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SOLVING NUMERICALLY THE TWO-DIMENSIONAL TIME HARMONIC MAXWELL PROBLEM WITH SIGN-CHANGING COEFFICIENTS

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Abstract. We are investigating the numerical solution to the 2D time-harmonic Maxwell equations in the presence of a classical medium and a metamaterial, that is with sign-changing coefficients. As soon as the problem has a (unique) solution, we are able to build a converging numerical approximation based on the finite element method, for which there is no constraint on the meshes related to the sign-changing behavior. To that aim, we use Lagrange finite elements to approximate the scalar potentials appearing in the Helmholtz decomposition of the vector-valued electromagnetic fields. Convergence in strong norm is proven for the fields. Numerical examples illustrate the theory.

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1. Introduction

In electromagnetism, negative materials have been a major research focus over the past two decades, due to their potential applications. These materials can either be metals at optical frequencies or negative metamaterials, which are engineered materials with unique optical properties, see for instance [35, 36]. Unlike traditional materials, the real part of the permittivity and/or permeability of these materials is negative within a certain frequency range. The combination of classical dielectrics with such negative materials offers exciting potential applications, such as plasmonic waveguides, perfect lenses [30, 33], invisibility cloaks [34]. However, due to the sign-changing coefficients, this leads to new challenges regarding the well-posedness and approximability of the resulting models. For instance, even in the case where the coefficients take constant values in the dielectric and in the negative material, the resulting model may be ill-posed in the Fredholm sense. It has been proven [5] that this occurs whenever the ratio between the positive and negative values of the coefficients belongs to a certain critical set, known as the critical interval. This critical interval is a subset of $\mathbb{R}_{<0}$ and depends only on the geometry and the regularity of the interface separating the two media. In a 2D geometry, it is further proven that, as soon as the interface includes a flat part, the value -1 (also known as the supercritical value) belongs to the critical interval.

Keywords and phrases. Maxwell's equations, metamaterial, sign-changing coefficients, optimal control.

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In this manuscript, we consider the numerical modeling of electromagnetic problems within a 2D geometry that involves structures combining both positive and negative materials. Such studies have already been carried out, both theoretically and numerically, see for instance [7, 25]. Compared to these references, the model that we study is more general, in the sense that the geometrical setting and the equations governing the fields are identical, but we allow for non-zero charge densities; in addition, we only assume uniqueness and existence of a solution (compared to well-posedness). In those references, the mathematical tool for the analysis is T-coercivity in [7], and weak T-coercivity in [25]: we propose instead to use a volume-based control method in the spirit of [19]. Regarding the numerical methods, none is proposed in [7], while the one in [25] relies on T-conform meshes, that is specifically designed meshes. In contrast, with our method, no constraint related to the sign-changing behavior of the coefficients needs to be imposed on the meshes for the approximation to converge [2, 19].

The volume-based control method for solving scalar problems with one or two sign-changing coefficients was introduced in [19]. We recall that for solving those problems, there exist other methods, either relying on the T-coercivity theory [4, 8, 11, 17, 24, 26, 32] and T-conform meshes [8, 11, 15, 24], or on *a posteriori* error estimators [18].

The outline is as follows. First, in Section 2, we enumerate the main mathematical ingredients. Then, in Section 3, we introduce the 2D time-harmonic electromagnetic model, with the vector-valued electric field as the main unknown. In Section 4, we propose a Helmholtz decomposition of the electric field, which allows to recast equivalently the model in three variational formulations, with scalar unknowns. Among the three variational formulations, two involve one or two sign-changing coefficients. To solve those two formulations (one with Dirichlet boundary condition, the other one with Neumann boundary condition), we choose the control-based method. One of the two (Dirichlet boundary condition) has already been dealt with in [2, 19]: in [2], a surface-based control was proposed, while a volume-based control is chosen in [19]. Finally, we discuss which assumption has to be imposed on the model (well-posedness, existence of a solution) to apply the control-based method. Next, we develop the theory with a volume-based control for the second variational formulation (Neumann boundary condition) in Sections 5 and 6 (see also [13]): this is the main part of this manuscript, which covers both the theoretical solution, and a numerical approximation, of the control-based method. We note that, since we are solving variational formulations with scalar unknowns, we can choose Lagrange finite elements for the approximation. Importantly, convergence of discrete solutions to the exact vector-valued electric field is proven. Then, in Section 7 we propose some numerical experiments, which we compare with direct solves on standard meshes (the plain method), or on T-conform meshes (the T-coercivity method). Since error estimates (*i.e. a priori* convergence rates) are not available for the control-based method, we provide some results regarding the approximation with T-conform meshes, and use them as a benchmark. Some concluding remarks are finally given.

2. MATHEMATICAL SETTING

Vector-valued function spaces are written in boldface characters. Unless otherwise specified, we consider spaces of real-valued functions. The index *zmv* indicates zero-mean-value fields. Duality brackets between a Banach space X and its topological dual X' are denoted by $\langle \cdot, \cdot \rangle_{X',X}$.

The symbol C is used to denote a generic positive constant which may depend only on the geometry, or on the coefficients defining the model. We use the notation $A \lesssim B$ for the inequality $A \leq CB$, where A and B are two scalar quantities, and C is a generic constant independent of A and B.

In $\mathbb{R}^d$, a domain is a non-empty, open, connected, and bounded subset with a Lipschitz-continuous boundary. We recall that, in $\mathbb{R}^2$, a domain is simply connected if, and only if, its boundary is connected.

Given a non-empty open subset O of $\mathbb{R}^2$ with a Lipschitz boundary ∂O, we denote by $\boldsymbol{\nu} = (\nu_x, \nu_y)^t$ the unit outward normal vector field to ∂O, and by $\boldsymbol{\tau} = (\tau_x, \tau_y)^t = (\nu_y, -\nu_x)^t$ the vector such that $(\boldsymbol{\tau}, \boldsymbol{\nu})$ is an orthonormal basis (see Fig. 1 for an example).

Let $(\cdot, \cdot)_{0,O}$ and $\| \cdot \|_{0,O}$ denote the inner product and norm, respectively, for both Hilbert spaces $\mathrm{L}^2(O)$ and $\mathbf{L}^2(O) := \mathrm{L}^2(O)^2$. More generally, $(\cdot, \cdot)_{s,O}$ and $\| \cdot \|_{s,O}$ (resp. $(| \cdot |_{s,O})$ refer to the inner product and norm (resp.

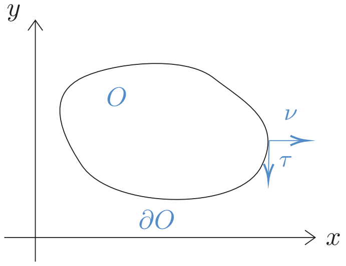

Figure 1. Example of an open subset O in 2D with notation.

semi-norm) of the Sobolev spaces $\mathrm{H}^s(O)$ and $\mathbf{H}^s(O) := \mathrm{H}^s(O)^2$ for $s \in \mathbb{R}$ (resp. for $s > 0$). Given Γ a Lipschitz submanifold of ∂O, we use the function space $\mathrm{H}^1_{0,\Gamma}(O) := \{u \in \mathrm{H}^1(O)$ such that $u_{|\Gamma} = 0\}$.

In a domain O, we say that $\xi \in L^2(O)$ is piecewise smooth if there exists a partition[1] $(O_p)_{p=1,P}$ of O such that $\xi_{|O_p} \in W^{1,\infty}(O_p)$ for $p = 1, P$. In this case, the partition is called compatible with respect to ξ. Finally, for $\mathtt{s} > 0$, we introduce

$$\mathrm{PH}^{\mathrm{s}}(O) := \{u \in \mathrm{L}^2(O) \text{ such that } u_{|O_p} \in \mathrm{H}^{\mathrm{s}}(O_p), \text{ for all } p = 1, P\}.$$

We recall next that there are two $2D$ curl operators:

$$u \mapsto \mathbf{curl}\, u = \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right)^t \text{ and } \mathbf{u} = (u_x, u_y)^t \mapsto \operatorname{curl} \mathbf{u} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y},$$

and that, given u, v and η sufficiently smooth in $\overline{O}$:

$$\begin{aligned} \mathbf{curl}\, u \cdot \mathbf{curl}\, v &= \nabla u \cdot \nabla v \text{ pointwise in } O\,; \\ \operatorname{curl}(\eta\, \mathbf{curl}\, u) &= -\operatorname{div}(\eta\, \nabla u) \text{ pointwise in } O\,; \\ \mathbf{curl}\, u \cdot \boldsymbol{\tau} &= \nabla u \cdot \boldsymbol{\nu} \text{ pointwise on } \partial O. \end{aligned}$$

We use classical functional spaces related to the 2D Maxwell's equations whose definitions are recalled below:

$$\begin{aligned} \mathbf{H}(\mathrm{curl}; O) &:= \{\mathbf{u} \in \mathbf{L}^2(O) \mid \operatorname{curl} \mathbf{u} \in \mathrm{L}^2(O)\}, \\ \mathbf{H}(\mathrm{div}; O) &:= \{\mathbf{u} \in \mathbf{L}^2(O) \mid \operatorname{div} \mathbf{u} \in \mathrm{L}^2(O)\}, \\ \mathbf{H}_0(\mathrm{curl}; O) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{curl}; O) \mid \mathbf{u} \cdot \boldsymbol{\tau} = 0 \text{ on } \partial O\}, \\ \mathbf{H}_0(\mathrm{div}; O) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{div}; O) \mid \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \partial O\}, \\ \mathbf{H}(\mathrm{div}\, 0; O) &:= \{\mathbf{u} \in \mathbf{H}(\mathrm{div}; O) \mid \operatorname{div} \mathbf{u} = 0 \text{ in } O\}. \end{aligned}$$

A priori, $\mathbf{H}(\mathrm{curl}; O)$ is endowed with the "natural" norm $\mathbf{u} \mapsto (\|\mathbf{u}\|^2_{0,O} + \|\operatorname{curl} \mathbf{u}\|^2_{0,O})^{1/2}$, etc. We refer to the monographs [3, 22, 28, 31] for details. Next, we recall a few useful results in 2D, namely integration by parts formulas and existence of scalar potentials (see *e.g.* [3, 22]).

Theorem 2.1. *Let O be a domain of $\mathbb{R}^2$. The following Green formulas hold:*

$$(\mathbf{v}, \nabla\varphi)_{0,O} + (\operatorname{div} \mathbf{v}, \varphi)_{0,O} = \langle \mathbf{v} \cdot \boldsymbol{\nu}, \varphi \rangle_{\mathrm{H}^{-1/2}(\partial O), \mathrm{H}^{1/2}(\partial O)}, \quad \forall (\mathbf{v}, \varphi) \in \mathbf{H}(\mathrm{div}; O) \times \mathrm{H}^1(O). \tag{1}$$

$$(\mathbf{v}, \mathbf{curl}\, \varphi)_{0,O} - (\operatorname{curl} \mathbf{v}, \varphi)_{0,O} = \langle \mathbf{v} \cdot \boldsymbol{\tau}, \varphi \rangle_{\mathrm{H}^{-1/2}(\partial O), \mathrm{H}^{1/2}(\partial O)}, \quad \forall (\mathbf{v}, \varphi) \in \mathbf{H}(\mathrm{curl}; O) \times \mathrm{H}^1(O). \tag{2}$$

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[1] A partition of O is $(O_p)_{p=1,P}$ such that O_p is a domain, for $p = 1, P$; $O_p \cap O_q = \emptyset$ for $p \neq q$; $\overline{O} = \cup_{p=1,P} \overline{O_p}$.

Theorem 2.2. *Let O be a domain of $\mathbb{R}^2$ with a connected boundary.*

Let $\mathbf{v} \in \mathbf{L}^2(O)$, then

$$\operatorname{div} \mathbf{v} = 0 \text{ in } O \iff \exists \psi \in \mathrm{H}^1(O) \text{ such that } \mathbf{v} = \mathbf{curl}\, \psi \text{ in } O. \tag{3}$$

$$\operatorname{curl} \mathbf{v} = 0 \text{ in } O,\ \ \mathbf{v}\cdot\boldsymbol{\tau} = 0 \text{ on } \partial O \iff \exists \varphi \in \mathrm{H}^1_0(O) \text{ such that } \mathbf{v} = \nabla\, \varphi \text{ in } O. \tag{4}$$

Moreover, the scalar potential ψ is unique up to a constant, and the scalar potential φ is unique.

3. Model

We are interested in solving the 2D Maxwell's equations, assuming that the magnetic permeability and the electric permittivity have a sign-change. Broadly speaking, we are considering the case where there is a limiting amplitude principle, namely that if the data is harmonic in time, then so is the solution for large times. In particular, the electromagnetic energy remains bounded (in bounded regions) of $\mathbb{R}^2$. For an extended discussion on the physical validity or relevance of this assumption, we refer to [12]. Note that the numerical solution of Maxwell's equations with sign changing coefficients using a classical discretization with Nédélec finite elements have difficulties to converge (see [16] for some numerical illustrations).

3.1. Equations

In this subsection, fields and function spaces are complex-valued, unless otherwise specified.

We study the time-harmonic Maxwell equations for a given pulsation $\omega > 0$, that is with known time dependence $\exp(-\imath\omega t)$. The model is set in a domain Ω of $\mathbb{R}^2$ with a connected boundary,[2] where the medium is isotropic and surrounded by a perfect conductor. We assume that the electromagnetic fields and the current density are measurable and square integrable: $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}, \mathbf{J} \in \mathbf{L}^2(\Omega)$, and that the magnetic permeability μ and the electric permittivity ε are real-valued, measurable and bounded as well as their inverse: $\mu, \mu^{-1}, \varepsilon, \varepsilon^{-1} \in \mathrm{L}^\infty(\Omega)$. Consequently, one can use $\varepsilon\mathbf{E}$ for $\mathbf{D}$, or vice versa $\varepsilon^{-1}\mathbf{D}$ for $\mathbf{E}$, respectively $\mu\mathbf{H}$ for $\mathbf{B}$, or vice versa $\mu^{-1}\mathbf{B}$ for $\mathbf{H}$. Using Ampère's and Faraday's laws[3] and the perfect conductor boundary conditions in terms of $(\mathbf{E}, H_z)$:

$$\begin{cases} \imath\omega\varepsilon\mathbf{E} + \mathbf{curl}\, H_z = \mathbf{J} & \text{in } \Omega,\\ -\imath\omega\mu H_z + \operatorname{curl} \mathbf{E} = 0 & \text{in } \Omega,\\ \mathbf{E}\cdot\boldsymbol{\tau} = 0 & \text{on } \partial\Omega, \end{cases}$$

we find that $\mathbf{E} \in \mathbf{H}_0(\operatorname{curl}; \Omega)$, and $\mathbf{curl}(\mu^{-1} \operatorname{curl} \mathbf{E}) \in \mathrm{L}^2(\Omega)$. Furthermore, $\mathbf{E}$ is governed by the vector second order PDE

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{H}_0(\operatorname{curl}; \Omega) \text{ such that:}\\ \mathbf{curl}(\mu^{-1} \operatorname{curl} \mathbf{E}) - \omega^2\, \varepsilon\, \mathbf{E} = \imath\, \omega\, \mathbf{J} \qquad \text{in } \Omega. \end{cases} \tag{5}$$

This can be recast equivalently as

$$\begin{cases} \text{Find } \mathbf{E} \in \mathbf{H}_0(\operatorname{curl}; \Omega) \text{ such that:}\\ (\mu^{-1} \operatorname{curl} \mathbf{E}, \operatorname{curl} \mathbf{v})_{0,\Omega} - \omega^2 (\varepsilon\, \mathbf{E}, \mathbf{v})_{0,\Omega} = \imath\, \omega\, (\mathbf{J}, \mathbf{v})_{0,\Omega}, \qquad \forall \mathbf{v} \in \mathbf{H}_0(\operatorname{curl}; \Omega). \end{cases} \tag{6}$$

Then, we observe that for any $\varphi' \in \mathrm{H}^1_0(\Omega)$, one can choose the test-field $\mathbf{v} = \nabla\varphi'$ in (6). So, it holds that

$$-\omega^2 (\varepsilon\, \mathbf{E}, \nabla\varphi')_{0,\Omega} = \imath\, \omega\, (\mathbf{J}, \nabla\varphi')_{0,\Omega}, \quad \forall \varphi' \in \mathrm{H}^1_0(\Omega), \tag{7}$$

i.e. $\operatorname{div} \mathbf{J} \in \mathrm{H}^{-1}(\Omega)$, with the relation $\operatorname{div}(\varepsilon\mathbf{E}) = -\imath\, \omega^{-1} \operatorname{div} \mathbf{J}$. If $\mathbf{J} \in \mathbf{H}(\operatorname{div}; \Omega)$, we find that $\operatorname{div}(\varepsilon\mathbf{E}) = -\imath\, \omega^{-1} \operatorname{div} \mathbf{J}$ a.e. in Ω. In the manuscript, we do not assume that $\operatorname{div} \mathbf{J} = 0$: physically, this means that the charge density may not vanish.

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[2] For the case of a boundary that is not connected, we refer to Appendix A.

[3] In the time-harmonic regime, Gauss' and magnetic Gauss' laws are consequences of Ampère's and Faraday's laws and the charge conservation equation.

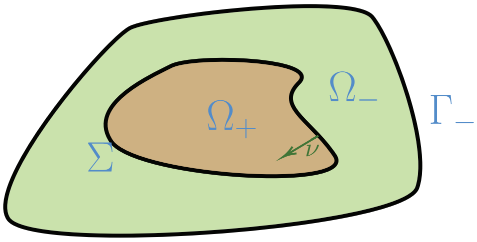

Figure 2. Example of a 2D geometry.

3.2. Real-valued fields and function spaces

Note that one can split the problem into two parts, where $\Re e(\mathbf{E})$ is related to $-\Im m(\mathbf{J})$, respectively $-\Im m(\mathbf{E})$ is related to $-\Re e(\mathbf{J})$. So, we carry on with $\mathbf{E}$ standing either for $\Re e(\mathbf{E})$ or $\Im m(\mathbf{E})$, respectively $\mathbf{f}$ standing for $-\omega^{-1}\,\Im m(\mathbf{J})$ or $\omega^{-1}\,\Re e(\mathbf{J})$, that is with real-valued fields. For this reason, we use real-valued function spaces. With these notation, one can check that problem (5) can be recast as:

$$\begin{cases}\text{Find } \mathbf{E}\in \mathbf{H}_0(\mathrm{curl};\Omega) \text{ such that:}\\ \mathbf{curl}(\mu^{-1}\,\mathrm{curl}\,\mathbf{E}) - \omega^2\,\varepsilon\,\mathbf{E} = \omega^2\,\mathbf{f} \qquad \text{in } \Omega.\end{cases} \tag{8}$$

One has $\mathbf{f}\in \mathbf{L}^2(\Omega)$ in (8) with the relation $(\varepsilon\,\mathbf{E},\nabla\varphi')_{0,\Omega} = -(\mathbf{f},\nabla\varphi')_{0,\Omega}$ for all $\varphi'\in \mathrm{H}^1_0(\Omega)$. It is convenient to define $g := \mathrm{div}\ \mathbf{f}\in \mathrm{H}^{-1}(\Omega)$ for later use.

4. Helmholtz decompositions

As before, Ω is a domain of $\mathbb{R}^2$ with a connected boundary.

4.1. Assumptions on well-posedness

Since we are dealing with 2D electromagnetic fields, we need assumptions on the companion scalar problems (see Sect. 4.2 for more details). Regarding the properties of the permeability μ and the permittivity ε, we assume in addition that they exhibit a sign-change within the domain. Precisely, we suppose that Ω can be divided into two disjoint subdomains Ω_+ and Ω_- ($\overline{\Omega} = \overline{\Omega_+}\cup\overline{\Omega_-}$, $\Omega_+\cap\Omega_- = \emptyset$), which are such that μ and ε are positive in Ω_+ and negative in Ω_-. In other words, the subscripts $'+'$ and $'-'$ respectively refer to the positive and negative subdomains. Given a field z in Ω, we let $z_+ = z_{|\Omega_+}$, respectively $z_- = z_{|\Omega_-}$. We introduce the interface Σ, defined as $\Sigma := \partial\Omega_+\cap\partial\Omega_-$ and subsequently, $\Gamma_\pm = \partial\Omega_\pm\setminus\Sigma$ and we assume that $\mathrm{mes}_{\partial\Omega}(\Gamma_-) > 0$, and we denote by $\boldsymbol{\nu}$ the unit normal vector field to Σ, pointing to Ω_+ (see Fig. 2 for an example that summarizes these notation).

Throughout the manuscript, we make some assumptions regarding the permittivity ε and the permeability μ. These assumptions are not optimal as far as the volume-based control method is concerned, however they are proposed to facilitate comparison with existing methods. A discussion on how these assumptions can be weakened is proposed in Section 4.4. We make a first assumption on the companion scalar problem with homogeneous Dirichlet boundary condition:

$$\begin{cases}\text{Find } \varphi\in \mathrm{H}^1_0(\Omega) \text{ such that:}\\ (\varepsilon\nabla\varphi,\nabla\varphi')_{0,\Omega} = \langle g,\varphi'\rangle_{\mathrm{H}^{-1}(\Omega),\mathrm{H}^1_0(\Omega)}, \qquad \forall\varphi'\in \mathrm{H}^1_0(\Omega).\end{cases} \tag{9}$$

Let us call $A_D\in \mathcal{L}(\mathrm{H}^1_0(\Omega),\mathrm{H}^{-1}(\Omega))$ the operator defined by

$$\langle A_D u, v\rangle_{\mathrm{H}^{-1}(\Omega),\mathrm{H}^1_0(\Omega)} = (\varepsilon\nabla u,\nabla v)_{0,\Omega},\quad \forall u,v\in \mathrm{H}^1_0(\Omega).$$

Assumption 1. The coefficient ε is such that the operator A_D is an isomorphism.

Assumption 1 amounts to saying that the companion scalar problem (9) is well-posed. For a characterization of classes of permittivities that lead to a well-posed companion scalar problem (9), we refer to [4,5,32].

