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DEEP RELU NEURAL NETWORK EMULATION IN HIGH-FREQUENCY ACOUSTIC SCATTERING[★]

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Abstract. We obtain wavenumber-robust approximation error bounds for the deep neural network (DNN) emulation of the solution to the time-harmonic, sound-soft acoustic scattering problem in the exterior of a smooth, convex obstacle in two physical dimensions. The approximation error bounds are proved *via* a boundary reduction of the scattering problem in the unbounded exterior region to its smooth, curved boundary Γ using the so-called combined field integral equation (CFIE), a well-posed, second-kind boundary integral equation (BIE) for the field's Neumann datum on Γ. In this setting, the continuity and stability constants of this formulation are explicit in terms of the (non-dimensional) wavenumber κ. Using known, wavenumber-explicit asymptotics of the Neumann datum of the BIE, we develop a wavenumber-explicit DNN approximation rate bound for the surface density of the BIE. Our DNN approximation is based on fixed-width, deep, fully connected feedforward NNs with strict Rectified Linear Unit (ReLU) activation: the approximation architecture is, in particular, geometry- and wavenumber-independent. We prove the existence of DNNs with an ϵ-error bound in the $L^\infty(\Gamma)$-norm having fixed width and depth that increases *algebraically, with arbitrary small exponent* in terms of the target accuracy $\epsilon > 0$. We also show that for fixed target accuracy $\epsilon > 0$, the depth of these NNs should increase *poly-logarithmically* with respect to the wavenumber κ. Unlike current computational approaches such as wavenumber-adapted versions of the Galerkin Boundary Element Method (BEM) with shape- and wavenumber-tailored solution *ansatz* spaces, the presently considered DNN approximation architectures do not require prior analytic information about the scatterer's shape: wavenumber- and geometry-dependent solution features can be *learned* accurately with moderate-sized, feedforward ReLU NNs.

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1. Introduction

In recent years, the use of DNNs for the numerical approximation of PDEs has emerged as an alternative to well-established, "classical" discretization techniques such as Finite Element-, Finite Difference- and Finite Volume methods. The success of DNNs in the field of computational mathematics is arguably based on three

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[★]Dedicated to Professor Wolfgang L. Wendland on the occasion of his 90th birthday.

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main pillars: (i) universality results (see, *e.g.*, [29] and references therein), (ii) considerably stronger approximation properties compared to classical linear approximation methods or nonlinear (adaptive) approximation schemes, and (iii) the intense development of dedicated machine learning algorithms and computer hardware for DNN-based computations.

Mounting computational evidence together with theoretical insights shows practical relevance of superior expressivity properties of DNNs. This comes at the price of a *nonlinear dependence of the DNNs upon their hidden parameters*. This renders the solution approximation by DNNs of linear PDEs and BIEs a nonlinear, finite-parametric problem which must be solved numerically in an iterative fashion during the so-called "DNN training" phase. Therefore, the question arises if, in particular applications, the potential gain afforded by employing DNN-based approximation architectures for numerical approximation of solutions is so high as to justify the use of nonlinear optimization in the computation of the DNN approximation (as compared to a linear system solve in, *e.g.*, Galerkin type methods based on linear subspaces).

1.1. Previous work on numerical high-frequency scattering

Due to its large importance in applications, the numerical solution of exterior, time-harmonic acoustic and electromagnetic scattering problems for large wavenumber κ has received considerable attention by engineers and by numerical analysts. A large body of literature is available. First, as $\kappa \to \infty$, it is classical that the solution of the problem tends to the so-called "geometric optics" limit, the solution of which is governed by the Eikonal equation, see, *e.g.*, Chapter X of [35] and [26] and the references there. The Eikonal equation is a Hamilton–Jacobi PDE which, in general, must be treated numerically. For large, but finite κ, asymptotic expansions w.r. to $1/\kappa$ of the solution have been obtained, and justified, for smooth, convex scatterers [22, 24]. These works provide recursive expressions for the terms appearing in the asymptotic expansions w.r. to $1/\kappa \downarrow 0$ of the solution, in terms of powers of $\kappa^{\frac{1}{3}}$, for the solution of the exterior problem, in any space dimension $d \geq 2$. These results indicate that in the case of convex scatterers, and for large κ, the solution comprises a piecewise smooth part plus oscillatory terms whose specific structure amounts to a smooth modulation of oscillatory functions depending in a nontrivial way on the scatterer's boundary Γ. This indicates the multiscale nature of the problem, and immediately implies a *scale resolution* requirement for the FEM and BEM. These conditions are typically of the type $\kappa^2 h < 1$ for low order FE or of the type $\kappa h < 1$ for BE. This kind of requirement is prohibitive for large values of κ, especially in space dimension $d = 3$. High order methods of polynomial degree $p > 1$ mitigate scale resolution requirements to $\kappa h/p < 1$. See, *e.g.*, [5, 18].

Early attempts to alleviate this scale resolution requirement by incorporating information on the asymptotic expansions into the numerical schemes include [7] for an approach in connection with Galerkin BEM. We refer to [5] for a survey of subsequent developments and references.

The past decade has seen intense developments in the numerical analysis of FE and BE methods for the Helmholtz equation at high wavenumber. For further details, we refer to *e.g.*, [19, 20], and also to [5] and the references there. For boundary integral formulations, it was shown that several critical issues in FE discretizations such as dispersion, and scale resolution are either absent, or strongly mitigated. We refer to [5, 14] and the references therein for a detailed discussion. Boundary reduction and BIE-reformulation of the acoustic scattering problem is also the basis for the present results.

1.2. Contributions

We investigate the wavenumber-robustness of a NN-based numerical approximation for time-harmonic, sound-soft, acoustic scattering in the exterior $\mathbb{R}^2\backslash\overline{\mathrm{D}}$ of a bounded scatterer occupying a smooth, bounded domain D. Adopting a (standard, for this type of problem, see, *e.g.*, [5, 31]) suitable (in particular, resonance-free) reduction to a boundary integral equation on the closed curve $\Gamma = \partial\mathrm{D}$ implies satisfaction of the Sommerfeld radiation condition at infinity, and of the homogeneous Helmholtz PDE in $\mathbb{R}^2\backslash\overline{\mathrm{D}}$. Continuity and coercivity of the boundary integral operator (BIO) with κ-explicit coercivity and continuity bounds renders Ritz-Galerkin variational approximations of the unknown density on Γ well-posed, and quasi-optimal. In the present paper, we extend [1] and address *wavenumber-explicit convergence rates* of deep-BEM approximations in $L^2(\Gamma)$ of the corresponding boundary integral equations (BIEs for short). This is reminiscent of *e.g.*, the so-called "Deep Ritz Method" NN approximation for variational elliptic PDEs in [9]. In the low-frequency case, in [1], we showed that on polygonal Γ, ReLU-NN approximations can resolve the corner singularities of the surface densities in BIEs at an optimal (even exponential) rate, in terms of the "number of degrees of freedom", being the NN weights that are active in the DNN approximation.

The results in the present paper are based on the high wavenumber asymptotics of the unknown surface density solution to a second kind BIE reformulation of the exterior scattering problem. NN approximation is preceded by boundary reduction whereby the governing equation and the radiation condition at infinity are satisfied exactly. The surface density is unknown, and we study neural networks to approximate it in a wavenumber-robust fashion, based on high wavenumber asymptotics proposed in [13] for the density on the surface of the scatterer. Numerical approximations based on this idea have been developed in recent years in [7, 8, 10, 11], among others. These methods were based on earlier work on high-wavenumber asymptotics [21, 22, 24, 26] in PDE theory. While in [10], local geometric asymptotics have been used to develop geometry- and wavenumber-adapted trial spaces for Galerkin boundary integral equations, such spaces necessarily depend on the wavenumber and the geometry, with corresponding challenges to accurate numerical integration of the Galerkin matrix for the (likewise wavenumber-dependent) fundamental solution. While affording wavenumber-robust approximations these highly problem-adapted trial spaces for the Galerkin BEM method entail locally different approximation spaces, depending on the illuminated, shadow and glancing regions on the boundary Γ, with wavenumber-explicit localization of these regions of Γ and strongly differing functional forms of the trial functions in them. While it was argued (and corroborated by sophisticated numerical tests) in [11] that this results in a κ-robust approximation scheme, there is a significant analytic overhead in specifying the $\kappa \to \infty$ asymptotic expansion of the solution in dependence on the geometry of Γ. This is used in [7,8,10,11] to *a-priori* design "problem adapted, non-polynomial, approximation spaces" on Γ. This design, being strongly geometry dependent, could be viewed as a significant obstruction to the versatility of the approach proposed in [10, 11] and in similar, earlier work [8].

In the main results of the present paper, Theorems 4.4 and 4.6, we show that deep feedforward, strict ReLU-activated NNs of *moderate, fixed width* (of about 40 neurons/layer) can achieve ϵ-accurate approximation of the scattered farfield (a) without any *a-priori* information on the (assumed convex) scatterer geometry Γ, (b) subject to mild constraints on the NN depth L w.r. to the wavenumber κ and the target accuracy $\epsilon \in (0, 1]$. Specifically, there are positive constants $a, b > 0$ independent of κ and of $\epsilon \in (0, 1]$ such that the NN depth $L(\kappa, \epsilon)$ is subject to a requirement of the type $L(\kappa, \epsilon) \geq C_n(\log(\kappa)^a + \epsilon^{-b/n})$, for κ large enough and for any integer $n \in \mathbb{N}$ with a constant $C_n > 0$ that is independent of κ and of ϵ. *I.e.*, with the NN depth growing poly-logarithmic w.r. to κ and polynomially with arbitrary small order w.r. to the approximation accuracy $\epsilon \in (0, 1]$ (the latter is due to the $\mathscr{C}^\infty$-regularity used in the large κ asymptotics of the solution; improving this to poly-log would require invoking Gevrey-type regularity here).

In contrast to the recent work on "asymptotics-augmented Galerkin BEM" (see *e.g.*, [11]), the present expression rate results hold without any *a-priori* adaptation of the NN feature space to the scatterers' shape. The main results of this work are Theorems 4.4 and 4.6, ReLU-NN emulation results which contain ReLU NN emulation rates with wavenumber robust expression rates. Although we only analyze a model problem, we remark that technical results presented in Section 4.1 on the DNN emulation of the so-called Fock's integral and its derivatives (naturally appearing in geometric optics based asymptotics, also on surfaces [22]), will be relevant in possible extensions of the present results to three space dimensions.

The present DNN emulation rate results are established for deep, fixed-width NNs with ReLU activations. Standard arguments (indicated in various places of the text) allow to extend our proofs to other, non-polynomial activation functions (ReLU^k with $k \geq 2$ or tanh() activations; we expect the values of the constants a and b in the above NN depth bounds, generally, to be smaller). The present ReLU DNN approximation rate results will hold for spiking DNN architectures, *via* the ReLU-to-spiking conversion [33].

1.3. Layout

This work is structured as follows. In Section 2 we introduce the sound-soft acoustic scattering and describe its boundary reduction *via* the combined field boundary integral equation. In addition, we recall κ-explicit stability and coercivity bounds together with the well-posedness of the Galerkin formulation in $L^2(\Gamma)$. A key component of the ensuing wavenumber-explicit DNN analysis are known asymptotic expansions of the solution with respect to the wavenumber κ. These are thoroughly reviewed in Section 2.5. In Section 3 we introduce definitions, notation, and auxiliary results concerning DNNs which are required for proving the ensuing expression rate bounds. In Section 4 we present the main results of this work: the wavenumber-explicit ReLU-NN emulation of the solution to the CFIE, the problem's Neumann trace. Finally, in Section 5 we provide concluding remarks.

Remark 1.1. In several appendices, which may be found as online/electronic supplementary material, we provide complete proofs of auxiliary results from Section 3, which could also be of interest elsewhere. In addition, complete proofs of Propositions 4.1 and 4.2 in Section 4 may also be found in the appendices, which are part of the electronic supplementary material.

