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CONVERGENCE RATE FOR A SEMIDISCRETE APPROXIMATION OF SCALAR CONSERVATION LAWS

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Abstract. We propose a semidiscrete scheme for approximation of entropy solutions of one-dimensional scalar conservation laws with nonnegative initial data. The scheme is based on the concept of particle paths for conservation laws and can be interpreted as a finite-particle discretization. A convergence rate of order 1/2 with respect to initial particle spacing is proved. As a special case, this covers the convergence of the Follow-the-Leader model to the Lighthill–Whitham–Richards model for traffic flow.

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1. Introduction

We consider the one-dimensional, scalar conservation law

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(0) = u_0, \end{cases} \tag{1}$$

where the initial data is a nonnegative function $u_0 \in \mathrm{BV} \cap L^1(\mathbb{R})$ and the flux $f\colon \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz continuous with a well-defined derivative at zero ($f'(0) \in \mathbb{R}$).

The present work builds on the equivalence between unique particle paths and entropy solutions for conservation laws established in [12]. This result is based on interpreting the conservation law (1) as a continuity equation

$$\partial_t u + \partial_x\big(a(u)u\big) = 0, \quad \text{where} \quad a(u) \coloneqq \begin{cases} \frac{f(u)-f(0)}{u} & \text{if } u \neq 0, \\ f'(0) & \text{if } u = 0, \end{cases} \tag{2}$$

leading to particle paths

$$\begin{cases} \dot{x}_t = a(u(x_t, t)) \\ x_0 = x. \end{cases} \tag{3}$$

In particular, it states that if u is the entropy solution of (1), then the ODE (3) has a unique solution $t \mapsto x_t$ for all initial conditions $x \in \mathbb{R}$. (Here, $\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}x$, and x_t denotes x evaluated at time t). Moreover, the collection

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of all such solutions x_t for initial conditions $x \in \mathbb{R}$ forms a continuous forward flow $(x,t) \mapsto X_t(x)$, and the entropy solution satisfies the pushforward formula

$$u(t) = (X(t))_{\#}u_0, \quad \text{i.e.} \quad \int_{\mathbb{R}} \vartheta(x)u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} \vartheta(X_t(x))u_0(x)\,\mathrm{d}x \tag{4}$$

for all $\vartheta \in C_c(\mathbb{R})$.

We will formulate a novel finite-particle approximation of entropy solutions of the conservation law (1), centered around discretized versions of the particle path ODE (3) and the pushforward representation (4).

For concave flux functions f, the proposed scheme coincides exactly with the well-known Follow-the-Leader (FtL) model for traffic flow. This connection provides a new theoretical underpinning for the FtL model, and enables us to systematically analyze its rate of convergence to the macroscopic Lighthill–Whitham–Richards (LWR) model. To our knowledge, a convergence proof of this nature for the FtL model has been previously lacking.

A simplified version of the scheme, hereafter referred to as the *particle path scheme*, is as follows. A complete detailed specification will be given in Section 3.

Step 1. *Approximation of initial data.* Initialize a collection of N *particles* $x_0^1 < x_0^2 < \cdots < x_0^N$ with arbitrary spacing, and approximate the initial data u_0 by a function ρ_0 which is piecewise constant with *local densities*

$$\rho_0^i = \frac{1}{\Delta x_0^i} \int_{x_0^i}^{x_0^{i+1}} u_0(x)\,\mathrm{d}x$$

on each interval (x_0^i, x_0^{i+1}), where $\Delta x_0^i \coloneqq x_0^{i+1} - x_0^i$.

Step 2. *Particle dynamics.* Let each particle $t \mapsto x_t^i$ evolve according to the ODE $\dot{x}_t^i = V(\rho_t^{i-1}, \rho_t^i)$, where V is a velocity function given by

$$V(\rho_l, \rho_r) \coloneqq \begin{cases} \min_{\rho \in [\rho_l, \rho_r]} a(\rho) & \text{if } \ 0 \le \rho_l \le \rho_r \\ \max_{\rho \in [\rho_r, \rho_l]} a(\rho) & \text{if } \ 0 \le \rho_r \le \rho_l. \end{cases} \tag{5}$$

Simultaneously update local densities $t \mapsto \rho_t^i$ as

$$\rho_t^i = \frac{\Delta x_0^i}{\Delta x_t^i} \rho_0^i,$$

where similarly $\Delta x_t^i \coloneqq x_t^{i+1} - x_t^i$. Define an approximate solution ρ from the local densities as

$$\rho(x,t) \coloneqq \sum_{i=0}^{N} \rho_t^i \chi_{(x_t^i, x_t^{i+1})}(x), \tag{6}$$

where $\chi_{(x^i, x^{i+1})}$ is an indicator function on the interval (x^i, x^{i+1}).

Step 3. *Resolution of collisions.* If two or more particles collide, delete the leftmost particles and local densities in the collision, and continue the system. Repeat Steps 2 and 3 iteratively up to time T.

1.1. Main results

Rather than analyzing Steps 1–3 directly, we show that the function ρ defined in (6) satisfies a continuity equation

$$\begin{cases} \partial_t \rho + \partial_x (\overline{V} \rho) = 0 \\ \rho(0) = \rho_0, \end{cases} \tag{7}$$

where $\overline{V}$ is the piecewise linear interpolation of particle velocities with interpolation nodes given by particle positions. That is, $\overline{V}$ is (up to the first collision, cf. (31)) defined as

$$\overline{V}(x,t) = \frac{x_t^{i+1} - x}{x_t^{i+1} - x_t^i} V(\rho_t^{i-1}, \rho_t^i) + \frac{x - x_t^i}{x_t^{i+1} - x_t^i} V(\rho_t^i, \rho_t^{i+1}) \tag{8}$$

for $x \in (x_t^i, x_t^{i+1})$. This formulation makes it possible to derive an accurate relationship between the approximation ρ and the entropy solution of the conservation law (1). In particular, we prove that ρ satisfies the entropy-like inequality

$$\partial_t |\rho - k| + \partial_x \big((\overline{V}\rho - f(k)) \operatorname{sgn}(\rho - k) \big) \leq 0 \tag{9}$$

for all nonnegative constants $k \in \mathbb{R}$. As a consequence, we obtain the following theorem.

Theorem 1.1 (Main Theorem)**.** *Let u_0 be a nonnegative function in $\mathrm{BV} \cap L^1(\mathbb{R})$ and assume that f is Lipschitz continuous with a well-defined derivative $f'(0)$ at zero. Then the particle path scheme generates a unique approximation ρ which satisfies*

$$\|\rho(T) - u(T)\|_{L^1(\mathbb{R})} \leq \|\rho_0 - u_0\|_{L^1(\mathbb{R})} + 2\sqrt{2|u_0|_{\mathrm{BV}(\mathbb{R})} \|\overline{V}\rho - f(\rho)\|_{L^1(\mathbb{R}\times(0,T))}}, \tag{10}$$

where u is the entropy solution of the conservation law (1).

Remark 1.2. The stability estimate (10) is independent of the number of collision events (which typically increases as the number of particles N increases). Since the scheme is total variation diminishing (Prop. 4.2), collisions are dissipative interactions that reduce the total variation. Consequently, the increasing frequency of collisions as $N \to \infty$ does not introduce instability or degrade the convergence rate.

For a precise definition of the total variation, $|\cdot|_{\mathrm{BV}}$, see *e.g.* Appendix A of [16]. One should note that the function $\overline{V}$ in the stability estimate (10) is the global-in-time version of (8), defined in (31) (see also Sect. 3.3). It follows the same principle but is more involved to write down due to particle collisions.

Theorem 1.1 is a general result, imposing no assumptions on the distribution of initial particles. As the following corollary exemplifies, explicit convergence rates can be derived under additional conditions.

Corollary 1.3. *In addition to the assumptions of Theorem 1.1, assume that f' is Lipschitz and that u_0 has compact support. Further, assume that $\operatorname{supp}(u_0) \subset [x_0^1, x_0^N]$ and $\Delta x_0^i \leq \Delta x^*$ for all $i \in \{1, \dots, N-1\}$. Then the approximation error is bounded by*

$$\|\rho(T) - u(T)\|_{L^1(\mathbb{R})} \leq |u_0|_{\mathrm{BV}(\mathbb{R})} \left(\Delta x^* + 2\sqrt{T[f']_{\mathrm{Lip}} \|u_0\|_{L^\infty(\mathbb{R})} \Delta x^*} \right).$$

That is, the scheme is of order $1/2$ with respect to initial distance between particles.

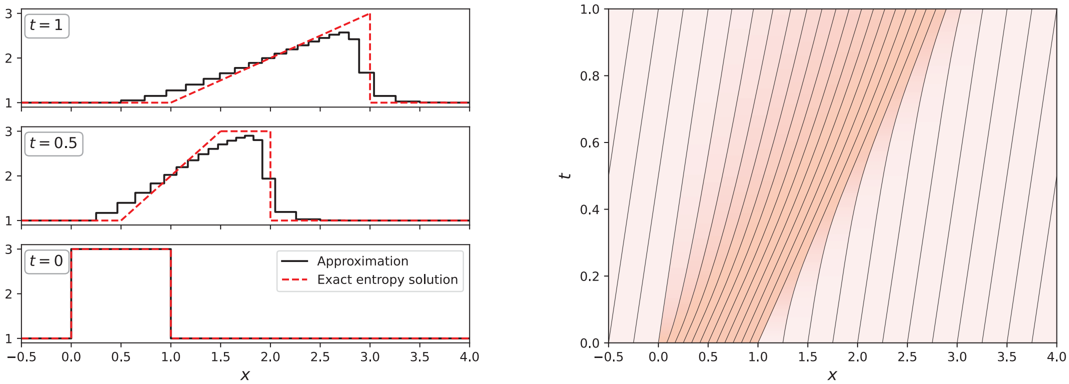
1.2. Illustrative example

To illustrate the nature of the approximation, we consider the following problem for Burgers' equation:

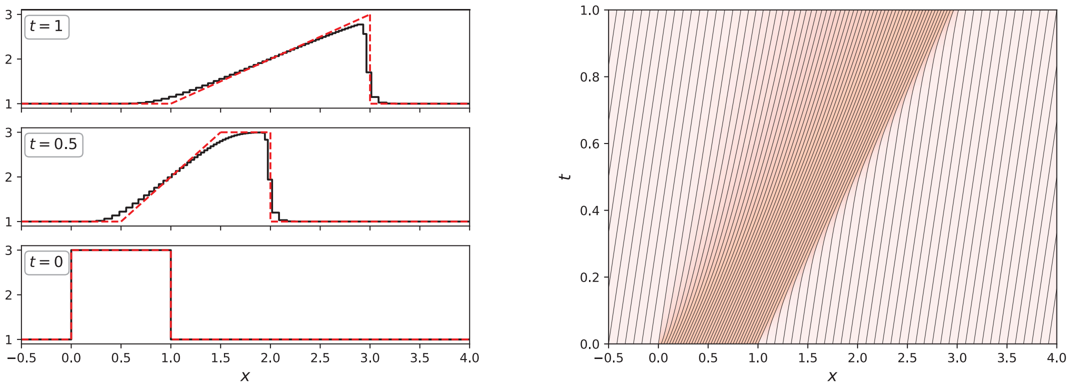
$$\partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0, \qquad u_0(x) = \begin{cases} 3 & \text{for } 0 < x < 1 \\ 1 & \text{else.} \end{cases}$$

The unique entropy solution is given by

$$u(x,t) = \begin{cases} \frac{x}{t} & \text{if } \ t < x < 3t \\ 3 & \text{if } \ 3t < x < 1 + 2t \\ 1 & \text{else,} \end{cases} \tag{11}$$



(A) Coarse approximation.



