ESAIM: M2AN 60 (2026) 1667–1714
https://doi.org/10.1051/m2an/2026041

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE PRESCRIBED MEAN CURVATURE EQUATION

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Abstract. We construct a non-polynomial local discontinuous Galerkin (LDG) scheme for the prescribed mean curvature equation to approximate boundary gradient blow-up solutions and obtain error estimates.

Mathematics Subject Classification. 65N30, 65N15, 35J93, 35K93.

Received July 28, 2025. Revised April 23, 2026. Accepted April 28, 2026.

1. Introduction

Consider the prescribed mean curvature type equation with Dirichlet boundary condition

$$-\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(\boldsymbol{x}, u, \nabla u), \quad \boldsymbol{x} \in \Omega, \tag{1}$$

$$u = b_0, \qquad \boldsymbol{x} \in \partial\Omega, \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with $\partial\Omega \in C^{2,\alpha}$ for some $\alpha, 0 < \alpha < 1$. The most significant challenge is the absence of a gradient estimate; such an estimate is not universally applicable, and boundary gradient blow-up is a potential issue.

There is an interesting example, the Jang's equation [25], which can be written as

$$\nabla_i \left(\frac{u^i}{\sqrt{1+|\nabla u|_g^2}}\right) = \left(g^{ij} - \frac{u^i u^j}{1+|\nabla u|_g^2}\right) K_{ij}, \tag{3}$$

where g_{ij} $(i,j=1,2,3)$ are the components of the metric and K_{ij} are the components of the extrinsic curvature; $u^i = g^{ij}u_j$ (where $u_j = \partial_j u$, g^{ij} is the inverse of metric g_{ij}) are the components of the gradient and $|\nabla u|_g^2 = g_{ij}u^i u^j$ is the square of its norm. Near the boundary (apparent horizon), the gradient $|\nabla u|$ blows up. Schoen and

Keywords and phrases. LDG scheme, prescribed mean curvature equation, Jang's equation.

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Yau [25] developed the application of the Jang's equation in their famous proof of the positive mass theorem in general relativity in the early 1980s. We are interested in studying the blow-up solutions numerically of the Jang's equation inside black holes, which is a very challenging task. We will complete this task in the following three steps: First, we will study the prescribed mean curvature equation in flat space to obtain a good numerical scheme; second, we will study the Jang's equation under certain symmetry assumptions (such as spherical and axial symmetry); and third, we will study the general Jang's equation. This paper aims to complete the first step, in which we will consider the prescribed mean curvature equation in flat space, rather than curved space. For simplicity, in this paper, we assume that the source term f is a function of $\boldsymbol{x}$ only. We hope to construct a local discontinuous Galerkin (LDG) scheme with non-polynomial basis to solve the boundary gradient blow-up problem.

A simple property of (1) is that if the integration of the source term $f(\boldsymbol{x})$ is "too large" on Ω, then Eqs. (1)–(2) will not have a classical solution of Ω. As $|\nabla u| \to \infty$ at the boundary $\partial\Omega$, the term $\frac{|\nabla u|}{\sqrt{1+|\nabla u|^2}} \to 1$. Integrating by parts, we obtain

$$\int_{\partial\Omega} \frac{\nabla u \cdot n}{\sqrt{1+|\nabla u|^2}}\, \mathrm{d}s = \int_{\Omega} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

where n is the outward unit normal vector of $\partial\Omega$ and $\mathrm{d}s$ is the measure on $\partial\Omega$. A necessary condition for the existence of a classical solution is

$$\left| \int_{\Omega} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \right| < |\partial\Omega|\,. \tag{4}$$

When the equality $\left|\int_\Omega f(x)\,\mathrm{d}x\right| = |\partial\Omega|$ holds in (4), classical solutions cease to exist; instead, Giusti [18] proved the existence of an "extreme solution", defined as a generalized solution whose gradient completely blows up ($|\nabla u| \to \infty$) everywhere on the boundary $\partial\Omega$. In fact, the gradient blow-up solution of the Jang's equation corresponds to such an "extreme solution".

Moreover, what if "$>$" appears in (4)? Obviously, Eqs. (1) and (2) does not have a solution. In this case, when studying the parabolic equation

$$u_t = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{5}$$

$$u = g, \qquad \boldsymbol{x} \in \partial\Omega \tag{6}$$

we will find that $u_t \nrightarrow 0$ when $t \to +\infty$.

In this paper, we study "extreme solutions" and aim to design an LDG scheme to approximate these blow-up solutions. We use the parabolic equations (5) and (6) to find the steady-state solution instead of solving the equations (1) and (2) directly; the theoretical guarantee that the solution of this parabolic equation converges to the extreme solution as $t \to \infty$ was rigorously established by Lichnewsky and Temam [21]. However, when we use the LDG scheme with polynomial basis functions to approximate the "extreme solutions", we observe an interesting phenomenon on coarse grids: the time derivative of the numerical solution does not decay to zero, that is $(u_h)_t \nrightarrow 0$ when $t \to +\infty$, which is due to the large numerical errors near the boundary. This indicates that we need a better finite element space with suitable non-polynomial basis functions to approximate the blow-up solutions. In the study of singular perturbation problems, exponentially fitted schemes were employed, as referenced in [20] and [23]. To solve boundary layer and highly oscillatory problems, Yuan and Shu [27] constructed discontinuous Galerkin (DG) schemes with non-polynomial basis functions, such as trigonometric functions and exponential functions. Moreover, the non-polynomial DG method was to solve elliptic multiscale problems [28, 29].

The LDG method, initially presented by Cockburn and Shu [13], extends the DG framework for convection equations and builds on the foundational work of Bassi and Rebay [6] on the compressible Navier–Stokes equations. This method further develops the Runge–Kutta discontinuous Galerkin methodology [10–12, 14], which was constructed for nonlinear hyperbolic systems. The LDG method has become a significant technique among DG approaches, especially for convection–diffusion problems, due to its wide range of applicability and its characteristics of local conservativity and high degree of locality. When dealing with purely elliptic problems, the LDG method and the technique by Baumann and Oden [7] are closely linked to the interior penalty (IP) methods, primarily investigated by Babuška and Zlâmal [3], Douglas and Dupont [15], Baker [4], Wheeler [26], Arnold [1], and further by Baker *et al.* [5], Rusten *et al.* [24]. These DG methods for elliptic equations fit into one comprehensive framework, as shown by Arnold *et al.* [2], which helps in understanding their relationships and aids in a comprehensive error analysis. As for the *a priori* error estimate of LDG methods for elliptic equations, Castillo *et al.* [9] show that if the stabilization parameters of order $\frac{1}{h}$ are taken, the optimal convergence order can be achieved.

Returning to the above mean curvature equation, several numerical works exist for the non-extreme solution (the smooth solution). Johnson and Thomée [19] studied a finite element method for the minimal surface problem (with the source term $f = 0$), employing a piecewise linear basis and obtaining the optimal H^1 and L^p (for $1 \le p < 2$) error estimates. Dziuk and Hutchinson [16, 17] gave an algorithm for finding finite element approximations to surfaces of prescribed variable mean curvature. Optimal error estimates in the H^1 norm was given. Brenner *et al.* [8] used a finite element method to study the one-dimensional prescribed curvature problem.

The rest of the paper is organized as follows. In Section 2, we consider the extreme solution for one-dimensional prescribed mean curvature equation. We construct a non-polynomial basis LDG scheme and obtain an error estimate. In Section 3, we study the LDG scheme for smooth solution. In Section 4, we show the non-polynomial LDG scheme for the multi-dimensional case. In Section 5, we present a numerical test to confirm our error estimates.

2. LDG scheme for one-dimensional prescribed mean curvature equation; the nonsmooth case

To illustrate our ideas, we start with the one-dimensional problem

$$\begin{aligned} 0 &= \left(\frac{u_x}{\sqrt{1+u_x^2}}\right)_x + f(x) \quad \text{in } [a,b], \\ u(a) &= u(b) = 0. \end{aligned} \tag{7}$$

As mentioned above, if $f(x)$ is "too big" on $[a,b]$, then there exists no solution. This is because $\left|\frac{u_x}{\sqrt{1+u_x^2}}\right| \le 1$. Defining $q = \frac{u_x}{\sqrt{1+u_x^2}}$ and integrating by parts, we obtain

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le |q(a) - q(b)| \le 2.$$

For simplicity, we assume $f(x) = -1$ and we know that

$$b - a \le 2.$$

If $b - a = 2$, as shown by Giusti [18], there is an extreme solution $u(x)$ on $[a,b]$. Assuming $a = -1, b = 1$, we have

$$u(x) = -\sqrt{1 - x^2} \tag{8}$$

and

$$u_x = \frac{x}{\sqrt{1-x^2}}. \tag{9}$$

As $x \to \pm 1$, $u_x \to \infty$. To construct a non-polynomial LDG scheme, the main idea is that, if we know the singular part of the exact solution, denoted as g_1 (which can be obtained using an *a priori* estimate), we can construct a group of basis functions $\phi_i = g_1 g_{2i}$, where g_{2i} is a regular function. This allows us to effectively approximate the blow-up solution.

In this section, we consider only the one-dimensional elliptic problem (7) with condition

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| = 2. \tag{10}$$

We will analyze the singular properties of u. For simplicity, we assume $f(x) < 0$. Define $F(x) := \int_a^x f(s)\,\mathrm{d}s$, since $-\int_a^b f(x)\,\mathrm{d}x = 2$. A Taylor expansion of $F(x)$ at $x = b$ yields $F(x) = 2 - f(b)(x-b) - \frac{f'(\theta)}{2}(x-b)^2$, hence $q(x) = -1 + \int_a^x f(s)\,\mathrm{d}s$. Then

$$q(x) = 1 - f(b)(x-b) + O((x-b)^2).$$

Using $u_x = \frac{q}{\sqrt{1-q^2}}$,

$$u_x(x) = \frac{1}{\sqrt{-2f(b)}\sqrt{b-x}} + O((b-x)^{1/2}). \tag{11}$$

Similarly, near $x = a$, we have

$$u_x(x) = \frac{1}{\sqrt{-2f(a)}\sqrt{x-a}} + O((x-a)^{1/2}). \tag{12}$$

We consider the parabolic problem

$$\begin{aligned} u_t &= \left(\frac{u_x}{\sqrt{1+u_x^2}} \right)_x + f(x) \quad \text{in } [a,b], \\ u(a) &= u(b) = 0. \end{aligned} \tag{13}$$

We define

$$r = u_x, \quad q = \frac{r}{\sqrt{1+r^2}}.$$

We assume the following uniform mesh to cover the computational domain $[a,b]$, consisting of cells $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], 1 \le i \le N$, where

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b.$$

We denote $h := x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})$ and the finite element space consisting of piecewise polynomials

$$V_h^k := \left\{ v : v|_{I_i} \in P^k(I_i); \quad 1 \le i \le N \right\}, \tag{14}$$

where $P^k(I_k)$ denotes the set of polynomials of degree up to k defined on the cell I_i.

We again denote

$$S^k := \left\{v : v|_{I_i} \in \text{span}\{1, g(x)(x-x_i)^0, g(x)(x-x_i)^1, \ldots, g(x)(x-x_i)^k\}\right\} \tag{15}$$

and

$$\tilde{S}^k := \left\{v : v|_{I_i} \in \text{span}\left\{1, \frac{1}{g(x)}(x-x_i)^0, \frac{1}{g(x)}(x-x_i)^1, \ldots, \frac{1}{g(x)}(x-x_i)^k\right\}\right\}, \tag{16}$$

where $g(x) = \sqrt{(x-a)(b-x)}$, because of (11) and (12).

The LDG scheme for Eq. (13) can be described as: find $u_h \in S^k, r_h \in \tilde{S}^k, q_h \in V_h^{k+1}$, such that $\forall \phi \in S^k$, $\forall \gamma \in \tilde{S}^k$, $\forall \psi \in V_h^{k+1}$; there holds

$$\int_{I_i} r_h \psi \, \mathrm{d}x = -\int_{I_i} u_h \psi_x \, \mathrm{d}x + (\hat{u}_h)_{i+\frac{1}{2}} \, \psi^-_{i+\frac{1}{2}} - (\hat{u}_h)_{i-\frac{1}{2}} \, \psi^+_{i-\frac{1}{2}}, \tag{17}$$

$$\int_{I_i} q_h \gamma \, \mathrm{d}x = \int_{I_i} \frac{r_h}{\sqrt{1+r_h^2}} \gamma \, \mathrm{d}x, \tag{18}$$

$$\int_{I_i} (u_h)_t \phi \, \mathrm{d}x = \int_{I_i} f\phi \, \mathrm{d}x - \int_{I_i} q_h \phi_x \, \mathrm{d}x + (\hat{q}_h)_{i+\frac{1}{2}} \, \phi^-_{i+\frac{1}{2}} - (\hat{q}_h)_{i-\frac{1}{2}} \, \phi^+_{i-\frac{1}{2}}, \tag{19}$$

where the numerical flux at the interior nodes are given by

$$(\hat{q}_h)_{i+\frac{1}{2}} = (q_h)^+_{i+\frac{1}{2}}\,, \quad (\hat{u}_h)_{i+\frac{1}{2}} = (u_h)^-_{i+\frac{1}{2}}\,, \quad 1 \le i \le N-1. \tag{20}$$

The numerical flux at the left boundary is

$$(\hat{q}_h)_{\frac{1}{2}} = (q_h)^+_{\frac{1}{2}}\,, \quad (\hat{u}_h)_{\frac{1}{2}} = (u_h)^-_{\frac{1}{2}} = u(a), \tag{21}$$

and the numerical flux at the right boundary is given by

$$(\hat{u}_h)_{N+\frac{1}{2}} = (u_h)^-_{N+\frac{1}{2}}\,, \quad (\hat{q}_h)_{N+\frac{1}{2}} = (q_h)^-_{N+\frac{1}{2}} - \frac{\alpha}{h}\left(u_h^-(b) - u(b)\right), \quad \alpha \ge 1. \tag{22}$$

2.1. Stability

We will consider the L^2 stability of the LDG scheme (17)–(19).

Theorem 1. *For the LDG scheme (17)–(19) approximating the problem with homogeneous Dirichlet boundary condition $u(a) = u(b) = 0$, we have the following stability estimate:*

$$\int_a^b u_h^2 \, \mathrm{d}x + \int_0^t \int_a^b \frac{2r_h^2}{\sqrt{1+r_h^2}} \, \mathrm{d}x \, \mathrm{d}t \le Ce^t(1+t)\left(\int_a^b u_0^2 \, \mathrm{d}x + \int_a^b f^2 \, \mathrm{d}x\right). \tag{23}$$

Proof. Taking $\phi = u_h, \psi = q_h$ in (17)–(19), we obtain

$$\int_{I_i} \left(\left(\frac{1}{2}u_h^2\right)_t + q_h r_h\right) \mathrm{d}x = u^-_{i+\frac{1}{2}} (\hat{q}_h)_{i+\frac{1}{2}} - u^-_{i-\frac{1}{2}} (q_h^+)_{i-\frac{1}{2}} + \int_{I_i} f u_h \, \mathrm{d}x.$$

Using (18) and taking $\gamma = r_h$, we obtain

$$\int_{I_i} q_h r_h \, \mathrm{d}x = \int_{I_i} \frac{r_h^2}{\sqrt{1+r_h^2}} \, \mathrm{d}x.$$

Then,

$$\int_{I_i}\left(\frac{1}{2}(u_h^2)_t+\frac{r_h^2}{\sqrt{1+r_h^2}}\right)\mathrm{d}x=u^-_{i+\frac{1}{2}}(\hat{q}_h)_{i+\frac{1}{2}}-u^-_{i-\frac{1}{2}}(q_h^+)_{i-\frac{1}{2}}+\int_{I_i}fu_h\,\mathrm{d}x. \tag{24}$$

Summing over with $i=1,\ldots,N-1$, and using $(\hat{q}_h)_{i+\frac{1}{2}}=(q_h^+)_{i+\frac{1}{2}},\ i\le N-1$, we have

$$\sum_{i=1}^{N-1}\int_{I_i}\frac{1}{2}(u_h^2)_t+\frac{r_h^2}{\sqrt{1+r_h^2}}\,\mathrm{d}x=\sum_{i=1}^{N-1}\int_{I_i}fu_h\,\mathrm{d}x+u^-_{N-\frac{1}{2}}q^+_{N-\frac{1}{2}}-u^-_{\frac{1}{2}}q^+_{\frac{1}{2}}. \tag{25}$$

Since $u^-_{\frac{1}{2}}=0$, the last term vanishes. In the final cell I_N, we have

$$\int_{I_N}\frac{1}{2}(u_h^2)_t+\frac{r_h^2}{\sqrt{1+r_h^2}}\,\mathrm{d}x=(\hat{q}_h)_{N+\frac{1}{2}}u^-_{N+\frac{1}{2}}-(q_h^+)_{N-\frac{1}{2}}(u_h^-)_{N-\frac{1}{2}}+\int_{I_N}fu_h\,\mathrm{d}x. \tag{26}$$

Using (22) and $u(b)=0$, we obtain

$$(\hat{q}_h)_{N+\frac{1}{2}}=(q_h^-)_{N+\frac{1}{2}}-\frac{\alpha}{h}u^-_{N+\frac{1}{2}}.$$

Adding (25) and (26), we get

$$\int_a^b\left(\frac{1}{2}(u_h^2)_t+\frac{r_h^2}{\sqrt{1+r_h^2}}\right)\mathrm{d}x+\frac{\alpha}{h}(u^-_{N+\frac{1}{2}})^2=(q_h^-)_{N+\frac{1}{2}}(u_h^-)_{N+\frac{1}{2}}+\int_a^b fu_h\,\mathrm{d}x. \tag{27}$$

Using Young's inequality $ab\le\frac{\alpha}{h}a^2+\frac{h}{4\alpha}b^2$,

$$(q_h^-)_{N+\frac{1}{2}}(u_h^-)_{N+\frac{1}{2}}\le\frac{\alpha}{h}(u_h^-)^2_{N+\frac{1}{2}}+\frac{h}{4\alpha}(q_h^-)^2_{N+\frac{1}{2}}. \tag{28}$$

Inserting (28) into (27), we have

$$\int_a^b\left(\frac{1}{2}(u_h^2)_t+\frac{r_h^2}{\sqrt{1+r_h^2}}\right)\mathrm{d}x\le\frac{h}{4\alpha}(q_h^-)^2_{N+\frac{1}{2}}+\frac{1}{2}\int_a^b f^2\,\mathrm{d}x+\frac{1}{2}\int_a^b u_h^2\,\mathrm{d}x, \tag{29}$$

where q_h requires a uniform bound estimate. We state this result in Lemma 3:

$$\|q_h\|_{L^\infty([a,b])}\le C.$$

To avoid interrupting the flow of the stability proof, the technical derivation of Lemma 3 is provided separately after Lemma 2. Then,

$$\frac{h}{4\alpha}(q_h^-)^2_{N+\frac{1}{2}}\le\frac{hC}{4\alpha},$$

and inserting this into (29), we have

$$\int_a^b(u_h^2)_t+\frac{2r_h^2}{\sqrt{1+r_h^2}}\,\mathrm{d}x\le\frac{Ch}{4\alpha}+\int_a^b f^2\,\mathrm{d}x+\int_a^b u_h^2\,\mathrm{d}x. \tag{30}$$