1.4. Notation

We denote by $\mathbb{N}$ the set of natural numbers starting with 1, and for any $C > 1$ we set $\mathbb{N}_{\geq C} \coloneqq \mathbb{N} \cap [C, \infty)$. Let D be a bounded, Lipschitz domain with boundary $\partial\mathrm{D}$ and exterior complement $\mathrm{D}^{\mathrm{c}} \coloneqq \mathbb{R}^d \backslash \overline{\mathrm{D}}$ in $\mathbb{R}^d$, $d \in \mathbb{N}$. We denote by $L^p(\mathrm{D})$, $p \in [1, \infty)$ the Lebesgue space of measurable, p-integrable functions in D, with the appropriate extension to $p = \infty$, and by $H^s(\mathrm{D})$, $s \geq 0$, standard Sobolev spaces in D.

In addition, let $H^s(\partial\mathrm{D})$, for $s \in [0,1]$, be the Sobolev space of traces on $\partial\mathrm{D}$ ([31], Sect. 2.4). As is customary, we identify $H^0(\partial\mathrm{D})$ with $L^2(\partial\mathrm{D})$ and, for $s \in [0,1]$, $H^{-s}(\partial\mathrm{D})$ with the dual space of $H^s(\partial\mathrm{D})$. Exterior Dirichlet and Neumann trace operators are denoted by

$$\gamma^{\mathrm{c}} \colon H^1_{\mathrm{loc}}(\mathrm{D}^{\mathrm{c}}) \to H^{\frac{1}{2}}(\Gamma) \quad \text{and} \quad \frac{\partial}{\partial \mathbf{n}} \colon H^1_{\mathrm{loc}}(\Delta, \mathrm{D}^{\mathrm{c}}) \to H^{-\frac{1}{2}}(\Gamma),$$

where the spaces $H^1_{\mathrm{loc}}(\mathrm{D}^{\mathrm{c}})$ and $H^1_{\mathrm{loc}}(\Delta, \mathrm{D}^{\mathrm{c}})$ are defined in Definition 2.6.1 of [31] and Equation (2.108) of [31], respectively. These operators are continuous as stated in Theorems 2.6.8 and 2.7.7 of [31]. For a Banach space X, we denote by $\mathscr{L}(X)$ the space of bounded linear operators from X into itself equipped with the standard operator norm, which we denote by $\|\cdot\|_{\mathscr{L}(X)}$. Let $f(s)$ be a ℓ-times continuously differentiable function depending on a single variable s. We denote by $f'(s)$ the first derivative of $f(s)$ with respect to s. For higher-order derivatives, we use both the notation $f^{(\ell)}(s)$ and $D^\ell_s f(s)$ to denote the ℓ-th order differential of $f(s)$ with respect to the argument s.

2. High-frequency acoustic scattering formulation

2.1. Sound-soft scattering problem

Let $\mathrm{D} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\Gamma \coloneqq \partial\mathrm{D}$. We denote by $\mathrm{D}^{\mathrm{c}} \coloneqq \mathbb{R}^2 \backslash \overline{\mathrm{D}}$ the corresponding exterior domain. Given $\kappa > 0$ and $\hat{\mathbf{d}} \in \mathbb{R}^2$ a unit vector, we define $u^{\mathrm{i}}(\mathbf{x}) \coloneqq \exp(\imath\kappa \mathbf{x} \cdot \hat{\mathbf{d}})$, $\mathbf{x} \in \mathbb{R}^2$, and consider the following problem: Find the scattered field $u^{\mathrm{s}} \in H^1_{\mathrm{loc}}(\mathrm{D}^{\mathrm{c}})$ such that the total field $u \coloneqq u^{\mathrm{i}} + u^{\mathrm{s}}$ satisfies

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathrm{D}^{\mathrm{c}}, \quad \text{and} \tag{2.1a}$$

$$\gamma^{\mathrm{c}} u = 0 \quad \text{on } \Gamma. \tag{2.1b}$$

In addition to (2.1a) and (2.1b), as it is customary, we impose the Sommerfeld radiation condition on the scattered field u^{s}

$$\frac{\partial u^{\mathrm{s}}}{\partial r}(\mathbf{x}) - \imath\kappa u^{\mathrm{s}}(\mathbf{x}) = o\left(\|\mathbf{x}\|^{-\frac{1}{2}}\right) \tag{2.2}$$

as $\|\mathbf{x}\| \to \infty$, uniformly in $\hat{\mathbf{x}} \coloneqq \mathbf{x}/\|\mathbf{x}\|$.

2.2. Boundary integral formulation

We reduce the exterior acoustic scattering problem (2.1) to an integral equation on the boundary Γ. In doing so, we satisfy exactly the radiation condition (2.2) and the homogeneous PDE (2.1a).

Let $\mathrm{G}_\kappa(\mathbf{z}) \coloneqq \frac{\imath}{4} H_0^{(1)}(\kappa\|\mathbf{z}\|)$ for $\mathbf{z} \in \mathbb{R}^2\backslash\{\mathbf{0}\}$ denote the fundamental solution of the Helmholtz equation in two spatial dimensions and of wavenumber $\kappa \in \mathbb{R}_+$. Here $H_0^{(1)}$ corresponds to the Hankel function of the first kind and of order zero. We proceed to recast (2.1) using BIOs by means of boundary potentials. To this end, we introduce *single layer potential*

$$(\mathcal{S}_\kappa\,\psi)(\mathbf{x}) \coloneqq \int_\Gamma \mathrm{G}_\kappa(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}}, \quad \mathbf{x} \in \mathrm{D}^{\mathrm{c}},$$

and the *double layer potential*

$$(\mathcal{D}_\kappa\phi)(\mathbf{x}) \coloneqq \int_\Gamma \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \mathrm{G}_\kappa(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}}, \quad \mathbf{x} \in \mathrm{D}^{\mathrm{c}},$$

where $\psi, \phi \in L^1(\Gamma)$ are referred to as *surface densities*, which are defined on the boundary Γ. The application of exterior Dirichlet and Neumann traces to $\mathcal{S}_\kappa$ and $\mathcal{D}_\kappa$ yields the single layer operator

$$\mathsf{V}_\kappa \coloneqq \gamma^{\mathrm{c}}\mathcal{S}_\kappa \colon H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma),$$

the double layer operator

$$\frac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa \coloneqq \gamma^{\mathrm{c}}\mathcal{D}_\kappa \colon H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma),$$

and the adjoint double layer operator

$$\frac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa \coloneqq \frac{\partial}{\partial \mathbf{n}}\mathcal{S}_\kappa \colon H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma).$$

The double layer operator and the adjoint double layer operator are mutually adjoint in the bilinear $L^2(\Gamma)$-duality pairing (*e.g.*, [31]).

The starting point of the reformulation of (2.1) is Green's representation formula, which provides the following representation of the total field u:

$$u(\mathbf{x}) = u^{\mathrm{i}}(\mathbf{x}) - \mathcal{S}_\kappa\left(\frac{\partial u}{\partial \mathbf{n}}\right)(\mathbf{x}), \quad \mathbf{x} \in \mathrm{D}^{\mathrm{c}}, \tag{2.3}$$

where $\frac{\partial u}{\partial \mathbf{n}}$ corresponds to the Neumann trace of the scattered field u. We remark that Sommerfeld radiation conditions are built-in in the single layer potential. The application of both Dirichlet and Neumann traces to (2.3) yields the BIEs

$$\mathsf{V}_\kappa \frac{\partial u}{\partial \mathbf{n}} = \gamma^{\mathrm{c}} u^{\mathrm{i}}, \quad \text{and,} \tag{2.4}$$

$$\left(\frac{1}{2}\mathsf{I} + \mathsf{K}'_\kappa\right)\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial u^{\mathrm{i}}}{\partial \mathbf{n}} \cdot \tag{2.5}$$

Both BIEs (2.4) and (2.5) can be proven to be well-posed, except for those κ that are eigenvalues of the Laplace operator in D with homogeneous Dirichlet and Neumann boundary conditions, respectively. A linear combination of (2.4) and (2.5) leads to the resonance-free, well-posed BIE

$$\mathsf{A}'_{\kappa,\eta}\varphi_\kappa = \frac{\partial u^{\mathrm{i}}}{\partial \mathbf{n}} - \imath\eta\gamma^{\mathrm{c}}u^{\mathrm{i}} =: f_{\kappa,\eta} \in L^2(\Gamma), \tag{2.6}$$

where $\varphi_\kappa := \frac{\partial u}{\partial \mathbf{n}}$, and $\mathsf{A}_{\kappa,\eta}$ and its adjoint are defined as

$$\mathsf{A}_{\kappa,\eta} := \frac{1}{2}\mathsf{I} + \mathsf{K}_\kappa - \imath\eta\mathsf{V}_\kappa \quad \text{and} \quad \mathsf{A}'_{\kappa,\eta} := \frac{1}{2}\mathsf{I} + \mathsf{K}'_\kappa - \imath\eta\mathsf{V}_\kappa, \tag{2.7}$$

respectively. Here $\eta \in \mathbb{R}\backslash\{0\}$ is a coupling parameter to be specified. It is well established that $\frac{\partial u}{\partial \mathbf{n}} \in H^{-\frac{1}{2}}(\Gamma)$. However, if D is a domain of class $\mathscr{C}^2$ standard regularity results assert that $\frac{\partial u}{\partial \mathbf{n}} \in L^2(\Gamma)$. Indeed, this also holds true even if D is a Lipschitz domain (see *e.g.*, [27], Sect. 5.1.2). Therefore, as it is customary, we consider (2.6) as an operator equation posed in $L^2(\Gamma)$.

2.3. Well-posedness of the CFIE and κ-explicit bounds

The following two propositions address the boundedness and coercivity of $\mathsf{A}_{\kappa,\eta}$ and its adjoint, with bounds that are explicit in the wavenumber.

Proposition 2.1 ([4], Thm. 3.6)**.** *It holds*

$$\|\mathsf{A}_{\kappa,\eta}\|_{\mathscr{L}(L^2(\Gamma))} = \left\|\mathsf{A}'_{\kappa,\eta}\right\|_{\mathscr{L}(L^2(\Gamma))} \lesssim 1 + \kappa^{\frac{1}{2}} + |\eta|\kappa^{-\frac{1}{2}},$$

where the implied constant is independent of η and κ.

Proposition 2.2 ([32], Thm. 1.2)**.** *Let* D *be a convex domain in either two dimensions or three dimensions whose boundary, Γ, has a strictly positive curvature and is both $\mathscr{C}^3$ and piecewise analytic. Then there exists a constant $\eta_0 > 0$ such that, given $\delta \in (0, 1/2)$, there exists κ_0 (depending on δ) such that for $\kappa \geq \kappa_0$ and $\eta \geq \eta_0\kappa$,*

$$\mathfrak{Re}\left\{\left(\mathsf{A}'_{\kappa,\eta}\phi, \phi\right)_{L^2(\Gamma)}\right\} \geq \left(\frac{1}{2} - \delta\right)\|\phi\|^2_{L^2(\Gamma)}$$

for all $\phi \in L^2(\Gamma)$. The bound also holds with $\mathsf{A}'_{\kappa,\eta}$ replaced by $\mathsf{A}_{\kappa,\eta}$.

As a consequence of Propositions 2.1 and 2.2, together with the Lax–Milgram lemma, we may conclude well-posedness of the CFIE, stated before in (2.6), as an operator equation set in $L^2(\Gamma)$. In addition, as thoroughly discussed in Section 1.3 of [32], there holds κ-explicit quasi-optimality of the Galerkin discretization method. More precisely, given a finite-dimensional subspace V_N of $L^2(\Gamma)$, we consider the following Galerkin approximation of the variational form of (2.6): Find $\varphi_{\kappa,N} \in V_N$ such that

$$\left(\mathsf{A}'_{\kappa,\eta}\varphi_{\kappa,N}, \psi_N\right)_{L^2(\Gamma)} = (f_{\kappa,\eta}, \psi_N)_{L^2(\Gamma)} \quad \forall\psi_N \in V_N, \tag{2.8}$$

with $f_{\kappa,\eta}$ as in the right-hand side of (2.6). Quasi-optimality of the Galerkin BEM reads as follows: Provided that η is chosen such that $\eta_0\kappa \leq \eta \lesssim \kappa$ with η_0 and κ_0 as in Proposition 2.2, then for all $\kappa \geq \kappa_0$ it holds

$$\|\varphi_\kappa - \varphi_{\kappa,N}\|_{L^2(\Gamma)} \lesssim \kappa^{\frac{d-1}{2}} \inf_{\phi_N \in V_N} \|\varphi_\kappa - \phi_N\|_{L^2(\Gamma)}, \tag{2.9}$$

where $d \in \mathbb{N}$ corresponds to the problem's physical dimension.