We make a second assumption on the companion scalar problem with homogeneous Neumann boundary condition:

$$\begin{cases}\text{Find } \psi \in \mathrm{H}^1(\Omega) \text{ such that:}\\ (\varepsilon^{-1}\nabla\psi, \nabla\psi')_{0,\Omega} - \omega^2\,(\mu\,\psi, \psi')_{0,\Omega} = \langle g, \psi'\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)}, \qquad \forall\psi' \in \mathrm{H}^1(\Omega).\end{cases} \tag{10}$$

Let us call $A_N(\omega^2) \in \mathcal{L}(\mathrm{H}^1(\Omega), (\mathrm{H}^1(\Omega))')$ the operator defined by

$$\langle A_N(\omega^2)\,u, v\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)} = (\varepsilon^{-1}\nabla u, \nabla v)_{0,\Omega} - \omega^2\,(\mu\,u, v)_{0,\Omega}, \quad \forall u, v \in \mathrm{H}^1(\Omega).$$

Assumption 2. The coefficients ε, μ and the pulsation ω are such that the operator $A_N(\omega^2)$ is an isomorphism.

Assumption 2 amounts to saying that the companion scalar problem (10) is well-posed. We recall that, according to Theorem 5.1 in [7], Assumption 1 implies that $A_N(0)$ is a Fredholm operator, so that $A_N(\omega^2)$ is also a Fredholm operator. In other words, Assumption 2 means that the kernel of $A_N(\omega^2)$ is reduced to $\{0\}$. Note that, Assumptions 1 and 2 can be relaxed (see Sect. 4.4 for more details).

Remark 4.1. In [7], one proceeds by building an equivalent variational formulation to problem (8), and then by proving that the bilinear form is T-coercive. In [25], a similar approach is taken, except that it is proven that the form is weakly T-coercive. We proceed differently hereafter, using a decomposition of the electric field $\mathbf{E}$ governed by (8).

4.2. A splitting of the electric field

One has a first Helmholtz decomposition of the space $\mathbf{L}^2(\Omega)$.

Proposition 4.2. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathrm{H}^1_0(\Omega)] \oplus \mathbf{A}_0(\Omega, \varepsilon), \tag{11}$$

with $\mathbf{A}_0(\Omega,\varepsilon) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \forall\varphi \in \mathrm{H}^1_0(\Omega),\ (\varepsilon\mathbf{w}, \nabla\varphi)_{0,\Omega} = 0\} = \{\mathbf{w} \in \mathbf{L}^2(\Omega) \mid \operatorname{div}(\varepsilon\mathbf{w}) = 0 \textit{ in } \Omega\}$. *In addition, the linear map* $\mathbf{v} \mapsto (\varphi, \mathbf{w})$ *with* $\mathbf{v} = \nabla\varphi + \mathbf{w}$ *is continuous from* $\mathbf{L}^2(\Omega)$ *to* $\mathrm{H}^1_0(\Omega) \times \mathbf{A}_0(\Omega, \varepsilon)$.

Proof. It is clear that $\nabla[\mathrm{H}^1_0(\Omega)] + \mathbf{A}_0(\Omega,\varepsilon) \subseteq \mathbf{L}^2(\Omega)$. Let us now show the converse imbedding. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. Since problem (9) is well-posed (*cf.* Assumption 1), there exists $\varphi \in \mathrm{H}^1_0(\Omega)$ such that:

$$(\varepsilon\nabla\varphi, \nabla\varphi')_{0,\Omega} = (\varepsilon\mathbf{v}, \nabla\varphi')_{0,\Omega}, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

Then, we set $\mathbf{w} = \mathbf{v} - \nabla\varphi$. By construction, $\mathbf{w} \in \mathbf{A}_0(\Omega,\varepsilon)$. It follows that $\mathbf{L}^2(\Omega) = \nabla[\mathrm{H}^1_0(\Omega)] + \mathbf{A}_0(\Omega,\varepsilon)$.

Next, let $z \in \nabla[\mathrm{H}^1_0(\Omega)] \cap \mathbf{A}_0(\Omega,\varepsilon)$ be given. There exists $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $z = \nabla\varphi$ and, by definition of $\mathbf{A}_0(\Omega,\varepsilon)$, φ is governed by (9) with zero right-hand side. By uniqueness of the solution, one has $\varphi = 0$ and so $z = 0$: the sum is direct.

Continuity of the linear map is a straightforward consequence of the well-posedness of the companion scalar problem (9). ☐

Remark 4.3. Because ε has a sign-change, the decomposition is not orthogonal, since $(\mathbf{u}, \mathbf{v}) \mapsto (\varepsilon\mathbf{u}, \mathbf{v})_{0,\Omega}$ does not define an inner product in $\mathbf{L}^2(\Omega)$.

According to Theorem 2.2, one has $\mathbf{A}_0(\Omega,\varepsilon) = \varepsilon^{-1}\,\mathbf{curl}[\mathrm{H}^1_{zmv}(\Omega)]$, because $\mathbf{v} \in \mathbf{A}_0(\Omega,\varepsilon)$ is equivalent to $\varepsilon\mathbf{v} \in \mathbf{H}(\operatorname{div} 0;\Omega)$. One can then write a second Helmholtz decomposition of the space $\mathbf{L}^2(\Omega)$.

Proposition 4.4. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathrm{H}^1_0(\Omega)] \oplus \varepsilon^{-1}\,\mathbf{curl}[\mathrm{H}^1_{zmv}(\Omega)]. \tag{12}$$

In addition, the linear map $\mathbf{v} \mapsto (\varphi, \psi)$ *with* $\mathbf{v} = \nabla\varphi + \varepsilon^{-1}\,\mathbf{curl}\,\psi$ *is continuous from* $\mathbf{L}^2(\Omega)$ *to* $\mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_{zmv}(\Omega)$.

Note that, for all $\varphi \in \mathrm{H}^1_0(\Omega)$ and $\psi \in \mathrm{H}^1_{zmv}(\Omega)$, using Green formula (2) we find that since $\nabla\varphi \cdot \boldsymbol{\tau} = 0$ on $\partial\Omega$:

$$(\varepsilon\nabla\varphi, \varepsilon^{-1}\,\mathbf{curl}\,\psi)_{0,\Omega} = (\nabla\varphi, \mathbf{curl}\,\psi)_{0,\Omega} = 0. \tag{13}$$

As a by-product, one can decompose the electric field $\mathbf{E}$ that is governed by (8):

$$\exists!(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}}) \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_{zmv}(\Omega) \text{ such that } \mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}} \text{ in } \Omega. \tag{14}$$

Our next step is the characterization of the two scalar potentials $(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}})$.

Proposition 4.5. *Let* $\mathbf{E}$*, a solution of* (8)*, be decomposed as in* (14)*. Then its scalar potential* $\varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega)$ *is governed by the Dirichlet problem:*

$$\operatorname{div}(\varepsilon\nabla\varphi_{\mathbf{E}}) = -g \quad \text{in } \Omega. \tag{15}$$

The equivalent variational formulation writes:

$$\begin{cases} \text{Find } \varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega) \text{ such that :} \\ (\varepsilon\nabla\varphi_{\mathbf{E}}, \nabla\varphi')_{0,\Omega} = \langle g, \varphi' \rangle_{\mathrm{H}^{-1}(\Omega), \mathrm{H}^1_0(\Omega)}, \qquad \forall \varphi' \in \mathrm{H}^1_0(\Omega). \end{cases} \tag{16}$$

Proof. Let $\varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega)$ in the decomposition (14) of $\mathbf{E}$. According to the problem (8) and the definition of g, we find that $\varphi_{\mathbf{E}}$ satisfies (15).

The fact that (15) is equivalent to (16) is classical. □

Proposition 4.6. *Let* $\mathbf{E}$*, a solution of* (8)*, be decomposed as in* (14)*. Then there exists a constant* $c \in \mathbb{R}$ *such that the shifted scalar potential* $\psi_{c,\mathbf{E}} := (\psi_{0,\mathbf{E}} + c) \in \mathrm{H}^1(\Omega)$ *is governed by the Neumann problem*

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}) - \omega^2\mu\,\psi_{c,\mathbf{E}} = \omega^2\mu\,s_0 & \text{in } \Omega, \\ \varepsilon^{-1}\nabla\psi_{c,\mathbf{E}} \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where the right-hand side is characterized by

$$s_0 \in \mathrm{H}^1_{zmv}(\Omega) \text{ such that } \mathbf{curl}\,s_0 = \mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}} \text{ in } \Omega. \tag{18}$$

The equivalent variational formulation writes:

$$\begin{cases} \text{Find } \psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega) \text{ such that :} \\ (\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}, \nabla\psi')_{0,\Omega} - \omega^2\,(\mu\,\psi_{c,\mathbf{E}}, \psi')_{0,\Omega} = \omega^2\,(\mu\,s_0, \psi')_{0,\Omega}, \qquad \forall \psi' \in \mathrm{H}^1(\Omega). \end{cases} \tag{19}$$

Proof. Let $\varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega)$ and $\psi_{0,\mathbf{E}} \in \mathrm{H}^1_{zmv}(\Omega)$ be the two scalar potentials of a solution $\mathbf{E}$ to (8), see (14). Since $\operatorname{div}(\mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}) = 0$ in Ω, we observe that the existence and uniqueness of $s_0 \in \mathrm{H}^1_{zmv}(\Omega)$ fulfilling (18) is guaranteed by Theorem 2.2.

On the other hand, using problem (8), we find:

$$\begin{aligned} \mathbf{curl}(\mu^{-1}\operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}})) - \omega^2\,\mathbf{curl}\,\psi_{0,\mathbf{E}} &= \mathbf{curl}(\mu^{-1}\operatorname{curl}\mathbf{E}) - \omega^2\varepsilon\,\mathbf{E} + \omega^2\,\varepsilon\,\nabla\varphi_{\mathbf{E}} \\ &= \omega^2(\mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}) \text{ in } \Omega. \end{aligned}$$

Hence, we obtain:

$$\mathbf{curl}(\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}})) - \omega^2\,\mathbf{curl}\,\psi_{0,\mathbf{E}} = \omega^2\,\mathbf{curl}\,s_0 \text{ in } \Omega.$$

Consequently, we derive:

$$\mathbf{curl}\big(\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}}) - \omega^2\,\psi_{0,\mathbf{E}} - \omega^2\,s_0\big) = 0 \text{ in } \Omega.$$

Because Ω is connected, the scalar field $\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}}) - \omega^2\,\psi_{0,\mathbf{E}} - \omega^2\,s_0$ is equal to some constant, *i.e.* there exists a (unique) $c \in \mathbb{R}$ such that:

$$\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}}) - \omega^2\,\psi_{0,\mathbf{E}} - \omega^2\,s_0 = \omega^2\,c \text{ in } \Omega.$$

We deduce that:

$$\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}(\psi_{0,\mathbf{E}} + c)) - \omega^2\,(\psi_{0,\mathbf{E}} + c) = \omega^2\,s_0 \text{ in } \Omega,$$

so we have:

$$\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}(\psi_{0,\mathbf{E}} + c)) - \omega^2\,\mu\,(\psi_{0,\mathbf{E}} + c) = \omega^2\,\mu\,s_0 \text{ in } \Omega.$$

Let $\psi_{c,\mathbf{E}} := (\psi_{0,\mathbf{E}} + c) \in \mathrm{H}^1(\Omega)$. We note that $\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}) = -\,\mathrm{div}(\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}})$ in Ω, hence

$$-\,\mathrm{div}(\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}) - \omega^2\,\mu\,\psi_{c,\mathbf{E}} = \omega^2\,\mu\,s_0 \text{ in } \Omega.$$

According to (14), one has

$$\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}(\psi_{0,\mathbf{E}} + c) = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}} \text{ in } \Omega,$$

and because $\mathbf{E}$ is a solution to the problem (8), we infer the boundary condition:

$$\varepsilon^{-1}\nabla\,\psi_{c,\mathbf{E}}\cdot\boldsymbol{\nu} = \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}\cdot\boldsymbol{\tau} = \mathbf{E}\cdot\boldsymbol{\tau} = 0 \text{ on } \partial\Omega.$$

We conclude that the equations governing $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ are indeed (17). In particular $\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}$ belongs to $\mathbf{H}(\mathrm{div};\Omega)$ with vanishing normal trace. Then, for $\psi' \in \mathrm{H}^1(\Omega)$, we find by integration by parts (1) that

$$(\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}, \nabla\,\psi')_{0,\Omega} - \omega^2\,(\mu\,\psi_{c,\mathbf{E}}, \psi')_{0,\Omega} = \omega^2\,(\mu\,s_0, \psi')_{0,\Omega}, \forall\psi' \in \mathrm{H}^1(\Omega).$$

In other words, $\psi_{c,\mathbf{E}}$ is governed by the variational formulation (19).

The fact that (19) implies (17) is classical: in particular, one uses the fact that the trace mapping is surjective from $\mathrm{H}^1(\Omega)$ to $\mathrm{H}^{1/2}(\partial\Omega)$ to derive the boundary condition. □

How to characterize the auxiliary potential s_0 variationally? We note that if s_0 is given by (18), then obviously it solves

$$\begin{cases} \text{Find } s_0 \in \mathrm{H}^1_{zmv}(\Omega) \text{ such that:} \\ (\mathbf{curl}\,s_0, \mathbf{curl}\,\psi')_{0,\Omega} = (\mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}, \mathbf{curl}\,\psi')_{0,\Omega}, \qquad \forall\psi' \in \mathrm{H}^1_{zmv}(\Omega), \end{cases} \tag{20}$$

which is a well-posed variational formulation.

Lemma 4.7. *The auxiliary potential s_0 solves (18) if, and only if, it solves (20).*

Proof. We just proved that if s_0 solves (18), then it solves (20).

Reciprocally, if s_0 solves (20) which holds for all $\psi' \in \mathrm{H}^1(\Omega)$, then choosing $\psi' \in \mathcal{D}(\Omega)$, we find first that $\mathrm{curl}(\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}}) = 0$ in $\mathcal{D}'(\Omega)$. Next, using $\psi' \in \mathrm{H}^1(\Omega)$, we find after integration by parts (2) that:

$$\langle(\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}})\cdot\boldsymbol{\tau}, \psi'_{|\partial\Omega}\rangle_{\mathrm{H}^{-1/2}(\partial\Omega),\mathrm{H}^{1/2}(\partial\Omega)} = 0.$$

Since the trace mapping is surjective from $\mathrm{H}^1(\Omega)$ to $H^{1/2}(\partial\Omega)$, we conclude that $(\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}})\cdot\boldsymbol{\tau} = 0$ on $\partial\Omega$. According to (4), there exists $z \in \mathrm{H}^1_0(\Omega)$ such that $\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}} = \nabla z$ in Ω. Also,

$$\Delta z = \mathrm{div}(\mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}}) = 0 \text{ in } \Omega,$$

and we conclude that $z = 0$ so that $\mathbf{curl}\,s_0 = \mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}$, *i.e.* s_0 is characterized by (18). □

As we observed in the proof of Proposition 4.6, the presence of the constant c does not affect the resolution of the problem (8). In fact, one has

$$\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}} \text{ in } \Omega.$$

We conclude that to solve the problem (8), one should be looking for the two scalar potentials $\varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega)$ and $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ respectively governed by (15) and (17), or their equivalent variational formulations (16) and (19). We summarize the results in the next theorem.

Theorem 4.8. *Let* $\mathbf{E} \in \mathbf{L}^2(\Omega)$ *be governed by* (8)*. Then there exist two scalar potentials* $\varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega)$ *and* $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ *such that* $\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}$ *in* Ω*, where* $\varphi_{\mathbf{E}}$ *is a solution to* (15)*, resp.* $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ *is a solution to* (17)*.*

Conversely, let $\varphi_{\mathbf{E}} \in \mathrm{H}^1_0(\Omega)$ *be governed by* (15)*, the auxiliary field* s_0 *be characterized by* (18)*, and* $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ *be governed by* (17)*. Then* $\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}} \in \mathbf{L}^2(\Omega)$ *is a solution to* (8)*. Finally, the problem* (8) *is well-posed.*

Proof. The first part of the theorem has already been established, see Propositions 4.5 and 4.6.

Conversely, let $\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}$, where $\varphi_{\mathbf{E}}$, s_0, $\psi_{c,\mathbf{E}}$ are defined as in the statement. One finds that

$$\operatorname{curl}\mathbf{E} = \operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}) = -\operatorname{div}(\varepsilon^{-1}\nabla\psi_{c,\mathbf{E}}) \overset{(17)}{=} \omega^2\mu\,\psi_{c,\mathbf{E}} + \omega^2\mu\, s_0 \text{ in } \Omega.$$

In particular, $\mathbf{E} \in \mathbf{H}(\operatorname{curl};\Omega)$. Furthermore,

$$\mathbf{E}\cdot\boldsymbol{\tau} = \nabla\varphi_{\mathbf{E}}\cdot\boldsymbol{\tau} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}\cdot\boldsymbol{\tau} = \nabla\varphi_{\mathbf{E}}\cdot\boldsymbol{\tau} + \varepsilon^{-1}\nabla\,\psi_{c,\mathbf{E}}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega,$$

that is $\mathbf{E} \in \mathbf{H}_0(\operatorname{curl};\Omega)$. Finally,

$$\begin{aligned}
\mathbf{curl}(\mu^{-1}\operatorname{curl}\mathbf{E}) &= \omega^2\,\mathbf{curl}\,\psi_{c,\mathbf{E}} + \omega^2\,\mathbf{curl}\,s_0\\
&\overset{(18)}{=} \omega^2\,\mathbf{curl}\,\psi_{c,\mathbf{E}} + \omega^2\mathbf{f} + \omega^2\varepsilon\,\nabla\varphi_{\mathbf{E}}\\
&= \omega^2\varepsilon\,\mathbf{E} + \omega^2\mathbf{f} \text{ in } \Omega.
\end{aligned}$$

So we conclude that $\mathbf{E}$ is a solution to problem (8). Finally, we note that well-posedness of problem (8) is a direct consequence of Assumptions 1 and 2. ☐

4.3. Assumptions on regularity

At some point, it is worthwhile to have some *a priori* knowledge of the smoothness of the quantities of interest (data, solution, auxiliary unknowns...). Also, when one is using finite element discretizations, error estimates, if any, are expected to be driven by the *a priori* piecewise regularity of the solution. So, we introduce the limit regularity exponents for the problems (9) and (10). We refer to Section 2 in [16] and the references therein for more details.

We assume that a first shift theorem holds for problem (9). Namely, there exists $\tau_D(\varepsilon) \in]0,1]$ depending only on the geometry and on ε such that

$$\begin{aligned}
&\forall \mathsf{s} \in [0,\tau_D(\varepsilon)[\setminus\{1/2\},\ \forall g \in \mathrm{H}^{-1+\mathsf{s}}(\Omega),\ \ \varphi \in \mathrm{PH}^{1+\mathsf{s}}(\Omega), \text{ and } \|\varphi\|_{\mathrm{PH}^{1+\mathsf{s}}(\Omega)} \lesssim \|g\|_{-1+\mathsf{s},\Omega}\,;\\
&\text{if } \tau_D(\varepsilon) = 1,\ \varphi \in \mathrm{PH}^2(\Omega), \text{ and } \|\varphi\|_{\mathrm{PH}^2(\Omega)} \lesssim \|g\|_{0,\Omega}.
\end{aligned}$$

We also assume that a second shift theorem holds for problem (10) with $\omega^2 = 0$, regarding the *a priori* regularity of the solutions with $\mathrm{L}^2(\Omega)$ data g. Namely, there exists $\tau_N(\varepsilon^{-1}) \in]0,1]$ depending only on the geometry and on ε^{-1} such that

$$\begin{aligned}
&\forall \mathsf{s} \in [0,\tau_N(\varepsilon^{-1})[\setminus\{1/2\},\ \forall g \in \mathrm{L}^2(\Omega),\ \ \psi \in \mathrm{PH}^{1+\mathsf{s}}(\Omega), \text{ and } \|\psi\|_{\mathrm{PH}^{1+\mathsf{s}}(\Omega)} \lesssim \|g\|_{0,\Omega}\,;\\
&\text{if } \tau_N(\varepsilon^{-1}) = 1,\ \psi \in \mathrm{PH}^2(\Omega), \text{ and } \|\psi\|_{\mathrm{PH}^2(\Omega)} \lesssim \|g\|_{0,\Omega}.
\end{aligned}$$

Above, the constant hidden in $\lesssim$ may depend on $\mathbf{s}$, but not on g. In the analysis, we keep the largest value of $\tau_D(\varepsilon)$ and of $\tau_N(\varepsilon^{-1})$, which is called the limit regularity exponent.

One can define similarly limit regularity exponents for problems set in Ω or in subdomains of Ω, with either Dirichlet, Neumann or mixed boundary conditions, and possibly with other coefficients.

4.4. Relaxing the assumptions

In this section, we show that the Assumptions 1 and 2 on the well-posedness of the scalar problems can be weakened when one uses the volume-based control method. Specifically, the theory remains valid as long as we only require:

- [injectivity] *uniqueness* of solutions for problems (8) and (15), which means that, for a zero r.h.s., the only solution is the trivial one;
- *existence* of the solutions for problems (8) and (15), *with the given data* $\mathbf{f}$.

Let us work under these assumptions.

