{R}^3)}\bigg(1 + \frac14\int_\Omega \varrho_\tau^m|\boldsymbol{v}_\tau^m|^2\,\mathrm{d}\boldsymbol{x}\bigg).\end{aligned} \tag{4.12}$$

Furthermore, we need to know that $\theta_\tau^k \geq 0$, which can be seen by testing (3.1d) with $(\theta_\tau^k)^-$ and by relying on that $\theta_\tau^{k-1} \geq 0$ so that also $\mathscr{U}(\theta_\tau^{k-1}) \geq 0$. Then, relying on $L^\infty(I\times\Omega;\mathbb{R}^3)$-bound of $\boldsymbol{g}$, we can use the discrete Gronwall inequality to obtain the *a-priori* estimates

$$\bigg\|\frac{\bar{\boldsymbol{p}}_\tau}{\sqrt{\bar\varrho_\tau}}\bigg\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))} \leq C, \tag{4.13a}$$

and a similar $L^\infty(I;L^1(\Omega))$-estimate of $\mathscr{E}(\overline{\boldsymbol{E}}_\tau,\bar\theta_\tau)$. By using the coercivity assumption in (4.6b), we also obtain

$$\big\|\overline{\boldsymbol{E}}_\tau\big\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{3\times3}))} \leq C \qquad \text{and} \qquad \big\|\bar\theta_\tau\big\|_{L^\infty(I;L^{1+\alpha}(\Omega))} \leq C. \tag{4.13b}$$

Thus, by exploiting (4.6d), we obtain an estimate for $\overline{\boldsymbol{T}}_\tau = \mathscr{T}(\overline{\boldsymbol{E}}_\tau,\bar\theta_\tau)$:

$$\big\|\overline{\boldsymbol{T}}_\tau\big\|_{L^\infty(I;L^1(\Omega;\mathbb{R}^{3\times3}))} \leq C. \tag{4.13c}$$

Moreover, from the first estimate in (4.13a), we have also

$$\|\bar{\boldsymbol{p}}_\tau\|_{L^\infty(I;L^1(\Omega;\mathbb{R}^3))} \leq \underbrace{\Big\|\sqrt{\bar\varrho_\tau}\Big\|_{L^\infty(I;L^2(\Omega))}}_{=M^{1/2}}\bigg\|\frac{\bar{\boldsymbol{p}}_\tau}{\sqrt{\bar\varrho_\tau}}\bigg\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^3))} \leq C. \tag{4.13d}$$

It should also be emphasized that all the estimates (4.13) hold only for sufficiently small $\tau > 0$ for which the discrete Gronwall inequality can be applied. Specifically they hold for all $\tau \le 1/(M^{1/2}\|\boldsymbol{g}\|_{L^\infty(I\times\Omega;\mathbb{R}^3)})$.

Step 2: Further estimates from the mechanical energy dissipation. We will now exploit (4.9) itself. Due to (4.6d) and the already obtained bounds, we have that $\mathscr{T}(\boldsymbol{E}_\tau^m,\theta_\tau^m)$ is bounded in $L^1(\Omega;\mathbb{R}^{3\times3})$ uniformly in m. Moreover, the growth of $\mathscr{E}(\cdot,0)$ in (4.6b) with (4.13b) ensures also that $\mathscr{T}(\boldsymbol{E}_\tau^m,0)$ is bounded in $L^1(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$. Altogether, $\mathcal{T}_\tau^m := {\rm tr}(\mathscr{T}(\boldsymbol{E}_\tau^m,\theta_\tau^m)-\mathscr{T}(\boldsymbol{E}_\tau^m,0))$ is bounded in $L^1(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$ uniformly in m. Thus we can estimate the power of the adiabatic effects as

$$
\begin{aligned}
-\int_\Omega \mathcal{T}_\tau^m\,{\rm div}\,\boldsymbol{v}_\tau^m\,{\rm d}\boldsymbol{x} &\le \|\mathcal{T}_\tau^m\|_{L^1(\Omega)}\|{\rm div}\,\boldsymbol{v}_\tau^m\|_{L^\infty(\Omega)}\\
&\le C\|\mathcal{T}_\tau^m\|_{L^1(\Omega;\mathbb{R}^{3\times3})}\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^m)\|_{L^2(\Omega;\mathbb{R}^{3\times3})}+\|\nabla^2\boldsymbol{v}_\tau^m\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}\Big)\\
&\le C_\delta \max_{n=1,\dots,T/\tau}\|\mathcal{T}_\tau^n\|_{L^1(\Omega;\mathbb{R}^{3\times3})}+\delta\|\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^m)\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})}+\delta\|\nabla^2\boldsymbol{v}_\tau^m\|^p_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}
\end{aligned}
\tag{4.14}
$$

with $\delta > 0$ sufficiently small with respect to $\inf_{|E|=1}\mathbb{D}E:E > 0$ and to $\mu > 0$. The power of the gravity in (4.9) can be treated by emploing the information about the total mass as in (4.12). Thus, from (4.9), we obtain the estimate for the dissipation rate. From this, we can read the estimates

$$
\|\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau)\|_{L^2(I\times\Omega;\mathbb{R}^{3\times3})}\le C \quad\text{ and }\quad \|\nabla^2\bar{\boldsymbol{v}}_\tau\|_{L^p(I\times\Omega;\mathbb{R}^{3\times3\times3})}\le C. \tag{4.15}
$$

Then (3.1a) can be tested by $|\varrho_\tau^k|^{s-2}\varrho_\tau^k = (\varrho_\tau^k)^{s-1}$ with some $s > 1$. Using the Green formula with the boundary condition $\boldsymbol{n}\cdot\boldsymbol{v}_\tau^k = 0$, the convective term can be handled as

$$
\begin{aligned}
\int_\Omega {\rm div}\big(\varrho_\tau^k\boldsymbol{v}_\tau^k\big)\Big(\big|\varrho_\tau^k\big|^{s-2}\varrho_\tau^k\Big)\,{\rm d}\boldsymbol{x} &= \int_\Omega\big({\rm div}\,\boldsymbol{v}_\tau^k\big)\big|\varrho_\tau^k\big|^s + \big(\boldsymbol{v}_\tau^k\cdot\nabla\varrho_\tau^k\big)\Big(\big|\varrho_\tau^k\big|^{s-2}\varrho_\tau^k\Big)\,{\rm d}\boldsymbol{x}\\
&= \int_\Omega\big({\rm div}\,\boldsymbol{v}_\tau^k\big)\big|\varrho_\tau^k\big|^s - \varrho_\tau^k\,{\rm div}\Big(\big|\varrho_\tau^k\big|^{s-2}\varrho_\tau^k\boldsymbol{v}_\tau^k\Big)\,{\rm d}\boldsymbol{x} = \Big(1-\frac{1}{s}\Big)\int_\Omega\big({\rm div}\,\boldsymbol{v}_\tau^k\big)\big|\varrho_\tau^k\big|^s{\rm d}\boldsymbol{x},
\end{aligned}
\tag{4.16}
$$

so that this test gives the inequality

$$
\int_\Omega\frac{\big|\varrho_\tau^k\big|^s-\big|\varrho_\tau^{k-1}\big|^s}{\tau}\,{\rm d}\boldsymbol{x} \le \frac{1}{s'}\int_\Omega\big({\rm div}\,\boldsymbol{v}_\tau^k\big)\big|\varrho_\tau^k\big|^s\,{\rm d}\boldsymbol{x} \le \frac{1}{s'}\big\|{\rm div}\,\boldsymbol{v}_\tau^k\big\|_{L^\infty(\Omega)}\int_\Omega\big|\varrho_\tau^k\big|^s\,{\rm d}\boldsymbol{x}. \tag{4.17}
$$

Due to (4.13a) with $p > 3$, we have $\tau\sum_{k=1}^{T/\tau}\|{\rm div}\,\boldsymbol{v}_\tau^k\|^p_{L^\infty(\Omega)} \le C$, from which we can see that $\max_{1\le k\le T/\tau}\|{\rm div}\,\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega)} \le (C/\tau)^{1/p}$. This allows us to use the discrete Gronwall inequality which, for a sufficiently small τ, say for $\tau \le \frac{1}{2}C^{1/(p+1)}$, gives the estimate

$$
\|\bar{\varrho}_\tau\|_{L^\infty(I;L^s(\Omega))}\le C. \tag{4.18}
$$

Furthermore, we test (3.1a) by ${\rm div}(|\nabla\varrho_\tau^k|^{r-2}\nabla\varrho_\tau^k)$ with some $r > 1$. Using the Green formula with the boundary condition $\boldsymbol{n}\cdot\boldsymbol{v}_\tau^k = 0$, the convective term can be handled as

$$
\begin{aligned}
&\int_\Omega\nabla\big(\boldsymbol{v}_\tau^k\cdot\nabla\varrho_\tau^k\big)\cdot\Big(\big|\nabla\varrho_\tau^k\big|^{r-2}\nabla\varrho_\tau^k\Big)\,{\rm d}\boldsymbol{x}\\
&\quad= \int_\Omega\big|\nabla\varrho_\tau^k\big|^{r-2}\big(\nabla\varrho_\tau^k\otimes\nabla\varrho_\tau^k\big):\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big) + \big(\boldsymbol{v}_\tau^k\cdot\nabla\big)\nabla\varrho_\tau^k\cdot\Big(\big|\nabla\varrho_\tau^k\big|^{r-2}\nabla\varrho_\tau^k\Big)\,{\rm d}\boldsymbol{x}\\
&\quad= \int_\Gamma\big|\nabla\varrho_\tau^k\big|^r\,\boldsymbol{v}_\tau^k\cdot\boldsymbol{n}\,{\rm d}S + \int_\Omega\Big(\big|\nabla\varrho_\tau^k\big|^{r-2}\big(\nabla\varrho_\tau^k\otimes\nabla\varrho_\tau^k\big):\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big)\\
&\qquad-\big({\rm div}\,\boldsymbol{v}_\tau^k\big)\big|\nabla\varrho_\tau^k\big|^r - (r-1)\big|\nabla\varrho_\tau^k\big|^{r-2}\big|\nabla\varrho_\tau^k\big|^{r-2}\nabla\varrho_\tau^k\cdot\big(\boldsymbol{v}_\tau^k\cdot\nabla\big)\nabla\varrho_\tau^k\Big)\,{\rm d}\boldsymbol{x}
\end{aligned}
$$

$$= \int_\Gamma \frac{1}{r}\big|\nabla\varrho_\tau^k\big|^r \underbrace{\boldsymbol{v}_\tau^k\cdot\boldsymbol{n}}_{=0}\,\mathrm{d}S + \int_\Omega \big|\nabla\varrho_\tau^k\big|^{r-2}\big(\nabla\varrho_\tau^k\otimes\nabla\varrho_\tau^k\big):\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big) - \frac{1}{r}\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big|\nabla\varrho_\tau^k\big|^r\,\mathrm{d}\boldsymbol{x}. \tag{4.19}$$

Thus, counting also with $\nabla(\varrho_\tau^k\,\mathrm{div}\,\boldsymbol{v}_\tau^k)\cdot(|\nabla\varrho_\tau^k|^{r-2}\nabla\varrho_\tau^k) = (\mathrm{div}\,\boldsymbol{v}_\tau^k)|\nabla\varrho_\tau^k|^r + \varrho_\tau^k|\nabla\varrho_\tau^k|^{r-2}\nabla\varrho_\tau^k\cdot\nabla(\mathrm{div}\,\boldsymbol{v}_\tau^k)$, we obtain the inequality

$$\begin{aligned}\frac{1}{r}\int_\Omega \frac{\big|\nabla\varrho_\tau^k\big|^r - \big|\nabla\varrho_\tau^{k-1}\big|^r}{\tau}\,\mathrm{d}\boldsymbol{x} &\le \int_\Omega \bigg(\Big(\frac{1}{r}-1\Big)\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big|\nabla\varrho_\tau^k\big|^r - \big|\nabla\varrho_\tau^k\big|^{r-2}\big(\nabla\varrho_\tau^k\otimes\nabla\varrho_\tau^k\big):\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big)\\ &\quad - \varrho_\tau^k\big|\nabla\varrho_\tau^k\big|^{r-2}\nabla\varrho_\tau^k\cdot\nabla\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\bigg)\,\mathrm{d}\boldsymbol{x}\\ &\le C\big\|\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big)\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}\int_\Omega\big|\nabla\varrho_\tau^k\big|^r\,\mathrm{d}\boldsymbol{x} + \big\|\varrho_\tau^k\big\|_{L^{pr/(p-r)}}\big\|\nabla\varrho_\tau^k\big\|_{L^r(\Omega;\mathbb{R}^3)}^{r-1}\big\|\nabla\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big\|_{L^p(\Omega;\mathbb{R}^3)}.\end{aligned} \tag{4.20}$$