2.4. Galerkin *vs.* DNN approximations

In view of (2.9), and as it is usual in the Galerkin BEM, the convergence analysis of (2.8) boils down to the choices of the subspaces V_N and the regularity of the solution φ_κ to the CFIE (2.6). Concerning the latter aspect, in Section 2.5 ahead we recall relevant results addressing the asymptotic regularity of φ_κ in (2.7), which are explicit in the wavenumber κ.

These results have been used in [11] and the references therein to design *solution-adapted subspaces* V_N affording wavenumber-robust spectral convergence in the approximation of the discrete Galerkin BEM problem stated in (2.8). The subspaces V_N obtained in [11] depend on the wavenumber κ, the incident angle $\hat{\mathbf{d}}$, and on Γ itself in a nontrivial way. In the words of [2], pg. 2: "…frequency independent methods can only be devised provided the Melrose–Taylor expansions are properly incorporated into integral equation formulations…"

In the present work, we also leverage the wavenumber-explicit, asymptotic expansion by Melrose and Taylor for the solution φ_κ of the CFIE (2.6). Rather than incorporating analytic, κ-explicit expressions from the work of Melrose and Taylor into Galerkin approximation spaces, we leverage wavenumber-explicit asymptotic expansions as developed, *e.g.*, in [2,11,22], to prove the existence of DNN approximations of φ_κ with κ-explicit expression-rate bounds. Specifically, *without a-priori* incorporation of asymptotic expansion information into the DNN architecture or feature space, we prove that a fixed width, deep ReLU-NN architecture can realize "neural Galerkin spaces" on Γ which afford, uniformly over a class of (convex) geometries, spectral error bounds for DNN expression rates, with constants that depend only logarithmically on the wavenumber κ. As in other recent analyses of NN emulations of solutions to PDEs with small or large parameters (*e.g.*, [28]), we find that *deep NN architectures with "standard activations" (ReLU in the present work) can express "neural Galerkin subspaces" with κ-robust approximation rates.* Again, all NNs in this work are deep feedforward NNs with strict ReLU activation and *fixed width*.

Contrary to the constructions in [2,11], the presently proved DNN emulation results do not require *a-priori* inclusion of explicit asymptotic information of φ_κ into the DNN approximation architecture, while enjoying, as we shall show, κ-robust spectral expression rates, uniformly over a class of domains.

Instead, *NN training* would account *via* feature optimization for the dependence of the approximation on the problem's wavenumber κ and on the scatterer's geometry. The process of tuning the NN feature space to specific geometries and wavenumbers may, in turn, be realized by meta-learning algorithms. In effect, the present, constructive proofs of NN expression rates may be viewed as a roadmap to mathematical justification of meta-learning algorithms [17] which map instances of problem (2.1), (2.2) to the NN feature spaces constructed in the present expression rate proofs. As usual in many DNN based PDE solvers, this enhanced approximation capability comes with the price of a nonlinear dependence of the DNN features on the DNN parameters. In practice, for the numerical computation of these parameters, iterative training algorithms are required, as opposed to the solution of the Galerkin approximation (2.8), which merely requires assembling and solving a linear system of equations.

2.5. Asymptotics of the normal derivative of the total field

We assume that the scatterer's boundary Γ is given by

$$\Gamma := \left\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \boldsymbol{\gamma}(s) : \ s \in [0, 2\pi]\right\},$$

where $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ is a 2π-periodic parametrization of Γ. Following Section 5 of [8] (which in turn relies on the work by Melrose and Taylor [24], and references therein), the solution $\varphi_\kappa \in L^2(\Gamma)$ to (2.6) admits the following decomposition

$$\hat{\varphi}_\kappa(s) := (\tau_{\boldsymbol{\gamma}}\varphi_\kappa)(s) = \kappa V_\kappa(s) \exp(\imath\kappa\boldsymbol{\gamma}(s) \cdot \hat{\mathbf{d}}), \quad s \in [0, 2\pi], \tag{2.10}$$

where $\tau_{\boldsymbol{\gamma}} : L^2(\Gamma) \to L^2((0, 2\pi)) : u \mapsto u \circ \boldsymbol{\gamma}$ defines a bounded linear operator with a bounded inverse.

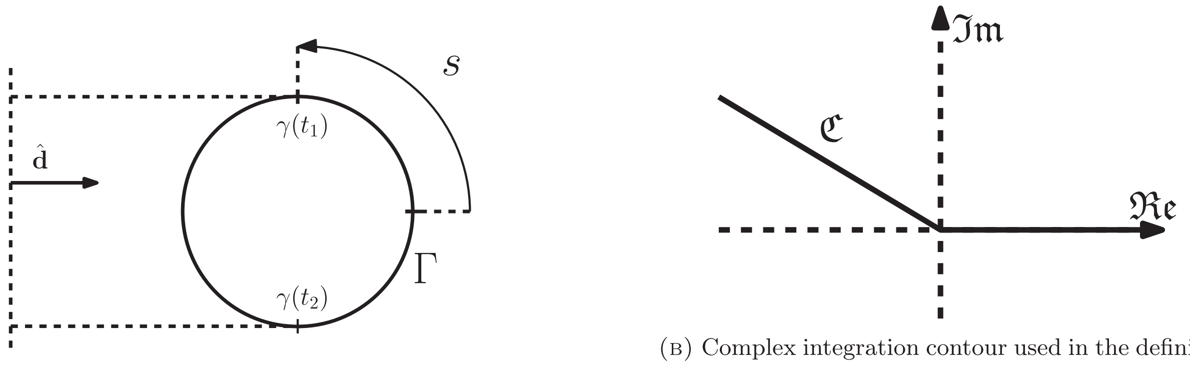
Throughout this manuscript, we work under the assumption stated below.

Assumption 2.3. D *is a convex domain with a* $\mathscr{C}^\infty$ *boundary* Γ *of strictly positive curvature.*

Assumption 2.3 provides an asymptotic expansion of V_κ that is explicit in the wavenumber.

Theorem 2.4 ([8], Thm. 5.1, [24], Thm. 9.27)**.** *Let Assumption 2.3 be satisfied. There exists* $\delta > 0$ *such that* V_κ *has the asymptotic expansion:*

$$V_\kappa(s) \sim \sum_{\ell,m\geq 0} \kappa^{-\frac{1}{3}-\frac{2}{3}\ell-m} b_{\ell,m}(s) \Psi^{(\ell)}\left(\kappa^{\frac{1}{3}} Z(s)\right) \tag{2.11}$$



(A) Geometrical setting.

(B) Complex integration contour used in the definition of Fock's integral.

FIGURE 1. Figure 1a portrays the geometrical setting under consideration, whereas Figure 1b depicts the complex integration contour used in the definition of Fock's integral stated in (2.12).

valid for $s \in I_\delta := (t_1-\delta, t_1+\delta) \cup (t_2-\delta, t_2+\delta)$, *where* $\boldsymbol{\gamma}(t_1)$ *and* $\boldsymbol{\gamma}(t_2)$ *are the tangency points (see Fig. 1a). The functions* $b_{\ell,m}$, Ψ *and* Z *have the following properties:*

(i) $b_{\ell,m}$ *are* $\mathscr{C}^\infty$ *complex-valued functions on* I_δ.
(ii) Z *is a* $\mathscr{C}^\infty$ *real-valued function on* I_δ *with simple zeros at* t_1, *and* t_2, *which is positive-valued on* $(t_1, t_2) \cap I_\delta$ *and negative-valued on* $(t_2 - 2\pi, t_1) \cap I_\delta$.
(iii) $\Psi : \mathbb{C} \to \mathbb{C}$ *is an entire function specified by*

$$\Psi(\tau) := \exp\big(-\imath\tau^3/3\big) \int_{\mathfrak{C}} \frac{\exp(-\imath z \tau)}{\mathrm{Ai}(\exp(2\pi\imath/3)z)} \, \mathrm{d}z, \tag{2.12}$$

where Ai *is the Airy function and* $\mathfrak{C}$ *is the contour depicted in Figure 1b. This function is often referred to as "Fock's integral", see e.g., [3, 13].*

The asymptotics of $\Psi(\tau)$ *for large* τ *is known. We have that*

$$\Psi(\tau) = \sum_{j=0}^{n} a_j \tau^{1-3j} + \mathcal{O}\Big(\tau^{1-3(n+1)}\Big), \quad \text{as} \quad \tau \to \infty, \tag{2.13}$$

where $a_0 \neq 0$. *This expansion remains valid for all derivatives of* Ψ *by formally differentiating each term on the right-hand side, including the error term. Furthermore, there exists* $\beta > 0$ *and* $c_0 \neq 0$ *such that for any* $n \in \mathbb{N}_0$

$$D^n_\tau \Psi(\tau) = c_0 D^n_\tau \big\{\exp(-\imath\tau^3/3 - \imath\tau\alpha_1)\big\}(1 + \mathcal{O}(\exp(-\beta|\tau|)), \quad \text{as} \quad \tau \to -\infty,$$

where $\alpha_1 = \exp(-2\pi/3\imath)\nu_1$ *and* $\nu_1 \in \mathbb{R}_-$ *is the right-most root of* Ai.

Remark 2.5. We make the following observations concerning Theorem 2.4.

(i) In (2.11), the symbol "$\sim$" refers to the symbol classes of Hörmander. For details we refer to Definition 7.8.1 of [16], p. 236, and we do not elaborate on this any further.
(ii) The function Z can be extended (non-uniquely) to a $\mathscr{C}^\infty$, 2π-periodic function that is positive-valued on (t_1, t_2) and negative-valued on $(t_2 - 2\pi, t_1)$. From now on we assume that this extension has been made.

Proposition 2.6 ([8], Cor. 5.3)**.** *The functions* $b_{\ell,m}$ *can be extended to* 2π*-periodic* $\mathscr{C}^\infty$ *functions such that, for all* L, $M \in \mathbb{N}_0$, *the decomposition*

$$V_\kappa(s) = \sum_{\ell,m=0}^{L,M} \kappa^{-\frac{1}{3}-2\ell/3-m} b_{\ell,m}(s) \Psi^{(\ell)}\Big(\kappa^{\frac{1}{3}} Z(s)\Big) + R_{L,M,\kappa}(s) \tag{2.14}$$

holds for all $s \in [0, 2\pi]$, with the remainder term satisfying, for all $n \in \mathbb{N}_0$,

$$|D_s^n R_{L,M,\kappa}(s)| \leq C_{L,M,n}(1+\kappa)^{\mu+\frac{n}{3}}, \tag{2.15}$$

where

$$\mu := -\min\left\{\frac{2}{3}(L+1), (M+1)\right\}, \tag{2.16}$$

and $C_{L,M,n}$ is independent of κ.

Remark 2.7. Proposition 2.6 and Remark 2.5 allow us to conclude that the remainder $R_{L,M,\kappa}(s)$ can also be extended (non-uniquely) to a $\mathscr{C}^\infty$, 2π-periodic function.

Remark 2.8. For $\tau \in \mathbb{R}$ it holds

$$|\Psi(\tau)| \leq C_0(1+|\tau|), \quad |\Psi'(\tau)| \leq C_1, \quad \left|\Psi^{(\ell)}(\tau)\right| \leq C_\ell(1+|\tau|)^{-2-\ell}, \quad \text{for } \ell \geq 2,$$

with C_ℓ independent of κ.

The next result provides wavenumber-explicit bounds for the derivatives of V_κ.

Lemma 2.9 ([8], Cor. 5.5)**.** *For all $n \geq 1$ there exists a constant $C_n > 0$ independent of κ such that, for κ sufficiently large, $|D_s^n V_\kappa(s)| \leq C_n(1+\kappa)^{\frac{(n-1)}{3}}$, $s \in [0, 2\pi]$.*

Remark 2.10. As originally stated in Corollary 5.5 of [8], Lemma 2.9 holds for $n \in \mathbb{N}$. An inspection of Corollary 5.3 from [8] reveals that one has $|V_\kappa(s)| \leq C$, $s \in [0, 2\pi]$, for some constant $C > 0$ independent of κ.

