ESAIM: M2AN 60 (2026) 1769–1803
https://doi.org/10.1051/m2an/2026042

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

HYBRID FEM/IPDG SEMI-IMPLICIT SCHEMES FOR TIME DOMAIN ELECTROMAGNETIC WAVE PROPAGATION IN NON-CYLINDRICAL COAXIAL CABLES

Akram Beni Hamad[1,2], Sébastien Imperiale[3,*] and Patrick Joly[1]

Abstract. In this work, we develop an efficient numerical method for solving 3D Maxwell's equations in non-cylindrical coaxial cables. The main challenge arises from the elongated geometry of the computational domain, which induces strong anisotropy between the longitudinal direction (along the cable) and the transverse directions (within the cross-sections). This leads to the use of highly anisotropic meshes, where the longitudinal mesh size is much larger than the transverse one. Our objective is to design a numerical scheme that is explicit in the longitudinal direction, with a CFL stability condition depending only on the longitudinal mesh size. In a previous work, we achieved this for cylindrical cables by employing prismatic edge elements, 1D quadrature for longitudinal mass lumping, and a hybrid explicit/implicit time discretization. The present paper extends this approach to non-cylindrical cables, addressing several new difficulties with the following key ingredients: (1) representing the cable as a deformation of a reference cylindrical cable and employing mapping techniques between the physical and reference domains; (2) using an anisotropic space discretization that combines an interior penalty discontinuous Galerkin (IPDG) method in the transverse directions with a conforming finite element method in the longitudinal direction; (3) utilizing prismatic edge elements on a prismatic mesh of the reference cable; and (4) adapting the construction of the hybrid explicit–implicit time discretization to the new structure of the semi-discrete problem. From a theoretical perspective, the main difficulty lies in the stability analysis, which requires extending and adapting standard techniques for discontinuous Galerkin methods in space and energy methods in time.

Mathematics Subject Classification. 35Q61, 78M10, 65M60.

Received November 07, 2025. Revised March 18, 2026. Accepted April 28, 2026.

1. Introduction and motivation

The propagation of electromagnetic waves is governed by Maxwell's equations. Solving these equations in three dimensions is a challenging task due to complex geometries, possible structural defects, and the need for fine meshes, especially in the case of thin coaxial cables. This work has been motivated by a project from the ANR (the French Research Agency). This project, called SODDA, concerned the non-destructive control of

Keywords and phrases. Non-cylindrical coaxial cables, Maxwell's equations, hybrid numerical method, edge elements, discontinuous Galerkin method, numerical simulation.

[1] POEMS (UMR CNRS-INRIA-ENSTA Paris) Institut Polytechnique de Paris, Paris, France.

[2] LAMMDA-ESST Hammam Sousse, Université de Sousse, Sousse, Tunisie.

[3] Inria Saclay, Ecole Polytechnique, CNRS, Institut Polytechnique de Paris, Paris, France.

*Corresponding author: sebastien.imperiale@inria.fr

networks of coaxial cables, like the one of SNCF (the national train company), using electromagnetic waves. For the direct problem, one needs to simulate the propagation of such waves and in particular to take into account defects in the cable (whose detection is the object of the inverse problem), consisting of local damages or local crushing that result in local changes in the geometry and the material properties of the cable (breaking its original cylindrical structure).

A coaxial cable is an electrical cable—its transverse dimensions are much smaller than its longitudinal dimension—that consists of a dielectric material surrounding a metallic inner wire, which enables wave propagation at the frequencies of interest, all encased in an outer sheath. The inner and outer structures are made of perfect conductors, confining the electromagnetic field within the dielectric region. The dielectric region can be modeled as a longitudinal succession of 2D transverse cross-sections with a central hole. Mathematically, the propagation domain Ω for the electric field is the union of these cross-sections along the longitudinal axis x_3. In industrial settings, cables are thin, and their transverse dimension is proportional to a small parameter $\eta \ll 1$. A typical simulation problem involves three characteristic lengths: the diameter of the cable η, the wavelength λ along the cable, and the length of the cable itself. Due to the elongated nature of the propagation domain Ω, it is natural to use anisotropic meshes with stepsize h in the longitudinal direction x_3 and stepsize h in the transverse directions. These parameters are constrained by the problem's requirements: $h \ll \eta$ to accurately represent the internal structure of the cable and the electromagnetic field variations within each cross-section, and $\mathrm{h} \sim \lambda$ to capture the solution's variations in the longitudinal direction. Since $\eta \ll \lambda$, we have $h \ll \mathrm{h}$, leading to a large number of degrees of freedom in the transverse direction compared to the longitudinal one.

Time discretization further complicates the problem. When using a time step Δt, one typically faces two options: either a fully explicit time discretization, where stability (CFL condition) requires that Δt be constrained by the small space step h, resulting in very small time steps, or a fully implicit time discretization, where there are no stability constraints on Δt, but large linear systems must be solved at each iteration. Both approaches are computationally prohibitive: either the number of time steps is too large, or each iteration is too expensive. Therefore, the goal is to find a compromise where Δt is constrained only by the longitudinal space step h, and the cost of each iteration remains manageable. This can be achieved through a hybrid time discretization, implicit in the transverse directions and explicit in the longitudinal direction.

In [1], we have constructed—in cylindrical coaxial cables—an efficient numerical method for time domain electromagnetic wave propagation to solve 3D Maxwell's equations in coaxial cables. Our strategy is based on the use of edge elements on prisms combined with numerical quadrature. In order to address more realistic situations, we aim to extend this approach to non-cylindrical cables. Therefore, this paper presents a generalization of the method proposed in [1].

In this work, we have not employed the classical interior penalty discontinuous Galerkin (IPDG) method (see, *e.g.*, [2] or [18]), for the entire problem domain, despite its widespread use for solving wave equations and Maxwell's equations, as seen in [12–14, 27]. The IPDG method, which is particularly effective for handling complex boundary conditions and discontinuities, computes fluxes over the entire domain. While this provides flexibility and accuracy in capturing the solution, it significantly increases the computational cost, especially in the case of thin cables like the ones considered here.

Our approach is based on two main components. First, we use a section-by-section matching technique between the non-cylindrical cable and a cylindrical reference cable. Second, we apply an anisotropic Piola transformation to define new unknowns in the reference cable. The new problem formulated in the reference domain introduces terms that depend on the cable's geometry, which vanish in the cylindrical case. These terms are incompatible with the finite element approach used in [1]. To address this issue, we propose a hybrid finite element/discontinuous Galerkin (DG) method, where the DG part handles the new terms. The resulting method extends the cylindrical case method and maintains similar stability properties.

The outline of this paper is as follows:

– Section 2 introduces the 3D Maxwell problem in non-cylindrical cables, discusses the limitations of the classical 3D Piola transform for this setting, and presents the anisotropic Piola transform as a suitable alternative. The weak formulation in the reference domain is then derived.

– Section 3 applies the IPDG method to the 3D model, but restricts its use to the transverse direction, resulting in an interior penalty weak formulation. This formulation is then mapped to the reference domain using the anisotropic Piola transform.
– Section 4 details the spatial discretization of the problem using a hybrid finite element/IPDG approach. The use of 1D numerical quadrature in the longitudinal direction is emphasized, as it is crucial for achieving an explicit scheme in that direction.
– Section 5 presents the construction of a hybrid explicit–implicit time discretization. This relies on a specific splitting of the stiffness matrix, based on its decomposition into seven submatrices that distinguish between couplings of transverse and longitudinal fields, as well as between different cross-sections. The stability analysis is then carried out using energy techniques, leading to the main theoretical results (Theorem 5.1 and Corollary 5.1).

2. The Maxwell's equations in cables with varying cross-section

2.1. Setting of the problem

We consider a cable for which the cross-section depends on x_3 (see Fig. 1) and we assume, for simplicity of exposure, that it is infinite so that

$$\Omega := \bigcup_{x_3 \in \mathbb{R}} S(x_3), \tag{1}$$

where the cross-section $S(x_3)$ is the difference between two connected open sets of $\mathbb{R}^2$:

$$S(x_3) = \mathcal{O}_e(x_3) \backslash \overline{\mathcal{O}_i(x_3)}, \quad \mathcal{O}_e(x_3), \mathcal{O}_i(x_3) \subset \mathbb{R}^2 \text{ connected}, \quad \overline{\mathcal{O}_i(x_3)} \subset \mathcal{O}_e(x_3), \tag{2}$$

which means that the boundary of $S(x_3)$ has two connected components: $\partial S(x_3) = \Gamma_e(x_3) \cup \Gamma_i(x_3)$, where $\Gamma_e(x_3) := \partial \mathcal{O}_e(x_3)$ (resp. $\Gamma_i(x_3) := \partial \mathcal{O}_i(x_3)$) will be called the exterior boundary (resp. the interior boundary) of $S(x_3)$.