(B) Finer approximation.

FIGURE 1. Schematic illustration of approximations of (11) using different numbers of initial particles. The particle paths were computed with the forward Euler method using a sufficiently small time step to resolve the dynamics. In the particle path plots (*right*), the background shading represents the magnitude of the approximate solution.

which features both a rarefaction region and a shock. Figure 1 compares the approximation ρ generated by the particle path scheme with the entropy solution for two different spacings of initial particles. The underlying idea of the approximation becomes evident in the particle plots to the right: Particles $(x_t^i)_{i=1}^N$ divide the mass of u into $N-1$ parts $(\rho_t^i)_{i=1}^{N-1}$. The mass between adjacent particles remains constant over time and is continually reaveraged, making ρ a piecewise constant function. See Example 6.1 of [12] for the exact particle paths of the entropy solution in this example.

1.3. Connection to existing literature

Historically, the approximation of scalar conservation laws has been dominated by Eulerian finite volume-type schemes [14], while constructive analytical approximations – most notably the front tracking method (see *e.g.* [8, 16]) – have played a central role in the theoretical development of entropy solutions [8, 16]. The particle path scheme proposed here belongs to this latter class of constructive methods and adopts a Lagrangian perspective.

This work builds on the theoretical framework developed in [12], where an equivalence was established between entropy solutions of scalar conservation laws and well-posedness of a Filippov flow associated with the corresponding continuity equation (see Sect. 2.2). While the results of [12] are purely analytical and formulated at the level of infinite-dimensional flows, the present paper develops a constructive approximation scheme grounded in this theory. Our main contribution is the translation of this theory into a stable finite-particle discretization, together with the derivation of an explicit convergence rate (10).

From a conceptual standpoint, the proposed scheme may be regarded as a complementary Lagrangian construction to the front tracking method, as the two approaches constitute fundamental representations of the scalar conservation law (see Sect. 1.4 for further discussion on this).

The proposed scheme also shares a conceptual link with Godunov's method [15]. In Godunov's method, the spatial grid is fixed and the mass contained in each cell is updated through fluxes across the cell interfaces. In the present particle path scheme, the roles are reversed: The mass associated with each interval is conserved, while the interfaces – namely, the particle positions – evolve in time.

It is important to distinguish our approach from the "sticky particle" methods found in the literature, such as those by Brenier and Grenier [5]. In those models, the solution is typically represented as a sum of Dirac masses governed by collision-sticking rules. In contrast, the "particles" in our scheme act as moving boundaries that define a piecewise constant density function, ensuring a representation that remains in $L^1 \cap \mathrm{BV}$.

In the specific case where the flux f is concave (*i.e.* the velocity field a is nonincreasing), our scheme coincides with the Follow-the-Leader (FtL) model (see Sect. 1.5 for further details). The convergence of the FtL model to the macroscopic Lighthill–Whitham–Richards (LWR) model [20, 24] is well-known; it was first proved by Di Francesco and Rosini [9] and has been studied extensively by others [3, 4, 6, 10, 13, 17, 18].

While most previous results relied on compactness arguments to establish qualitative convergence, our result is based on a direct constructive framework. Crucially, this allows us to derive an explicit convergence rate of order 1/2.

Furthermore, existing results often require specific structural conditions on the velocity field. For instance, the recent paper by Ancona *et al.* [3] establishes convergence for concave fluxes but notes that the convergence of FtL schemes for non-concave fluxes "has not yet been properly understood, and needs further investigation". Since our stability estimate (10) requires only that the flux f be Lipschitz, our result provides a theoretical justification for the convergence of particle schemes in the general non-concave setting.

We note that recently, Marconi *et al.* [22] established the same convergence rate of order 1/2, with respect to the number of particles, for a different particle scheme applied to non-local conservation laws (specifically, traffic models with congestion). Their analysis relies on a general stability framework for quasi-entropy solutions involving space-time dependent fluxes. In contrast, our work focuses on the local setting and derives the rate directly from the geometry of the Filippov particle paths.

1.4. Comparison to front tracking

The difference between our approach and the front tracking method can be understood through the interpretation of the conservation law (1). Standard methods, such as front tracking, often rely on the quasi-linear form $\partial_t u + f'(u)\partial_x u = 0$. In this view, approximating the characteristic speed $f'(u)$ by a piecewise constant function (which corresponds to approximating the flux $f(u)$ by a piecewise linear function) effectively assigns a specific velocity to the values of the state variable u. The dynamics are then governed by rules for how these constant states interact, such as rarefaction waves and the Rankine–Hugoniot jump conditions which are used in front tracking to construct approximate entropy solutions.

In contrast, the particle path scheme stems from interpreting (1) as a continuity equation (2) with a density-dependent velocity field $a(u)$. We then approximate the velocity field $a(u(x,t))$, viewed as a function of the spatial variable x, by a piecewise linear function $\overline{V}$. This assigns a specific velocity to particles, and thus a rate of compression or expansion to the densities in the spatial intervals between particles. A fundamental difference between our approach and front tracking is therefore that front tracking approximates the flux f as a function of the state variable u, while our method approximates the velocity field $a(u(x,t))$ as a function of the spatial variable x.

Since u is generally discontinuous, the velocity field $a(u)$ will also have discontinuities. This necessitates a careful choice for the velocity at these spatial jumps. As we will demonstrate, the specific choice leading to the entropy solution is the velocity defined in (5).

We emphasize that unlike the front-tracking algorithm, where the number of fronts can increase over time, the proposed particle path scheme maintains a nonincreasing number of particles, which strictly decreases only upon collision.

1.5. Application to traffic modelling

In the particular case of concave flux f, the proposed scheme coincides with the Follow-the-Leader (FtL) model for traffic flow (see *e.g.* [9]), and establishes its convergence rate to the Lighthill–Whitham–Richards (LWR) model [20, 24].

The LWR model describes average vehicular density ρ on a macroscopic level through the conservation law $\partial_t\rho + \partial_x(v(\rho)\rho) = 0$, where v is a nonincreasing Lipschitz velocity field. On the other hand, the FtL model is a microscopic model for individual vehicles $t \mapsto z_t^i$, each of which is assigned a velocity $\dot{z}_t^i = v\big(\rho_t^i\big)$, where $\rho_t^i \coloneqq l/(z_t^{i+1} - z_t^i)$ and $l > 0$ denotes the length of the vehicles.

Assume that a, which corresponds to the velocity v in the LWR model, is nonincreasing (*i.e.* f is concave). If $\rho_t^{i-1} \le \rho_t^i$, then $V(\rho_t^{i-1}, \rho_t^i) = \min_{\rho\in[\rho_t^{i-1},\rho_t^i]} a(\rho) = a(\rho_t^i)$, and if on the other hand $\rho_t^{i-1} \ge \rho_t^i$, then $V(\rho_t^{i-1}, \rho_t^i) = \max_{\rho\in[\rho_t^i,\rho_t^{i-1}]} a(\rho) = a(\rho_t^i)$. In both cases, our velocity $V = a(\rho_t^i)$ coincides with the velocity $v(\rho_t^i)$ for the FtL model, showing that in this case, our scheme is equivalent to the FtL model.

1.6. Outline of the paper

Section 2 covers theory on conservation laws, particle paths, and continuity equations. Section 3 defines the particle path scheme in full detail, and Section 4 establishes existence and uniqueness of solutions to the system of ODEs in Step 2. The core of the paper is Section 5, wherein justification of the continuity equation (7) and proof of the approximate entropy inequality (9) is given. Theorem 1.1 is proved in Section 6. In Section 7 we mention possible future work.

2. Preliminaries

We begin by reviewing key concepts and results needed for the paper.

2.1. Scalar conservation laws

A solution of the conservation law (1) means a weak solution, that is, a function $u \in L^\infty(\mathbb{R}\times(0,T))$ which satisfies

$$\int_0^T \int_{\mathbb{R}} u\partial_t\varphi + f(u)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\,\mathrm{d}x = 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times[0,T))$. An entropy solution is a solution which satisfies the entropy inequality

$$\int_0^T \int_{\mathbb{R}} \eta(u)\partial_t\varphi + q(u)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x,0)\,\mathrm{d}x \ge 0 \tag{12}$$

for all nonnegative $\varphi \in C_c^\infty(\mathbb{R}\times[0,T))$ and all entropy pairs (η, q) (where $\eta\colon \mathbb{R}\to\mathbb{R}$ is a convex function and $q\colon \mathbb{R}\to\mathbb{R}$ is such that $q' = \eta' f'$). Kruzkhov [19] proved that there is a unique entropy solution of (1) for any $u_0 \in L^\infty(\mathbb{R})$. It is enough that (12) holds for entropy pairs $\eta_k(u) = |u-k|$ and $q_k(u) = \operatorname{sgn}(u-k)(f(u)-f(k))$ for all $k \in \mathbb{R}$. (Here and throughout, sgn denotes the signum function. It returns the sign of its argument or 0 if the argument is zero.) If u_0 is an absolutely integrable function of bounded variation, then so is the entropy solution for all positive times.

2.2. Particle paths for conservation laws

The concept of particle paths for the conservation law (1) comes from its interpretation as a continuity equation (2). The following theorem describes the link between entropy solutions and particle paths.

Theorem 2.1 (Fjordholm *et al.* [12])**.** *Let* $f \in C^1(\mathbb{R})$ *and* $u_0 \in \mathrm{BV}_{\mathrm{loc}} \cap L^\infty(\mathbb{R})$. *If* $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ *is a weak solution of the conservation law* (1) *with* $u(t) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ *for a.e.* $t \geq 0$, *then the following are equivalent:*

(i) *u is the entropy solution.*
(ii) *The ODE*

$$\begin{cases} \dot{x}_t = \frac{f(u(x_t,t))-f(k)}{u(x_t,t)-k} & \text{for } t > s \\ x_s = x \end{cases} \tag{13}$$

is well-posed in the Filippov sense for all $x \in \mathbb{R}$, $s \geq 0$ *and* $k \in \mathbb{R}$.

Moreover, for any $k \in \mathbb{R}$, *the entropy solution satisfies* $u(t) = k + (X_t^k)_\#(u_0 - k)$ *for* $t \geq 0$, *where* $X^k = X_t^k(x)$ *is the flow of the ODE* (13).