3. Deep ReLU neural networks

We recall notation, terminology, and results concerning feedforward, fully connected DNNs, as subsequently required, consistent with [12].

Definition 3.1 (Deep Neural Network)**.** Let $L \in \mathbb{N}$, $\ell_0, \dots, \ell_L \in \mathbb{N}$ and let $\varrho : \mathbb{R} \to \mathbb{R}$ be a non-linear function, referred to in the following as the *activation function*. Given $(\mathbf{W}_k, \mathbf{b}_k)_{k=1}^L$, $\mathbf{W}_k \in \mathbb{R}^{\ell_k \times \ell_{k-1}}$, $\mathbf{b}_k \in \mathbb{R}^{\ell_k}$, define the affine transformation $\mathcal{A}_k : \mathbb{R}^{\ell_{k-1}} \to \mathbb{R}^{\ell_k} : \mathbf{x} \mapsto \mathbf{W}_k\mathbf{x} + \mathbf{b}_k$ for $k \in \{1, \dots, L\}$. We define a *Deep Neural Network (DNN)* with activation function $\varrho : \mathbb{R} \to \mathbb{R}$ as a map $\Phi_{\mathcal{NN}} : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_L}$ with

$$\Phi_{\mathcal{NN}}(\mathbf{x}) := \begin{cases} \mathcal{A}_1(\mathbf{x}), & L = 1, \\ (\mathcal{A}_L \circ \varrho \circ \mathcal{A}_{L-1} \circ \varrho \cdots \circ \varrho \circ \mathcal{A}_1)(\mathbf{x}), & L \geq 2, \end{cases}$$

where the activation function ϱ is applied component-wise to vector inputs. We define the width and depth of a DNN as

$$\mathcal{M}(\Phi_{\mathcal{NN}}) := \max\{\ell_1, \dots, \ell_L\} \quad \text{and} \quad \mathcal{L}(\Phi_{\mathcal{NN}}) := L$$

together with the weight bounds

$$\mathcal{B}(\Phi_{\mathcal{NN}}) = \max_{\ell=1,\dots,L} \max\{\|\mathbf{W}_k\|_\infty, \|\mathbf{b}_k\|_\infty\}.$$

In the above definition, ℓ_0 is the dimension of the *input layer*, ℓ_L denotes the dimension of the *output layer*.

In addition, we denote by $\mathcal{NN}^{\varrho}_{L,M,\ell_0,\ell_L}$ the set of all DNNs $\Phi_{\mathcal{NN}} : \mathbb{R}^{\ell_0} \to \mathbb{R}^{\ell_L}$ with input dimension ℓ_0, output dimension ℓ_L, a depth of at most L layers, maximum width M, and activation function ϱ.

Unless explicitly stated otherwise, we assume that the activation is by the ReLU defined as $\varrho(x) \coloneqq \max\{x, 0\}$, and we drop the explicit dependence on the activation function in $\mathcal{NN}^{\varrho}_{L,M,\ell_0,\ell_L}$.

We proceed to recall results concerning the approximation properties of ReLU-NNs following [12]. Firstly, we introduce the multiplication ReLU-NN.

Proposition 3.2 ([12], Prop. III.3)**.** *There exists a constant $C > 0$ such that for all $D \in \mathbb{R}_+$ and $\epsilon \in (0, 1/2)$ there are networks $\Phi_{D,\epsilon} \in \mathcal{NN}_{L,5,2,1}$ with*

$$L \leq C\left(\log(\lceil D \rceil) + \log\left(\frac{1}{\epsilon}\right)\right)$$

such that

$$\|\Phi_{D,\epsilon}(x, y) - xy\|_{L^\infty((-D,D)^2)} \leq \epsilon.$$

The weights of $\Phi_{D,\epsilon}$ are bounded according to $\mathcal{B}(\Phi_{D,\epsilon}) \leq 1$.

Observe that $p_0(x) = 1$ and $p_1(x) = x$ can be exactly represented using ReLU-NNs as follows

$$p_0(x) = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \varrho\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix}\right) \in \mathcal{NN}_{2,4,1,1}, \quad \text{and} \tag{3.1a}$$

$$p_1(x) = \begin{pmatrix} 1 & -1 \end{pmatrix} \varrho\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} x\right) \in \mathcal{NN}_{2,2,1,1}. \tag{3.1b}$$

The next result addresses the ReLU-NN emulation of polynomials of higher order.

Proposition 3.3 ([12], Prop. III.5)**.** *There exists a constant $C > 0$ such that for $m \in \mathbb{N}$, $A \geq 1$, $D \in \mathbb{R}_+$, $\epsilon \in (0, 1/2)$ and*

$$p_m(x) = \sum_{i=0}^{m} a_i x^i \quad \text{with} \quad \max_{i=0,\ldots,m} |a_i| \leq A,$$

there are ReLU-NNs $\Phi_{p_m,D,\epsilon} \in \mathcal{NN}_{L,9,1,1}$ with

$$L \leq C\left(m \log\left(\frac{1}{\epsilon}\right) + m^2 \log(\lceil D \rceil) + m \log(m) + m \log(A)\right) \tag{3.2}$$

such that

$$\|\Phi_{p_m,D,\epsilon}(x) - p_m(x)\|_{L^\infty((-D,D))} \leq \epsilon.$$

The weights of $\Phi_{p_m,D,\epsilon}$ are bounded according to $\mathcal{B}(\Phi_{p_m,D,\epsilon}) \leq 1$.

Remark 3.4 (ReLU-NN Emulation of Polynomials with Complex Coefficients)**.** An inspection of the proof of Proposition 3.3 reveals that in order to accommodate for polynomials with complex coefficients the architecture of the ReLU-NN should be modified as follows: two outputs should be considered instead of one in order to account for the real and imaginary parts separately, and the width of the ReLU-NN should be increased to 11. The depth of the ReLU-NN remains unchanged, *i.e.*, it behaves exactly as in (3.2).

Proposition 3.5 ([12], Thm. III.8 and Cor. III.9)**.** *There exists a constant $C > 0$ such that for each $a, D \in \mathbb{R}_+$, $\epsilon \in (0, 1/2)$ there are ReLU-NNs $\Phi^{\cos}_{a,D,\epsilon} \in \mathcal{NN}_{L,9,1,1}$ and $\Phi^{\sin}_{a,D,\epsilon} \in \mathcal{NN}_{L,9,1,1}$ with*

$$L \leq C\left(\left(\log\left(\frac{1}{\epsilon}\right)\right)^2 + \log(\lceil aD \rceil)\right)$$

such that

$$\left\|\cos(ax) - \Phi^{\cos}_{a,\epsilon}(x)\right\|_{L^\infty((-D,D))} \leq \epsilon \quad \text{and} \quad \left\|\sin(ax) - \Phi^{\sin}_{a,\epsilon}(x)\right\|_{L^\infty((-D,D))} \leq \epsilon.$$

The weights of $\Phi^{\cos}_{a,D,\epsilon}$ and $\Phi^{\sin}_{a,D,\epsilon}$ are bounded according to $\mathcal{B}(\Phi^{\cos}_{a,\epsilon}) \leq 1$ and $\mathcal{B}(\Phi^{\sin}_{a,\epsilon}) \leq 1$.

3.1. ReLU-NN emulation of decaying exponentials

We approximate decaying exponentials, *i.e.*, maps of the form $x \mapsto \exp(-ax)$ with $a \in \mathbb{R}_+$, using ReLU-NNs within the interval $[0, D]$, where $D \in \mathbb{R}_+$. A proof of the next result may be found in Appendix A.1.

Proposition 3.6. *There exist constants C_1, $C_2 > 0$ such that for each a, $D \in \mathbb{R}_+$ there exist ReLU-NNs $\Phi^{\exp}_{a,D,\epsilon} \in \mathcal{NN}_{L,9,1,1}$ with*

$$L \leq C_1\left(\lceil \log(aD)\rceil^2 + \left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right), \quad \text{as } \epsilon \to 0,$$

such that $\|\exp(-ax) - \Phi^{\exp}_{a,D,\epsilon}(x)\|_{L^\infty((0,D))} \leq \epsilon$. The weights of $\Phi^{\exp}_{a,D,\epsilon}$ are bounded in absolute value by a constant independent of a and D.

3.2. ReLU-NN emulation of $x \mapsto \frac{1}{x}$

We proceed to approximate the function $x \mapsto \frac{1}{x}$ using ReLU neural networks within the interval $[1, D]$ with $D > 1$. The construction of these neural networks is performed in two steps. Firstly, we approximate $x \mapsto \frac{1}{x}$, for $x \geq 1$, by an appropriate combination of decaying exponentials. The key ingredient of this construction corresponds to the so-called "sinc"-quadrature rule developed by F. Stenger [34]. The second step consists in approximating each of these exponentials using the results from Section 3.1. The ReLU-NN that approximates $x \mapsto \frac{1}{x}$ will be a suitable linear combination of the ReLU-NNs approximating each of the decaying exponentials.

Lemma 3.7 ([15], Lem. 4.2)**.** *Let $z \in \mathbb{C}$ with $\Re\{z\} \geq 1$. Then for each $k \in \mathbb{N}$ the (positive) quadrature points and weights*

$$t_j := \log\left(\exp\left(jk^{-\frac{1}{2}}\right) + \sqrt{1 + \exp\left(2jk^{-\frac{1}{2}}\right)}\right),$$
$$\omega_j := \left(k + k\exp\left(-2jk^{-\frac{1}{2}}\right)\right)^{-\frac{1}{2}}, \quad j = -k, \ldots, k,$$

fulfil

$$\left|\int_0^\infty \exp(-zt)\,\mathrm{d}t - \sum_{j=-k}^{k} \omega_j \exp(-t_j z)\right| \leq C_S \exp\left(\frac{|\Im\{z\}|}{\pi}\right)\exp\left(-\sqrt{k}\right),$$

where $C_S > 0$ does not depend upon k or z.

Equipped with Lemma 3.7, one can show the following result.

Lemma 3.8. *There exists a constant $C > 0$ such that for each $D > 1$ there exist ReLU-NNs $\psi_{n,D,\epsilon} \in \mathcal{NN}_{L,13,1,1}$ with*

(i) *for $n = 1$*

$$L \leq C\left(\log\left(\frac{1}{\epsilon}\right)\right)^2\left(\lceil \log(D)\rceil^2 + \left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right) \quad \text{as } \epsilon \to 0,$$

(ii) *for* $n \in \mathbb{N}_{\geq 2}$

$$
\begin{aligned}
L \leq Cn\left(\log\left(\frac{1}{\epsilon}\right) + \log(2n+1)\right) \\
+ C\left(\log\left(\frac{1}{\epsilon}\right) + \log(2n+1)\right)^2 \left((\log(D))^2 + \left(\log\left(\frac{1}{\epsilon}\right) + \log(2n+1)\right)^2\right) \quad \text{as } \epsilon \to 0,
\end{aligned}
$$

such that

$$
\left\| \frac{1}{x^n} - \psi_{n,D,\epsilon}(x) \right\|_{L^\infty((1,D))} \leq \epsilon.
$$

The weights of $\psi_{n,D,\epsilon}$ *are bounded according to* $\mathcal{B}(\psi_{n,D,\epsilon}) \leq 1$.

Proof. The proof of this result may be found in Appendix A.2 for $n = 1$ and in Appendix A.3 for $n \in \mathbb{N}_{\geq 2}$. $\square$

4. Wavenumber-robust deep ReLU-NN emulation

In this section, we establish the existence of ReLU-NNs emulating $\varphi_\kappa \in L^2(\Gamma)$ with a robust dependence[1] of the depth, width and weights bounds on the wavenumber, and spectral convergence with respect to the target accuracy.

A result in this direction is established in the following proposition, which is in turn based on Corollary 2.9 and Proposition B.3.