Using

$$\int_a^b (u_h^2)_t \,\mathrm{d}x \le \frac{Ch}{4\alpha} + \int_a^b f^2 \,\mathrm{d}x + \int_a^b u_h^2 \,\mathrm{d}x,$$

we have

$$\int_a^b u_h^2 \,\mathrm{d}x \le e^t \left(\int_a^b u_0^2 \,\mathrm{d}x + t \left(\frac{Ch}{4\alpha} + \int_a^b f^2 \,\mathrm{d}x \right) \right),$$

$$\int_0^t \int_a^b u_h^2 \,\mathrm{d}x \,\mathrm{d}t \le (e^t - 1) \int_a^b u_0^2 \,\mathrm{d}x + (e^t(t-1)+1) \left(\frac{Ch}{4\alpha} + \int_a^b f^2 \,\mathrm{d}x \right). \tag{31}$$

Using (30) and (31), we obtain

$$\begin{aligned} &\int_a^b u_h^2 \,\mathrm{d}x + \int_0^t \int_a^b \frac{2r_h^2}{\sqrt{1+r_h^2}} \,\mathrm{d}x \,\mathrm{d}t \\ &\le e^t \int_a^b u_0^2 \,\mathrm{d}x + (e^t(t-1)+1+t) \left(\frac{Ch}{4\alpha} + \int_a^b f^2 \,\mathrm{d}x \right). \end{aligned}$$

□

2.2. Error estimate

Note that, using Lagrange mean value theorem, there is an r_θ between r and r_h such that

$$\frac{r}{\sqrt{1+r^2}} - \frac{r_h}{\sqrt{1+r_h^2}} = \frac{r - r_h}{(1+r_\theta^2)^{\frac{3}{2}}},$$

We define the errors e^u, e^r, e^q as

$$e^u = u - u_h = \eta^u - \xi^u, \tag{32}$$

$$e^r = r - r_h = \eta^r - \xi^r, \tag{33}$$

$$e^q = q - q_h = \eta^q - \xi^q, \tag{34}$$

where

$$\begin{aligned} \eta^u &= u - \Pi_1 u, \quad \xi^u = u_h - \Pi_1 u, \\ \eta^r &= r - \Pi_2 r, \quad \xi^r = r_h - \Pi_2 r, \\ \eta^q &= q - \Pi_3 q, \quad \xi^q = q_h - \Pi_3 q, \end{aligned}$$

For the projection of q, we define

$$\widehat{S}^k(I_i) := \frac{1}{g(x)} P^k(I_i) = \operatorname{span}\left\{ \frac{1}{g(x)}, \frac{x - x_i}{g(x)}, \dots, \frac{(x-x_i)^k}{g(x)} \right\}.$$

On each cell $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, we define $\Pi_3 q|_{I_i} \in P^{k+1}(I_i)$ by

$$\int_{I_i} \eta^q \gamma \,\mathrm{d}x = 0, \quad \forall \gamma \in \widehat{S}^k(I_i), \quad \eta^{q,+}_{i-\frac{1}{2}} = 0. \tag{35}$$

The projection Π_1, Π_2 are defined in Lemmas 1 and 2, respectively. Then, we have $\xi^q = -e^q + \eta^q$,

$$\int_{I_i} (\xi^q - \eta^q)\gamma \,\mathrm{d}x = \int_{I_i} \frac{\xi^r - \eta^r}{(1+r_\theta^2)^{\frac{3}{2}}}\gamma \,\mathrm{d}x, \quad \xi^q \in V_h^{k+1}, \ \forall \gamma \in \tilde{S}^k. \tag{36}$$

Here the test space in (36) remains $\tilde{S}^k$, since this equation is inherited from the LDG formulation for q_h, whose test functions lie in $\tilde{S}^k$. The auxiliary space $\widehat{S}^k$ is used only in the definition of the projection $\Pi_3 q$. We need the following projection error estimate. We employ a bootstrap argument. Assume

$$\|g\xi^r\|_{L^\infty[a,b]} \le h^{\frac{1}{8}}. \tag{B1}$$

This ensures gr_h remains close to gr, implying the uniform bounds

$$c_1 \le g^2(1+r_\theta^2) \le c_2, \tag{B2}$$

where $c_1, c_2 > 0$. Applying Lemma 4 yields $\|g\xi^r\|_{L^\infty} \le Ch^{k+\frac{1}{4}}$, which is strictly smaller than the assumed bound for sufficiently small h, closing the bootstrap (the details are in Lemma 4). Thus, we have

$$\lim_{x\to a^-} g^2(x)r_\theta^2(x) \ge c_1 > 0, \quad \lim_{x\to b^+} g^2(x)r_\theta^2(x) \ge c_1 > 0. \tag{B3}$$

Under assumption (B1), we verify the properties (B2)–(B3) of r_θ. Recall that r_θ is a convex combination of r and r_h. Multiplying by g, we have:

$$gr_\theta = \lambda(gr) + (1-\lambda)(gr_h) = gr + (\lambda - 1)g(r - r_h), \quad \lambda \in [0,1].$$

Note that $|g(r-r_h)| \le \|g\eta^r\|_\infty + \|g\xi^r\|_\infty$. By Lemma 1 (39), we have $\|g\eta^r\|_\infty \le Ch^{k+1}$, and using Lemma 4, we can control $\|g\xi^r\|_\infty \le Ch^{k+\frac{1}{4}}$. So, $|(\lambda-1)g(r-r_h)| \le Ch^{k+\frac{1}{4}}$. We assume $|gr| \ge c_0 \gg h^{k+\frac{1}{4}}$ near the boundary, if h is small enough. The term gr_θ preserves the sign and magnitude of gr, which is regular. Consequently, there exist uniform positive constants c_1 and c_2, such that

$$c_1 \le g^2(1+r_\theta^2) \le c_2, \lim_{x\to a^- \text{ or } b^+} g^2(x)r_\theta^2(x) \ge c_1 > 0.$$

Lemma 1. *Assume that the function r has the form $r(x,t) = \frac{1}{g(x)}R(x,t)$, where $R(\cdot,t) \in C^{k+1}([a,b])$ and $g(x)$ is the weight function defined in (16). Let $\Pi_2 r \in \tilde{S}^k$ be the projection of r defined by the following conditions for each cell I_i:*

$$\int_{I_i} (r - \Pi_2 r)\psi \,\mathrm{d}x = 0, \quad (r - \Pi_2 r)^+_{i-\frac{1}{2}} = 0, \quad \forall \psi \in V_h^k, \tag{37}$$

then, we have

$$\left(\int_a^b (\eta^r)^2 g^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \le Ch^{k+1} \tag{38}$$

and

$$\|g\eta^r\|_{L^\infty([a,b])} \le Ch^{k+1}. \tag{39}$$

A direct corollary of (38) is

$$\left(\int_a^b \frac{(\eta^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x\right)^{\frac{1}{2}} \le Ch^{k+1}, \tag{40}$$

where C is a positive constant independent of the mesh size h, and r_θ lies between r and r_h.

Proof. Without loss of generality, we perform the analysis on the reference interval $[-1,1]$ by the affine mapping $x \mapsto \frac{2x-(a+b)}{b-a}$. The general case on $[a,b]$ follows directly by scaling, which introduces a constant factor dependent on the domain size but does not affect the convergence order. We assume that the local weight function behaves as $g(x) \sim \sqrt{1-x^2}$ (or $\sqrt{1-x}$ near the boundary). We need to define a set of weighted orthogonal polynomial bases, $\psi_0, \psi_1, \ldots, \psi_k$, that satisfy the following relationships:

$$\int_{I_i} \psi_\ell \psi_j \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \delta_{j\ell}, \quad j,\ell = 0,1,\ldots,k.$$

First, we need to define a set of simple polynomial basis functions:

$$\begin{aligned}
&\beta_0 = 1\\
&\beta_1 = x - x_i,\\
&\cdots\\
&\beta_k = (x-x_i)^k,
\end{aligned}$$

where $x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})$. Define the inner product as $(A,B)_i := \int_{I_i} AB\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$. Using the Gram–Schmidt orthogonalization process, we can construct the following orthogonal basis functions $\{\psi_j\}$, $j = 0,1,\ldots,k$. Assuming the $x_i = 1-\varepsilon, 0 < \frac{h}{2} \le \varepsilon < 1$, we have (see Appendix A)

$$\begin{aligned}
&\psi_0 = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}},\\
&\psi_1 = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}}\\
&\psi_2 = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}}\\
&\cdots\\
&\psi_k = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}}.
\end{aligned}$$

Assume $r = \frac{1}{g}R(x,t)$, where $R(x,t) \in C^\infty(\mathbb{R}\times\mathbb{R})$. Using Taylor expansion of $R(x,t)$ at x_i, we obtain

$$\begin{aligned}
R(x,t) &= R(x_i) + R'(x_i)(x-x_i) + \frac{R''(x_i)}{2}(x-x_i)^2 + \cdots + e_i(x-x_i)^{k+1}\\
&= b_0\psi_0 + b_1\psi_1 + \cdots + b_k\psi_k + e_i(x-x_i)^{k+1}.
\end{aligned}$$

We define $\Pi r := \frac{1}{g}(a_0\psi_0 + a_1\psi_1 + \cdots + a_k\psi_k)$. Then

$$\int_{I_i} \Pi r \psi_j \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{I_i} r\psi_j \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x, \quad j = 0,1,\ldots,k.$$

Substituting the expansions for Πr and r, and denoting $C_j := a_j - b_j$, the orthogonality property yields

$$C_j = \int_{I_i} e_i(x)(x-x_i)^{k+1}\psi_j \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Since $R(x,t)$ is smooth, the remainder coefficient $e_i(x) = \frac{R^{(k+1)}(\xi)}{(k+1)!}$ is uniformly bounded on I_i. Utilizing the boundedness of $e_i(x)$ and the estimate $\psi_j = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}}$, we obtain the following bound:

$$
\begin{aligned}
|C_j| &\le \max_{x\in I_i} |e_i(x)| \int_{I_i} |x - x_i|^{k+1} |\psi_j| \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \\
&\le Ch^{k+1} \cdot h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}} \int_{I_i} \frac{1}{\sqrt{1-x}} \, \mathrm{d}x \\
&\le Ch^{k+\frac{1}{2}} \varepsilon^{\frac{1}{4}} \cdot \frac{h}{\sqrt{\varepsilon} + \sqrt{\varepsilon - h}} \\
&\le Ch^{k+\frac{3}{2}} \varepsilon^{-\frac{1}{4}}.
\end{aligned}
$$

We note that the explicit integral estimation above focuses on the singular behavior near the right boundary $(x \to 1^-)$*, where the weight function is dominated by the term* $(1-x)^{-1/2}$*. For elements near the left boundary* $(x \to -1^+)$*, the weight function exhibits the symmetric singularity* $(1+x)^{-1/2}$*. Due to the symmetry of the domain and the weight function* $g(x) = \sqrt{1-x^2}$*, the derivation is strictly analogous, yielding identical asymptotic bounds for the coefficients* C_j. Thus, the estimate $|C_j| \le Ch^{k+3/2}\varepsilon^{-1/4}$ holds uniformly for all elements I_i across the domain. Using the boundary condition $\eta^{r+}_{i-1/2} = 0$, we obtain the linear constraint

$$
\frac{1}{g(x_{i-1/2})} \left(\mathcal{R}_{\mathrm{rem}}(x_{i-1/2}) - \sum_{j=0}^{k} C_j \psi_j(x_{i-1/2}) \right) = 0.
$$

This relation uniquely determines C_0. Given the uniform boundedness of the remainder term $\mathcal{R}_{\mathrm{rem}}$ and the asymptotic behavior of the basis functions ψ_j derived in Appendix A, the coefficient C_0 satisfies the same asymptotic bound as C_j $(j \ge 1)$. Thus, we have:

$$
|C_0| \le Ch^{k+\frac{3}{2}} \varepsilon^{-\frac{1}{4}}.
$$

Combining the estimates for all coefficients C_j $(j = 0, \dots, k)$, we estimate the weighted L^2 error norm. Using the fact that $|\mathcal{R}_{\mathrm{rem}}(x)| \le Ch^{k+1}$ and the pointwise estimates for ψ_j, the local error on element I_i satisfies

$$
\begin{aligned}
\int_{I_i} (r - \Pi r)^2 g^2 \, \mathrm{d}x &= \int_{I_i} \left(\mathcal{R}_{\mathrm{rem}}(x) - \sum_{j=0}^{k} C_j \psi_j(x) \right)^2 \mathrm{d}x \\
&\le 2 \int_{I_i} \mathcal{R}_{\mathrm{rem}}^2 \, \mathrm{d}x + 2 \sum_{j=0}^{k} C_j^2 \int_{I_i} \psi_j^2 \, \mathrm{d}x \\
&\le Ch \cdot h^{2k+2} + C \sum_{j=0}^{k} (h^{2k+3} \varepsilon^{-\frac{1}{2}}) \cdot (h^{-1} \varepsilon^{\frac{1}{2}} \cdot h) \\
&\le Ch^{2k+3}.
\end{aligned}
$$

Note that we used the estimate $\int_{I_i} \psi_j^2 \, \mathrm{d}x \approx h$, which follows from the normalization in the weighted space (see Appendix A). Summing over all $N \sim 1/h$ elements, we obtain the global bound as follows:

$$
\begin{aligned}
&\int_a^b (r - \Pi r)^2 g^2 \, \mathrm{d}x \\
&\le \sum_{i=1}^{N} \int_{I_i} (r - \Pi r)^2 g^2 \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le N \cdot Ch^{2k+3} \\
&\le Ch^{2k+2}.
\end{aligned}
$$

This implies

$$\left(\int_a^b (\eta^r)^2 g^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \le Ch^{k+1}.$$

Finally, we convert this to the target norm. Using the boundedness of the domain (which implies $g(x)$ is bounded), we have

$$
\begin{aligned}
&\int_a^b \frac{(\eta^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}} \, \mathrm{d}x \\
&= \int_a^b \frac{g^3(\eta^r)^2}{(g^2+g^2 r_\theta^2)^{\frac{3}{2}}} \, \mathrm{d}x \\
&\le C \int_a^b g^3 (\eta^r)^2 \, \mathrm{d}x \quad \text{(using (B2))} \\
&\le C \sup_{x\in[a,b]} |g(x)| \int_a^b g^2 (\eta^r)^2 \, \mathrm{d}x \\
&\le Ch^{2k+2}.
\end{aligned}
$$

Taking the square root completes the proof

$$\left(\int_a^b \frac{(\eta^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}} \, \mathrm{d}x\right)^{\frac{1}{2}} \le Ch^{k+1}.$$

Pointwise estimate for $\|g\eta^r\|_\infty$:

Recall the expression for the weighted projection error derived from the expansion of $R(x,t)$:

$$g(x)\eta^r(x) = g(x)(r - \Pi r) = R(x) - \sum_{j=0}^{k} a_j \psi_j = \sum_{j=0}^{k} (b_j - a_j)\psi_j + e_i (x - x_i)^{k+1}.$$

Substituting $C_j = a_j - b_j$, we have $g\eta^r = -\sum_{j=0}^k C_j \psi_j + \mathcal{O}(h^{k+1})$. We now apply the bounds derived above: $|C_j| \le Ch^{k+\frac{3}{2}}\varepsilon^{-\frac{1}{4}}$ and $|\psi_j| \le Ch^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}$. A key observation is that the singular factors $\varepsilon^{-\frac{1}{4}}$ and $\varepsilon^{\frac{1}{4}}$ cancel out perfectly:

$$
\begin{aligned}
\|g\eta^r\|_{L^\infty(I_i)} &\le \sum_{j=0}^{k} |C_j| \|\psi_j\|_{L^\infty(I_i)} + \max_{x\in I_i} |e_i (x-x_i)^{k+1}| \\
&\le (k+1) \cdot \left(Ch^{k+\frac{3}{2}}\varepsilon^{-\frac{1}{4}}\right) \cdot \left(Ch^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}\right) + Ch^{k+1} \\
&\le Ch^{k+1} + Ch^{k+1} \\
&\le Ch^{k+1}.
\end{aligned}
$$

Since this bound is independent of ε and holds for any element I_i, we conclude globally:

$$\|g\eta^r\|_{L^\infty([a,b])} \leq Ch^{k+1}.$$

$\square$

Lemma 2. *Assume that*

$$u(x,t) = g(x)U(x,t), \quad U(\cdot,t) \in C^{k+1}([a,b]).$$

Let $\Pi_1 u \in S^k$ *be defined cell wise by*

$$\int_{I_i} (u - \Pi_1 u)\phi \, \mathrm{d}x = 0, \quad \forall \phi \in V_h^k,$$

with condition

$$(u - \Pi_1 u)^-_{i+\frac{1}{2}} = 0.$$

Then,

$$\|\eta^u\|_{L^2(a,b)} \leq Ch^{k+1},$$

where C *is independent of* h.

Proof. We need to define a set of weighted orthogonal polynomial bases $\zeta_0, \zeta_1, \ldots, \zeta_k$, that satisfy the following relationships:

$$\int_{I_i} \zeta_\ell \zeta_j g^2 \, \mathrm{d}x = \delta_{j\ell}, \quad j, \ell = 0, 1, \ldots, k.$$

First, we need to define a set of simple polynomial basis functions

$$\begin{aligned}
\beta_0 &= 1, \\
\beta_1 &= x - x_i, \\
&\cdots \\
\beta_k &= (x - x_i)^k,
\end{aligned}$$

where $x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})$. Define the inner product as $(A,B)_i := \int_{I_i} ABg^2 \, \mathrm{d}x$. Using the Gram–Schmidt orthogonalization process, we can construct the following orthogonal basis functions. Assuming $x_i = 1 - \varepsilon, 0 < \frac{h}{2} \leq \varepsilon < 1$, we have (see Appendix B.1)

$$\begin{aligned}
\zeta_0 &= \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}), \\
\zeta_1 &= \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}), \\
\zeta_2 &= \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}), \\
&\cdots \\
\zeta_k &= \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}).
\end{aligned}$$

Assume $u = g(x)R(x,t)$, where $R(x,t) \in C^\infty(\mathbb{R} \times \mathbb{R})$. Using Taylor expansion of $R(x,t)$ at x_i, we obtain

$$\begin{aligned}
R(x,t) &= R(x_i) + R'(x_i)(x - x_i) + \frac{R''(x_i)}{2}(x - x_i)^2 + \cdots + e_i(x - x_i)^{k+1} \\
&= b_0\zeta_0 + b_1\zeta_1 + b_2\zeta_2 + \cdots + e_i(x - x_i)^{k+1}.
\end{aligned}$$

We define $\Pi u := g(x)(a_0\zeta_0 + a_1\zeta_1 + \cdots + a_k\zeta_k)$, then

$$\int_{I_i} \Pi u \zeta_j g \, \mathrm{d}x = \int_{I_i} u\zeta_j g \, \mathrm{d}x, \quad j = 0, 1, \ldots, k-1,$$
$$c_j = \int_{I_i} e_i g^2(x)(x - x_i)^{k+1}\zeta_j \, \mathrm{d}x,$$

where $c_j := a_j - b_j$. Using $\eta^{u-}_{i+\frac{1}{2}} = 0$, we obtain

$$g_{i+\frac{1}{2}} \left(a_0\zeta_{0,i+\frac{1}{2}} + \cdots + a_k\zeta_{k,i+\frac{1}{2}}\right) = g_{i+\frac{1}{2}} \left(b_0\zeta_{0,i+\frac{1}{2}} + \cdots + b_k\zeta_{k,i+\frac{1}{2}} + \mathcal{O}(h^{k+1})\right). \tag{41}$$

If $g_{i+\frac{1}{2}} \neq 0$, we have

$$c_0\zeta_{0,i+\frac{1}{2}} + \cdots + c_k\zeta_{k,i+\frac{1}{2}} = \mathcal{O}(h^{k+1}). \tag{42}$$

If $g_{i+\frac{1}{2}} = 0$ (on the last cell $i = N$), we need another condition (43) to replace (42)

$$c_k = \int_{I_i} e_i(x - x_i)^{k+1} g^2 \zeta_k \, \mathrm{d}x. \tag{43}$$

We also have the following estimate (details in Appendix B.2):

$$\begin{aligned}
&c_0 \le ch^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}, \\
&c_1 \le ch^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}, \\
&c_2 \le ch^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}, \\
&\cdots \\
&c_k \le ch^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

$$\begin{aligned}
\int_{I_i} (\eta^u)^2 \, \mathrm{d}x &= \int_{I_i} g^2 \left(c_0\zeta_0 + \cdots + c_k\zeta_k + \mathcal{O}(h^{k+1})\right)^2 \mathrm{d}x \\
&\le c_0^2 \int_{I_i} g^2\zeta_0^2 \, \mathrm{d}x + \cdots c_k^2 \int_{I_i} g^2\zeta_k^2 \, \mathrm{d}x + \mathcal{O}(h^{2k+2}) \int_{I_i} g^2 \, \mathrm{d}x,
\end{aligned}$$

where we have the estimates

$$\int_{I_i} g^2\zeta_j^2 \, \mathrm{d}x = 1,$$
$$\cdots$$
$$\int_{I_i} g^2 \, \mathrm{d}x = \mathcal{O}(h\varepsilon),$$

then

$$\int_{I_i} (\eta^u)^2 \, \mathrm{d}x \le \mathcal{O}(h^{2k+3}\varepsilon) \le \mathcal{O}(h^{2k+3})$$

and

$$\sum_{i=1}^{N} \int_{I_i} (\eta^u)^2 \,\mathrm{d}x \leq \mathcal{O}(h^{2k+2}).$$

□

To estimate the terms involving the time derivatives of the projection errors, such as $\|\eta_t^u\|$, we rely on the fact that the mesh is fixed. Consequently, the projection operator Π commutes with the time differentiation, yielding $\eta_t^u = u_t - \Pi u_t$. Assuming that the time derivative of the exact solution satisfies the same regularity conditions as the solution itself—namely, $u_t \in H^{k+1}(\Omega)$ and r_t satisfy the structure $r_t = g^{-1} R_t$ with $R_t \in C^{k+1}$—the estimates derived in Lemmas 1 and 2 apply directly to η_t^u and η_t^r, that is, $\|\eta_t^u\| \leq Ch^{k+1}$.