Here, we needed $p > r$ to be able then to choose $s = pr/(p-r) < +\infty$ for (4.18). Using the discrete Gronwall inequality again, relying on (4.13a), we obtain the estimate for sufficiently small time steps τ for the mass density:

$$\|\bar\varrho_\tau\|_{L^\infty(I;W^{1,r}(\Omega))} \le C. \tag{4.21}$$

Furthermore, by testing (3.1a) with the nonpositive part $(\varrho_\tau^k)^- = \min(0,\varrho_\tau^k) \in W^{1,r}(\Omega)$ of ϱ_τ^k, we obtain

$$\begin{aligned}\int_\Omega \Big(\big(\varrho_\tau^k\big)^-\Big)^2\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \underbrace{\varrho_\tau^{k-1}\big(\varrho_\tau^k\big)^-}_{\le 0 \text{ if } \varrho_\tau^{k-1}\ge 0} - \tau\mathrm{div}\big(\varrho_\tau^k\boldsymbol{v}_\tau^k\big)\big(\varrho_\tau^k\big)^-\,\mathrm{d}\boldsymbol{x} \le -\tau\int_\Omega \varrho_\tau^k\boldsymbol{v}_\tau^k\cdot\nabla\big(\varrho_\tau^k\big)^-\,\mathrm{d}\boldsymbol{x}\\ &= -\tau\int_\Omega\big(\varrho_\tau^k\big)^-\boldsymbol{v}_\tau^k\cdot\nabla\big(\varrho_\tau^k\big)^-\,\mathrm{d}\boldsymbol{x} = \frac{\tau}{2}\int_\Omega\Big(\big(\varrho_\tau^k\big)^-\Big)^2\mathrm{div}\,\boldsymbol{v}_\tau^k\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{4.22}$$

where we also used the calculus like (4.16) for $s = 2$. From this, for each $\tau > 0$ sufficiently small, namely for $\tau < 2/\|\mathrm{div}\,\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega)}$, we can see that $\varrho_\tau^k \ge 0$ provided $\varrho_\tau^{k-1} \ge 0$ on Ω. Even, we can prove positivity of ϱ_τ^k. This can be seen *via* a contradiction argument: Assuming that $\varrho_\tau^{k-1} > 0$ on $\overline{\Omega}$ and that the minimum of a (momentarily smooth) solution ϱ_τ^k is attained at some $\boldsymbol{x}\in\Omega$ and $\varrho_\tau^k(\boldsymbol{x}) = 0$, we have that $\varrho_\tau^k(\boldsymbol{x})-\varrho_\tau^{k-1}(\boldsymbol{x}) < 0$ and $\nabla\varrho_\tau^k(\boldsymbol{x}) = \boldsymbol{0}$. This yields that $\varrho_\tau^k(\boldsymbol{x}) - \varrho_\tau^{k-1}(\boldsymbol{x}) = -\tau\boldsymbol{v}_\tau^k(\boldsymbol{x})\cdot\nabla\varrho_\tau^k(\boldsymbol{x}) - \tau\varrho_\tau^k(\boldsymbol{x})\,\mathrm{div}\,\boldsymbol{v}_\tau^k(\boldsymbol{x}) < 0$ but, in view of (3.1a) written at $\boldsymbol{x}$, it should be equal to 0. Thus we obtain the contradiction, showing that $\varrho_\tau^k(\boldsymbol{x}) > 0$. This a.e. positivity allows for a test with $\sigma_\tau^k := 1/\varrho_\tau^k$. Using the convexity of the function $\varrho \mapsto 1/\varrho$ on $(0,+\infty)$ for the convective time differences and (3.1a), we obtain the inequality a.e. on Ω:

$$\begin{aligned}\frac{\sigma_\tau^k - \sigma_\tau^{k-1}}{\tau} + \boldsymbol{v}_\tau^k\cdot\nabla\sigma_\tau^k &= \frac{1}{\tau}\bigg(\frac{1}{\varrho_\tau^k} - \frac{1}{\varrho_\tau^{k-1}}\bigg) + \boldsymbol{v}_\tau^k\cdot\nabla\frac{1}{\varrho_\tau^k}\\ &\le -\frac{1}{\big(\varrho_\tau^k\big)^2}\bigg(\frac{\varrho_\tau^k - \varrho_\tau^{k-1}}{\tau} + \boldsymbol{v}_\tau^k\cdot\nabla\varrho_\tau^k\bigg) = \frac{1}{\varrho_\tau^k}\mathrm{div}\,\boldsymbol{v}_\tau^k = \big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\,\sigma_\tau^k,\end{aligned} \tag{4.23}$$

where $\sigma_\tau^{k-1} := 1/\varrho_\tau^{k-1}$. Testing (4.23) by $|\sigma_\tau^k|^{s-2}\sigma_\tau^k$ for some $s > 1$ and using the slightly modified calculus (4.16), this test gives the following inequality

$$\begin{aligned}\frac{1}{s}\int_\Omega \frac{\big(\sigma_\tau^k\big)^s - \big(\sigma_\tau^{k-1}\big)^s}{\tau}\,\mathrm{d}\boldsymbol{x} &\le \int_\Omega\Big(1+\frac{1}{s}\Big)\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big(\sigma_\tau^k\big)^s\,\mathrm{d}\boldsymbol{x}\\ &\le \Big(1+\frac{1}{s}\Big)\big\|\mathrm{div}\,\boldsymbol{v}_\tau^k\big\|_{L^\infty(\Omega)}\int_\Omega\big(\sigma_\tau^k\big)^s\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{4.24}$$

For a sufficiently small $\tau>0$, say for $\tau\le 1/((1{+}s)\max_{1\le k\le T/\tau}\|{\rm div}\,\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega)})$, we obtain a uniform bound for $\|\sigma_\tau^k\|_{L^s(\Omega)}$ by using the discrete Gronwall inequality. Here note that we have $\max_{1\le k\le T/\tau}\|{\rm div}\,\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega)}\le\sqrt{NC/\tau}$ with C from the second estimate in (4.15) and with N the norm of the embedding $L^2(I;L^2(\Omega))\cap L^p(I;W^{1,p}((\Omega))\subset L^2(I;L^\infty((\Omega))$. Thus, choosing $\tau\le 1/((1{+}s)^2NC)$ is sufficient. Having $\|\sigma_\tau^k\|_{L^s(\Omega)}$ bounded and using (4.13a) while realizing that $\boldsymbol{v}_\tau^k=\sqrt{\sigma_\tau^k}\,\boldsymbol{p}_\tau^k/\sqrt{\varrho_\tau^k}$, we obtain the bound

$$\big\|\boldsymbol{v}_\tau^k\big\|_{L^{2s/(s+1)}(\Omega;\mathbb{R}^3)}\le C_s\Big\|\sqrt{\sigma_\tau^k}\Big\|_{L^{2s}(\Omega)}\bigg\|\frac{\boldsymbol{p}_\tau^k}{\sqrt{\varrho_\tau^k}}\bigg\|_{L^2(\Omega;\mathbb{R}^3)}. \tag{4.25}$$

From (4.25) with an arbitrarily large $s>1$, we have obtained an estimate

$$\|\bar{\boldsymbol{v}}_\tau\|_{L^\infty(I;L^a(\Omega;\mathbb{R}^3))}\le C_a\quad\text{with an arbitrary }\ 1\le a<2. \tag{4.26}$$

Together with (4.13a), we thus obtain the bound for $\bar{\boldsymbol{v}}_\tau$ in $L^a(I;W^{2,p}(\Omega;\mathbb{R}^3))$ with $1\le a<2$. In particular,

$$\|\bar{\boldsymbol{v}}_\tau\|_{L^p(I;W^{1,\infty}(\Omega;\mathbb{R}^3))}\le C. \tag{4.27}$$

Moreover, (4.26) together with (4.13a) and (4.21) also allow us to augment (4.13d) by an estimate for $\nabla\bar{\boldsymbol{p}}_\tau=\nabla(\bar\varrho_\tau\bar{\boldsymbol{v}}_\tau)=\bar\varrho_\tau\nabla\bar{\boldsymbol{v}}_\tau+\nabla\bar\varrho_\tau\otimes\bar{\boldsymbol{v}}_\tau$, namely

$$\|\bar{\boldsymbol{p}}_\tau\|_{L^p(I;W^{1,r}(\Omega;\mathbb{R}^3))}\le C. \tag{4.28}$$

By the comparison from (3.5a), using (4.28), we obtain

$$\bigg\|\frac{\partial\varrho_\tau}{\partial t}\bigg\|_{L^p(I;L^r(\Omega))}\le C. \tag{4.29a}$$

Moreover, by the comparison from (3.1b) when using (4.13b), (4.21), (4.27), and (4.28), we obtain

$$\begin{aligned}\bigg\|\frac{\partial\boldsymbol{p}_\tau}{\partial t}\bigg\|_{L^{p'}(I;W^{2,p}(\Omega;\mathbb{R}^3)^*)}=\sup_{\|\tilde{\boldsymbol{v}}\|_{L^p(I;W^{2,p}(\Omega;\mathbb{R}^3))}\le1}\int_0^T\!\!\int_\Omega\big(\overline{\boldsymbol{T}}_\tau{+}\mathbb{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau){-}\bar{\boldsymbol{p}}_\tau{\otimes}\bar{\boldsymbol{v}}_\tau\big){:}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})\\ +\bar{\mathfrak{H}}_\tau\vdots\nabla^2\tilde{\boldsymbol{v}}-\bar\varrho_\tau\bar{\boldsymbol{g}}_\tau\cdot\tilde{\boldsymbol{v}}\,{\rm d}\boldsymbol{x}\,{\rm d}t\le C\,;\end{aligned} \tag{4.29b}$$

note that, as $p\ge3$ is assumed, $\bar\varrho_\tau{\otimes}\bar{\boldsymbol{v}}_\tau\in L^{p/2}(I;L^\infty(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$ is indeed in duality with $\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})\in L^p(I;L^\infty(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$.

Step 3: A test towards a generalized entropy. The next test is for (3.1d) by $(1/\theta_\tau^k)^\lambda$ for some $0<\lambda$. Note that, for $\lambda=1$, such test leads to the standard entropy balance. To this aim, reminding that $\theta_\tau^k\ge0$, we can also show that θ_τ^k is positive a.e. on Ω. More specifically, let us assume that $\theta_\tau^{k-1}>0$ on $\overline\Omega$ and that the minimum of a (momentarily smooth) solution θ_τ^k is attained at some $\boldsymbol{x}\in\Omega$ and $\theta_\tau^k(\boldsymbol{x})=0$. Thus $\theta_\tau^k(\boldsymbol{x})-\theta_\tau^{k-1}(\boldsymbol{x})<0$ and $\nabla\theta_\tau^k(\boldsymbol{x})=\boldsymbol{0}$ and also ${\rm div}(\kappa(\theta_\tau^k(\boldsymbol{x}))\nabla\theta_\tau^k(\boldsymbol{x}))\ge0$. The heat equation (3.1d) written as $u_\tau^k{-}u_\tau^{k-1}=\tau({\rm div}\,\boldsymbol{v}_\tau^k)u_\tau^k+\tau\boldsymbol{v}_\tau^k\cdot\nabla u_\tau^k+\tau{\rm div}(\kappa(\theta_\tau^k)\nabla\theta_\tau^k)+\frac13\tau{\rm tr}(\mathscr{T}(\boldsymbol{E}_\tau^k,\theta_\tau^k){-}\mathscr{T}(\boldsymbol{E}_\tau^k,0))\,{\rm div}\,\boldsymbol{v}_\tau^k+\tau\xi_{\rm ext}(\theta_\tau^k;\boldsymbol{v}_\tau^k,\boldsymbol{\Pi}_\tau^k)\ge0$ at this point $\boldsymbol{x}$. Since $\mathscr{U}$ is increasing on $[0,+\infty)$, we have $u_\tau^k(\boldsymbol{x}){-}u_\tau^{k-1}(\boldsymbol{x})<0$ and we thus obtain the contradiction, showing that $\theta_\tau^k(\boldsymbol{x})>0$.

We now use the assumption that the function $1/[\mathscr{U}^{-1}](\cdot)^\lambda$ is convex, so that we have the inequality

$$\frac{1}{(\theta_\tau^k)^\lambda}\big(u_\tau^k-u_\tau^{k-1}\big)=\frac{1}{\mathscr{U}^{-1}(u_\tau^k)^\lambda}\big(u_\tau^k-u_\tau^{k-1}\big)\le\tilde\eta_\lambda\big(u_\tau^k\big)-\tilde\eta_\lambda\big(u_\tau^{k-1}\big), \tag{4.30}$$

where $\tilde{\eta}_\lambda$ is a primitive function of $1/[\mathscr{U}^{-1}](\cdot)^\lambda$. Note that, up to a possible additive constant which can be calibrated as 0, it holds that $\tilde{\eta}_\lambda \circ \mathscr{U} = \eta_\lambda$, where

$$\eta_\lambda(\theta) := -\int_0^\theta \vartheta^{1-\lambda}\gamma''(\vartheta)\,\mathrm{d}\vartheta; \tag{4.31}$$

note that we have $[\tilde{\eta}_\lambda\circ\mathscr{U}]'(\theta) = ((1/\mathscr{U}^{-1})^\lambda \circ \mathscr{U}(\theta))\mathscr{U}'(\theta) = (1/\theta^\lambda)\mathscr{U}'(\theta) = -\theta^{1-\lambda}\gamma''(\theta) = \eta_\lambda'(\theta)$. Further, we will exploit the discrete version of the calculus

$$\begin{aligned}\frac{1}{\theta^\lambda}\Big(\frac{\partial u}{\partial t} + \mathrm{div}(u\boldsymbol{v})\Big) &= \frac{1}{\theta^\lambda}\Big(\frac{\partial}{\partial t}\mathscr{U}(\theta) + \mathrm{div}(\mathscr{U}(\theta)\boldsymbol{v})\Big)\\ &= \frac{1}{\theta^\lambda}\frac{\partial}{\partial t}(\gamma(\theta){-}\theta\gamma'(\theta)) + \frac{1}{\theta^\lambda}\mathrm{div}((\gamma(\theta){-}\theta\gamma'(\theta))\boldsymbol{v})\\ &= \frac{\partial}{\partial t}\eta_\lambda(\theta) + \mathrm{div}(\eta_\lambda(\theta)\boldsymbol{v}) + \Big(\frac{\gamma(\theta){-}\theta\gamma'(\theta)}{\theta^\lambda} - \eta_\lambda(\theta)\Big)\,\mathrm{div}\,\boldsymbol{v};\end{aligned} \tag{4.32}$$

note that, for $\lambda = 1$, it reduces to the calculus (4.4) for η_1 as defined in (4.3). Thus, using (4.30), we have now the inequality

$$\begin{aligned}\frac{1}{\big(\theta_\tau^k\big)^\lambda}\Big(\frac{u_\tau^k{-}u_\tau^{k-1}}{\tau} + \mathrm{div}\big(u_\tau^k\boldsymbol{v}_\tau^k\big)\Big) \le \frac{\eta_\lambda\big(\theta_\tau^k\big) - \eta_\lambda\big(\theta_\tau^{k-1}\big)}{\tau}&\\ + \mathrm{div}\big(\eta_\lambda\big(\theta_\tau^k\big)\boldsymbol{v}_\tau^k\big) + \bigg(\frac{\gamma\big(\theta_\tau^k\big){-}\theta_\tau^k\gamma'\big(\theta_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda} - \eta_\lambda\big(\theta_\tau^k\big)\bigg)\,\mathrm{div}\,\boldsymbol{v}_\tau^k.&\end{aligned} \tag{4.33}$$

Multiplying (3.1d) by $(\theta_\tau^k)^{-\lambda}$ and integrating it over Ω while using the Green formula with the boundary condition for the normal velocity, we obtain

$$\begin{aligned}\int_\Omega \frac{\xi_{\rm ext}\big(\theta_\tau^k;\boldsymbol{v}_\tau^k,\boldsymbol{\Pi}_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda} + \kappa\big(\theta_\tau^k\big)\frac{\big|\nabla\theta_\tau^k\big|^2}{\big(\theta_\tau^k\big)^{1+\lambda}}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{\kappa\big(\theta_\tau^k\big)\nabla\theta_\tau^k}{\big(\theta_\tau^k\big)^\lambda}\cdot\boldsymbol{n}\,\mathrm{d}S&\\ = \int_\Omega \frac{1}{\big(\theta_\tau^k\big)^\lambda}\Big(\frac{u_\tau^k - u_\tau^{k-1}}{\tau} + \mathrm{div}\big(u_\tau^k\boldsymbol{v}_\tau^k\big) - \tfrac{1}{3}\mathcal{T}_\tau^m\mathrm{div}\,\boldsymbol{v}_\tau^m\Big)\,\mathrm{d}\boldsymbol{x}.&\end{aligned} \tag{4.34}$$