The ODE (13) (and (3)) is interpreted in the Filippov sense (see [11]), as weak solutions of (1) are generally discontinuous. In particular, when a particle encounters a shock, its velocity must typically be assigned from an infinite set of possible values. The prototypical example of this situation is the Riemann problem: Let u be the entropy solution of (1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases}$$

for $u_l, u_r \in \mathbb{R}$. As shown in Theorem 1.6 of [12], the unique (Fillipov) solution of (3) starting at $x = 0$ is the straight line $x_t = Vt$, with velocity $V = V(u_l, u_r)$ given by

$$V(u_l, u_r) = \begin{cases} \min_{w \in [u_l, u_r]} a(w) & \text{if } 0 < u_l \leq u_r \\ \max_{w \in [u_r, u_l]} a(w) & \text{if } 0 < u_r \leq u_l \\ (f_\smile)'(0) & \text{if } u_l \leq 0 \leq u_r \\ (f_\frown)'(0) & \text{if } u_r \leq 0 \leq u_l \\ \max_{w \in [u_l, u_r]} a(w) & \text{if } u_l \leq u_r < 0 \\ \min_{w \in [u_r, u_l]} a(w) & \text{if } u_r \leq u_l < 0, \end{cases} \tag{14}$$

where $f_\smile$ and $f_\frown$ are the convex and concave envelopes of f between u_l and u_r (definitions can be found in [16], Chap. 2). Comparing with (5), it is clear that our choice of particle velocity in Step 2 is simply a special case of (14) for nonnegative left and right states u_l and u_r.

2.3. Continuity equations

Consider the continuity equation

$$\partial_t u + \partial_x(bu) = h \tag{15}$$

with initial data $u_0 \in L^\infty(\mathbb{R})$, velocity field $b \in L^\infty(\mathbb{R} \times (0,T))$, and a source $h \in L^\infty((0,T); L^1(\mathbb{R}))$. A weak solution of (15) is a function $u \in L^\infty(\mathbb{R} \times (0,T))$ which satisfies

$$\int_0^T \int_{\mathbb{R}} \big(\partial_t \varphi + b \partial_x \varphi\big) u \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}} \varphi h \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} u_0(x) \varphi(x,0) \,\mathrm{d}x = 0 \tag{16}$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times [0,T))$.

Remark 2.2. Although measure-valued solutions of (15) are more natural in this context (see *e.g.* [2]), we restrict ourselves here to the material relevant for our purposes.

The continuity equation can be solved *via* the system of ODEs

$$\begin{cases} \dot{x}_t = b(x_t, t) & \text{for a.e. } \ t > s \\ x_s = x. \end{cases} \tag{17}$$

Assume that b is continuous in space for a.e. t, so that (17) can be understood in the usual sense (in particular, we will not need the Filippov solution concept). The following lemma is a standard extension of the Cauchy–Lipschitz theory for ODEs; see *e.g.* [7, 21].

Lemma 2.3 (Forward flow)**.** *Assume that* $x \mapsto b(x,t)$ *is continuous and one-sided Lipschitz, i.e. there is a constant* $C > 0$ *such that*

$$\big(b(x,t) - b(y,t)\big)(x-y) \le C(x-y)^2 \tag{18}$$

for all $x, y \in \mathbb{R}$*, for a.e.* $t \in (0,T)$*. Then the system* (17) *generates a unique forward Lipschitz flow* $X = X_t(x,s)$*:*

(i) *For all* $(x,s) \in \mathbb{R} \times [0,T]$*, the ODE* (17) *has a unique continuously differentiable solution* $t \mapsto x_t$ *on* (s,T)*.*
(ii) *The function* $(x,s,t) \mapsto X_t(x,s)$ *is jointly Lipschitz for* $0 < s \le t < T$ *and* $x \in \mathbb{R}$*.*
(iii) *The identity* $X_{s,s}(x) = x$ *holds for all* $(x,s) \in \mathbb{R} \times [0,T]$*. Moreover,*

$$X_t(X_r(s,x),r) = X_t(x,s)$$

for all $0 \le s \le r \le t \le T$ *and* $x \in \mathbb{R}$*.*

We denote $X_t(x,0)$ by $X_t(x)$ for simplicity.

Lemma 2.4 (Representation formula)**.** *Let* b *satisfy the assumptions of Lemma 2.3 and assume that* $u \in L^\infty(\mathbb{R} \times (0,T))$ *is a weak solution of* (15)*. Then* u *satisfies*

$$u(t) = (X_t)_\# u_0 + \int_0^t (X_t(r))_\# h(r)\,\mathrm{d}r,$$

in the sense of

$$\int_{\mathbb{R}} \vartheta(x) u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} \vartheta(X_t(x)) u_0(x)\,\mathrm{d}x + \int_0^t \int_{\mathbb{R}} \vartheta(X_t(x,r)) h(x,r)\,\mathrm{d}x\,\mathrm{d}r \tag{19}$$

for all $t \in [0,T]$ *and* $\vartheta \in C_c(\mathbb{R})$*, where* X *is the unique forward Lipschitz flow generated by* (17)*.*

This is a special case of Theorem 6.1, Remark 6.2 from [12] (see also [23]). Though the proof is much simpler in this case, we omit it.

3. The scheme

In this section, we provide a complete specification of the particle path scheme introduced in Steps 1–3, and fix notation which will be used throughout.

3.1. Approximation of initial data

Let $(x_0^i)_{i=1}^N$ be a strictly increasing tuple of real values representing the initial positions of N arbitrarily spaced particles. Extend the tuple symbolically by $x_0^0 \coloneqq -\infty$ and $x_0^{N+1} \coloneqq \infty$. Recalling that u_0 is assumed to be nonnegative, we define

$$\rho_0^i \coloneqq \frac{1}{\Delta x_0^i} \int_{x_0^i}^{x_0^{i+1}} u_0(x)\,\mathrm{d}x$$

for $i \in \{0, \ldots, N\}$, where $\Delta x_0^i \coloneqq x_0^{i+1} - x_0^i$. Note in particular that $\rho_0^0 = 0$ and $\rho_0^N = 0$. Let χ_I denote the indicator function on a set $I \subset \mathbb{R}$. Then

$$\rho_0(x) \coloneqq \sum_{i=0}^{N} \rho_0^i \chi_{(x_0^i, x_0^{i+1})}(x)$$

is a piecewise constant approximation of u_0.

3.2. Particle dynamics

Fix $x_t^0 = -\infty$ and $x_t^{N+1} = \infty$, along with $\rho_t^0 = 0$ and $\rho_t^N = 0$, for all $t \in [0, T]$. Let particle positions $(x_t^i)_{i=1}^N$ and local densities $(\rho_t^i)_{i=1}^{N-1}$ evolve according to the system

$$\begin{cases} \dot{x}_t^i = V\left(\rho_t^{i-1}, \rho_t^i\right) & \text{for } 0 < t < t_1 \\ x^i|_{t=0} = x_0^i, \end{cases} \tag{20}$$

coupled with

$$\rho_t^i = \frac{\Delta x_0^i}{\Delta x_t^i} \rho_0^i, \tag{21}$$

where t_1 is the first *collision* time of two or more particles, defined by

$$t_1 = \sup\left\{t \in (0, T) \colon x_t^i < x_t^{i+1} \text{ for all } i \in \{1, \ldots, N-1\}\right\} \wedge T, \tag{22}$$

and V is the velocity field from (5). (Here, $x \wedge y$ denotes $\min\{x, y\}$). For brevity, we shall sometimes write $(x_t^i)_{i=1}^N$ and $(\rho_t^i)_{i=1}^{N-1}$ as (x_t^i) and (ρ_t^i).

An approximate solution of (1) can now be defined as

$$\rho(x, t) = \sum_{i=0}^{N} \rho_t^i \chi_{\left(x_t^i, x_t^{i+1}\right)}(x) \tag{23}$$

on $\mathbb{R} \times (0, t_1)$.

3.3. Resolution of collisions

When particles collide, the system (20) and (21) is stopped. The leftmost particles and corresponding local densities are deleted, and then the system is continued (see Fig. 2 for an illustration).

More specifically, let $\mathcal{I}_1$ be the index set of remaining particles after t_1. That is,

$$\mathcal{I}_1 \coloneqq \left\{i \in \{1, \ldots, N\} \colon x_{t_1}^i \neq x_{t_1}^{i+1}\right\}.$$

Remaining particles and local densities can thus be written as $(x^i)_{i \in \mathcal{I}_1}$ and $(\rho^i)_{i \in \mathcal{I}_1}$. Let $N_1 \coloneqq |I_1|$ and define π_1 to be the increasing, one-to-one function mapping the index set $\{1, \ldots, N_1\}$ to $\mathcal{I}_1$. Then remaining particles can equivalently be written as $\left(x^{\pi_1(i)}\right)_{i=1}^{N_1}$.

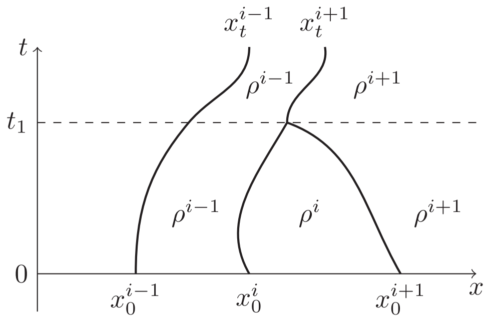


FIGURE 2. Particles x^i and x^{i+1} collide at time t_1. The particle x^i and the local density ρ^i are deleted before the system is restarted.

After the collision at t_1, the system (20) and (21) is continued with updated indices $\mathcal{I}_1$ and initial data $(x_{t_1}^{\pi_1(i)})_{i=1}^{N_1}$ and $(\rho_{t_1}^{\pi_1(i)})_{i=1}^{N_1-1}$.

Define collision times recursively as

$$t_{j+1} := \sup\Big\{t \in (t_j, T) \colon x_t^{\pi_j(i)} < x_t^{\pi_j(i+1)} \text{ for all } i \in \{1, \dots, N_j\}\Big\}.$$

The set of all collision times $(t_j)_{j\geq 0}$ will be denoted by $\mathcal{T}_c$. The index set of remaining particles after t_{j+1} is then

$$\mathcal{I}_{j+1} := \Big\{\pi_j(i) \colon x_{t_{j+1}}^{\pi_j(i)} \neq x_{t_{j+1}}^{\pi_j(i+1)},\ i \in \{1, \dots, N_j\}\Big\}.$$

Let $N_{j+1} := |\mathcal{I}_{j+1}|$ and let π_{j+1} be the increasing, one-to-one function mapping $\{1, \dots, N_{j+1}\}$ to $\mathcal{I}_{j+1}$. Set $t_0 = 0$ and let $\pi_0(i) = i$ for all $i \in \{0, \dots, N\}$. In view of the above definitions, we denote by (x^π) and (ρ^π) the global-in-time particles and local densities, given by

$$(x_t^\pi) = \Big(x_t^{\pi(i)}\Big)_{i=1}^{N_j}, \qquad (\rho_t^\pi) = \Big(\rho_t^{\pi(i)}\Big)_{i=1}^{N_j-1} \tag{24}$$

for $t \in (t_j, t_{j+1})$.

This allows us to build a global approximate solution of (1), defined piecewise in time as

$$\rho(x,t) := \sum_{i=0}^{N_j+1} \rho_t^{\pi_j(i)} \chi_{\left(x_t^{\pi_j(i)}, x_t^{\pi_j(i+1)}\right)}(x) \tag{25}$$

for $(x,t) \in \mathbb{R} \times (t_j, t_{j+1})$.

4. Existence and uniqueness of the approximation

In this section we consider the system of ODEs (20) and (21) which is at the core of the particle path scheme. We establish first some *a priori* properties of solutions, assuming their existence. We then use these *a priori* properties and classical Cauchy–Lipschitz theory to show that solutions indeed exist and are unique.