Lemma 3. *Let $q_h \in V_h^{k+1}$ be defined by the scheme (18). The following estimate holds:*

$$\|q_h\|_{L^\infty([a,b])} \leq C,$$

where C is a constant independent of h.

The proof of this lemma is given in Appendix C. To obtain the estimate for $\|\eta^q\|_{L^\infty}$, we present Lemma C.7 in Appendix C, namely,

$$\|\eta^q\|_{L^\infty} \leq Ch^{k+2}. \tag{44}$$

Note that this is because q is smooth and $q_h \in V_h^{k+1}$.

We define the local energy norm as $\|\xi^r\|_{E,I_i}^2 := \int_{I_i} (1 + r_\theta^2)^{-\frac{3}{2}} (\xi^r)^2 \,\mathrm{d}x$. Then, we have the following:

(1) Estimate for the cross-term volume integral $\left|\int_{I_i} \eta^r \xi^q \,\mathrm{d}x\right|$:

$$\left|\int_{I_i} \eta^r \xi^q \,\mathrm{d}x\right| \leq \frac{C^2}{4\epsilon} h^{2k+2} + \epsilon \,\|\xi^r\|_{E,I_i}^2 + Ch^{2k+2}, \quad \forall \epsilon > 0. \tag{45}$$

(2) Estimate for the boundary trace term $\frac{h}{4\alpha}(\xi_{N+1/2}^{q-})^2$:

$$\frac{h}{4\alpha}(\xi_{N+1/2}^{q-})^2 \leq \frac{C}{4\alpha} \|\xi^r\|_{E,I_N}^2 + \frac{C}{4\alpha} h^{2k+2}. \tag{46}$$

Remark 1. The ϵ in (45) is a constant, which is different from the ε defined in Lemma 1.

Remark 2. The proofs of (45) and (46) are given by Lemma C.10 in Appendix C.

Finally, we have the error estimate.

Theorem 2. *For the LDG scheme (17)–(19) with boundary condition $u(a) = u(b) = 0$, we have the following error estimate: for $k \geq 0$,*

$$\int_a^b (u - u_h)^2 \,\mathrm{d}x + \int_0^t \int_a^b 2\left(1 - \frac{c_1}{2\alpha}\right) \frac{(r - r_h)^2}{(1 + r_\theta^2)^{\frac{3}{2}}} \,\mathrm{d}x \,\mathrm{d}t \leq c_2(1 + t)e^t h^{2k+2},$$

where $\alpha > 0, c_1, c_2 > 0$ are constants which satisfy $1 - \frac{c_1}{2\alpha} > 0$.

Proof. The error equations are

$$\int_{I_i} e_t^u \phi \,\mathrm{d}x = -\int_{I_i} e^q \phi_x \,\mathrm{d}x + (q-\hat{q}_h)_{i+\frac{1}{2}}\phi^-_{i+\frac{1}{2}} - (q-\hat{q}_h)_{i-\frac{1}{2}}\phi^+_{i-\frac{1}{2}}, \tag{47}$$

$$\int_{I_i} (q-q_h)\gamma \,\mathrm{d}x = \int_{I_i} \frac{r-r_h}{(1+r_\theta^2)^{\frac{3}{2}}}\gamma \,\mathrm{d}x, \tag{48}$$

$$\int_{I_i} e^r \psi \,\mathrm{d}x = -\int_{I_i} e^u \psi_x \,\mathrm{d}x + (u-\hat{u}_h)_{i+\frac{1}{2}}\psi^-_{i+\frac{1}{2}} - (u-\hat{u}_h)_{i-\frac{1}{2}}\psi^+_{i-\frac{1}{2}}. \tag{49}$$

Considering $\phi = -\xi^u, \psi = -\xi^q, \gamma = \xi^u_x$, and for $i \le N-1$, and assuming $\hat{q}_h = q_h^+, \hat{u}_h = u_h^-$, we obtain

$$\begin{aligned}\int_{I_i} \frac{1}{2}(\xi^u)^2_t \,\mathrm{d}x = &-\int_{I_i} (-\eta^q + \xi^q)\xi^u_x \,\mathrm{d}x + \int_{I_i} \eta^u_t \xi^u \,\mathrm{d}x \\ &+ (-\eta^{q+} + \xi^{q+})_{i+\frac{1}{2}}\xi^{u-}_{i+\frac{1}{2}} - (-\eta^{q+} + \xi^{q+})_{i-\frac{1}{2}}\xi^{u+}_{i-\frac{1}{2}}.\end{aligned}$$

Using Gauss–Radau projection $\int_{I_i} \eta^q \gamma \,\mathrm{d}x = 0,\ \eta^{q+}_{i-\frac{1}{2}} = 0,\ \forall \gamma \in \hat{S}^k$, then (47) reduces to

$$\int_{I_i} \frac{1}{2}(\xi^u)^2_t = -\int_{I_i} \xi^q \xi^u_x \,\mathrm{d}x + \xi^{q+}_{i+\frac{1}{2}}\xi^{u-}_{i+\frac{1}{2}} - \xi^{q+}_{i-\frac{1}{2}}\xi^{u+}_{i-\frac{1}{2}} + \int_{I_i} (\eta^u_t)\xi^u \,\mathrm{d}x, \quad (i \le N-1). \tag{50}$$

Similarly, using Gauss–Radau projection $\int_{I_i} \eta^u \xi^q_x \,\mathrm{d}x = 0,\ \eta^{u-}_{i+\frac{1}{2}} = 0$, (49) reduces to

$$\int_{I_i} \xi^r \xi^q \,\mathrm{d}x = -\int_{I_i} \xi^u \xi^q_x \,\mathrm{d}x + \int_{I_i} \eta^r \xi^q \,\mathrm{d}x + \xi^{u-}_{i+\frac{1}{2}}\xi^{q-}_{i+\frac{1}{2}} - \xi^{u-}_{i-\frac{1}{2}}\xi^{q+}_{i-\frac{1}{2}}, \quad (1 \le i \le N). \tag{51}$$

Summing (50) and (51) over i from 1 to $N-1$, we obtain

$$\sum_{i=1}^{N-1} \frac{1}{2}(\xi^u)^2_t + \xi^r \xi^q \,\mathrm{d}x = \sum_{i=1}^{N-1} \int_{I_i} (\eta^u_t)\xi^u + \eta^r \xi^q \,\mathrm{d}x + \xi^{q+}_{N-\frac{1}{2}}\xi^{u-}_{N-\frac{1}{2}} - \xi^{q+}_{\frac{1}{2}}\xi^{u-}_{\frac{1}{2}}, \tag{52}$$

where $\xi^{u-}_{\frac{1}{2}} = 0$.

If $i = N$, (47) + (49) reduces to

$$\int_{I_N} \frac{1}{2}(\xi^u)^2_t + \xi^r \xi^q \,\mathrm{d}x = (q-\hat{q}_h)_{N+\frac{1}{2}}(-\xi^{u-}_{N+\frac{1}{2}}) - \xi^{q+}_{N-\frac{1}{2}}\xi^{u-}_{N-\frac{1}{2}} + \int_{I_N} (\eta^u_t)\xi^u + \eta^r \xi^q \,\mathrm{d}x,$$

where

$$\hat{q}_h = q_h^- - \frac{\alpha}{h}(u_h^-)_{N+\frac{1}{2}},$$

$$(q_{N+\frac{1}{2}} - (\hat{q}_h)_{N+\frac{1}{2}})(-\xi^{u-}_{N+\frac{1}{2}}) = \xi^{q-}_{N+\frac{1}{2}}\xi^{u-}_{N+\frac{1}{2}} - \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 \quad \left(\text{using } \eta^{q-}_{N+\frac{1}{2}} = 0,\ \eta^{u-}_{N+\frac{1}{2}} = 0\right).$$

Then (47) + (49) becomes

$$\int_{I_N} \frac{1}{2}(\xi^u)_t + \xi^r \xi^q \,\mathrm{d}x = \xi^{q-}_{N+\frac{1}{2}}\xi^{u-}_{N+\frac{1}{2}} - \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 - \xi^{q+}_{N-\frac{1}{2}}\xi^{u-}_{N-\frac{1}{2}} + \int_{I_N} (\eta^u_t)\xi^u + \eta^r \xi^q \,\mathrm{d}x. \tag{53}$$

Summing (53) and (52), we have

$$\int_{I} \frac{1}{2}(\xi^u)^2_t + \xi^r \xi^q \,\mathrm{d}x = \int_{I} (\eta^u_t)\xi^u + \eta^r \xi^q \,\mathrm{d}x + \xi^{q-}_{N+\frac{1}{2}}\xi^{u-}_{N+\frac{1}{2}} - \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2. \tag{54}$$

Using the definition of ξ^q in (36), the second term on the left-hand side in (54) becomes

$$\int_a^b \xi^r \xi^q \,\mathrm{d}x = \int_a^b \frac{(\xi^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x + \underbrace{\int_a^b \eta^q \xi^r \,\mathrm{d}x}_{P_1} - \underbrace{\int_a^b \frac{\eta^r \xi^r}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x}_{P_2}. \tag{55}$$

To obtain the estimate for $|P_1|$, we present Lemma C.8 in Appendix C, namely,

$$\left|\int_a^b \eta^q \xi^r \,\mathrm{d}x\right| \le \frac{c_1}{4\alpha}\int_a^b \frac{(\xi^r)^2}{(1+r_\theta^2)^{3/2}}\,\mathrm{d}x + Ch^{2k+2},$$

and P_2 can be controlled by Lemma 1:

$$|P_2| \le \frac{\alpha}{c_1}\int_a^b \frac{(\eta^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x + \frac{c_1}{4\alpha}\int_a^b \frac{(\xi^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x \le Ch^{2k+2} + \frac{c_1}{4\alpha}\int_a^b \frac{(\xi^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x.$$

Then, (54) becomes

$$\int_I \frac{1}{2}(\xi^u)_t^2 + \left(1-\frac{c_1}{2\alpha}\right)\frac{(\xi^r)^2}{(1+r_\theta^2)^{3/2}}\,\mathrm{d}x \le \int_I (\eta_t^u)\xi^u + \eta^r\xi^q\,\mathrm{d}x + \xi^{q-}_{N+\frac{1}{2}}\xi^{u-}_{N+\frac{1}{2}} - \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 + Ch^{2k+2}. \tag{56}$$

By Young's inequality,

$$\begin{aligned}
\xi^{q-}_{N+\frac{1}{2}}\xi^{u-}_{N+\frac{1}{2}} &\le \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 + \frac{h}{4\alpha}(\xi^{q-}_{N+\frac{1}{2}})^2, \\
&\le \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 + \frac{c_1}{4\alpha}\int_a^b \left(\frac{\xi^r}{(1+r_\theta^2)^{\frac{3}{2}}}\right)^2\mathrm{d}x + Ch^{2k+2} &(57)\\
&\le \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 + \frac{c_1}{4\alpha}\int_a^b \frac{1}{(1+r_\theta^2)^{\frac{3}{2}}}(\xi^r)^2\frac{1}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x + Ch^{2k+2} \\
&\le \frac{\alpha}{h}(\xi^{u-}_{N+\frac{1}{2}})^2 + \frac{c_1}{4\alpha}\int_a^b \frac{1}{(1+r_\theta^2)^{\frac{3}{2}}}(\xi^r)^2\,\mathrm{d}x + Ch^{2k+2}. &(58)
\end{aligned}$$

Let us provide a brief explanation of equation (57): From the error relation (46), we have $\frac{h}{4\alpha}(\xi^{q-}_{N+1/2})^2 \le \frac{C}{4\alpha}\|\xi^r\|^2_{E,I_N} + \frac{C}{4\alpha}h^{2k+2}$. Then, using (58) and (56), we obtain

$$\int_a^b \frac{1}{2}(\xi^u)_t^2 + \frac{1}{(1+r_\theta^2)^{\frac{3}{2}}}(\xi^r)^2\,\mathrm{d}x \le \int_a^b (\eta_t^u)\xi^u + \eta^r\xi^q\,\mathrm{d}x + \frac{c_1}{4\alpha}\int_a^b \frac{1}{(1+r_\theta^2)^{\frac{3}{2}}}(\xi^r)^2\,\mathrm{d}x. \tag{59}$$

By (45) of Lemma C.10 in Appendix C, we have

$$\left|\int_a^b \eta^r\xi^q\,\mathrm{d}x\right| \le c_3h^{2k+2} + \frac{c_1}{4\alpha}\int_a^b \frac{(\xi^r)^2}{(1+r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x. \tag{60}$$

The $\int_a^b (\eta_t^u)\xi^u\,\mathrm{d}x$ can be controlled as

$$2\int_a^b (\eta_t^u)\xi^u \le c_4h^{2k+2} + \int_a^b (\xi^u)^2\,\mathrm{d}x.$$

We take $0 \leq \alpha$ large enough such that $1 - \frac{c_1}{2\alpha} > 0$; then, (59) reduces to

$$\int_a^b (\xi^u)_t^2 + 2\left(1 - \frac{c_1}{2\alpha}\right) \frac{1}{(1+r_\theta^2)^{\frac{3}{2}}} (\xi^r)^2 \,\mathrm{d}x \leq \int_a^b (\xi^u)^2 \,\mathrm{d}x + c_2 h^{2k+2}. \tag{61}$$

Finally, we have

$$\int_a^b (\xi^u)^2 \,\mathrm{d}x + \int_0^t \int_a^b 2\left(1 - \frac{c_1}{2\alpha}\right) \frac{1}{(1+r_\theta^2)^{\frac{3}{2}}} (\xi^r)^2 \,\mathrm{d}x \,\mathrm{d}t \leq \left(e^t(t-1) + 1 + t\right) c_2 h^{2k+2}. \tag{62}$$

□

We now verify that the bootstrap assumption (B1) is never violated by deriving a sharp pointwise bound. With the bounds (B2), the error estimate (62) yields the time-integrated estimate

$$\int_0^t \int_a^b g^3 (\xi^r)^2 \,\mathrm{d}x \,\mathrm{d}s \leq C h^{2k+2}. \tag{63}$$

Set

$$Q(x,s) := g(x)\xi^r(x,s).$$

For each fixed time $s \in [0,t]$, we apply only the spatial estimate (66) in Lemma 4 to the function $Q(\cdot,s)$:

$$\|Q(\cdot,s)\|_{L^\infty(a,b)} \leq C h^{-3/4} \|\sqrt{g}\, Q(\cdot,s)\|_{L^2(a,b)} = C h^{-3/4} \left(\int_a^b g^3 (\xi^r)^2 \,\mathrm{d}x\right)^{1/2}.$$

Squaring this inequality and integrating in s, we obtain

$$\|g\xi^r\|_{L^2(0,t;L^\infty(a,b))} \leq C h^{-3/4} \left(\int_0^t \int_a^b g^3 (\xi^r)^2 \,\mathrm{d}x \,\mathrm{d}s\right)^{1/2} \leq C h^{k+\frac{1}{4}}. \tag{64}$$

Next, we differentiate the error system (47)–(49) with respect to t. Since the mesh is fixed, the projection operators commute with time differentiation, and the same projection estimates apply to η_t^u, η_t^r, and η_t^q. Repeating the proof of Theorem 2 for the differentiated system yields

$$\int_0^t \int_a^b g^3 (\xi_t^r)^2 \,\mathrm{d}x \,\mathrm{d}s \leq C h^{2k+2}. \tag{65}$$

For each fixed time $s \in [0,t]$, apply again only the spatial estimate (66) in Lemma 4 to

$$Q_t(\cdot,s) := g(\cdot)\xi_t^r(\cdot,s).$$

Since g is independent of t, we have $Q_t = g\xi_t^r$, and hence

$$\|Q_t(\cdot,s)\|_{L^\infty(a,b)} \leq C h^{-3/4} \|\sqrt{g}\, Q_t(\cdot,s)\|_{L^2(a,b)} = C h^{-3/4} \left(\int_a^b g^3 (\xi_t^r)^2 \,\mathrm{d}x\right)^{1/2}.$$

Squaring this inequality and integrating in s, we obtain

$$\|g\xi_t^r\|_{L^2(0,t;L^\infty(a,b))} \leq C h^{-3/4} \left(\int_0^t \int_a^b g^3 (\xi_t^r)^2 \,\mathrm{d}x \,\mathrm{d}s\right)^{1/2} \leq C h^{k+\frac{1}{4}}. \tag{66}$$

Under the standard projected initialization, $\xi^r(\cdot,0)=0$, and therefore

$$g(x)\xi^r(x,0)=0.$$

Hence, by the fundamental theorem of calculus in time and the Cauchy–Schwarz inequality,

$$\begin{aligned}\sup_{0\le s\le t}\|g\xi^r(\cdot,s)\|_{L^\infty(a,b)} &\le \|g\xi^r(\cdot,0)\|_{L^\infty(a,b)}+\sup_{0\le s\le t}\int_0^s\|g\xi_t^r(\cdot,\tau)\|_{L^\infty(a,b)}\,\mathrm{d}\tau\\ &\le \int_0^t\|g\xi_t^r(\cdot,\tau)\|_{L^\infty(a,b)}\,\mathrm{d}\tau\\ &\le t^{1/2}\|g\xi_t^r\|_{L^2(0,t;L^\infty(a,b))}\le C_Th^{k+\frac14}.\end{aligned}\tag{67}$$

Since $k+\frac14>\frac18$ for all $k\ge 0$, the bound (67) is strictly smaller than the bootstrap threshold in (B1) for sufficiently small h. Therefore, the bootstrap assumption (B1) is closed, and the bounds in (B2) remain valid.