The dielectric material is characterized by the electrical permittivity ε and the magnetic permeability μ which are both functions of the space variable $\mathbf{x} \in \Omega$ that satisfy the usual assumptions

$$0 < \mu_- \leq \mu(\mathbf{x}) \leq \mu_+, \quad 0 < \varepsilon_- \leq \varepsilon(\mathbf{x}) \leq \varepsilon_+, \quad \text{a.e. } \mathbf{x} = (x_1, x_2, x_3) \in \Omega.$$

We shall also consider the possibility that the cable can be, at least locally, conducting, which is modelled through the conductivity σ,

$$0 \leq \sigma(\mathbf{x}) \leq \sigma_+, \quad \text{a.e. } \mathbf{x} = (x_1, x_2, x_3) \in \Omega. \tag{3}$$

The propagation of waves in the cable Ω, through the unknowns $\mathbf{E}(\mathbf{x}, t)$ (the electric field) and $\mathbf{H}(\mathbf{x}, t)$ (the magnetic field), is governed by 3D Maxwell's equations completed with perfectly conducting boundary conditions on $\Gamma := \partial\Omega$,

$$\begin{cases} \varepsilon \partial_t \mathbf{E} + \sigma \mathbf{E} - \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \mu \partial_t \mathbf{H} + \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{E} \times \mathbf{n} = \mathbf{0}, & \text{on } \Gamma \times \mathbb{R}^+, \end{cases} \tag{4}$$

where t is the time, $\boldsymbol{\nabla} \times$ the 3D curl operator, and $\mathbf{n}$ stands for the unit outward normal. The model is completed with initial conditions (for simplicity again we have supposed the absence of source terms, without any loss of generality)

$$\mathbf{E}(\cdot, 0) = \mathbf{E}_0, \quad \mathbf{H}(\cdot, 0) = \mathbf{H}_0. \tag{5}$$

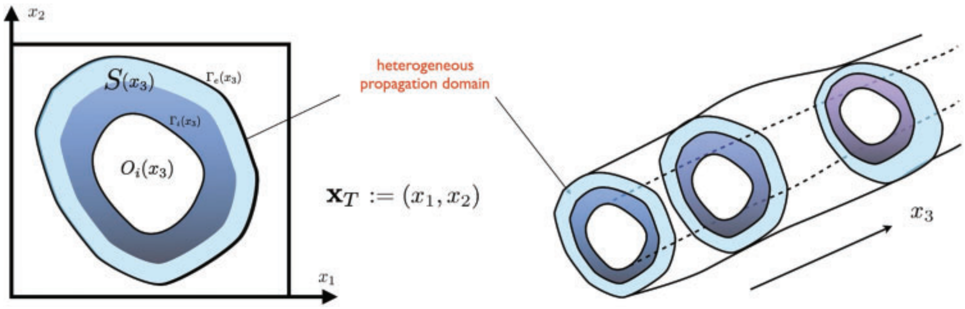

FIGURE 1. Left: slice of the domain; right: the geometry of the domain Ω.

Our method will be developed for the second-order electric field formulation of the problem, obtained after elimination of the magnetic field,

$$\left| \begin{array}{ll} \varepsilon \partial_t^2 \mathbf{E} + \sigma \partial_t \mathbf{E} + \boldsymbol{\nabla} \times \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}, & \text{in } \Omega \times \mathbb{R}^+, \\ \mathbf{E} \times \mathbf{n} = \mathbf{0}, & \text{on } \Gamma \times \mathbb{R}^+, \\ \mathbf{E}(\cdot, 0) = \mathbf{E}_0, \quad \partial_t \mathbf{E}(\cdot, 0) = \varepsilon^{-1}(\boldsymbol{\nabla} \times \mathbf{H}_0 - \sigma \mathbf{E}_0), & \text{in } \Omega. \end{array} \right. \tag{6}$$

Our approach will be based on a particular rewriting that well separates the roles of the longitudinal and transverse space variables (resp. longitudinal and transverse electric fields). For simplicity, without any loss of generality, we shall present our method in the case $\sigma = 0$.

We introduce the functional space

$$H(\mathbf{rot}; \Omega) := \left\{ (\mathbf{E}_T, E_3) \in L^2(\Omega)^2 \times L^2(\Omega) / \mathbf{rot}_T E_3 + \mathbf{e}_3 \times \partial_3 \mathbf{E}_T \in L^2(\Omega)^2, \mathrm{rot}_T \mathbf{E}_T \in L^2(\Omega) \right\}.$$

In that case, the weak formulation of (6) is: find $\mathbf{E}(\cdot, t) = (\mathbf{E}_T(\cdot, t), E_3(\cdot, t)) \in H(\mathbf{rot}; \Omega)$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_\Omega \varepsilon \mathbf{E} \cdot \mathbf{F} \,\mathrm{d}\mathbf{x} + \int_\Omega \mu^{-1} \boldsymbol{\nabla} \times \mathbf{E} \cdot \boldsymbol{\nabla} \times \mathbf{F} \,\mathrm{d}\mathbf{x} = 0, \quad \forall \mathbf{F} \in H(\mathbf{rot}; \Omega). \tag{7}$$

To conclude this introductory section, we rewrite Maxwell's equations (6) as a system that distinguishes the longitudinal and transverse space variables as well as the longitudinal and transverse fields, according to

$$\mathbf{x} := (\mathbf{x}_T, x_3), \quad \mathbf{x}_T := (x_1, x_2), \quad \mathbf{E} := (\mathbf{E}_T, E_3), \quad \mathbf{E}_T := (E_1, E_2). \tag{8}$$

To rewrite (6), we shall use the following transverse curl operators and the transverse divergence operator (note that the index T refers to transverse derivatives),

$$\mathrm{rot}_T \mathbf{E}_T = \partial_1 E_2 - \partial_2 E_1, \quad \mathbf{rot}_T E_3 = (\partial_2 E_3, -\partial_1 E_3), \quad \mathrm{div}_T \mathbf{E}_T = \partial_1 E_1 + \partial_2 E_2. \tag{9}$$

The first one is a scalar rotational operator whereas the second is a vectorial rotational operator which can be seen as a "rotated" gradient

$$\mathbf{rot}_T E_3 = -\mathbf{e}_3 \times \nabla_T E_3, \quad \text{with} \quad \mathbf{e}_3 \times \mathbf{E}_T = (-E_2, E_1) \quad \text{and} \quad \mathbf{e}_3 = (0, 0, 1)^t. \tag{10}$$

These rotational operators are related to the 3D curl operator via

$$\boldsymbol{\nabla} \times \mathbf{E} = \begin{pmatrix} \mathbf{rot}_T E_3 + \mathbf{e}_3 \times \partial_3 \mathbf{E}_T \\ \mathrm{rot}_T \mathbf{E}_T \end{pmatrix}. \tag{11}$$

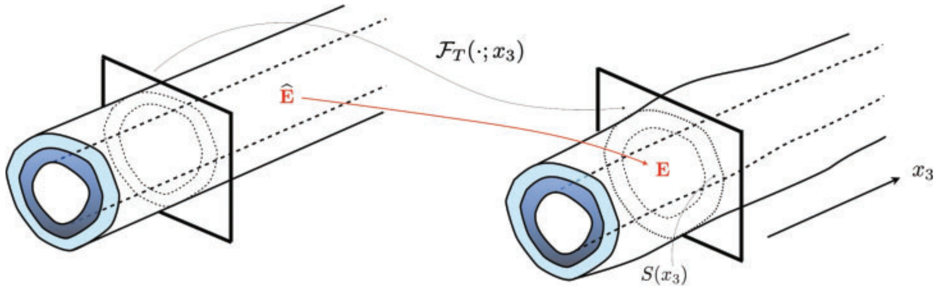

FIGURE 2. An (adapted) mapping approach.

Thus, the Maxwell's equations (6) are rewritten as (these are straightforward computations)

$$\left| \begin{array}{l} \varepsilon\partial_t^2\mathbf{E}_T - \partial_3(\mu^{-1}\partial_3\mathbf{E}_T) + \mathbf{rot}_T(\mu^{-1}\mathrm{rot}_T\mathbf{E}_T) + \partial_3(\mu^{-1}\nabla_T E_3) = 0, \\ \varepsilon\partial_t^2 E_3 + \mathrm{rot}_T(\mu^{-1}\mathbf{rot}_T E_3) + \mathrm{div}_T(\mu^{-1}\partial_3\mathbf{E}_T) = 0. \end{array} \right. \tag{12}$$

2.2. The principle of the mapping approach

In [1], the propagation domain of electromagnetic waves is a cylindrical cable

$$\widehat{\Omega} := \widehat{S} \times \mathbb{R},$$

with $\widehat{S}$ an invariant cross-section in the longitudinal direction. In this section we introduce the case of the variable cross-section, and the idea is to come back to the cylindrical case [1], via a mapping (see Fig. 2),

$$\mathcal{F} : \widehat{\Omega} = \widehat{S} \times \mathbb{R} \quad \longrightarrow \quad \Omega = \bigcup_{x_3\in\mathbb{R}} S(x_3), \quad \mathcal{F}(\widehat{\mathbf{x}}_T, x_3) = (\mathcal{F}_T(\widehat{\mathbf{x}}_T; x_3), x_3), \tag{13}$$

with $\mathcal{F}_T := (\mathcal{F}_1, \mathcal{F}_2)$ a diffeomorphism,

$$\mathcal{F}_T(\cdot; x_3) : \widehat{S} \longrightarrow S(x_3). \tag{14}$$

The specificity of the mapping $\mathcal{F}$ is to keep invariant the transverse planes, so, in some sense, it is a series of 2D transformations from reference cross-section $\widehat{S}$ to each cross-section $S(x_3)$.