Proposition 4.9. *For the given data* $\mathbf{f}$*, if problem* (8) *with r.h.s.* $\omega^2\mathbf{f}$ *admits a solution* $\mathbf{E}$*, respectively problem* (15) *with r.h.s.* $-\operatorname{div}\mathbf{f}$ *admits a solution* $\varphi_{\mathbf{E}}$*, then problem* (17) *with r.h.s.* $\omega^2\mu\, s_0$*, where* s_0 *is characterized by* (18)*, has a solution* $\psi_{c,\mathbf{E}}$*.*

Proof. By construction: $\operatorname{div}(\varepsilon(\mathbf{E}-\nabla\varphi_{\mathbf{E}}))=0$. According to Theorem 2.2, there exists $\psi_{0,\mathbf{E}}\in \mathrm{H}^1_{zmv}(\Omega)$ such that $\varepsilon(\mathbf{E}-\nabla\varphi_{\mathbf{E}})=\mathbf{curl}\,\psi_{0,\mathbf{E}}$. This implies $\mathbf{E}=\nabla\varphi_{\mathbf{E}}+\varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}}$. Following the proof of Proposition 4.6 and defining the constant c as before, one then checks that $\psi_{c,\mathbf{E}}=\psi_{0,\mathbf{E}}+c$ solves problem (17), with the data s_0 given by (18). $\square$

One can demonstrate the equivalence between the injectivity of problem (8) and the injectivity of problem (17).

Proposition 4.10. *Problem* (8) *is injective if and only if problem* (17) *is injective.*

Proof. Suppose that problem (17) is injective. We aim to show that problem (8) is injective: Let $\mathbf{E}\in\mathbf{H}_0(\mathbf{curl};\Omega)$ be such that:

$$\mathbf{curl}(\mu^{-1}\operatorname{curl}\mathbf{E})-\omega^2\,\varepsilon\,\mathbf{E}=0 \text{ in } \Omega.$$

Taking the divergence of both sides, we obtain: $\operatorname{div}(\varepsilon\,\mathbf{E})=0$ in Ω. By Theorem 2.2, $\exists\psi\in\mathrm{H}^1(\Omega)$ such that $\mathbf{E}=\varepsilon^{-1}\,\mathbf{curl}\,\psi$. First, we have $\varepsilon^{-1}\,\mathbf{curl}\,\psi\cdot\boldsymbol{\tau}=0$ on $\partial\Omega$. Then, it holds that $\mathbf{curl}(\mu^{-1}\operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi))-\omega^2\,\mathbf{curl}\,\psi=0$ in Ω, *i.e.* $\mathbf{curl}(\mu^{-1}\operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi)-\omega^2\,\psi)=0$ in Ω. Then, there exists a unique constant $c\in\mathbb{R}$ such that: $\mu^{-1}\operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi)-\omega^2\,\psi=\omega^2\,c$ in Ω. Hence, $\mu^{-1}\operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}(\psi+c))-\omega^2\,(\psi+c)=0$ in Ω. Noting that $\operatorname{curl}(\varepsilon^{-1}\,\mathbf{curl}(\psi+c))=-\operatorname{div}(\varepsilon^{-1}\nabla(\psi+c))$ in Ω, then $\psi+c\in\mathrm{H}^1(\Omega)$ is governed by:

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1}\nabla(\psi+c))-\omega^2\,\mu\,(\psi+c)=0 & \text{in } \Omega,\\ \varepsilon^{-1}\nabla(\psi+c)\cdot\boldsymbol{\nu}=0 & \text{on } \partial\Omega.\end{cases}$$

Since problem (17) is injective, it follows that $(\psi+c)=0$ in Ω. Thus, $\mathbf{E}=\varepsilon^{-1}\,\mathbf{curl}\,\psi=0$.

Conversely, assume that problem (8) is injective. Our goal is to prove the injectivity of problem (17). Let $\psi\in\mathrm{H}^1(\Omega)$ satisfy the following problem:

$$\begin{cases} -\operatorname{div}(\varepsilon^{-1}\nabla\psi)-\omega^2\,\mu\,\psi=0 & \text{in } \Omega,\\ \varepsilon^{-1}\nabla\psi\cdot\boldsymbol{\nu}=0 & \text{on } \partial\Omega.\end{cases}$$

The equation satisfied by ψ in Ω can be written as: $\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi) = \omega^2\,\psi$. By applying the vector rotational to both sides, we obtain: $\mathbf{curl}(\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi)) = \omega^2\,\mathbf{curl}\,\psi$ in Ω. Then, $\varepsilon^{-1}\,\mathbf{curl}\,\psi \in \mathbf{H}(\mathbf{curl};\Omega)$ is governed by:

$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathrm{curl}(\varepsilon^{-1}\,\mathbf{curl}\,\psi)) - \omega^2\varepsilon(\varepsilon^{-1}\,\mathbf{curl}\,\psi) = 0 & \text{in } \Omega, \\ \varepsilon^{-1}\,\mathbf{curl}\,\psi\cdot\boldsymbol{\tau} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since problem (8) is injective, it follows that $\varepsilon^{-1}\,\mathbf{curl}\,\psi = 0$ in Ω. Then, there exists a constant $c' \in \mathbb{R}$ such that: $\psi = c'$ in Ω. Furthermore, $-\omega^2\mu\,c' = 0$ in Ω, it follows that $\psi = 0$ in Ω. □

The proof of Proposition 4.10 is independent of the assumptions on problem (15) with solution $\varphi_{\mathbf{E}}$. Hence, one may replace the weakened assumptions on problem (8) by similar ones on problem (17), where s_0 is characterized by (18). Namely, *in addition to* the injectivity of problem (15) and the existence of a solution with r.h.s. $-\operatorname{div}\mathbf{f}$, that:

- [injectivity] *uniqueness* of solution for problem (17);
- *existence* of the solution for problem (17) *with r.h.s.* $\omega^2\mu\,s_0$.

Finally, in the case where only the weakened assumptions hold, one assumes extra regularity of the solutions $\varphi_{\mathbf{E}}$ and $\psi_{c,\mathbf{E}}$. For simplicity, the extra regularity exponents are still denoted by $\tau_D(\varepsilon)$, resp. $\tau_N(\varepsilon^{-1})$.

Remark 4.11. We emphasize that with these weakened assumptions, one can handle configurations that cannot be treated with T-conform meshes (the T-coercivity method). For instance, in the case of a $2D$ non-symmetric cavity with a flat interface and piecewise constant ε, as presented in Section 3.3 of [15], it has been shown that for a super-critical contrast $\varepsilon_-/\varepsilon_+ = -1$ and $\omega = 0$, the operators associated with the scalar problems are injective but not Fredholm. As a result of the loss of the Fredholm property, the T-coercivity method cannot be applied, and numerical resolution based on T-conform meshes is also not feasible. We refer to Section 7.4 for a numerical illustration. Note that two numerical strategies have recently been tested for this configuration with $\varepsilon_-/\varepsilon_+ = -1$ (see [1,10]). However, both are limited by constraints, as they require strong extra-regularity assumptions of the solution and the use of structured meshes.

5. A VOLUME OPTIMAL CONTROL METHOD FOR SOLVING SCALAR PROBLEMS

In what follows, we will adopt an *optimal control method* (introduced originally in [21]) to solve numerically the variational formulations (16) and (19), resp. corresponding to the Dirichlet problem, and the Neumann problem.

As demonstrated in Section 4.2, these two scalar problems are coupled. As a matter of fact, to solve the variational formulation (19), we need to solve first the variational formulation (16), then to calculate the source term s_0, as characterized by (18). For simplicity, we will use the notations φ and ψ in place of $\varphi_{\mathbf{E}}$ and $\psi_{c,\mathbf{E}}$, respectively. The method for solving the Dirichlet problem with $\omega^2 = 0$ has been thoroughly investigated in [19]. In the subsequent sections, we will focus on the method for solving the Neumann problem with $\omega^2 \neq 0$.

5.1. A first reformulation of the Neumann problem

We propose below a reformulation of problem (17) with a right-hand side $f^N = \omega^2\mu\,s_0$, in which the unknown $\psi \in \mathrm{H}^1(\Omega)$ is split into the two unknowns ψ_+ and ψ_-. Given that $f^N \in \mathrm{L}^2(\Omega)$, the solution ψ to (17) is such that $\varepsilon^{-1}\nabla\psi$ belongs to $\mathbf{H}(\mathrm{div};\Omega)$. Consequently, finding ψ as a solution to problem (17) is equivalent to finding $(\psi_+,\psi_-) \in \mathrm{H}^1(\Omega_+)\times\mathrm{H}^1(\Omega_-)$, which is governed by

$$\begin{cases} -\operatorname{div}(\varepsilon_+^{-1}\nabla\psi_+) - \omega^2\mu_+\psi_+ = f_+^N & \text{in } \Omega_+, \\ -\operatorname{div}(\varepsilon_-^{-1}\nabla\psi_-) - \omega^2\mu_-\psi_- = f_-^N & \text{in } \Omega_-, \\ \psi_+ = \psi_- \text{ and } \varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+ = \varepsilon_-^{-1}\partial_{\boldsymbol{\nu}}\psi_- & \text{on } \Sigma, \\ \varepsilon_+^{-1}\nabla\psi_+\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega_+\setminus\Sigma, \quad \varepsilon_-^{-1}\nabla\psi_-\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\Omega_-\setminus\Sigma. \end{cases} \tag{21}$$

The equations governing ψ_+ and ψ_- are elliptic. However, we cannot solve them independently, because they are coupled through the transmission conditions on Σ. To overcome this difficulty, we will apply the same strategy as in [19], by introducing an optimization-based formulation with volume control, which will be explained in the following section.

5.2. An optimal control reformulation of the Neumann problem

Instead of looking for $(\psi_+, \psi_-) \in \mathrm{H}^1(\Omega_+) \times \mathrm{H}^1(\Omega_-)$ as a solution to (21), the approach below focuses on finding a pair of functions $(\tilde{\psi}, \psi_-) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-)$, where $\tilde{\psi}$ is a continuous extension of the unknown ψ_+ to the entire domain Ω. This extension will be characterized by solving an optimal control problem, detailed below.

Remark 5.1. It is worth noting that one could choose an extension $\tilde{\psi}$ from Ω_- to Ω so that $(\psi_+, \tilde{\psi}_{|\Omega_-})$ is a solution to (21). Or, one could extend the function ψ_+ to a strict sub-domain of Ω_- only [13].

To proceed, we denote by $\tilde{\varepsilon}$ and $\tilde{\mu}$ extensions of the functions ε_+ and μ_+ to Ω. Concerning their values in Ω_-, we choose them as follows:

$$\tilde{\varepsilon}, \tilde{\mu} \in \mathrm{L}^\infty(\Omega), \text{ and } \exists\, c_1 > 0,\ \exists\, c_2 < 0 \text{ such that } \tilde{\varepsilon} > c_1 > 0 \text{ and } \tilde{\mu} < c_2 < 0 \text{ a.e. in } \Omega_-. \tag{22}$$

With this choice (22), we can define the following inner product on the space $\mathrm{H}^1(\Omega_-)$:

$$\begin{aligned} (\cdot,\cdot)_{\tilde{\varepsilon},\tilde{\mu}} : \mathrm{H}^1(\Omega_-) \times \mathrm{H}^1(\Omega_-) &\to \mathbb{R} \\ (u, v) &\mapsto (u,v)_{\tilde{\varepsilon},\tilde{\mu}} := (|\tilde{\varepsilon}^{-1}|\, \nabla u, \nabla v)_{0,\Omega_-} + \omega^2 (|\tilde{\mu}|\, u, v)_{0,\Omega_-}. \end{aligned}$$

The corresponding norm is denoted by $\|\cdot\|_{\tilde{\varepsilon},\tilde{\mu}}$.

We then define the bilinear forms:

$$\begin{aligned} &\tilde{a} : \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbb{R}, \qquad \tilde{a}(u,v) := (\tilde{\varepsilon}^{-1}\, \nabla u, \nabla v)_{0,\Omega} - \omega^2 (\tilde{\mu}\, u, v)_{0,\Omega},\ \forall u, v \in \mathrm{H}^1(\Omega), \\ &a_- : \mathrm{H}^1(\Omega_-) \times \mathrm{H}^1(\Omega_-) \to \mathbb{R},\ a_-(u,v) := (\varepsilon_-^{-1} \nabla u, \nabla v)_{0,\Omega_-} - \omega^2 (\mu_- u, v)_{0,\Omega_-},\ \forall u, v \in \mathrm{H}^1(\Omega_-). \end{aligned}$$

Now, let $E \in \mathcal{L}(\mathrm{H}^1(\Omega_+), \mathrm{H}^1(\Omega))$ be an arbitrary continuous extension operator. We have the

Lemma 5.2. *It holds that*

$$\begin{cases} \tilde{a}(E(\psi_+), v) = (f_+^N, v)_{0,\Omega_+} + (E(\psi_+), v)_{\tilde{\varepsilon},\tilde{\mu}} - \langle \varepsilon_+^{-1} \partial_{\boldsymbol{\nu}} \psi_+, v \rangle_{\mathrm{H}^{-1/2}(\Sigma), \mathrm{H}^{1/2}(\Sigma)}, & \forall v \in \mathrm{H}^1(\Omega), \\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0,\Omega_-} + \langle \varepsilon_+^{-1} \partial_{\boldsymbol{\nu}} \psi_+, v_- \rangle_{\mathrm{H}^{-1/2}(\Sigma), \mathrm{H}^{1/2}(\Sigma)}, & \forall v_- \in \mathrm{H}^1(\Omega_-). \end{cases} \tag{23}$$

Proof. $\forall v \in \mathrm{H}^1(\Omega)$, we have by integration by parts

$$\begin{aligned} \tilde{a}(E(\psi_+), v) &= (\tilde{\varepsilon}^{-1}\, \nabla E(\psi_+), \nabla v)_{0,\Omega} - \omega^2 (\tilde{\mu}\, E(\psi_+), v)_{0,\Omega} \\ &= (\varepsilon_+^{-1}\, \nabla \psi_+, \nabla v)_{0,\Omega_+} - \omega^2 (\mu_+\, \psi_+, v)_{0,\Omega_+} \\ &\quad + (\tilde{\varepsilon}^{-1}\, \nabla E(\psi_+), \nabla v)_{0,\Omega_-} - \omega^2 (\tilde{\mu}\, E(\psi_+), v)_{0,\Omega_-} \\ &= -(\mathrm{div}(\varepsilon_+^{-1}\, \nabla \psi_+), v)_{0,\Omega_+} - \langle \varepsilon_+^{-1}\, \partial_{\boldsymbol{\nu}} \psi_+, v \rangle_{\mathrm{H}^{-1/2}(\Sigma), \mathrm{H}^{1/2}(\Sigma)} \\ &\quad - \omega^2 (\mu_+\, \psi_+, v)_{0,\Omega_+} + (\tilde{\varepsilon}^{-1} \nabla E(\psi_+), \nabla \overline{v})_{0,\Omega_-} - \omega^2 (\tilde{\mu}\, E(\psi_+), \overline{v})_{0,\Omega_-} \\ &= (f_+^N, v)_{0,\Omega_+} - \langle \varepsilon_+^{-1}\, \partial_{\boldsymbol{\nu}} \psi_+, v \rangle_{\mathrm{H}^{-1/2}(\Sigma), \mathrm{H}^{1/2}(\Sigma)} \\ &\quad + (\tilde{\varepsilon}^{-1} \nabla E(\psi_+), \nabla v)_{0,\Omega_-} - \omega^2 (\tilde{\mu}\, E(\psi_+), v)_{0,\Omega_-}. \end{aligned}$$

The second equality is obtained similarly. □

By design, $(\cdot,\cdot)_{\tilde\varepsilon,\tilde\mu}$ is an inner product in $\mathrm{H}^1(\Omega_-)$. Hence, for each $E(\psi_+) \in \mathrm{H}^1(\Omega)$, the Riesz representation theorem ensures the existence of a unique $z_{E(\psi_+)} \in \mathrm{H}^1(\Omega_-)$ such that

$$(z_{E(\psi_+)}, v_-)_{\tilde\varepsilon,\tilde\mu} = (E(\psi_+), v_-)_{\tilde\varepsilon,\tilde\mu} - \langle \varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+, v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, \quad \forall v_- \in \mathrm{H}^1(\Omega_-). \tag{24}$$

Consequently, we find that

$$\begin{cases} \tilde a(E(\psi_+), v) = (f_+^N, v)_{0,\Omega_+} + (z_{E(\psi_+)}, v)_{\tilde\varepsilon,\tilde\mu}, & \forall v \in \mathrm{H}^1(\Omega), \\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0,\Omega_-} + (E(\psi_+) - z_{E(\psi_+)}, v_-)_{\tilde\varepsilon,\tilde\mu}, & \forall v_- \in \mathrm{H}^1(\Omega_-). \end{cases} \tag{25}$$

Let us now proceed backwards. Given an auxiliary function $z \in \mathrm{H}^1(\Omega_-)$, we introduce the pair of functions $(\psi^z, \psi_-^z) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-)$ governed by

$$\begin{cases} \tilde a(\psi^z, v) = (f_+^N, v)_{0,\Omega_+} + (z, v)_{\tilde\varepsilon,\tilde\mu}, & \forall v \in \mathrm{H}^1(\Omega), \\ a_-(\psi_-^z, v_-) = (f_-^N, v_-)_{0,\Omega_-} + (\psi^z - z, v_-)_{\tilde\varepsilon,\tilde\mu}, & \forall v_- \in \mathrm{H}^1(\Omega_-). \end{cases} \tag{26}$$

For the volume optimal control method to operate, we investigate the well-posedness of the system of equations (26). First, we consider the Fredholm operator $\mathrm{A}_-^- \in \mathcal{L}(\mathrm{H}^1(\Omega_-), (\mathrm{H}^1(\Omega_-))')$ defined by:

$$\langle \mathrm{A}_-^- u, v\rangle_{(\mathrm{H}^1(\Omega_-))',\mathrm{H}^1(\Omega_-)} := a_-(u,v), \quad \forall u, v \in \mathrm{H}^1(\Omega_-).$$

One has the:

Lemma 5.3. *Except for a discrete set of* $\omega \in \mathbb{R}_{>0}$*, the operator* A_-^- *is an isomorphism.*

Proof. Let us consider the operator $\mathtt{A}_0 \in \mathcal{L}(\mathrm{H}^1(\Omega_-))$, and the related bilinear form, defined by

$$(\mathtt{A}_0\, u, v)_{1,\Omega_-} = (\varepsilon_-^{-1}\nabla u, \nabla v)_{0,\Omega_-}, \quad \forall u, v \in \mathrm{H}^1(\Omega_-).$$

The bilinear form is coercive and thus $\mathtt{A}_0$ an isomorphism. On the other hand, introducing the compact operator $\mathtt{C} \in \mathcal{L}(\mathrm{H}^1(\Omega_-))$ defined by

$$(\mathtt{C}\, u, v)_{1,\Omega_-} = (\mu_- u, v)_{0,\Omega_-}, \quad \forall u, v \in \mathrm{H}^1(\Omega_-),$$

we note that $(\omega^2\mathtt{C})_{\omega\in\mathbb{R}_{>0}}$ is a family of compact operators. By applying the analytic Fredholm theorem (see *e.g.* [29], Thm. 1.1.1), we conclude that, except for a discrete set of $\omega \in \mathbb{R}_{>0}$, the operator $\mathrm{A}_-^- = \mathtt{A}_0 - \omega^2\mathtt{C}$ is an isomorphism. ☐

We assume from now on that A_-^- is an isomorphism.

Next, we introduce the second Fredholm operator $\widetilde{\mathrm{A}} \in \mathcal{L}(\mathrm{H}^1(\Omega), (\mathrm{H}^1(\Omega))')$

$$\langle \widetilde{\mathrm{A}}\, u, v\rangle_{(\mathrm{H}^1(\Omega))',\mathrm{H}^1(\Omega)} = \tilde a(u,v) \quad \forall u, v \in \mathrm{H}^1(\Omega). \tag{27}$$

Similarly to the assumption on A_-^-, we assume from now on that the operator $\widetilde{\mathrm{A}}$ is an isomorphism (recall that the coefficients $\tilde\varepsilon$ and $\tilde\mu$ can be chosen among any scalar fields satisfying the condition (22)).

Proposition 5.4. *The system of equations (26) is well-posed, and*

$$\|\psi^z\|_{1,\Omega} + \|\psi_-^z\|_{1,\Omega_-} \lesssim \|f^N\|_{0,\Omega} + \|z\|_{1,\Omega_-}.$$

Proof. The operator $\widetilde{\mathrm{A}}$ is an isomorphism. This implies that the first equation of system (26) is well-posed, *i.e.* $\exists\, C_1 > 0$ (independent of z and f^N) such that:

$$\|\psi^z\|_{1,\Omega} \leq C_1(\|f_+^N\|_{0,\Omega_+} + \|z\|_{1,\Omega_-}).$$

Regarding the second equation of this system, since the operator A_-^- is an isomorphism, this equation is well-posed, *i.e.* $\exists\, C_2 > 0$ (independent of z, ψ^z, and f_-^N) such that:

$$\begin{aligned}\|\psi_-^z\|_{1,\Omega_-} &\leq C_2(\|f_-^N\|_{0,\Omega_-} + \|\psi^z\|_{1,\Omega} + \|z\|_{1,\Omega_-})\\ &\leq C_2(\|f_-^N\|_{0,\Omega_-} + C_1(\|f_+^N\|_{0,\Omega_+} + \|z\|_{1,\Omega_-}) + \|z\|_{1,\Omega_-})\\ &\leq C(\|f_-^N\|_{0,\Omega_-} + \|f_+^N\|_{0,\Omega_+} + \|z\|_{1,\Omega_-})\end{aligned}$$

Finally, we conclude that there exists $C' > 0$ such that:

$$\|\psi^z\|_{1,\Omega} + \|\psi_-^z\|_{1,\Omega_-} \leq C'(\|f^N\|_{0,\Omega} + \|z\|_{1,\Omega_-}).$$

□

As we proceed, we show that introducing a control $z \in \mathrm{H}^1(\Omega_-)$ allows us to construct a pair of functions that satisfy the same equation as ψ in both Ω_+ and Ω_-, with their respective boundary conditions and the vanishing jump of the normal derivative on the interface, although in general without vanishing jump of the trace on the interface. One can easily check the following properties of the solution $(\psi^z, \psi_-^z) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-)$ to (26).