In Lemma 4, we shall prove that for any $k\ge 0$ and sufficiently small h,

$$\|g\xi^r\|_{L^\infty([a,b])}\le Ch^{k+\frac14}\ll h^{\frac18}.$$

Lemma 4. *Let $\xi^r=r_h-\Pi_2 r\in\tilde S^k$ and $g(x)=\sqrt{(x-a)(b-x)}$. Assume the global weighted L^2 error estimate (63) holds. Then, for the function $Q(x):=g(x)\xi^r(x)$, the global pointwise bound satisfies*

$$\|Q\|_{L^\infty([a,b])}\le Ch^{-\frac34}\|\sqrt{g}Q\|_{L^2([a,b])}\tag{68}$$

and

$$\|Q\|_{L^\infty([a,b])}\le Ch^{k+\frac14}.\tag{69}$$

Consequently, for any $k\ge0$ and small enough h, the bootstrap assumption $\|g\xi^r\|_{L^\infty}\ll h^{1/8}$ is satisfied.

The proof of this lemma is given in Appendix C.

3. One-dimensional smooth case

In this section, we discuss the case with smooth solutions, and when using the same basis function S^k, we can achieve higher accuracy $O(h^{k+2})$. The essential difference in the error estimation is obtaining different projection error estimates, which means deriving new versions of Lemmas 1 and 2.

Lemma 5. *Assuming (4) holds strictly, which means there exists a sufficiently smooth solution on the interval $[a,b]$, we define a Gauss–Radau projection $\Pi_2 r\in\tilde S^k$ to satisfy the following conditions for each cell I_i:*

$$\int_{I_i}(r-\Pi_2 r)\psi\,\mathrm{d}x=0,\quad (r-\Pi_2 r)^+_{i-\frac12}=0,\quad \forall\psi\in V_h^k,\tag{70}$$

then, we have

$$\left(\int_a^b\frac{(\eta^r)^2}{(1+r_\theta^2)^{\frac32}}\,\mathrm{d}x\right)^{\frac12}\le Ch^{k+2},\tag{71}$$

where C is a positive constant independent of h.

Proof. This proof fundamentally differs from that of Lemma 1 in that $\frac{1}{g}$ is smooth. Using Taylor expansion, we obtain

$$\frac{1}{g(x)} = \frac{1}{g_i} + \left(\frac{1}{g}\right)'_{\theta_i}(x - x_i), \tag{72}$$

where θ_i is between $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ and $\left(\frac{1}{g}\right)'_{\theta_i}$ is bounded. Then, the basis $\tilde{S}^k$ becomes

$$\left\{1, \frac{1}{g_i} + \left(\frac{1}{g}\right)'_{\theta_i}(x - x_i), \dots, \frac{1}{g_i}(x - x_i)^k + \left(\frac{1}{g}\right)'_{\theta_i}(x - x_i)^{k+1}\right\}.$$

This is equivalent to a polynomial basis function space of order $k+1$. Using Taylor expansion, we have

$$r(x) = \frac{1}{g}S(x) = \left(\frac{S}{g}\right)_{i-\frac{1}{2}} + \left(\frac{S}{g}\right)'_{i-\frac{1}{2}}(x - x_{i-\frac{1}{2}}) + \cdots + \frac{(\frac{S}{g})^{(k+2)}_{\theta_i}}{(k+2)!}(x - x_{i-\frac{1}{2}})^{k+2}.$$

We define the projection $\Pi_2 r$ as

$$\Pi_2 r = \left(\frac{S}{g}\right)_{i-\frac{1}{2}} + \left(\frac{S}{g}\right)'_{i-\frac{1}{2}}(x - x_{i-\frac{1}{2}}) + \cdots + \frac{1}{(k+1)!}\left(\frac{S}{g}\right)^{(k+1)}_{i-\frac{1}{2}}(x - x_{i-\frac{1}{2}})^{k+1}.$$

Then,

$$\eta^r = r - \Pi_2 r = C(x - x_{i-\frac{1}{2}})^{k+2}$$

and

$$\left(\int_a^b \frac{(\eta^r)^2}{(1 + r_\theta^2)^{\frac{3}{2}}}\,\mathrm{d}x\right)^{\frac{1}{2}} \le Ch^{k+2}.$$

□

Similarly, we can obtain the following projection error estimate, which is a new version of Lemma 2.

Lemma 6. *Assuming (4) holds strictly, which means there exists a sufficiently smooth solution on the interval $[a, b]$. The Gauss–Radau projection $\Pi_1 u$ satisfies*

$$\int_{I_i} \eta^u \phi\,\mathrm{d}x = 0, \quad \eta^{u-}_{i+\frac{1}{2}} = 0, \quad \forall \phi \in V_h^{k-1}, \tag{73}$$

then, we have

$$\|\eta^u\| \le Ch^{k+2}, \tag{74}$$

where C is a positive constant independent of h.

Proof. This proof fundamentally differs from that of Lemma 2 in that g is smooth. Using Taylor expansion, we obtain

$$g(x) = g_i + g'(\theta_i)(x - x_i), \tag{75}$$

where θ_i is between $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ and g'_{θ_i} is bounded. Then, the basis S^k becomes

$$\left\{1, \frac{1}{g_i} + \left(\frac{1}{g}\right)'_{\theta_i}(x-x_i), \ldots, \frac{1}{g_{i-\frac{1}{2}}}(x-x_i)^k + \left(\frac{1}{g}\right)'_{\theta_i}(x-x_i)^{k+1}\right\}.$$

This is equivalent to a polynomial basis function space of order $k+1$. Using Taylor expansion, we have

$$u(x,t) = g(x)S(x,t) = (gS)_{i+\frac{1}{2}} + (gS)'_{i+\frac{1}{2}}(x-x_{i+\frac{1}{2}}) + \cdots + \frac{(gS)^{(k+2)}_{\theta_i}}{(k+2)!}(x-x_{i+\frac{1}{2}})^{k+2}.$$

We define the projection $\Pi_1 u$ as

$$\Pi_1 u = (gS)_{i+\frac{1}{2}} + (gS)'_{i+\frac{1}{2}}(x-x_{i+\frac{1}{2}}) + \cdots + \frac{1}{(k+1)!}(gS)^{(k+1)}_{i+\frac{1}{2}}(x-x_{i+\frac{1}{2}})^{k+1}.$$

Then

$$\eta^u = u - \Pi_1 u = C(x-x_{i+\frac{1}{2}})^{k+2}$$

and

$$\|\eta^u\| \le Ch^{k+2}.$$

□

Finally, we can prove the following theorem using the same method as for Theorem 2.

Theorem 3.

$$\int_a^b (u-u_h)^2 \,\mathrm{d}x \le c_2(1+t)e^t h^{2k+4}.$$

4. The local discontinuous Galerkin method for the multidimensional case

We introduce the auxiliary variable $\boldsymbol{r} = \nabla u, \boldsymbol{q} = \frac{\boldsymbol{r}}{\sqrt{1+|\boldsymbol{r}|^2}}$ and obtain the equations

$$\begin{aligned}
\boldsymbol{r} &= \nabla u && \text{in } \Omega, && (76)\\
\boldsymbol{q} &= \frac{\boldsymbol{r}}{\sqrt{1+|\boldsymbol{r}|^2}} && \text{in } \Omega, && (77)\\
u_t &= \nabla\cdot\boldsymbol{q} + f(\boldsymbol{x}) && \text{in } \Omega, && (78)\\
u &= b_0 && \text{on } \Gamma_D, && (79)
\end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^d, d \ge 2$ is an integer. To facilitate the illustration of our idea in multidimensional case, we assume that $f(\boldsymbol{x})$ is radially symmetric and that Ω is a region enclosed by a unite sphere, *i.e.* $\Omega := \{\boldsymbol{x} : |\boldsymbol{x}| \le 1\}$ and the boundary condition b_0 is a constant. The extreme solution is given by a source term which satisfies

$$\left|\int_\Omega f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right| = |\partial\Omega|. \qquad (80)$$

To construct the non-polynomial LDG scheme, as the first step, we need the boundary gradient estimate. Consider the spherically symmetric $u(r)$ satisfying the equation

$$-\frac{1}{r^{d-1}}\left(\frac{r^{d-1}u_r}{\sqrt{1+u_r^2}}\right)_r = f(r), \tag{81}$$

$$u_r(0) = 0, \tag{82}$$

$$u(1) = b_0, \tag{83}$$

where $r = \sqrt{|\boldsymbol{x}|^2}$. We analyze u_r near $r = 1$ for the equation

$$-\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1}\frac{u_r}{\sqrt{1+u_r^2}}\right) = f(r),$$

with $u_r(0) = 0$ and $u(1) = b_0$. We rewrite and integrate

$$F(r) := r^{d-1}\frac{u_r}{\sqrt{1+u_r^2}} = -\int_0^r s^{d-1}f(s)\,\mathrm{d}s.$$

Let $y = u_r$. Solving, we get

$$y = \frac{F(r)}{\sqrt{r^{2(d-1)} - F(r)^2}}.$$

For $\varepsilon = 1 - r$ and $F(1-\varepsilon) = 1 + f(1)\varepsilon + \mathcal{O}(\varepsilon^2)$, we obtain

$$u_r = \frac{1}{\sqrt{-2(d-1+f(1))\varepsilon}} + \mathcal{O}(\varepsilon).$$

For simplicity, we assume that the source term satisfies $f(1) < -(d-1)$ to ensure the existence of the gradient blow-up solution. We define $C = \frac{1}{\sqrt{-2(d-1+f(1))}}$; thus,

$$u_r(r) \sim \frac{C}{\sqrt{1-r}} \quad \text{as } r \to 1^-.$$

So, we can define, as before,

$$g(\boldsymbol{x}) := \sqrt{1-r^2}.$$

Then, we can introduce the LDG scheme. We consider a triangulation $\mathcal{T}$, with elements K. We first define the non-polynomial basis function space S^k and $\tilde{S}^k$ (for simplicity, we demonstrate only the two-dimensional case, the higher-dimensional case is analogous). Let (x_K, y_K) be the center coordinates of the Kth cell and $g(x,y) := \sqrt{1-x^2-y^2}$, then we can define

$$\begin{aligned} S^k := \big\{v : v|_K \in \operatorname{span}\big\{&1, g(x,y), g(x,y)(x-x_K), g(x,y)(y-y_K),\\ &g(x,y)(x-x_K)^2, g(x,y)(x-x_K)(y-y_K), g(x,y)(y-y_K)^2,\\ &g(x,y)(x-x_K)^k, \ldots, g(x,y)(y-y_K)^k, \big\},\\ &(x,y) \in K\big\}, \end{aligned}$$

$$\tilde{S}^k := \Big\{ v : v|_K \in \operatorname{span}\Big\{1, \frac{1}{g}(x-x_K), \frac{1}{g}(y-y_K), \frac{1}{g}(x-x_K)^2, \frac{1}{g}(x-x_K)(y-y_K), \frac{1}{g}(y-y_K)^2, \frac{1}{g}(x-x_K)^k, \ldots, \frac{1}{g}(y-y_K)^k\Big\}, (x,y)\in K\Big\},$$

$$V_h^k := \big\{ v : v|_K \in \operatorname{span}\big\{1, (x-x_K), (y-y_K), (x-x_K)^2, (x-x_K)(y-y_K), (y-y_K)^2, \ldots (x-x_K)^k, \ldots, (y-y_K)^k\big\}, (x,y)\in K\big\}$$

The LDG scheme is defined as: find $\boldsymbol{r}_h \in \tilde{S}^k, \boldsymbol{q}_h \in V_h^k$ and $u_h \in S^k$, such that $\forall \boldsymbol{\psi} \in V_h^k, \boldsymbol{\gamma} \in \tilde{S}^k, \phi \in S^k$:

$$\int_K \boldsymbol{r_h} \cdot \boldsymbol{\psi}\, \mathrm{d}\boldsymbol{x} = -\int_K u_h \nabla \cdot \boldsymbol{\psi}\, \mathrm{d}\boldsymbol{x} + \int_{\partial K} \hat{u}_h \boldsymbol{\psi} \cdot \boldsymbol{n}\, \mathrm{d}s, \tag{84}$$

$$\int_K \boldsymbol{q}_h \cdot \boldsymbol{\gamma}\, \mathrm{d}\boldsymbol{x} = \int_K \frac{\boldsymbol{r}_h \cdot \boldsymbol{\gamma}}{\sqrt{1+|\boldsymbol{r}_h|^2}}\, \mathrm{d}\boldsymbol{x}, \tag{85}$$

$$\int_K (u_h)_t \phi\, \mathrm{d}\boldsymbol{x} = -\int_K \boldsymbol{q}_h \cdot \nabla\phi\, \mathrm{d}\boldsymbol{x} + \int_{\partial K} \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\phi\, \mathrm{d}s + \int_K f\phi\, \mathrm{d}\boldsymbol{x}, \tag{86}$$

where the numerical fluxes $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$ are defined as follows. We define $\mathscr{E}^i$ as the collection of all interior faces and Γ as the collection of all boundary faces. An edge e is considered to be in $\mathscr{E}^i$ if there exist two elements K^+, K^- in Ω_h such that $e = \partial K^+ \bigcap \partial K^-$, and an edge e is considered to be in Γ if there exists an element in Ω_h such that $e = \partial K \bigcap \partial\Omega_h$. We set $\mathscr{E} = \mathscr{E}^i \bigcup \Gamma$. Let e be an interior face shared by elements K_1 and K_2, and define the unit normal vectors $\boldsymbol{n}_1, \boldsymbol{n}_2$ on e pointing outward from K_1 and K_2, respectively. We define the average $\{\{\cdot\}\}$ and jump $[\![\cdot]\!]$ at $\boldsymbol{x} \in e$ as follows:

$$\{\{u\}\} := \frac{1}{2}(u_1 + u_2), \qquad \{\{\boldsymbol{q}\}\} := \frac{1}{2}(\boldsymbol{q}_1 + \boldsymbol{q}_2),$$
$$[\![u]\!] := u_1\boldsymbol{n}_1 + u_2\boldsymbol{n}_2, \qquad [\![\boldsymbol{q}]\!] := \boldsymbol{q}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{q}_2 \cdot \boldsymbol{n}_2,$$

where $u_i = u|_{\partial K_i}$. We define a vector value function $\boldsymbol{\beta}_0$ on $\mathscr{E}$ such that, for $e \in \partial K \cap \mathscr{E}$,

$$\boldsymbol{\beta_0} \cdot n_K(e) = \frac{1}{2}\operatorname{sign}(v_0 \cdot n_K(e)),$$

where v_0 is any nonzero constant vector and

$$\Gamma_{v_0}^- = \{e \in \Gamma : v_0 \cdot n_e < 0\}, \quad \Gamma_{v_0}^+ = \Gamma \backslash \Gamma_{v_0}^-.$$

Next, we are going to introduce the expressions of the numerical fluxes:

$$\hat{u}_h = \left\{ \begin{array}{ll} \{\{u_h\}\} + \boldsymbol{\beta_0} \cdot [\![u_h]\!], & \text{if } e \in \mathscr{E}^i \\ g, & \text{if } e \in \partial\Omega_h \cap \Gamma_{v_0}^-, \\ u_h, & \text{if } e \in \partial\Omega_h \cap \Gamma_{v_0}^+, \end{array} \right\}. \tag{87}$$

$$\hat{\boldsymbol{q}}_h = \begin{Bmatrix} \{\{\boldsymbol{q}_h\}\} - \boldsymbol{\beta_0}[\![\boldsymbol{q}_h]\!], & \text{if } e \in \mathscr{E}^i \\ \boldsymbol{q}_h, & \text{if } e \in \partial\Omega_h \cap \Gamma^-_{v_0}, \\ \boldsymbol{q}_h - \frac{\alpha}{h}(u_h - g)\boldsymbol{n}, & \text{if } e \in \partial\Omega_h \cap \Gamma^+_{v_0}, \end{Bmatrix}. \tag{88}$$

where α is a positive parameter.

We still have the following stability result.

Theorem 4. *For the LDG scheme (84)–(86) approximating the problem with homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ (*i.e.*, setting the parameter $b_0 = 0$ in the numerical fluxes), we have the following stability estimate*

$$\int_{\Omega_h} u_h^2 \,\mathrm{d}\boldsymbol{x} + \int_0^t \int_{\Omega_h} \frac{2\boldsymbol{r_h} \cdot \boldsymbol{r_h}}{\sqrt{1 + |\boldsymbol{r_h}|^2}} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \le C(t+1)e^t \left(\int_{\Omega_h} u_0^2 + f^2 \,\mathrm{d}\boldsymbol{x} \right). \tag{89}$$

Proof. Taking $\boldsymbol{\psi} = \boldsymbol{q}_h, \phi = u_h$ in (84) and (86), we have

$$\int_K \frac{1}{2}(u_h^2)_t + \boldsymbol{r}_h \cdot \boldsymbol{q}_h \,\mathrm{d}\boldsymbol{x} = \int_{\partial K} -u_h \boldsymbol{q}_h \cdot \boldsymbol{n} + \hat{u}_h \boldsymbol{q}_h \cdot \boldsymbol{n} + u_h \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \,\mathrm{d}s + \int_K f u_h \,\mathrm{d}\boldsymbol{x},$$

Taking $\gamma = \boldsymbol{r}_h$ in Eq. (85), we have

$$\int_K \frac{1}{2}(u_h^2)_t + \frac{\boldsymbol{r}_h \cdot \boldsymbol{r}_h}{\sqrt{1+|\boldsymbol{r}_h|^2}} \,\mathrm{d}\boldsymbol{x} = \int_{\partial K} -u_h \boldsymbol{q}_h \cdot \boldsymbol{n} + \hat{u}_h \boldsymbol{q}_h \cdot \boldsymbol{n} + u_h \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \,\mathrm{d}s + \int_K f u_h \,\mathrm{d}\boldsymbol{x}.$$

Summing up over K, with the numerical fluxes (87)–(88), we obtain

$$\begin{aligned} \int_\Omega \frac{1}{2}(u_h^2)_t + \frac{\boldsymbol{r}_h \cdot \boldsymbol{r}_h}{\sqrt{1+|\boldsymbol{r}_h|^2}} \,\mathrm{d}\boldsymbol{x} &= \int_{\partial\Omega_h \cap \Gamma^-_{v_0}} b_0 \boldsymbol{q}_h \cdot \boldsymbol{n} \,\mathrm{d}s + \int_{\partial\Omega_h \cap \Gamma^+_{v_0}} u_h \boldsymbol{q}_h \cdot \boldsymbol{n} - \frac{\alpha}{h} u_h(u_h - b_0) \,\mathrm{d}s + \int_\Omega f u_h \,\mathrm{d}\boldsymbol{x}, \\ &= \int_{\partial\Omega_h \cap \Gamma^+_{v_0}} u_h \boldsymbol{q}_h \cdot \boldsymbol{n} - \frac{\alpha}{h} u_h^2 \,\mathrm{d}s + \int_\Omega u_h f \,\mathrm{d}\boldsymbol{x}. \quad \text{(To simplify, assume } b_0 = 0.) \end{aligned}$$

Using Young's inequality, we have

$$\int_{\partial\Omega_h \cap \Gamma^+_{v_0}} u_h \boldsymbol{q}_h \cdot \boldsymbol{n} \,\mathrm{d}s \le \frac{\alpha}{h} \int_{\partial\Omega_h \cap \Gamma^+_{v_0}} (u_h)^2 \,\mathrm{d}s + \frac{h}{4\alpha} \int_{\partial\Omega_h \cap \Gamma^+_{v_0}} (\boldsymbol{q}_h \cdot \boldsymbol{n})^2 \,\mathrm{d}s.$$

Then,

$$\begin{aligned} \int_\Omega \frac{1}{2}(u_h^2)_t + \frac{\boldsymbol{r}_h \cdot \boldsymbol{r}_h}{\sqrt{1+|\boldsymbol{r}_h|^2}} \,\mathrm{d}\boldsymbol{x} &\le \frac{h}{4\alpha} \int_{\partial\Omega_h \cap \Gamma^+_{v_0}} (\boldsymbol{q}_h \cdot \boldsymbol{n})^2 \,\mathrm{d}s + \int_\Omega f^2 + u_h^2 \,\mathrm{d}\boldsymbol{x} \\ &\le \frac{h}{4\alpha}|\partial\Omega_h| + \int_\Omega f^2 + u_h^2 \,\mathrm{d}\boldsymbol{x}. \quad (\text{using } \|\boldsymbol{q} \cdot \boldsymbol{n}\|_\infty \le 1). \end{aligned}$$

By Gronwall's inequality, we obtain Theorem 4. ☐

5. Numerical test

In this section, we show some numerical evidence to confirm our *a priori* error estimate for the non-polynomial LDG scheme. To verify the spatial accuracy of our scheme, we evolve the parabolic systems to a steady state using the SSP-RK3. The iteration stops when $\|u_h^{n+1} - u_h^n\|_{L^2}/\Delta t < 10^{-12}$. At steady state, the temporal error vanishes, so the reported $\|u - u_h\|_{L^2}$ represents the pure spatial discretization error. This allows us to directly confirm our theoretical $O(h^{k+1})$ and $O(h^{k+2})$ bounds.