Proposition 4.1 (κ-Explicit ReLU-NN Emulation of V_κ). *Let Assumption 2.3 be satisfied. For each* $n \in \mathbb{N}_{\geq 2}$ *there exist* $C_n > 0$ *independent of* κ *and ReLU-NNs* $V_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,15,1,2}$*, i.e., one input and two outputs accounting for the real and imaginary parts of* V_κ *separately, and*

$$
L \leq C_n (1+\kappa)^{\frac{2}{3}} \left((1+\kappa)^{\frac{2}{3}} \epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) \right) \quad \text{as } \epsilon \to 0
$$

satisfying

$$
\left\| V_\kappa - (V_{\kappa,n,\epsilon})_1 - \imath (V_{\kappa,n,\epsilon})_2 \right\|_{L^\infty((0,2\pi))} \leq \epsilon,
$$

where $(V_{\kappa,n,\epsilon})_1$ *and* $(V_{\kappa,n,\epsilon})_2$ *represent the first and second outputs of* $V_{\kappa,n,\epsilon}$*, respectively, with weights bounded in absolute value by* $C_n(1+\kappa)^{\frac{(n-1)}{3}}$ *for some constant* $C_n > 0$ *independent of* κ *and dependent on* $n \in \mathbb{N}_{\geq 2}$.

Proof. According to Lemma 2.9 the function V_κ in (2.10) belongs to $\mathcal{P}_{\boldsymbol{\lambda}_\kappa, 2\pi, \mathbb{C}}$ (cf. Def. B.2) with $D = 2\pi$ and $\boldsymbol{\lambda}_\kappa = \{\lambda_{n,\kappa}\}_{n \in \mathbb{N}_0}$ with $\lambda_{n,\kappa} = C_n (1+\kappa)^{\frac{(n-1)}{3}}$ for each $n \in \mathbb{N}_0$, where each $C_n > 0$ does not depend on κ.

It follows from Proposition B.3 that for each $n \in \mathbb{N}_{\geq 2}$ there exist $\widetilde{C}_{n,\kappa} > 0$, (which in this case does depend on κ), and ReLU-NNs $V_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,15,1,2}$ with

$$
L \leq \widetilde{C}_{n,\kappa} \left(\widetilde{C}_{n,\kappa} \epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) \right) \quad \text{as } \epsilon \to 0,
$$

and satisfying

$$
\left\| V_\kappa - (V_{\kappa,n,\epsilon})_1 - \imath (V_{\kappa,n,\epsilon})_2 \right\|_{L^\infty((0,2\pi))} \leq \epsilon.
$$

Here, according to the bound in (B.9), $\widetilde{C}_{n,\kappa} \leq C_n (1+\kappa)^{\frac{2}{3}}$. This concludes the proof. $\square$

[1]We use the terminology 'robust' here in the sense of sub-polynomial or poly-logarithmic dependence on κ.

The result presented in Proposition 4.1 provides a ReLU-NN emulation of the Neumann datum with spectral accuracy, fixed width, and increasing depth with an explicit dependence on the problem's wavenumber. However, this result is not entirely satisfactory as the required number of layers increases algebraically with the wavenumber, as opposed to a wavenumber-robust dependence that would be obtained by having a number of layers that increases poly-logarithmically with the wavenumber.

The reason for this behaviour is that Lemma 2.9 provides uniform bounds for the derivatives of V_κ over the entire interval $[0, 2\pi]$. It is well-understood that close to the tangency points t_1 and t_2 (cf. Fig. 1a) the appearance of transition layers of size $\mathcal{O}(\kappa^{-\frac{1}{3}})$ thwarts the convergence of traditional BEM discretizations.

In order to obtain wavenumber-robust ReLU-NN emulation of the solution to the BIE (2.6) one needs to resort to Theorem 2.4. In particular, the asymptotic expansion stated in (2.11) uses the composition operation to describe the behaviour of the normal derivative. This operation is a key building block in the construction of ReLU-NNs, regardless of the chosen activation function.

The above observation naturally suggests the use of ReLU-NNs for the approximation of V_κ, the normal derivative, and, in turn, the approximation of the scattered field.

4.1. Wavenumber robust ReLU-NN emulation of Fock's integral and its derivatives

We consider the approximation of Fock's integral as in (2.12) within the interval

$$\mathcal{I}_\kappa := \left(-\kappa^{\frac{1}{3}} A, \kappa^{\frac{1}{3}} A\right), \tag{4.1}$$

for $A > 0$ that does not depend on the wavenumber κ.

The next result addresses the wavenumber-robust ReLU-NN emulation rate bounds of Fock's integral with a fixed width and a depth depending poly-logarithmically on the wavenumber. The proof of this result may be found in Appendix D.

Proposition 4.2 (Wavenumber-Robust ReLU-NN Emulation of Fock's Integral Ψ)**.** *For each $n \in \mathbb{N}_{\geq 2}$, there exist a constant $C_n > 0$ and ReLU-NNs $\Psi_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,30,1,2}$, i.e., one input and two outputs accounting for the real and imaginary parts of Ψ, and*

$$\begin{aligned} L \leq C_n \left(\log\left(\frac{1}{\epsilon}\right)\right)^2 \left(\log\left(\kappa^{\frac{1}{3}} A\right)^2 + \left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right) \\ + B_{n,\epsilon}\left(B_{n,\epsilon}\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{2}{n-1}} \log\left(\frac{1}{\epsilon}\right)\right) \quad \text{as } \epsilon \to 0, \end{aligned} \tag{4.2}$$

with $B_{n,\epsilon} \leq C_n \epsilon^{-\frac{n+1}{(n-1)(3n+2)}}$ and for

$$\kappa \geq \left(\frac{2}{A}\right)^3 \max\left\{\left(2\frac{C_n}{\epsilon}\right)^{\frac{3}{3(n+1)-1}}, 1\right\}, \tag{4.3}$$

satisfying

$$\left\| \Psi - (\Psi_{\kappa,n,\epsilon})_1 - \imath (\Psi_{\kappa,n,\epsilon})_2 \right\|_{L^\infty(\mathcal{I}_\kappa)} \leq \epsilon.$$

The weights of $\Psi_{\kappa,n,\epsilon}$ are bounded in absolute value by a constant independent of κ.

The next result addresses the ReLU-NN emulation of the higher-order derivatives of Fock's integral. The proof of this result follows similar steps to that of Proposition 4.2, thus here we only highlight the main differences. Details may also be found in Appendix D.

Proposition 4.3 (Wavenumber-Robust ReLU-NN Emulation of $\Psi^{(\ell)}$)**.** *For each $\ell \in \mathbb{N}$ and $n \in \mathbb{N}_{\geq 2}$ there exists a constant $C_{n,\ell} > 0$ and ReLU-NNs $\Psi_{\kappa,n,\ell,\epsilon} \in \mathcal{NN}_{L,30,1,2}$, i.e., one input and two outputs accounting for the real and imaginary parts of $\Psi^{(\ell)}$, and*

$$L \leq C_{n,\ell}\left(\log\left(\frac{1}{\epsilon}\right)\right)^2\left(\log(\kappa) + \log\left(\kappa^{\frac{1}{3}}A\right)^2 + \log\left(\frac{1}{\epsilon}\right)\right) + B_{n,\ell,\epsilon}\left(B_{n,\ell,\epsilon}\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{2}{n-1}}\log\left(\frac{1}{\epsilon}\right)\right) \tag{4.4}$$

with

$$B_{n,\ell,\kappa,\epsilon} \leq C_{n,\ell}\left(\log\left(\kappa^{\frac{1}{3}}A\right) + \log\left(\frac{1}{\epsilon}\right)\right)^{\frac{n+1}{n-1}}\epsilon^{-\frac{n+1}{(n-1)(3n+\ell+2)}},$$

as $\epsilon \to 0$, and for

$$\kappa \geq \left(\frac{2}{A}\right)^3 \max\left\{\left(2\frac{C_n}{\epsilon}\right)^{\frac{3}{3(n+1)+\ell-1}}, 1\right\}, \tag{4.5}$$

satisfying

$$\left\|\Psi^{(\ell)} - (\Psi_{\kappa,n,\ell,\epsilon})_1 - \imath(\Psi_{\kappa,n,\ell,\epsilon})_2\right\|_{L^\infty(\mathcal{I}_\kappa)} \leq \epsilon.$$

The weights of $\Psi_{\kappa,n,\ell,\epsilon}$ are bounded in absolute value by a constant independent of κ.

4.2. Wavenumber-robust approximation of the Neumann trace

We present three different results that aim at approximating the Neumann trace of the sound-soft acoustic scattering problem in two dimensions. Our approach relies on the decomposition stated in (2.10). We proceed to describe these three different statements:

(i) **Wavenumber-Robust ReLU-NN Emulation of V_κ.** In Theorem 4.4, we provide a constructive wavenumber-robust ReLU-NN emulation of the function V_κ appearing in (2.10). The construction of such ReLU-NN makes use of the following ingredients. Proposition 2.6 provides a wavenumber-explicit asymptotic expansion of V_κ. Propositions 4.2 and 4.3 provide wavenumber robust ReLU-NN emulation of Fock's integral and its derivatives. By combining these two results, we obtain a wavenumber robust ReLU-NN emulation of V_κ.

(ii) **Partial Wavenumber-Robust Approximation of the Neumann Trace.** The second result, presented in Corollary 4.5, aims at providing a first, preliminary wavenumber-robust ReLU-NN emulation result of the Neumann trace. Therein, however, we still incorporate explicitly the oscillatory component of the solution.

(iii) **Full Wavenumber-Robust Approximation of the Neumann Trace.** Finally, in Theorem 4.6, we construct a fully wavenumber robust approximation of the Neumann trace φ_κ. As opposed to Corollary 4.5, we provide a complete ReLU-NN emulation including, in particular, NN emulation of the oscillatory component of the Neumann trace.

Theorem 4.4 (Wavenumber-Robust ReLU-NN Emulation of V_κ)**.** *Let Assumption 2.3 be satisfied. For each $n \in \mathbb{N}_{\geq 2}$ there exist a constant $C_n > 0$ and ReLU-NNs $V_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,38,1,2}$, i.e., one input and two output channels accounting for the real and imaginary part of V_κ separately, such that for every $0 < \epsilon \leq 1$ and for every $\kappa \geq C_n\epsilon^{-\frac{3}{4n+2}}$, with*

$$L \leq C_n\left(\log\left(\frac{1}{\epsilon}\right)\right)^2\left(\log(\kappa)^2 + \left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right) + B_{n,\kappa,\epsilon}\left(B_{n,\kappa,\epsilon}\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}}\log\left(\frac{2}{\epsilon}\right)\right) \tag{4.6}$$

where

$$B_{n,\kappa,\epsilon} \le C_n \left(\log(\kappa) + \log\left(\frac{1}{\epsilon}\right)\right)^{\frac{n+1}{n-1}} \epsilon^{-\frac{n+1}{(n-1)(4n+2)}} \quad \epsilon \to 0,$$

there holds

$$\left\| V_\kappa - (V_{\kappa,n,\epsilon})_1 - \imath (V_{\kappa,n,\epsilon})_2 \right\|_{L^\infty((0,2\pi))} \le \epsilon.$$

The weights of $V_{\kappa,n,\epsilon}$ *are bounded in absolute value by* C_n*.*

Proof. Assumption 2.3 allows to use the asymptotic expansions in Section 2.5.

Let us consider Proposition 2.6, in particular the decomposition stated in (2.14), which reads as follows: for any L, $M \in \mathbb{N}_0$ it holds

$$V_\kappa(s) = \sum_{\ell,m=0}^{L,M} \kappa^{-\frac{1}{3}-\frac{2}{3}\ell-m} b_{\ell,m}(s) \Psi^{(\ell)}\left(\kappa^{\frac{1}{3}} Z(s)\right) + R_{L,M,\kappa}(s), \quad s \in [0, 2\pi], \tag{4.7}$$

with remainder term $R_{L,M,\kappa}$ satisfying (2.15).

In view of (2.16) for each $n \in \mathbb{N}$ we select $L = M = n$, thus yielding for each $j \in \mathbb{N}_0$

$$\left| D_s^j R_{L,M,\kappa}(s) \right| \le C_{L,M,j} (1+\kappa)^{-\frac{2}{3}(n+1)+\frac{j}{3}}, \quad s \in [0, 2\pi], \tag{4.8}$$

where $C_{L,M,j}$ does not depend on κ.