Proposition 5.5. *$\forall z \in \mathrm{H}^1(\Omega_-)$, the functions $\psi_+^z := \psi^z{}_{|\Omega_+}$ and ψ_-^z are such that*

$$\begin{cases} -\operatorname{div}(\varepsilon_+^{-1}\,\nabla\psi_+^z) - \omega^2\,\mu_+\,\psi_+^z = f_+^N & \text{in } \Omega_+,\\ -\operatorname{div}(\varepsilon_-^{-1}\,\nabla\psi_-^z) - \omega^2\,\mu_-\,\psi_-^z = f_-^N & \text{in } \Omega_-,\\ \varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+^z = \varepsilon_-^{-1}\,\partial_{\boldsymbol{\nu}}\psi_-^z & \text{on } \Sigma,\\ \varepsilon_+^{-1}\nabla\psi_+\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\Omega_+\setminus\Sigma,\\ \varepsilon_-^{-1}\nabla\psi_-\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\Omega_-\setminus\Sigma. \end{cases} \tag{28}$$

Consequently, it remains to prove that there exists $z^* \in \mathrm{H}^1(\Omega_-)$ such that the solution of problem (26) satisfies $\psi^{z^*} = \psi_-^{z^*}$ on Σ, so that $(\psi^{z^*}_{|\Omega_+}, \psi_-^{z^*})$ solves (21). Next, we show the existence of such a control in $\mathrm{H}^1(\Omega_-)$.

Lemma 5.6. *There exists $z^* \in \mathrm{H}^1(\Omega_-)$ such that the solution of problem (26) satisfies $\psi^{z^*} = \psi_-^{z^*}$ on Σ.*

Proof. According to Assumption 2, the problem (21) is well posed. Therefore, the solutions ψ_+ and ψ_- are well defined. We choose the extension $\psi = E(\psi_+)$, in particular, its restriction to the domain Ω_- is ψ_-. With this choice, the system (23) can be written as

$$\begin{cases} \tilde{a}(\psi, v) = (f_+^N, v)_{0,\Omega_+} + (\psi_-, v)_{\tilde{\varepsilon},\tilde{\mu}} - \langle\varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+, v\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, & \forall v \in \mathrm{H}^1(\Omega),\\ a_-(\psi_-, v_-) = (f_-^N, v_-)_{0,\Omega_-} + \langle\varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+, v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, & \forall v_- \in \mathrm{H}^1(\Omega_-). \end{cases}$$

Since $\psi_- \in \mathrm{H}^1(\Omega_-)$, we know that according to the Riesz representation theorem, there exists a unique $z^* = z_{\psi_-} \in \mathrm{H}^1(\Omega_-)$ such that

$$(z_{\psi_-}, v_-)_{\tilde{\varepsilon},\tilde{\mu}} = (\psi_-, v_-)_{\tilde{\varepsilon},\tilde{\mu}} - \langle\varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+, v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, \quad \forall v_- \in \mathrm{H}^1(\Omega_-).$$

By inspection, we notice that the solution to (26) is $(\psi^{z^*}, \psi_-^{z^*}) = (E(\psi_+), \psi_-) = (\psi, \psi_-)$. Using this definition, and since the traces on Σ of $\psi_{|\Omega_+}$ and ψ_- match, the claim is proven. □

This leads us to the next result.

Lemma 5.7. *Let* (ψ_+,ψ_-) *be the solution of* (21). *Then, the set of* $z^*\in \mathrm{H}^1(\Omega_-)$ *such that the solution of* (26) *satisfying* $\psi^{z^*}=\psi_-^{z^*}$ *on* Σ *in addition to* (28) *is isomorphic to the set of all possible continuous extensions of* ψ_+ *over* Ω. *Moreover,* z^* *and* ψ^{z^*} *are linked by the relation*

$$(z^*,v_-)_{\tilde\varepsilon,\tilde\mu}=(\psi^{z^*},v_-)_{\tilde\varepsilon,\tilde\mu}-\langle\varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_+,v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)},\quad \forall v_-\in \mathrm{H}^1(\Omega_-). \tag{29}$$

Proof. Let $E:\mathrm{H}^1(\Omega_+)\to\mathrm{H}^1(\Omega)$ be an arbitrary continuous extension operator. According to Lemma 5.2, $(E(\psi_+),\psi_-)$ satisfies the following problem

$$\begin{cases}\tilde a(E(\psi_+),v)=(f_+^N,v)_{0,\Omega_+}+(E(\psi_+),v)_{\tilde\varepsilon,\tilde\mu}-\langle\varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_+,v\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, & \forall v\in\mathrm{H}^1(\Omega),\\ a_-(\psi_-,v_-)=(f_-^N,v_-)_{0,\Omega_-}+\langle\varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_+,v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, & \forall v_-\in\mathrm{H}^1(\Omega_-).\end{cases}$$

We proceed as in the proof of Lemma 5.6. Since $E(\psi_+)\in\mathrm{H}^1(\Omega)$, the Riesz representation theorem guarantees the existence of a unique $z^*=z_{E(\psi_+)}\in\mathrm{H}^1(\Omega_-)$ such that

$$(z_{E(\psi_+)},v_-)_{\tilde\varepsilon,\tilde\mu}=(E(\psi_+),v_-)_{\tilde\varepsilon,\tilde\mu}-\langle\varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_+,v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)},\quad\forall v_-\in\mathrm{H}^1(\Omega_-).$$

Then, we conclude as before that $(\psi^{z^*},\psi_-^{z^*})=(E(\psi_+),\psi_-)$ (and that the traces on Σ match). Replacing $z_{E(\psi_+)}$ by z^*, we find (29). Finally, if one chooses two extensions such that $E_1(\psi_+)\neq E_2(\psi_+)$, looking once more at (29) we note that the corresponding controls z_1^* and z_2^* are different. Conversely, assume that one is given a z^* such that $\psi_+^{z^*}{}_{|\Sigma}=\psi_-^{z^*}{}_{|\Sigma}$. It follows that $\psi_+^{z^*}$ and $\psi_-^{z^*}$, which are already governed by (28), fulfill (21): one concludes by uniqueness that $\psi_+^{z^*}=\psi_+$, and $\psi_-^{z^*}=\psi_-$, so ψ^{z^*} is an extension of ψ_+. □

We will explain in the section below how to choose a control function $z^*\in\mathrm{H}^1(\Omega_-)$.

5.3. Reformulation of the problem as a minimization problem

To find a function $z^*\in\mathrm{H}^1(\Omega_-)$, such that $\psi^{z^*}=\psi_-^{z^*}$ on Σ, one can solve the following optimal control problem

$$\text{Find } z^*=\underset{z\in\mathrm{H}^1(\Omega_-)}{\operatorname{argmin}}\,J(z)\quad\text{with}\quad J(z)=\frac12\|\psi^z-\psi_-^z\|^2_{\mathrm{L}^2(\Sigma)}, \tag{30}$$

where (ψ^z,ψ_-^z) is the solution of (26).

Thanks to Lemma 5.7, one can show without any difficulty, that the optimal control problem (30) has an infinite number of solutions. Then, the set of solutions of (30), denoted M_J, is infinite.

Below, we show some properties of the functional J and its set of minimizers M_J.

5.3.1. Properties of the functional J and its set of minimizers M_J

The proofs of the following propositions are similar to those presented in [19].

Proposition 5.8. *The functional* J *satisfies the following properties:*

1. *Let* $(z_n)_n$ *be a sequence of elements in* $\mathrm{H}^1(\Omega_-)$ *that weakly converges to* $z_0\in\mathrm{H}^1(\Omega_-)$. *Then,* $(J(z_n))_n$ *converges to* $J(z_0)$.
2. *The functional* J *is continuous and convex on* $\mathrm{H}^1(\Omega_-)$.

Proposition 5.9. *One has:*

1. $M_J=\{z\in\mathrm{H}^1(\Omega_-)$ *such that* $J(z)=0\}$.
2. *The set* M_J *is a convex and closed subset of* $\mathrm{H}^1(\Omega_-)$.

Consequently, as a result of Proposition 5.9, the minimization problem

$$\text{Find } z_J^* = \underset{z \in M_J}{\operatorname{argmin}} \|z\|^2_{\tilde{\varepsilon},\tilde{\mu}}, \tag{31}$$

has a unique solution.

The goal of the following is to find a characterization of $E_{z_J^*}(\psi_+)$, the continuous extension of ψ_+ associated with z_J^*, which is essential for the discretization process. To proceed, we define $E_{z_H}(\psi_+) \in \mathrm{H}^1(\Omega)$, the continuous extension of ψ_+, that satisfies

$$\begin{cases} -\operatorname{div}(\tilde{\varepsilon}^{-1}\,\nabla E_{z_H}(\psi_+)) - \omega^2\,\tilde{\mu}\,E_{z_H}(\psi_+) = 0, & \text{in } \Omega_-, \\ \tilde{\varepsilon}^{-1}\partial_{\boldsymbol{\nu}} E_{z_H}(\psi_+) = 0, & \text{on } \partial\Omega_- \setminus \Sigma, \end{cases} \tag{32}$$

and we denote by $z_H \in M_J$ the minimizer associated with $E_{z_H}(\psi_+)$ through (29).

Proposition 5.10. *The minimizers z_H and z_J^* coincide.*

Proof. Starting from (32), we have by integration by parts

$$\begin{aligned}(E_{z_H}(\psi_+), v_-)_{\tilde{\varepsilon},\tilde{\mu}} = (\tilde{\varepsilon}^{-1}\,\nabla E_{z_H}(\psi_+), \nabla v_-)_{0,\Omega_-} \\ - \omega^2(\tilde{\mu}\,E_{z_H}(\psi_+), v_-)_{0,\Omega_-} = 0, \quad \forall v_- \in \mathrm{H}^1_{0,\Sigma}(\Omega_-).\end{aligned}$$

On the other hand, we know that, for all $v_- \in \mathrm{H}^1(\Omega_-)$,

$$\begin{aligned}(z_H, v_-)_{\tilde{\varepsilon},\tilde{\mu}} = (\tilde{\varepsilon}^{-1}\,\nabla E_{z_H}(\psi_+), \nabla v_-)_{0,\Omega_-} - \omega^2(\tilde{\mu}\,E_{z_H}(\psi_+), v_-)_{0,\Omega_-} \\ - \langle \varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+, v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}.\end{aligned}$$

We infer that

$$(z_H, v_-)_{\tilde{\varepsilon},\tilde{\mu}} = 0, \quad \forall v_- \in \mathrm{H}^1_{0,\Sigma}(\Omega_-).$$

In particular, for $v_- = E_{z_H}(\psi_+) - E_{z_J^*}(\psi_+) \in \mathrm{H}^1_{0,\Sigma}(\Omega_-)$, we have

$$(E_{z_H}(\psi_+) - E_{z_J^*}(\psi_+), z_H)_{\tilde{\varepsilon},\tilde{\mu}} = 0.$$

However, according to (24), we also have

$$(z_J^* - z_H, v_-)_{\tilde{\varepsilon},\tilde{\mu}} = (E_{z_H}(\psi_+) - E_{z_J^*}(\psi_+), v_-)_{\tilde{\varepsilon},\tilde{\mu}}.$$

Hence, taking $v_- = z_H$, we obtain $(z_J^* - z_H, z_H)_{\tilde{\varepsilon},\tilde{\mu}} = 0$. It follows by orthogonality that

$$\|z_J^*\|^2_{\tilde{\varepsilon},\tilde{\mu}} = \|z_J^* - z_H + z_H\|^2_{\tilde{\varepsilon},\tilde{\mu}} = \|z_J^* - z_H\|^2_{\tilde{\varepsilon},\tilde{\mu}} + \|z_H\|^2_{\tilde{\varepsilon},\tilde{\mu}} \geq \|z_H\|^2_{\tilde{\varepsilon},\tilde{\mu}},$$

so that $z_H = z_J^*$, due to the characterization (31) of z_J^*. □

The main goal of the rest of Section 5.3 is to apply a regularization technique to ensure the uniqueness of the solution to problem (30).

We know that $z_J^* = z_H$, where $z_H \in \mathrm{H}^1(\Omega_-)$ is associated with $E_{z_H}(\psi_+)$ through (29), and $E_{z_H}(\psi_+)$ is itself characterized by (32). So, $z_J^* \in \mathrm{H}^1(\Omega_-)$ and, according to (29),

$$(z_J^*, v_-)_{\tilde{\varepsilon},\tilde{\mu}} = (E_{z_H}(\psi_+), v_-)_{\tilde{\varepsilon},\tilde{\mu}} - \langle \varepsilon_+^{-1}\partial_{\boldsymbol{\nu}}\psi_+, v_-\rangle_{\mathrm{H}^{-1/2}(\Sigma),\mathrm{H}^{1/2}(\Sigma)}, \quad \forall v_- \in \mathrm{H}^1(\Omega_-). \tag{33}$$

It follows that one can characterize z_J^*.

Corollary 5.11. *The minimizer* $z_J^* \in \mathrm{H}^1(\Omega_-)$ *is governed by*

$$\begin{cases} -\operatorname{div}(\tilde\varepsilon^{-1}\,\nabla z_J^*) - \omega^2\,\tilde\mu\, z_J^* = 0, & \text{in } \Omega_-,\\ \tilde\varepsilon^{-1}\partial_{\boldsymbol\nu} z_J^* = 0, & \text{on } \partial\Omega_-\setminus\Sigma,\\ \tilde\varepsilon^{-1}\partial_{\boldsymbol\nu} z_J^* = \tilde\varepsilon^{-1}\,\partial_{\boldsymbol\nu} E_{z_J^*}(\psi_+) - \varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_+, & \text{on } \Sigma. \end{cases} \tag{34}$$

An interesting consequence of Corollary 5.11 is that one can gather further knowledge on the minimizer $z_J^* \in \mathrm{H}^1(\Omega_-)$, namely determine its *a priori* regularity. Let us give an illustration (with simplifying assumptions). Specifically, let us assume that: ε_+ is constant in Ω_+ ; $\tilde\varepsilon$ is constant in Ω_- ; $\partial\Omega\cap\Sigma=\emptyset$; and $\tau_N(\varepsilon^{-1}) \in (1/2,1)$. Starting from (34), and since $\tilde\varepsilon$ is constant, z_J^* solves a Laplace problem with a right-hand side $-\omega^2\,\tilde\varepsilon\,\tilde\mu\,z_J^*$ that belongs to $\mathrm{L}^2(\Omega_-)$, and Neumann boundary condition (equal to $(\tilde\varepsilon^{-1}\,\partial_{\boldsymbol\nu} E_{z_J^*}(\psi_+) - \varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_+)_{|\Sigma}$ on Σ, vanishing on $\partial\Omega_-\setminus\Sigma$). What is the *a priori* regularity of this boundary data ?

On the one hand, the regularity of $\partial_{\boldsymbol\nu}\psi_{+|\Sigma}$ is driven by the *a priori* regularity of ψ: going back to Section 4.1 and thanks to the above assumption on $\tau_N(\varepsilon^{-1})$, we find in particular that $\partial_{\boldsymbol\nu}\psi_{+|\Sigma} \in \mathrm{L}^2(\Sigma)$, so that $\varepsilon_+^{-1}\partial_{\boldsymbol\nu}\psi_{+|\Sigma} \in \mathrm{L}^2(\Sigma)$.

On the other hand, the regularity of $\tilde\varepsilon^{-1}\,\partial_{\boldsymbol\nu} E_{z_J^*}(\psi_+)$ is driven by the *a priori* regularity of $E_{z_J^*}(\psi_+)$: according to (32), it is governed by a Laplace problem with a right-hand side that belongs to $\mathrm{L}^2(\Omega_-)$, and mixed boundary conditions, with Dirichlet data on Σ, where the trace of $E_{z_J^*}(\psi_+)$ is by definition equal to $\psi_{+|\Sigma}$, and vanishing Neumann data on $\partial\Omega_-\setminus\Sigma$. According again to the above assumption on $\tau_N(\varepsilon^{-1})$, we find that $\psi_{+|\Sigma}$ belongs to $\mathrm{H}^1(\Sigma)$. Adapting the result of [27] (specifically, using that $\partial\Omega\cap\Sigma=\emptyset$ to handle separately the two boundary conditions), we now find that $E_{z_J^*}(\psi_+) \in \mathrm{H}^{3/2}(\Omega_-)$, and $\tilde\varepsilon^{-1}\,\partial_{\boldsymbol\nu} E_{z_J^*}(\psi_+)_{|\Sigma} \in \mathrm{L}^2(\Sigma)$.

Hence, the Neumann data for the Laplace problem with unknown z_J^* belongs to $\mathrm{L}^2(\Sigma)$: using again [27], we conclude that the minimizer z_J^* belongs to $\mathrm{H}^{3/2}(\Omega_-)$.

In the general case, that is without the above simplifying assumptions, we note that one can nevertheless derive improved regularity results with the help of other limit regularity exponents.

5.3.2. Tikhonov regularization of the minimization problem

To ensure a unique and well-posed solution, we will apply the classical Tikhonov regularization to the problem (30). Our objective is to analyze the convergence of this regularization. To achieve this, we consider for any $\lambda > 0$ the minimization problem

$$\min_{z\in\mathrm{H}^1(\Omega_-)} J^\lambda(z) := J(z) + \lambda\|z\|^2_{\tilde\varepsilon,\tilde\mu}. \tag{35}$$

Given that the function J^λ is strictly convex and coercive for all $\lambda > 0$, it follows that the problem (35) has a unique solution, which we denote z_λ.

Proposition 5.12. *As* $\lambda \to 0^+$, *the sequence* $(z_\lambda)_\lambda$ *converges to* z_J^* *in* $\mathrm{H}^1(\Omega_-)$.

Proof. By the definitions of z_λ and of z_J^*, we have

$$\forall\lambda>0,\quad \lambda\|z_\lambda\|^2_{\tilde\varepsilon,\tilde\mu} \le J^\lambda(z_\lambda) \le J^\lambda(z_J^*) = J(z_J^*) + \lambda\|z_J^*\|^2_{\tilde\varepsilon,\tilde\mu} = \lambda\|z_J^*\|^2_{\tilde\varepsilon,\tilde\mu}. \tag{36}$$

So, for all $\lambda > 0$, we have $\|z_\lambda\|^2_{\tilde\varepsilon,\tilde\mu} \le \|z_J^*\|^2_{\tilde\varepsilon,\tilde\mu}$, and as a consequence $(z_\lambda)_\lambda$ is bounded in $\mathrm{H}^1(\Omega_-)$. Using *e.g.* Theorem 3.18 of [9], we infer that there exists a subsequence that we also denote $(z_\lambda)_\lambda$ which converges weakly to some z_0 in $\mathrm{H}^1(\Omega_-)$, when λ tends to 0. Let's now show that z_0 is a minimizer of J. According to (36), we have

$$\forall\lambda>0,\quad 0 \le J(z_\lambda) \le \lambda\|z_J^*\|^2_{\tilde\varepsilon,\tilde\mu}.$$

This shows that $(J(z_\lambda))_\lambda$ converges to zero when λ tends to zero.

On the other hand, using the result from Proposition 5.8, we know that $(J(z_\lambda))_\lambda$ converges to $J(z_0)$. Consequently, $J(z_0) = 0$ and then z_0 is a minimizer of J.

Let's now show that $(z_\lambda)_\lambda$ converges strongly to z_0 and that $z_0 = z_J^*$. First, we have

$$\begin{cases} \|z_\lambda\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}, \ \forall \lambda > 0 \Longrightarrow \limsup\limits_{\lambda\to 0} \|z_\lambda\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}, \\ \\ z_\lambda \rightharpoonup z_0 \text{ in } \mathrm{H}^1(\Omega_-) \Longrightarrow \|z_0\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \liminf\limits_{\lambda\to 0} \|z_\lambda\|_{\tilde{\varepsilon},\tilde{\mu}}, \end{cases}$$

where the second assertion is proven *e.g.* in Proposition 3.13 of [9]. This implies that $\|z_0\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}$. From the definition of z_J^*, we have that $z_0 = z_J^*$: in other words, $(z_\lambda)_\lambda$ converges weakly to z_J^*. Also,

$$\limsup_{\lambda\to 0} \|z_\lambda\|_{\tilde{\varepsilon},\tilde{\mu}} = \liminf_{\lambda\to 0} \|z_\lambda\|_{\tilde{\varepsilon},\tilde{\mu}} = \|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}.$$

Now, as $\mathrm{H}^1(\Omega_-)$ is a Hilbert space, we get that $z_\lambda \to z_J^*$ in $\mathrm{H}^1(\Omega_-)$ (see *e.g.* [9], Prop. 3.32). Noting that z_J^* is independent of the subsequence considered, the result is then shown. □

The purpose of the next section is to calculate the gradient of the functional J so that we can use it in the numerical part.

5.3.3. *Gradient of the functional J*

To do this, let's start by recalling that the functional J is differentiable, because it can be written as a composition of the two differentiable maps j_1 and j_2 defined by: $j_1 : \mathrm{H}^1(\Omega_-) \to \mathrm{L}^2(\Sigma)$ and $j_2 : \mathrm{L}^2(\Sigma) \to \mathbb{R}$ such that for all $z \in \mathrm{H}^1(\Omega_-)$, $g \in \mathrm{L}^2(\Sigma)$, we have

$$\begin{cases} j_1(z) = (\psi^z - \psi_-^z)_{|\Sigma}, \text{ where } (\psi^z, \psi_-^z) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-) \text{ is the solution to (26)}, \\ j_2(g) = \dfrac{1}{2}\|g\|^2_{\mathrm{L}^2(\Sigma)}. \end{cases}$$

Now, since the functional J is scalar valued, its differential in $z \in \mathrm{H}^1(\Omega_-)$ can be represented by its gradient $J'(z) \in \mathrm{H}^1(\Omega_-)$

$$\forall h \in \mathrm{H}^1(\Omega_-), \quad (J'(z), h)_{\tilde{\varepsilon},\tilde{\mu}} = \lim_{t\to 0} \frac{J(z+th) - J(z)}{t}. \tag{37}$$

In our context, to compute the gradient of the cost function J, the most appropriate method appears to be the adjoint state approach, see *e.g.* [14]. In particular, this will allow us to compute the gradient of the functional J which depends in a non-explicit way on the main variable of the problem.