The integer N will be fixed throughout the section. We assume that initial data $(\rho_0^i)_{i=1}^{N-1}$, $(x_0^i)_{i=1}^N$, as constructed in Step 1 from nonnegative $u_0 \in \mathrm{BV} \cap L^1(\mathbb{R})$, is given. Since there is only N particles there can be at most $N-1$ collisions (note that new particles are never created), so it will suffice to analyze the scheme locally in time. Thus, we consider only the interval $(0, t_1)$.

4.1. *A priori* properties

We begin by defining suitable solutions of (20) and (21).

Definition 4.1. A solution of the system (20) and (21) on an interval $(0,t')$ is a pair $(x^i)_{i=1}^N$, $(\rho^i)_{i=1}^{N-1}$ such that

(i) $(x^i)_{i=1}^N\colon (0,t') \to \mathbb{R}^N$ is continuously differentiable and satisfies (20),
(ii) $x_t^1 < x_t^2 < \cdots < x_t^N$ for all $t \in (0,t')$,
(iii) $(\rho^i)_{i=1}^{N-1}\colon (0,t') \to \mathbb{R}^{N-1}$ satisfies (21) for all $t \in (0,t')$.

The following proposition gathers essential characteristics of such solutions. To simplify notation, let $\rho_0^* := \max_{i\in\{0,\dots,N\}} \rho_0^i$ and moreover $a_{\min} := \min_{\rho\in[0,\rho_0^*]} a(\rho)$ and $a_{\max} := \max_{\rho\in[0,\rho_0^*]} a(\rho)$, where we recall the defintition of a from (2).

Proposition 4.2. *Let $(x^i)_{i=1}^N$, $(\rho^i)_{i=1}^{N-1}$ be a solution of (20) and (21) on a given interval $(0,t')$ according to Definition 4.1. Then, for all $t \in (0,t')$:*

(i) *(Mass conservation) The mass between particles is conserved, i.e.*

$$\Delta x_t^i \rho_t^i = \Delta x_0^i \rho_0^i$$

for all $i \in \{0,\dots,N\}$.
(ii) *(Maximum principle) The local densities satisfy*

$$\frac{\Delta x_0^i}{\Delta x_0^i + t(a_{\max} - a_{\min})}\rho_0^i \le \rho_t^i \le \rho_0^* \tag{26}$$

for all $i \in \{1,\dots,N-1\}$.
(iii) *(Particle separation) The distances between particles satisfy*

$$\frac{\rho_0^i}{\rho_0^*}\Delta x_0^i \le \Delta x_t^i \le \Delta x_0^i + t(a_{\max} - a_{\min}) \tag{27}$$

for all $i \in \{1,\dots,N-1\}$.
(iv) *(Diminishing total variation) The total variation of the approximate solution (23) is nonincreasing, i.e.*

$$|\rho(t)|_{\mathrm{BV}(\mathbb{R})} \le |\rho_0|_{\mathrm{BV}(\mathbb{R})}, \tag{28}$$

where $|\cdot|_{\mathrm{BV}(\mathbb{R})}$ denotes the total variation on $\mathbb{R}$.

Proof. (i) The conservation of mass follows directly from (21).
(ii) We first claim that the local densities (ρ^i) satisfy

$$\begin{cases} \dot\rho_t^i \le 0 & \text{if } \ \rho_t^{i-1} \le \rho_t^i \ge \rho_t^{i+1} \\ \dot\rho_t^i \ge 0 & \text{if } \ \rho_t^{i-1} \ge \rho_t^i \le \rho_t^{i+1} \end{cases} \tag{29}$$

for all $t \in (0,t')$ and $i \in \{1,\dots,N-1\}$. Indeed, if $\rho_t^i = 0$ for some $t \in (0,t')$, then $\rho_t^i \equiv 0$ on $[0,t')$ due to (21), and (29) trivially holds. Let therefore ρ_t^i be strictly positive and assume that $\rho_t^{i-1} \le \rho_t^i \ge \rho_t^{i+1}$ for some $t \in (0,t')$. Then

$$\dot x_t^i = V\big(\rho_t^{i-1},\rho_t^i\big) = \min_{\rho\in[\rho_t^{i-1},\rho_t^i]} a(\rho) \le a\big(\rho_t^i\big) \le \max_{\rho\in[\rho_t^{i+1},\rho_t^i]} a(\rho) = V\big(\rho_t^i,\rho_t^{i+1}\big) = \dot x_t^{i+1}$$

which implies that

$$\dot{\rho}_t^i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x_0^{i+1} - x_0^i}{x_t^{i+1} - x_t^i}\rho_0^i\right) = -\rho_t^i \frac{\dot{x}_t^{i+1} - \dot{x}_t^i}{x_t^{i+1} - x_t^i} \le 0$$

whenever $x_t^{i+1} > x_t^i$. The second inequality in (29) follows by a similar argument.
It is now straightforward to infer from (29) that $0 \le \rho_t^i \le \rho_0^*$ for all $t \in (0, t')$. Since x_t^i satisfies (20), we have moreover that

$$x_0^i + ta_{\min} \le x_t^i \le x_0^i + ta_{\max}$$

for all $t \in (0, t')$ and $i \in \{1, \dots, N\}$, where we have used $a_{\min} \le V(\rho_t^{i-1}, \rho_t^i) \le a_{\max}$ on $(0, t')$. This yields $\Delta x_t^i \le \Delta x_0^i + t(a_{\max} - a_{\min})$, which inserted in (21) proves (26).

(iii) Since $\Delta x_t^i = \frac{\rho_0^i}{\rho_t^i}\Delta x_0^i$, the inequality (27) is a direct consequence of $\rho_t^i \le \rho_0^*$ and the first inequality from (26).

(iv) For given $t \in (0, t')$, let $(\mu_t^i)_{i=0}^{N'}$ be a subset of $(\rho_t^i)_{i=0}^N$ such that either

$$\mu_t^{j-1} \le \mu_t^j \ge \mu_t^{j+1} \qquad \text{or} \qquad \mu_t^{j-1} \ge \mu_t^j \le \mu_t^{j+1}$$

for all $i \in \{1, \dots, N'-1\}$, and moreover

$$|\rho(t)|_{\mathrm{BV}(\mathbb{R})} = \sum_{i=0}^{N-1} \left|\rho_t^{i+1} - \rho_t^i\right| = \sum_{i=0}^{N'-1} \left|\mu_t^{i+1} - \mu_t^i\right|.$$

In other words, the tuple (μ_t^i) is a collection of peaks and troughs of (ρ_t^i). Suppose further that $\mu_t^i \ne \mu_t^{i+1}$ for all $i \in \{1, \dots, N'-1\}$ (otherwise, one of the values could be removed from the tuple without changing the total variation). Let us further separate the index set $\{1, \dots, N'\}$ into two sets $P = \{i \colon \mu_t^i < \mu_t^{i+1}\}$ and $N = \{i \colon \mu_t^i > \mu_t^{i+1}\}$. Then the total variation of ρ is

$$|\rho(t)|_{\mathrm{BV}(\mathbb{R})} = \sum_{i \in P}\left(\mu_t^{i+1} - \mu_t^i\right) - \sum_{i \in N}\left(\mu_t^{i+1} - \mu_t^i\right).$$

In view of (29) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)|_{\mathrm{BV}(\mathbb{R})} = \sum_{i \in P}\left(\dot{\mu}_t^{i+1} - \dot{\mu}_t^i\right) - \sum_{i \in N}\left(\dot{\mu}_t^{i+1} - \dot{\mu}_t^i\right) \le 0,$$

which proves (28) upon integrating over $(0, t)$.

□

4.2. Existence and uniqueness

Existence and uniqueness for (20) and (21) depend on the regularity of the velocity function V. It is not Lipschitz on all of $\mathbb{R}$, so we restrict the class of solutions. For $\delta > 0$, let $\mathcal{S}_\delta^N$ be the collection of all tuples $(x^i)_{i=1}^N$ and $(\rho^i)_{i=1}^{N-1}$ such that

$$x^i < x^{i+1} \qquad \text{and} \qquad \rho^i = 0 \;\text{ or }\; \delta \le \rho^i \le \frac{1}{\delta}$$

for all $i \in \{1, \dots, N-1\}$. Using the *a priori* properties from the previous section, we will show that local solutions of (20) and (21) remain in $\mathcal{S}_\delta^N$ for some $\delta > 0$ which can be chosen independently of the interval of existence. The following lemma proves that V is Lipschitz in this setting.

Lemma 4.3. *Assume that f is Lipschitz on $\mathbb{R}$. Then the particle velocity function $(\rho_l, \rho_r) \mapsto V(\rho_l, \rho_r)$ is locally Lipschitz on $(0, \infty)^2$, and similarly the functions $\rho_r \mapsto V(0, \rho_r)$ and $\rho_l \mapsto V(\rho_l, 0)$ are locally Lipschitz on $(0, \infty)$.*

Proof. Since f is Lipschitz on $\mathbb{R}$, the function $\rho \mapsto a(\rho)$ is at least locally Lipschitz on $(0, \infty)$. The particle velocity $V(\rho_l, \rho_r)$ is defined using the minimum or the maximum of $a(\rho)$ over the interval between ρ_l and ρ_r (see (5)). Since min and max operations preserve Lipschitz continuity, the function V inherits the local Lipschitz property of a when both ρ_l, ρ_r are bounded away from zero. When one argument is zero, *e.g.* $\rho_r \mapsto V(0, \rho_r)$, this equals $\min_{\rho\in[0,\rho_r]} a(\rho)$. Thus, the rate of change of $V(0, \rho_r)$ is bounded by the rate of change of a around ρ_r. A similar argument applies to $\rho_l \mapsto V(\rho_l, 0) = \max_{\rho\in[0,\rho_l]} a(\rho)$. A detailed case analysis easily confirms these claims. □

With the Lipschitz regularity of the velocity function established, we can now prove the main existence and uniqueness result for the particle path scheme. The following theorem ensures that the scheme produces a unique solution, valid globally in time until the first collision of particles.