With respect to the cylindrical case, the deformation is characterized by the fact that $\mathcal{F}_T$ does depend on x_3. We shall assume that this dependence is smooth enough, typically of class C^2 and to measure deformations, we shall use the quantity

$$\delta := \sup_{\mathbf{x}\in\widehat{\Omega}} \left|\partial_3\mathcal{F}_T(\mathbf{x})\right|. \tag{15}$$

Our strategy is then to deduce the solution $\mathbf{E}(\mathbf{x}, t)$ of Maxwell's equations in Ω from an auxiliary unknown $\widehat{\mathbf{E}}(\widehat{\mathbf{x}}, t)$ defined in $\widehat{\Omega}$. More precisely, $\widehat{\mathbf{E}}(\widehat{\mathbf{x}}, t)$ will be defined as the solution of a modified Maxwell-like problem in the reference domain $\widehat{\Omega}$ to which we will apply—more precisely adapt—the method presented in [1]. Then, $\mathbf{E}(\mathbf{x}, t)$ will be deduced via a simple linear transformation

$$\mathbf{E} \circ \mathcal{F}(\widehat{\mathbf{x}}, t) = \mathbf{T}(\widehat{\mathbf{x}})\widehat{\mathbf{E}}(\widehat{\mathbf{x}}, t) \quad (\mathbf{T} \text{ is a matrix}). \tag{16}$$

The key issue is to find the appropriate transformation of the field $\mathbf{E}$ that will allow us to adapt the method applicable in the cylindrical case [1].

Remark 2.1. In what follows, the differential operators, like gradient, curl, or divergence, with respect to $\widehat{\mathbf{x}}$, will be denoted with the same symbols as those with respect to $\mathbf{x}$ except that we add the symbol $\widehat{\cdot}$ in order to distinguish them, like for $\widehat{\boldsymbol{\nabla}}$, $\widehat{\mathrm{rot}}_T$, or $\widehat{\mathbf{rot}}_T$. However, to avoid a too heavy notation, for the derivative with respect to one single scalar variable, $\widehat{x}_i$ or x_i, we shall use the same symbol ∂_i. The function which it is applied to will make the difference: if $g(\mathbf{x})$ is a function of $\mathbf{x}$, $\partial_i g(\mathbf{x})$ denotes the derivative with respect to x_i while, if $\widehat{g}(\widehat{\mathbf{x}})$ is of $\widehat{\mathbf{x}}$ (which is indicated by the presence of the $\widehat{\cdot}$ over g), $\partial_i \widehat{g}(\widehat{\mathbf{x}})$ denotes the derivative with respect to $\widehat{x}_i$.

Why the usual Piola transformation is not adapted. For Maxwell's equations, the most commonly used transformation is the so-called 3D Piola transform [26] given by

$$\mathbf{E}(\mathbf{x}) = D\mathcal{F}(\widehat{\mathbf{x}})^{-*}\widehat{\mathbf{E}}(\widehat{\mathbf{x}}), \quad \text{for } \mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}}), \tag{17}$$

where $D\mathcal{F}(\widehat{\mathbf{x}})$ is the differential matrix of the mapping F and, for any invertible matrix $\mathbf{M}$, $\mathbf{M}^{-*} := (\mathbf{M}^{-1})^* \equiv (\mathbf{M}^*)^{-1}$. Note that (17) corresponds to (16) with $\mathbf{T} = D\mathcal{F}^{-*}$. This transformation is inspired by the usual transformation of the 3D gradient operator $\boldsymbol{\nabla}$ of a scalar field $u(\mathbf{x})$ via the simple change of variable $\mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}})$,

$$\boldsymbol{\nabla} u(\mathbf{x}) = D\mathcal{F}(\widehat{\mathbf{x}})^{-*}\widehat{\boldsymbol{\nabla}}\widehat{u}(\widehat{\mathbf{x}}), \quad \text{if } u(\mathbf{x}) = \widehat{u}(\widehat{\mathbf{x}}).$$

This shows that $\widehat{\mathbf{E}}$ in (17) is curl-free when $\mathbf{E}$ is curl-free. More generally the main interest of the Piola transform is that it preserves the curl operator up to a linear mapping. More precisely, see [26],

$$(\boldsymbol{\nabla} \times \mathbf{E})(\mathbf{x}) = J^{-1}(\widehat{\mathbf{x}}) D\mathcal{F}(\widehat{\mathbf{x}})(\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})(\widehat{\mathbf{x}}) \quad \text{for } \mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}}), \quad \text{with } J(\widehat{\mathbf{x}}) = \det(D\mathcal{F}(\widehat{\mathbf{x}})), \tag{18}$$

where of course, we have as in (11),

$$\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}} = \begin{pmatrix} \widehat{\mathbf{rot}}_T \widehat{E}_3 + \mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T \\ \widehat{\mathrm{rot}}_T \widehat{\mathbf{E}}_T \end{pmatrix}. \tag{19}$$

In particular, the space $H(\mathbf{rot}; \widehat{\Omega})$ is preserved by the Piola transformation; this is the reason why it is adapted to finite elements. For the case of the transformation $\mathcal{F}$ given in (13), the differential $D\mathcal{F}$ has the following structure, where $D_T\mathcal{F}_T$ holds for the partial differential in the transverse variables:

$$D\mathcal{F}(\widehat{\mathbf{x}}_T, x_3) = \begin{pmatrix} D_T\mathcal{F}_T(\widehat{\mathbf{x}}_T, x_3) & \partial_3\mathcal{F}_T(\widehat{\mathbf{x}}_T, x_3) \\ 0 & 1 \end{pmatrix}, \quad D_T\mathcal{F}_T = \begin{pmatrix} \partial_1\mathcal{F}_1 & \partial_2\mathcal{F}_1 \\ \partial_1\mathcal{F}_2 & \partial_2\mathcal{F}_2 \end{pmatrix}. \tag{20}$$

The problem comes from the presence of the off-diagonal term $\partial_3\mathcal{F}_T$ in (20). Indeed, by defining $\widehat{\mathbf{E}}(\widehat{\mathbf{x}}, t)$ from $\mathbf{E}(\mathbf{x}, t)$ via (17), it is easy to obtain the weak formulation for $\widehat{\mathbf{E}}(\widehat{\mathbf{x}}, t)$ from (7). Transforming also the test field $\mathbf{F}$ into $\widehat{\mathbf{F}}$ via (17), using (18) for the transformation of the 3D curl operator and $\mathrm{d}\mathbf{x} = |J(\widehat{\mathbf{x}})|\,\mathrm{d}\widehat{\mathbf{x}}$, one obtains, for any $\widehat{\mathbf{F}} \in H(\mathbf{rot}; \widehat{\Omega})$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\widehat{\Omega}} \widehat{\varepsilon}\mathbf{A}\widehat{\mathbf{E}} \cdot \widehat{\mathbf{F}}\,\mathrm{d}\widehat{\mathbf{x}} + \int_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}^{-1}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}} \cdot \widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{F}}\,\mathrm{d}\widehat{\mathbf{x}} = 0,$$

where the matrix $\mathbf{A}$ and its inverse depend on $\mathbf{x}$ via $D\mathcal{F}$ as follows:

$$\mathbf{A} = |J| D\mathcal{F}^{-1} D\mathcal{F}^{-*}, \quad \mathbf{A}^{-1} = |J|^{-1} D\mathcal{F}^* D\mathcal{F} = |J|^{-1} \begin{pmatrix} (D_T\mathcal{F}_T)^*(D_T\mathcal{F}_T) & (D_T\mathcal{F}_T)^*\partial_3\mathcal{F}_T \\ (\partial_3\mathcal{F}_T)^*(D_T\mathcal{F}_T) & 1 + |\partial_3\mathcal{F}_T|^2 \end{pmatrix}. \tag{21}$$

When $\partial_3\mathcal{F}_T \neq 0$, the problem is the presence of off-diagonal terms in $\mathbf{A}$ and $\mathbf{A}^{-1}$, which introduces new couplings between longitudinal and transverse fields and longitudinal and transverse derivatives.

The method of [1] relies on a mass-lumping strategy possible only when the mass bilinear form does not couple longitudinal and transverse fields. Without mass lumping, performances are drastically reduced at the point where it is then preferable to use a fully implicit time discretization. To circumvent this problem, we are going to introduce a mapping approach that corresponds to a modified Piola transform.