Let us introduce the Lagrangian function associated with the constrained optimization problem from the variational formulation of (26). Let $\mathtt{V} := \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-) \times \mathrm{H}^1(\Omega_-) \times \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-)$ and $\mathscr{L} : \mathtt{V} \to \mathbb{R}$ such that

$$\begin{aligned} \mathscr{L}(\psi, \psi_-, z, d, d_-) = \frac{1}{2}\|\psi - \psi_-\|^2_{\mathrm{L}^2(\Sigma)} + \tilde{a}(\psi, d) - (f_+^N, d)_{0,\Omega_+} - (z, d)_{\tilde{\varepsilon},\tilde{\mu}} \\ + a_-(\psi_-, d_-) - (f_-^N, d_-)_{0,\Omega_-} - (\psi - z, d_-)_{\tilde{\varepsilon},\tilde{\mu}}. \end{aligned}$$

The functions $d \in \mathrm{H}^1(\Omega)$, $d_- \in \mathrm{H}^1(\Omega_-)$ are the adjoint variables associated to ψ, ψ_- respectively. Given $z \in \mathrm{H}^1(\Omega_-)$, we recall that (ψ^z, ψ_-^z) is the solution of the problem (26). By construction,

$$\begin{cases} \tilde{a}(\psi^z, d) - (f_+^N, d)_{0,\Omega_+} - (z, d)_{\tilde{\varepsilon},\tilde{\mu}} = 0, & \forall d \in \mathrm{H}^1(\Omega), \\ a_-(\psi_-^z, d_-) - (f_-^N, d_-)_{0,\Omega_-} - (\psi^z - z, d_-)_{\tilde{\varepsilon},\tilde{\mu}} = 0, & \forall d_- \in \mathrm{H}^1(\Omega_-). \end{cases}$$

So, we obtain:

$$\mathscr{L}(\psi^z, \psi_-^z, z, d, d_-) = J(z), \quad \forall d \in \mathrm{H}^1(\Omega),\ \forall d_- \in \mathrm{H}^1(\Omega_-). \tag{38}$$

On the other hand, it is clear that the functional $\mathscr{L}$ is differentiable with respect to all its variables.

For all $\mathtt{v} = (\psi, \psi_-, z, d, d_-) \in \mathtt{V}$, the partial derivatives of $\mathscr{L}$ at $\mathtt{v}$ belong respectively to $\partial_\psi \mathscr{L}(\mathtt{v}) \in (\mathrm{H}^1(\Omega))'$, $\partial_{\psi_-}\mathscr{L}(\mathtt{v}) \in (\mathrm{H}^1(\Omega_-))'$, $\partial_z \mathscr{L}(\mathtt{v}) \in (\mathrm{H}^1(\Omega_-))'$, $\partial_d \mathscr{L}(\mathtt{v}) \in (\mathrm{H}^1(\Omega))'$, and $\partial_{d_-}\mathscr{L}(\mathtt{v}) \in (\mathrm{H}^1(\Omega_-))'$. Below, we write duality products without mentioning Banach spaces.

Let $d \in \mathrm{H}^1(\Omega)$ and $d_- \in \mathrm{H}^1(\Omega_-)$ be given and $\mathtt{v}^z = (\psi^z, \psi_-^z, z, d, d_-) \in \mathtt{V}$. Taking the derivative of the relation (38) with respect to z and applying the formula of the chain rule, we obtain

$$(J'(z), h)_{\tilde{\varepsilon},\tilde{\mu}} = \langle \partial_\psi \mathscr{L}(\mathtt{v}^z), \frac{d\psi^z}{dz}(h)\rangle + \langle \partial_{\psi_-}\mathscr{L}(\mathtt{v}^z), \frac{d\psi_-^z}{dz}(h)\rangle + \langle \partial_z \mathscr{L}(\mathtt{v}^z), h\rangle,\ \forall h \in \mathrm{H}^1(\Omega_-). \tag{39}$$

Now, if there exists $(d^z, d_-^z) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-)$ for which the equations

$$\partial_\psi \mathscr{L}(\psi^z, \psi_-^z, z, d^z, d_-^z) = 0 \qquad \text{and} \qquad \partial_{\psi_-}\mathscr{L}(\psi^z, \psi_-^z, z, d^z, d_-^z) = 0,$$

are satisfied for all $z \in \mathrm{H}^1(\Omega_-)$, then

$$(J'(z), h)_{\tilde{\varepsilon},\tilde{\mu}} = \langle \partial_z \mathscr{L}(\psi^z, \psi_-^z, z, d^z, d_-^z), h\rangle.$$

Our goal now is to study the existence of (d^z, d_-^z). For that, we calculate $\partial_\psi \mathscr{L}(\mathtt{v}^z)$ and $\partial_{\psi_-}\mathscr{L}(\mathtt{v}^z)$:

$$\begin{aligned} \langle \partial_\psi \mathscr{L}(\mathtt{v}^z), v\rangle &= (\psi^z - \psi_-^z, v)_{\mathrm{L}^2(\Sigma)} + \tilde{a}(d, v) - (d_-, v)_{\tilde{\varepsilon},\tilde{\mu}}, && \forall\, v \in \mathrm{H}^1(\Omega),\\ \langle \partial_{\psi_-}\mathscr{L}(\mathtt{v}^z), v_-\rangle &= -(\psi^z - \psi_-^z, v_-)_{\mathrm{L}^2(\Sigma)} + a_-(d_-, v_-), && \forall\, v_- \in \mathrm{H}^1(\Omega_-). \end{aligned}$$

Consequently, the functions $(d^z, d_-^z) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega_-)$ should verify the system of equations

$$\begin{cases} \tilde{a}(d^z, v) = (d_-^z, v)_{\tilde{\varepsilon},\tilde{\mu}} - (\psi^z - \psi_-^z, v)_{\mathrm{L}^2(\Sigma)}, & \forall v \in \mathrm{H}^1(\Omega),\\ a_-(d_-^z, v_-) = (\psi^z - \psi_-^z, v_-)_{\mathrm{L}^2(\Sigma)}, & \forall v_- \in \mathrm{H}^1(\Omega_-). \end{cases} \tag{40}$$

As a straightforward consequence of (38), (39) and (40), one has the

Lemma 5.13. *$\forall z \in \mathrm{H}^1(\Omega_-)$, there holds $J'(z) = d_-^z - d^z_{|\Omega_-}$ where (d^z, d_-^z) solves (40).*

By assumption, the operators A_-^- and $\widetilde{\mathrm{A}}$ are isomorphisms, so we easily obtain a stability bound.

Proposition 5.14. *The system of equations (40) is well posed, and*

$$\|d^z\|_{1,\Omega} + \|d_-^z\|_{1,\Omega_-} \lesssim \|\psi^z\|_{1,\Omega} + \|\psi_-^z\|_{1,\Omega_-}.$$

5.4. A volume optimal control reformulation of the problem satisfied by φ

This problem has already been addressed in [19]. The steps are similar to those in the previous section, but differ in two ways: the use of Dirichlet boundary conditions and the absence of zero-order term. The main distinction is that for the scalar problem associated with φ, we simply extend ε_+ like in (22) and easily recover the results of Section 5.2.

6. Discretization of the volume optimal control method

For simplicity, we assume that Assumptions 1 and 2 hold, *i.e.* that A_D and $A_N(\omega^2)$ are isomorphisms. See Section 6.3 for the case of the weakened assumptions. We present a numerical approximation of our problems, based on Lagrange finite elements. To further simplify the presentation, we consider that Ω, Ω_+ and Ω_- are polygons. The notation h denotes the discretisation parameter (classically, for the finite element method, h is equal to the meshsize). Next, we first propose an approximation of the minimisation problem (35), and prove that the sequence of discrete controls $(z_h^*)_h$ converges to z_J^* in $\mathrm{H}^1(\Omega_-)$. In addition, by approximating (26), we prove that one can build a sequence $(\psi_h^{z_h^*})_h$ that converges to $E_{z_J^*}(\psi_+)$ in $\mathrm{H}^1(\Omega)$, respectively a sequence $(\psi_{-,h}^{z_h^*})_h$ that converges to ψ_- in $\mathrm{H}^1(\Omega_-)$. We also refer to Section 7 for a comparison of convergence rates between the method and a method using T-conform meshes (for which convergence rates are known, *cf.* [8,15]).

To discretize our problems, we use a shape regular family of standard meshes of $\overline{\Omega}$, made of triangles $(\mathcal{T}_h)_h$, which are conforming with the interface Σ: for all h, for all $K \in \mathcal{T}_h$, there exists $i \in \{+,-\}$ such that $K \subset \overline{\Omega_i}$. We denote $(\mathcal{T}_h^i)_h$ the corresponding meshes of Ω_i for $i \in \{+,-\}$. Additionally, we let S_h be the family of edges of $\mathcal{T}_h$ such that $\overline{\Sigma} := \cup_{S\in S_h}\overline{S}$.

6.1. Discretization of the problem satisfied by ψ

We outline how to discretize the problem associated with ψ, using again an arbitrary source term $f^N \in \mathrm{L}^2(\Omega)$, instead of the source term $\omega^2\,\mu\, s_0$.

For any $k \in \mathbb{N}^*$, we define

$$\mathrm{V}_h^k(\Omega) := \{v_h \in \mathrm{H}^1(\Omega) |\, v_{h|T} \in \mathrm{P}^k(T), \forall\, T \in \mathcal{T}_h\},$$

where $\mathrm{P}^k(T)$ represents the space of polynomials defined on T of degree at most k.

As a preamble, we recall that, even under Assumption 2, there is no guarantee that the problem

$$\text{Find } \psi_h \in \mathrm{V}_h^k(\Omega) \text{ such that } (\varepsilon^{-1}\nabla\psi_h, \nabla v_h)_{0,\Omega} - \omega^2(\mu\,\psi_h, v_h)_{0,\Omega} = (f^N, v_h)_{0,\Omega}, \quad \forall v_h \in \mathrm{V}_h^k(\Omega),$$

is well-posed, even for h small enough. This is the reason why we proceed with the discretization of the volume optimal control approach. We refer to Section 7.1 for the T-coercivity method, that is a discretisation with specifically designed meshes, the so-called T-conform meshes. To that aim, we define

$$\begin{aligned}\mathrm{V}_h^k(\Omega_i) &:= \{v_{i,h} \in \mathrm{H}^1(\Omega_i) \,|\, v_{i,h|T} \in \mathrm{P}^k(T), \forall\, T \in \mathcal{T}_h^i\}\,, i = \pm,\\ \mathrm{V}_h^k(\Sigma) &:= \{g \in \mathrm{L}^2(\Sigma) \,|\, \exists v_h \in \mathrm{V}_h^k(\Omega),\ g = v_{h|\Sigma}\}.\end{aligned}$$

Moreover, we recall the basic approximation property

$$\begin{cases}\forall v \in \mathrm{H}^1(\Omega), & \lim\limits_{h\to 0}\left(\inf\limits_{v_h\in \mathrm{V}_h^k(\Omega)} \|v - v_h\|_{1,\Omega}\right) = 0,\\ \forall v_- \in \mathrm{H}^1(\Omega_-), & \lim\limits_{h\to 0}\left(\inf\limits_{v_{-,h}\in \mathrm{V}_h^k(\Omega_-)} \|v_- - v_{-,h}\|_{\tilde{\varepsilon},\tilde{\mu}}\right) = 0.\end{cases} \tag{41}$$

Mimicking (26), for given $h > 0$ and $z \in \mathrm{H}^1(\Omega_-)$, we define the functions $\psi_h^z \in \mathrm{V}_h^k(\Omega)$ and $\psi_{-,h}^z \in \mathrm{V}_h^k(\Omega_-)$ as solutions to the following well-posed discrete problems: Find $(\psi_h^z, \psi_{-,h}^z) \in \mathrm{V}_h^k(\Omega) \times \mathrm{V}_h^k(\Omega_-)$ such that

$$\begin{cases}\tilde{a}(\psi_h^z, v_h) = (f_+^N, v_h)_{0,\Omega_+} + (z, v_h)_{\tilde{\varepsilon},\tilde{\mu}}, & \forall v_h \in \mathrm{V}_h^k(\Omega),\\ a_-(\psi_{-,h}^z, v_{-,h}) = (f_-^N, v_{-,h})_{0,\Omega_-} + (\psi_h^z - z, v_{-,h})_{\tilde{\varepsilon},\tilde{\mu}}, & \forall v_{-,h} \in \mathrm{V}_h^k(\Omega_-).\end{cases} \tag{42}$$

First, we introduce the projection operator $\pi_h^k : \mathrm{H}^1(\Omega_-) \to \mathrm{V}_h^k(\Omega_-)$ such that: $\forall z \in \mathrm{H}^1(\Omega_-)$, $\pi_h^k z$ is defined as the unique element of $\mathrm{V}_h^k(\Omega_-)$ that satisfies the problem

$$(\pi_h^k z, v_{-,h})_{\tilde{\varepsilon},\tilde{\mu}} = (z, v_{-,h})_{\tilde{\varepsilon},\tilde{\mu}}, \quad \forall v_{-,h} \in \mathrm{V}_h^k(\Omega_-). \tag{43}$$

In other words, π_h^k is the orthogonal projection operator from $\mathrm{H}^1(\Omega_-)$ to $\mathrm{V}_h^k(\Omega_-)$, with respect to the inner product $(\cdot,\cdot)_{\tilde{\varepsilon},\tilde{\mu}}$. In particular,

$$\forall z \in \mathrm{H}^1(\Omega_-) \setminus \{0\},\ \forall h, \quad \|\pi_h^k z\|_{\tilde{\varepsilon},\tilde{\mu}} \le \|z\|_{\tilde{\varepsilon},\tilde{\mu}}. \tag{44}$$

From the definitions (43) and (42), we notice that $\forall z \in \mathrm{H}^1(\Omega_-)$, we have the identities

$$\psi_h^{\pi_h^k z} = \psi_h^z \qquad \text{and} \qquad \psi_{-,h}^{\pi_h^k z} = \psi_{-,h}^z. \tag{45}$$

Now, we are interested in the numerical approximation of the optimization problem (30) by the finite element method. We consider the discrete problem

$$\min_{z_h \in \mathrm{V}_h^k(\Omega_-)} J_h(z_h) \quad \text{with} \quad J_h(z_h) = \frac{1}{2}\|\psi_h^{z_h} - \psi_{-,h}^{z_h}\|^2_{\mathrm{L}^2(\Sigma)}. \tag{46}$$

We proceed as before to show that the functional $J_h : \mathrm{V}_h^k(\Omega_-) \to \mathbb{R}_+$ is convex and continuous. Next, we introduce the functional

$$J_h^{\lambda_h}(z_h) := \frac{1}{2}\|\psi_h^{z_h} - \psi_{-,h}^{z_h}\|^2_{\mathrm{L}^2(\Sigma)} + \lambda_h \|z_h\|^2_{\tilde{\varepsilon},\tilde{\mu}}, \quad \forall z_h \in \mathrm{V}_h^k(\Omega_-),$$

with λ_h a strictly positive function that depends on h such that $\lim\limits_{h\to 0} \lambda_h = 0$.

As $\lambda_h > 0$ for all $h > 0$, the functional $J_h^{\lambda_h}$ is continuous, strictly convex and coercive. Therefore, the finite dimensional minimization problem

$$\text{Find } z_h^* = \operatorname*{argmin}_{z_h \in \mathrm{V}_h^k(\Omega_-)} J_h^{\lambda_h}(z_h) \tag{47}$$

admits a unique solution.

6.2. Convergence of the method

In this section, our goal is to choose the parameter λ_h such that the sequence of solutions of (42) with $z = z_h^*$, more precisely $(\psi_h^{z_h^*}{}_{|\Omega_+}, \psi_{-,h}^{z_h^*})_h$, converges to the solution of (21), when $h \to 0$.

Lemma 6.1. *We have the following estimate*

$$J_h^{\lambda_h}(z_h^*) \le \frac{1}{2}\|\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}\|^2_{\mathrm{L}^2(\Sigma)} + \lambda_h \|z_J^*\|^2_{\tilde{\varepsilon},\tilde{\mu}}, \tag{48}$$

where z_J^* is defined in (31).

Proof. On the one hand, since $\pi_h^k z_J^* \in \mathrm{V}_h^k(\Omega_-)$ and using the fact that $z_{k,h}^*$ is the unique solution to the optimization problem (47), we deduce that $J_h^{\lambda_h}(z_h^*) \le J_h^{\lambda_h}(\pi_h^k z_J^*)$.

On the other hand, the identity (45) gives us that $\psi_h^{\pi_h^k z_J^*} - \psi_{-,h}^{\pi_h^k z_J^*} = \psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}$. Therefore,

$$J_h^{\lambda_h}(\pi_h^k z_J^*) = \frac{1}{2}\|\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}\|^2_{\mathrm{L}^2(\Sigma)} + \lambda_h \|\pi_h^k z_J^*\|^2_{\tilde{\varepsilon},\tilde{\mu}} \le \frac{1}{2}\|\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}\|^2_{\mathrm{L}^2(\Sigma)} + \lambda_h \|z_J^*\|^2_{\tilde{\varepsilon},\tilde{\mu}},$$

thanks to the estimate (44). The two bounds put together lead to (48). □

To simplify the notations, for $h > 0$ and $z \in \mathrm{H}^1(\Omega_-)$, we denote by $\mathrm{A}_h(z)$ the real number

$$\mathrm{A}_h(z) = \frac{1}{2}\|\psi_h^z - \psi_{-,h}^z\|^2_{\mathrm{L}^2(\Sigma)}.$$

Importantly, *the value of* $\mathrm{A}_h(z)$ *is independent of the value of* λ_h. Also, using (45)-(46), we know that for any $z \in \mathrm{H}^1(\Omega_-)$, we have

$$\mathrm{A}_h(z) = J_h(\pi_h^k z).$$

Theorem 6.2. *Suppose we can choose* λ_h *such that the sequences* $(\lambda_h)_h$ *and* $(\mathrm{A}_h(z_J^*)/\lambda_h)_h$ *converge to* 0 *when* $h \to 0$. *Then, when* h *tends to* 0, *we have that*

- *the sequence* $(z_h^*)_h$ *converges to* z_J^* *in* $\mathrm{H}^1(\Omega_-)$ *;*
- *the sequence* $(\psi_h^{z_h^*})_h$ *converges to* $E_{z_J^*}(\psi_+)$ *in* $\mathrm{H}^1(\Omega)$, *respectively the sequence* $(\psi_{-,h}^{z_h^*})_h$ *converges to* ψ_- *in* $\mathrm{H}^1(\Omega_-)$, *where* (ψ_+, ψ_-) *is the solution of* (21).

Proof. To simplify notation, we refer to $\psi_h^* \in \mathrm{V}_h^k(\Omega)$ and $\psi_{-,h}^* \in \mathrm{V}_h^k(\Omega_-)$ as the functions

$$\psi_h^* = \psi_h^{z_h^*} \qquad \text{and} \qquad \psi_{-,h}^* = \psi_{-,h}^{z_h^*}.$$

First step: Weak convergence of $(z_h^*)_h, (\psi_h^*)_h$ and $(\psi_{-,h}^*)_h$

According to (48),

$$\|z_h^*\|^2_{\tilde{\varepsilon},\tilde{\mu}} \leq J_h^{\lambda_h}(z_h^*)/\lambda_h \leq \mathrm{A}_h(z_J^*)/\lambda_h + \|z_J^*\|^2_{\tilde{\varepsilon},\tilde{\mu}}. \tag{49}$$

Using the fact that $(\mathrm{A}_h(z_J^*)/\lambda_h)_h$ goes to 0 when $h \to 0$, we deduce that $(z_h^*)_h$ is bounded in $\mathrm{H}^1(\Omega_-)$.

Hence, there exists a subsequence, also called $(z_h^*)_h$ that converges weakly to a $z_0 \in \mathrm{H}^1(\Omega_-)$. As the system of equations (42) fits the coercive + compact framework with A_-^- and $\widetilde{\mathrm{A}}$ being isomorphisms, one has a uniform discrete inf-sup condition (*cf.* [4]). Consequently, we know that the sequence $(\psi_h^*)_h$ converges weakly in $\mathrm{H}^1(\Omega)$ to a $\psi \in \mathrm{H}^1(\Omega)$ and the sequence $(\psi_{-,h}^*)_h$ converges weakly in $\mathrm{H}^1(\Omega_-)$ to a $\psi_- \in \mathrm{H}^1(\Omega_-)$. On the other hand, according the basic approximation property (41), given any $v \in \mathrm{H}^1(\Omega)$, there exists a sequance $(v_h)_h$ with $v_h \in \mathrm{V}_h^k(\Omega)$ for all h that converges strongly to v in $\mathrm{H}^1(\Omega)$. Thanks to (42), we conclude that $\psi = \psi^{z_0}$. Similarly, one finds that $\psi_- = \psi_-^{z_0}$.

Second step: z_0 is a minimizer of J

The continuity of the trace operator and the compact inclusion $\mathrm{H}^{1/2}(\Sigma) \subset \mathrm{L}^2(\Sigma)$ give us

$$\psi^*_{h|\Sigma} - \psi^*_{-,h|\Sigma} \to \psi^{z_0}{}_{|\Sigma} - \psi_-^{z_0}{}_{|\Sigma} \text{ in } \mathrm{L}^2(\Sigma),$$

when $h \to 0$. Noting that

$$\frac{1}{2}\|\psi_h^* - \psi_{-,h}^*\|^2_{\mathrm{L}^2(\Sigma)} = J_h(z_h^*) \leq J_h^{\lambda_h}(z_h^*) \leq \lambda_h(\mathrm{A}_h(z_J^*)/\lambda_h + \|z_J^*\|^2_{\tilde{\varepsilon},\tilde{\mu}}), \tag{50}$$

and using the fact that $\lambda_h, \mathrm{A}_h(z_J^*)/\lambda_h$ go to 0 when $h \to 0$, we deduce that $\psi^{z_0}{}_{|\Sigma} - \psi_-^{z_0}{}_{|\Sigma} = 0$. Therefore, z_0 is a minimizer of J.