TABLE 1. Error table of numerical solution for Example 1 with S^k non-polynomial basis.

k	N	$\|u-u_h\|_{L^2}$	Order
	10	8.91E−04	–
	20	4.58E−04	0.96
0	40	2.32E−04	0.98
	80	1.15E−04	1.01
	160	5.76E−05	1.00
	10	3.13E−06	–
	20	8.27E−07	1.92
1	40	2.08E−07	1.99
	80	5.21E−08	2.00
	160	1.31E−08	1.99

Example 1. We compute the one-dimensional problem (13) in $[a,b]=[-1,1]$ with boundary condition $u(a)=u(b)=0$. The source term is

$$f(x)=-1+\frac{1}{10}\sin(x)$$

and satisfies

$$\int_{-1}^{1} f(x)\,\mathrm{d}x=-2.$$

Then, the gradient $|u_x|\to\infty$ near boundary. Since we do not know the exact solution, we use the numerical solution corresponding to the grid number $N=640$ as the "exact solution". Table 1 provides the error table. This confirms the error estimate of Theorem 2 as $O(h^{k+1})$.

Example 2. We consider the same source term defined in Example 1. The computation interval is $[-\frac{4}{5},\frac{4}{5}]$, with boundary conditions $u(-\frac{4}{5})=u(\frac{4}{5})=0$. We can check that

$$\int_{-\frac{4}{5}}^{\frac{4}{5}} f(x)\,\mathrm{d}x=-\frac{8}{5},$$

so the solution is smooth. Since we do not know the exact solution, we use the numerical solution corresponding to the grid number $N=640$ as the "exact solution". Table 2 provides the error table. This confirms the error estimate of Theorem 3 as $O(h^{k+2})$.

Example 3. We compute the one-dimensional problem (13) in $[a,b]=[-1,1]$ with boundary condition $u(a)=u(b)=0$. We define the source term as

$$f(x)=\frac{\left(\left(x^2-1\right)^2+1\right)\cos(x)+2x\left(x^2-1\right)\sin(x)}{\left(\left(x\cos(x)-(x^2-1)\sin(x)\right)^2-x^2+1\right)^{3/2}}.$$

The source term $f(x)$ is chosen such that the exact solution is given by

$$u(x)=\sqrt{1-x^2}\cos(x).$$

TABLE 2. Error table of numerical solution for Example 2 with S^k non-polynomial basis.

k	N	$\|u-u_h\|_{L^2}$	Order
0	10	2.10E−02	–
	20	6.24E−03	1.75
	40	1.58E−03	1.98
	80	3.92E−04	2.02
	160	9.62E−05	2.02
1	10	2.15E−07	–
	20	3.29E−08	2.71
	40	4.60E−09	2.84
	80	5.71E−10	3.01
	160	7.14E−11	3.00

TABLE 3. Error table of numerical solution for Example 3 with S^k non-polynomial basis.

k	N	$\|u-u_h\|_{L^2}$	Order
1	10	1.08E−04	–
	20	3.24E−05	1.74
	40	8.92E−06	1.86
	80	2.36E−06	1.91
	160	5.99E−07	1.98

We can check that $f(x)$ is smooth and $\int_{-1}^{1} f(x)\,\mathrm{d}x = 2$, the gradient blow-up near the boundary. Table 3 provides the error table. This confirms the error estimate of Theorem 2 as $O(h^{k+1})$.

Example 4. We consider the same source term defined in Example 3. The computation interval is $[-\frac{4}{5}, \frac{4}{5}]$, with boundary conditions $u(-\frac{4}{5}) = u(\frac{4}{5}) = 0$. The exact solution is given by

$$u(x) = \sqrt{1-x^2}\cos(x) - \frac{3}{5}\cos\left(\frac{4}{5}\right).$$

which is a smooth solution. Table 4 provides the error table. This confirms the error estimate of Theorem 3 as $O(h^{k+2})$.

Example 5. We take $f(x) = -1, u(x) = -\sqrt{1-x^2}$. We show the error table in Table 5.

Example 6. This is a two-dimensional case. Define $r = \sqrt{1-x^2-y^2}$ and source term $f(x,y)$ as

$$\begin{aligned}\sqrt{1-x^2-y^2}f(x,y) = &-\frac{\left(r^2-1\right)^3\sin^3(r)}{(-r^2+((\sin(r)+r(\cos(r)-r\sin(r)))+r)^2+1)^{3/2}}\\ &+\frac{3r\left(r^2-1\right)^2\sin^2(r)(\cos(r)+1)}{(-r^2+((\sin(r)+r(\cos(r)-r\sin(r)))+r)^2+1)^{3/2}}\end{aligned}$$

TABLE 4. Error table of numerical solution for Example 4 with S^k non-polynomial basis.

k	N	$\|u-u_h\|_{L^2}$	Order
	10	2.10E−05	–
	20	6.24E−06	1.75
0	40	1.58E−06	1.98
	80	3.92E−07	2.02
	160	9.62E−08	2.02
	10	2.39E−06	–
	20	3.40E−07	2.81
1	40	4.23E−08	3.01
	80	5.32E−09	2.99
	160	6.61E−10	3.01

TABLE 5. Error table of numerical solution for Example 5 with S^2 non-polynomial basis.

k	N	$\|u-u_h\|_{L^2}$	Order
	10	1.35E–15	–
	20	2.50E–14	–
2	40	1.66E–13	–
	80	4.89E–12	–
	160	7.15E–12	–

$$- \frac{\left(r^2-1\right)\sin(r)\left(3r^2\cos(r)(\cos(r)+2)+1\right)}{\left(-r^2+((\sin(r)+r(\cos(r)-r\sin(r)))+r)^2+1\right)^{3/2}} + \frac{r\left(\cos(r)\left(r^4+r^2\cos(r)(\cos(r)+3)+3\right)+2\right)}{\left(-r^2+((\sin(r)+r(\cos(r)-r\sin(r)))+r)^2+1\right)^{3/2}}. \tag{90}$$

We solve (76)–(79) in $\Omega = (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2})$ with Dirichlet boundary conditions. The source term $f(x,y)$ is chosen such that the exact solution is given by

$$u(x,y) = \sqrt{1-x^2-y^2}\,(1+\cos(r)). \tag{91}$$

We present the numerical results using uniform rectangular meshes. Table 6 provides the error table.

Example 7. We define Ω as a circular region with a radius of 1, $\Omega = \{(x,y) : \sqrt{x^2+y^2} \le 1\}$. We solve the (76)–(79) in Ω with Dirichlet boundary conditions. The source term and exact solution are given by (90) and (91). We can check that

$$\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = -2\pi.$$

In this case, the gradient of the exact solution will blow up on the boundary.

We present numerical results using sequences of triangular meshes $\{\mathcal{T}_i\}, i = 1,2,\ldots,$ where the mesh size of $\mathcal{T}_{i+1}$ is half of $\mathcal{T}_i$. The triangular mesh sequence is generated through iterative global refinement of an initial coarse mesh, with each triangle being subdivided into four similar triangles at every refinement stage. The number of triangles of the meshes are 8, 32, 128, and 512. Table 7 provides the error table.

TABLE 6. Error table of numerical solution for Example 6 with S^k non-polynomial basis in rectangular mesh.

k	N	$\|u-u_h\|_{L^2}$	Order
	10	1.89E−01	–
	20	4.79E−02	1.98
0	40	1.19E−02	2.01
	80	2.97E−03	2.00
	10	8.75E−03	–
	20	1.12E−03	2.97
1	40	1.39E−04	3.01
	80	1.73E−05	3.00

TABLE 7. Error table of numerical solution for Example 7 with S^k non-polynomial basis in triangular mesh.

k	N	$\|u-u_h\|_{L^2}$	Order
	8	1.23E−01	–
	32	6.54E−02	0.91
0	128	3.29E−02	0.99
	512	1.63E−02	1.01
	8	5.88E−02	–
	32	1.58E−02	1.89
1	128	4.04E−03	1.97
	512	1.02E−03	1.99

6. Concluding remarks

In this paper, we present *a priori* error analysis for a non-polynomial LDG method, which is used to solve the prescribed mean curvature type equation. Through boundary gradient estimation, we can determine the blow-up rate of the exact solution along the boundary (for simplicity, we considered only a very simple boundary in this paper; gradient estimation for general boundaries is also feasible, see [22]). We can design a set of non-polynomial basis functions to well-approximate such singularities. We have obtained the error estimates.

For the Jang's equation, we do not intend to discuss it in this paper because the metric g_{ij} and the extrinsic curvature K_{ij} require solving the Einstein equations, which is very complex and deserves a separate paper for discussion. In this paper, we establish only a general framework.

Funding

Research of the second author Chi-Wang Shu is supported in part by NSF Grant DMS-2309249.

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A. Detailed calculation of $\psi_0, \psi_1, \ldots, \psi_k$ and order estimation

We begin by defining a set of simple polynomial basis functions on each cell I_i:

$$\begin{aligned}
\beta_0 &= 1, \\
\beta_1 &= x - x_i, \\
\beta_2 &= (x - x_i)^2, \\
&\vdots \\
\beta_k &= (x - x_i)^k.
\end{aligned}$$

Following the paper, we define the inner product

$$(A, B)_i := \int_{I_i} AB \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x. \tag{92}$$

We define $\varepsilon := 1 - x_i$, where $\frac{h}{2} \leq \varepsilon < 1$.

Calculation of ψ_0

First, we compute the normalization factor:

$$\begin{aligned}
(\beta_0, \beta_0)_i &= \int_{I_i} \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x \\
&= \int_{1-\varepsilon-h/2}^{1-\varepsilon+h/2} \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x \\
&= \int_{1-\varepsilon-h/2}^{1-\varepsilon+h/2} \frac{1}{\sqrt{2(1 - x)}} \frac{1}{\sqrt{1 + \frac{1-x}{2}}} \,\mathrm{d}x \\
&= c_1 \int_{1-\varepsilon-h/2}^{1-\varepsilon+h/2} \frac{1}{\sqrt{2(1 - x)}} \,\mathrm{d}x,
\end{aligned}$$

where $c_1 = \frac{1}{\sqrt{1+\frac{1-\xi}{2}}}, 1 - \varepsilon - \frac{h}{2} \leq \xi \leq 1 - \varepsilon + \frac{h}{2}$, then c_1 is a bounded constant:

$$\sqrt{\frac{2}{2 + \epsilon}} \leq c_1 \leq \sqrt{\frac{2}{2 + \epsilon - h}}.$$

$$
\begin{aligned}
(\beta_0, \beta_0)_i &= c_1 \int_{1-\varepsilon-\frac{h}{2}}^{1-\varepsilon+\frac{h}{2}} \frac{1}{\sqrt{2(1-x)}} \,\mathrm{d}x \\
&= \sqrt{2} c_1 \int_{1-\varepsilon-\frac{h}{2}}^{1-\varepsilon+\frac{h}{2}} (1-x)^{-\frac{1}{2}} \,\mathrm{d}x \\
&= 2\sqrt{2} c_1 \left(\sqrt{\varepsilon + \frac{h}{2}} - \sqrt{\varepsilon - \frac{h}{2}} \right),
\end{aligned}
$$

$$
\begin{aligned}
(\beta_0, \beta_0)_i^{-\frac{1}{2}} &= 2^{-\frac{1}{4}} c_1^{-\frac{1}{2}} \left(\frac{1}{\sqrt{\varepsilon} - \sqrt{\varepsilon - h}} \right)^{\frac{1}{2}} \\
&= 2^{-\frac{1}{4}} c_1^{-\frac{1}{2}} \left(\frac{\sqrt{\varepsilon} + \sqrt{\varepsilon - h}}{h} \right)^{\frac{1}{2}} \\
&\sim C h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}.
\end{aligned}
$$

Finally, we have

$$
\psi_0 = \beta_0 (\beta_0, \beta_0)_i^{-\frac{1}{2}} \sim C h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}.
$$

Calculation of ψ_1

Following the Gram–Schmidt orthogonalization process

$$
\bar{\psi}_1 = \beta_1 - (\psi_0, \beta_1)_i \psi_0,
$$

we need to compute $(\psi_0, \beta_1)_i$:

$$
(\psi_0, \beta_1)_i = \int_{I_i} \psi_0 (x - x_i) \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x,
$$

$$
\begin{aligned}
(\psi_0, \beta_1)_i &= \int_{I_i} \psi_0 (x - x_i) \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x, \\
&\sim C (h^{-\frac{1}{2}}) \varepsilon^{\frac{1}{4}} \left(\frac{1}{24} h^3 \varepsilon^{-\frac{3}{2}} + R \right), \quad \text{where } |R| \le c h^4 \varepsilon^{-\frac{5}{2}} \\
&\sim C h^{\frac{5}{2}} \varepsilon^{-\frac{5}{4}}.
\end{aligned}
$$

Now we can compute $\bar{\psi}_1$:

$$
\begin{aligned}
\bar{\psi}_1 &= \beta_1 - (\psi_0, \beta_1)_i \psi_0, \\
&= (x - x_i) - \mathcal{O}(h^{\frac{5}{2}} \varepsilon^{-\frac{5}{4}}).
\end{aligned}
$$

Next, we compute $(\bar{\psi}_1, \bar{\psi}_1)_i$:

$$
\begin{aligned}
(\bar{\psi}_1, \bar{\psi}_1)_i &= \int_{I_i} \bar{\psi}_1^2 \frac{1}{\sqrt{1 - x^2}} \,\mathrm{d}x \\
&\sim C h^3 \varepsilon^{-\frac{1}{2}}.
\end{aligned}
$$

We need to consider two cases, $h \le \varepsilon/2$ and $h = \varepsilon$, since ε has a relation $\varepsilon = h, 2h, \ldots$.

If $h \leq \frac{\varepsilon}{2}$,

$$\begin{aligned}(\bar{\psi}_1, \bar{\psi}_1)_i &= \int_{I_i} \bar{\psi}_1^2 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x, \\ &= \frac{1}{\sqrt{1-\xi^2}} \int_{I_i} \bar{\psi}^2 \, \mathrm{d}x, \quad \text{where } 1-\varepsilon \leq \xi \leq 1-\varepsilon+h,\end{aligned}$$

where

$$\frac{h^3}{12} \leq \int_{I_i} \bar{\psi}^2 \, \mathrm{d}x \leq h\left(\frac{h^2}{12} + Mh^5\varepsilon^{-\frac{5}{2}}\right),$$

and

$$\frac{1}{\sqrt{2\varepsilon}} \leq \frac{1}{\sqrt{1-\xi^2}} \leq \sqrt{2}\frac{1}{\sqrt{\varepsilon}}.$$

Finally, we have

$$c_1 \frac{h^3}{\sqrt{\varepsilon}} \leq (\bar{\psi}_1, \bar{\psi}_1)_i \leq c_2 \frac{h^3}{\sqrt{\varepsilon}}, \quad \text{where } c_1, c_2 > 0.$$

If $h = \varepsilon$,

$$\begin{aligned}(\bar{\psi}_1, \bar{\psi}_1)_i &= \int_{I_i} \bar{\psi}_1^2 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{1+\xi_2}} \int_{I_i} \frac{\bar{\psi}_1^2}{\sqrt{1-x}} \, \mathrm{d}x, \quad \text{where } 1-\varepsilon \leq \xi_2 \leq 1,\end{aligned}$$

where

$$\frac{8}{45}\varepsilon^{\frac{5}{2}} = \frac{8}{45}h^3\varepsilon^{-\frac{1}{2}} \leq \int_{I_i} \frac{\bar{\psi}_1^2}{\sqrt{1-x}} \, \mathrm{d}x \leq \varepsilon^{\frac{11}{4}} + \frac{7}{30}\varepsilon^{\frac{5}{2}},$$

and

$$\frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1+\xi_2}} \leq 1,$$

then,

$$c_3 \frac{h^3}{\sqrt{\varepsilon}} \leq (\bar{\psi}_1, \bar{\psi}_1)_i \leq c_4 \frac{h^3}{\sqrt{\varepsilon}}, \quad \text{where } c_3, c_4 > 0.$$

Combining the two cases, we have

$$c_5 \frac{h^3}{\sqrt{\varepsilon}} \leq (\bar{\psi}_1, \bar{\psi}_1)_i \leq c_6 \frac{h^3}{\sqrt{\varepsilon}}, \quad \text{where } c_5, c_6 > 0.$$

Therefore,

$$\begin{aligned}\psi_1 &= \bar{\psi}_1 (\bar{\psi}_1, \bar{\psi}_1)_i^{-\frac{1}{2}} \\ &= ((x-x_i) - \mathcal{O}(h^{\frac{5}{2}}\varepsilon^{-\frac{5}{4}})) \cdot (\mathcal{O}(h^3\varepsilon^{-\frac{1}{2}}))^{-\frac{1}{2}} \\ &= ((x-x_i) - \mathcal{O}(h^{\frac{5}{2}}\varepsilon^{-\frac{5}{4}})) \cdot \mathcal{O}(h^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}) \\ &= \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}).\end{aligned}$$

Calculation of ψ_2

Following the Gram–Schmidt process,

$$\bar{\psi}_2 = \beta_2 - (\beta_2, \psi_0)_i \psi_0 - (\beta_2, \psi_1)_i \psi_1.$$

We compute $(\beta_2, \psi_0)_i$ and $(\beta_2, \psi_1)_i$ using similar techniques. After simplification,

$$c_1 h^5 \varepsilon^{-\frac{1}{2}} \leq (\bar{\psi}_2, \bar{\psi}_2)_i \leq c_2 h^5 \varepsilon^{-\frac{1}{2}}, \quad \text{where } c_1, c_2 > 0.$$

Thus,

$$\begin{aligned} \psi_2 &= \bar{\psi}_2 (\bar{\psi}_2, \bar{\psi}_2)_i^{-\frac{1}{2}} \\ &= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}). \end{aligned}$$