Theorem 4.4. *For any set of initial conditions $(x_0^i)_{i=1}^N$ and $(\rho_0^i)_{i=1}^{N-1}$ generated as in Step 1, there exists a unique set of solutions $(x_t^i)_{i=1}^N$ and $(\rho_t^i)_{i=1}^{N-1}$ of the system (20) and (21) on the interval $(0, t_1)$, where $t_1 \in (0, T]$ is a uniquely determined collision time given by (22).*

Proof. We will prove that for any (x_0^i), (ρ_0^i) in $S^N_{\delta_0}$, there is a unique solution (x_t^i), (ρ_t^i) of (20) and (21) on $(0, t_1)$ which belongs to $S^N_{\delta_1}$, where

$$0 \le \delta_1 = \delta_0 \min_{i\in\{1,N-1\}} \frac{\Delta x_0^i}{\Delta x_0^i + T(a_{\max} - a_{\min})}. \tag{30}$$

To this end, consider (x_0^i), (ρ_0^i) in $S^N_{\delta_0}$ for some $\delta_0 > 0$. The equations (20) and (21) can be written compactly as

$$\dot{x}_t^i = V\big(\rho_t^{i-1}, \rho_t^i\big) = V\left(\frac{\Delta x_0^{i-1}}{\Delta x_t^{i-1}}\rho_0^{i-1}, \frac{\Delta x_0^i}{\Delta x_t^i}\rho_0^i,\right) =: F^i\Big(\big(x_t^i\big)_{i=1}^N; \big(x_0^i\big)_{i=1}^N, \big(\rho_0^i\big)_{i=1}^{N-1}\Big)$$

for $i \in \{1, \ldots, N\}$. The right-hand side is a function $F^i \colon \mathbb{R}^N \to \mathbb{R}^N$ which takes parameters (x_0^i) and (ρ_0^i). In light of Lemma 4.3, we see that F^i is locally Lipschitz around the initial condition (x_0^i) for all $i \in \{1, \ldots, N\}$. This gives a unique local solution (x_t^i), (ρ_t^i) on an interval $(0, t')$ for some $t' > 0$. Since $x_0^1 < x_0^2 < \cdots < x_0^N$ and the solution is continuous with respect to time, we may assume that $x_t^1 < x_t^2 < \cdots < x_t^N$ for all $(0, t')$. Moreover, by (ii) from Proposition 4.2 the local densities (ρ_t^i) which are nonzero are bounded from below by a constant as in (30), independently of the existence time t'. Thus, the local solution belongs to $S^N_{\delta_1}$.

To obtain a global solution, it suffices to iterate this argument finitely many times up to a possible collision time t_1 when $x_{t_1}^i = x_{t_1}^{i+1}$ for some $i \in \{1, \ldots, N\}$. □

We have so far considered solutions only up to the first collision time t_1. To extend the solution to the entire interval $[0, T]$, we observe that the scheme, in Step 3, explicitly removes at least one particle and its associated local density at every collision event. Since the initial number of particles N is finite and particles are never created, the total number of collision times is strictly bounded by $N - 1$. Consequently, the sequence of collision times cannot form an accumulation point in finite time. The system can be iteratively restarted after each collision, ensuring the existence and uniqueness of a global solution on $[0, T]$. Moreover, the bounds (29) hold for all $t \in [0, T] \setminus \mathcal{T}_c$, which implies that the properties from Proposition 4.2 are valid for all $t \in [0, T]$. Finally, since particles and local densities are removed at collision times, the total variation after a collision can only have decreased, guaranteeing that the total variation estimate (28) holds on $[0, T]$.

5. PDE formulation

In this section, we will show that the approximation ρ from (25) is the weak solution of a continuity equation of the form (7), with a velocity field $\overline{V}$ given by the linear interpolation of particle velocities. Ultimately, this formulation allows us to prove that ρ satisfies an approximate entropy inequality.

Recall from Section 4 that the particle path scheme yields a unique global solution defined on $[0,T]$. This solution consists of a finite set of collision times $(t_j)_{j\geq 0} \subset [0,T]$ with corresponding particle paths (x_t^π) and local densities (ρ_t^π) evolving according to (20) and (21) between collisions.

5.1. Approximate continuity equation

We begin by showing that ρ is a weak solution of a continuity equation. Define

$$\begin{aligned}\overline{V}((x_t^\pi),(\rho_t^\pi);x,t) \coloneqq \; &\frac{x_t^{\pi_j(i+1)} - x}{x_t^{\pi_j(i+1)} - x_t^{\pi_j(i)}} V\left(\rho_t^{\pi_j(i-1)}, \rho_t^{\pi_j(i)}\right) \\ &+ \frac{x - x_t^{\pi_j(i)}}{x_t^{\pi_j(i+1)} - x_t^{\pi_j(i)}} V\left(\rho_t^{\pi_j(i)}, \rho_t^{\pi_j(i+1)}\right)\end{aligned} \tag{31}$$

for $t \in (t_j, t_{j+1})$ and $x \in \left[x_t^{\pi_j(i)}, x_t^{\pi_j(i+1)}\right]$. We will mainly write this as $\overline{V}(x,t)$, but included (x^π) and (ρ^π) as arguments in the definition above to stress this dependence. The function $\overline{V}$ is a global-in-time piecewise linear interpolation of particle velocities V, with interpolation nodes given by particle positions (x^π).

Proposition 5.1. *Let ρ be constructed by the particle path scheme and defined as in (25). Then it satisfies*

$$\int_0^T \int_{\mathbb{R}} \big(\partial_t\varphi(x,t) + \overline{V}(x,t)\partial_x\varphi(x,t)\big)\rho(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} \varphi(x,0)\rho_0(x)\,\mathrm{d}x = 0 \tag{32}$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times[0,T))$.

Remark 5.2. We will treat (32) as a linear equation with a prescribed velocity $\overline{V}$. However, since both $\overline{V}$ and ρ depend on the underlying output (x_t^π) and (ρ_t^π) of the particle path scheme, the equation can also be viewed as nonlinear. From this perspective, it is also a nonlocal equation, since $\overline{V}$ is a piecewise linear interpolation with interpolation nodes (x_t^π).

Proof. There can only be finitely many collisions of particles, so it suffices to prove that the equation holds for ρ given by (23) on the interval $(0,t_1)$, for all test functions $\varphi \in C_c^\infty(\mathbb{R}\times[0,t_1])$. To that end, let $\varphi \in C_c^\infty(\mathbb{R}\times[0,t_1])$ such that $\varphi = \gamma\vartheta$ for $\vartheta \in C_c^\infty(\mathbb{R})$ and $\gamma \in C_c^\infty([0,t_1])$. Then

$$\begin{aligned}\int_0^{t_1}\int_{\mathbb{R}} \rho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t &= \sum_{i=0}^N \int_0^{t_1} \rho_t^i\partial_t\gamma(t)\left(\int_{x_t^i}^{x_t^{i+1}} \vartheta(x)\,\mathrm{d}x\right)\mathrm{d}t \\ &= \sum_{i=0}^N \gamma(t_1)\rho_{t_1}^i \int_{x_{t_1}^i}^{x_{t_1}^{i+1}} \vartheta(x)\,\mathrm{d}x - \sum_{i=0}^N \gamma(0)\rho_0^i \int_{x_0^i}^{x_0^{i+1}} \vartheta(x)\,\mathrm{d}x \\ &\quad - \sum_{i=0}^N \int_0^{t_1} \gamma(t)\partial_t\left(\rho_t^i \int_{x_t^i}^{x_t^{i+1}} \vartheta(x)\,\mathrm{d}x\right)\mathrm{d}t\end{aligned} \tag{33}$$

where we have used integration by parts in the last equality. Adopting the shorthand notation $V_t^i = V(\rho_t^{i-1}, \rho_t^i)$, the temporal derivative in the last term can be written as

$$\begin{aligned}
&\partial_t\left(\rho_t^i \int_{x_t^i}^{x_t^{i+1}} \vartheta(x)\,\mathrm{d}x\right)\\
&\quad= -\rho_t^i \frac{V_t^{i+1}-V_t^i}{x_t^{i+1}-x_t^i}\int_{x_t^i}^{x_t^{i+1}} \vartheta(x)\,\mathrm{d}x + \rho_t^i\big[\vartheta\big(x_t^{i+1}\big)V_t^{i+1} - \vartheta\big(x_t^i\big)V_t^i\big]\\
&\quad= \rho_t^i\left[V_t^i\left(\int_{x_t^i}^{x_t^{i+1}} \frac{\vartheta(x)}{x_t^{i+1}-x_t^i}\,\mathrm{d}x - \vartheta\big(x_t^i\big)\right) - V_t^{i+1}\left(\int_{x_t^i}^{x_t^{i+1}} \frac{\vartheta(x)}{x_t^{i+1}-x_t^i}\,\mathrm{d}x - \vartheta\big(x_t^{i+1}\big)\right)\right]\\
&\quad= \rho_t^i\left(V_t^i\int_{x_t^i}^{x_t^{i+1}} \frac{x_t^{i+1}-x}{x_t^{i+1}-x_i}\partial_x\vartheta(x)\,\mathrm{d}x + V_t^{i+1}\int_{x_t^i}^{x_t^{i+1}} \frac{x-x_t^i}{x_t^{i+1}-x_i}\partial_x\vartheta(x)\,\mathrm{d}x\right)\\
&\quad= \rho_t^i\int_{x_t^i}^{x_t^{i+1}} \overline{V}(x,t)\partial_x\vartheta(x)\,\mathrm{d}x,
\end{aligned}$$

for all $i \in \{1,\dots,N-1\}$ (recall that $\rho_t^0 = \rho_t^N = 0$ for all t). Here, we have used (20) in the first equality and integration by parts in the third equality. Inserting this in (33) yields

$$\begin{aligned}
\int_0^{t_1}\int_{\mathbb{R}} \rho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t &= \sum_{i=0}^N \gamma(t_1)\rho_{t_1}^i \int_{x_{t_1}^i}^{x_{t_1}^{i+1}} \vartheta(x)\,\mathrm{d}x - \sum_{i=0}^N \gamma(0)\rho_0^i\int_{x_0^i}^{x_0^{i+1}}\vartheta(x)\,\mathrm{d}x\\
&\quad- \sum_{i=0}^N \int_0^{t_1}\gamma(t)\rho_t^i\left(\int_{x_t^i}^{x_t^{i+1}} \overline{V}(x,t)\partial_x\vartheta(x)\,\mathrm{d}x\right)\mathrm{d}t\\
&= \int_{\mathbb{R}} \rho(t_1)\varphi(t_1)\,\mathrm{d}x - \int_{\mathbb{R}} \rho_0\varphi(0)\,\mathrm{d}x - \int_0^{t_1}\int_{\mathbb{R}} \rho\overline{V}\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The general case now follows by approximation of arbitrary $\varphi \in C_c^\infty(\mathbb{R}\times[0,t_1])$ by finite linear combinations of $\gamma(t)\vartheta(x)$. ☐

Proposition 5.1 implies some temporal regularity of the function ρ.

Lemma 5.3. *Let ρ be given by* (25)*. Then*

$$\|\rho(t)-\rho(s)\|_{L^1(\mathbb{R})} \le 4[f]_{\mathrm{Lip}}|\rho_0|_{\mathrm{BV}(\mathbb{R})}|t-s| \tag{34}$$

for all $s,t \in [0,T]$.