Third step: Strong convergence of $(z_h^*)_h$ to z_J^*

Since $\mathrm{A}_h(z_J^*)/\lambda_h$ goes to 0 when $h \to 0$ and using the inequality (49), we obtain that

$$\limsup_{h\to 0} \|z_h^*\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}.$$

On the other hand, $z_h^* \rightharpoonup z_0$ in $\mathrm{H}^1(\Omega_-)$ when $h \to 0$, so (*cf.* again [9], Prop. 3.13)

$$\|z_0\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \liminf_{h\to 0} \|z_h^*\|_{\tilde{\varepsilon},\tilde{\mu}}.$$

This implies that $\|z_0\|_{\tilde{\varepsilon},\tilde{\mu}} \leq \|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}$ and, since z_0 is a minimizer of J, we conclude thanks to (31) that $z_0 = z_J^*$. Furthermore, we deduce that

$$\lim_{h\to 0} \|z_h^*\|_{\tilde{\varepsilon},\tilde{\mu}} = \|z_0\|_{\tilde{\varepsilon},\tilde{\mu}}.$$

Thanks once more to Proposition 3.32 of [9], we conclude that $(z_h^*)_h$ converges strongly to $z_0 = z_J^*$ in $\mathrm{H}^1(\Omega_-)$.

Fourth step: Strong convergence of $(\psi_h^*)_h$ and $(\psi_{-,h}^*)_h$

Using the basic approximation property (41), the fact that the system of equations (42) fits the coercive + compact framework, and that $(z_h^*)_h$ converges strongly to z_J^*, we obtain the convergence of $(\psi_h^*)_h$ in $\mathrm{H}^1(\Omega)$ to $\psi^{z_J^*}$ and of $(\psi_{-,h}^*)_h$ in $\mathrm{H}^1(\Omega_-)$ to $\psi_-^{z_J^*}$. Observing that $\psi^{z_J^*} = E_{z_J^*}(\psi_+)$ ends the proof. □

Remark 6.3. A consequence of Theorem 6.2 is that, if one can prove that $\mathrm{A}_h(z_J^*)$ goes to 0 when $h \to 0$ then, taking $(\lambda_h)_h$ in the order of $\{\mathrm{A}_h(z_J^*)\}^{\mathtt{t}}$ for some $\mathtt{t} \in (0,1)$ yields convergence. It should also be noted that, in practice, the control z_J^* is not accessible. Nevertheless, if one can show that $J_h^{\lambda_h}(z_h^*)/\lambda_h$ is bounded as expressed in (49), and that $\mathrm{A}_h(z_h^*)/\lambda_h \to 0$ as stated in (50), then convergence is verified.

Now we have to show that we can find a family of parameters $(\lambda_h)_h$ such that $(\lambda_h)_h$ and $(\mathrm{A}_h(z_J^*)/\lambda_h)_h$ converge to 0 when $h \to 0$.

Let us introduce limit regularity exponents (with values in $(0,1]$) for some scalar problems with coefficient $\tilde{\varepsilon}^{-1}$: with homogeneous Neumann boundary condition, set in Ω ($\tau_N(\tilde{\varepsilon}^{-1})$); in Ω_-, with mixed homogeneous Dirichlet (on Σ) and Neumann (on $\partial\Omega_- \setminus \Sigma$) boundary conditions ($\tau_{-,M}(\tilde{\varepsilon}^{-1})$). We introduce

$$\tau = \min(\tau_N(\varepsilon^{-1}), \tau_{-,M}(\tilde{\varepsilon}^{-1})) \text{ and } \tilde{\tau} := \min(\tau_N(\tilde{\varepsilon}^{-1}), \tau_{-,N}(\varepsilon_-^{-1})).$$

Following [23], if $\tilde{\varepsilon}$ is smooth, or if it is piecewise smooth without triple point in the domain and without double point at the boundary, it holds that $\tilde{\tau} > 1/2$. In this situation, $\tau_{-,M}(\tilde{\varepsilon}^{-1}) > 1/4$ and, if $\partial\Omega \cap \Sigma = \emptyset$, $\tau_{-,M}(\tilde{\varepsilon}^{-1}) > 1/2$.

Proposition 6.4. *For all* $\mathtt{s}' \in [0,\tau)$, *it holds that*

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{1,\Omega} \lesssim h^{\mathtt{s}'} \|f_N\|_{0,\Omega} \text{ and } \|\psi_-^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{\tilde{\varepsilon},\tilde{\mu}} \lesssim h^{\mathtt{s}'} \|f_N\|_{0,\Omega}.$$

Moreover, for all $\mathtt{s}'' \in [0, \tau + \tilde{\tau})$, *it holds that*

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{0,\Omega} \lesssim h^{\mathtt{s}''} \|f_N\|_{0,\Omega} \text{ and } \|\psi_-^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{0,\Omega_-} \lesssim h^{\mathtt{s}''} \|f_N\|_{0,\Omega}.$$

Proof. Let $\mathtt{s}' \in [0,\tau)$. Recall that $\psi^{z_J^*} = E_{z_H}(\psi_+)$: one has $(\psi^{z_J^*})_{|\Omega_+} = \psi_+ \in \mathrm{PH}^{1+\mathtt{s}'}(\Omega_+)$, because $\mathtt{s}' < \tau_N(\varepsilon^{-1})$. Also, $(\psi^{z_J^*})_{|\Omega_-} \in \mathrm{H}^1(\Omega_-)$ solves equation (32): it is governed by

$$\begin{cases} -\operatorname{div}(\tilde{\varepsilon}^{-1}\, \nabla(\psi^{z_J^*})_{|\Omega_-}) = \omega^2\, \tilde{\mu}\, (\psi^{z_J^*})_{|\Omega_-} & \text{in } \Omega_-, \\ \tilde{\varepsilon}^{-1} \partial_{\boldsymbol{\nu}} (\psi^{z_J^*})_{|\Omega_-} = 0 & \text{on } \partial\Omega_- \setminus \Sigma, \\ (\psi^{z_J^*})_{|\Omega_-} = \psi_+ & \text{on } \Sigma. \end{cases}$$

This is a scalar problem with mixed boundary conditions. Since $\omega^2\, \tilde{\mu}\, (\psi^{z_J^*})_{|\Omega_-} \in \mathrm{L}^2(\Omega_-)$ and $(\psi_+)_{|\Sigma} \in \mathrm{PH}^{1/2+\mathtt{s}'}(\Sigma)$, and because $\mathtt{s}' < \tau$, we find that $(\psi^{z_J^*})_{|\Omega_-} \in \mathrm{PH}^{1+\mathtt{s}'}(\Omega_-)$. Thanks to the regularity of $\psi_+^{z_J^*} = \psi_+$, one has $\psi^{z_J^*} \in \mathrm{PH}^{1+\mathtt{s}'}(\Omega)$. On the other hand, $\psi_-^{z_J^*} = \psi_- \in \mathrm{PH}^{1+\mathtt{s}'}(\Omega_-)$.

The system of equations in (26) fits the coercive + compact framework, so we can compare the exact and discrete solutions with the estimates

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{1,\Omega} \lesssim h^{\mathtt{s}'}\|\psi^{z_J^*}\|_{\mathrm{PH}^{1+\mathtt{s}'}(\Omega)} \lesssim h^{\mathtt{s}'}\|f_N\|_{0,\Omega} \text{ and}$$
$$\|\psi_-^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{\tilde{\varepsilon},\tilde{\mu}} \lesssim h^{\mathtt{s}'}\|\psi_-^{z_J^*}\|_{\mathrm{PH}^{1+\mathtt{s}'}(\Omega_-)} \lesssim h^{\mathtt{s}'}\|f_N\|_{0,\Omega}.$$

This proves the first part of the claim.

Let $\mathtt{s}'' \in [0, \tau + \tilde{\tau})$. Applying the classic Aubin–Nitsche lemma to (26) (scalar problems with homogeneous Neumann boundary condition), we find that

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{0,\Omega} \lesssim h^{\mathtt{s}''}\|f_N\|_{0,\Omega} \text{ and } \|\psi_-^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{0,\Omega_-} \lesssim h^{\mathtt{s}''}\|f_N\|_{0,\Omega},$$

which is the second part of the claim. □

Corollary 6.5. *For all* $\mathtt{s} \in [0, 2\tau + \tilde{\tau})$, *it holds that*

$$\mathrm{A}_h(z_J^*) \lesssim h^{\mathtt{s}}\|f_N\|_{0,\Omega}^2.$$

Proof. Let $\mathtt{s} \in [0, 2\tau + \tilde{\tau})$: there exists $\mathtt{s}' \in [0, \tau)$ and $\mathtt{s}'' \in [0, \tau + \tilde{\tau})$ such that $\mathtt{s} = \mathtt{s}' + \mathtt{s}''$. By applying the multiplicative trace inequality and using the estimates of Proposition 6.4, we obtain

$$\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{\mathrm{L}^2(\Sigma)}^2 \lesssim h^{\mathtt{s}}\|f_N\|_{0,\Omega}^2 \text{ and } \|\psi_-^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{\mathrm{L}^2(\Sigma)}^2 \lesssim h^{\mathtt{s}}\|f_N\|_{0,\Omega}^2.$$

Now, we have $\psi^{z_J^*}{}_{|\Sigma} = \psi_-^{z_J^*}{}_{|\Sigma}$. Using the triangle inequality

$$\|\psi_h^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{\mathrm{L}^2(\Sigma)}^2 \leq 2(\|\psi^{z_J^*} - \psi_h^{z_J^*}\|_{\mathrm{L}^2(\Sigma)}^2 + \|\psi_-^{z_J^*} - \psi_{-,h}^{z_J^*}\|_{\mathrm{L}^2(\Sigma)}^2)$$

proves the claim. □

This enables us to choose $(\lambda_h)_h$ such that both $(\lambda_h)_h$ and $(A_h(z_J^*)/\lambda_h)_h$ converge to 0 as $h \to 0$.

Corollary 6.6. *Any sequence* $(\lambda_h)_h$ *such that* $\lambda_h \lesssim h^q$ *for some* $q \in (0, 2\tau + \tilde{\tau})$ *fulfills the conditions of theorem 6.2: the sequences* $(\lambda_h)_h$ *and* $(\mathrm{A}_h(z_J^*)/\lambda_h)_h$ *converge to* 0 *when* $h \to 0$.

We observe that the convergence to 0 of the traces $\big((\psi_h^{z_h^*} - \psi_{-,h}^{z_h^*})_{|\Sigma}\big)_h$ can be proven in a stronger norm than the $\mathrm{L}^2(\Sigma)$-norm. Let us give an illustration: if one can choose $q > 1$ in Corollary 6.6, then both sequences $(\lambda_h/h)_h$ and $(\mathrm{A}_h(z_J^*)/h)_h$ converge to 0 when $h \to 0$. On the other hand, we recall that if the sequence of meshes of the interface Σ is quasi-uniform in the sense of Definition 22.20 in [20], one can use global inverse inequalities. In particular (Eq. (22.39) loc. cit.), it holds that

$$\forall h,\ \forall g_h \in \mathrm{V}_h^k(\Sigma), \quad \|g_h\|_{\mathrm{H}^{1/2}(\Sigma)}^2 \lesssim \frac{1}{h}\|g_h\|_{\mathrm{L}^2(\Sigma)}^2.$$

Since using the bound (48), we already know that

$$\|\psi_h^{z_h^*} - \psi_{-,h}^{z_h^*}\|_{\mathrm{L}^2(\Sigma)}^2 \leq 2\, J_h^{\lambda_h}(z_h^*) \leq 2\mathrm{A}_h(z_J^*) + 2\lambda_h\|z_J^*\|_{\tilde{\varepsilon},\tilde{\mu}}^2,$$

it follows that $\|\psi_h^{z_h^*} - \psi_{-,h}^{z_h^*}\|_{\mathrm{H}^{1/2}(\Sigma)}$ converges to 0 when $h \to 0$. More generally (using still Eq. (22.39) loc. cit.), depending on the admissible range of values for q given by Corollary 6.6, convergence in $\mathrm{H}^\eta(\Sigma)$-norm can always be recovered for some $\eta \in (0, 1]$.

6.3. Relaxing the assumptions

In this case, the concept of limit regularity exponents is not meaningful anymore. On the other hand, if the solution exhibits extra regularity, one can still achieve results like those of Proposition 6.4, that is with bounds like $h^{\mathsf{s}'}$, for ad hoc $\mathsf{s}' > 0$. Then, mimicking the proof of Corollary 6.5, it follows that $\mathrm{A}_h(z_J^*)$ goes to 0 when $h \to 0$. Finally, taking $(\lambda_h)_h$ as in Remark 6.3 yields convergence.

In practice, we observe that, given $(\lambda_h)_h$ that goes to 0, one can solve numerically the discrete minimization problems (47) for h small enough: this is possible because we assume that (42) fits the coercive + compact framework with A_-^- and $\widetilde{\mathrm{A}}$ being isomorphisms. In this situation, if $(J_h(z_h^*))_h$ goes to 0 and if $(\|z_h^*\|_{\tilde{\varepsilon},\tilde{\mu}})_h$ remains bounded, then we can conclude exactly as in the proof of Theorem 6.2 that: $(z_h^*)_h$ converges to a minimizer of J, and that $(\psi_h^{z_h^*}, \psi_{-,h}^{z_h^*})_h$, converges to a pair $(\psi^\sharp, \psi_-^\sharp)$ such that $((\psi^\sharp)_{|\Omega_+}, \psi_-^\sharp) \in \mathrm{H}^1(\Omega_+) \times \mathrm{H}(\Omega_-)$ solves (21). In other words, $\psi \in \mathrm{H}^1(\Omega)$ defined by $\psi_{|\Omega_+} = (\psi^\sharp)_{|\Omega_+}$ and $\psi_{|\Omega_-} = \psi_-^\sharp$ actually solves (17), which proves (numerically) existence of a solution to this problem.

6.4. Numerical approximation of the electric field E

For the numerical approximation of the problem satisfied by φ, the steps are similar to the ones that we proposed to approximate the problem satisfied by ψ. The main difference lies in the choice of the inner product, which plays a crucial role in choosing the coefficient λ_h. Further details can be found in [19]. With these numerical tools, one can build an approximation of $\mathbf{E}$, called $\mathbf{E}_h$, in the form $\mathbf{E}_h = \nabla\varphi_h + \varepsilon^{-1}\,\mathbf{curl}\,\psi_h$, where $\varphi_h \in \mathrm{V}_h^k(\Omega) \cap \mathrm{H}_0^1(\Omega)$ is an approximation of $\varphi_{\mathbf{E}}$, and $\psi_h \in \mathrm{V}_h^k(\Omega)$ is an approximation of $\psi_{c,\mathbf{E}}$. For that, we approximate (16) first. Second, we compute an approximation $s_0^h \in \mathrm{V}_h^k(\Omega)$ of s_0 by discretizing the variational formulation (20). To that aim we use the discrete spaces $(\mathrm{V}_{h,zmv}^k(\Omega))_h$, where $\mathrm{V}_{h,zmv}^k(\Omega) = \mathrm{V}_h^k(\Omega) \cap \mathrm{H}_{zmv}^1(\Omega)$:

$$\begin{cases} \text{Find } s_0^h \in \mathrm{V}_{h,zmv}^k(\Omega) \text{ such that:} \\ (\mathbf{curl}\, s_0^h, \mathbf{curl}\,\psi_h')_{0,\Omega} = (\mathbf{f} + \varepsilon\,\nabla\varphi_h, \mathbf{curl}\,\psi_h')_{0,\Omega}, \quad \forall \psi_h' \in \mathrm{V}_{h,zmv}^k(\Omega). \end{cases} \tag{51}$$

Third, we approximate (19) with s_0 replaced by s_0^h in the right-hand side. According to Section 6.2, this yields convergence of $(\mathbf{E}_h)_h$ to $\mathbf{E}$ in $\mathbf{L}^2(\Omega)$-norm: $\lim_{h\to 0} \|\mathbf{E} - \mathbf{E}_h\|_{0,\Omega} = 0$.

In addition, with the help of the 2D Maxwell's equations, we know that $H_z = -\imath\omega^{-1}\mu^{-1}\,\mathrm{curl}\,\mathbf{E}$ is governed by another scalar problem with sign-changing coefficients. Applying the same machinery, it is possible to build approximations $(H_z^h)_h$ that converge to H_z in $\mathrm{H}^1(\Omega)$-norm. Hence, we find that $\lim_{h\to 0} \|\imath\omega\mu H_z^h - \mathrm{curl}\,\mathbf{E}\|_{1,\Omega} = 0$, which implies convergence for the vector-valued field $\mathbf{E}$ in the strong $\mathbf{H}(\mathrm{curl};\Omega)$-norm.

7. NUMERICAL ILLUSTRATIONS

In this section, we present some numerical illustrations of the proposed method, generated using the Freefem++ library. From an implementation perspective, we can solve the discrete minimization problems associated to ψ (and φ) in two ways: either by applying iterative algorithms like BFGS (Broyden–Fletcher–Goldfarb–Shanno) and NLCG (Nonlinear Conjugate Gradient), as proposed by [19], or by using an exact solver [1,13]. However, building the linear system is costly, so we prefer to use an iterative algorithm. Details and discussions can be found in [13]. We will employ the BFGS function to solve the minimization problems associated to $\varphi_{\mathbf{E}}$ and $\psi_{\mathbf{E}}$. In the chosen numerical experiments, we emphasize that both contributions $\nabla\varphi_{\mathbf{E}}$ and $\mathbf{curl}\,\psi_{\mathbf{E}}$ are non-vanishing.

A relevant question is why the direct approach, that is the discretization of problems (16) and (19), is not used to solve the problem. To address this issue, we will compare the results with the plain method (use of standard meshes) and with the T-coercivity method (use of specifically designed meshes). In particular, the numerical analysis is available for the latter method [8,15]. For the numerical tests proposed next, we assume that ε and μ are piecewise constant, *i.e.* constant in Ω_+ and in Ω_-. We introduce the contrast $\kappa_\varepsilon = \varepsilon_-/\varepsilon_+ < 0$.

7.1. *T*-coercivity and plain methods

As mentioned above, both methods rely on the discretization of problems (16), (20) and (19).

For the so-called T-coercivity method, that is with the use of *locally T-conform meshes*, one can derive estimates on the error between the exact and discrete electric fields. Locally T-conform meshes $(\mathcal{T}_h)_h$ fulfill the same assumptions on the meshes as in Section 6, with *additional geometrical constraints*: detailed meshing rules can be found in [8, 15], with illustrations; an example is presented in Figure 3. A limitation of the T-conform meshes is that they can be difficult to build in the presence of corners.

We note that if the data in (8) is such that $\mathbf{f} \in \mathbf{H}^{\mathtt{s}}(\Omega)$, then $g := \operatorname{div} \mathbf{f} \in \mathrm{H}^{-1+\mathtt{s}}(\Omega)$ with $\|g\|_{-1+\mathtt{s},\Omega} \le \|\mathbf{f}\|_{\mathtt{s},\Omega}$. So one can replace the assumption $g \in \mathrm{H}^{-1+\mathtt{s}}(\Omega)$ by $\mathbf{f} \in \mathbf{H}^{\mathtt{s}}(\Omega)$, and use $\|\mathbf{f}\|_{\mathtt{s},\Omega}$ in the bounds. To simplify further the presentation, we assume that $\mathbf{f} \in \mathbf{H}^{\tau_D(\varepsilon)}(\Omega)$, with the notation $|||\mathbf{f}||| = \|\mathbf{f}\|_{\tau_D(\varepsilon),\Omega}$.[4]

We use the discrete spaces $(\mathrm{V}^k_{h,0}(\Omega))_h$, where $\mathrm{V}^k_{h,0}(\Omega) = \mathrm{V}^k_h(\Omega) \cap \mathrm{H}^1_0(\Omega)$, to discretize (16):

$$\begin{cases} \text{Find } \varphi_h \in \mathrm{V}^k_{h,0}(\Omega) \text{ such that:} \\ (\varepsilon \nabla \varphi_h \nabla \varphi'_h)_{0,\Omega} = \langle g, \varphi'_h \rangle_{\mathrm{H}^{-1}(\Omega),\mathrm{H}^1_0(\Omega)}, \quad \forall \varphi'_h \in \mathrm{V}^k_{h,0}(\Omega). \end{cases} \tag{52}$$

One can perform the numerical analysis [8, 16] to find that, for all $\mathtt{t} \in [0, \tau_D(\varepsilon))$,

$$\exists h_0,\ \forall h < h_0, \quad \|\varphi_{\mathbf{E}} - \varphi_h\|_{1,\Omega} \lesssim h^{\mathtt{t}}\, |||\mathbf{f}|||. \tag{53}$$

We turn next to problem (20). Regarding the data, we observe that because ε has a jump at the interface, one gets only that $\varepsilon \nabla \varphi_{\mathbf{E}}$ belongs to $\mathrm{H}^{\mathtt{t}'}(\Omega)$ for $\mathtt{t}' \in [0, \tau_D^-(\varepsilon))$, where $\tau_D^-(\varepsilon) = \min(\tau_D(\varepsilon), 1/2)$. In this setting, since $\mathbf{f} \in \mathbf{H}^{\mathtt{t}'}(\Omega)$, one has $\mathbf{f} + \varepsilon \nabla \varphi_{\mathbf{E}} \in \mathbf{H}^{\mathtt{t}'}(\Omega)$. Because the auxiliary potential s_0 governed by (20) is also characterized by (18), we infer that $s_0 \in \mathrm{H}^{1+\mathtt{t}'}(\Omega)$. One is solving a classical Laplace problem with homogeneous Dirichlet boundary conditions, so the numerical analysis is completely standard. Comparing (51) to (20), for all $\mathtt{t} \in [0, \tau_D(\varepsilon))$, the consistency error is bounded by

$$\|\varepsilon(\nabla \varphi_{\mathbf{E}} - \nabla \varphi_h)\|_{0,\Omega} \le \|\varepsilon\|_{\mathrm{L}^\infty(\Omega)}\, \|\nabla \varphi_{\mathbf{E}} - \nabla \varphi_h\|_{0,\Omega} \lesssim h^{\mathtt{t}}\, |||\mathbf{f}|||.$$

Recall that $\tau_D^-(\varepsilon) \le \tau_D(\varepsilon)$, so the consistency error in the right-hand side (order $\tau_D(\varepsilon)$) is dominated by the interpolation error on the solution (order $\tau_D^-(\varepsilon)$), and we find that, for all $\mathtt{t}' \in [0, \tau_D^-(\varepsilon))$,

$$\forall h < h_0, \quad \|s_0 - s_0^h\|_{1,\Omega} \lesssim h^{\mathtt{t}'}\, |||\mathbf{f}|||.$$

Let $\tau_N \in]1/2, 1]$ be the limit regularity exponent for this classical problem with $\mathrm{L}^2(\Omega)$ right-hand sides. Using the Aubin-Nitsche lemma, we conclude that, for all $\mathtt{t}'' \in [0, \tau_N + \tau_D^-(\varepsilon))$,

$$\forall h < h_0, \quad \|s_0 - s_0^h\|_{0,\Omega} \lesssim h^{\mathtt{t}''}\, |||\mathbf{f}|||. \tag{54}$$

Finally, to discretize problem (19), we use the discrete spaces $(\mathrm{V}^k_h(\Omega))_h$:

$$\begin{cases} \text{Find } \psi_h \in \mathrm{V}^k_h(\Omega) \text{ such that:} \\ (\varepsilon^{-1} \nabla \psi_h, \nabla \psi'_h)_{0,\Omega} - \omega^2\, (\mu\, \psi_h, \psi'_h)_{0,\Omega} = \omega^2\, (\mu\, s_0^h, \psi'_h)_{0,\Omega}, \quad \forall \psi'_h \in \mathrm{V}^k_h(\Omega). \end{cases} \tag{55}$$

In the right-hand side s_0 is replaced by s_0^h, and the consistency error is bounded by, for all $\mathtt{t}'' \in [0, \tau_N + \tau_D^-(\varepsilon))$,

$$\|\mu(s_0 - s_0^h)\|_{0,\Omega} \lesssim h^{\mathtt{t}''}\, |||\mathbf{f}|||.$$

---

[4] Obviously, the numerical analysis is simpler if $g = 0$. In this case, $\varphi_{\mathbf{E}} = 0$, one chooses $\varphi^h_{\mathbf{E}} = 0$ as its (best) approximation, and one can consider that $\tau_D(\varepsilon) = 1$ in the results below.