General Pattern for $\psi_k, k \geq 2$

We use mathematical induction to prove the following conclusion: $\psi_k = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{2}}$.
We assume

$$\begin{aligned} \psi_0 &= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}), \\ &\cdots \\ \psi_{k-1} &= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}), \end{aligned}$$

and define

$$\bar{\psi}_k = \beta_k - (\beta_k, \psi_0)_i \psi_0 - \cdots - (\beta_k, \psi_{k-1})_i \psi_{k-1}.$$

Then,

$$\bar{\psi}_k = (x - x_i)^k - \mathcal{O}(h^k)\varepsilon^{\frac{1}{2}}.$$

If $h \leq \frac{\varepsilon}{2}$,

$$\begin{aligned} (\bar{\psi}_k, \bar{\psi}_k)_i &= \int_{I_i} \frac{\bar{\psi}_k^2}{\sqrt{1 - x^2}} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{1 - \xi^2}} \int_{I_i} \bar{\psi}_k^2 \, \mathrm{d}x, \quad \text{where } 1 - \varepsilon \leq \xi \leq 1 - \varepsilon + h, \end{aligned}$$

where

$$\frac{h^{2k+1}}{4^k(2k+1)} \leq \int_{I_i} \bar{\psi}_k^2 \, \mathrm{d}x \leq \frac{h^{2k+1}}{4^k(2k+1)} + M h^{2k+1} \varepsilon,$$

and

$$\frac{1}{\sqrt{2\varepsilon}} \leq \frac{1}{\sqrt{1 - \xi^2}} \leq \sqrt{2} \frac{1}{\sqrt{\varepsilon}}.$$

Then,

$$c_1 h^{2k+1} \varepsilon^{-\frac{1}{2}} \leq (\bar{\psi}_k, \bar{\psi}_k)_i \leq c_2 h^{2k+1} \varepsilon^{-\frac{1}{2}}, \quad \text{where } c_1, c_2 > 0.$$

If $h = \varepsilon$,

$$
\begin{aligned}
(\bar{\psi}_k, \bar{\psi}_k)_i &= \int_{I_i} \bar{\psi}_k^2 \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{1+\xi_2}} \int_{I_i} \frac{\bar{\psi}_k^2}{\sqrt{1-x}} \, \mathrm{d}x, \quad \text{where } 1-\varepsilon \leq \xi_2 \leq 1.
\end{aligned}
$$

Similarly, we have

$$
c_3 h^{2k+\frac{1}{2}} \leq \int_{I_i} \frac{\bar{\psi}_k^2}{\sqrt{1-x}} \, \mathrm{d}x \leq c_4 h^{2k+\frac{1}{2}}, \quad \text{where } c_3, c_4 > 0,
$$

and

$$
\frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1+\xi_2}} \leq 1.
$$

Finally,

$$
\psi_k = \bar{\psi}_k (\bar{\psi}_k, \bar{\psi}_k)_i^{-\frac{1}{2}} = \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}).
$$

Detailed calculation of C_0

We need to compute C_0, which is defined as

$$
\begin{aligned}
C_0 &:= a_0 - b_0 \\
&= \int_{I_i} \frac{1}{g} (x - x_i)^{k+1} \psi_0 \, \mathrm{d}x,
\end{aligned}
$$

where

- $g(x) = \sqrt{1-x^2}$ is the function capturing the boundary singularity;
- $\psi_0 = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}}$ is the first orthogonal basis function;
- $I_i = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ with $x_{i-\frac{1}{2}} = 1-\varepsilon$ and $x_{i+\frac{1}{2}} = 1-\varepsilon+h$; and
- $h \leq \varepsilon$ is the mesh size parameter.

Substituting $\psi_0 = \mathcal{O}(h^{-\frac{1}{2}})\varepsilon^{\frac{1}{4}}$ into the integral and using the first mean value theorem for definite integrals, we have

$$
\begin{aligned}
C_0 &= \int_{I_i} \frac{1}{g} (x - x_i)^{k+1} \psi_0 \, \mathrm{d}x \\
&= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}) \int_{I_i} \frac{1}{g} (x - x_i)^{k+1} \, \mathrm{d}x \\
&= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{\frac{1}{4}}) \mathcal{O}(h^{k+1}) \int_{I_i} \frac{1}{\sqrt{1-x}} \, \mathrm{d}x \\
&= \mathcal{O}(h^{k+\frac{1}{2}} \varepsilon^{\frac{1}{4}}) \frac{h}{\sqrt{\varepsilon} + \sqrt{\varepsilon - h}} \\
&= \mathcal{O}(h^{k+\frac{3}{2}} \varepsilon^{-\frac{1}{4}}).
\end{aligned}
$$

Detailed calculation of C_1

Following the definition, C_1 is given by

$$
\begin{aligned}
C_1 &:= a_1 - b_1 \\
&= \int_{I_i} \frac{1}{g} (x - x_i)^{k+1} \psi_1 \, \mathrm{d}x.
\end{aligned}
$$

From our previous calculations,

$$\psi_1 = (x - x_i) \cdot O(h^{-\frac{3}{2}}\varepsilon^{\frac{1}{4}}). \tag{93}$$

Substituting this into the integral for C_1 and using the first mean value theorem for definite integrals, we have

$$\begin{aligned} C_1 &= \int_{I_i} \frac{1}{g}(x - x_i)^{k+1}\psi_1 \,\mathrm{d}x \\ &= \mathcal{O}(h^{k+1})\mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}) \int_{I_i} \frac{1}{\sqrt{1-x}} \,\mathrm{d}x \\ &= \mathcal{O}(h^{k+\frac{1}{2}}\varepsilon^{\frac{1}{4}}) \frac{h}{\sqrt{\varepsilon} + \sqrt{\varepsilon - h}} \\ &= \mathcal{O}(h^{k+\frac{3}{2}}\varepsilon^{-\frac{1}{4}}). \end{aligned}$$

Similarly, for $C_k, k \geq 1$, using $\psi_k = \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}})$ and the first mean value theorem for definite integrals, we have

$$\begin{aligned} C_k &= \int_{I_i} \frac{1}{g}(x - x_i)^{k+1}\psi_k \,\mathrm{d}x \\ &= \mathcal{O}(h^{k+1})\mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{\frac{1}{4}}) \int_{I_i} \frac{1}{\sqrt{1-x}} \,\mathrm{d}x \\ &= \mathcal{O}(h^{k+\frac{1}{2}}\varepsilon^{\frac{1}{4}}) \frac{h}{\sqrt{\varepsilon} + \sqrt{\varepsilon - h}} \\ &= \mathcal{O}(h^{k+\frac{3}{2}}\varepsilon^{-\frac{1}{4}}). \end{aligned}$$

B. Detailed analysis of the Gauss–Radau projection error estimates

In this section, we provide a comprehensive analysis of the error estimates for the Gauss–Radau projection stated in Lemma 2. We present detailed calculations for the orthogonal polynomial bases ζ_j, analyze the coefficients c_j, and derive precise order estimates for key integrals.

B.1. Construction and order analysis of orthogonal basis functions $\boldsymbol{\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_k}$

We begin by analyzing the weighted orthogonal polynomial bases constructed using the Gram–Schmidt orthogonalization process. Recall that these bases satisfy

$$\int_{I_i} \zeta_\ell \zeta_j g^2 \,\mathrm{d}x = \delta_{j\ell}, \quad j, \ell = 0, 1, \ldots, k. \tag{94}$$

Starting with the monomial basis functions

$$\begin{aligned} &\beta_0 = 1, \\ &\beta_1 = x - x_i, \\ &\beta_2 = (x - x_i)^2, \\ &\quad \vdots \\ &\beta_k = (x - x_i)^k, \end{aligned}$$

where $x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}})$, and we assume $x_i = 1 - \varepsilon$ with $0 < h/2 \leq \varepsilon < 1$.

We define the inner product as $(A, B)_i := \int_{I_i} ABg^2 \,\mathrm{d}x$. The orthogonal basis functions are constructed as follows.

Calculation of $\boldsymbol{\zeta_0}$:

$$\zeta_0 = \beta_0(\beta_0, \beta_0)_i^{-\frac{1}{2}} \tag{95}$$

$$= 1 \cdot \left(\int_{I_i} g^2 \, \mathrm{d}x \right)^{-\frac{1}{2}}. \tag{96}$$

We have

$$\begin{aligned} \frac{1}{2} h\varepsilon \leq \int_{I_i} g^2 \, \mathrm{d}x \leq 2h\varepsilon, \\ \zeta_0 = \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}), \end{aligned}$$

Calculation of $\boldsymbol{\zeta_1}$:

$$\begin{aligned} \bar{\zeta}_1 &= \beta_1 - (\zeta_0, \beta_1)_i \zeta_0 \\ &= (x - x_i) - \int_{I_i} \zeta_0 (x - x_i) g^2 \, \mathrm{d}x \cdot \zeta_0. \end{aligned}$$

The inner product $(\zeta_0, \beta_1)_i$ is given by

$$\begin{aligned} (\zeta_0, \beta_1)_i &= \int_{I_i} \zeta_0 (x - x_i) g^2 \, \mathrm{d}x \\ &= \mathcal{O}(h^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}}) \int_{I_i} g^2 \, \mathrm{d}x \\ &= \mathcal{O}(h^{\frac{3}{2}} \varepsilon^{\frac{1}{2}}) \\ \Rightarrow \bar{\zeta}_1 &= (x - x_i) - \mathcal{O}(h). \end{aligned}$$

Computing the norm

$$\begin{aligned} (\bar{\zeta}_1, \bar{\zeta}_1)_i &= \int_{I_i} \bar{\zeta}_1^2 g^2 \, \mathrm{d}x \\ &= \int_{I_i} (x - x_i - \mathcal{O}(h))^2 g^2 \, \mathrm{d}x \\ \frac{\varepsilon h^3}{24} &\leq (\bar{\zeta}_1, \bar{\zeta}_1)_i \leq \frac{\varepsilon h^3}{6}. \end{aligned}$$

Therefore,

$$\begin{aligned} \zeta_1 &= \bar{\zeta}_1 (\bar{\zeta}_1, \bar{\zeta}_1)_i^{-\frac{1}{2}} \\ &= \mathcal{O}(h) \cdot (h^3 \varepsilon)^{-\frac{1}{2}} \\ &= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}). \end{aligned}$$

Calculation of $\boldsymbol{\zeta_2}$: Following the same procedure,

$$\begin{aligned} \bar{\zeta}_2 &= \beta_2 - (\beta_2, \zeta_0)_i \zeta_0 - (\beta_2, \zeta_1)_i \zeta_1 \\ &= (x - x_i)^2 - \mathcal{O}(h^2). \end{aligned}$$

Computing the norm

$$(\bar{\zeta}_2, \bar{\zeta}_2)_i = \int_{I_i} \bar{\zeta}_2^2 g^2 \,\mathrm{d}x,$$
$$\frac{\varepsilon h^5}{40} \leq (\bar{\zeta}_2, \bar{\zeta}_2)_i \leq \varepsilon h^5.$$

Therefore

$$\begin{aligned} \zeta_2 &= \bar{\zeta}_2(\bar{\zeta}_2, \bar{\zeta}_2)_i^{-\frac{1}{2}} \\ &= \mathcal{O}(h^2) \cdot (h^5 \varepsilon)^{-\frac{1}{2}} \\ &= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}). \end{aligned}$$

General pattern for ζ_j: By induction, we can show that for all $j = 0, 1, \ldots, k$,

$$\zeta_j = \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}). \tag{97}$$

We proceed by mathematical induction to verify the proposition. We assume that

$$\zeta_j = \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}), \quad j = 0, 1, \ldots, m-1.$$

Then,

$$\bar{\zeta}_m = \beta_m - (\beta_m, \zeta_0)_i \zeta_0 - (\beta_m, \zeta_1)_i \zeta_1 - \cdots - (\beta_m, \zeta_{m-1})_i \zeta_{m-1},$$

where the inner product $(\beta_m, \zeta_j)_i,\ 0 \leq j \leq m-1$ is given by

$$\begin{aligned} (\beta_m, \zeta_j)_i &= \int_{I_i} (x - x_i)^m g^2 \zeta_j \,\mathrm{d}x \\ &= \mathcal{O}(h^m) \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}) \int_{I_i} g^2 \,\mathrm{d}x \\ &= \mathcal{O}(h^{m-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}) \mathcal{O}(h\varepsilon) \\ &= \mathcal{O}(h^{m+\frac{1}{2}} \varepsilon^{\frac{1}{2}}), \\ \bar{\zeta}_m &= (x - x_i)^m - \mathcal{O}(h^{m+\frac{1}{2}} \varepsilon^{\frac{1}{2}}) \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}) \\ &= (x - x_i)^m - \mathcal{O}(h^m). \end{aligned}$$

Then, the inner product $(\bar{\zeta}_m, \bar{\zeta}_m)_i$ is

$$(\bar{\zeta}_m, \bar{\zeta}_m)_i = \int_{I_i} \bar{\zeta}_m^2 g^2 \,\mathrm{d}x,$$
$$c_1 h^{2m+1} \varepsilon \leq (\bar{\zeta}_m, \bar{\zeta}_m)_i \leq c_2 h^{2m+1} \varepsilon.$$

Finally, we have

$$\begin{aligned} \zeta_m &= \bar{\zeta}_m (\bar{\zeta}_m, \bar{\zeta}_m)^{-\frac{1}{2}} \\ &= \mathcal{O}(h^m) \mathcal{O}(h^{-m-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}) \\ &= \mathcal{O}(h^{-\frac{1}{2}} \varepsilon^{-\frac{1}{2}}). \end{aligned}$$

B.2. Calculation and order analysis of coefficients $c_0, c_1, c_2, \ldots, c_k$

Let us now analyze the coefficients c_j that appear in the error expression. Given $u = g(x)R(x,t)$ where $R(x,t) \in C^\infty(\mathbb{R} \times \mathbb{R})$, we can express $R(x,t)$ using Taylor expansion around x_i:

$$\begin{aligned} R(x,t) &= R(x_i) + R'(x_i)(x - x_i) + \frac{R''(x_i)}{2}(x - x_i)^2 + \cdots + e_i(x - x_i)^{k+1} \\ &= b_0\zeta_0 + b_1\zeta_1 + b_2\zeta_2 + \cdots + e_i(x - x_i)^{k+1}. \end{aligned}$$

We define the projection $\Pi u := g(x)(a_0\zeta_0 + a_1\zeta_1 + \cdots + a_k\zeta_k)$ such that

$$\int_{I_i} \Pi u \zeta_j g \, \mathrm{d}x = \int_{I_i} u \zeta_j g \, \mathrm{d}x, \quad j = 0, 1, \ldots, k.$$

The coefficients of interest are $c_j := a_j - b_j$, which can be expressed as:

$$c_j = \int_{I_i} e_i g^2(x)(x - x_i)^{k+1}\zeta_j \, \mathrm{d}x, \quad j = 0, \ldots, k-1$$

where $e_i(x) = \frac{u^{(k+1)}(\xi_x)}{(k+1)!}$ is the coefficient of the remainder term in the Taylor expansion. We analyze the order of magnitude for each term in the integrand on the element I_i (with length h and distance to singularity ε):

- The solution is assumed to be regular enough locally, so $|e_i(x)| \le C$.
- The weight function satisfies $g^2(x) = 1 - x^2 \approx 2(1 - x)$, thus $\|g^2\|_{L^\infty(I_i)} \le C\varepsilon$.
- The polynomial term is bounded by the element size: $\|(x - x_i)^{k+1}\|_{L^\infty(I_i)} \le h^{k+1}$.
- The normalized basis functions ζ_j (orthonormal with respect to weight g^2) satisfy $\|\zeta_j\|_{L^\infty(I_i)} \le Ch^{-1/2}\varepsilon^{-1/2}$.

Applying the integral inequality yields, for $0 \le j \le k-1$,

$$\begin{aligned} |c_j| &\le \int_{I_i} |e_i(x)| \cdot |g^2(x)| \cdot |(x - x_i)^{k+1}| \cdot |\zeta_j(x)| \mathrm{d}x \\ &\le C \cdot (C\varepsilon) \cdot h^{k+1} \cdot (Ch^{-1/2}\varepsilon^{-1/2}) \cdot h \\ &= Ch^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}. \end{aligned}$$

To compute c_k, we need the condition $\eta^{u-}_{i+\frac{1}{2}} = 0$:

$$g_{i+\frac{1}{2}} \left(a_0\zeta_{0,i+\frac{1}{2}} + \cdots + a_k\zeta_{k,i+\frac{1}{2}}\right) = g_{i+\frac{1}{2}} \left(b_0\zeta_{0,i+\frac{1}{2}} + \cdots + b_k\zeta_{k,i+\frac{1}{2}} + \mathcal{O}(h^{k+1})\right). \tag{98}$$

If $g_{i+\frac{1}{2}} \neq 0$, this implies

$$c_0\zeta_{0,i+\frac{1}{2}} + \cdots + c_k\zeta_{k,i+\frac{1}{2}} = \mathcal{O}(h^{k+1}). \tag{99}$$

Since $c_j = \mathcal{O}(h^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}),\ j = 0, 1, \ldots, k-1$ and $\zeta_j = \mathcal{O}(h^{-\frac{1}{2}}\varepsilon^{-\frac{1}{2}}), j = 0, \ldots k$,

$$c_k = \mathcal{O}(h^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}).$$

If $g_{i+\frac{1}{2}} = 0$ (on the last cell $i = N$), we need another condition (43) to replace (42), then

$$c_k = \int_{I_i} e_i(x - x_i)^{k+1} g^2 \zeta_k \, \mathrm{d}x = \mathcal{O}(h^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}).$$

Combining these results, we conclude as follows:

$$|c_j| \le Ch^{k+\frac{3}{2}}\varepsilon^{\frac{1}{2}}, \quad \text{for } j = 0, 1, \ldots, k.$$

C. Lemmas

Proof of Lemma 3

Proof. Let $F(x) = \frac{r_h}{\sqrt{1+r_h^2}}$. Then, we have $\|F\|_{L^\infty(I_i)} \le 1$.