Proof. Since ρ satisfies (32), for all $\vartheta \in C_c^\infty(\mathbb{R})$ the function $t \mapsto \int_{\mathbb{R}} \rho\vartheta\,\mathrm{d}x$ is absolutely continuous on $(0,T)$ with weak derivative $\int_{\mathbb{R}} \overline{V}\rho\partial_x\vartheta\,\mathrm{d}x$. Equivalently stated,

$$\int_{\mathbb{R}} \rho(x,t)\vartheta(x)\,\mathrm{d}x - \int_{\mathbb{R}} \rho(x,s)\vartheta(x)\,\mathrm{d}x = \int_s^t\int_{\mathbb{R}} \overline{V}(x,r)\rho(x,r)\partial_x\vartheta(x)\,\mathrm{d}x\,\mathrm{d}r$$

for all $t < s$. Using integration by parts on the right-hand side and taking the supremum over all $|\vartheta| \le 1$ yields

$$\int_{\mathbb{R}} |\rho(x,t)-\rho(x,s)|\,\mathrm{d}x = \int_s^t\int_{\mathbb{R}} \big|\partial_x\big(\overline{V}(x,r)\rho(x,r)\big)\big|\,\mathrm{d}x\,\mathrm{d}r,$$

from which we infer

$$\|\rho(t)-\rho(s)\|_{L^1(\mathbb{R})} \le \sup_{r\in(s,t)}\big|\overline{V}(r)\rho(r)\big|_{\mathrm{BV}(\mathbb{R})}|t-s|.$$

It remains to estimate the total variation of $\overline{V}\rho$. It is a piecewise linear function with breakpoints at (x^i) and values

$$\lim_{x\to(x_t^i)^-}\overline{V}(x,t)\rho(x,t)=a(\tilde\rho_t^i)\rho_t^{i-1},\qquad \lim_{x\to(x_t^i)^+}\overline{V}(x,t)\rho(x,t)=a(\tilde\rho_t^i)\rho_t^i,$$

for all $i\in\{1,\dots,N\}$, where $\tilde\rho_t^i\in[\rho_t^{i-1},\rho_t^i]$ is defined as the point at which a attains the value $V(\rho_t^{i-1},\rho_t^i)$ (note that $\tilde\rho_t^i$ is not necessarily unique). The total variation is

$$\left|\overline{V}(t)\rho(t)\right|_{\mathrm{BV}(\mathbb{R})}=\sum_{i=1}^{N}\left|a\left(\tilde\rho_t^i\right)\rho_t^{i-1}-a\left(\tilde\rho_t^i\right)\rho_t^i\right|+\sum_{i=1}^{N-1}\left|a\left(\tilde\rho_t^i\right)\rho_t^i-a\left(\tilde\rho_t^{i+1}\right)\rho_t^i\right|. \tag{35}$$

Since $|a(\rho)|\le[f]_{\mathrm{Lip}}$, the first sum in (35) can be bounded by

$$\sum_{i=1}^{N}\left|a\left(\tilde\rho_t^i\right)\rho_t^{i-1}-a\left(\tilde\rho_t^i\right)\rho_t^i\right|\le[f]_{\mathrm{Lip}}\sum_{i=1}^{N}\left|\rho_t^{i-1}-\rho_t^i\right|\le[f]_{\mathrm{Lip}}|\rho(t)|_{\mathrm{BV}(\mathbb{R})}.$$

For the terms in the second sum of (35), we have

$$\begin{aligned}&\left|a\left(\tilde\rho_t^i\right)\rho_t^i-a\left(\tilde\rho_t^{i+1}\right)\rho_t^i\right|\\&\quad\le\left|a\left(\tilde\rho_t^i\right)\rho_t^i-a\left(\tilde\rho_t^i\right)\tilde\rho_t^i\right|+\left|a\left(\tilde\rho_t^i\right)\tilde\rho_t^i-a\left(\tilde\rho_t^{i+1}\right)\tilde\rho_t^{i+1}\right|+\left|a\left(\tilde\rho_t^{i+1}\right)\tilde\rho_t^{i+1}-a\left(\tilde\rho_t^{i+1}\right)\rho_t^i\right|.\end{aligned} \tag{36}$$

Using the definition of a from (2), we find that

$$\left|a\left(\tilde\rho_t^i\right)\tilde\rho_t^i-a\left(\tilde\rho_t^{i+1}\right)\tilde\rho_t^{i+1}\right|\le[f]_{\mathrm{Lip}}\left|\tilde\rho_t^i-\tilde\rho_t^{i+1}\right|,$$

which inserted in (36) yields

$$\begin{aligned}&\left|a\left(\tilde\rho_t^i\right)\rho_t^i-a\left(\tilde\rho_t^{i+1}\right)\rho_t^i\right|\\&\quad\le\left|a\left(\tilde\rho_t^i\right)\right|\left|\rho_t^i-\tilde\rho_t^i\right|+[f]_{\mathrm{Lip}}\left|\tilde\rho_t^i-\tilde\rho_t^{i+1}\right|+\left|a\left(\tilde\rho_t^{i+1}\right)\right|\left|\tilde\rho_t^{i+1}-\rho_t^i\right|\\&\quad\le[f]_{\mathrm{Lip}}\left(\left|\rho_t^i-\tilde\rho_t^i\right|+\left|\tilde\rho_t^i-\tilde\rho_t^{i+1}\right|+\left|\tilde\rho_t^{i+1}-\rho_t^i\right|\right).\end{aligned}$$

This implies

$$\begin{aligned}\sum_{i=1}^{N-1}\left|a\left(\tilde\rho_t^i\right)\rho_t^i-a\left(\tilde\rho_t^{i+1}\right)\rho_t^i\right|&\le[f]_{\mathrm{Lip}}\sum_{i=1}^{N-1}\left(\left|\rho_t^i-\tilde\rho_t^i\right|+\left|\tilde\rho_t^i-\tilde\rho_t^{i+1}\right|+\left|\tilde\rho_t^{i+1}-\rho_t^i\right|\right)\\&\le[f]_{\mathrm{Lip}}\sum_{i=1}^{N-1}\left(\left|\rho_t^i-\rho_t^{i-1}\right|+\left|\tilde\rho_t^i-\tilde\rho_t^{i+1}\right|+\left|\rho_t^{i+1}-\rho_t^i\right|\right)\\&\le3[f]_{\mathrm{Lip}}|\rho(t)|_{\mathrm{BV}(\mathbb{R})}.\end{aligned}$$

Plugging this into (35) and using that the total variation of ρ is nonincreasing from part (iv) of Proposition 4.2, we obtain (34). □

Towards an entropy-like inequality for ρ, we consider the vertically shifted function $\rho_k:=\rho-k$ for constants $k\in\mathbb{R}$. Set also $\rho_{0,k}:=\rho_0-k$. Since ρ is a weak solution of (32), it follows easily that ρ_k is a weak solution to

$$\begin{cases}\partial_t\rho_k+\partial_x\left(\overline{V}\rho_k\right)=-k\partial_x\overline{V}\\ \rho_k(0)=\rho_{0,k}\end{cases} \tag{37}$$

in the sense of (16). Note that $\partial_x \overline{V}$ is finite except at collision times $t_j \in \mathcal{T}_c$, and satisfies

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_{\mathbb{R}} |\partial_x \overline{V}(x,t)| \,\mathrm{d}x = \operatorname*{ess\,sup}_{t\in(0,T)} |\overline{V}(t)|_{\mathrm{BV}(\mathbb{R})} \le (N+1)a_{\max},$$

where we recall that $a_{\max} = \max_{\rho\in[0,\rho_0^*]} a(\rho)$. The next lemma shows that the equation (37) can be studied within the one-sided Lipschitz framework from Section 2.3.

Lemma 5.4. *The function $x \mapsto \overline{V}(x,t)$ is continuous on $\mathbb{R}$ for all $t \in [0,T] \setminus \mathcal{T}_c$. Furthermore, it is one-sided Lipschitz (cf. (18)) for all $t \in [0,T]$.*

Proof. Since $x \mapsto \overline{V}(x,t)$ is a linear interpolation of values $V(\rho^{i-1},\rho^i)$ on interpolation nodes (x_t^i), it is Lipschitz except possibly at collision times, towards which two particles move arbitrarily close. Consider without loss of generality the interval $[0,t_1]$. From (27), two particles x^i and x^{i+1} can only collide if $\rho^i = 0$. But then

$$\overline{V}\big(x_t^{i+1},t\big) = V\big(0,\rho_t^{i+1}\big) = \min_{\rho\in[0,\rho_t^{i+1}]} a(\rho) \le \max_{\rho\in[0,\rho_t^{i-1}]} a(\rho) = V\big(\rho_t^{i-1},0\big) = \overline{V}\big(x_t^i,t\big),$$

for all $t \in [0,t_1]$, and the one-sided Lipschitz condition (18) holds. □

Consider the system of ODEs

$$\begin{cases} \dot{x}_t = \overline{V}(x_t,t) & \text{for a.e. } t \in (s,T) \\ x_s = x. \end{cases} \tag{38}$$

Using the theory for continuity equations outlined in Section 2.3, we obtain a representation formula for ρ_k.

Proposition 5.5. *The system of ODEs (38) generates a unique forward Lipschitz flow $X = X_t(x,s)$. Let ρ be defined by (25). Then for all $k \in \mathbb{R}$, the function $\rho_k = \rho - k$ is the unique weak solution of (37). Moreover, for all $s \in [0,T]$, it satisfies*

$$\rho_k(t) = (X_t(s))_\# \rho_k(s) - k \int_s^t X_t(r)_\# (\partial_x \overline{V}(r)) \,\mathrm{d}r.$$

5.2. Entropy inequality

We prove that ρ satisfies the approximate entropy inequality (9), a key step in our analysis. The inequality in the following lemma turns out to be the cornerstone of this proof. It does not hold in general for any other choice of particle velocity other than (5). In this sense, it confirms that (5) is the correct entropic velocity. Recall that we denote $\rho - k$ by ρ_k.

Lemma 5.6. *Let ρ be given by (25). Then*

$$\int_{\mathbb{R}} \vartheta(x) \operatorname{sgn}(\rho_k(x,t)) \partial_x \big(\overline{V}(x,t) - a(k)\big) \,\mathrm{d}x \ge - \int_{\mathbb{R}} \partial_x \vartheta(x) \operatorname{sgn}(\rho_k(x,t)) \big(\overline{V}(x,t) - a(k)\big) \,\mathrm{d}x \tag{39}$$

for all $t \in (0,T) \setminus \mathcal{T}_c$, all $k \in \mathbb{R}$ and all nonnegative $\vartheta \in C_c^\infty(\mathbb{R})$.