For the sake of simplicity, we write the double sum in (2.14) as a single one. To this end, we uniquely enumerate the pair of indices in the double sum through a bijection $\{1,\dots,(n+1)^2\} \ni \ell \mapsto (k_{1,\ell}, k_{2,\ell}) \in \{0,\dots,n\} \times \{0,\dots,n\}$ and obtain

$$V_\kappa(s) = \sum_{\ell=1}^{(n+1)^2} \kappa^{-\frac{1}{3}-\frac{2}{3}k_{1,\ell}-k_{2,\ell}} b_{k_{1,\ell},k_{2,\ell}}(s) \Psi^{(k_{1,\ell})}\left(\kappa^{\frac{1}{3}} Z(s)\right) + R_{L,M,\kappa}(s), \quad s \in [0, 2\pi].$$

The following claims follow from Proposition B.3 in Appendix B (the definition of $\mathcal{P}_{\boldsymbol{\lambda},2\pi,\mathbb{C}}$ has been introduced in Def. B.2):

(i) The bound stated in (4.8) implies that for each $n \in \mathbb{N}$ one has $R_{L,M,\kappa} \in \mathcal{P}_{\boldsymbol{\lambda},2\pi,\mathbb{C}}$ with

$$\boldsymbol{\lambda}_n := \{\lambda_{j,n}\}_{j\in\mathbb{N}_0} \quad \text{and} \quad \lambda_{j,n} := C_{L,M,j}(1+\kappa)^{-\frac{2}{3}(n+1)+\frac{j}{3}}, \quad j \in \mathbb{N}_0.$$

For $j = 0,\dots,2(n+1)$ it holds $\left| D_s^j R_{L,M,\kappa}(s) \right| \le C_{L,M,j}$, where $C_{L,M,j}$ is as in (4.8) and does not depend on κ. According to Proposition B.3, for each $n \in \mathbb{N}_{\ge 2}$ there exist ReLU-NNs $\phi_{n,\epsilon} \in \mathcal{NN}_{L_R(\epsilon),15,1,2}$ with

$$L_R(\epsilon) \le C_n \left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}} \right) \quad \text{as } \epsilon \to 0$$

such that

$$\left\| R_{L,M,\kappa}(s) - \left((\phi_{n,\epsilon}(s))_1 + \imath (\phi_{n,\epsilon}(s))_2 \right) \right\|_{L^\infty((0,2\pi))} \le \epsilon,$$

with $C_n > 0$ independent of κ. The weights of $\phi_{n,\epsilon}$ are bounded in absolute value by a constant depending on n but not on κ or ϵ.

(ii) Item (ii) in Theorem 2.4 and Proposition 2.6 imply that for each $\ell, m \in \mathbb{N}_0 \cap [0,\dots,n]$ one has $b_{\ell,m} \in \mathcal{P}_{\boldsymbol{\lambda},2\pi,\mathbb{C}}$ with $\boldsymbol{\lambda} := \{\lambda_n\}_{n\in\mathbb{N}_0}$ independent of κ. According to Proposition B.3 for each $n \in \mathbb{N}_{\ge 2}$ there exist $C_n > 0$ independent of κ and ReLU-NNs $\beta_{\ell,m,n,\epsilon} \in \mathcal{NN}_{L,15,1,2}$ with

$$L_{b_{\ell,m}}(\epsilon) \le B_n \left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}} \right) \quad \text{as } \epsilon \to 0$$

such that

$$\left\| b_{\ell,m} - \left((\beta_{\ell,m,n,\epsilon})_1 + \imath (\beta_{\ell,m,n,\epsilon})_2 \right) \right\|_{L^\infty((0,2\pi))} \leq \epsilon.$$

Again, the weights of $\beta_{\ell,m,n,\epsilon}$ are bounded in absolute value by a constant depending on n but not on κ.

(iii) Item (ii) in Remark 2.5 and item (ii) in Theorem 2.4 imply that $Z \in \mathcal{P}_{\boldsymbol{\lambda},2\pi,\mathbb{R}}$ with a sequence $\boldsymbol{\lambda}_\kappa \coloneqq \{\lambda_n\}_{n\in\mathbb{N}_0}$ independent of κ. According to Proposition B.3, for each $n \in \mathbb{N}_{\geq 2}$ there exist $C_n > 0$ independent of κ and ReLU-NNs $\Phi_{Z,n,\epsilon} \in \mathcal{NN}_{L_Z(\epsilon),13,1,2}$ with

$$L_Z(\epsilon) \leq C_n \left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}} \right) \quad \text{as } \epsilon \to 0$$

such that

$$\| Z - \Phi_{Z,n,\epsilon} \|_{L^\infty((0,2\pi))} \leq \epsilon.$$

Define $T_\kappa(x) = \kappa^{\frac{1}{3}} x$, which according to (3.1a) belongs to $\mathcal{NN}_{2,2,1,1}$. Furthermore, according to Proposition C.5 there exists a ReLU-NN (which we still refer to as T_κ) emulating exactly this map. Indeed, $T_\kappa \in \mathcal{NN}_{L,3,1,2}$ with $L \leq 5 + \frac{1}{3}\log(\kappa)$. Define the ReLU-NN $\Lambda_{\kappa,n,\epsilon} = T_\kappa \circ \Phi_{Z,n,\epsilon}$. For each $n \in \mathbb{N}$ one has $\Lambda_{\kappa,n,\epsilon} \in \mathcal{NN}_{L_{\Lambda_{\kappa,n,\epsilon}}(\epsilon),13,1,1}$ with

$$L_{\Lambda_{\kappa,n,\epsilon}}(\epsilon) \leq 5 + \frac{1}{3}\log(\kappa) + C_n \left(\left(\frac{\kappa^{\frac{1}{3}}}{\epsilon}\right)^{\frac{2}{n-1}} + \left(\frac{\kappa^{\frac{1}{3}}}{\epsilon}\right)^{\frac{1}{n-1}} \log\left(\frac{\kappa^{\frac{1}{3}}}{\epsilon}\right) + \left(\frac{\kappa^{\frac{1}{3}}}{\epsilon}\right)^{\frac{1}{n-1}} \right), \tag{4.9}$$

as $\epsilon \to 0$, satisfying

$$\left\| \kappa^{\frac{1}{3}} Z - \Lambda_{\kappa,n,\epsilon} \right\|_{L^\infty((0,2\pi))} \leq \epsilon,$$

and with weights bounded in absolute value by a constant independent of κ, yet still depending on $n \in \mathbb{N}_{\geq 2}$. The polynomial dependence on the wavenumber in (4.9) at this point does not directly lead to wavenumber-robust bounds. However, ahead we will see how the polynomial dependence on the wavenumber can be mitigated.

(iv) For each $n \in \mathbb{N}_{\geq 2}$, Propositions 4.2 and 4.3 assert the existence of ReLU-NN $\Psi_{\kappa,n,\epsilon}$ and $\Psi_{\kappa,\ell,n,\epsilon}$ emulating Fock's integral Ψ and its derivatives, respectively. We set $A \coloneqq 2\max_{s\in[0,2\pi]}|Z(s)| > 0$ in the definition of $\mathcal{I}_\kappa$ and denote by $L_{\Psi^{(\ell)}}(\epsilon)$ the depth of the ReLU-NN approximating $\Psi_{\kappa,n,\epsilon}$ and $\Psi_{\kappa,\ell,n,\epsilon}$, as described in Propositions 4.2 and 4.3, with accuracy $\epsilon > 0$.

We are interested in the wavenumber-robust ReLU-NN emulation of terms of the form $b_{\ell,m}\Psi^{(\ell)} \circ Z_\kappa$, for $\ell, m \in \mathbb{N}_0$, which appear in (4.7). We recall Lemma C.4 in Appendix C. One can readily verify that items (ii) through (iv) satisfy the assumptions of said result. Therefore, given $\ell, m \in \mathbb{N}_0$, we conclude that for each $n \in \mathbb{N}_{\geq 2}$ there exist ReLU-NNs $\Sigma_{n,\ell,m,\epsilon} \in \mathcal{NN}_{L,32,1,2}$ with

$$\begin{aligned} L \leq\ & C\left(\log(\lceil U_{\ell,m,\kappa} \rceil) + \log\left(\frac{12(n+1)^2}{\epsilon\kappa^{\frac{1}{3}}} \right) \right) \\ & + 2\, \underbrace{L_{b_{\ell,m}}\left(\frac{\epsilon\kappa^{\frac{1}{3}}}{24(n+1)^2 \|b_{\ell,m}\|_{L^\infty(\mathcal{I}_{2\pi})} \left\|\Psi^{(\ell+1)}\right\|_{L^\infty(\mathcal{I}_\kappa)}} \right)}_{\text{(I)}} \\ & + 2\, \underbrace{L_{\Psi^{(\ell)}}\left(\frac{\epsilon\kappa^{\frac{1}{3}}}{(n+1)^2 \|b_{\ell,m}\|_{L^\infty(\mathcal{I}_{2\pi})}} \right)}_{\text{(II)}} \\ & + 2\, \underbrace{L_{\Lambda_{\kappa,n,\epsilon}}\left(\frac{\epsilon\kappa^{\frac{1}{3}}}{24(n+1)^2 \|b_{\ell,m}\|_{L^\infty(\mathcal{I}_{2\pi})} \left\|\Psi^{(\ell+1)}\right\|_{L^\infty(\mathcal{I}_\kappa)}} \right)}_{\text{(III)}}, \end{aligned} \tag{4.10}$$

where

$$U_{\ell,m,\kappa} = 2\max\left\{\|b_{\ell,m}\|_{L^\infty(\mathcal{I}_{2\pi})}, \left\|\Psi^{(\ell+1)}\right\|_{L^\infty(\mathcal{I}_\kappa)}\right\}$$

satisfying

$$\left\|b_{\ell,m}\Psi^{(\ell)} \circ Z_\kappa - \Sigma_{n,\ell,m,\epsilon}\right\|_{L^\infty(\mathcal{I}_D)} \leq \frac{\epsilon\kappa^{\frac{1}{3}}}{(n+1)^2}. \tag{4.11}$$

We proceed to expand the terms on the right-hand side of (4.10) term-by-term. Firstly, by recalling that $b_{\ell,m}$ does not depend on κ and that, according to Remark 2.8, for any $\ell \in \mathbb{N}$ one has $\left\|\Psi^{(\ell)}\right\|_{L^\infty(\mathcal{I}_\kappa)} \leq C_\ell$, with C_ℓ independent of κ for each $\ell \in \mathbb{N}$. Therefore, for each $n, \ell \in \mathbb{N}$ there exists $B_{n,\ell} > 0$ independent of κ such that

$$\text{(I)} \leq B_{n,\ell}\left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}}\log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}}\right) \quad \text{as } \epsilon \to 0.$$

The second term, labelled (II) in (4.10), is bounded as in (4.2) and (4.4), for $\ell = 0$ and $\ell \in \mathbb{N}$, respectively, and effectively remains unchanged with possible different constants. The third term in (4.10), *i.e.*, (III), is bounded according to

$$\text{(III)} \leq 5 + \frac{1}{3}\log(\kappa) + C_m\left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}}\log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}}\log(\kappa)\right),$$

with C_n independent of κ. Observe that the presence of the wavenumber in the terms of the form $\epsilon^{-\frac{1}{n-1}}$ and $\epsilon^{-\frac{2}{n-1}}$ is cancelled due to the required accuracy stated in (4.11). Next, set

$$V_{\kappa,n,\epsilon}(s) := \sum_{\ell=1}^{(n+1)^2} \kappa^{-\frac{1}{3}-\frac{2}{3}k_{1,\ell}-k_{2,\ell}}\Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon}(s) + \phi_{n,\epsilon}(s), \quad s \in [0, 2\pi].$$