We represent the numerical solution in the trial space V_h^{k+1} using the $\frac{1}{g}$-weighted orthogonal basis $\{\psi_j\}_{j=0}^{k+1}$ derived in Appendix A:

$$q_h(x) = \sum_{j=0}^{k+1} a_j \psi_j(x).$$

Substituting the above relation into (18) yields the solution for a_j:

$$\int_{I_i} q_h \gamma \, \mathrm{d}x = \int_{I_i} \frac{r_h}{\sqrt{1+r_h^2}} \gamma \, \mathrm{d}x.$$

Step 1: Decoupling the first $k+1$ modes using singular test functions.
Since $\frac{\psi_j}{g}$ is a polynomial of degree j scaled by $1/g$, it strictly belongs to the test space $\tilde{S}^k$ for $j = 0, \ldots, k$. Testing the scheme with $\gamma_j = \frac{\psi_j}{g}$ and invoking orthogonality $\int_{I_i} \psi_m \psi_j \frac{1}{g} \, \mathrm{d}x = \delta_{mj}$, the system (18) perfectly decouples:

$$a_j = \int_{I_i} F(x) \frac{\psi_j(x)}{g(x)} \, \mathrm{d}x, \quad \text{for } j = 0, \ldots, k.$$

We extract the uniform bound for the coefficients by applying Hölder's inequality:

$$|a_j| \le \|F\|_{L^\infty} \|\psi_j\|_{L^\infty} \int_{I_i} \frac{1}{g} \, \mathrm{d}x.$$

Using the explicit sharp asymptotic bounds from Appendix A, $\|\psi_j\|_{L^\infty} \sim \mathcal{O}(h^{-1/2}\varepsilon^{1/4})$ and $\int_{I_i} \frac{1}{g} \, \mathrm{d}x \sim \mathcal{O}(h\varepsilon^{-1/2})$, we obtain the coefficient scaling

$$|a_j| \le 1 \cdot \mathcal{O}(h^{-1/2}\varepsilon^{1/4}) \cdot \mathcal{O}(h\varepsilon^{-1/2}) = \mathcal{O}(h^{1/2}\varepsilon^{-1/4}).$$

Therefore, the L^∞ contribution of each mode is a pure, dimensionless constant:

$$\|a_j \psi_j\|_{L^\infty(I_i)} \le |a_j| \|\psi_j\|_{L^\infty} \le \mathcal{O}(h^{1/2}\varepsilon^{-1/4}) \cdot \mathcal{O}(h^{-1/2}\varepsilon^{1/4}) \le C.$$

Step 2: Securing the highest degree mode using the constant test function.
The test space $\tilde{S}^k$ has dimension $k+2$. The remaining linearly independent test function is the constant $1 \in \tilde{S}^k$. Testing with $\gamma = 1$ yields

$$\sum_{j=0}^{k+1} a_j \int_{I_i} \psi_j \, \mathrm{d}x = \int_{I_i} F \, \mathrm{d}x \implies a_{k+1} \int_{I_i} \psi_{k+1} \, \mathrm{d}x = \int_{I_i} F \, \mathrm{d}x - \sum_{j=0}^{k} a_j \int_{I_i} \psi_j \, \mathrm{d}x.$$

We bound the right-hand side. Clearly, $|\int_{I_i} F \, \mathrm{d}x| \le \|F\|_{L^\infty} h \le h$. Furthermore, using the coefficient bounds from Step 1,

$$\left| a_j \int_{I_i} \psi_j \, \mathrm{d}x \right| \le |a_j| \|\psi_j\|_{L^\infty} h \le Ch.$$

Since $|\int_{I_i} \psi_{k+1} \, \mathrm{d}x| \sim \|\psi_{k+1}\|_{L^\infty} h \sim \mathcal{O}(h^{1/2}\varepsilon^{1/4})$,

$$|a_{k+1}| \le \frac{\mathcal{O}(h)}{\mathcal{O}(h^{1/2}\varepsilon^{1/4})} = \mathcal{O}(h^{1/2}\varepsilon^{-1/4}).$$

This forces its spatial contribution to be bounded as well:

$$\|a_{k+1}\psi_{k+1}\|_{L^\infty(I_i)} \leq \mathcal{O}(h^{1/2}\varepsilon^{-1/4}) \cdot \mathcal{O}(h^{-1/2}\varepsilon^{1/4}) \leq C.$$

Applying the triangle inequality, q_h is uniformly bounded:

$$\|q_h\|_{L^\infty(I_i)} \leq \sum_{j=0}^{k+1} \|a_j\psi_j\|_{L^\infty(I_i)} \leq \sum_{j=0}^{k+1} C \leq \mathcal{C}_{\text{global}}.$$

□

Proof of Lemma 4

Proof. We analyze the function $Q(x) = g(x)\xi^r(x)$ on an arbitrary physical element $I_i = [x_{i-1/2}, x_{i+1/2}]$ of size h. The space $V_h(I_i)$ is finite-dimensional with dimension $K = k+2$. We introduce an orthonormal basis $\{\phi_{i,j}(x)\}_{j=1}^K$, for $V_h(I_i)$ with respect to the weighted inner product:

$$\langle u, v\rangle_{g,I_i} := \int_{I_i} g(x)u(x)v(x)\,\mathrm{d}x. \tag{100}$$

By definition, these basis functions satisfy $\int_{I_i} g(x)\phi_{i,j}(x)\phi_{i,l}(x)\,\mathrm{d}x = \delta_{jl}$.

Any function $Q \in V_h(I_i)$ can be uniquely expanded in the orthonormal basis as

$$Q(x) = \sum_{j=1}^{K} c_{i,j}\phi_{i,j}(x). \tag{101}$$

First, utilizing the orthonormality of $\{\phi_{i,j}\}_{j=1}^K$ with respect to the weight $g(x)$, the weighted L^2 norm satisfies Parseval's identity

$$\|\sqrt{g}Q\|_{L^2(I_i)} = \left(\int_{I_i} g(x)\left(\sum_{j=1}^{K} c_{i,j}\phi_{i,j}(x)\right)^2 \mathrm{d}x\right)^{1/2} = \left(\sum_{j=1}^{K} c_{i,j}^2\right)^{1/2}. \tag{102}$$

Next, to estimate the L^∞ norm, we fix an arbitrary point $x \in I_i$ and apply the discrete Cauchy–Schwarz inequality to the sum

$$\begin{aligned} |Q(x)| &= \left|\sum_{j=1}^{K} c_{i,j}\phi_{i,j}(x)\right| \\ &\leq \left(\sum_{j=1}^{K} c_{i,j}^2\right)^{1/2} \left(\sum_{j=1}^{K} \phi_{i,j}^2(x)\right)^{1/2} \\ &= \|\sqrt{g}Q\|_{L^2(I_i)} \left(\sum_{j=1}^{K} \phi_{i,j}^2(x)\right)^{1/2}. \end{aligned} \tag{103}$$

Taking the supremum over all $x \in I_i$, we obtain

$$\|Q\|_{L^\infty(I_i)} = \sup_{x\in I_i} |Q(x)| \leq \Lambda_i \|\sqrt{g}Q\|_{L^2(I_i)}, \tag{104}$$

where the constant is defined as

$$\Lambda_i := \sup_{x\in I_i} \left(\sum_{j=1}^{K} \phi_{i,j}^2(x)\right)^{1/2}. \tag{105}$$

We estimate $|\Lambda_i|$ using a unified scaling argument. Let $\varepsilon_i := |x_i - a|$ denote the distance from the **center** of element I_i to the singularity a. Since the mesh is uniform, we have the geometric constraint $\varepsilon_i \geq h/2$, for all $i = 1, \ldots, N$, and $\int_{I_i} g \, \mathrm{d}x \sim h\sqrt{\varepsilon_i}$. Scaling arguments consistent with Appendix A confirm that for all $\varepsilon_i \geq h/2$, the basis functions satisfy the unified scaling law

$$|\phi_{i,j}| \leq Ch^{-1/2}\varepsilon_i^{-1/4}. \tag{106}$$

We show the details in Remark C.3. Consequently, the local stability constant is bounded by

$$\Lambda_i \leq Ch^{-1/2}\varepsilon_i^{-1/4},$$

then,

$$\max_{1\leq i\leq N} \Lambda_i \leq Ch^{-1/2}(h/2)^{-1/4} \leq Ch^{-3/4}. \tag{107}$$

This establishes the global inverse inequality:

$$\|Q\|_{L^\infty(\Omega)} \leq Ch^{-3/4}\|\sqrt{g}Q\|_{L^2(\Omega)}. \tag{108}$$

Bootstrap Closing:
We invoke the global energy estimate from Theorem 2: $\|\sqrt{g}Q\|_{L^2(\Omega)} = \|g^{3/2}\xi^r\|_{L^2(\Omega)} \leq Ch^{k+1}$. Substituting this into the inverse inequality yields

$$\|Q\|_{L^\infty(\Omega)} \leq Ch^{-3/4} \cdot Ch^{k+1} = Ch^{k+1/4}. \tag{109}$$

Finally, since $Q = g\xi^r$, we have $\|g\xi^r\|_{L^\infty(\Omega)} \leq Ch^{k+1/4}$. For any $k \geq 0$, the decay rate $k + 1/4$ is strictly greater than the required $1/8$, thus validating the bootstrap assumption. □

Remark C.3. We will prove the $|\phi_{i,j}| \leq Ch^{-1/2}\varepsilon_i^{-1/4}$. Consider a physical element $I_i = [x_i - h/2, x_i + h/2]$ and let $t = b - x$ denote the distance to the singularity at $x = b$. (Without loss of generality, we restrict our analysis to the neighborhood of the singularity b, as the argument for the other half is symmetric.) The element maps to the interval $[\varepsilon - h/2, \varepsilon + h/2]$ in the t-space, where $\varepsilon = b - x_i \geq h/2$. The weighted inner product is defined as

$$\langle u, v\rangle_g = \int_{\varepsilon-h/2}^{\varepsilon+h/2} u(t)v(t)\sqrt{t}\,\mathrm{d}t. \tag{110}$$

First, we derive the scaling for the constant basis function ψ_0. Let the unnormalized basis be $\beta_0 = 1$. Its squared norm is given by exact integration:

$$\|\beta_0\|_g^2 = \int_{\varepsilon-h/2}^{\varepsilon+h/2} \sqrt{t}\,\mathrm{d}t = \left[\frac{2}{3}t^{3/2}\right]_{\varepsilon-h/2}^{\varepsilon+h/2} = \frac{2}{3}\left((\varepsilon+h/2)^{3/2} - (\varepsilon-h/2)^{3/2}\right). \tag{111}$$

By the mean value theorem, there exists $\xi \in (\varepsilon - h/2, \varepsilon + h/2)$ such that

$$(\varepsilon + h/2)^{3/2} - (\varepsilon - h/2)^{3/2} = h \cdot \frac{3}{2}\xi^{1/2}. \tag{112}$$

We use the notation $A \asymp B$ to denote that there exist positive constants C_1, C_2 independent of h and ε such that $C_1B \leq A \leq C_2B$. Since $\varepsilon \geq h/2$, we have the bounds $\varepsilon/2 \leq \xi \leq 3\varepsilon/2$, which implies $\xi^{1/2} \asymp \varepsilon^{1/2}$. Thus, there exist constants C_1, C_2 independent of h, ε such that

$$C_1h\varepsilon^{1/2} \leq \|\beta_0\|_g^2 \leq C_2h\varepsilon^{1/2}. \tag{113}$$

The normalized basis function $\psi_0 = \beta_0/\|\beta_0\|_g$ satisfies the L^∞ bound

$$C_1h^{-1/2}\varepsilon^{-1/4} \leq \|\psi_0\|_{L^\infty} \leq C_2h^{-1/2}\varepsilon^{-1/4}. \tag{114}$$

Next, we consider the linear basis function ψ_1. Let the unnormalized basis be $\beta_1 = t - \varepsilon$. Its squared norm is:

$$\|\beta_1\|_g^2 = \int_{\varepsilon-h/2}^{\varepsilon+h/2} (t-\varepsilon)^2 \sqrt{t}\,\mathrm{d}t. \tag{115}$$

Applying the change of variable $y = t - \varepsilon$, the integral becomes

$$\|\beta_1\|_g^2 = \int_{-h/2}^{h/2} y^2 \sqrt{\varepsilon + y}\,\mathrm{d}y. \tag{116}$$

For $y \in [-h/2, h/2]$ and $\varepsilon \geq h/2$, the term $\sqrt{\varepsilon+y}$ is bounded by $\sqrt{\varepsilon/2} \leq \sqrt{\varepsilon+y} \leq \sqrt{3\varepsilon/2}$, implying $\sqrt{\varepsilon+y} \asymp \varepsilon^{1/2}$. Factoring this term out of the integral yields

$$C_3 \varepsilon^{1/2} \int_{-h/2}^{h/2} y^2\,\mathrm{d}y \leq \|\beta_1\|_g^2 \leq C_4 \varepsilon^{1/2} \int_{-h/2}^{h/2} y^2\,\mathrm{d}y. \tag{117}$$

Substituting the exact polynomial integral $\int_{-h/2}^{h/2} y^2\,\mathrm{d}y = \frac{h^3}{12}$, we obtain

$$C_5 h^3 \varepsilon^{1/2} \leq \|\beta_1\|_g^2 \leq C_6 h^3 \varepsilon^{1/2}. \tag{118}$$

The normalized basis function is $\psi_1(t) = (t-\varepsilon)/\|\beta_1\|_g$. The maximum absolute value of the numerator on the interval is $h/2$. Therefore, the L^∞ norm satisfies

$$\|\psi_1\|_{L^\infty} = \frac{h/2}{\|\beta_1\|_g} \implies C_7 h^{-1/2} \varepsilon^{-1/4} \leq \|\psi_1\|_{L^\infty} \leq C_8 h^{-1/2} \varepsilon^{-1/4}. \tag{119}$$

In summary, the first two orthonormal basis functions admit the uniform upper bound:

$$\|\psi_k\|_{L^\infty} \leq C h^{-1/2} \varepsilon^{-1/4}, \quad k = 0, 1. \tag{120}$$

For higher-order basis functions, the same conclusion is verified using Python symbolic computation.

We now provide a rigorous L^∞ error estimate for the Gauss–Radau projection $\eta^q = q - \Pi_3 q$.

Lemma C.7. *Let η^q be the projection error defined as*

$$\int_{I_i} \eta^q \gamma\,\mathrm{d}x = 0, \eta^{q+}_{i-\frac{1}{2}} = 0, \quad \forall \gamma \in \hat{S}^k,$$

where

$$\hat{S}^k := \frac{1}{g(x)} \operatorname{span}\{1, x - x_i, \ldots, (x-x_i)^k\},$$

then, we have L^∞ estimate

$$\|\eta^q\|_{L^\infty} \leq C h^{k+2}.$$

Recall that the exact flux $q(x) = u_x/\sqrt{1+u_x^2}$ is smooth up to the boundary, admitting a regular Taylor expansion such that $q \in H^{k+2}(I_i)$.

Proof. Let $\{\psi_j\}_{j=0}^{k+1}$ be the weighted orthogonal polynomial basis functions on I_i with respect to the weight $1/g(x)$, satisfying $\int_{I_i} \psi_m \psi_j \frac{1}{g}\,\mathrm{d}x = \delta_{mj}$. According to Lemma 1, these basis functions satisfy the uniform bound

$$\|\psi_j\|_{L^\infty(I_i)} \leq C h^{-1/2} \varepsilon^{1/4}.$$

We expand the smooth function $q(x)$ up to degree $k+1$:

$$q(x) = \sum_{j=0}^{k+1} b_j \psi_j(x) + E_{k+2}(x),$$

where the truncation error satisfies $\|E_{k+2}\|_{L^\infty(I_i)} \le Ch^{k+2}$. Let the numerical projection be $\Pi_3 q = \sum_{j=0}^{k+1} a_j \psi_j(x) \in V_h^{k+1}$. The projection error is then expressed as

$$\eta^q(x) = \sum_{j=0}^{k+1} C_j \psi_j(x) + E_{k+2}(x),$$

where $C_j = b_j - a_j$.

By definition, the $(k+1)$-dimensional test space $\hat{S}^k := \frac{1}{g(x)} \operatorname{span}\{1, x - x_i, \dots, (x - x_i)^k\}$ is exactly spanned by $\{\psi_j/g\}_{j=0}^k$. Taking the test function $\gamma = \psi_j/g \in \hat{S}^k$ for $0 \le j \le k$, the internal orthogonality condition $\int_{I_i} \eta^q \gamma \, \mathrm{d}x = 0$ yields

$$\int_{I_i} \left(\sum_{m=0}^{k+1} C_m \psi_m + E_{k+2} \right) \frac{\psi_j}{g} \, \mathrm{d}x = 0.$$

Due to the orthogonality of ψ_j, this directly simplifies to

$$C_j = -\int_{I_i} E_{k+2}(x) \frac{\psi_j(x)}{g(x)} \, \mathrm{d}x, \quad \text{for } 0 \le j \le k.$$

Using the basis bound and the weight integral property $\int_{I_i} \frac{1}{g} \, \mathrm{d}x \le Ch\varepsilon^{-1/2}$, we bound the first $k+1$ coefficients:

$$|C_j| \le \|E_{k+2}\|_{L^\infty} \|\psi_j\|_{L^\infty} \int_{I_i} \frac{1}{g} \, \mathrm{d}x \le (Ch^{k+2})(Ch^{-1/2}\varepsilon^{1/4})(Ch\epsilon^{-1/2}) \le Ch^{k+2.5}\epsilon^{-1/4}.$$

To determine the final coefficient C_{k+1}, we enforce the boundary condition $\eta^q(x_{i-1/2}) = 0$:

$$C_{k+1}\psi_{k+1}(x_{i-1/2}) = -E_{k+2}(x_{i-1/2}) - \sum_{j=0}^{k} C_j \psi_j(x_{i-1/2}).$$

Given the asymptotic behavior $\psi_{k+1}(x_{i-1/2}) \sim Ch^{-1/2}\varepsilon^{1/4}$, solving for C_{k+1} yields the same asymptotic bound:

$$|C_{k+1}| \le Ch^{k+2.5}\varepsilon^{-1/4}.$$

Finally, substituting the coefficient bounds back into the expression for η^q, the geometric singularity factors $\varepsilon^{-1/4}$ and $\varepsilon^{1/4}$ perfectly cancel out:

$$\begin{aligned} \|\eta^q\|_{L^\infty(I_i)} &\le \sum_{j=0}^{k+1} |C_j| \|\psi_j\|_{L^\infty} + \|E_{k+2}\|_{L^\infty} \\ &\le (k+2) \left(Ch^{k+2.5}\varepsilon^{-1/4} \right) \left(Ch^{-1/2}\varepsilon^{1/4} \right) + Ch^{k+2} \le Ch^{k+2}. \end{aligned}$$

Since this bound holds independently of ε, we achieve the global optimal estimate $\|\eta^q\|_{L^\infty} \le Ch^{k+2}$. □

Lemma C.8.

$$\left| \int_a^b \eta^q \xi^r \, \mathrm{d}x \right| \le \frac{c_1}{4\alpha} \int_a^b \frac{(\xi^r)^2}{(1 + r_\theta^2)^{3/2}} \, \mathrm{d}x + Ch^{2k+2}.$$

Proof. To rigorously estimate the residual term $I_q := \int_a^b \eta^q \xi^r \,\mathrm{d}x$ uniformly over the domain without artificially splitting the boundary and interior cells, we proceed as follows. By the scaling argument and the inverse inequality established in Lemma 4, the local L^∞ norm and L^2 norm satisfy a uniform relation on any element I_i. Recalling that the energy norm weight satisfies $1 + r_\theta^2 \sim g^{-2}$, the pointwise value of $\xi^r \in \tilde{S}^k$ can be uniformly bounded by the local energy norm

$$\|\xi^r\|_{E,I_i} := \left(\int_{I_i} \frac{(\xi^r)^2}{(1 + r_\theta^2)^{3/2}} \,\mathrm{d}x \right)^{1/2},$$

$$|\xi^r(x)| \le \frac{1}{g(x)} \|g\xi^r\|_{L^\infty(I_i)} \le \frac{C}{g(x)} h^{-3/4} \|\xi^r\|_{E,I_i}.$$

Given that the projection space is V_h^{k+1}, the projection error satisfies $\|\eta^q\|_{L^\infty} \le Ch^{k+2}$. Integrating the absolute value of the product $\eta^q \xi^r$ over a single element I_i yields

$$\left| \int_{I_i} \eta^q \xi^r \,\mathrm{d}x \right| \le \|\eta^q\|_{L^\infty(I_i)} \int_{I_i} |\xi^r(x)| \,\mathrm{d}x \le Ch^{k+5/4} \|\xi^r\|_{E,I_i} \int_{I_i} \frac{1}{g(x)} \,\mathrm{d}x.$$

We introduce a local geometric weight sequence $w_i := \int_{I_i} g(x)^{-1} \,\mathrm{d}x$. Summing the integrals over all elements and applying the discrete Cauchy–Schwarz inequality, we obtain a global bound

$$|I_q| \le \sum_{i=1}^{N} \left| \int_{I_i} \eta^q \xi^r \,\mathrm{d}x \right| \le Ch^{k+5/4} \sum_{i=1}^{N} (w_i \|\xi^r\|_{E,I_i}) \le Ch^{k+5/4} \left(\sum_{i=1}^{N} w_i^2 \right)^{1/2} \|\xi^r\|_E.$$