The handling of the boundary term in (4.34) is quite tricky. Since $h(\cdot)$ is strictly monotone, cf. (4.6i), we can identify an external temperature

$$\theta_{{\rm ext},\tau}^k := h^{-1}\big(h_{{\rm ext},\tau}^k\big).$$

Together with the non-negativity of h, this yields the following estimate

$$\begin{aligned}\frac{\kappa\big(\theta_\tau^k\big)\nabla\theta_\tau^k}{\big(\theta_\tau^k\big)^\lambda}\cdot\boldsymbol{n} = \frac{h_{{\rm ext},\tau}^k{-}h\big(\theta_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda} = \frac{h\big(\theta_{{\rm ext},\tau}^k\big) - h\big(\theta_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda} \pm \frac{h\big(\theta_\tau^k\big) - h\big(\theta_{{\rm ext},\tau}^k\big)}{\big(\theta_{{\rm ext},\tau}^k\big)^\lambda}&\\ = \frac{h\big(\theta_\tau^k\big) - h\big(\theta_{{\rm ext},\tau}^k\big)}{\big(\theta_{{\rm ext},\tau}^k\big)^\lambda} + \frac{\big(h\big(\theta_\tau^k\big) - h\big(\theta_{{\rm ext},\tau}^k\big)\big)\Big(\big(\theta_\tau^k\big)^\lambda - \big(\theta_{{\rm ext},\tau}^k\big)^\lambda\Big)}{\big(\theta_\tau^k\big)^\lambda\big(\theta_{{\rm ext},\tau}^k\big)^\lambda} \ge -\frac{h\big(\theta_{{\rm ext},\tau}^k\big)}{\big(\theta_{{\rm ext},\tau}^k\big)^\lambda}.&\end{aligned}$$

By merging it with (4.33), realizing (4.2), and summing it over the time levels $k = 1,\dots,T/\tau$, we obtain the inequality

$$\sum_{k=1}^{T/\tau}\int_\Omega \frac{\xi_{\rm ext}\big(\theta_\tau^k;\boldsymbol{v}_\tau^k,\boldsymbol{\Pi}_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda} + \kappa\big(\theta_\tau^k\big)\frac{\big|\nabla\theta_\tau^k\big|^2}{\big(\theta_\tau^k\big)^{1+\lambda}}\,\mathrm{d}\boldsymbol{x} \le \int_\Omega \frac{\eta_\lambda\big(\theta_\tau^{T/\tau}\big) - \eta_\lambda(\theta_0)}{\tau}\,\mathrm{d}\boldsymbol{x} + \sum_{k=1}^{T/\tau}\int_\Gamma \frac{h\big(\theta_{{\rm ext},\tau}^k\big)}{\big(\theta_{{\rm ext},\tau}^k\big)^\lambda}\,\mathrm{d}S$$

$$-\sum_{k=1}^{T/\tau}\int_\Omega\bigg(\theta_\tau^k\frac{\phi'\big(\mathrm{tr}\boldsymbol{E}_\tau^k\big)+\phi\big(\mathrm{tr}\boldsymbol{E}_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda}+\frac{\theta_\tau^k\gamma'\big(\theta_\tau^k\big)+\gamma\big(\theta_\tau^k\big)}{\big(\theta_\tau^k\big)^\lambda}+\eta_\lambda\big(\theta_\tau^k\big)\bigg)\,\mathrm{div}\,\boldsymbol{v}_\tau^k\,\mathrm{d}\boldsymbol{x} \tag{4.35}$$

with $\xi_{\rm ext}=\xi_{\rm ext}(\theta;\boldsymbol{v},\boldsymbol{\Pi})$ from (2.22d). This inequality written in terms of the interpolants yields the variant of the thermal entropy balance (4.5) integrated over the time integral I and over Ω as an inequality, *i.e.*

$$\int_0^T\!\!\int_\Omega\frac{\xi_{\rm ext}\big(\bar\theta_\tau;\bar{\boldsymbol{v}}_\tau,\bar{\boldsymbol{\Pi}}_\tau\big)}{\bar\theta_\tau^\lambda}+\kappa\big(\bar\theta_\tau\big)\frac{\big|\nabla\bar\theta_\tau\big|^2}{\bar\theta_\tau^{1+\lambda}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\le\int_\Omega\eta_\lambda(\theta_\tau(T))-\eta_\lambda(\theta_0)\,\mathrm{d}\boldsymbol{x}+\int_0^T\!\!\int_\Gamma\frac{h\big(\bar\theta_{{\rm ext},\tau}\big)}{\bar\theta_{{\rm ext},\tau}^\lambda}\,\mathrm{d}S\,\mathrm{d}t$$
$$-\int_0^T\!\!\int_\Omega\bar\theta_\tau^{1-\lambda}\big(\phi'\big(\mathrm{tr}\bar{\boldsymbol{E}}_\tau\big)+\phi\big(\mathrm{tr}\bar{\boldsymbol{E}}_\tau\big)\big)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau+\big(\bar\theta_\tau^{1-\lambda}\gamma'\big(\bar\theta_\tau\big)+\bar\theta_\tau^\lambda\gamma\big(\bar\theta_\tau\big)+\eta_\lambda\big(\bar\theta_\tau\big)\big)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.36}$$

The last integral in (4.36) is bounded due to (4.15) and (4.13b). In more detail, we first improve the former estimate in (4.13b) by testing (3.1c) with $|\boldsymbol{E}_\tau^k|^{r-2}\boldsymbol{E}_\tau^k$, where r is arbitrarily large. Reminding the abbreviation $\mathscr{R}$ from (2.20) and the assumption (4.6e), we can estimate

$$\frac1r\int_\Omega\frac{\big|\boldsymbol{E}_\tau^k\big|^r-\big|\boldsymbol{E}_\tau^{k-1}\big|^r}{\tau}\,\mathrm{d}\boldsymbol{x}\le\int_\Omega\big|\boldsymbol{E}_\tau^k\big|^{r-2}\boldsymbol{E}_\tau^k:\big(\boldsymbol{\varepsilon}\big(\boldsymbol{v}_\tau^k\big)-\boldsymbol{B}_{\rm ZJ}\big(\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k\big)-\mathscr{R}\big(\boldsymbol{E}_\tau^k,\theta_\tau^k\big)\big)\,\mathrm{d}\boldsymbol{x}$$
$$\le\Big(1+\big\|\nabla\boldsymbol{v}_\tau^k\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}+\big\|\boldsymbol{E}_\tau^k\big\|_{L^r(\Omega;\mathbb{R}^{3\times3})}\Big)\big\|\boldsymbol{E}_\tau^k\big\|_{L^r(\Omega;\mathbb{R}^{3\times3})}^{r-1}; \tag{4.37}$$

here, for the term $(\boldsymbol{v}_\tau^k\cdot\nabla\boldsymbol{E}_\tau^k)$ involved in $\boldsymbol{B}_{\rm ZJ}(\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k)$, we have used the Green formula with the boundary condition $\boldsymbol{v}_\tau^k\cdot\boldsymbol{n}=0$ on Γ as

$$\int_\Omega\big(\boldsymbol{v}_\tau^k\cdot\nabla\boldsymbol{E}_\tau^k\big):\Big(\big|\boldsymbol{E}_\tau^k\big|^{r-2}\boldsymbol{E}_\tau^k\Big)\,\mathrm{d}\boldsymbol{x}=\int_\Gamma\big|\boldsymbol{E}_\tau^k\big|^r\big(\boldsymbol{v}_\tau^k\cdot\boldsymbol{n}\big)\,\mathrm{d}S$$
$$-\int_\Omega(r-1)\big|\boldsymbol{E}_\tau^k\big|^{r-2}\boldsymbol{E}_\tau^k:\big(\boldsymbol{v}_\tau^k\cdot\nabla\boldsymbol{E}_\tau^k\big)+\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big|\boldsymbol{E}_\tau^k\big|^r\mathrm{d}\boldsymbol{x}=-\int_\Omega\frac{\mathrm{div}\,\boldsymbol{v}_\tau^k}{r}\big|\boldsymbol{E}_\tau^k\big|^r\mathrm{d}\boldsymbol{x}. \tag{4.38}$$

By the assumed regularity (4.6k) of the initial condition $\boldsymbol{E}_0$ with the growth assumption (4.6e) on $\mathscr{R}$, by the discrete Gronwall inequality, for sufficiently small $\tau>0$, we obtain

$$\big\|\bar{\boldsymbol{E}}_\tau\big\|_{L^\infty(I;L^r(\Omega;\mathbb{R}^{3\times3}))}\le C_r. \tag{4.39}$$

Here, we have used that $\tau\sum_{k=1}^{T/\tau}\|\nabla\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}^2$ is bounded due to (4.15) with Korn's inequality, so that $\tau\max_{k=1,\dots,T/\tau}\|\nabla\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}=\mathscr{O}(\sqrt{\tau})$ is sufficiently small for τ small enough, which allows us to use of the mentioned discrete Gronwall inequality.

This allows estimation of the terms on the right-hand side of (4.36). Using (4.39) with r sufficiently large and the growth assumption (4.6c) when realizing that $\eta'_E(E,0)=\phi'(\mathrm{tr}E)\mathbb{I}$ due to the ansatz (4.1), the term $\bar\theta_\tau^{1-\lambda}\phi'(\mathrm{tr}\bar{\boldsymbol{E}}_\tau)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau$ can be estimated as

$$-\int_\Omega\bar\theta_\tau^{1-\lambda}\phi'\big(\mathrm{tr}\bar{\boldsymbol{E}}_\tau\big)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau\,\mathrm{d}\boldsymbol{x}\le\big\|\bar\theta_\tau\big\|_{L^{(1+\alpha)/(1-\lambda)}(\Omega)}^{1-\lambda}\big\|\phi'\big(\mathrm{tr}\bar{\boldsymbol{E}}_\tau\big)\big\|_{L^{r/C_1}(\Omega)}\|\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau)\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}.$$

The term $\bar\theta_\tau^{1-\lambda}\phi(\mathrm{tr}\bar{\boldsymbol{E}}_\tau)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau$ can be estimated similarly. To estimate the term $\bar\theta_\tau^{1-\lambda}\gamma'(\bar\theta_\tau)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau$, we use the growth $|\gamma'(\theta)|=|\int_0^\theta\gamma''(\vartheta)\,\mathrm{d}\vartheta|=\mathscr{O}(\theta^\alpha)$. Here note the growth $|\gamma''(\theta)|=|\mathscr{U}'(\theta)/\theta|=|\mathscr{E}'_\theta(E,\theta)/\theta|=\mathscr{O}(\theta^{\alpha-1})$ due to the assumption (4.6b). Finally, in view of (4.31), when using the ansatz (4.1) and the estimate (4.13b), we can see that $\eta_\lambda(\bar\theta_\tau)$ with $\eta_\lambda(\theta)=\int_0^\theta\mathscr{U}'(\vartheta)/\vartheta^\lambda\,\mathrm{d}\vartheta$ is bounded in $L^\infty(I;L^{1/(1-\lambda)}(\Omega))$. Therefore, $\eta_\lambda(\theta_\tau(T))$ is bounded in $L^{1/(1-\lambda)}(\Omega)$ and $\eta_\lambda(\bar\theta_\tau)\,\mathrm{div}\,\bar{\boldsymbol{v}}_\tau$ is bounded in $L^p(I;L^{1/(1-\lambda)}(\Omega))$. Eventually, the boundary term $h(\bar\theta_{{\rm ext},\tau})/\bar\theta_{{\rm ext},\tau}^\lambda=\bar h_{{\rm ext},\tau}/h^{-1}(\bar h_{{\rm ext},\tau})^\lambda$ in (4.36) is uniformly bounded in $L^1(I{\times}\Gamma)$ under the assumption $h_{\rm ext}/h^{-1}(h_{\rm ext})^\lambda\in L^1(I{\times}\Gamma)$.

With the left-hand side of (4.36) estimated, we therefore obtain a bound for $\kappa(\bar\theta_\tau)|\nabla\bar\theta_\tau|^2/\bar\theta_\tau^{1+\lambda}$ in $L^1(I{\times}\Omega)$, which can be used to estimate $\nabla\bar\theta_{\varepsilon\delta\tau}$ in $L^\mu(I{\times}\Omega;\mathbb{R}^3)$ for some $1\le\mu<2$. To this end, we apply Hölder's inequality to $|\nabla\theta_\varepsilon|^\mu$ written as the product of $\bar\theta_\tau^{\mu(1+\lambda)/2}/\kappa(\bar\theta_\tau)^{\mu/2}$ and $\kappa(\bar\theta_\tau)^{\mu/2}|\nabla\bar\theta_\tau|^\mu/\bar\theta_\tau^{\mu(1+\lambda)/2}$:

$$\int_0^T\!\!\int_\Omega|\nabla\bar\theta_\tau|^\mu\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\le C_{\mu,\lambda}\bigg(\int_0^T\!\!\int_\Omega\frac{\bar\theta_\tau^{\mu(1+\lambda)/(2-\mu)}}{\kappa(\bar\theta_\tau)^{\mu/(2-\mu)}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\bigg)^{1-\mu/2}\bigg(\int_0^T\!\!\int_\Omega\frac{\kappa(\bar\theta_\tau)|\nabla\bar\theta_\tau|^2}{\bar\theta_\tau^{1+\lambda}}\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\bigg)^{\mu/2} \tag{4.40}$$

with a constant $C_{\mu,\lambda}$ dependent on μ and λ. The last integral in (4.40) is estimated from (4.36), while the penultimate integral is to be estimated using the latter estimate in (4.13b). The first integral on the right-hand side is to be estimated by interpolation with the latter estimate in (4.13b), specifically

$$\begin{aligned}\int_0^T\!\!\int_\Omega\frac{\bar\theta_\tau^{\mu(1+\lambda)/(2-\mu)}}{\kappa(\bar\theta_\tau)^{\mu/(2-\mu)}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t&=\int_0^T\!\!\int_\Omega\bigg(\frac{\bar\theta_\tau^{1+\lambda}}{\kappa(\bar\theta_\tau)}\bigg)^{\mu/(2-\mu)}\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\overset{(4.6g)}{\le}C\int_0^T\!\!\int_\Omega\Big(1+\bar\theta_\tau^{\mu(1+\lambda-\beta^+)/(2-\mu)}\Big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\&\le C'\bigg(1+\int_0^T\!\!\int_\Omega|\nabla\bar\theta_\tau|^\mu\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\bigg).\end{aligned} \tag{4.41}$$