One can perform the numerical analysis in the spirit of [8,16] (the Neumann boundary condition replacing the Dirichlet one). We find that, for all $\mathtt{t}''' \in [0, \min(\tau_N(\varepsilon^{-1}), \tau_N + \tau_D^-(\varepsilon)))$,

$$\exists h_0',\ \forall h < \min(h_0, h_0'),\quad \|\psi_{c,\mathbf{E}} - \psi_h\|_{1,\Omega} \lesssim h^{\mathtt{t}'''}\, |||\mathbf{f}|||. \tag{56}$$

Finally, with the T-conform meshes, the approximation of $\mathbf{E}$ is equal to $\mathbf{E}_h^T = \nabla\varphi_h + \varepsilon^{-1}\,\mathbf{curl}\,\psi_h \in \mathbf{L}^2(\Omega)$. Then it holds the

Theorem 7.1. *For all* $\mathtt{t} \in [0, \min(\tau_D(\varepsilon), \tau_N(\varepsilon^{-1})))$,

$$\exists h_0'',\ \forall h < h_0'',\quad \|\mathbf{E} - \mathbf{E}_h^T\|_{0,\Omega} \lesssim h^{\mathtt{t}}\, |||\mathbf{f}|||.$$

Proof. Observe that for all $\tau_D(\varepsilon) \in (0,1]$, one has $\tau_N + \tau_D^-(\varepsilon) > \tau_D(\varepsilon)$. So one just gathers the previous error estimates (53) and (56) and the claim follows. ☐

To conclude, the expected order of convergence is $\min(\tau_D(\varepsilon), \tau_N(\varepsilon^{-1})) \in (0,1]$.

Briefly, we call the *plain method* the same sequence of numerical algorithms with discrete spaces supported by standard meshes, meaning that are not necessarily T-conform (see Fig. 3 below for an illustration in a practical case). Up to our knowledge, when the coefficients are sign-changing, the numerical analysis is not available “in all generality”. In particular, it has not been proven that convergence is always achieved. This can be verified in some configurations that are proposed below.

7.2. Flat interface

We test the performance of our method in the case where the interface Σ is flat, setting the coefficients $\mu_- = -2$ and $\varepsilon_+ = \mu_+ = 1$, with their extensions defined as $\tilde{\varepsilon}|_{\Omega_-} = 10, \tilde{\mu}|_{\Omega_-} = -10$. We adjust ε_- to modify the value of the contrast κ_ε. We consider below the configuration where the domain Ω is a symmetric cavity, formed by the union of:

$$\Omega_+ = \{(x,y) \in (0;1/2) \times (0;1)\} \quad \text{and} \quad \Omega_- = \{(x,y) \in (1/2;1) \times (0;1)\}.$$

For such a configuration, problem (15) is well posed as soon as $\kappa_\varepsilon \neq -1$. We assume that is it also the case for problem (17) for the set of values $\kappa_\varepsilon \in \{-2, -1.01, -1.001\}$ and $\omega \in \{2, 9\}$ that we consider below for the experiments.

We define $\omega^2\mathbf{f}$ as the source term associated to the solution:

$$\mathbf{E}(x,y)|_{\Omega_+} = \begin{bmatrix} (2x+b)\sin(\pi y) + y^3(1-y)^3 \\ \pi(x^2+bx)\cos(\pi y) + \sin(2\pi x)^3 \end{bmatrix}, \quad \mathbf{E}(x,y)|_{\Omega_-} = \begin{bmatrix} a\sin(\pi y) + y^3(1-y)^3 \\ \pi a(x-1)\cos(\pi y) + \sin(2\pi x)^3 \end{bmatrix},$$

where $a = \dfrac{1}{2(\kappa_\varepsilon + 1)}$ and $b = -\dfrac{\kappa_\varepsilon + 2}{2(\kappa_\varepsilon + 1)}$.

How to choose λ_h? Taking into account the geometry of the domain and the piecewise constant value of $\tilde{\varepsilon}$, we note that the limit regularity exponents τ and $\tilde{\tau}$ defined in Section 6.2 are both equal to -1. Hence, we observe that our discretized optimal control method should converge if one chooses $\lambda_h = c\,h^q$ with $q \in (0;3)$ (*cf.* Cor. 6.6). Specifically, for $\omega = 2$, we set $\lambda_h = 0.05h^2, 0.05h^{2.3}, 0.05h^{2.5}$ respectively for $\kappa_\varepsilon = -2, -1.01, -1.001$; and for $\omega = 9$, we set $\lambda_h = 0.05h^{2.5}, 0.05h^2, 0.05h^2$ respectively for $\kappa_\varepsilon = -2, -1.01, -1.001$. The relative error between the exact and numerical solutions, equal to

$$e_h = \frac{\|\mathbf{E} - \mathbf{E}_h\|_{0,\Omega}}{\|\mathbf{E}\|_{0,\Omega}} \tag{57}$$

is plotted in log-log scale in Figure 4.

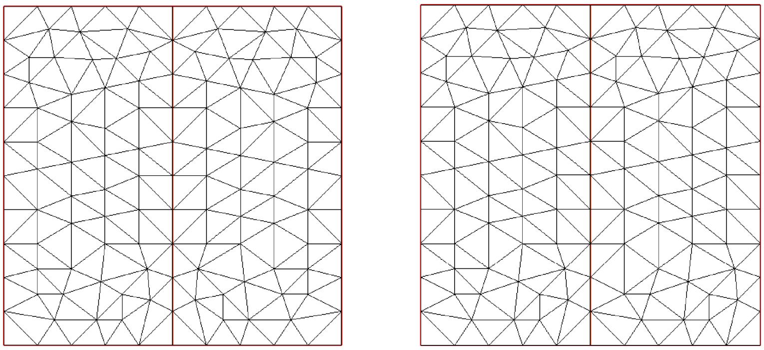

FIGURE 3. *Left*: T-conform mesh (symmetric with respect to Σ); *Right*: plain mesh.

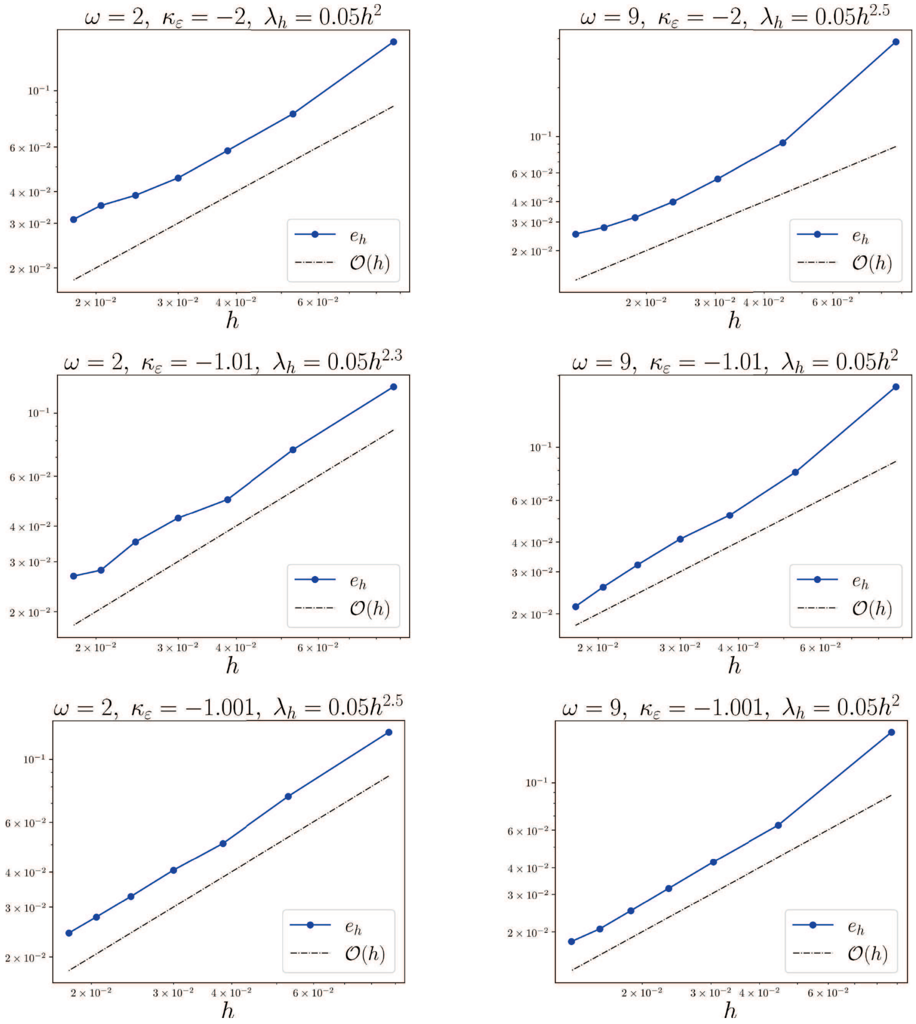

FIGURE 4. Behavior of the relative error with respect to the mesh size h.

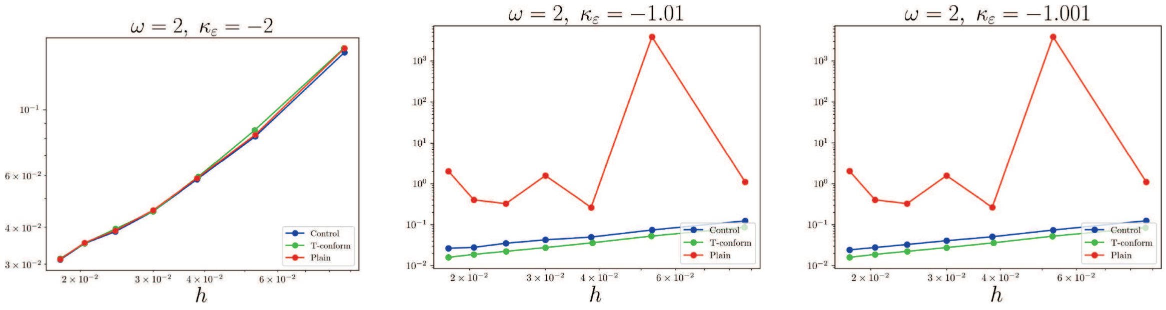

FIGURE 5. Comparison of the three relative errors (control, T-conform, plain) with respect to the mesh size h.

For our method, we use the same standard meshes as in the plain method. On the other hand, T-conform meshes are meshes that are (at least locally) symmetric with respect to the interface Σ (see Fig. 3 for an example). For this method, the expected order of convergence of Theorem 7.1 is equal to 1, since $\tau_D(\varepsilon) = \tau_N(\varepsilon^{-1}) = 1$, *cf.* [15].

For $\omega = 2$, we remark that our method converges at roughly the same rate as the T-conforming meshes method for $\kappa_\varepsilon = -2$, although with a slightly higher error level when the contrast is closer to -1 (see Fig. 5). In these cases (and although the meshes presented in Fig. 3 look quite similar) we note that the plain method fails to converge for $\kappa_\varepsilon \in \{-1.01, -1.001\}$.

7.3. Circular interface

We now test the performance of our method in the case where the interface Σ is circular. We consider the case where the domains Ω_+, Ω_- are equal to

$$\Omega_+ = \{(x,y) \in \mathbb{R}^2 \mid 1 < |(x,y)|_2 < 2\} \text{ and } \Omega_- = \{(x,y) \in \mathbb{R}^2 \mid 2 < |(x,y)|_2 < 3\}.$$

Specifically, Ω_+ refers to the smaller ring with radii ranging from 1 to 2, while Ω_- represents the larger ring with radii between 2 and 3. We set again the coefficients $\mu_- = -2$ and $\varepsilon_+ = \mu_+ = 1$, with extensions defined as $\tilde{\varepsilon}|_{\Omega_-} = 10, \tilde{\mu}|_{\Omega_-} = -10$. In this case, the boundary $\partial\Omega$ is not connected. This situation is handled in Appendix A. We assume below that problem (15) and problem (17) (with $\omega = 2$) are well-posed for the set of values $\kappa_\varepsilon \in \{-2, -1.001\}$.

We define $\omega^2\mathbf{f}$ as the source term associated to the solution:

$$\mathbf{E}(x,y)|_{\Omega_+} = \begin{bmatrix} (2a_1 r + a_2)\cos(\theta) - (r-1)(r-3)(r-2)^3\sin(\theta) \\ (2a_1 r + a_2)\sin(\theta) + (r-1)(r-3)(r-2)^3\cos(\theta) \end{bmatrix},$$

$$\mathbf{E}(x,y)|_{\Omega_-} = \begin{bmatrix} 2a_3(r-3)\cos(\theta) - (r-1)(r-3)(r-2)^3\sin(\theta) \\ 2a_3(r-3)\sin(\theta) + (r-1)(r-3)(r-2)^3\cos(\theta) \end{bmatrix},$$

where (r,θ) are the polar coordinates, and $a_1 = \dfrac{-1-2\kappa_\varepsilon}{3+4\kappa_\varepsilon}$, $a_2 = 1 - a_1$, and $a_3 = 4a_1 + 2a_2 - 1$. An example of mesh is given in Figure 6. We note that, due to the geometry, triangular meshes are non-conforming with the two components of the boundary, nor with the interface.

How to choose λ_h? Because of the smoothness of the boundary and interface, and because the coefficients are piecewise smooth, we find again that $\tau = \tilde{\tau} = 1$. Hence, $\lambda_h = c\,h^q$ with $q \in (0;3)$ (*cf.* Cor. 6.6) is admissible. We choose $\lambda_h = 0.002h^2$ for $\kappa_\varepsilon = -2$, and $\lambda_h = 0.05h^{2.5}$ for $\kappa_\varepsilon = -1.001$. We report in Figure 7 the behavior of the relative error (57), with convergence rates exceeding expectations.

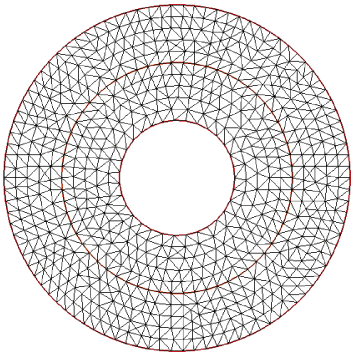

FIGURE 6. An example of standard mesh.

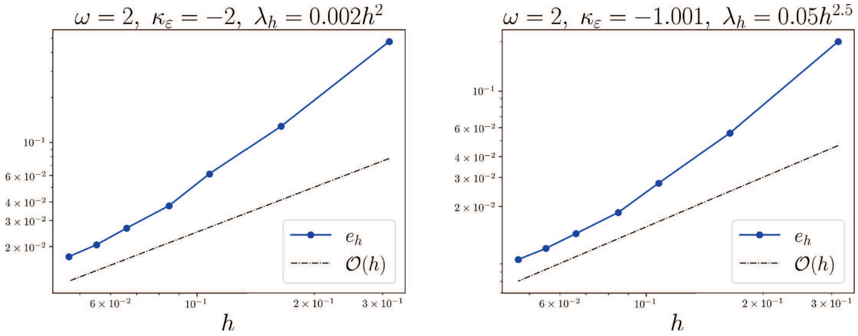

FIGURE 7. Behavior of the relative error of the volume optimal control method with respect to the mesh size h.

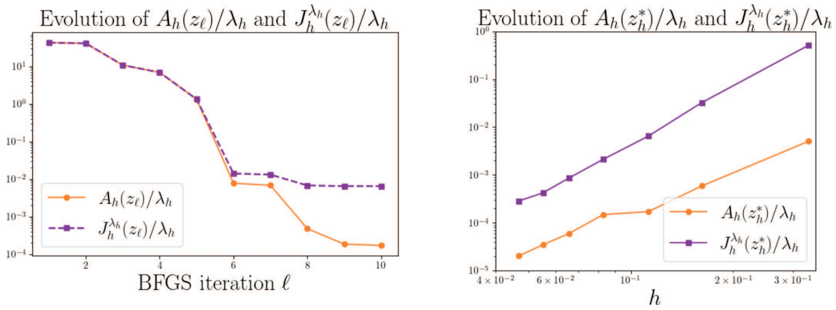

FIGURE 8. Numerical convergence. *Left*: For fixed h, performing the BFGS iterations. *Right*: For varying h.

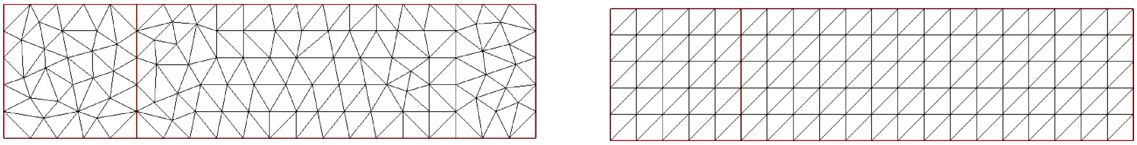

FIGURE 9. *Left*: unstructured mesh; *Right*: structured mesh.

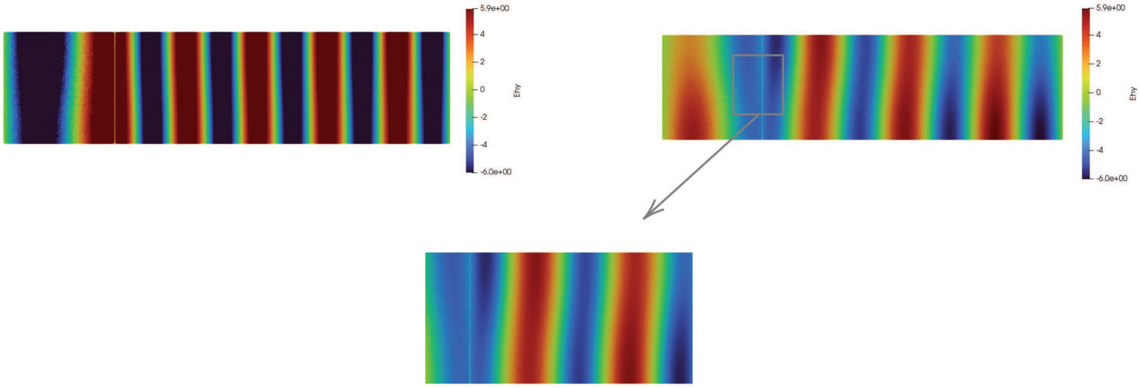

FIGURE 10. The y-component of the approximate solution $\mathbf{E}_h$ obtained with the volume optimal control method (*Left*: $\kappa_\varepsilon = -2$; *Right*: $\kappa_\varepsilon = -1$).

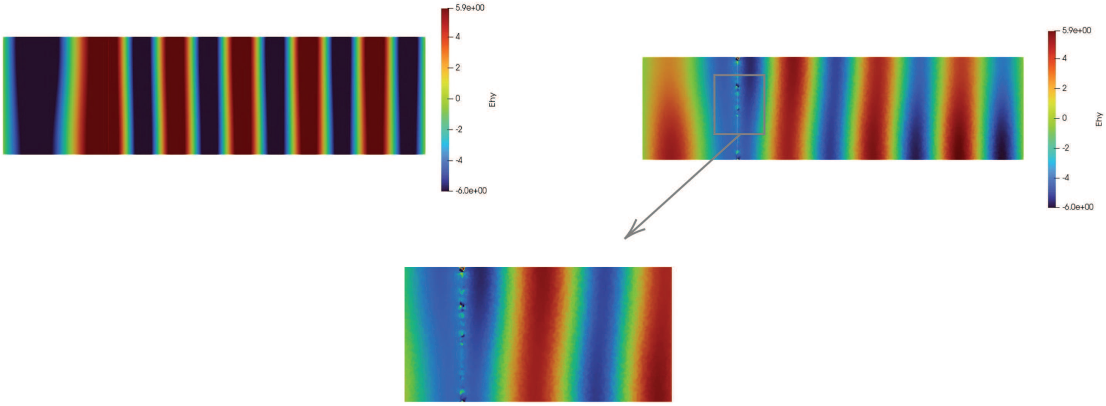

FIGURE 11. The y-component of the approximate solution $\mathbf{E}_h$ obtained with the T-coercivity method (*Left*: $\kappa_\varepsilon = -2$; *Right*: $\kappa_\varepsilon = -1$).