Therefore, we have

$$\begin{aligned}
&V_\kappa(s) - (V_{\kappa,n,\epsilon}(s))_1 - \imath(V_{\kappa,n,\epsilon}(s))_2 \\
&\qquad = \sum_{\ell=1}^{(n+1)^2} \kappa^{-\frac{1}{3}-\frac{2}{3}k_{1,\ell}-k_{2,\ell}}\left(b_{k_{1,\ell},k_{2,\ell}}(s)\Psi^{(k_{1,\ell})}\left(\kappa^{\frac{1}{3}}Z(s)\right) - \Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon}\right) \\
&\qquad\quad + R_{L,M}(s,\kappa) - \left((\phi_{n,\epsilon}(s))_1 + \imath(\phi_{n,\epsilon}(s))_2\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|V_\kappa(s) - (V_{\kappa,n,\epsilon}(s))_1 - \imath(V_{\kappa,n,\epsilon}(s))_2\right\|_{L^\infty((0,2\pi))} \\
&\qquad \leq \sum_{\ell=1}^{(n+1)^2} \kappa^{-\frac{1}{3}-\frac{2}{3}k_{1,\ell}-k_{2,\ell}}\left\|\Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon} - b_{k_{1,\ell},k_{2,\ell}}\Psi^{(k_{1,\ell})}\left(\kappa^{\frac{1}{3}}Z\right)\right\|_{L^\infty((0,2\pi))} \\
&\qquad\quad + \|\phi_{n,\epsilon}(s) - R_{L,M}(s,\kappa)\|_{L^\infty((0,2\pi))} \\
&\qquad \leq \epsilon.
\end{aligned}$$

We express $V_{\kappa,n,\epsilon}$ as a ReLU-NN of fixed width. To this end, we define

$$\Theta_{0,n,\epsilon}(x) := \begin{pmatrix} (\phi_{n,\epsilon}(x))_1 \\ (\phi_{n,\epsilon}(x))_2 \end{pmatrix}$$

and

$$\Theta_{n,\ell,\epsilon}(x_1, x_2, x_3) := A_\ell \begin{pmatrix} (\Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon}(x_1))_1 \\ (\Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon}(x_1))_2 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \ell = 1, \ldots, (n+1)^2,$$

with $A_\ell \in \mathbb{R}^{3\times 5}$ given by

$$A_\ell = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ \kappa^{-\frac{1}{3}-\frac{2}{3}k_{1,\ell}-k_{2,\ell}} & 0 & 0 & 1 & 0 \\ 0 & \kappa^{-\frac{1}{3}-\frac{2}{3}k_{1,\ell}-k_{2,\ell}} & 0 & 0 & 1 \end{pmatrix}.$$

One can readily see that

$$V_{\kappa,n,\epsilon}(s) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \big(\Theta_{n,(n+1)^2,\epsilon} \circ \cdots \circ \Theta_{n,1,\epsilon} \circ \Theta_{n,0,\epsilon}\big)(s).$$

Therefore

$$\begin{aligned} \mathcal{M}(V_{\kappa,n,\epsilon}) &= \max_{\ell \in \{0,\dots,(n+1)^2\}} \mathcal{M}(\Theta_{n,\ell,\epsilon}) \\ &= 6 + \max_{\ell \in \{0,\dots,(n+1)^2\}} \mathcal{M}\big(\Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon}\big) = 38 \end{aligned}$$

and

$$\mathcal{L}(V_{\kappa,n,\epsilon}) = \sum_{\ell=0}^{(n+1)^2} \mathcal{L}(\Theta_{\ell,\epsilon}) = \mathcal{L}(\phi_{n,\epsilon}) + \sum_{\ell=1}^{(n+1)^2} \mathcal{L}\big(\Sigma_{n,k_{1,\ell},k_{2,\ell},\epsilon}\big),$$

which yields (4.6). □

Corollary 4.5 (Partial Wavenumber-Robust ReLU-NN Emulation of the Neumann Trace)**.** *Let Assumption 2.3 be satisfied. For each $n \in \mathbb{N}_{\geq 2}$ there exist $C_n > 0$ and ReLU-NNs $V_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,38,1,2}$ with L as in (4.6), and $\Phi_{\gamma,\hat{\mathbf{d}},n,\epsilon} \in \mathcal{NN}_{L,13,1,1}$ with*

$$L \leq C_n\left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}}\right)$$

satisfying

$$\left\| \hat{\varphi}_\kappa - \kappa\big((V_{\kappa,n,\epsilon})_1 + \imath (V_{\kappa,n,\epsilon})_2\big) \exp\Big(\imath\kappa \Phi_{\gamma,\hat{\mathbf{d}},n,\epsilon}\Big) \right\|_{L^\infty((0,2\pi))} \leq \epsilon,$$

where $(V_{\kappa,n,\epsilon})_1$ and $(V_{\kappa,n,\epsilon})_2$ represent the first and second outputs of $V_{\kappa,n,\epsilon}$, respectively, C_n is independent of κ, and $\hat{\varphi}_\kappa$ is as in (2.10). The weights of $\Phi_{\boldsymbol{\gamma},\hat{\mathbf{d}},n,\epsilon}$ are bounded in absolute value by a constant independent of κ.

Proof. It follows from Proposition B.3 that for each $n \in \mathbb{N}_{\geq 2}$ there exist ReLU-NNs $\Phi_{\boldsymbol{\gamma},\hat{\mathbf{d}},n,\epsilon} \in \mathcal{NN}_{L,13,1,1}$ with

$$L \leq C_n\left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}} \log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}}\right) \quad \text{as } \epsilon \to 0 \tag{4.12}$$

such that

$$\left\| \boldsymbol{\gamma} \cdot \hat{\mathbf{d}} - \Phi_{\boldsymbol{\gamma},\hat{\mathbf{d}},n,\epsilon} \right\|_{L^\infty(\mathcal{I}_{2\pi})} \leq \epsilon,$$

with weights bounded in absolute value by a constant depending on $n \in \mathbb{N}$, but not on the wavenumber κ. □

Theorem 4.6 (Full Wavenumber-Robust ReLU-NN Emulation of the Neumann Trace)**.** *There exists a constant $C > 0$ such that for each $n \in \mathbb{N}_{n \geq \max\{C\log(\kappa),2\}}$ there exist $C_n > 0$ and ReLU-NNs $\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,42,1,2}$, i.e., one input and two outputs for real and imaginary parts of $\hat{\varphi}_\kappa$, of depth*

$$\begin{aligned} L \leq\ & C_n\left(\log(\kappa) + \log\left(\frac{1}{\epsilon}\right)\right)^2 \left(\log(\kappa)^2 + \left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right) \\ & + B_{n,\kappa,\epsilon}\left(1 + B_{n,\kappa,\epsilon} + \log\left(\frac{1}{\epsilon}\right)\right)\epsilon^{-\frac{2}{n-1}} \quad \text{as } \epsilon \to 0 \end{aligned} \tag{4.13}$$

and

$$B_{n,\kappa,\epsilon} \leq C_n \left(\log(\kappa) + \log\left(\frac{1}{\epsilon}\right)\right)^{\frac{n+1}{n-1}} \epsilon^{-\frac{n+1}{(n-1)(4n+2)}} \quad \text{as } \epsilon \to 0$$

satisfying for $\kappa \geq C_n \epsilon^{-\frac{3}{4n+2}}$

$$\left\| \hat{\varphi}_\kappa - \left(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\right)_1 - \imath \left(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\right)_2 \right\|_{L^\infty((0,2\pi))} \leq \epsilon,$$

where $\left(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\right)_1$ *and* $\left(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\right)_2$ *represent the first and second outputs of* $\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}$*, respectively. The weights of* $\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}$ *are bounded in absolute value by a constant independent of* κ*, however still depending on* $n \in \mathbb{N}$.

Proof. Firstly, it follows from the decomposition stated in (2.10) that

$$\hat{\varphi}_\kappa(s) = V_\kappa(s) g_\kappa\left(\boldsymbol{\gamma} \cdot \hat{\mathbf{d}}(s)\right), \quad s \in [0, 2\pi], \tag{4.14}$$

with $g_\kappa(x) = \kappa \exp(\imath \kappa x)$.

The proof of this theorem relies on the application of Lemma C.4 to (4.14). We proceed to verify the hypothesis of this result item-by-item.

(i) According to Theorem 4.4, for each $n \in \mathbb{N}_{\geq 2}$ there exist $C_n > 0$ and ReLU-NNs $V_{\kappa,n,\epsilon} \in \mathcal{NN}_{L_{V_\kappa}(\epsilon),38,1,2}$, *i.e.*, with one input and two outputs (real and imaginary part, respectively), such that

$$\left\| V_\kappa - (V_{\kappa,n,\epsilon})_1 - \imath (V_{\kappa,n,\epsilon})_2 \right\|_{L^\infty((0,2\pi))} \leq \epsilon,$$

with depth $L_{V_\kappa}(\epsilon)$ depending on $\epsilon > 0$ as in (4.6).

(ii) Set $\zeta_k = \exp(\imath \kappa x)$. It follows from Proposition 3.5 that there exist ReLU-NNs $\Phi_{\zeta_k,\epsilon} \in \mathcal{NN}_{L_{\zeta_\kappa}(\epsilon),18,1,2}$ with

$$L_{\zeta_k}(\epsilon, B) \leq C\left(\left(\log\left(\frac{1}{\epsilon}\right)\right)^2 + \log(\kappa B)\right) \quad \text{as } \epsilon \to 0,$$

with $C > 0$ independent of κ, such that

$$\left\| \zeta_\kappa - (\Phi_{\zeta_\kappa,\epsilon})_1 - \imath (\Phi_{\zeta_\kappa,\epsilon})_2 \right\|_{L^\infty((-B,B))} \leq \epsilon,$$

and with weights bounded in absolute value by a constant independent of κ. According to Proposition C.5, the map $T_\kappa(x) = \kappa x$ can be exactly expressed by a ReLU-NN belonging to $\mathcal{NN}_{L,3,1,1}$ with $L \leq 5 + \log(\kappa)$, and with weights bounded in absolute value by one. Consequently, the ReLU-NN

$$\Phi_{g_\kappa,\epsilon}(x) = \begin{pmatrix} T_\kappa \circ (\Phi_{\zeta_k,\frac{\epsilon}{\kappa}}(x))_1 \\ T_\kappa \circ (\Phi_{\zeta_k,\frac{\epsilon}{\kappa}}(x))_2 \end{pmatrix} \in \mathcal{NN}_{L_{g_k}(\epsilon,B),18,1,2}$$

emulates g_κ according to

$$\left\| g_\kappa - (\Phi_{g_\kappa,\epsilon})_1 - \imath (\Phi_{g_\kappa,\epsilon})_2 \right\|_{L^\infty((-B,B))} \leq \epsilon \quad \text{as } \epsilon \to 0,$$

with

$$L_{g_k}(\epsilon, B) \leq 5 + \log(\kappa) + C\left(\left(\log\left(\frac{\kappa}{\epsilon}\right)\right)^2 + \log(\kappa B)\right) \quad \text{as } \epsilon \to 0,$$

where $C > 0$ does not depend on κ, B or $\epsilon > 0$.

(iii) As in the proof of Corollary 4.5, it follows from Proposition B.3 that for each $n \in \mathbb{N}_{\geq 2}$ there exist ReLU-NNs $\Phi_{\gamma,\hat{\mathbf{d}},n,\epsilon} \in \mathcal{NN}_{L_{\gamma,\hat{\mathbf{d}}}(\epsilon),13,1,1}$ with

$$L_{\gamma,\hat{\mathbf{d}}}(\epsilon) \leq C_n\left(\epsilon^{-\frac{2}{n-1}} + \epsilon^{-\frac{1}{n-1}}\log\left(\frac{1}{\epsilon}\right) + \epsilon^{-\frac{1}{n-1}}\right), \quad \text{as } \epsilon \to 0, \tag{4.15}$$

such that $\|\boldsymbol{\gamma}\cdot\hat{\mathbf{d}} - \Phi_{\gamma,\hat{\mathbf{d}},n,\epsilon}\|_{L^\infty(\mathcal{I}_{2\pi})} \leq \epsilon$, with weights bounded in absolute value by a constant depending on $n \in \mathbb{N}$ but not on κ.