The last inequality relies on the latter estimate in (4.13b) *via* the Gagliardo–Nirenberg inequality. Specifically, for each time instant t (not explicitly denoted), we have

$$\begin{aligned}&\big\|1{+}\bar\theta_\tau\big\|_{L^{\mu(1+\lambda-\beta^+)/(2-\mu)}(\Omega)}\le C\big\|1{+}\bar\theta_\tau\big\|_{L^{1+\alpha}(\Omega)}\Big(\big\|1{+}\bar\theta_\tau\big\|_{L^{1+\alpha}(\Omega)}+\big\|\nabla\bar\theta_\tau\big\|_{L^\mu(\Omega;\mathbb{R}^3)}\Big)\\&\qquad\text{for}\quad\frac{2-\mu}{\mu(1{+}\lambda{-}\beta^+)}\ge a\Big(\frac1\mu-\frac13\Big)+\frac{1-a}{1+\alpha}\quad\text{with}\quad 0<a\le1.\end{aligned} \tag{4.42}$$

We raise it to the power $\mu(1{+}\lambda{-}\beta^+)/(2{-}\mu)$ and choose a to obtain the desired exponent $a\mu(1{+}\lambda{-}\beta^+)/(2{-}\mu)=\mu$, *i.e.* $a=(2{-}\mu)/(1{+}\lambda{-}\beta^+)$, which allows for the last inequality in (4.41). Since the exponent $1-\mu/2$ is less than 1, merging (4.40) and (4.41) yields the bound

$$\big\|\nabla\bar\theta_\tau\big\|_{L^\mu(I\times\Omega;\mathbb{R}^3)}\le C. \tag{4.43}$$

After some algebra, substituting the mentioned choice $a=(2{-}\mu)/(1{+}\lambda{-}\beta^+)$ into (4.42) yields the bound (4.8). In more detail, note that, for this choice of a, the inequality in (4.42) simply becomes $a/3\ge(1{-}a)/(1{+}\alpha)$, *i.e.* $a\ge3/(4{+}\alpha)$. Realizing $a=(2{-}\mu)/(1{+}\lambda{-}\beta^+)$, we eventually obtain (4.8).
According to the Sobolev embedding theorem, (4.43) yields a bound for $\bar\theta_\tau$ in $L^\mu(I;L^{\mu^*}(\Omega))$ and, by interpolation with the bound in $L^\infty(I;L^{1+\alpha}(\Omega))$ with the weights $3/(4{+}\alpha)$ and $(1{+}\alpha)/(4{+}\alpha)$, also the bound

$$\big\|\bar\theta_\tau\big\|_{L^{(4+\alpha)\,\mu/3}(I\times\Omega)}\le C. \tag{4.44}$$

From this, realizing the bounds $c(\theta)=\mathscr{O}(\theta^\alpha)$ and $\mathscr{U}(\theta)=\mathscr{O}(\theta^{1+\alpha})$ and that $\nabla\bar u_\tau=\nabla\mathscr{U}(\bar\theta_\tau)=c(\bar\theta_\tau)\nabla\bar\theta_\tau$, we can read also the estimates

$$\|\bar u_\tau\|_{L^{(4+\alpha)\,\mu/(3+3\alpha)}(I\times\Omega)}\le C\quad\text{and}\quad\|\nabla\bar u_\tau\|_{L^{(4+\alpha)\,\mu/(4+4\alpha)}(I\times\Omega;\mathbb{R}^3)}\le C. \tag{4.45}$$

Recalling $\varkappa$ as a primitive function of κ as used in (2.38b), we realize that $\varkappa(\theta)=\mathscr{O}(\theta^{1+\beta})$ and thus, from (4.44), we obtain the estimate

$$\big\|\varkappa(\bar\theta_\tau)\big\|_{L^{(4+\alpha)\,\mu/(3+3\beta)}(I\times\Omega)}\le C. \tag{4.46}$$

Let us specify the feasible exponents α and β depending on λ. First, we assume that $\alpha\ge0$ in order to work with the conventional $L^{1+\alpha}$-Lebesgue space. Recalling (4.8), the above-specified bounds $1\le\mu<2$ needed for (4.40) lead respectively to the restrictions $1+\alpha+3\beta^+ \geq 3\lambda$ and $\beta^+ < 1+\lambda$. The latter restriction is involved in (4.7) while the former restriction is automatically satisfied if $\alpha \geq 0$ and if the exponents in (4.45) are greater than (or equal to) 1. This last requirement means the restriction $1+3\beta^+ \geq 2\alpha+3\lambda$, which is also included in (4.7). Furthermore, also the exponent in (4.44) should be greater than (or equal to) 1, which means another restriction $2+2\alpha+3\beta^+ \geq 2\lambda$. This is automatically satisfied by the previous restrictions. Eventually, also the exponent in (4.46) should be greater than (or equal to) 1, which needs $2+2\alpha \geq 3\lambda+3(\beta-\beta^+)$. This restricts $\alpha \geq 0$ further as $\alpha \geq \frac{3}{2}\lambda - 1$ when $\lambda > \frac{2}{3}$. Altogether, we obtain the conditions in (4.7). Cf. also Remark 4.4 below.

Step 4: An estimate of $\nabla \boldsymbol{E}_\tau$. For a mere Kelvin–Voigt rheology, *i.e.* when the Maxwellian viscosity vanishes by considering $\boldsymbol{\Pi} \equiv 0$, the estimate of $\nabla \boldsymbol{E}_\tau$ would follow from the obtained regularity of the velocity field (4.15) together with the assumed regularity of the initial condition $\boldsymbol{E}_0$. For the general temperature-dependent Jeffreys rheology, the estimation of $\nabla \boldsymbol{E}_\tau$ is more involved and we also rely on the previously obtained estimate for $\nabla\theta_\tau$.

To obtain an L^s-estimate $\nabla \boldsymbol{E}_\tau$ with some $s > 1$, we can test (3.1c) by $\mathrm{div}(|\nabla \boldsymbol{E}_\tau^k|^{s-2}\nabla \boldsymbol{E}_\tau^k)$ and choose r in (4.39) sufficiently large so that $1/r+1/\mu+1/s' \leq 1$. Therefore, we need $1/r+1/\mu \leq 1/s$, which requires that $s < \mu$. For this, we can choose $r \geq s\mu/(\mu{-}s)$. Additionally, we also need $1/p+1/s' \leq 1$, which requires that $s \leq p$, a condition that is always met. This yields

$$\begin{aligned}\frac{1}{s}\int_\Omega \frac{\big|\nabla \boldsymbol{E}_\tau^k\big|^s - \big|\nabla \boldsymbol{E}_\tau^{k-1}\big|^s}{\tau}\,\mathrm{d}\boldsymbol{x} &\leq \int_\Omega \big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\nabla \boldsymbol{E}_\tau^k \vdots \big(\nabla\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^k) - \nabla\mathscr{R}(\boldsymbol{E}_\tau^k,\theta_\tau^k) - \nabla\boldsymbol{B}_{\text{ZJ}}(\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k)\big)\,\mathrm{d}\boldsymbol{x}\\ &\leq \int_\Omega \big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\nabla \boldsymbol{E}_\tau^k \vdots \big(\nabla\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^k) - \mathscr{R}'_{\boldsymbol{E}}(\boldsymbol{E}_\tau^k,\theta_\tau^k)\cdot\nabla \boldsymbol{E}_\tau^k - \mathscr{R}'_\theta(\boldsymbol{E}_\tau^k,\theta_\tau^k)\otimes\nabla\theta_\tau^k - \nabla\boldsymbol{B}_{\text{ZJ}}(\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k)\big)\,\mathrm{d}\boldsymbol{x}\\ &\leq C\Big(1+\big\|\nabla\boldsymbol{v}_\tau^k\big\|_{W^{1,p}(\Omega;\mathbb{R}^{3\times3})} + \big\|\boldsymbol{E}_\tau^k\big\|_{L^r(\Omega;\mathbb{R}^{3\times3})}\big\|\nabla\theta_\tau^k\big\|_{L^\mu(\Omega;\mathbb{R}^3)}\Big)\big\|\nabla \boldsymbol{E}_\tau^k\big\|_{L^s(\Omega;\mathbb{R}^{3\times3\times3})}^{s-1}\end{aligned} \tag{4.47}$$

with C depending here on (p,r,s,μ). Here we have also used the assumption (4.6e), so that, in particular, we have $|\nabla \boldsymbol{E}_\tau^k|^{s-2}\nabla \boldsymbol{E}_\tau^k \vdots (\mathscr{R}'_{\boldsymbol{E}}(\boldsymbol{E}_\tau^k,\theta_\tau^k)\cdot\nabla \boldsymbol{E}_\tau^k) \geq 0$, and for the term $\nabla((\boldsymbol{v}_\tau^k\cdot\nabla)\boldsymbol{E}_\tau^k)$ contained in $\nabla\boldsymbol{B}_{\text{ZJ}}(\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k)$, we have used the Green formula:

$$\begin{aligned}&\int_\Omega \nabla\big((\boldsymbol{v}_\tau^k\cdot\nabla)\boldsymbol{E}_\tau^k\big)\vdots\big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\nabla \boldsymbol{E}_\tau^k\,\mathrm{d}\boldsymbol{x}\\ &\quad= \int_\Omega \big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\big(\nabla \boldsymbol{E}_\tau^k\boxtimes\nabla \boldsymbol{E}_\tau^k\big):\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^k) + \big(\boldsymbol{v}_\tau^k\cdot\nabla\big)\nabla \boldsymbol{E}_\tau^k\vdots\big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\nabla \boldsymbol{E}_\tau^k\,\mathrm{d}\boldsymbol{x}\\ &\quad= \int_\Gamma \big|\nabla \boldsymbol{E}_\tau^k\big|^s\underbrace{\boldsymbol{v}_\tau^k\cdot\boldsymbol{n}}_{=0}\,\mathrm{d}S + \int_\Omega \Big(|\nabla \boldsymbol{E}_\tau^k|^{s-2}(\nabla \boldsymbol{E}_\tau^k\boxtimes\nabla \boldsymbol{E}_\tau^k):\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^k)\\ &\qquad\quad - \big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big|\nabla \boldsymbol{E}_\tau^k\big|^s - (s{-}1)\big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\nabla \boldsymbol{E}_\tau^k\vdots\big(\boldsymbol{v}_\tau^k\cdot\nabla\big)\nabla \boldsymbol{E}_\tau^k\Big)\,\mathrm{d}\boldsymbol{x}\\ &\quad= \int_\Omega \big|\nabla \boldsymbol{E}_\tau^k\big|^{s-2}\big(\nabla \boldsymbol{E}_\tau^k\boxtimes\nabla \boldsymbol{E}_\tau^k\big):\boldsymbol{\varepsilon}(\boldsymbol{v}_\tau^k) - \frac{1}{s}\big(\mathrm{div}\,\boldsymbol{v}_\tau^k\big)\big|\nabla \boldsymbol{E}_\tau^k\big|^s\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{4.48}$$

where the product $\boxtimes$ of the 3rd-order tensors is defined as $[\boldsymbol{G}\boxtimes\boldsymbol{G}]_{ij} = \sum_{k,l=1}^3 G_{ikl}G_{jkl}$. Relying on the already obtained estimate (4.39) and using the discrete Gronwall inequality for sufficiently small $\tau > 0$, from (4.47) we obtain

$$\big\|\nabla\overline{\boldsymbol{E}}_\tau\big\|_{L^\infty(I;L^s(\Omega;\mathbb{R}^{3\times3\times3}))} \leq C_{p,s} \quad \text{with any } \ 1 \leq s < \mu; \tag{4.49}$$

here we have used the assumption $\boldsymbol{E}_0 \in W^{1,s}(\Omega;\mathbb{R}^{3\times3}_{\text{sym}})$, cf. (4.6k). Here, for the aforementioned discrete Gronwall inequality, we have used that both $\tau\sum_{k=1}^{T/\tau}\|\nabla\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}^p$ is bounded for $p > 1$ and

$\tau\sum_{k=1}^{T/\tau}\|\nabla\theta_\tau^k\|_{L^\mu(\Omega;\mathbb{R}^3)}^\mu$ is bounded for $\mu>1$. Thus both $\tau\max_{k=1,\dots,T/\tau}\|\nabla\boldsymbol{v}_\tau^k\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}=\mathscr{O}(\tau^{1-1/p})$ and $\tau\max_{k=1,\dots,T/\tau}\|\nabla\theta_\tau^k\|_{L^\mu(\Omega;\mathbb{R}^3)}=\mathscr{O}(\tau^{1-1/\mu})$ are sufficiently small for τ small enough.

Moreover, by comparison, from $\frac{\partial}{\partial t}\boldsymbol{E}_\tau=\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau)-\boldsymbol{B}_{\rm ZJ}(\bar{\boldsymbol{v}}_\tau,\bar{\boldsymbol{E}}_\tau)-\mathscr{R}(\bar{\boldsymbol{E}}_\tau,\bar\theta_\tau)$, we also obtain

$$\left\|\frac{\partial\boldsymbol{E}_\tau}{\partial t}\right\|_{L^p(I;L^s(\Omega;\mathbb{R}^{3\times3}))}\le C_{r,s}. \tag{4.50}$$

Step 5: Existence of a solution to (3.1). The rigorous proof of the existence of a solution $(\varrho_\tau^k,\boldsymbol{p}_\tau^k,\boldsymbol{E}_\tau^k,\theta_\tau^k)$ and thus also $\boldsymbol{v}_\tau^k$ of the system (3.1) is slightly delicate. Convincing arguments may rely on a suitable regularization towards a quasilinear elliptic system for which standard theory can be used, and then making a limit passage.