2.3. Anisotropic Piola transform and its properties

An adequate alternative to (17) is obtained by treating differently the transverse and longitudinal fields via what we shall call the *anisotropic* Piola transform,

$$\widehat{\mathbf{E}}(\widehat{\mathbf{x}}_T, x_3) = (\widehat{\mathbf{E}}_T, \widehat{E}_3)(\widehat{\mathbf{x}}_T, x_3) \quad \longrightarrow \quad \mathbf{E}(\mathbf{x}_T, x_3) = (\mathbf{E}_T, E_3)(\mathbf{x}_T, x_3),$$

defined as follows:

$$\mathbf{E}_T(\mathbf{x}_T, x_3) := D_T\mathcal{F}_T(\widehat{\mathbf{x}}_T, x_3)^{-*}\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}_T, x_3), \quad E_3(\mathbf{x}_T, x_3) := \widehat{E}_3(\widehat{\mathbf{x}}_T, x_3), \quad \text{for } \mathbf{x}_T = \mathcal{F}_T(\widehat{\mathbf{x}}_T), \tag{22}$$

where $D_T\mathcal{F}_T$ is defined in (20). The formula (22) can be rewritten as

$$\mathbf{E}(\mathbf{x}) = \mathbf{D}^{-*}(\widehat{\mathbf{x}})\widehat{\mathbf{E}}(\widehat{\mathbf{x}}), \quad \mathbf{D}(\widehat{\mathbf{x}}) = \begin{pmatrix} \mathbf{D}_T(\widehat{\mathbf{x}}) & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}_T(\widehat{\mathbf{x}}) := D_T\mathcal{F}_T(\widehat{\mathbf{x}}). \tag{23}$$

In other words, the passage from $\mathbf{E}_T$ to $\widehat{\mathbf{E}}_T$ results from a 2D Piola transform, while the one from E_3 to $\widehat{E}_3$ results from a simple change of variable. For the following, we shall introduce the transverse Jacobian

$$J_T(\widehat{\mathbf{x}}) := \det(D_T\mathcal{F}_T(\widehat{\mathbf{x}})) \equiv \det \mathbf{D}_T(\widehat{\mathbf{x}}). \tag{24}$$

The rest of this section is devoted to describing how $\nabla \times \mathbf{E}$ is transformed through the transformation (22) for $\mathbf{E}$; in other words how the formula (18) is modified (see Remark 2.3), which will result in Theorem 2.1. For this we shall use the expression (11) for $\nabla \times \mathbf{E}$. Lemmas 2.1, 2.2, and 2.4 explain how the various terms appearing in (11), namely $\mathbf{rot}_T E_3$, $\mathrm{rot}_T \mathbf{E}_T$, and $\partial_3 \mathbf{E}_T$, are transformed. In the following, for shortening the writing, we shall introduce

$$\mathbf{B}_T^\delta(\widehat{\mathbf{x}}) := (\partial_3\mathbf{D}_T^*)(\widehat{\mathbf{x}})\mathbf{D}_T^{-*}(\widehat{\mathbf{x}}), \quad \mathbf{v}_T^\delta(\widehat{\mathbf{x}}) = \partial_3\mathcal{F}_T(\widehat{\mathbf{x}}), \quad \mathbf{b}_T^\delta(\widehat{\mathbf{x}}) = \mathbf{D}_T^{-1}(\widehat{\mathbf{x}})\mathbf{v}_T^\delta(\widehat{\mathbf{x}}). \tag{25}$$

Remark 2.2. In (25), as well as in the rest of the paper, the superscript δ is used to identify terms due to the deformation of the cable via $\partial_3\mathcal{F}_T$ (cf. (15)). In particular, these quantities vanish when $\delta = 0$.

The first two lemmas are classical 2D variations of the formula in the 3D case.

Lemma 2.1. *Through the transformation (22), the 2D vector curl operator is transformed according to*

$$\mathbf{rot}_T E_3(\mathbf{x}) = J_T^{-1}(\widehat{\mathbf{x}})\mathbf{D}_T(\widehat{\mathbf{x}})\widehat{\mathbf{rot}}_T\widehat{E}_3(\widehat{\mathbf{x}}). \tag{26}$$

Lemma 2.2. *Through the transformation (22), the 2D scalar curl operator is transformed according to*

$$\mathrm{rot}_T\mathbf{E}_T(\mathbf{x}) = J_T^{-1}(\widehat{\mathbf{x}})\widehat{\mathrm{rot}}_T\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}). \tag{27}$$

For our next result, we shall need the technical lemma below:

Lemma 2.3. *For any* $\mathbf{u}_T \in L^2(\widehat{S})^2$, $\mathbf{e}_3 \times (\mathbf{D}_T^{-*}\mathbf{u}_T) = J_T^{-1}\mathbf{D}_T(\mathbf{e}_3 \times \mathbf{u}_T)$.

Proof. First note that

$$\mathbf{D}_T^{-*} := ((D_T\mathcal{F}_T)^*)^{-1} = J_T^{-1}\begin{pmatrix} \partial_2\mathcal{F}_2 & -\partial_1\mathcal{F}_2 \\ -\partial_2\mathcal{F}_1 & \partial_1\mathcal{F}_1 \end{pmatrix},$$

so that

$$\mathbf{D}_T^{-*}\mathbf{u}_T = J_T^{-1}\begin{pmatrix} (\partial_2\mathcal{F}_2)u_1 - (\partial_1\mathcal{F}_2)u_2 \\ -(\partial_2\mathcal{F}_1)u_1 + (\partial_1\mathcal{F}_1)u_2 \end{pmatrix}.$$

Thus, using $\mathbf{e_3} \times (v_1, v_2)^t = (-v_2, v_1)^t$, we have

$$\mathbf{e}_3 \times \mathbf{D}_T^{-*}\mathbf{u}_T = J_T^{-1}\begin{pmatrix} (\partial_2\mathcal{F}_1)u_1 - (\partial_1\mathcal{F}_1)u_2 \\ (\partial_2\mathcal{F}_2)u_1 - (\partial_1\mathcal{F}_2)u_2 \end{pmatrix} = J_T^{-1}\begin{pmatrix} \partial_1\mathcal{F}_1 & \partial_2\mathcal{F}_1 \\ \partial_1\mathcal{F}_2 & \partial_2\mathcal{F}_2 \end{pmatrix}\begin{pmatrix} -u_2 \\ u_1 \end{pmatrix},$$

which is nothing but the announced result. □

The transformation of $\partial_3\mathbf{E}_T$ is not classical. This is the object of the next lemma.

Lemma 2.4. *Through the transformation (22), the x_3-derivative is transformed according to*

$$\partial_3\mathbf{E}_T(\mathbf{x}) = \mathbf{D}_T^{-*}(\widehat{\mathbf{x}})(\partial_3\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}) - \mathbf{B}_T^\delta\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}})) - (\mathbf{D}_T^{-1}(\widehat{\mathbf{x}})\mathbf{v}_T^\delta \cdot \widehat{\nabla}_T)(\mathbf{D}_T^{-*}\widehat{\mathbf{E}}_T)(\widehat{\mathbf{x}}). \tag{28}$$

Proof. We recall the chain rule, which we rewrite using $\partial_3\mathcal{F}_3(\widehat{\mathbf{x}}) = 1$ and $\partial_3\mathcal{F}_T(\widehat{\mathbf{x}}) = \mathbf{v}_T^\delta(\widehat{\mathbf{x}})$. In the following, we use Remark 2.1 for the notation.

$$\text{With } \mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}}), \quad \widehat{u}(\widehat{\mathbf{x}}) = u(\mathbf{x}) \quad \Longrightarrow \quad \partial_3\widehat{u}(\widehat{\mathbf{x}}) = \partial_3 u(\mathbf{x}) + \mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T u(\mathbf{x}).$$

We use this formula for differentiating in $\widehat{x}_3$ the formulas (22). This gives

$$\begin{cases} \partial_3\widehat{E}_1(\widehat{\mathbf{x}}) = (\partial_1\mathcal{F}_1)(\partial_3 E_1(\mathbf{x}) + \mathbf{v}_T^\delta(\widehat{\mathbf{x}})\cdot\nabla_T E_1(\mathbf{x})) \\ \qquad +(\partial_1\mathcal{F}_2)(\partial_3 E_2(\mathbf{x}) + \mathbf{v}_T^\delta(\widehat{\mathbf{x}})\cdot\nabla_T E_2(\mathbf{x})) \\ \qquad +\partial_{31}^2\mathcal{F}_1(\widehat{\mathbf{x}})E_1(\mathbf{x}) + \partial_{31}^2\mathcal{F}_2(\widehat{\mathbf{x}})E_2(\mathbf{x}), \end{cases} \quad \begin{cases} \partial_3\widehat{E}_2(\widehat{\mathbf{x}}) = (\partial_2\mathcal{F}_1)(\partial_3 E_1(\mathbf{x}) + \mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_1(\mathbf{x})) \\ \qquad +(\partial_2\mathcal{F}_2)(\partial_3 E_2(\mathbf{x}) + \mathbf{v}_T^\delta(\widehat{\mathbf{x}})\cdot\nabla_T E_2(\mathbf{x})) \\ \qquad + \partial_{32}^2\mathcal{F}_1(\widehat{\mathbf{x}})E_1(\mathbf{x}) + \partial_{32}^2\mathcal{F}_2(\widehat{\mathbf{x}})E_2(\mathbf{x}), \end{cases}$$

which we can write in vector form as

$$\left| \begin{aligned} \partial_3\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}) = & \begin{pmatrix} \partial_1\mathcal{F}_1(\widehat{\mathbf{x}})\partial_3 E_1(\mathbf{x}) + \partial_1\mathcal{F}_2(\widehat{\mathbf{x}})\partial_3 E_2(\mathbf{x}) \\ \partial_2\mathcal{F}_1(\widehat{\mathbf{x}})\partial_3 E_1(\mathbf{x}) + \partial_2\mathcal{F}_2(\widehat{\mathbf{x}})\partial_3 E_2(\mathbf{x}) \end{pmatrix} \\ & + \begin{pmatrix} \partial_1\mathcal{F}_1(\widehat{\mathbf{x}})\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_1(\mathbf{x}) + \partial_1\mathcal{F}_2(\widehat{\mathbf{x}})\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_2(\mathbf{x}) \\ \partial_2\mathcal{F}_1(\widehat{\mathbf{x}})\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_1(\mathbf{x}) + \partial_2\mathcal{F}_2(\widehat{\mathbf{x}})\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_2(\mathbf{x}) \end{pmatrix} \\ & + \begin{pmatrix} \partial_{31}^2\mathcal{F}_1(\widehat{\mathbf{x}})E_1(\mathbf{x}) + \partial_{31}^2\mathcal{F}_2(\widehat{\mathbf{x}})E_2(\mathbf{x}) \\ \partial_{32}^2\mathcal{F}_1(\widehat{\mathbf{x}})E_1(\mathbf{x}) + \partial_{32}^2\mathcal{F}_2(\widehat{\mathbf{x}})E_2(\mathbf{x}) \end{pmatrix}, \end{aligned} \right. \tag{29}$$