Proof. Let $t \in (0, t_1)$ without loss of generality. Splitting the integral on the left-hand side of (39) and using integration by parts, we see that

$$\begin{aligned}
&\int_{\mathbb{R}} \vartheta(x) \operatorname{sgn}(\rho_k(x,t)) \partial_x \big(\overline{V}(x,t) - a(k)\big) \,\mathrm{d}x \\
&\quad = \sum_{i=0}^{N} \int_{x_t^i}^{x_t^{i+1}} \vartheta(x) \operatorname{sgn}\big(\rho_t^i - k\big) \partial_x \big(\overline{V}(x,t) - a(k)\big) \,\mathrm{d}x \\
&\quad = \sum_{i=0}^{N} \operatorname{sgn}\big(\rho_t^i - k\big) \big[\vartheta\big(x_t^{i+1}\big)\big(\overline{V}\big(x_t^{i+1},t\big) - a(k)\big) - \vartheta\big(x_t^i\big)\big(\overline{V}\big(x_t^i,t\big) - a(k)\big)\big] \\
&\qquad - \sum_{i=0}^{N} \int_{x_t^i}^{x_t^{i+1}} \partial_x \vartheta(x) \operatorname{sgn}\big(\rho_t^i - k\big)\big(\overline{V}(x,t) - a(k)\big) \,\mathrm{d}x.
\end{aligned} \tag{40}$$

In the third line, the series is telescoping except when $\rho_t^i - k$ and $\rho_t^{i+1} - k$ have different signs. This becomes apparent when writing

$$\begin{aligned}
&\sum_{i=0}^{N} \operatorname{sgn}\big(\rho_t^i - k\big) \big[\vartheta\big(x_t^{i+1}\big)\big(\overline{V}\big(x_t^{i+1},t\big) - a(k)\big) - \vartheta\big(x_t^i\big)\big(\overline{V}\big(x_t^i,t\big) - a(k)\big)\big] \\
&\quad = \sum_{i=0}^{N-1} \big(\operatorname{sgn}\big(\rho_t^i - k\big) - \operatorname{sgn}\big(\rho_t^{i+1} - k\big)\big) \vartheta\big(x_t^{i+1}\big)\big(\overline{V}\big(x_t^{i+1},t\big) - a(k)\big),
\end{aligned}$$

where we used $\vartheta(x_t^0) = \vartheta(x_t^{N+1}) = 0$. If $\operatorname{sgn}(\rho_t^i - k) < \operatorname{sgn}(\rho_t^{i+1} - k)$, then $\rho_t^i \le k \le \rho_t^{i+1}$, and consequently

$$\overline{V}(x_t^{i+1}, t) = \min_{\rho \in [\rho_t^i, \rho_t^{i+1}]} a(\rho) \le a(k).$$

On the other hand, if $\operatorname{sgn}(\rho_t^i - k) > \operatorname{sgn}(\rho_t^{i+1} - k)$, then $\rho_t^i \ge k \ge \rho_t^{i+1}$, and

$$\overline{V}(x_t^{i+1}, t) = \max_{\rho \in [\rho_t^{i+1}, \rho_t^i]} a(\rho) \ge a(k).$$

In both cases, the inequality

$$\big(\operatorname{sgn}\big(\rho_t^i - k\big) - \operatorname{sgn}\big(\rho_t^{i+1} - k\big)\big) \vartheta\big(x_t^{i+1}\big)\big(\overline{V}\big(x_t^{i+1},t\big) - a(k)\big) \ge 0$$

holds. Inserting this in (40) yields (39). □

Next, we use the previous lemma to prove the approximate entropy inequality.

Theorem 5.7. *The function $\rho_k = \rho - k$, where ρ is given by (25), satisfies*

$$\int_0^T \int_{\mathbb{R}} |\rho_k| \partial_t \varphi + (\overline{V}\rho - f(k)) \operatorname{sgn}(\rho_k) \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}t - \int_{\mathbb{R}} \varphi(T) |\rho_k(T)| \,\mathrm{d}x + \int_{\mathbb{R}} \varphi(0) |\rho_{0,k}| \,\mathrm{d}x \ge 0 \tag{41}$$

for all $k \ge 0$ and nonnegative $\varphi \in C_c^\infty(\mathbb{R} \times [0,T])$.

Proof. Let $0 \le \varphi \in C_c^\infty(\mathbb{R} \times [0,T])$ and set $\varepsilon > 0$. We begin by writing

$$\int_0^T \int_{\mathbb{R}} \varphi |\rho_k| \,\mathrm{d}x \,\mathrm{d}t = \int_0^\varepsilon \int_{\mathbb{R}} \varphi |\rho_k| \,\mathrm{d}x \,\mathrm{d}t + \int_0^{T-\varepsilon} \int_{\mathbb{R}} \varphi(t+\varepsilon) \operatorname{sgn}(\rho_k(t+\varepsilon)) \rho_k(t+\varepsilon) \,\mathrm{d}x \,\mathrm{d}t. \tag{42}$$

For the last term, we use the formula

$$\rho_k(t+\varepsilon) = (X_{t+\varepsilon}(t))_{\#}\rho_k(t) - k\int_t^{t+\varepsilon} (X_{t+\varepsilon}(r))_{\#}(\partial_x \overline{V}(r))\,\mathrm{d}r$$

from Proposition 5.5, which gives

$$\begin{aligned}
&\int_0^{T-\varepsilon}\int_{\mathbb{R}} \varphi(t+\varepsilon)\,\mathrm{sgn}\,(\rho_k(t+\varepsilon))\rho_k(t+\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^{T-\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(t), t+\varepsilon\big)\,\mathrm{sgn}\,\big(\rho_k(X_{t+\varepsilon}(t), t+\varepsilon)\big)\rho_k(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - k\int_0^{T-\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(t), t+\varepsilon\big)\,\mathrm{sgn}\,\big(\rho_k(X_{t+\varepsilon}(r), t+\varepsilon)\big)\partial_x\overline{V}(r)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t
\end{aligned}$$

(see Def. 1.70 from [1] for justification of using the push-forward formula (19) in this setting). In the above integrals, we have dropped the spatial argument for readability. Since

$$\begin{aligned}
&\int_0^{T-\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(t), t+\varepsilon\big)\,\mathrm{sgn}\,\big(\rho(X_{t+\varepsilon}(t), t+\varepsilon)\big)\rho_k(t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_0^{T-\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(t), t+\varepsilon\big)|\rho_k(t)|\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

inserting the above in (42) yields

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}} \varphi|\rho_k|\,\mathrm{d}x\,\mathrm{d}t \le \int_0^{\varepsilon}\int_{\mathbb{R}} \varphi|\rho_k|\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T-\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(t), t+\varepsilon\big)|\rho_k(t)|\,\mathrm{d}x\,\mathrm{d}t \\
- k\int_0^{T-\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(r), t+\varepsilon\big)\,\mathrm{sgn}\,\big(\rho_k(X_{t+\varepsilon}(r), t+\varepsilon)\big)\partial_x\overline{V}(r)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned}$$

Dividing by ε and rearranging the terms, we have

$$\begin{aligned}
&\int_0^{T-\varepsilon}\int_{\mathbb{R}} \frac{\varphi\big(X_{t+\varepsilon}(x,t), t+\varepsilon\big) - \varphi(x,t)}{\varepsilon}|\rho_k(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{1}{\varepsilon}\int_{T-\varepsilon}^{T}\int_{\mathbb{R}} \varphi(x,t)|\rho_k(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^{\varepsilon}\int_{\mathbb{R}} \varphi(x,t)|\rho_k(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \ge \frac{k}{\varepsilon}\int_0^{T-\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{R}} \varphi\big(X_{t+\varepsilon}(x,r), t+\varepsilon\big)\,\mathrm{sgn}\,\big(\rho_k(X_{t+\varepsilon}(x,r), t+\varepsilon)\big)\partial_x\overline{V}(x,r)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}t.
\end{aligned} \tag{43}$$

Towards passing $\varepsilon \to 0$, note that

$$\begin{aligned}
&\frac{\varphi\big(X_{t+\varepsilon}(x,t), t+\varepsilon\big) - \varphi(x,t)}{\varepsilon} \\
&\quad = \int_0^1 \partial_t\varphi(x, t+r\varepsilon) + \partial_x\varphi\big(x + r(X_{t+\varepsilon}(x,t) - x), t+\varepsilon\big)\left(\frac{X_{t+\varepsilon}(x,t) - x}{\varepsilon}\right)\mathrm{d}r.
\end{aligned} \tag{44}$$

Since furthermore

$$\frac{X_{t+\varepsilon}(x,t) - x}{\varepsilon} = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \overline{V}(X_r(x,t), r)\,\mathrm{d}r$$

and the velocity $\overline{V}$ is continuous for all $t \in [0,T]\setminus\mathcal{T}_c$, we see that (44) converges to $\partial_t\varphi+\overline{V}\partial_x\varphi$ in $L^1(\mathbb{R}\times(0,T))$ as $\varepsilon\to 0$. Taking the limit in (43) therefore yields

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\big(\partial_t\varphi+\overline{V}\partial_x\varphi\big)|\rho_k|\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}}\varphi(T)|\rho_k(T)|\,\mathrm{d}x+\int_{\mathbb{R}}\varphi(0)|\rho_{0,k}|\,\mathrm{d}x\,\mathrm{d}t\\ \geq k\int_0^T\int_{\mathbb{R}}\varphi\,\mathrm{sgn}\,(\rho_k)\partial_x\overline{V}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We now insert $k\partial_x\overline{V}=\partial_x(k\overline{V}-f(k))$ and use the result of Lemma 5.6, by which we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\big(\partial_t\varphi+\overline{V}\partial_x\varphi\big)|\rho_k|\,\mathrm{d}x\,\mathrm{d}t-\int_{\mathbb{R}}\varphi(T)|\rho_k(T)|\,\mathrm{d}x+\int_{\mathbb{R}}\varphi(0)|\rho_{0,k}|\,\mathrm{d}x\,\mathrm{d}t\\ \geq -\int_0^T\int_{\mathbb{R}}\partial_x\varphi\,\mathrm{sgn}\,(\rho_k)\big(k\overline{V}-f(k)\big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all nonnegative $k\in\mathbb{R}$. Finally, in view of

$$\overline{V}|\rho_k|+\mathrm{sgn}\,(\rho_k)\big(k\overline{V}-f(k)\big)=\big(\overline{V}\rho-f(k)\big)\,\mathrm{sgn}\,(\rho_k),$$

we arrive at (41). □

6. Convergence rate

By comparing (41) to the exact entropy inequality, it is possible to estimate how far ρ is from being an entropy solution of the conservation law (1). Following Chapter 3.3 of [16], we define the Kruzkhov form

$$\begin{aligned}\Lambda_T(\rho,\varphi,k)=\int_0^T\int_{\mathbb{R}}|\rho_k|\partial_t\varphi+\big(f(\rho)-f(k)\big)\,\mathrm{sgn}\,(\rho_k)\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\ -\int_{\mathbb{R}}|\rho_k(T)|\varphi(T)\,\mathrm{d}x+\int_{\mathbb{R}}|\rho_{k,0}|\varphi(0)\,\mathrm{d}x,\end{aligned}$$

where $k\in\mathbb{R}$ and $\varphi\in C_c^\infty(\mathbb{R}\times[0,T])$. Let $(\omega_\varepsilon)_{\varepsilon\geq 0}$ be a family of standard mollifiers, and set

$$\Omega^{\varepsilon,\tau}(x,y,t,s)=\omega^\varepsilon(x-y)\omega^\tau(t-s).$$

Furthermore define

$$\Lambda_T^{\varepsilon,\tau}(\rho,u):=\int_0^T\int_{\mathbb{R}}\Lambda_T(\rho(\cdot,\cdot),\Omega^{\varepsilon,\tau}(\cdot,y,\cdot,s),u(y,s))\,\mathrm{d}y\,\mathrm{d}s. \tag{45}$$

To prove Theorem 1.1, we will invoke Kuznetsov's lemma ([16], Thm. 3.14): If u is the entropy solution of (1) and ρ is an approximation which satisfies

$$\|\rho(t)\|_{L^\infty}\leq\|\rho_0\|_{L^\infty}\quad\text{and}\quad|\rho(t)|_{\mathrm{BV}}\leq|\rho_0|_{\mathrm{BV}} \tag{46}$$

for all $t\in[0,T]$, then

$$\|\rho(T)-u(T)\|_{L^1(\mathbb{R})}\leq\|\rho_0-u_0\|_{L^1(\mathbb{R})}+|u_0|_{\mathrm{BV}(\mathbb{R})}(2\varepsilon+\tau[f]_{\mathrm{Lip}})+\nu(\rho,\tau)-\Lambda^{\varepsilon,\tau}(\rho,u) \tag{47}$$

for all $\varepsilon>0$ and $\tau\in(0,T)$, where

$$\nu(\rho,\tau):=\sup_{t\in(0,T)}\sup_{r\in(0,\tau)}\|\rho(t+r)-\rho(t)\|_{L^1(\mathbb{R})}.$$

Note in particular that the function ρ generated by the the particle path scheme satisfies the conditions in (46).