Given $(\varrho_\tau^{k-1},\boldsymbol{p}_\tau^{k-1},\boldsymbol{E}_\tau^{k-1},\theta_\tau^{k-1})\in W^{1,r}(\Omega)\times W^{1,r}(\Omega;\mathbb{R}^3)\times W^{1,s}(\Omega;\mathbb{R}^{3\times3}_{\rm sym})\times L^{1+\alpha}(\Omega)$ with $\varrho_\tau^{k-1}>0$ and $\theta_\tau^{k-1}>0$ a.e., we seek a solution $(\varrho_{\varepsilon\delta\tau}^k,\boldsymbol{v}_{\varepsilon\delta\tau}^k,\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k)$ and thus also $\boldsymbol{p}_{\varepsilon\delta\tau}^k$ and $u_{\varepsilon\delta\tau}^k$ of the regularized quasilinear elliptic system

$$\varrho_{\varepsilon\delta\tau}^k+{\rm div}\big(\tau\boldsymbol{p}_{\varepsilon\delta\tau}^k-\delta|\nabla\varrho_{\varepsilon\delta\tau}^k|^{r-2}\nabla\varrho_{\varepsilon\delta\tau}^k\big)=\varrho_\tau^{k-1}\qquad\text{with}\quad \boldsymbol{p}_{\varepsilon\delta\tau}^k=\varrho_{\varepsilon\delta\tau}^k\boldsymbol{v}_{\varepsilon\delta\tau}^k, \tag{4.51a}$$

$$\begin{aligned}\boldsymbol{p}_{\varepsilon\delta\tau}^k+\tau{\rm div}\big(\boldsymbol{p}_{\varepsilon\delta\tau}^k\otimes\boldsymbol{v}_{\varepsilon\delta\tau}^k-\boldsymbol{D}_{\varepsilon\delta\tau}^k\big)&=\boldsymbol{p}_\tau^{k-1}+\tau\varrho_{\varepsilon\delta\tau}^k\boldsymbol{g}_\tau^k+\tau{\rm div}\,\mathscr{T}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k\big)\\&\quad-\varepsilon\big|\boldsymbol{v}_{\varepsilon\delta\tau}^k\big|^{p-2}\boldsymbol{v}_{\varepsilon\delta\tau}^k-\delta\big|\nabla\varrho_{\varepsilon\delta\tau}^k\big|^{r-2}\big(\nabla\boldsymbol{v}_{\varepsilon\delta\tau}^k\big)\nabla\varrho_{\varepsilon\delta\tau}^k,\\ \text{where}\ \ \boldsymbol{D}_{\varepsilon\delta\tau}^k=\mathbb{D}\boldsymbol{\varepsilon}\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k\big)-{\rm div}\mathfrak{H}_{\varepsilon\delta\tau}^k&\ \ \text{with}\ \ \mathfrak{H}_{\varepsilon\delta\tau}^k=\mu\big|\nabla^2\boldsymbol{v}_{\varepsilon\delta\tau}^k\big|^{p-2}\nabla^2\boldsymbol{v}_{\varepsilon\delta\tau}^k,\end{aligned} \tag{4.51b}$$

$$\begin{aligned}\boldsymbol{E}_{\varepsilon\delta\tau}^k=\boldsymbol{E}_\tau^{k-1}+\tau\boldsymbol{\varepsilon}(\boldsymbol{v}_{\varepsilon\delta\tau}^k)-\tau\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k-\tau\boldsymbol{B}_{\rm ZJ}\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k,\boldsymbol{E}_{\varepsilon\delta\tau}^k\big)+{\rm div}\Big(\varepsilon\big|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\big|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\Big)\\ \text{with}\ \ \boldsymbol{\Pi}_{\varepsilon\delta\tau}^k=\mathscr{R}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k\big),\end{aligned} \tag{4.51c}$$

$$\begin{aligned}u_{\varepsilon\delta\tau}^k+\tau{\rm div}\big(u_{\varepsilon\delta\tau}^k\boldsymbol{v}_{\varepsilon\delta\tau}^k-\kappa\big(\theta_{\varepsilon\delta\tau}^k\big)\nabla\theta_{\varepsilon\delta\tau}^k\big)&=u_\tau^{k-1}+\tau\,\xi_{\rm ext}\big(\theta_{\varepsilon\delta\tau}^k;\boldsymbol{v}_{\varepsilon\delta\tau}^k,\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k\big)\\+\tfrac{\tau}{3}{\rm tr}\big(\mathscr{T}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k\big)-\mathscr{T}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,0\big)\big)\,{\rm div}\,\boldsymbol{v}_{\varepsilon\delta\tau}^k&\qquad\text{with}\quad u_{\varepsilon\delta\tau}^k=\mathscr{U}\big(\theta_{\varepsilon\delta\tau}^k\big),\end{aligned} \tag{4.51d}$$

considered with $r>3$ and $s>3$. For the δ-terms in (4.51a) and (4.51b) in the case $r=2$ (not used here) we refer to [39]. Of course, we consider the corresponding boundary conditions (3.2) now written for the (ε,δ)-regularization and augmented also by $\boldsymbol{n}\cdot\nabla\varrho_{\varepsilon\delta\tau}^k=0$ and by $(\boldsymbol{n}\cdot\nabla)\boldsymbol{E}_{\varepsilon\delta\tau}^k=\boldsymbol{0}$ on Γ.

Like (4.10), we obtain the inequality

$$\begin{aligned}\int_\Omega\Bigg(\frac{\big|\boldsymbol{p}_{\varepsilon\delta\tau}^k\big|^2}{2\varrho_{\varepsilon\delta\tau}^k}+\varphi\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k\big)+\mathscr{U}\big(\theta_{\varepsilon\delta\tau}^k\big)+\tau[\zeta_{\rm p}]_{\boldsymbol{\Pi}}'\big(\theta_{\varepsilon\delta\tau}^k,\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k\big):\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k\\+\varepsilon\varphi''\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k\big)\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\vdots\Big(\big|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\big|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\Big)+\varepsilon|\boldsymbol{v}_{\varepsilon\delta\tau}^k|^p\Bigg)\,{\rm d}\boldsymbol{x}+\tau\int_\Gamma h\big(\theta_{\varepsilon\delta\tau}^k\big)\,{\rm d}S\\ \le\int_\Omega\frac{\big|\boldsymbol{p}_\tau^{k-1}\big|^2}{2\varrho_\tau^{k-1}}+\varphi\big(\boldsymbol{E}_\tau^{k-1}\big)+\mathscr{U}\big(\theta_\tau^{k-1}\big)+\tau\varrho_{\varepsilon\delta\tau}^k\boldsymbol{g}_\tau^k\cdot\boldsymbol{v}_{\varepsilon\delta\tau}^k\,{\rm d}\boldsymbol{x}+\int_\Gamma h_{{\rm ext},\tau}^k\,{\rm d}S.\end{aligned} \tag{4.52}$$

Here we used the cancellation of the two δ-regularizing terms as in [34, 39]. From (4.52), treating the term $\tau\varrho_{\varepsilon\delta\tau}^k\boldsymbol{g}_\tau^k\cdot\boldsymbol{v}_{\varepsilon\delta\tau}^k$ as in (4.12), we can directly read the *a-priori* estimates for $\boldsymbol{v}_{\varepsilon\delta\tau}^k\in L^p(\Omega;\mathbb{R}^3)$, $\boldsymbol{E}_{\varepsilon\delta\tau}^k\in L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$, and $\theta_{\varepsilon\delta\tau}^k\in L^{1+\alpha}(\Omega)$. Here we also used that $\theta_{\varepsilon\delta\tau}^k\ge0$, which can be seen by testing (4.51d) by $(\theta_{\varepsilon\delta\tau}^k)^-$.

Then, test of (4.51b) by $\boldsymbol{v}_{\varepsilon\delta\tau}^k$ while using (4.51a) multiplied by $|\boldsymbol{v}_{\varepsilon\delta\tau}^k|^2/2$ and (4.51c) multiplied by $\varphi(\boldsymbol{E}_{\varepsilon\delta\tau}^k)$ gives

$$\int_\Omega\Bigg(\frac{\big|\boldsymbol{p}_{\varepsilon\delta\tau}^k\big|^2}{2\varrho_{\varepsilon\delta\tau}^k}+\varphi\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k\big)+\varepsilon\big|\boldsymbol{v}_{\varepsilon\delta\tau}^k\big|^p+\tau\mathbb{D}\boldsymbol{\varepsilon}\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k\big):\boldsymbol{\varepsilon}\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k\big)+\tau\mu|\nabla^2\boldsymbol{v}_{\varepsilon\delta\tau}^k|^p\Bigg)\,{\rm d}\boldsymbol{x}$$

$$\leq \int_\Omega \frac{\big|\boldsymbol{p}_\tau^{k-1}\big|^2}{2\varrho_\tau^{k-1}} + \varphi\big(\boldsymbol{E}_\tau^{k-1}\big) + \tau\varrho_{\varepsilon\delta\tau}^k\boldsymbol{g}_\tau^k\cdot\boldsymbol{v}_{\varepsilon\delta\tau}^k + \frac{1}{3}\mathrm{tr}\big(\mathscr{T}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k\big) - \mathscr{T}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,0\big)\big)\,\mathrm{div}\,\boldsymbol{v}_{\varepsilon\delta\tau}^k\,\mathrm{d}\boldsymbol{x}. \tag{4.53}$$

Here the terms $[\zeta_\mathrm{p}]'_{\boldsymbol{\Pi}}(\theta_{\varepsilon\delta\tau}^k,\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k):\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k \geq 0$ and $\varepsilon\varphi''(\boldsymbol{E}_{\varepsilon\delta\tau}^k)\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\vdots(|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k) \geq 0$ in (4.52) are omitted because they would not yield a useful information. Treating the right-hand side as in (4.14), this gives the further *a-priori* estimates $\nabla\boldsymbol{v}_{\varepsilon\delta\tau}^k \in W^{1,p}(\Omega;\mathbb{R}^{3\times3})$. Then, testing (4.51a) by $\varrho_{\varepsilon\delta\tau}^k$ leads to an estimate (4.17) augmented by the left-hand-side term $\delta|\nabla\varrho_{\varepsilon\delta\tau}^k|^r$ and thus an estimate of $\varrho_{\varepsilon\delta\tau}^k \in W^{1,r}(\Omega)$ and, from (4.20) augmented correspondingly, we still obtain an estimate for $\mathrm{div}(\delta|\nabla\varrho_{\varepsilon\delta\tau}^k|^{r-2}\nabla\varrho_{\varepsilon\delta\tau}^k)$ in $L^2(\Omega)$. Moreover, by the same arguments as in Step 3, we can show that $\theta_{\varepsilon\delta\tau}^k > 0$ a.e. on Ω. Then, we can test (4.51d) by $(\theta_{\varepsilon\delta\tau}^k)^{-\lambda}$, and thus we arrive at a direct analogue of (4.35), namely

$$\int_\Omega \frac{\xi_\mathrm{ext}\big(\theta_{\varepsilon\delta\tau}^k;\boldsymbol{v}_{\varepsilon\delta\tau}^k,\boldsymbol{\Pi}_{\varepsilon\delta\tau}^k\big)}{\big(\theta_\tau^k\big)^\lambda} + \kappa\big(\theta_\tau^k\big)\frac{\big|\nabla\theta_{\varepsilon\delta\tau}^k\big|^2}{\big(\theta_{\varepsilon\delta\tau}^k\big)^{1+\lambda}}\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega \frac{\eta_\lambda\big(\theta_{\varepsilon\delta\tau}^k\big) - \eta_\lambda\big(\theta_\tau^{k-1}\big)}{\tau}\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \frac{h\big(\theta_{\mathrm{ext},\tau}^k\big)}{\big(\theta_{\mathrm{ext},\tau}^k\big)^\lambda}\,\mathrm{d}S$$
$$- \int_\Omega \Bigg(\theta_{\varepsilon\delta\tau}^k\frac{\phi'\big(\mathrm{tr}\boldsymbol{E}_{\varepsilon\delta\tau}^k\big) + \phi\big(\mathrm{tr}\boldsymbol{E}_{\varepsilon\delta\tau}^k\big)}{\big(\theta_\tau^k\big)^\lambda} + \frac{\theta_{\varepsilon\delta\tau}^k\gamma'\big(\theta_{\varepsilon\delta\tau}^k\big) + \gamma\big(\theta_{\varepsilon\delta\tau}^k\big)}{\big(\theta_{\varepsilon\delta\tau}^k\big)^\lambda} + \eta_\lambda\big(\theta_{\varepsilon\delta\tau}^k\big)\Bigg)\,\mathrm{div}\,\boldsymbol{v}_{\varepsilon\delta\tau}^k\,\mathrm{d}\boldsymbol{x}. \tag{4.54}$$

Then, by testing (4.51c) with $|\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{r-2}\boldsymbol{E}_{\varepsilon\delta\tau}^k$, by the calculus (4.37)–(4.38) for $\tau > 0$ sufficiently small, we obtain a bound of $\boldsymbol{E}_{\varepsilon\delta\tau}^k$ in $L^r(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym})$ for r arbitrarily large. Note that $\int_\Omega \mathrm{div}(\varepsilon|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k):(|\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{r-2}\boldsymbol{E}_{\varepsilon\delta\tau}^k)\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \varepsilon|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\vdots\nabla(|\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{r-2}\boldsymbol{E}_{\varepsilon\delta\tau}^k)\,\mathrm{d}\boldsymbol{x} = (1-r)\int_\Omega \varepsilon|\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{r-2}|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^s\,\mathrm{d}\boldsymbol{x} \leq 0$. Then, by interpolation calculus as in (4.40)–(4.42), we obtain a bound of $\nabla\theta_{\varepsilon\delta\tau}^k$ as in (4.43), *i.e.* here in $L^\mu(\Omega;\mathbb{R}^3)$ with μ from (4.8).

Also, testing (4.51c) by $\mathrm{div}(|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k)$ gives an additional estimate for $\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k \in L^s(\Omega;\mathbb{R}^{3\times3\times3})$ and for $\mathrm{div}(|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k)$ itself. More specifically,

$$\big\|\varrho_{\varepsilon\delta\tau}^k\big\|_{W^{1,r}(\Omega)} \leq C \quad\text{and}\quad \Big\|\mathrm{div}\Big(\big|\nabla\varrho_{\varepsilon\delta\tau}^k\big|^{r-2}\nabla\varrho_{\varepsilon\delta\tau}^k\Big)\Big\|_{L^2(\Omega)} \leq C/\sqrt{\delta}, \tag{4.55a}$$

$$\big\|\nabla\boldsymbol{v}_{\varepsilon\delta\tau}^k\big\|_{W^{1,p}(\Omega;\mathbb{R}^{3\times3})} \leq C, \quad \big\|\boldsymbol{v}_{\varepsilon\delta\tau}^k\big\|_{L^p(\Omega;\mathbb{R}^3)} \leq C/\sqrt[p]{\varepsilon}, \quad\text{and}\quad \Bigg\|\frac{\boldsymbol{p}_{\varepsilon\delta\tau}^k}{\sqrt{\varrho_{\varepsilon\delta\tau}^k}}\Bigg\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \tag{4.55b}$$

$$\big\|\boldsymbol{E}_{\varepsilon\delta\tau}^k\big\|_{W^{1,s}(\Omega;\mathbb{R}^{3\times3})} \leq C \quad\text{and}\quad \Big\|\mathrm{div}\Big(\big|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\big|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k\Big)\Big\|_{L^2(\Omega;\mathbb{R}^{3\times3\times3})} \leq C/\sqrt{\varepsilon}, \quad\text{and} \tag{4.55c}$$

$$\big\|\theta_{\varepsilon\delta\tau}^k\big\|_{L^{1+\alpha}(\Omega)\cap W^{1,\mu}(\Omega)} \leq C. \tag{4.55d}$$

This allows for the existence of weak solutions to (4.51) by rather standard methods for quasilinear elliptic problems when realizing that the highest-order terms in each equations in (4.51a)–(4.51c) are monotone and when realizing that the equation (4.51d) is semilinear, albeit with the L^1-right-hand side. Here, the compactness of $\boldsymbol{E}$'s and θ's is important together with the continuity of the nonlinearities $\mathscr{T}$ and $\mathscr{R}$ in the lower-order terms.