or in more compact form, by definition of $\mathbf{D}_T$

$$\partial_3\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}) = \mathbf{D}_T^*(\widehat{\mathbf{x}})[\partial_3\mathbf{E}_T(\mathbf{x}) + (\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T)\mathbf{E}_T(\mathbf{x})] + (\partial_3\mathbf{D}_T^*)(\widehat{\mathbf{x}})\mathbf{E}_T(\mathbf{x}), \tag{30}$$

from which we deduce $\partial_3\mathbf{E}_T(\mathbf{x})$ after multiplication by $\mathbf{D}_T^{-*}(\widehat{\mathbf{x}})$

$$\partial_3\mathbf{E}_T(\mathbf{x}) = \mathbf{D}_T^{-*}(\widehat{\mathbf{x}})\partial_3\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}) - \mathbf{D}_T^{-*}(\widehat{\mathbf{x}})(\partial_3\mathbf{D}_T^*)(\widehat{\mathbf{x}})\mathbf{E}_T(\mathbf{x}) - (\widehat{\mathbf{v}}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T)\mathbf{E}_T(\mathbf{x}). \tag{31}$$

It remains to reexpress the second and third terms of the right-hand side of (31) with the help of $\widehat{\mathbf{E}}_T$. For the second one, as by definition $\mathbf{E}_T(\mathbf{x}) = \mathbf{D}_T^{-*}(\widehat{\mathbf{x}})\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}})$, we get, by definition of $\mathbf{B}_T^\delta$,

$$(\partial_3 \mathbf{D}_T^*)(\widehat{\mathbf{x}})\mathbf{E}_T(\mathbf{x}) = (\partial_3 \mathbf{D}_T^*)(\widehat{\mathbf{x}})\mathbf{D}_T^{-*}(\widehat{\mathbf{x}})\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}) \equiv \mathbf{B}_T^\delta(\widehat{\mathbf{x}})\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}). \tag{32}$$

The third term of (31) is the most delicate one because $\widehat{E}_T \neq E_T \circ \mathcal{F}$, *i.e.*, $\widehat{E}_j \neq E_j \circ \mathcal{F}$ for $j = 1, 2$.

First by the chain rule, $\widehat{\nabla}_T(E_j \circ \mathcal{F})(\widehat{\mathbf{x}}) = \mathbf{D}_T^*(\widehat{\mathbf{x}})\nabla_T E_j(\mathbf{x})$ with $\mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}})$ or equivalently

$$\nabla_T E_j(\mathbf{x}) = \mathbf{D}_T^{-*}(\widehat{\mathbf{x}})\widehat{\nabla}_T(E_j \circ \mathcal{F})(\widehat{\mathbf{x}}),$$

and consequently, by definition (25) of $\mathbf{b}_T^\delta(\widehat{\mathbf{x}})$,

$$\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_j(\mathbf{x}) = \mathbf{D}_T^{-1}(\widehat{\mathbf{x}})\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \widehat{\nabla}_T(E_j \circ \mathcal{F})(\widehat{\mathbf{x}}) = \mathbf{b}_T^\delta(\widehat{\mathbf{x}}) \cdot \widehat{\nabla}_T(E_j \circ \mathcal{F})(\widehat{\mathbf{x}}).$$

Since by the anisotropic Piola transform (22), $\mathbf{E}_T \circ \mathcal{F}(\widehat{\mathbf{x}}) = \mathbf{D}_T^{-*}(\widehat{\mathbf{x}})\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}})$, we have finally

$$\mathbf{v}_T^\delta(\widehat{\mathbf{x}}) \cdot \nabla_T E_j(\mathbf{x}) = \mathbf{b}_T^\delta(\widehat{\mathbf{x}}) \cdot \widehat{\nabla}_T(\mathbf{D}_T^{-*}\widehat{E}_T \cdot \mathbf{e}_j)(\widehat{\mathbf{x}}). \tag{33}$$

The formula is obtained by substituting (33) and (32) into (31). □

For our final result, we introduce the first-order vectorial transverse differential operator $\mathbf{P}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)$,

$$\mathbf{P}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}} := \begin{pmatrix} \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_T \\ 0 \end{pmatrix}, \quad \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_T := \mathbf{e}_3 \times [(\mathbf{b}_T^\delta \cdot \widehat{\nabla}_T)(\mathbf{D}_T^{-*}\widehat{\mathbf{E}}_T) + \mathbf{D}_T^{-*}\, \mathbf{B}_T^\delta\, \widehat{\mathbf{E}}_T]. \tag{34}$$

Theorem 2.1. *Through the transformation* (22), *one has the formula*

$$(\boldsymbol{\nabla} \times \boldsymbol{E})(\mathbf{x}) = J_T(\widehat{\mathbf{x}})^{-1}\mathbf{D}(\widehat{\mathbf{x}})(\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})(\widehat{\mathbf{x}}) - \mathbf{P}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}(\widehat{\mathbf{x}}) \quad \textit{with} \;\; \mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}}). \tag{35}$$

Proof. For the first line of $\boldsymbol{\nabla} \times \mathbf{E}$ in (11), we first use (26), that is to say

$$\mathbf{rot}_T E_3(\mathbf{x}) = J_T^{-1}(\widehat{\mathbf{x}})\; \mathbf{D}_T(\widehat{\mathbf{x}})\; \widehat{\mathbf{rot}}_T \widehat{E}_3(\widehat{\mathbf{x}}). \tag{36}$$

Next, using the formula (28) for $\partial_3 \mathbf{E}_T$ (see Lemma 2.4), we also have

$$\mathbf{e}_3 \times \partial_3 \mathbf{E}_T(\mathbf{x}) = [\mathbf{e}_3 \times \mathbf{D}_T^{-*}\partial_3 \widehat{\mathbf{E}}_T - \mathbf{e}_3 \times (\; \mathbf{D}_T^{-*}\; \mathbf{B}_T^\delta\; \widehat{\mathbf{E}}_T + (\mathbf{b}_T^\delta \cdot \widehat{\nabla}_T)(\mathbf{D}_T^{-*}\widehat{\mathbf{E}}_T)](\widehat{\mathbf{x}}). \tag{37}$$

In order to make $\mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T$ appear as in the first line of (19) for $\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}$, we use Lemma 2.3,

$$\forall\, \mathbf{u}_T \in L^2(\widehat{S})^2, \quad \mathbf{e}_3 \times \mathbf{D}_T^{-*}\; \mathbf{u}_T = J_T^{-1}\; \mathbf{D}_T(\mathbf{e}_3 \times \mathbf{u}_T), \tag{38}$$

that we apply with $\mathbf{u}_T = \partial_3 \widehat{\mathbf{E}}_T$ to obtain

$$\mathbf{e}_3 \times \partial_3 \mathbf{E}_T(\mathbf{x}) = [J_T^{-1}\; \mathbf{D}_T(\mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T) - \mathbf{e}_3 \times (\mathbf{D}_T^{-*}\; \mathbf{B}_T^\delta\; \widehat{\mathbf{E}}_T + (\mathbf{b}_T^\delta \cdot \widehat{\nabla}_T)(\mathbf{D}_T^{-*}\widehat{\mathbf{E}}_T))](\widehat{\mathbf{x}}). \tag{39}$$

Joining (36) and (39), we obtain

$$\left|\begin{aligned} [\mathbf{rot}_T E_3 + \mathbf{e}_3 \times \partial_3 \mathbf{E}_T](\mathbf{x}) \;&=\; [J_T^{-1}\; \mathbf{D}_T(\widehat{\mathbf{rot}}_T \widehat{E}_3 + \mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T) \\ &\quad - \mathbf{e}_3 \times (\mathbf{D}_T^{-*}\; \mathbf{B}_T^\delta\; \widehat{\mathbf{E}}_T + (\mathbf{b}_T^\delta \cdot \widehat{\nabla}_T)(\mathbf{D}_T^{-*}\widehat{\mathbf{E}}_T))](\widehat{\mathbf{x}}). \end{aligned}\right. \tag{40}$$

On the other hand, the second line of the expression of $\boldsymbol{\nabla}\times\mathbf{E}$ is directly given by Lemma 2.2,

$$\mathrm{rot}_T\mathbf{E}_T(\mathbf{x}) = J_T^{-1}(\widehat{\mathbf{x}})\ \widehat{\mathrm{rot}}_T\widehat{\mathbf{E}}_T(\widehat{\mathbf{x}}). \tag{41}$$

Finally, gathering (40) and (41) leads to

$$\boldsymbol{\nabla}\times\mathbf{E} = J_T^{-1}\begin{pmatrix}\mathbf{D}_T(\widehat{\mathbf{rot}}_T\widehat{E}_3+\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\\ \widehat{\mathrm{rot}}_T\widehat{\mathbf{E}}_T\end{pmatrix} - \begin{pmatrix}\mathbf{e}_3\times(\mathbf{D}_T^{-*}\mathbf{B}_T^{\delta}\widehat{\mathbf{E}}_T+(\mathbf{b}_T^{\delta}\cdot\widehat{\nabla}_T)(\mathbf{D}_T^{-*}\widehat{\mathbf{E}}_T))\\ 0\end{pmatrix}, \tag{42}$$

where we recognize (35), by definition (23) and (34) of $\mathbf{D}$ and $\mathbf{P}^{\delta}(\widehat{\mathbf{x}},\widehat{\nabla}_T)$ and the expression (19) for $\widehat{\boldsymbol{\nabla}}\times\widehat{\mathbf{E}}$. $\square$

Remark 2.3. According to Remark 2.2, $\mathbf{P}^{\delta}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}=0$ when $\delta=0$. In this case, one recovers (35) directly from the formula (18) since in this case $D\mathcal{F}(\widehat{\mathbf{x}})=\mathbf{D}(\widehat{\mathbf{x}})$ and $J(\widehat{\mathbf{x}})=J_T(\widehat{\mathbf{x}})$.