To bound the sum of the squared geometric weights, we analyze w_i. Near the boundary $x = a$, the singular weight behaves as $g(x) \sim \sqrt{x - a}$. Exact integration over $I_i = [x_{i-1/2}, x_{i+1/2}]$ gives

$$w_i = \int_{x_{i-1/2}}^{x_{i+1/2}} \frac{1}{\sqrt{x - a}} \,\mathrm{d}x = \frac{2h}{\sqrt{x_{i+1/2} - a} + \sqrt{x_{i-1/2} - a}}.$$

For the first element,where $i = 1$, we have $w_1 = 2\sqrt{h}$, which implies $w_1^2 = 4h$. For the remaining elements, where $i \ge 2$, we can strictly bound the squared weight by

$$w_i^2 \le \frac{4h^2}{x_{i-1/2} - a} = \frac{4h^2}{(i-1)h} = \frac{4h}{i-1}.$$

Summing the sequence over the entire mesh $N \sim h^{-1}$ yields

$$\sum_{i=1}^{N} w_i^2 \le 4h + 4h \sum_{i=1}^{N-1} \frac{1}{i} \le Ch(1 + |\ln h|) \le Ch|\ln h|.$$

Substituting this purely geometric bound back into the global estimate for I_q, we have

$$|I_q| \le Ch^{k+5/4} \left(Ch|\ln h| \right)^{1/2} \|\xi^r\|_E = Ch^{k+7/4} |\ln h|^{1/2} \|\xi^r\|_E.$$

Finally, we apply Young's inequality with a free parameter α to absorb the energy norm term:

$$\begin{aligned}
|I_q| &\le \frac{c_1}{4\alpha} \|\xi^r\|_E^2 + Ch^{2k+3.5} |\ln h| \\
&= \frac{c_1}{4\alpha} \int_a^b \frac{(\xi^r)^2}{(1 + r_\theta^2)^{3/2}} \,\mathrm{d}x + Ch^{2k+3.5} |\ln h| \\
&\le \frac{c_1}{4\alpha} \int_a^b \frac{(\xi^r)^2}{(1 + r_\theta^2)^{3/2}} \,\mathrm{d}x + Ch^{2k+2}.
\end{aligned}$$

This confirms that the residual term I_q remains of higher order, modulo a mild logarithmic factor, and can be strictly absorbed into the left-hand side of the stability estimate without requiring a piecewise domain decomposition. $\square$

In this section, we establish the fundamental control inequalities linking the numerical flux error and the numerical gradient error. Let the approximation errors be decomposed as $e^q = q - q_h = \eta^q - \xi^q$ and $e^r = r - r_h = \eta^r - \xi^r$. We define the local energy norm as $\|\xi^r\|_{E,I_i}^2 := \int_{I_i} w_\theta (\xi^r)^2 \, \mathrm{d}x$, where the nonlinear weight function is given by $w_\theta(x) = (1 + r_\theta^2)^{-3/2}$.

It has been established that the error estimate $\|\eta^q\|_{L^\infty} \le Ch^{k+2}$. However, the exact gradient blows up at the boundaries as $r \sim 1/g(x)$, where $g(x) = \sqrt{(x-a)(b-x)}$. Thus, its interpolation error strictly satisfies the weighted estimate $\|g\eta^r\|_{L^\infty} \le Ch^{k+1}$.

Lemma C.9. *Let $\xi^q \in V_h^{k+1}$ and $\xi^r \in \tilde{S}^k$. Assume that the projection error*

$$\eta^q := q - \Pi_3 q$$

is defined locally by

$$\int_{I_i} \eta^q \zeta \, \mathrm{d}x = 0, \qquad \forall \zeta \in \widehat{S}^k(I_i), \quad \eta^{q,+}_{i-\frac{1}{2}} = 0,$$

where

$$\widehat{S}^k(I_i) := \frac{1}{g(x)} P^k(I_i) = \operatorname{span}\left\{ \frac{1}{g(x)}, \frac{x - x_i}{g(x)}, \dots, \frac{(x - x_i)^k}{g(x)} \right\}.$$

For any local element I_i, there exists a constant $C > 0$ independent of the mesh size h such that

$$\|\xi^q\|_{L^2(I_i)} \le C \|\xi^r\|_{E,I_i} + \mathcal{O}(h^{k+1.5}). \tag{121}$$

Proof. The local error equation for q is

$$\int_{I_i} \xi^q \gamma \, \mathrm{d}x = \int_{I_i} \eta^q \gamma \, \mathrm{d}x + \int_{I_i} w_\theta \xi^r \gamma \, \mathrm{d}x - \int_{I_i} w_\theta \eta^r \gamma \, \mathrm{d}x, \quad \forall \gamma \in \tilde{S}^k. \tag{122}$$

According to the definition of the space $\tilde{S}^k$, any test function $\gamma \in \tilde{S}^k$ can be uniquely decomposed as

$$\gamma(x) = c_0 \cdot 1 + \frac{\psi(x)}{g(x)}, \qquad c_0 \in \mathbb{R}, \quad \psi \in P^k(I_i).$$

To match the finite element scaling, we introduce the composite measure norm on the local space $\tilde{S}^k$:

$$\|\gamma\|_{*,I_i} := |c_0| \, h^{\frac{1}{2}} + \|\psi\|_{L^2(I_i)}. \tag{123}$$

Let $F : \tilde{S}^k \to \mathbb{R}$ denote the residual functional defined by the right-hand side of (122):

$$F(\gamma) := \int_{I_i} \eta^q \gamma \, \mathrm{d}x + \int_{I_i} w_\theta \xi^r \gamma \, \mathrm{d}x - \int_{I_i} w_\theta \eta^r \gamma \, \mathrm{d}x.$$

By linearity, for

$$\gamma = \gamma_0 + \gamma_1, \qquad \gamma_0 := c_0, \qquad \gamma_1 := \frac{\psi}{g},$$

we have

$$\int_{I_i} \xi^q \gamma \, \mathrm{d}x = F(\gamma_0) + F(\gamma_1).$$

We now bound these two modes separately.

I. Control of the constant part $\boldsymbol{\gamma_0 = c_0}$.

For the truncation error term, using $w_\theta \sim g^3$ and

$$\|g\eta^r\|_{L^\infty} \le Ch^{k+1},$$

we have

$$\begin{aligned}\left|\int_{I_i} w_\theta \eta^r c_0 \, \mathrm{d}x\right| &\le |c_0| \int_{I_i} \left|\frac{w_\theta}{g}\right| |g\eta^r| \, \mathrm{d}x \\ &\le |c_0| \, \|g\eta^r\|_{L^\infty} \int_{I_i} Cg^2 \, \mathrm{d}x \\ &\le |c_0|(Ch^{k+1})(Ch) \le Ch^{k+1.5} \left(|c_0|h^{\frac{1}{2}}\right). \end{aligned} \tag{124}$$

For the energy term, by Cauchy–Schwarz inequality,

$$\left|\int_{I_i} w_\theta \xi^r c_0 \, \mathrm{d}x\right| \le \|\xi^r\|_{E,I_i} \|w_\theta^{\frac{1}{2}}\|_{L^\infty(I_i)} \|c_0\|_{L^2(I_i)} \le \|\xi^r\|_{E,I_i} \left(|c_0|h^{\frac{1}{2}}\right).$$

For the projection error, since the constant mode $1 \notin \widehat{S}^k(I_i)$, we use the L^∞ estimate from Lemma C.7:

$$\left|\int_{I_i} \eta^q c_0 \, \mathrm{d}x\right| \le \|\eta^q\|_{L^\infty(I_i)} |c_0| h \le Ch^{k+2}|c_0|h = Ch^{k+2.5}\left(|c_0|h^{\frac{1}{2}}\right) \le Ch^{k+1.5}\left(|c_0|h^{\frac{1}{2}}\right).$$

Collecting these bounds, the total action on the constant mode is controlled by

$$|F(\gamma_0)| \le \left(\|\xi^r\|_{E,I_i} + Ch^{k+1.5}\right)\left(|c_0|h^{\frac{1}{2}}\right). \tag{125}$$

II. Control of the singular part $\gamma_1 = \psi/g$.
Since $\psi \in P^k(I_i)$, we have

$$\gamma_1 = \frac{\psi}{g} \in \widehat{S}^k(I_i).$$

Therefore, by the definition of the projection $\Pi_3 q$,

$$\int_{I_i} \eta^q \gamma_1 \, \mathrm{d}x = \int_{I_i} \eta^q \frac{\psi}{g} \, \mathrm{d}x = 0. \tag{126}$$

For the truncation error term, we group the weights as

$$\begin{aligned}\left|\int_{I_i} w_\theta \eta^r \frac{\psi}{g} \, \mathrm{d}x\right| &\le \int_{I_i} \left|\frac{w_\theta}{g^2}\right| |g\eta^r| \, |\psi| \, \mathrm{d}x \\ &\le C \, \|g\eta^r\|_{L^\infty(I_i)} \int_{I_i} g(x)|\psi| \, \mathrm{d}x \\ &\le Ch^{k+1} \left(\int_{I_i} 1 \, \mathrm{d}x\right)^{1/2} \|\psi\|_{L^2(I_i)} \\ &\le Ch^{k+1.5} \|\psi\|_{L^2(I_i)}. \end{aligned} \tag{127}$$

For the energy term, the Cauchy–Schwarz inequality gives

$$\left|\int_{I_i} w_\theta \xi^r \frac{\psi}{g} \, \mathrm{d}x\right| \le \|\xi^r\|_{E,I_i} \left\|\frac{w_\theta^{1/2}}{g}\psi\right\|_{L^2(I_i)}.$$

Since $w_\theta \sim g^3$, the ratio

$$\frac{w_\theta}{g^2} \sim g$$

is uniformly bounded. Hence,

$$\left|\int_{I_i} w_\theta \xi^r \frac{\psi}{g} \, \mathrm{d}x\right| \le C\|\xi^r\|_{E,I_i} \|\psi\|_{L^2(I_i)}.$$

Combining this with (126), we obtain

$$|F(\gamma_1)| \le \left(C \left\|\xi^r\right\|_{E,I_i} + Ch^{k+1.5}\right) \|\psi\|_{L^2(I_i)}. \tag{128}$$

III. Synthesis and inf–sup condition.

Summing (125) and (128), and using

$$|F(\gamma)| \le |F(\gamma_0)| + |F(\gamma_1)|,$$

we get

$$|F(\gamma)| \le \left(C \left\|\xi^r\right\|_{E,I_i} + Ch^{k+1.5}\right) \left(|c_0|h^{\frac{1}{2}} + \|\psi\|_{L^2(I_i)}\right).$$

By the definition of the composite norm,

$$\|\gamma\|_{*,I_i} = |c_0|h^{\frac{1}{2}} + \|\psi\|_{L^2(I_i)},$$

we obtain

$$|F(\gamma)| \le \left(C \left\|\xi^r\right\|_{E,I_i} + Ch^{k+1.5}\right) \|\gamma\|_{*,I_i}. \tag{129}$$

Dividing by $\|\gamma\|_{*,I_i}$ and invoking the uniform local L^2 stability, namely, the local inf–sup condition,

$$\|\xi^q\|_{L^2(I_i)} \le C_{\text{stab}} \sup_{\gamma\in\tilde{S}^k} \frac{|F(\gamma)|}{\|\gamma\|_{*,I_i}},$$

we obtain

$$\|\xi^q\|_{L^2(I_i)} \le C \left\|\xi^r\right\|_{E,I_i} + \mathcal{O}(h^{k+1.5}).$$

This completes the proof. □

Remark C.4 (On the local L^2 stability: an inf–sup perspective)**.** Let the local bilinear form be

$$B_i(\xi^q, \gamma) := \int_{I_i} \xi^q \gamma \, \mathrm{d}x.$$

We assert that there exists a universal constant $\beta > 0$, independent of h, such that

$$\inf_{\xi^q\in V_h^{k+1}\setminus\{0\}} \sup_{\gamma\in\tilde{S}^k\setminus\{0\}} \frac{B_i(\xi^q,\gamma)}{\|\xi^q\|_{L^2(I_i)}\|\gamma\|_{*,I_i}} \ge \beta > 0, \tag{130}$$

where

$$\|\gamma\|_{*,I_i} = |c_0|h^{1/2} + \|\psi\|_{L^2(I_i)}$$

for the unique decomposition

$$\gamma(x) = c_0 + \frac{\psi(x)}{g(x)}, \qquad c_0 \in \mathbb{R}, \quad \psi \in P^k(I_i).$$

Proof sketch. Observe that

$$\dim(V_h^{k+1}) = \dim(\tilde{S}^k) = k+2.$$

The trial space V_h^{k+1} is spanned by the $1/g$-weighted orthogonal basis $\{\psi_j\}_{j=0}^{k+1}$ satisfying

$$\int_{I_i} \psi_m \psi_j \frac{1}{g} \, \mathrm{d}x = \delta_{mj}.$$

Choose the test basis

$$\gamma_j = \frac{\psi_j}{g} \in \tilde{S}^k, \qquad 0 \le j \le k, \quad \gamma_{k+1} = 1 \in \tilde{S}^k.$$

With respect to these bases, the Gram matrix $\mathbf{G}$ associated with $B_i(\cdot,\cdot)$ has the block lower-triangular form

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}_{(k+1)\times(k+1)} & \mathbf{0} \\ \mathbf{v}^T & \displaystyle\int_{I_i} \psi_{k+1}\,\mathrm{d}x \end{pmatrix},$$

where

$$\mathbf{v}^T = \left(\int_{I_i} \psi_0\,\mathrm{d}x, \ldots, \int_{I_i} \psi_k\,\mathrm{d}x \right).$$

After the standard scaling to the reference element, the h-factors in

$$B_i(\xi^q,\gamma), \qquad \|\xi^q\|_{L^2(I_i)}, \qquad \|\gamma\|_{*,I_i}$$

cancel exactly. Since the scaled spaces are finite-dimensional and the above Gram matrix is non-singular on the reference element, the corresponding reference inf–sup constant is strictly positive. Hence, the physical inf–sup constant is bounded below uniformly in h, which proves (130).

Lemma C.10. *Under the conditions of Lemma C.9, the following estimates hold strictly:*

(1) *Estimate for the cross-term volume integral:*

$$\left| \int_{I_i} \eta^r \xi^q\,\mathrm{d}x \right| \le \frac{C^2}{4\epsilon} h^{2k+2} + \epsilon \|\xi^r\|_{E,I_i}^2 + Ch^{2k+2}, \quad \forall \epsilon > 0. \tag{131}$$

(2) *Estimate for the boundary trace term:*

$$\frac{h}{4\alpha}(\xi^{q-}_{N+1/2})^2 \le \frac{C}{4\alpha}\|\xi^r\|_{E,I_N}^2 + \frac{C}{4\alpha}h^{2k+2}. \tag{132}$$

Remark C.5. The ϵ in (131) is a constant, which is different from the ε defined in Lemma 1.

Proof. **1. Rigorous Proof of the Cross-Term Integral:**
Since the exact gradient blows up at the boundary ($r \sim 1/g$), its standard interpolation error η^r diverges in the L^2 space. Therefore, the Cauchy–Schwarz inequality must *not* be applied directly to $\int_{I_i} \eta^r \xi^q\,\mathrm{d}x$. Instead, we utilize the L^1–L^∞ Hölder's inequality coupled with the local inverse estimate $\|\xi^q\|_{L^\infty} \le Ch^{-1/2}\|\xi^q\|_{L^2}$ for the finite element polynomial $\xi^q \in V_h^{k+1}$:

$$\left| \int_{I_i} \eta^r \xi^q\,\mathrm{d}x \right| \le \|\eta^r\|_{L^1(I_i)} \|\xi^q\|_{L^\infty(I_i)} \le \|\eta^r\|_{L^1(I_i)} \left(Ch^{-1/2}\|\xi^q\|_{L^2(I_i)} \right). \tag{133}$$

To evaluate $\|\eta^r\|_{L^1}$, we use the weighted bound $\|g\eta^r\|_{L^\infty} \le Ch^{k+1}$ and the convergent singular integral $\int_{I_i} g^{-1}\,\mathrm{d}x \le Ch^{1/2}$:

$$\|\eta^r\|_{L^1(I_i)} = \int_{I_i} |g\eta^r| \frac{1}{g}\,\mathrm{d}x \le \|g\eta^r\|_{L^\infty} \int_{I_i} \frac{1}{g}\,\mathrm{d}x \le (Ch^{k+1})(Ch^{1/2}) = Ch^{k+1.5}. \tag{134}$$

Substituting this back, and employing the result $\|\xi^q\|_{L^2} \le C\|\xi^r\|_E + \mathcal{O}(h^{k+1.5})$ from Lemma C.9,

$$\begin{aligned} \left| \int_{I_i} \eta^r \xi^q\,\mathrm{d}x \right| &\le (Ch^{k+1.5})(Ch^{-1/2}\|\xi^q\|_{L^2(I_i)}) = Ch^{k+1}\|\xi^q\|_{L^2(I_i)} \\ &\le Ch^{k+1}\left(C\|\xi^r\|_{E,I_i} + \mathcal{O}(h^{k+1.5}) \right) \le Ch^{k+1}\|\xi^r\|_{E,I_i} + \mathcal{O}(h^{2k+2.5}). \end{aligned} \tag{135}$$

Applying Young's inequality ($ab \le \frac{a^2}{4\epsilon} + \epsilon b^2$) to the principal term yields

$$\left| \int_{I_i} \eta^r \xi^q\,\mathrm{d}x \right| \le \frac{C^2}{4\epsilon} h^{2k+2} + \epsilon\|\xi^r\|_{E,I_i}^2 + \mathcal{O}(h^{2k+2}). \tag{136}$$

The term $\epsilon \left\| \xi^r \right\|_{E,I_i}^2$ will ultimately be absorbed by the coercivity of the global energy equation, perfectly preserving the optimal order $\mathcal{O}(h^{2k+2})$ while rigorously circumventing the L^2-divergence crisis of η^r.

2. Proof of the Boundary Trace Term:
For the boundary element I_N, since $\xi^q \in P^{k+1}(I_N)$ is a standard polynomial, the local trace inequality $v_{\text{bdry}}^2 \leq Ch^{-1} \left\| v \right\|_{L^2}^2$ is applicable. Thus,

$$(\xi_{N+1/2}^{q-})^2 \leq Ch^{-1} \left\| \xi^q \right\|_{L^2(I_N)}^2 . \tag{137}$$

Squaring the estimate (121) (*i.e.*, $\left\| \xi^q \right\|_{L^2}^2 \leq C \left\| \xi^r \right\|_{E,I_N}^2 + \mathcal{O}(h^{2k+3})$) and multiplying by the measure weight $\frac{h}{4\alpha}$ accompanying the LDG penalty term,

$$\begin{aligned} \frac{h}{4\alpha}(\xi_{N+1/2}^{q-})^2 &\leq \frac{h}{4\alpha}\left(Ch^{-1} \left\| \xi^r \right\|_{E,I_N}^2 + Ch^{2k+2}\right) \\ &= \frac{C}{4\alpha} \left\| \xi^r \right\|_{E,I_N}^2 + \frac{C}{4\alpha} h^{2k+3} \\ &\leq \frac{C}{4\alpha} \left\| \xi^r \right\|_{E,I_N}^2 + \frac{C}{4\alpha} h^{2k+2}. \end{aligned} \tag{138}$$

$\square$