Set $D_{\boldsymbol{\gamma}} = \|\boldsymbol{\gamma}\|_{L^\infty((0,2\pi))}$. Therefore, according to Lemma C.4 there exist ReLU-NNs $\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon} \in \mathcal{NN}_{L,42,1,2}$

$$\begin{aligned} L \leq C\left(\log(\lceil U\rceil) + \log\left(\frac{12}{\epsilon}\right)\right) + 2L_{V_\kappa}\left(\frac{\epsilon}{24\|V_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}\|g'_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}}\right) \\ + 2L_{g_\kappa}\left(\frac{\epsilon}{24\|V_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}}, 2D_{\boldsymbol{\gamma}}\right) + 2L_{\gamma,\hat{\mathbf{d}}}\left(\frac{\epsilon}{24\|V_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}\|g'_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}}\right) \end{aligned} \tag{4.16}$$

satisfying

$$\left\|\hat{\varphi}_\kappa - \left(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\right)_1 - \imath\left(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\right)_2\right\|_{L^\infty((0,2\pi))} \leq \epsilon,$$

where

$$U = 2\max\left\{\|V_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}, \|g'_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}\right\}.$$

Clearly, $\|g'_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})} = \kappa^2$, and it follows from Lemma 2.9 and Remark 2.10 that $\|V_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}$ is bounded by a constant independent of κ.

Observe that

$$\begin{aligned} &L_{V_\kappa}\left(\frac{\epsilon}{24\|V_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}\|g'_\kappa\|_{L^\infty(\mathcal{I}_{2\pi})}}\right) \\ &\quad\leq C'_n\left(\log(\kappa) + \log\left(\frac{1}{\epsilon}\right)\right)^2\left(\log(\kappa)^2 + \log\left(\frac{\kappa}{\epsilon}\right) + \left(\log\left(\frac{\kappa}{\epsilon}\right)\right)^2\right) \\ &\quad\quad + B_{n,\kappa,\epsilon}\left(B_{n,\kappa,\epsilon}\left(\frac{\kappa^2}{\epsilon}\right)^{\frac{2}{n-1}} + \left(\frac{\kappa^2}{\epsilon}\right)^{\frac{1}{n-1}}\log\left(\frac{\kappa}{\epsilon}\right) + \left(\frac{\kappa^2}{\epsilon}\right)^{\frac{2}{n-1}}\log\left(\frac{\kappa}{\epsilon}\right) + \left(\frac{\kappa^2}{\epsilon}\right)^{\frac{1}{n-1}}\right) \end{aligned} \tag{4.17}$$

with

$$B_{n,\kappa,\epsilon} \leq C'_n\left(\log(\kappa) + \log\left(\frac{1}{\epsilon}\right)\right)^{\frac{n+1}{n-1}}\left(\frac{\kappa^2}{\epsilon}\right)^{\frac{n+1}{(n-1)(4n+2)}}.$$

This bound is not entirely wavenumber-robust. However, there exist $C, C' > 0$ independent of n and κ such that for $n \geq C\log(\kappa)$ we have $\kappa^{\frac{4}{n-1}} + \kappa^{\frac{2}{n-1}} + \kappa^{2\frac{n+1}{(n-1)(4n+2)}} \leq C'$, thus allowing us to remove these terms in (4.17). Similar considerations are valid for the last two terms in (4.16), thus yielding the bound stated in (4.13).

Finally, the weights are bounded in absolute value by a constant independent of κ and $\epsilon > 0$ (still depending on $n \in \mathbb{N}$). $\square$

4.3. Wavenumber-robust approximation of the far-field

We are interested in the effect of the ReLU-NN approximation of the Neumann trace on the far-field pattern of the scattered field. An asymptotic expansion of (2.3) yields the following expression for the far-field behaviour of u^{s}

$$u^{\mathrm{s}}(\mathbf{x}) \sim \frac{\exp(\imath\pi/4)}{2\sqrt{2\pi}}\frac{\exp(\imath\kappa r)}{\sqrt{\kappa r}}F(\hat{\mathbf{x}}) \quad \text{as } r := \|\mathbf{x}\| \to \infty,$$

where we have assumed that the origin of the Cartesian system of coordinates is contained in D, $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\| \in \mathbb{S}_1$, *i.e.*, the unit circle, and

$$\begin{aligned} F(\hat{\mathbf{x}}) &:= -\int_\Gamma \exp(-\imath\kappa\hat{\mathbf{x}}\cdot\mathbf{y})\varphi_\kappa(\mathbf{y})\,\mathrm{d}s_{\mathbf{y}} \\ &= -\int_0^{2\pi} \exp(-\imath\kappa\hat{\mathbf{x}}\cdot\boldsymbol{\gamma}(s))\hat{\varphi}_\kappa(s)\|\boldsymbol{\gamma}'(s)\|\,\mathrm{d}s, \quad \hat{\mathbf{x}} \in \mathbb{S}_1. \end{aligned}$$

Let $\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}$ be as in Theorem 4.6 and define

$$F_{\kappa,n,\epsilon}(\hat{\mathbf{x}}) = -\int_0^{2\pi} \exp(-\imath\kappa\hat{\mathbf{x}}\cdot\boldsymbol{\gamma}(s))\Big(\big(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\big)_1 + \imath\big(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\big)_2\Big)\|\boldsymbol{\gamma}'(s)\|\,\mathrm{d}s, \quad \hat{\mathbf{x}} \in \mathbb{S}_1. \tag{4.18}$$

Thus, it holds

$$\|F - F_{\kappa,n,\epsilon}\|_{L^\infty(\mathbb{S}^1)} \leq 2\pi\|\boldsymbol{\gamma}'\|_{L^\infty((0,2\pi))}\Big\|\hat{\varphi}_\kappa - \big(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\big)_1 - \imath\big(\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}\big)_2\Big\|_{L^\infty((0,2\pi))}.$$

Therefore, the constructed ReLU-NN $\hat{\varphi}^{\mathcal{NN}}_{\kappa,n,\epsilon}$ once is inserted in (4.18) produces an equally accurate approximation for the far-field pattern, of course, up to a constant which is independent of the wavenumber.

5. Concluding remarks

We proved wavenumber-robust ReLU-NN emulation rate bounds for the sound-soft acoustic scattering by a model smooth, strictly convex obstacle in two dimensions. Due to these assumptions on the scatterers' shapes issues like conical diffraction, trapping modes and characteristic boundary behaviors were avoided.

We present a constructive approximation rate bound that relies on two main tools. Firstly, we adopt a boundary reduction of the scattering problem using boundary potentials. This yields an *equivalent BIE equation of the second kind* for the unknown Neumann datum on the scatterers' surface. Second, using the asymptotic expansion by Melrose and Taylor [24], we provide a detailed construction of ReLU-NNs approximating the Neumann datum with NNs of moderate, fixed width, and with depth that increases spectrally with the target accuracy, while the depth bound depends poly-logarithmically on the wavenumber.

The present results concern only the *DNN approximation error*. As it is well-known (see, *e.g.*, [25]), in a DNN-based solution algorithm, the training and generalization errors must also be bounded. In the presently considered, exterior acoustic scattering problem, (2.1) and (2.2), the boundary reduction to the BIE (2.6) on the scatterer's profile Γ, similar to [1], reduces controlling all these errors in $L^2(\Gamma)$, rather than in the unbounded, exterior domain D^{c}. In addition, all quantities are periodic with respect to the arclength parametrization of Γ, for which efficient quadratures, such as the trapezoidal rule, are available.

We conclude this work by indicating extensions of the present results, and comment on aspects of possible DL algorithms based on the BIE (2.6). For $L, M \in \mathbb{N}$, let us consider the map

$$L^2((0,2\pi)) \supset \mathcal{NN}_{L,M,1,2} \ni \Phi_\kappa \mapsto \mathrm{Loss}(\Phi_\kappa) := \|\hat{\varphi}_\kappa - (\Phi_\kappa)_1 - \imath(\Phi_\kappa)_2\|^2_{L^2((0,2\pi))}. \tag{5.1}$$

Given a target accuracy $\epsilon > 0$, we aim to find a ReLU-NN $\Phi^\star_{\kappa,\epsilon} \in \mathcal{NN}_{L,M,1,2} \subset L^2((0,2\pi))$ such that

$$\mathrm{Loss}(\Phi^\star_\kappa) \leq \epsilon,$$

where the depth L and width M depend on κ and ϵ. Of course, the exact numerical evaluation of the loss function defined in (5.1) cannot be performed in practice, and numerical quadrature has to be used.

Assuming such quadrature at hand for now, and recalling Proposition 2.2, there exists a constant $\eta_0 > 0$ such that, given $\delta \in (0, 1/2)$, there exists κ_0 (depending on δ) such that for $\kappa \geq \kappa_0$ and $\eta \geq \eta_0\kappa$. Choosing η so that $\eta_0\kappa \leq \eta \lesssim \kappa$, we have that

$$\begin{aligned}
\|\hat{\varphi}_\kappa - (\Phi_\kappa)_1 - \imath(\Phi_\kappa)_2\|_{L^2((0,2\pi))} &\leq \frac{\sup_{s\in[0,2\pi]}\|\boldsymbol{\gamma}'(s)\|^{-1}}{\frac{1}{2}-\delta}\left\|\mathsf{A}'_{\kappa,\eta}\left(\varphi_\kappa - \tau_{\boldsymbol{\gamma}}^{-1}((\Phi_\kappa)_1 + \imath(\Phi_\kappa)_2)\right)\right\|_{L^2(\Gamma)} \\
&\leq \frac{\sup_{s\in[0,2\pi]}\|\boldsymbol{\gamma}'(s)\|^{-1}}{\frac{1}{2}-\delta}\left\|f_{\kappa,\eta} - \mathsf{A}'_{\kappa,\eta}\left(\tau_{\boldsymbol{\gamma}}^{-1}(\Phi_\kappa)_1 + \imath(\Phi_\kappa)_2\right)\right\|_{L^2(\Gamma)} \\
&\leq C(\delta,\boldsymbol{\gamma})\left\|\tau_{\boldsymbol{\gamma}} f_{\kappa,\eta} - \tau_{\boldsymbol{\gamma}}\mathsf{A}'_{\kappa,\eta}\tau_{\boldsymbol{\gamma}}^{-1}((\Phi_\kappa)_1 + \imath(\Phi_\kappa)_2)\right\|_{L^2((0,2\pi))}
\end{aligned}$$

with $\tau_{\boldsymbol{\gamma}} : L^2(\Gamma) \to L^2((0,2\pi))$ as in Section 2.5, and where the constant

$$C(\delta,\boldsymbol{\gamma}) \coloneqq \frac{\sup_{s\in[0,2\pi]}\|\boldsymbol{\gamma}'(s)\|^{-1}\|\boldsymbol{\gamma}'\|_{L^\infty((0,2\pi))}}{\frac{1}{2}-\delta} > 0$$

does not depend on the wavenumber κ. Set

$$\widetilde{\text{Loss}}(\Phi_\kappa) \coloneqq \left\|\tau_{\boldsymbol{\gamma}} f_{\kappa,\eta} - \tau_{\boldsymbol{\gamma}}\mathsf{A}'_{\kappa,\eta}\tau_{\boldsymbol{\gamma}}^{-1}((\Phi_\kappa)_1 + \imath(\Phi_\kappa)_2)\right\|^2_{L^2((0,2\pi))}.$$

We conclude that a NN $\Phi^\star_\kappa \in \mathcal{NN}_{L,M,1,2}$ computed by minimizing the loss function $\widetilde{\text{Loss}}$ within accuracy $\epsilon > 0$ approximates $\hat{\varphi}_\kappa$ with accuracy $\sqrt{C(\delta,\boldsymbol{\gamma})\epsilon}$ in the $L^2((0,2\pi))$-norm for any $\delta \in (0,1/2)$ as long as $\eta > 0$ is chosen as explained previously.

The presently employed NN constructions were of so-called "strict ReLU" type. This entailed various technicalities in our proofs due to the inability of ReLU-NNs to perform exact multiplication of reals. Similar results with simpler proofs could be expected with more general activation functions. The present ReLU-based approximations do imply, however, corresponding results for other, "neuromorphic" approximations. E.g. by so-called spiking NNs [23], as follows from the ReLU-to-spiking conversion algorithm in [33].

Supplementary Material

Supplementary Material is only available in electronic form at https://www.esaim-m2an.org/10.1051/m2an/2026010/olm.

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