Then we pass to the limit with $\delta \to 0$ by choosing a subsequence such that

$$\varrho_{\varepsilon\delta\tau}^k \to \varrho_{\varepsilon\tau}^k \qquad \text{weakly in } W^{1,r}(\Omega), \tag{4.56a}$$

$$\boldsymbol{p}_{\varepsilon\delta\tau}^k \to \boldsymbol{p}_{\varepsilon\tau}^k \qquad \text{weakly in } W^{1,r}(\Omega;\mathbb{R}^3), \tag{4.56b}$$

$$\boldsymbol{v}_{\varepsilon\delta\tau}^k \to \boldsymbol{v}_{\varepsilon\tau}^k \qquad \text{weakly in } W^{2,p}(\Omega;\mathbb{R}^3), \tag{4.56c}$$

$$\boldsymbol{E}_{\varepsilon\delta\tau}^k \to \boldsymbol{E}_{\varepsilon\tau}^k \qquad \text{weakly in } W^{1,s}(\Omega;\mathbb{R}^{3\times3}_\mathrm{sym}),\ \text{and} \tag{4.56d}$$

$$\theta_{\varepsilon\delta\tau}^k \to \theta_{\varepsilon\tau}^k \qquad \text{weakly in } W^{1,\mu}(\Omega). \tag{4.56e}$$

According to the latter estimate in (4.55a), the regularizing term $\mathrm{div}(\delta|\nabla\varrho_{\varepsilon\delta\tau}^k|^{r-2}\nabla\varrho_{\varepsilon\delta\tau}^k)$ in (4.51a) is $\mathscr{O}(\sqrt{\delta})$ in $L^2(\Omega)$ and thus vanishes in the limit. Also, the compensating force $\delta|\nabla\varrho_{\varepsilon\delta\tau}^k|^{r-2}(\nabla\boldsymbol{v}_{\varepsilon\delta\tau}^k)\nabla\varrho_{\varepsilon\delta\tau}^k$ in (4.51b) is $\mathscr{O}(\delta)$ in $L^{r'}(\Omega;\mathbb{R}^3)$ and thus vanishes in the limit. The strong convergence (in terms of subsequences) of $\boldsymbol{v}_{\varepsilon\delta\tau}^k$ in $W^{2,p}(\Omega;\mathbb{R}^3)$ can be proved due to the strong monotonicity of the operator $\boldsymbol{v}\mapsto\mathrm{div}(\mathrm{div}(\mu|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v})-\mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}))$. In more detail, using (4.51b), we obtain

$$
\begin{aligned}
&\Big(\inf_{|E|=1}\mathbb{D}E{:}E\Big)\big\|\boldsymbol{\varepsilon}\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k-\tilde{\boldsymbol{v}}\big)\big\|_{L^2(\Omega;\mathbb{R}^{3\times3})}^2+\mu c_p\big\|\nabla^2\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k-\tilde{\boldsymbol{v}}\big)\big\|_{L^p(\Omega;\mathbb{R}^{3\times3\times3})}^p\\
&\quad\le\int_\Omega\bigg(\bigg(\varrho_{\varepsilon\delta\tau}^k\boldsymbol{g}_\tau^k-\frac{\boldsymbol{p}_{\varepsilon\delta\tau}^k-\boldsymbol{p}_\tau^{k-1}}{\tau}-\tau\varepsilon\big|\boldsymbol{v}_{\varepsilon\delta\tau}^k\big|^{p-2}\boldsymbol{v}_{\varepsilon\delta\tau}^k-\tau\delta\big|\nabla\varrho_{\varepsilon\delta\tau}^k\big|^{r-2}\big(\nabla\boldsymbol{v}_{\varepsilon\delta\tau}^k\big)\nabla\varrho_{\varepsilon\delta\tau}^k\bigg)\cdot\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k-\tilde{\boldsymbol{v}}\big)\\
&\qquad-\big(\mathbb{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})+\mathscr{T}\big(\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k\big)-\boldsymbol{p}_{\varepsilon\delta\tau}^k\otimes\boldsymbol{v}_{\varepsilon\delta\tau}^k\big){:}\boldsymbol{\varepsilon}\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k-\tilde{\boldsymbol{v}}\big)-\mu\big|\nabla^2\tilde{\boldsymbol{v}}\big|^{p-2}\nabla^2\tilde{\boldsymbol{v}}\,\vdots\,\nabla^2\big(\boldsymbol{v}_{\varepsilon\delta\tau}^k-\tilde{\boldsymbol{v}}\big)\bigg)\,\mathrm{d}\boldsymbol{x}\\
&\quad\overset{\delta\to0}{\to}\int_\Omega\bigg(\bigg(\varrho_{\varepsilon\tau}^k\boldsymbol{g}_\tau^k-\frac{\boldsymbol{p}_{\varepsilon\tau}^k-\boldsymbol{p}_\tau^{k-1}}{\tau}-\tau\varepsilon\big|\boldsymbol{v}_{\varepsilon\tau}^k\big|^{p-2}\boldsymbol{v}_{\varepsilon\tau}^k\bigg)\cdot\big(\boldsymbol{v}_{\varepsilon\tau}^k-\tilde{\boldsymbol{v}}\big)\\
&\qquad-\big(\mathbb{D}\boldsymbol{\varepsilon}(\tilde{\boldsymbol{v}})+\mathscr{T}\big(\boldsymbol{E}_{\varepsilon\tau}^k,\theta_{\varepsilon\tau}^k\big)-\boldsymbol{p}_{\varepsilon\tau}^k\otimes\boldsymbol{v}_{\varepsilon\tau}^k\big){:}\boldsymbol{\varepsilon}\big(\boldsymbol{v}_{\varepsilon\tau}^k-\tilde{\boldsymbol{v}}\big)-\mu\big|\nabla^2\tilde{\boldsymbol{v}}\big|^{p-2}\nabla^2\tilde{\boldsymbol{v}}\,\vdots\,\nabla^2\big(\boldsymbol{v}_{\varepsilon\tau}^k-\tilde{\boldsymbol{v}}\big)\bigg)\,\mathrm{d}\boldsymbol{x}
\end{aligned}\tag{4.57}
$$

for some $c_p>0$ and for any $\tilde{\boldsymbol{v}}\in W^{2,p}(\Omega;\mathbb{R}^3)$. We have also used the strong convergence $\mathscr{T}(\boldsymbol{E}_{\varepsilon\delta\tau}^k,\theta_{\varepsilon\delta\tau}^k)\to\mathscr{T}(\boldsymbol{E}_{\varepsilon\tau}^k,\theta_{\varepsilon\tau}^k)$ in $L^1(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})$. Choosing $\tilde{\boldsymbol{v}}=\boldsymbol{v}_{\varepsilon\tau}^k$, we obtain the mentioned strong convergence $\nabla\boldsymbol{v}_{\varepsilon\delta\tau}^k\to\nabla\boldsymbol{v}_{\varepsilon\tau}^k$ in $W^{1,p}(\Omega;\mathbb{R}^3)$. Similarly, the monotonicity of the operator $\boldsymbol{E}\mapsto-\mathrm{div}(\varepsilon|\nabla\boldsymbol{E}|^{s-2}\nabla\boldsymbol{E})$ allows for making the convergence in (4.56d) strong.

We thus obtain a solution $(\varrho_{\varepsilon\tau}^k,\boldsymbol{v}_{\varepsilon\tau}^k,\boldsymbol{E}_{\varepsilon\tau}^k,\theta_{\varepsilon\tau}^k)$ and thus also $\boldsymbol{p}_{\varepsilon\tau}^k$ and $u_{\varepsilon\tau}^k$ of the system (4.51) with δ omitted. In particular, (4.51a) turns into $\varrho_{\varepsilon\tau}^k+\mathrm{div}(\tau\varrho_{\varepsilon\tau}^k\boldsymbol{v}_{\varepsilon\tau}^k)=\varrho_\tau^{k-1}$ so that, by the estimation as in (4.23)–(4.25) when using also the last estimate in (4.55b), we obtain a bound for $\boldsymbol{v}_{\varepsilon\tau}^k$ in $L^a(\Omega;\mathbb{R}^3)$ for any $1\le a<2$ independent of ε. Together with the first estimate in (4.55b), we have $\boldsymbol{v}_{\varepsilon\tau}^k$ bounded in $W^{2,p}(\Omega;\mathbb{R}^3)$.

This allows us to pass to the limit with $\varepsilon\to0$ in terms of subsequences. The ε-regularizing in (4.51b) with δ omitted, *i.e.* now $\varepsilon|\boldsymbol{v}_{\varepsilon\tau}^k|^{p-2}\boldsymbol{v}_{\varepsilon\tau}^k$, is $\mathscr{O}(\varepsilon)$ in $L^\infty(\Omega;\mathbb{R}^3)$ so that it vanishes in the limit for ε. The latter estimate in (4.55c) makes the regularizing term $\mathrm{div}(\varepsilon|\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k|^{s-2}\nabla\boldsymbol{E}_{\varepsilon\delta\tau}^k)$ as $\mathscr{O}(\sqrt{\varepsilon})$ in $L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})$, so that it vanishes in the limit for $\varepsilon\to0$. Altogether, we obtain $(\varrho_\tau^k,\boldsymbol{v}_\tau^k,\boldsymbol{E}_\tau^k,\theta_\tau^k)$ and thus also $\boldsymbol{p}_\tau^k$ and u_τ^k solving the boundary-value problem (3.1) and (3.2).

Step 6: Convergence of (3.5) *for* $\tau\to0$. By the Banach selection principle, we obtain a subsequence converging weakly* with respect to the topologies indicated in (4.13), (4.15), (4.21), (4.26)–(4.29), (4.45), (4.49), and (4.50), to some limit $(\varrho,\boldsymbol{p},\boldsymbol{v},\boldsymbol{E},u)$. Specifically,

$$\bar\varrho_\tau\to\varrho\qquad\text{weakly* in }\ L^\infty\big(I;W^{1,r}(\Omega)\big),\tag{4.58a}$$
$$\varrho_\tau\to\varrho\qquad\text{weakly* in }\ L^\infty\big(I;W^{1,r}(\Omega)\big)\cap W^{1,p}(I;L^r(\Omega)),\tag{4.58b}$$
$$\bar{\boldsymbol{p}}_\tau\to\boldsymbol{p}\qquad\text{weakly in }\ L^p\big(I;W^{1,r}(\Omega;\mathbb{R}^3)\big),\tag{4.58c}$$
$$\boldsymbol{p}_\tau\to\boldsymbol{p}\qquad\text{weakly in }\ L^p\big(I;W^{1,r}(\Omega;\mathbb{R}^3)\big)\cap W^{1,p'}\big(I;W^{2,p}(\Omega;\mathbb{R}^3)^*\big),\tag{4.58d}$$
$$\bar{\boldsymbol{v}}_\tau\to\boldsymbol{v}\qquad\text{weakly* in }\ L^\infty\big(I;L^a(\Omega;\mathbb{R}^3)\big)\cap L^p\big(I;W^{2,p}(\Omega;\mathbb{R}^3)\big)\text{ with }1\le a<2,\tag{4.58e}$$
$$\overline{\boldsymbol{E}}_\tau\to\boldsymbol{E}\qquad\text{weakly* in }\ L^\infty\big(I;W^{1,s}(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)\cap L^\infty\big(I;L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})\big),\tag{4.58f}$$
$$\boldsymbol{E}_\tau\to\boldsymbol{E}\qquad\text{weakly* in }\ L^\infty\big(I;W^{1,s}(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})\big)\cap W^{1,p}\big(I;L^s(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})\big),\text{ and}\tag{4.58g}$$
$$\bar u_\tau\to u\qquad\text{weakly in }\ L^{(4+\alpha)\,\mu/(3+3\alpha)}(I\times\Omega).\tag{4.58h}$$

Notably, the limits of $\bar\varrho_\tau$ and ϱ_τ are indeed the same due to the control of $\frac{\partial}{\partial t}\varrho_\tau$ in (4.29a); cf. Section 8.2 of [29]. The same holds true also for $\bar{\boldsymbol{p}}_\tau$ and $\boldsymbol{p}_\tau$, and for $\overline{\boldsymbol{E}}_\tau$ and $\boldsymbol{E}_\tau$, too.