2.4. Weak formulation in the reference domain and numerical difficulties

The correct functional space to work with for $\widehat{\mathbf{E}}$ is

$$\widehat{V} := \{\widehat{\mathbf{E}}=(\widehat{\mathbf{E}}_T,\widehat{E}_3)\in L^2(\widehat{\Omega})^3/\widehat{\mathbf{E}}=\mathbf{D}^*\mathbf{E}\text{ with }\mathbf{E}=(\mathbf{E}_T,E_3)\in H(\mathbf{rot},\Omega)\}, \tag{43}$$

where $\mathbf{D}$ is given by (23). According to Theorem 2.1, we have

$$\widehat{V} = \{\widehat{\mathbf{E}}\in L^2(\widehat{\Omega})^3/\ \widehat{\mathrm{rot}}_T\widehat{\mathbf{E}}_T\in L^2(\widehat{\Omega}),\mathbf{D}_T(\widehat{\mathbf{rot}}_T\widehat{E}_3+\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)-\mathbf{P}_T^{\delta}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}_T\in L^2(\widehat{\Omega})^2\}. \tag{44}$$

In the cylindrical case, $\mathbf{P}_T^{\delta}(\widehat{\mathbf{x}},\widehat{\nabla}_T)=0$ and one sees that $\widehat{V}=\widehat{V}_c$, where $\widehat{V}_c$ is defined by

$$\widehat{V}_c := \{\widehat{\mathbf{E}}\in L^2(\widehat{\Omega})^3/\widehat{\mathrm{rot}}_T\widehat{\mathbf{E}}_T\in L^2(\widehat{\Omega}),\widehat{\mathbf{rot}}_T\widehat{E}_3+\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T\in L^2(\widehat{\Omega})^2\}. \tag{45}$$

However, in the general case, $\widehat{V}\neq\widehat{V}_c$, which is the main reason why we cannot use the same finite element method as in the cylindrical case. In the following, for shortening the writing, we shall set,

$$\mathbf{A} := \begin{pmatrix}\mathbf{A}_T & 0\\ 0 & |J_T|\end{pmatrix}\quad\text{with}\quad \mathbf{A}_T := |J_T|\mathbf{D}_T^{-1}\mathbf{D}_T^{-*}. \tag{46}$$

Theorem 2.2 (Weak formulation for the problem in $\widehat{\mathbf{E}}$)**.** *If $\mathbf{E}$ is a solution of (7), $\widehat{\mathbf{E}}$ given by (22) is a solution of the problem: find $\widehat{\mathbf{E}}(.,t)=(\widehat{\mathbf{E}}_T,\widehat{E}_3)(.,t)\in\widehat{V}$ such that*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}})+\boldsymbol{k}(\widehat{\mathbf{E}},\widehat{\mathbf{F}})=0,\quad\forall\,\widehat{\mathbf{F}}=(\widehat{\mathbf{F}}_T,\widehat{F}_3)\in\widehat{V}, \tag{47}$$

where the mass bilinear form $\boldsymbol{m}(\cdot,\cdot)$ is given by

$$\begin{cases}\boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) := \displaystyle\int_{\widehat{\Omega}}\widehat{\varepsilon}(\mathbf{A}\widehat{\mathbf{E}}\cdot\widehat{\mathbf{F}})\,\mathrm{d}\widehat{\mathbf{x}} = \mathbf{m}(\widehat{\mathbf{E}}_T,\widehat{\mathbf{F}}_T)+m(\widehat{E}_3,\widehat{F}_3),\\ \mathbf{m}(\widehat{\mathbf{E}}_T,\widehat{\mathbf{F}}_T) := \displaystyle\int_{\widehat{\Omega}}\widehat{\varepsilon}(\mathbf{A}_T\widehat{\mathbf{E}}_T\cdot\widehat{\mathbf{F}}_T)\,\mathrm{d}\widehat{\mathbf{x}},\quad m(\widehat{E}_3,\widehat{F}_3) := \displaystyle\int_{\widehat{\Omega}}\widehat{\varepsilon}|J_T|\widehat{E}_3\cdot\widehat{F}_3\,\mathrm{d}\widehat{\mathbf{x}},\end{cases} \tag{48}$$

and the stiffness bilinear form is given by

$$\boldsymbol{k}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) = \int_{\widehat{\Omega}}\widehat{\mu}^{-1}|J_T|\ (J_T^{-1}\boldsymbol{D}(\widehat{\boldsymbol{\nabla}}\times\widehat{\mathbf{E}})-\mathbf{P}^{\delta}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}})\cdot(J_T^{-1}\boldsymbol{D}(\widehat{\boldsymbol{\nabla}}\times\widehat{\mathbf{F}})-\mathbf{P}^{\delta}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{F}})\,\mathrm{d}\widehat{\mathbf{x}}. \tag{49}$$

Proof. It suffices to start from the weak formulation of the original problem in $\mathbf{E}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_\Omega \varepsilon \mathbf{E}\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} + \int_\Omega \mu^{-1}\boldsymbol{\nabla}\times\mathbf{E}\cdot\boldsymbol{\nabla}\times\mathbf{F}\,\mathrm{d}\mathbf{x} = 0, \tag{50}$$

and to use the anisotropic transformation of both the unknown and the test fields

$$\mathbf{E}(\mathbf{x}) = \mathbf{D}^{-*}\,\widehat{\mathbf{E}}\,(\widehat{\mathbf{x}})\,, \quad \mathbf{F}(\mathbf{x}) = \mathbf{D}^{-*}\,\widehat{\mathbf{F}}(\widehat{\mathbf{x}}), \quad \text{where } \mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}}), \tag{51}$$

so that (50) is nothing but (47) where the bilinear forms $\boldsymbol{m}$ and $\boldsymbol{k}$ are given by

$$\boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) = \int_\Omega \varepsilon\mathbf{E}\cdot\mathbf{F}\,\mathrm{d}\mathbf{x}, \quad \boldsymbol{k}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) = \int_\Omega \mu^{-1}\boldsymbol{\nabla}\times\mathbf{E}\cdot\boldsymbol{\nabla}\times\mathbf{F}\,\mathrm{d}\mathbf{x},$$

where $(\mathbf{E},\mathbf{F})$ are related to $(\widehat{\mathbf{E}},\widehat{\mathbf{F}})$ via (51). For this reason, the above expressions are thus implicit with respect to $(\widehat{\mathbf{E}},\widehat{\mathbf{F}})$. For obtaining the expression (48) for $\widehat{\boldsymbol{m}}$, we use the change of variable and unknowns (51) to deduce, since $\mathrm{d}\mathbf{x} = |J_T(\widehat{\mathbf{x}})|\,\mathrm{d}\widehat{\mathbf{x}}$,

$$\boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) = \int_\Omega \varepsilon\mathbf{E}\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\widehat{\Omega}} \widehat{\varepsilon}|J_T|\mathbf{D}^{-*}\widehat{\mathbf{E}}\cdot\mathbf{D}^{-*}\widehat{\mathbf{F}}\,\mathrm{d}\widehat{\mathbf{x}}. \tag{52}$$

To obtain (48), it suffices to remark that, by definition (46) of $\mathbf{A}_T$,

$$\int_{\widehat{\Omega}} \widehat{\varepsilon}|J_T|\mathbf{D}^{-*}\widehat{\mathbf{E}}\cdot\mathbf{D}^{-*}\widehat{\mathbf{F}}\,\mathrm{d}\widehat{\mathbf{x}} = \int_{\widehat{\Omega}} \widehat{\varepsilon}\mathbf{A}_T\widehat{\mathbf{E}}_T\cdot\widehat{\mathbf{F}}_T\,\mathrm{d}\widehat{\mathbf{x}} + \int_{\widehat{\Omega}} \widehat{\varepsilon}|J_T|\widehat{E}_3\cdot\widehat{F}_3\,\mathrm{d}\widehat{\mathbf{x}}. \tag{53}$$

To obtain the expression (49) of $\boldsymbol{k}$, it suffices to proceed in the same manner and to use Theorem 2.1 for the transformation of $\boldsymbol{\nabla}\times\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{F}$. □

The fact that $\widehat{V}\neq\widehat{V}_c$ when the cable is not cylindrical prevents us from proceeding as in the cylindrical case for the space discretization, *i.e.* of using $H(\mathbf{rot};\widehat{\Omega})$ conforming finite elements associated to a prismatic mesh of the reference cylindrical cable. That is why we will introduce a hybrid finite element/DG approach based on DG in the transverse direction and conformal finite elements for longitudinal variables.