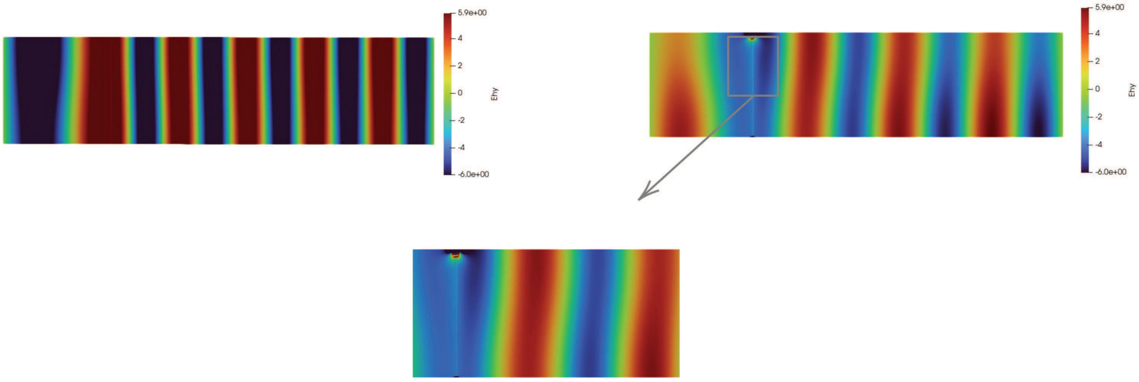

FIGURE 12. The y-component of the approximate solution $\mathbf{E}_h$ obtained with the plain method (*Left*: $\kappa_\varepsilon = -2$; *Right*: $\kappa_\varepsilon = -1$).

To numerically verify the statements in Remark 6.3, we plot in Figure 8(Left) the evolution of the quantities $A_h(z_\ell)/\lambda_h$ and $J_h^{\lambda_h}(z_\ell)/\lambda_h$ over the BFGS iterations ($\ell = 1, \ldots, 10$) for a fixed mesh size $h = 0.112656$. The plots show that these quantities start from relatively large values and decrease progressively, illustrating that the BFGS algorithm converges toward the control z_h^*. While in Figure 8(Right), we plot $A_h(z_h^*)/\lambda_h$ and $J_h^{\lambda_h}(z_h^*)/\lambda_h$ as h decreases. Both quantities tend to zero, confirming the behavior stated in Remark 6.3.

7.4. A non-symmetrical domain with a localized source

In this section, we provide a numerical validation of the volume optimal control method under the weakened assumptions of Section 4.4. We consider the case of a $2D$ non-symmetric cavity with a flat interface as presented in Section 3.3 of [15]. This cavity is formed by the union of two disjoint subdomains:

$$\Omega_+ = \{(x,y) \in (-1;0) \times (0;1)\} \quad \text{and} \quad \Omega_- = \{(x,y) \in (0;3) \times (0;1)\}.$$

We use either unstructured meshes (for the plain and the volume optimal control methods) or T-conform meshes (for the T-coercivity method) (see Fig. 9).

In this configuration, we choose the frequency $\omega = 5$ and the coefficients $\mu_- = -2$ and $\varepsilon_+ = \mu_+ = 1$, with their extensions defined as $\tilde{\varepsilon}|_{\Omega_-} = 1, \tilde{\mu}|_{\Omega_-} = -1$.

We also consider a localized source term $\omega^2\mathbf{f}$, given by:

$$\omega^2\mathbf{f}(x,y) = \begin{cases} \omega^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \text{if } (x,y) \in (-0.8;-0.5) \times (0;1), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

We recall that, as mentioned in Remark 4.11, the operators associated with the scalar problems are injective but not Fredholm for $\kappa_\varepsilon = -1$ and $\omega = 0$. We assume that this property also holds for $\kappa_\varepsilon \in \{-2, -1\}$ and $\omega = 5$. Similarly to the case discussed in Section 7.2, the volume optimal control method converges when $\lambda_h = C\,h^q$ with $q \in (0, 3)$. Specifically, we choose $\lambda_h = 0.002h^2$ for $\kappa_\varepsilon \in \{-2, -1\}$.

We plot respectively in Figures 10, 11 and 12, the y-component of the approximated solution $\mathbf{E}_h$ with the volume optimal control, T-coercivity, and plain methods, for $\kappa_\varepsilon \in \{-2, -1\}$. We observe that for $\kappa_\varepsilon = -2$, all methods exhibit similar and stable behavior. In contrast, for $\kappa_\varepsilon = -1$, the plain and T-coercivity methods show instabilities near the interface, while the volume optimal control method remains stable.

8. CONCLUDING REMARKS

In this paper, we generalized the volume based control method of [19] to 2D time harmonic Maxwell problem with sign-changing coefficients. As the method involves solving three scalar problems, classical Lagrange finite elements were used for discretization. Among others, we obtained convergence for the electric field $\mathbf{E}$ in the strong $\mathbf{H}(\text{curl}; \Omega)$-norm.

In addition, one of the key advantages of the volume-based control method is that it only requires uniqueness, and existence of the solution for the given data. A second one is that there is no *a priori* regularity assumption on the solutions, contrary to the surface-based control method of [1, 2].

For the numerical approximation, the claim that one can use any mesh is supported by the numerical experiments. In addition, the convergence rates match those obtained with a direct solver using T-conform meshes (for which the convergence rates are known).

Regarding the numerical analysis, it is still incomplete, although some partial results have been obtained (global convergence, convergence of the jump on the interface in $\mathrm{H}^\eta(\Sigma)$-norm for some $\eta > 0, \ldots$).

Finally, the natural continuation of this work is to apply this approach to the 3D time-harmonic Maxwell equations with sign changing coefficients. As a matter of fact, the numerical analysis has been carried out for one sign changing coefficient only [16]. However, unlike the 2D case, the decomposition of the electric field in 3D leads to a rotational problem with a vectorial unknown rather than a scalar one. Precisely, $\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{\psi}_{\mathbf{E}}$, where the vectorial unknown $\boldsymbol{\psi}_{\mathbf{E}}$ remains, like $\mathbf{E}$, governed by equations with sign-changing coefficients. Hence, a volume optimal control method has to be devised for the electric field itself. Consequently, the discretization ought to be performed with edge finite elements. We refer to [13].

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APPENDIX A. CASE OF A BOUNDARY THAT IS NOT CONNECTED

Let us consider the case of domain Ω of $\mathbb{R}^2$ with a boundary $\partial\Omega$ that is not connected: in other words, there exist at least two distinct maximal connected components $(\Gamma_k)_{k=0,K}$ of $\partial\Omega$. Define

$$\mathrm{H}^1_{\partial\Omega}(\Omega) := \{\varphi \in \mathrm{H}^1(\Omega) \,|\, \varphi_{|\Gamma_0} = 0 \text{ and } \varphi_{|\Gamma_k} \in \mathbb{R} \text{ for } k = 1, K\},$$

and, for $k = 1, K$, we let $q_k \in \mathrm{H}^1_{\partial\Omega}(\Omega)$ such that $\Delta q_k = 0$ in Ω and $q_{k\,|\Gamma_m} = \delta_{km}$ for $m = 0, K$. Classically,

$$\mathrm{H}^1_{\partial\Omega}(\Omega) = \mathrm{H}^1_0(\Omega) \oplus \mathrm{span}_{k=1,K}(q_k).$$

For a boundary that is not connected, the statement of Theorem 2.2 is replaced by

Theorem A.1. *Let Ω be a domain of $\mathbb{R}^2$.*

Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$, then

$$\operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \langle \mathbf{v}\cdot\boldsymbol{\nu}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma_k),\mathrm{H}^{1/2}(\Gamma_k)} = 0\ \forall k \iff \exists \psi \in \mathrm{H}^1(\Omega) \text{ s.t. } \mathbf{v} = \mathbf{curl}\,\psi \text{ in } \Omega. \tag{A.1}$$

$$\operatorname{curl} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\boldsymbol{\tau} = 0 \text{ on } \partial\Omega \iff \exists \varphi \in \mathrm{H}^1_{\partial\Omega}(\Omega) \text{ s.t. } \mathbf{v} = \nabla\varphi \text{ in } \Omega. \tag{A.2}$$

Moreover, the scalar potential ψ is unique up to a constant, and the scalar potential ϕ is unique.

In Subsection 3.1, going back to the model, one can now use the equivalent variational formulation (6) with the test-field $\mathbf{v} = \nabla\varphi'$ for any $\varphi' \in \mathrm{H}^1_{\partial\Omega}(\Omega)$:

$$-\omega^2(\varepsilon\,\mathbf{E}, \nabla\varphi')_{0,\Omega} = \imath\,\omega\,(\mathbf{J}, \nabla\varphi')_{0,\Omega}, \quad \forall\varphi' \in \mathrm{H}^1_{\partial\Omega}(\Omega). \tag{A.3}$$

In other words, $\varphi' \mapsto (\mathbf{J}, \nabla\varphi')_{0,\Omega}$ is an element of $\big(\mathrm{H}^1_{\partial\Omega}(\Omega)\big)'$.

If $\operatorname{div}\mathbf{J} \in \mathrm{L}^2(\Omega)$, then taking $\varphi' \in \mathrm{H}^1_0(\Omega)$, we find $\operatorname{div}(\varepsilon\mathbf{E}) = -\imath\,\omega^{-1}\operatorname{div}\mathbf{J}$ a.e. in Ω. Next, for $k = 1, K$, taking $\varphi' = q_k$ and integrating by parts yields

$$\imath\,\omega^{-1}(\mathbf{J}, \nabla q_k)_{0,\Omega} = -\imath\,\omega^{-1}(\operatorname{div}\mathbf{J}, q_k)_{0,\Omega} - \langle(\varepsilon\,\mathbf{E})\cdot\boldsymbol{\nu}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma_k),\mathrm{H}^{1/2}(\Gamma_k)},$$

that is $\langle(\varepsilon\,\mathbf{E})\cdot\boldsymbol{\nu}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma_k),\mathrm{H}^{1/2}(\Gamma_k)} = -\imath\,\omega^{-1}\big((\operatorname{div}\mathbf{J}, q_k)_{0,\Omega} + (\mathbf{J}, \nabla q_k)_{0,\Omega}\big)$. In other words, there are K additional relations. From the physics point of view, for $k = 1, K$, this corresponds to the flux of $\mathbf{D} = \varepsilon\,\mathbf{E}$ through the connected component Γ_k of the boundary, equal to the total surface charges on Γ_k.

To proceed, the assumption on the companion scalar problem (9) remains the same, namely that it is well-posed (*cf.* Assumption 1). Under this assumption, one has the (see *e.g.* [6], Prop. 8.9).

Lemma A.2. *Let $k = 1, K$, then there exists one, and only one, solution to the problem:*

$$\begin{cases}\text{Find } q_k^\varepsilon \in \mathrm{H}^1_{\partial\Omega}(\Omega) \text{ such that :}\\ \operatorname{div}(\varepsilon\nabla q_k^\varepsilon) = 0 \text{ in } \Omega, \quad q^\varepsilon_{k|\Gamma_m} = \delta_{km} \text{ for } m = 0, K.\end{cases} \tag{A.4}$$

In addition, $(q_k^\varepsilon)_{k=1,K}$ is a free family, and

$$\mathrm{H}^1_{\partial\Omega}(\Omega) = \mathrm{H}^1_0(\Omega) \oplus \operatorname{span}_{k=1,K}(q_k^\varepsilon). \tag{A.5}$$

Note that by reformulating (A.4) variationally, we find that $(\varepsilon\nabla q_k^\varepsilon, \nabla q_0)_{0,\Omega} = 0$, for all $q_0 \in \mathrm{H}^1_0(\Omega)$ and all $k = 1, K$. To complement the Assumption 1 on problem (9), we assume that the so-called capacitance matrix of $\mathbb{R}^{K\times K}$ with entries $(\varepsilon\nabla q_m^\varepsilon, \nabla q_k^\varepsilon)_{0,\Omega}$ for $k, m = 1, K$, is invertible.

Under those two assumptions, one can still achieve Helmholtz decompositions in a domain with a boundary that is not connected.

Proposition A.3. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathrm{H}^1_{\partial\Omega}(\Omega)] \oplus \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon), \tag{A.6}$$

with $\mathbf{A}_{\partial\Omega}(\Omega, \varepsilon) := \{\mathbf{w} \in \mathbf{L}^2(\Omega) \,|\, \forall\varphi \in \mathrm{H}^1_{\partial\Omega}(\Omega),\ (\varepsilon\mathbf{w}, \nabla\varphi)_{0,\Omega} = 0\}$. In addition, the linear map $\mathbf{v} \mapsto (\varphi, \mathbf{w})$ with $\mathbf{v} = \nabla\varphi + \mathbf{w}$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathrm{H}^1_{\partial\Omega}(\Omega) \times \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon)$.

Proof. Obviously, $\nabla[\mathrm{H}^1_{\partial\Omega}(\Omega)] + \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon) \subseteq \mathbf{L}^2(\Omega)$. Let us now show the converse imbedding. Let $\mathbf{v} \in \mathbf{L}^2(\Omega)$. Since problem (9) is well-posed, there exists $\varphi_0 \in \mathrm{H}^1_0(\Omega)$ such that:

$$(\varepsilon\nabla\varphi_0, \nabla\varphi')_{0,\Omega} = (\varepsilon\mathbf{v}, \nabla\varphi')_{0,\Omega}, \quad \forall\varphi' \in \mathrm{H}^1_0(\Omega).$$

And, since the capacitance matrix is invertible, there exists $\varphi_{\partial\Omega} \in \operatorname{span}_{k=1,K}(q_k^\varepsilon)$ such that:

$$(\varepsilon\nabla\varphi_{\partial\Omega}, \nabla q_m^\varepsilon)_{0,\Omega} = (\varepsilon\mathbf{v}, \nabla q_m^\varepsilon)_{0,\Omega}, \quad \forall m = 1, K.$$

Let $\varphi = \varphi_0 + \varphi_{\partial\Omega} \in \mathrm{H}^1_{\partial\Omega}(\Omega)$, we pose $\mathbf{w} = \mathbf{v} - \nabla\varphi$. Using the decomposition (A.5), we find that $\mathbf{w} \in \mathbf{A}_{\partial\Omega}(\Omega, \varepsilon)$, and the end of the proof is similar to that of the proof of Proposition 4.2. ☐

Using Theorem A.1, one obtains another Helmholtz decomposition.

Proposition A.4. *It holds that*

$$\mathbf{L}^2(\Omega) = \nabla[\mathrm{H}^1_{\partial\Omega}(\Omega)] \oplus \varepsilon^{-1}\,\mathbf{curl}[\mathrm{H}^1_{zmv}(\Omega)]. \tag{A.7}$$

In addition, the linear map $\mathbf{v} \mapsto (\varphi, \psi)$ with $\mathbf{v} = \nabla\varphi + \varepsilon^{-1}\,\mathbf{curl}\,\psi$ is continuous from $\mathbf{L}^2(\Omega)$ to $\mathrm{H}^1_{\partial\Omega}(\Omega) \times \mathrm{H}^1_{zmv}(\Omega)$.

Proof. According to Theorem A.1, one has $\mathbf{A}_{\partial\Omega}(\Omega,\varepsilon) = \varepsilon^{-1}\,\mathbf{curl}[\mathrm{H}^1_{zmv}(\Omega)]$, because $\mathbf{w} \in \mathbf{A}_{\partial\Omega}(\Omega,\varepsilon)$ is equivalent to $\operatorname{div}\varepsilon\mathbf{w} = 0$ in Ω, and $\langle \varepsilon\mathbf{w}\cdot\boldsymbol{\nu}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma_k),\mathrm{H}^{1/2}(\Gamma_k)} = 0$ for $k = 0, K$. The remainder of the proof is completely similar to that of the proof of Proposition 4.4. □

We note that the property (13) still holds for all $\varphi \in \mathrm{H}^1_{\partial\Omega}(\Omega)$ and $\psi \in \mathrm{H}^1_{zmv}(\Omega)$.

Moreover, one can again decompose the electric field $\mathbf{E}$ that is governed by (8):

$$\exists!(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}}) \in \mathrm{H}^1_{\partial\Omega}(\Omega) \times \mathrm{H}^1_{zmv}(\Omega) \text{ such that } \mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{0,\mathbf{E}} \text{ in } \Omega. \tag{A.8}$$

To characterize the two scalar potentials $(\varphi_{\mathbf{E}}, \psi_{0,\mathbf{E}})$, we proceed as before. First using the property (13), we find the

Proposition A.5. *Let* $\mathbf{E}$*, a solution of* (8)*, be decomposed as in* (A.8)*. Then its scalar potential* $\varphi_{\mathbf{E}} \in \mathrm{H}^1_{\partial\Omega}(\Omega)$ *is governed by:*

$$\begin{cases} \text{Find } \varphi_{\mathbf{E}} \in \mathrm{H}^1_{\partial\Omega}(\Omega) \text{ such that :} \\ (\varepsilon\nabla\varphi_{\mathbf{E}}, \nabla\varphi')_{0,\Omega} = -(\mathbf{f}, \nabla\varphi')_{0,\Omega} \quad \forall\varphi' \in \mathrm{H}^1_{\partial\Omega}(\Omega). \end{cases} \tag{A.9}$$

By construction, $\mathbf{w} = \varepsilon\nabla\varphi_{\mathbf{E}} + \mathbf{f}$ is such that

$$\operatorname{div}\mathbf{w} = 0 \text{ in } \Omega, \text{ and } \langle \mathbf{w}\cdot\boldsymbol{\nu}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma_k),\mathrm{H}^{1/2}(\Gamma_k)} = 0 \text{ for } k = 0, K. \tag{A.10}$$

Then, to characterize the scalar potential $\psi_{0,\mathbf{E}} \in \mathrm{H}^1_{zmv}(\Omega)$, the result is identical to that of Proposition 4.6, with the auxiliary potential s_0 given by (18). Finally, one has to check that the scalar potential s_0 can still be characterized variationally when the boundary is not connected.

Lemma A.6. *The auxiliary potential* s_0 *solves* (18) *if, and only if, it solves* (20)*.*

Proof. If s_0 solves (18), then it obviously solves (20).

Reciprocally, if s_0 solves (20) then proceeding as in the proof of Lemma 4.7, we find first that $\operatorname{curl}(\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}}) = 0$, and second $(\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}})\cdot\boldsymbol{\tau} = 0$ on $\partial\Omega$. According to (A.2), there exists $z \in \mathrm{H}^1_{\partial\Omega}(\Omega)$ such that $\mathbf{curl}\,s_0 - \mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}} = \nabla z$ in Ω. By construction,

$$\Delta z = \operatorname{div}(\mathbf{f} - \varepsilon\,\nabla\varphi_{\mathbf{E}}) = 0 \text{ in } \Omega,$$

so that $z \in \operatorname{span}_{k=1,K}(q_k)$. Integrating by parts (2) and (1), we find that for $k = 1, K$

$$(\nabla z, \nabla q_k)_{0,\Omega} = (\mathbf{curl}\,s_0, \nabla q_k)_{0,\Omega} - (\mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}, \nabla q_k)_{0,\Omega} = 0 - \langle \mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}\cdot\boldsymbol{\nu}, 1\rangle_{\mathrm{H}^{-1/2}(\Gamma_k),\mathrm{H}^{1/2}(\Gamma_k)} = 0,$$

where we used (A.10) to evaluate the last term. Hence, we conclude that $z = 0$ so that $\mathbf{curl}\,s_0 = \mathbf{f} + \varepsilon\,\nabla\varphi_{\mathbf{E}}$, *i.e.* s_0 remains characterized by (18). □

We can conclude as before.

Theorem A.7. *Let* $\mathbf{E} \in \mathbf{L}^2(\Omega)$ *be governed by* (8)*. Then there exist two scalar potentials* $\varphi_{\mathbf{E}} \in \mathrm{H}^1_{\partial\Omega}(\Omega)$ *and* $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ *such that* $\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}}$ *in* Ω*, where* $\varphi_{\mathbf{E}}$ *is a solution to* (A.9)*, resp.* $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ *is a solution to* (17)*.*

Conversely, Let $\varphi_{\mathbf{E}} \in \mathrm{H}^1_{\partial\Omega}(\Omega)$ *be governed by* (A.9)*, the auxiliary field* s_0 *be characterized by* (18)*, and* $\psi_{c,\mathbf{E}} \in \mathrm{H}^1(\Omega)$ *be governed by* (17)*. Then* $\mathbf{E} = \nabla\varphi_{\mathbf{E}} + \varepsilon^{-1}\,\mathbf{curl}\,\psi_{c,\mathbf{E}} \in \mathbf{L}^2(\Omega)$ *is a solution to* (8)*. Finally, the problem* (8) *is well-posed.*

In a domain Ω with a boundary that is not connected, we note that we have been able to characterize the two scalar potentials of the electric field, and the auxiliary potential s_0, variationally. It follows that one can discretize them as before, using either the volume optimal control method presented in Section 5, or the T-conform meshes presented in Section 7.1. The only difference lies in the definition of the discrete spaces for *approximating* $\varphi_{\mathbf{E}}$*, now equal to* $\mathrm{V}^k_h(\Omega) \cap \mathrm{H}^1_{\partial\Omega}(\Omega)$. The rest of the numerical analysis and implementation is as before.