By the compact embedding $W^{1,r}(\Omega) \subset C(\overline{\Omega})$ for $r > 3$ and the (generalized) Aubin–Lions theorem, cf. Corollary 7.9 of [29], we have also

$$\bar{\varrho}_\tau \to \varrho \qquad \text{strongly in } \ L^a\big(I; C(\overline{\Omega})\big) \quad \text{for any } \ 1 \le a < \infty, \tag{4.59a}$$

$$\bar{\boldsymbol{p}}_\tau \to \boldsymbol{p} \qquad \text{strongly in } \ L^p\big(I; C\big(\overline{\Omega}; \mathbb{R}^3\big)\big), \ \text{ and} \tag{4.59b}$$

$$\overline{\boldsymbol{E}}_\tau \to \boldsymbol{E} \qquad \text{strongly in } \ L^a\big(I; L^2\big(\Omega; \mathbb{R}^{3\times 3}_{\rm sym}\big)\big) \ \text{ for any } \ 1 \le a < \infty. \tag{4.59c}$$

Moreover, using the Arzelà–Ascoli-type theorem, cf. Lemma 7.10 of [29], we have also

$$\varrho_\tau \to \varrho \qquad \text{strongly in } \ C\big(I{\times}\overline{\Omega}\big). \tag{4.59d}$$

Moreover, from (4.59a,b), we know that, for a subsequence of $\tau \to 0$, $\bar{\boldsymbol{p}}_\tau/\bar{\varrho}_\tau$ converges to some $\tilde{\boldsymbol{v}}$ a.e. on $I{\times}\Omega$. Simultaneously, we know $\sup_{\tau>0} \int_0^T \int_\Omega |\tilde{\boldsymbol{v}} - \bar{\boldsymbol{p}}_\tau/\bar{\varrho}_\tau|^p \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t < \infty$ with $p > 1$ so that, by the de la Vallée Poussin theorem, $\{|\tilde{\boldsymbol{v}}{-}\bar{\boldsymbol{p}}_\tau/\bar{\varrho}_\tau|^q\}_{\tau>0}$ is relatively weakly compact in $L^1(I{\times}\Omega)$ for any $1 \le q < p$. Then, by the Dunford–Pettis theorem, it is uniformly integrable and, by the Vitali theorem, it converges a.e. to its limit which equals 0, *i.e.* it converges to 0 in $L^q(I{\times}\Omega)$. By (4.58e), we can identify $\tilde{\boldsymbol{v}} = \boldsymbol{v}$ so that all the (already chosen subsequence for (4.58)) converges to $\boldsymbol{v}$. This proves

$$\bar{\boldsymbol{v}}_\tau = \frac{\bar{\boldsymbol{p}}_\tau}{\bar{\varrho}_\tau} \to \frac{\boldsymbol{p}}{\varrho} = \boldsymbol{v} \qquad \text{strongly in } \ L^q(I{\times}\Omega; \mathbb{R}^3) \ \text{ for any } 1 \le q < p, \tag{4.60a}$$

Actually, by interpolation with (4.58e), we have the strong convergence $\bar{\boldsymbol{v}}_\tau \to \boldsymbol{v}$ even in a better space $L^s(I; L^a(\Omega;\mathbb{R}^3)) \cap L^q(I; C(\overline{\Omega};\mathbb{R}^3))$ with any $s < \infty$ and $1 \le a < 2$. Thus, by (4.59b), also

$$\bar{\boldsymbol{p}}_\tau{\otimes}\bar{\boldsymbol{v}}_\tau \to \boldsymbol{p}{\otimes}\boldsymbol{v} = \varrho\boldsymbol{v}{\otimes}\boldsymbol{v} \qquad \text{strongly in } \ L^q\big(I; L^a\big(\Omega; \mathbb{R}^{3\times 3}_{\rm sym}\big)\big), \quad 1 \le q < p, \ 1 \le a < 2. \tag{4.60b}$$

The bound (4.45) and, by comparison, the bound for $\frac{\partial}{\partial t} u_\tau$ in $L^1(I; W^{2,6}(\Omega)^*)$ allow us to use the (generalized) Aubin–Lions theorem, cf. Corollary 7.9 of [29], so that the convergence (4.58h) is even strong. Realizing $\bar{\theta}_\tau = \mathscr{U}^{-1}(\bar{u}_\tau)$ and the estimate (4.44), we use the continuity of the Nemytskiĭ mapping and obtain

$$\bar{\theta}_\tau \to \theta = \mathscr{U}^{-1}(u) \ \text{ strongly in } L^a(I{\times}\Omega) \text{ for any } 1 \le a < \frac{(4+\alpha)\,\mu}{3}. \tag{4.61a}$$

In view of (4.46), we thus also have

$$\varkappa(\bar{\theta}_\tau) \to \varkappa(\theta) \qquad \text{strongly in } \ L^{(4+\alpha)\,\mu/(3+3\beta)}(I{\times}\Omega). \tag{4.61b}$$

Furthermore, for the Cauchy stress and for the inelastic-rate mapping, we have

$$\mathscr{T}\big(\overline{\boldsymbol{E}}_\tau, \bar{\theta}_\tau\big) \to \mathscr{T}(\boldsymbol{E}, \theta) \qquad \text{strongly in } \ L^1(I{\times}\Omega; \mathbb{R}^{3\times 3}_{\rm sym}) \ \text{ and} \tag{4.61c}$$

$$\overline{\boldsymbol{\Pi}}_\tau = \mathscr{R}\big(\overline{\boldsymbol{E}}_\tau, \bar{\theta}_\tau\big) \to \mathscr{R}(\boldsymbol{E}, \theta) =: \boldsymbol{\Pi} \ \text{ strongly in } \ L^2\big(I{\times}\Omega; \mathbb{R}^{3\times 3}_{\rm dev}\big); \tag{4.61d}$$

here we have used the growth conditions (4.6d) and (4.6e) and the continuity of $\mathscr{T}$ and $\mathscr{R}$.
This already allows for the limit passage in (3.5). While the limit passage in (3.5a) is simple due to (4.58b,c), the limit passage in the quasilinear momentum equation (3.5b) is a bit more technical. To this aim, we use the uniform monotonicity of the operator $\boldsymbol{v} \mapsto \mathrm{div}(\mathrm{div}(\mu|\nabla^2\boldsymbol{v}|^{p-2}\nabla^2\boldsymbol{v}) - \mathbb{D}\boldsymbol{\varepsilon}(\boldsymbol{v}))$ with the boundary conditions (3.6) and, using (3.5b) tested by $\bar{\boldsymbol{v}}_\tau{-}\tilde{\boldsymbol{v}}$, we obtain

$$\Big(\inf_{|E|=1} \mathbb{D}E{:}E\Big) \|\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau{-}\tilde{\boldsymbol{v}})\|^2_{L^2(I\times\Omega;\mathbb{R}^{3\times3})} + \mu c_p \big\|\nabla^2(\bar{\boldsymbol{v}}_\tau{-}\tilde{\boldsymbol{v}})\big\|^p_{L^p(I\times\Omega;\mathbb{R}^{3\times3\times3})}$$
$$\le \int_0^T \!\! \int_\Omega \mathbb{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau{-}\tilde{\boldsymbol{v}}){:}\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}_\tau{-}\tilde{\boldsymbol{v}}) + \mu\Big(\big|\nabla^2\bar{\boldsymbol{v}}_\tau\big|^{p-2}\nabla^2\bar{\boldsymbol{v}}_\tau - \big|\nabla^2\tilde{\boldsymbol{v}}\big|^{p-2}\nabla^2\tilde{\boldsymbol{v}}\Big)\vdots\nabla^2(\bar{\boldsymbol{v}}_\tau{-}\tilde{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$= \int_0^T\!\!\int_\Omega \bigg(\Big(\bar\varrho_\tau \bar{\boldsymbol g}_\tau - \frac{\partial \boldsymbol p_\tau}{\partial t}\Big)\cdot(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v}) - \big(\mathscr{T}\big(\overline{\boldsymbol E}_\tau,\bar\theta_\tau\big) - \bar{\boldsymbol p}_\tau{\otimes}\bar{\boldsymbol v}_\tau\big){:}\boldsymbol\varepsilon(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v})$$
$$- \mathbb{D}\boldsymbol\varepsilon(\tilde{\boldsymbol v}){:}\boldsymbol\varepsilon(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v}) - \mu\big|\nabla^2\tilde{\boldsymbol v}\big|^{p-2}\nabla^2\tilde{\boldsymbol v}\vdots\nabla^2(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v})\bigg)\,\mathrm{d}\boldsymbol x\,\mathrm{d}t$$
$$\le \int_0^T\int_\Omega \bigg(\bar\varrho_\tau \boldsymbol g_\tau\cdot(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v}) + \frac{\partial \boldsymbol p_\tau}{\partial t}\cdot\tilde{\boldsymbol v} - (\bar{\boldsymbol p}_\tau{\otimes}\bar{\boldsymbol v}_\tau){:}\boldsymbol\varepsilon(\tilde{\boldsymbol v}) - \big(\mathbb{D}\boldsymbol\varepsilon(\tilde{\boldsymbol v}){+}\mathscr{T}\big(\overline{\boldsymbol E}_\tau,\bar\theta_\tau\big)\big){:}\boldsymbol\varepsilon(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v})$$
$$- \mu\big|\nabla^2\tilde{\boldsymbol v}\big|^{p-2}\nabla^2\tilde{\boldsymbol v}\vdots\nabla^2(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v})\bigg)\,\mathrm{d}\boldsymbol x\,\mathrm{d}t + \int_\Omega \frac{|\boldsymbol p_0|^2}{2\varrho_0} - \frac{|\boldsymbol p_\tau(T)|^2}{2\varrho_\tau(T)}\,\mathrm{d}\boldsymbol x \tag{4.62}$$

for some $c_p > 0$ and for any $\tilde{\boldsymbol v} \in L^p(I;W^{2,p}(\Omega;\mathbb{R}^3))$. The last inequality in (4.62) has again exploited the convexity of the kinetic energy $(\boldsymbol p,\varrho)\mapsto \frac12|\boldsymbol p|^2/\varrho$ in the calculus:

$$\int_\Omega \frac{|\boldsymbol p_\tau(T)|^2}{2\varrho_\tau(T)} - \frac{|\boldsymbol p_0|^2}{2\varrho_0}\,\mathrm{d}\boldsymbol x \le \int_0^T\!\!\int_\Omega \frac{\partial \boldsymbol p_\tau}{\partial t}\cdot\bar{\boldsymbol v}_\tau - \frac{|\bar{\boldsymbol v}_\tau|^2}{2}\frac{\partial\varrho_\tau}{\partial t}\,\mathrm{d}\boldsymbol x\,\mathrm{d}t$$
$$\overset{(3.5a)}{=} \int_0^T\!\!\int_\Omega \frac{\partial \boldsymbol p_\tau}{\partial t}\cdot\bar{\boldsymbol v}_\tau + \frac{|\bar{\boldsymbol v}_\tau|^2}{2}\,\mathrm{div}\,\bar{\boldsymbol p}_\tau\,\mathrm{d}\boldsymbol x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \frac{\partial \boldsymbol p_\tau}{\partial t}\cdot\bar{\boldsymbol v}_\tau + \bar{\boldsymbol v}_\tau\cdot\mathrm{div}\big(\bar{\boldsymbol p}_\tau{\otimes}\bar{\boldsymbol v}_\tau\big)\,\mathrm{d}\boldsymbol x\,\mathrm{d}t, \tag{4.63}$$

where, for the last equality, we have used (2.28). Now we want to pass to the limit in (4.62) or, more precisely, to estimate the limit superior from above. For this, we again use the kinetic-energy convexity, which causes the weak lower semicontinuity of $(\varrho,\boldsymbol p)\mapsto\int_\Omega|\boldsymbol p|^2/\varrho\,\mathrm{d}\boldsymbol x$ as a convex functional $\{\rho{\in}L^1(\Omega);\ \rho\ge 0\}\times L^1(\Omega;\mathbb{R}^3)\to[0,+\infty]$. Here, we rely also on that $|\boldsymbol p_\tau(T)|^2/\varrho_\tau(T)$ is bounded in $L^1(\Omega)$ due to the former estimate in (4.13a) and on that $\varrho_\tau(T)\to\varrho(T)$ even strongly in $C(\overline\Omega)$ due to (4.59d), and on that $\boldsymbol p_\tau(T)$ converges weakly* in $C(\overline\Omega)^*$, *i.e.* as measures on $\overline\Omega$ due to (4.13d) to its limit which is $\boldsymbol p(T)$ because simultaneously $\boldsymbol p_\tau(T)\to\boldsymbol p(T)$ weakly in $W^{2,p}(\Omega;\mathbb{R}^3)^*$ due to (4.58d). For the term $(\bar{\boldsymbol p}_\tau{\otimes}\bar{\boldsymbol v}_\tau){:}\boldsymbol\varepsilon(\tilde{\boldsymbol v})$, we use simply (4.60b). All of this allows us to estimate of the limit superior of the right-hand side in (4.62) from above, so that:

$$\limsup_{\tau\to0}\bigg(\Big(\inf_{|E|=1}\mathbb{D}E{:}E\Big)\|\boldsymbol\varepsilon(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v})\|^2_{L^2(I\times\Omega;\mathbb{R}^{3\times3})} + \mu c_p\big\|\nabla^2(\bar{\boldsymbol v}_\tau{-}\tilde{\boldsymbol v})\big\|^p_{L^2(I\times\Omega;\mathbb{R}^{3\times3\times3})}\bigg)$$
$$\le \int_0^T\Big\langle\frac{\partial\boldsymbol p}{\partial t},\tilde{\boldsymbol v}\Big\rangle + \int_\Omega\bigg(\varrho\boldsymbol g\cdot(\boldsymbol v{-}\tilde{\boldsymbol v}) - (\boldsymbol p{\otimes}\boldsymbol v){:}\boldsymbol\varepsilon(\tilde{\boldsymbol v}) - (\mathscr{T}(\boldsymbol E,\theta) + \mathbb{D}\boldsymbol\varepsilon(\tilde{\boldsymbol v})){:}\boldsymbol\varepsilon(\boldsymbol v{-}\tilde{\boldsymbol v})$$
$$- \mu\big|\nabla^2\tilde{\boldsymbol v}\big|^{p-2}\nabla^2\tilde{\boldsymbol v}\vdots\nabla^2(\boldsymbol v{-}\tilde{\boldsymbol v})\bigg)\,\mathrm{d}\boldsymbol x\,\mathrm{d}t + \int_\Omega\frac{|\boldsymbol p_0|^2}{2\varrho_0} - \frac{|\boldsymbol p(T)|^2}{2\varrho(T)}\,\mathrm{d}\boldsymbol x$$
$$= \int_0^T\Big\langle\frac{\partial\boldsymbol p}{\partial t},\tilde{\boldsymbol v}-\boldsymbol v\Big\rangle + \int_\Omega\bigg(\varrho\boldsymbol g\cdot(\boldsymbol v{-}\tilde{\boldsymbol v}) - (\mathscr{T}(\boldsymbol E,\theta) - \boldsymbol p{\otimes}\boldsymbol v){:}\boldsymbol\varepsilon(\boldsymbol v{-}\tilde{\boldsymbol v})$$
$$- \mathbb{D}\boldsymbol\varepsilon(\tilde{\boldsymbol v}){:}\boldsymbol\varepsilon(\boldsymbol v{-}\tilde{\boldsymbol v}) - \mu\big|\nabla^2\tilde{\boldsymbol v}\big|^{p-2}\nabla^2\tilde{\boldsymbol v}\vdots\nabla^2(\boldsymbol v{-}\tilde{\boldsymbol v})\bigg)\,\mathrm{d}\boldsymbol x\,\mathrm{d}t, \tag{4.64}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between $W^{2,p}(\Omega;\mathbb{R}^3)^*$ and $W^{2,p}(\Omega;\mathbb{R}^3)$ and where, for the last equality, we used the calculus like (4.63) but for the continuous-in-time limit which holds as an equality. Choosing $\tilde{\boldsymbol v}=\boldsymbol v$ and reminding also (4.60a), we obtained

$$\bar{\boldsymbol v}_\tau\to\boldsymbol v\quad\text{strongly in}\quad L^p\big(I;W^{2,p}\big(\Omega;\mathbb{R}^3\big)\big). \tag{4.65}$$

Thus, we can easily make the limit passage in (3.5b) and thus prove that $\boldsymbol v$ satisfies the momentum equation (2.22b) in the weak sense (2.38).
The limit passage in (3.5c) is even simpler. Specifically, due to (4.58f) and (4.65), we have

$$\boldsymbol B_{\rm ZJ}\big(\bar{\boldsymbol v}_\tau,\overline{\boldsymbol E}_\tau\big)\to\boldsymbol B_{\rm ZJ}(\boldsymbol v,\boldsymbol E)\quad\text{weakly in}\quad L^p\big(I;L^s\big(\Omega;\mathbb{R}^{3\times3}_{\rm sym}\big)\big). \tag{4.66}$$

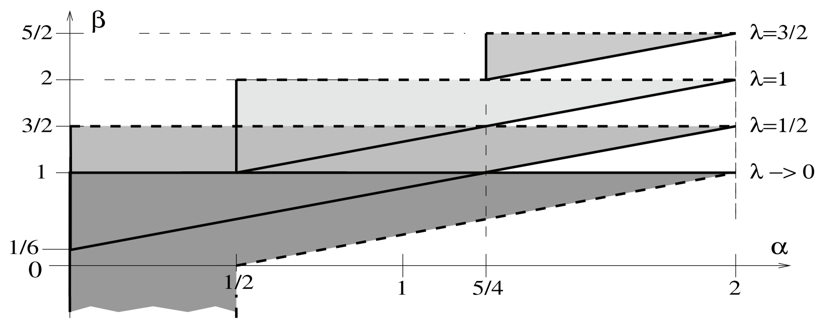

FIGURE 1. The (α, β)-pairs complying with the restrictions (4.7) for four values of $0<\lambda<2$.