3. A transverse IPDG approach

3.1. Strategy for the transverse semi-discretization

Our discretization strategy in the transverse variable $\mathbf{x}_T$ is based on the use of an IPDG approach found in several works in the literature [2, 12–14, 18]. For our construction, we have chosen the following approach (see also Remark 3.1):

(i) We first build a transverse mesh $\mathcal{T}_h(\widehat{\Omega})$ of the reference cylindrical domain $\widehat{\Omega}$ with infinite prismatic elements (Section 3.2.1) from which we build a corresponding mesh $\mathcal{T}_h(\Omega)$ of the deformed cable Ω via the transformation $\mathcal{F}$ (Section 3.2.2).

(ii) We write the DG weak formulation of Maxwell's equations in Ω, with solution $\mathbf{E}$, associated to the (non-cylindrical) mesh $\mathcal{T}_h(\Omega)$ (Section 3.4).

(iii) We write the DG weak formulation for the problem in $\widehat{\mathbf{E}}$, defined in Ω from $\mathbf{E}$ via the anisotropic Piola transform, from the formulation obtained from the previous terms (in the same way that we passed from (7) to (47) in Section 2.4). This formulation is then naturally associated to the cylindrical mesh $\mathcal{T}_h(\widehat{\Omega})$ (Section 3.3).

Remark 3.1. An alternative (and equivalent) approach would have consisted in writing in strong form the equations for $\widehat{\mathbf{E}}$ in the reference variable $\hat{\mathbf{x}}$ and write directly the DG formulation of the resulting Partial Differential Equations (PDEs) using the cylindrical mesh $\mathcal{T}_h(\widehat{\Omega})$.

3.2. Prismatic like meshes for the reference and deformed cables

3.2.1. Prismatic meshes in the reference domain

For the discretization in the transverse variables of the reference cable $\widehat{\Omega}$, we assume that $\widehat{S}$ is a polygon (that may be in practice a polygonal approximation of the true geometry); we introduce $\mathcal{T}_h(\widehat{S})$, a conformal triangular mesh of $\widehat{S}$. We shall denote

$$\begin{cases} h := \max\limits_{K\in\mathcal{T}_h(\widehat{S})} h_K, \quad h_K := \operatorname{diam} K, \quad \text{the transverse stepsize,} \\ N_e := \#\{\text{interior edges of } \mathcal{T}_h(\widehat{S})\}, \quad N := \#\{\text{interior nodes of } \mathcal{T}_h(\widehat{S})\}. \end{cases} \tag{54}$$

We shall more precisely consider a family of such triangulations $\mathcal{T}_h(\widehat{S})$, indexed by $h > 0$, which is regular in the following sense:

Definition 3.1. The family of triangulation $\mathcal{T}_h(\widehat{S})$ is said to be regular if:

(i) For any $K \in \mathcal{T}_h(\widehat{S})$, there exists $T_K \in \mathcal{L}(\mathbb{R}^2)$ invertible, such that $K = \widehat{x}_K + h_K\, T_K(K_r)$, where $\widehat{x}_K$ belongs to $\overline{K}$ and K_r the reference triangle:

$$K_r := \left\{x_r = (x_{r,1}, x_{r,2}) \in \mathbb{R}^2 / x_{r,1} \geq 0,\ x_{r,2} \geq 0,\ x_{r,1} + x_{r,2} \leq 1\right\}; \tag{55}$$

(ii) There exists $R > 0$ such that for any $h > 0$ and $K \in \mathcal{T}_h(\widehat{S})$, $\|T_K\| \leq R$ and $\|T_K^{-1}\| \leq R$, where $\|\cdot\|$ stands for any matrix norm.

Accordingly, we consider a corresponding partition $\mathcal{T}_h(\widehat{\Omega})$ of $\widehat{\Omega}$ into disjoint infinite elongated prisms $\widehat{P}_K$,

$$\mathcal{T}_h(\widehat{\Omega}) := \{\widehat{P}_K := K \times \mathbb{R}, K \in \mathcal{T}_h(\widehat{S})\}. \tag{56}$$

We shall also assume that the triangulations are $\widehat{\mu}$-compatible in the following sense:

$$\forall\, \mathcal{T}_h(\widehat{\Omega}), \quad \forall\, \widehat{P}_K \in \mathcal{T}_h(\widehat{\Omega}), \quad \widehat{\mu}|_{\widehat{P}_K} \in W^{1,\infty}(\widehat{P}_K). \tag{57}$$

In the following, we will denote the interface between two elements $\widehat{P}_K$ and $\widehat{P}_L$ by $\widehat{\Sigma}_{KL}$,

$$\widehat{\Sigma}_{KL} \equiv \widehat{\Sigma}_{LK} := \partial\widehat{P}_K \cap \partial\widehat{P}_L \quad \text{which form the skeleton} \quad \widehat{\Sigma}_h := \bigcup_{K\neq L} \widehat{\Sigma}_{KL}. \tag{58}$$

Note that each $\widehat{\Sigma}_{KL}$ is included in a plane parallel to the x_3 axis. In the same way, for any $K \in \mathcal{T}_h(\widehat{S})$, we set $\widehat{\Gamma}_K := \partial\widehat{P}_K \cap \widehat{\Gamma}$, which form when K describes $\mathcal{T}_h(\widehat{S})$, a partitioning of $\widehat{\Gamma}$,

$$\widehat{\Gamma} = \bigcup_{K\in\mathcal{T}_h(\widehat{S})} \widehat{\Gamma}_K.$$

3.2.2. Curved prismatic meshes in the deformed cable

According to the cylindrical partition $\mathcal{T}_h(\widehat{\Omega})$ of the cylindrical reference cable introduced in Section 3.2.1, we define a curved prismatic partition (Fig. 3) of the deformed cable Ω as

$$\mathcal{T}_h(\Omega) := \{P_K = \mathcal{F}(\widehat{P}_K), K \in \mathcal{T}_h(\widehat{S})\}. \tag{59}$$

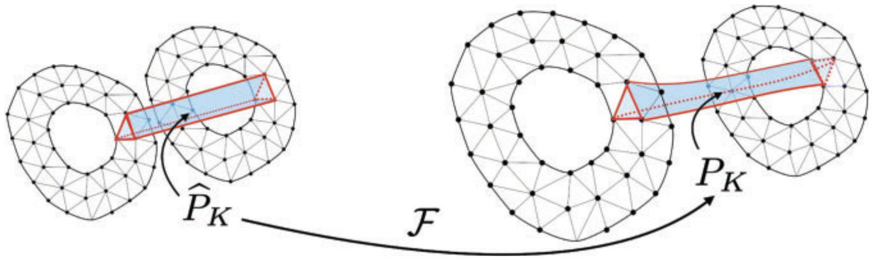

FIGURE 3. The transformation of the infinite reference prisms to an infinite deformed prism.

In the following, we denote the interface between two elements P_K and P_L by $\Sigma_{KL} := \mathcal{F}(\widehat{\Sigma}_{KL})$, so that

$$\Sigma_{KL} \equiv \Sigma_{LK} := \partial P_K \cap \partial P_L \quad \text{which form the skeleton} \quad \Sigma_h := \{\Sigma_{KL}\}. \tag{60}$$

Finally the sets $\Gamma_K := \mathcal{F}(\widehat{\Gamma}_K) \equiv \partial P_K \cap \Gamma$ form when K describes $\mathcal{T}_h(S)$, a partitioning of Γ,

$$\Gamma = \bigcup_{K \in \mathcal{T}_h(S)} \Gamma_K.$$

3.3. Transverse DG formulation in the deformed cable

As is usual with DG methods, we shall assume that the exact solution of the problem $\mathbf{E}(.,t)$ in our case is sufficiently regular inside each element of the mesh. In the following, for any $\mathbf{u}$ in $L^2(\Omega)^3$, we shall denote by $\mathbf{u}_K$ its restriction to P_K and we define the spaces

$$\begin{aligned} \mathbf{V}^h(\Omega) &:= \left\{\mathbf{u} \in L^2(\Omega)^3 / \forall\, K \in \mathcal{T}_h(S), \mathbf{u}_K \in H^1(P_K)^3\right\} \subset L^2(\Omega)^3, \\ \mathbf{R}^h(\Omega) &:= \left\{\mathbf{u} \in \mathbf{V}^h(\Omega) / \forall\, K \in \mathcal{T}_h(S), \mu^{-1} \boldsymbol{\nabla} \times \mathbf{u}_K \in H^1(P_K)^3\right\} \subset \mathbf{V}^h(\Omega). \end{aligned} \tag{61}$$

Vector fields $\mathbf{u}$ in $\mathbf{V}^h(\Omega)$ have L^2 traces on the interfaces Σ_{KL}. More precisely, on any interface Σ_{KL}, we can define the two traces

$$\mathbf{u}_K|_{\Sigma_{KL}} \in L^2(\Sigma_{KL}), \quad \mathbf{u}_L|_{\Sigma_{KL}} \in L^2(\Sigma_{KL}). \tag{62}$$