Proof of Theorem 1.1. Let ρ be the approximation generated by the particle path scheme as defined in (25). Using the approximate entropy inequality (41), we have

$$\Lambda_T(\rho,\varphi,k) \geq \int_0^T \int_{\mathbb{R}} \big(f(\rho) - \overline{V}\rho\big) \operatorname{sgn}(\rho_k) \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t$$

for all nonnegative $\varphi \in C_c^\infty(\mathbb{R}\times[0,T])$. Plugging this into (45) gives

$$\Lambda_T^{\varepsilon,\tau} \geq \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} \big(f(\rho(y,s)) - \overline{V}(y,s)\rho(y,s)\big) \times \operatorname{sgn}(\rho(y,s)-u(x,t))\partial_x \Omega^{\varepsilon,\tau}(x,y,t,s)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s. \tag{48}$$

In view of

$$\nu(\rho,\tau) \leq 4|\tau|[f]_{\mathrm{Lip}}[\rho_0]_{BV} \leq 4|\tau|[f]_{\mathrm{Lip}}[u_0]_{BV}$$

by Lemma 5.3, we can immediately pass $\tau \to 0$ to get rid of one of the temporal integrals. The remaining contribution to the right hand side of (48) can be estimated in the absolute value according to

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} &\big|f(\rho(y,t)) - \overline{V}(y,t)\rho(y,t)\big| |\partial_x \omega_\varepsilon(x-y)| \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\leq \frac{1}{\varepsilon}\int_0^T \int_{\mathbb{R}} \big|f(\rho(y,t)) - \overline{V}(y,t)\rho(y,t)\big| \,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

Inserting this in (47) yields

$$\|\rho(T)-u(T)\|_{L^1(\mathbb{R})} \leq \|\rho_0 - u_0\|_{L^1(\mathbb{R})} + 2\varepsilon|u_0|_{\mathrm{BV}(\mathbb{R})} + \frac{1}{\varepsilon}\|\overline{V}\rho - f(\rho)\|_{L^1(\mathbb{R}\times(0,T))}$$

which implies the estimate (10) upon minimization with respect to ε (*i.e.* setting $\varepsilon = \sqrt{\|\overline{V}\rho - f(\rho)\|_{L^1}/2|u_0|_{\mathrm{BV}}}$). □

Remark 6.1. Throughout the paper, the assumption that f is globally Lipschitz is only used for simplicity of exposition. In fact, it suffices to require f to be Lipschitz within the convex hull of $u_0(\mathbb{R})$. The constant $[f]_{\mathrm{Lip}}$ can then be replaced by the Lipschitz constant of f on this restricted interval.

Until now, no assumptions on the initial distribution of particles $(x_0^i)_{i=1}^N$ have been made. We specialize the result of Theorem 1.1 to the case when there is a bound on the maximal distance $\Delta x_0^* \coloneqq \max_i \Delta x_0^i$ between initial particles.

Proposition 6.2. *Let u the entropy solution of (1) and ρ the approximation generated by the the particle path scheme. Assume that f' is Lipschitz, and that there is a number $R_{u_0} > 0$ such that*

$$\int_{-\infty}^{x_0^1} u_0(x)\,\mathrm{d}x + \int_{x_0^N}^{\infty} u_0(x)\,\mathrm{d}x \leq R_{u_0}.$$

Then the approximation error is bounded by

$$\|\rho(T)-u(T)\|_{L^1(\mathbb{R})} \leq R_{u_0} + |u_0|_{\mathrm{BV}(\mathbb{R})}\left(\Delta x_0^* + 2\sqrt{T[f']_{\mathrm{Lip}}\|u_0\|_{L^\infty(\mathbb{R})}\Delta x_0^*}\right).$$

Proof. First, the standard estimate

$$\begin{aligned}\int_{x_0^i}^{x_0^{i+1}} |u_0 - \rho_0| \,\mathrm{d}x &= \int_{x_0^i}^{x_0^{i+1}} \left| u_0(x) - \frac{1}{\Delta x_0^i} \int_{x_0^i}^{x_0^{i+1}} u_0(y) \,\mathrm{d}y \right| \mathrm{d}x \\ &\le \frac{1}{\Delta x_0^i} \int_{x_0^i}^{x_0^{i+1}} \int_{x_0^i}^{x_0^{i+1}} |u_0(x) - u_0(y)| \,\mathrm{d}y \,\mathrm{d}x \\ &\le \Delta x_0^i \|u_0\|_{\mathrm{BV}([x_0^i, x_0^{i+1}])},\end{aligned}$$

where $|\cdot|_{\mathrm{BV}([x_0^i, x_0^{i+1}])}$ denotes the total variation on $[x_0^i, x_0^{i+1}]$, implies that

$$\|\rho_0 - u_0\|_{L^1(\mathbb{R})} = R_{u_0} + \sum_{i=1}^{N-1} \int_{x_0^i}^{x_0^{i+1}} |u_0 - \rho_0| \,\mathrm{d}x \le R_{u_0} + \Delta x_0^* |u_0|_{\mathrm{BV}(\mathbb{R})}.$$

It remains to estimate $\|\overline{V}\rho - f(\rho)\|_{L^1(\mathbb{R}\times(0,T))}$. Consider first $t \in (0, t_1)$, for which

$$\int_{\mathbb{R}} \left|\overline{V}(x,t)\rho(x,t) - f(\rho(x,t))\right| \mathrm{d}x = \sum_{i=1}^{N-1} \rho_t^i \int_{x_t^i}^{x_t^{i+1}} \left|\overline{V}(x,t) - a(\rho_t^i)\right| \mathrm{d}x. \tag{49}$$

Since $\overline{V}$ is defined as a linear interpolation, the supremum of the integrand can be bounded by

$$\sup_{x \in (x_t^i, x_t^{i+1})} \left|\overline{V}(x,t) - a(\rho_t^i)\right| \le \left|V(\rho_t^{i-1}, \rho_t^i) - a(\rho_t^i)\right| + \left|V(\rho_t^{i-1}, \rho_t^i) - a(\rho_t^i)\right|.$$

As in the proof of Lemma 5.3, let $\tilde{\rho}_t^i \in [\rho_t^{i-1}, \rho_t^i]$ be defined as the point at which a attains the value $V(\rho_t^{i-1}, \rho_t^i)$. Then

$$\begin{aligned}\sup_{x \in (x_t^i, x_t^{i+1})} \left|\overline{V}(x,t) - a(\rho_t^i)\right| &\le \left|a(\tilde{\rho}_t^i) - a(\rho_t^i)\right| + \left|a(\tilde{\rho}_t^{i+1}) - a(\rho_t^i)\right| \\ &\le [a]_{\mathrm{Lip}} \left(\left|\tilde{\rho}_t^i - \rho_t^i\right| + \left|\tilde{\rho}_t^{i+1} - \rho_t^i\right|\right).\end{aligned}$$

Plugging this into (49) gives

$$\|\overline{V}(t)\rho(t) - f(\rho(t))\|_{L^1(\mathbb{R})} \le [a]_{\mathrm{Lip}} \sum_{i=1}^{N-1} \rho_t^i \Delta x_t^i \left(|\tilde{\rho}_t^i - \rho_t^i| + |\tilde{\rho}_t^{i+1} - \rho_t^i|\right).$$

Note that $[a]_{\mathrm{Lip}} \le [f']_{\mathrm{Lip}}/2$ and $\rho_t^i \Delta x_t^i = \rho_0^i \Delta x_0^i \le \rho_0^* \Delta x_0^*$. This means that

$$\begin{aligned}\|\overline{V}(t)\rho(t) - f(\rho(t))\|_{L^1(\mathbb{R})} &\le \frac{[f']_{\mathrm{Lip}}}{2} \rho_0^* \Delta x_0^* \sum_{i=1}^{N-1} \left(|\tilde{\rho}_t^i - \rho_t^i| + |\tilde{\rho}_t^{i+1} - \rho_t^i|\right) \\ &\le \frac{[f']_{\mathrm{Lip}}}{2} \rho_0^* \Delta x_0^* |\rho_0|_{\mathrm{BV}(\mathbb{R})} \\ &\le \frac{[f']_{\mathrm{Lip}}}{2} \|u_0\|_{L^\infty(\mathbb{R})} \Delta x_0^* |u_0|_{\mathrm{BV}(\mathbb{R})}\end{aligned}$$

for all $t \in (0, t_1)$. A similar argument can be made for each interval (t_j, t_{j+1}) between collisions. Inserting this in (10) finishes the proof. □

Corollary 1.3 from the introduction follows immediately as a consequence of Proposition 6.2.

7. Conclusion and future work

In this paper, we introduced and analyzed the particle path scheme Steps 1–3, a novel method for approximating entropy solutions of one-dimensional scalar conservation laws with non-negative initial data. The scheme is motivated by the equivalence between entropy solutions and well-posed particle paths, using a specific particle velocity derived from the Filippov particle velocity at discontinuities of the solution.

The main contribution is the L^1-stability estimate presented in Theorem 1.1, which bounds the error between the approximation ρ and the true entropy solution u. This estimate directly leads to an explicit convergence rate of order $\mathcal{O}(\sqrt{\Delta x^*})$ under standard assumptions (Cor. 1.3).

Furthermore, we demonstrated that for concave flux functions, the Particle Path Scheme coincides precisely with the well-known Follow-the-Leader (FtL) traffic flow model. This connection provides a new theoretical foundation for the FtL model and yields a rigorous proof of its convergence rate towards the macroscopic LWR model.

While the management of particle collisions entails higher algorithmic complexity than fixed-grid methods, the primary value of this work lies in the theoretical framework. The scheme serves as a constructive bridge between microscopic dynamics and macroscopic entropy solutions, offering explicit convergence rates.

Throughout this work, we assumed non-negative initial data u_0, a standard assumption motivated by applications such as traffic modeling. To extend the scheme to signed initial data, one needs to handle the possible cancellation of mass where the solution changes sign. While the full particle velocity function (14) derived in Theorem 1.6 from [12] applies to signed solutions, incorporating it into the scheme is technically more involved, and it is not immediately clear whether an analogous PDE formulation, similar to the continuity equation (7) derived here, holds in this case.

A natural direction for future work is the development of fully discrete versions of the particle path scheme, obtained by applying numerical ODE solvers to the particle system (20). Although the stability and convergence of such schemes remains to be analyzed, a simple forward Euler discretization was employed to generate the numerical results shown in Figure 1. Related work exists for the Follow-the-Leader model, where convergence of a fully discrete scheme was established in [18], albeit without an explicit convergence rate.

Finally, the connection between the scheme and the particle path interpretation of entropy solutions suggests an investigation of convergence of the characteristic flow. Specifically, one could study whether the flow map $X_t(s,x)$ generated by the ODE $\dot{x}_t = \overline{V}(x_t, t)$ from Proposition 5.5 converges to the true Filippov flow associated with the entropy solution of the conservation law. Such convergence has interesting interpretations from a modelling perspective, and would provide further validation of the scheme's foundations.

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