The other terms in (3.5c) can be handled by (4.59c) with (4.59b).

Finally, for the limit passage in (3.5e), we exploit the strong convergences in (4.61).

Step 7: Energy balances. It is now important that the tests and then all the subsequent calculations leading to the energy balance (2.32) integrated over a current time interval $[0,t]$ are analytically legitimate.

First, note that, by (4.29), we have that $\frac{\partial}{\partial t}\varrho \in L^p(I;L^r(\Omega))$ and $\frac{\partial}{\partial t}\boldsymbol{p} \in L^{p'}(I;W^{2,p}(\Omega;\mathbb{R}^3)^*)$. By (4.50), we also have that $\frac{\partial}{\partial t}\boldsymbol{E} \in L^p(I;L^s(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$. Then, for the calculus (2.29) integrated over in time $[0,t]$, we rely on that $\boldsymbol{T}_0 = \varphi'(\boldsymbol{E})+\varphi(\boldsymbol{E})\mathbb{I} \in L^\infty(I;L^3(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$ is certainly in duality with $\boldsymbol{\varepsilon}(\boldsymbol{v}) \in L^p_{\rm w}(I;L^\infty(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$ and $\varphi'(\boldsymbol{E}) \in L^\infty(I;L^6(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$ is in duality both with $\mathring{\boldsymbol{E}} \in L^p(I;L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$ as well as with ${\rm dev}\,\boldsymbol{T}_0 \in L^\infty(I;L^6(\Omega;\mathbb{R}^{3\times3}_{\rm sym}))$, too. Thus the calculus (2.29) is indeed legitimate when integrated over time.

Furthermore, $\frac{\partial}{\partial t}\boldsymbol{p} \in L^{p'}(I;W^{2,p}(\Omega;\mathbb{R}^3)^*)$ and ${\rm div}(\varrho\boldsymbol{v}\otimes\boldsymbol{v}) \in L^{1+p/2}(I\times\Omega;\mathbb{R}^{3\times3}_{\rm sym})$ as well as ${\rm div}\boldsymbol{D} \in L^{p'}(I;W^{2,p}(\Omega;\mathbb{R}^3)^*)$ are in duality with $\boldsymbol{v} \in L^p(I;W^{2,p}(\Omega;\mathbb{R}^3))$, as used when testing the momentum equation by $\boldsymbol{v}$, in particular in (2.28). The calculus (2.27) also relies on that both $\frac{\partial}{\partial t}\varrho$ and ${\rm div}(\varrho\boldsymbol{v}) = \boldsymbol{v}\cdot\nabla\varrho+\varrho\,{\rm div}\boldsymbol{v}$ live in $L^p(I;L^r(\Omega))$ and are thus certainly in duality with $|\boldsymbol{v}|^2 \in L^{p/2}_{\rm w}(I;L^\infty(\Omega))$ for $p \geq 3$.

The right-hand side of (2.22d) is in $L^1(I\times\Omega)$, so that (2.22d) bears legitimately integration over $[0,t]\times\Omega$. Adding this to (2.32) integrating over $[0,t]$ then yields the total-energy balance (2.33) integrated over $[0,t]$. Thus the point (iv) is proven.

□

Remark 4.3 (Heat bulk sources)**.** The above analysis could easily be extended if the heat equation (2.22d) had a non-negative right-hand side in $L^1(I\times\Omega)$. Such sources arise in engineering from some exo- or endo-thermic chemical processes or from electric eddy currents etc. or in geophysics in the planetary mantle from radiogenic heating or absorption radiation from the Sun in the atmosphere.

Remark 4.4 (The restrictions on α and β in (4.7))**.** Depending on $\lambda > 0$, the bounds (4.7) restrict possible exponents α and β. For $\lambda > 2$, no pair is consistent with (4.7). Depending on $0 < \lambda \leq 2$, the admissible pairs (α, β) lie within the polyhedrons as in Figure 1. The usually considered situation that κ bounded and the test for $\lambda \to 0$ leads here to the restriction $0 \leq \alpha < 1/2$, as *e.g.* in [18,33]. Allowing for a growth of $\kappa = \kappa(\theta)$ as $\sim \theta^\beta$ with $\beta > 0$ and using a general test by some fixed positive λ opens possibilities for $1/2 \leq \alpha < 2$.

Remark 4.5 (The general $\boldsymbol{E}$-dependent heat conductivity)**.** When $\kappa = \kappa(\boldsymbol{E}, \theta)$, we can generalize $\varkappa$ in (2.38b) as $\varkappa(\boldsymbol{E},\theta) := \int_0^\theta \kappa(\boldsymbol{E},\vartheta)\,{\rm d}\vartheta$, so that $\kappa(\boldsymbol{E},\theta)\nabla\theta = \nabla\varkappa(\boldsymbol{E},\theta) - \varkappa'_{\boldsymbol{E}}(\boldsymbol{E},\theta)\nabla\boldsymbol{E}$. Thus, the integral identity (2.38b) augments by the term $\nabla\boldsymbol{E}\,\vdots\,(\varkappa'_{\boldsymbol{E}}(\boldsymbol{E},\theta)\otimes\nabla\widetilde{\theta})$ which, however, is not integrable if the Maxwellian viscosity $\zeta_{\rm p}$ is temperature dependent, as seen in the calculations in Step 4 above. Therefore, instead of $\varkappa(\theta)\Delta\widetilde{\theta}$ in (2.38b), one should use $-\kappa(\boldsymbol{E},\theta)\nabla\theta\cdot\nabla\tilde\theta$, for which we need the heat flux to be integrable. Realizing the bound $\kappa(\overline{\boldsymbol{E}}_\tau,\theta)=\mathscr{O}(\theta^\beta)$, we obtain the bound for the (negative) heat flux $\kappa(\overline{\boldsymbol{E}}_\tau,\bar\theta_\tau)\nabla\bar\theta_\tau$, namely

$$\big\|\kappa\big(\overline{\boldsymbol{E}}_\tau,\bar\theta_\tau\big)\nabla\bar\theta_\tau\big\|_{L^{(4+\alpha)\,\mu/(4+\alpha+3\beta)}(I\times\Omega;\mathbb{R}^3)}\le C. \tag{4.67}$$

To legitimate the weak formulation of the heat equation and to facilitate the limit passage, the exponent in (4.67) must be greater than (or equal to) 1. Recalling (4.8), this holds for $\alpha\ge3\lambda-1$. This is a slightly stronger restriction on the exponent α than $\alpha\ge\frac32\lambda-1$, which is needed for (4.46). For the convergence with $\tau\to0$, in view of (4.67), we also have

$$\kappa\big(\overline{\boldsymbol{E}}_\tau,\bar\theta_\tau\big)\nabla\bar\theta_\tau\to\kappa(\boldsymbol{E},\theta)\nabla\theta\ \text{ weakly in }\ L^{(4+\alpha)\,\mu/(4+\alpha+3\beta)}\big(I\times\Omega;\mathbb{R}^3\big). \tag{4.68}$$

Remark 4.6 (Rate-dependent plasticity)**.** The inelastic-strain rate $\mathscr{R}:(E,\theta)$ involving the conjugate dissipation potential $\zeta_{\rm p}^*(\theta,\cdot)$, considered here as continuously differentiable, can also cover the rate-dependent plasticity. However, since plasticity is an activated process, it typically involves a potential that is non-smooth at zero plastification rate. Nevertheless, some applications (particularly in geophysics, where the activated plastification is combined with so-called aseismic slip) combine non-smooth plastic-type and smooth creep-type (say, quadratic) potentials "in series"; such a serial arrangement is called an *extended Maxwell model*. This means that the original dissipation potential $\zeta_{\rm p}(\theta,\cdot)$ is a so-called infimal convolution of the two aforementioned potential and its conjugate $\zeta_{\rm p}^*(\theta,\cdot)$ is the sum of the conjugates of those two original potentials. To obtain a smooth $\zeta_{\rm p}^*(\theta,\cdot)$, it should still be combined with the Stokes viscosity in parallel. This can advantageously and directly be used in (2.20), avoiding an explicit (and often nontrivial) evaluation of the infimal convolution. Cf. [32].

Example 4.7 (Creep in thermally expanding materials)**.** The linear creep with the Maxwell viscosity modulus $M=M(\theta)>0$ is governed by the quadratic dissipation functional $\zeta_{\rm p}(\theta,\cdot)=\frac12M(\theta)|\cdot|^2$. Thus $\mathscr{R}(\boldsymbol{E},\theta)=M^{-1}(\theta)\,{\rm dev}\,\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta)=M^{-1}(\theta)\,{\rm dev}\,\varphi'(\boldsymbol{E})+\theta M^{-1}(\theta)\,{\rm dev}\,\phi'(\boldsymbol{E})$, where the ansatz (4.1) has been taken into account. When $\psi(\cdot,\theta)$ is convex, the positive semi-definiteness of $\mathscr{R}'_{\boldsymbol{E}}$ holds. For the specific case that complies with the ansatz (4.1), we can consider

$$\begin{aligned}\psi(\boldsymbol{E},\theta)&=\frac12K|{\rm tr}\,\boldsymbol{E}-\alpha_{\rm v}\theta|^2+G|{\rm dev}\,\boldsymbol{E}|^2-\frac{c_{\rm v}}{\alpha(1{+}\alpha)}\theta^{1+\alpha}-\tfrac12K\alpha_{\rm v}^2\theta^2\\&=\frac12K({\rm tr}\boldsymbol{E})^2+G|{\rm dev}\,\boldsymbol{E}|^2-\alpha_{\rm v}\theta K{\rm tr}\,\boldsymbol{E}-\frac{c_{\rm v}}{\alpha(1{+}\alpha)}\theta^{1+\alpha}\end{aligned} \tag{4.69}$$

with K and G as the bulk and the shear elastic moduli, respectively, and with $\alpha_{\rm v}$ denoting the *thermal volume expansibility*. Thus, we obtain

$$\begin{aligned}&\mathscr{E}(\boldsymbol{E},\theta)=\psi(\boldsymbol{E},\theta)-\theta\psi'_\theta(\boldsymbol{E},\theta)=\tfrac12K({\rm tr}\,E)^2+G|{\rm dev}\,\boldsymbol{E}|^2+\tfrac{c_{\rm v}}{1+\alpha}\theta^{1+\alpha},\\&\mathscr{E}'_E(E,\theta)=\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta)-\theta\psi''_{\boldsymbol{E}\theta}(\boldsymbol{E},\theta)=K{\rm tr}\,\boldsymbol{E}+2G\,{\rm dev}\,\boldsymbol{E},\\&\mathscr{E}'_\theta(\boldsymbol{E},\theta)=-\theta\psi''_{\theta\theta}(\boldsymbol{E},\theta)=c_{\rm v}\theta^\alpha,\quad\text{ and }\ \eta(\boldsymbol{E},\theta)=-\psi'_\theta(\boldsymbol{E},\theta)=\tfrac{c_{\rm v}}{\alpha}\theta^\alpha+\alpha_{\rm v}K\,{\rm tr}\,\boldsymbol{E},\\&\mathscr{T}(\boldsymbol{E},\theta)=\psi'_{\boldsymbol{E}}(\boldsymbol{E},\theta)+\psi(E,\theta)\mathbb{I}=(K({\rm tr}\,\boldsymbol{E}{-}\alpha_{\rm v}\theta)+\psi(\boldsymbol{E},\theta))\,\mathbb{I}+2G\,{\rm dev}\,\boldsymbol{E},\quad\text{ and}\\&\mathscr{R}(\boldsymbol{E},\theta)=[\zeta_{\rm p}(\theta,\cdot)^*]'({\rm dev}\mathscr{T}(\boldsymbol{E},\theta))=\tfrac{2G}{M(\theta)}\,{\rm dev}\,\boldsymbol{E}.\end{aligned}$$

This special case complies with (4.6b)–(4.6d). Also, the convexity of the function $1/[\mathscr{U}^{-1}](\cdot)^\lambda$, used for (4.30), is satisfied for any $0\le\lambda$. To see this, note that, from $u=\mathscr{U}(\theta)=\frac{c_{\rm v}}{1+\alpha}\theta^{1+\alpha}$, we can see $\theta=\mathscr{U}^{-1}(u)=(\frac{1+\alpha}{c_{\rm v}}u)^{1/(1+\alpha)}$ so that $1/\theta^\lambda=1/[\mathscr{U}^{-1}](u)^\lambda=(\frac{1+\alpha}{c_{\rm v}}u)^{-\lambda/(1+\alpha)}$. Also, note that $(u^{-\lambda/(1+\alpha)})''=-\frac{\lambda}{1+\alpha}(u^{-1-\lambda/(1+\alpha)})'=\frac{\lambda}{1+\alpha}(1+\frac{\lambda}{1+\alpha})u^{-2-\lambda/(1+\alpha)}>0$ so that $1/[\mathscr{U}^{-1}](\cdot)^\lambda$ is convex on $\mathbb{R}^+$. The function $\tilde\eta_\lambda=\tilde\eta_\lambda(u)$ from (4.30) is then $\tilde\eta_\lambda(u)=\frac{1+\alpha}{1+\alpha-\lambda}(\frac{1+\alpha}{c_{\rm v}}u)^{1-\lambda/(1+\alpha)}$.

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Data availability statement

The research data associated with this article are included in the article.

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