In the following, when it will be clear from the context that we look at traces on Σ_{KL}, we shall often simply write $\mathbf{u}_K$ for the first trace and $\mathbf{u}_L$ for the second one. With this convention we shall define respectively the tangential jump and mean trace of $\mathbf{u}$ on Σ_{KL} by

$$[\mathbf{n} \times \mathbf{u}]_{KL} = \mathbf{n}_K \times \mathbf{u}_K + \mathbf{n}_L \times \mathbf{u}_L, \quad \{\mathbf{u}\}_{KL} = (\mathbf{u}_K + \mathbf{u}_L)/2, \tag{63}$$

where $\mathbf{n}_K$ (resp. $\mathbf{n}_L$) is the unit normal vector along Σ_{KL}, outgoing with respect to P_K (resp. P_L). The reader will observe that the above quantities are invariant under the permutation of K and L. While it is clear that the second quantity in (63) is a mean value, the fact that the first one is called a jump is due to the fact that since $\mathbf{n}_K = -\mathbf{n}_L$, we also have

$$[\mathbf{n} \times \mathbf{u}]_{KL} = \mathbf{n}_K \times (\mathbf{u}_K - \mathbf{u}_L) \equiv \mathbf{n}_L \times (\mathbf{u}_L - \mathbf{u}_K)\,. \tag{64}$$

Note that $\mathbf{V}^h(\Omega)$ is not a subspace of $H(\mathbf{rot}, \Omega)$ since no continuity is imposed through the interfaces Σ_{KL}. Moreover we have the following characterization of functions $\mathbf{u}$ in $\mathbf{V}^h(\Omega) \cap H(\mathbf{rot}, \Omega)$,

$$\forall\, \Sigma_{KL}, \quad [\mathbf{n} \times \mathbf{u}]_{KL} = 0, \quad i.e., \quad \mathbf{n}_K \times \mathbf{u}_K + \mathbf{n}_L \times \mathbf{u}_L = 0. \tag{65}$$

Of course, when $\mathbf{u} \in \mathbf{R}^h(\Omega)$, all traces appearing with $\mu^{-1}\boldsymbol{\nabla} \times \mathbf{u}$ instead of $\mathbf{u}$ make sense in $L^2(\Sigma_{KL})$.

To write the DG formulation associated to the mesh $\mathcal{T}_h(\Omega)$, we assume that the solution $\mathbf{E}$ of (6) has sufficient regularity inside each element P_K, namely

$$\mathbf{E}(\cdot, t) \in C^2(\mathbb{R}^+; \mathbf{R}^h(\Omega)). \tag{66}$$

In this case it is easy to show that $\mathbf{E}$ satisfies the following variational problem (the so-called IPDG formulation): for any $\mathbf{F} \in \mathbf{R}^h(\Omega)$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{m}_\Omega(\mathbf{E},\mathbf{F}) + \boldsymbol{k}_{\Omega,h}(\mathbf{E},\mathbf{F}) + \boldsymbol{\sigma}_{\Sigma,h}(\mathbf{E},\mathbf{F}) + \boldsymbol{j}_{\Sigma,h}(\mathbf{E},\mathbf{F}) = 0, \tag{67}$$

where the various symmetric bilinear forms are defined as follows:

– The mass bilinear form and the stiffness $\boldsymbol{k}_{\Omega,h}(\mathbf{E},\mathbf{F})$ bilinear form in $\mathbf{V}^h(\Omega)$ are given by

$$\boldsymbol{m}_\Omega(\mathbf{E},\mathbf{F}) := \int_\Omega \varepsilon\mathbf{E}\cdot\mathbf{F}\,\mathrm{d}\mathbf{x}, \quad \text{and} \quad \boldsymbol{k}_{\Omega,h}(\mathbf{E},\mathbf{F}) := \int_\Omega \mu^{-1}\boldsymbol{\nabla}_h\times\mathbf{E}\cdot\boldsymbol{\nabla}_h\times\mathbf{F}\,\mathrm{d}\mathbf{x}, \tag{68}$$

where the broken curl, $\boldsymbol{\nabla}_h \times \mathbf{E}$, is classically defined in $\mathbf{V}^h(\Omega)$:

$$\forall\, K \in \mathcal{T}_h(S), \quad (\boldsymbol{\nabla}_h \times \mathbf{E})|_{P_K} = \boldsymbol{\nabla} \times \mathbf{E}_K; \tag{69}$$

– The coupling bilinear form, $\boldsymbol{\sigma}_{\Sigma,h}(\mathbf{E},\mathbf{F})$ in $\mathbf{R}^h(\Omega)$, supported by $\Sigma_h \cup \Gamma$, is given by

$$\boldsymbol{\sigma}_{\Sigma,h}(\mathbf{E},\mathbf{F}) := -\int_{\Sigma_h\cup\Gamma} \left(\left\{\mu^{-1}\boldsymbol{\nabla}_h\times\mathbf{E}\right\}_h\cdot[\mathbf{n}\times\mathbf{F}]_h + \left\{\mu^{-1}\boldsymbol{\nabla}_h\times\mathbf{F}\right\}_h\cdot[\mathbf{n}\times\mathbf{E}]_h\right)\mathrm{d}\sigma; \tag{70}$$

– The jump bilinear form $\boldsymbol{j}_{\Sigma,h}(\mathbf{E},\mathbf{F})$ in $\mathbf{R}^h(\Omega)$, supported by $\Sigma_h \cup \Gamma$, is given by

$$\boldsymbol{j}_{\Sigma,h}(\mathbf{E},\mathbf{F}) := \int_{\Sigma_h\cup\Gamma} \alpha_h[\mathbf{n}\times\mathbf{E}]_h\cdot[\mathbf{n}\times\mathbf{F}]_h\,\mathrm{d}\sigma, \tag{71}$$

where $\alpha_h \in L^\infty(\Sigma_h \cup \Gamma)$ is a positive weight function (at this stage arbitrary).

Note that, in (70) and (71), we used the following notations: for functions $\left\{\mu^{-1}\boldsymbol{\nabla}_h \times \mathbf{E}\right\}_h : \Sigma_h \to \mathbb{R}$ and $[\mathbf{n}\times\mathbf{F}]_h : \Sigma_h \to \mathbb{R}$, we set

$$\left\{\mu^{-1}\boldsymbol{\nabla}_h\times\mathbf{E}\right\}_h := \{\mu^{-1}\boldsymbol{\nabla}\times\mathbf{E}\}_{KL},\ [\mathbf{n}\times\mathbf{F}]_h = [\mathbf{n}\times\mathbf{F}]_{KL}\,,\ \text{on } \Sigma_{KL}, \tag{72}$$

and for any $f_h : \Sigma_h \to \mathbb{R}$,

$$\int_{\Sigma_h} f_h\,\mathrm{d}\sigma := \sum_{\Sigma_{KL}\in\Sigma_h}\int_{\Sigma_{KL}} f_{KL}\,\mathrm{d}\sigma \quad \text{with } f_{KL} := f_h|_{\Sigma_{KL}}. \tag{73}$$

Coming back to the definition of the bilinear forms, it is worth emphasizing that:

– The bilinear forms $\boldsymbol{m}_\Omega$ and $\boldsymbol{k}_{\Omega,h}$ do not couple mesh elements between themselves and are there to ensure Maxwell's equations inside each element;
– The coupling bilinear form $\boldsymbol{\sigma}_{\Sigma,h}$ ensures the consistency with Maxwell's equations across elements;
– The importance of the jump bilinear form $\boldsymbol{j}_{\Sigma,h}$ will be clear after space discretization; note that $\boldsymbol{j}_{\Sigma,h}(\mathbf{E},\mathbf{F})$ vanishes for the exact solution $\mathbf{E}$.

Remark 3.2. After space discretization, the function α_h will then be chosen in order to ensure a certain discrete coercivity inequality (see Section 4.1.2) which will imply the well-posedness and stability of the discrete problem (see Section 4.1.2 and more precisely Lemma 4.2).

3.4. Anisotropic IPDG formulation in the reference cylindrical cable

As indicated in Section 3.1, this formulation is obtained by applying the change of variable $\mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}})$ to the formulation (67) (Section 3.4.3). To do this, we first need to describe how the tangential jumps are transformed (Section 3.4.2), for which we shall use preliminary technical results (Section 3.4.1).

3.4.1. Basic linear algebra results

We shall denote by $\mathbf{u} \otimes \mathbf{v}$ the vector product in 2D as the bilinear form:

$$\forall\,(\mathbf{u},\mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad \mathbf{u} \otimes \mathbf{v} = \det(\mathbf{u},\mathbf{v}) \equiv u_1 v_2 - u_2 v_1, \tag{74}$$

which is linked to the 3D vector product via the following formulae:

$$\begin{cases} \forall\,(\mathbf{u},\mathbf{v}) \in \mathbb{R}^2 \times \mathbb{R}^2, & (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{e}_3 = \mathbf{u}_T \otimes \mathbf{v}_T, \qquad \text{(i)} \\ \mathbf{u} \cdot \mathbf{e}_3 = \mathbf{v} \cdot \mathbf{e}_3 = 0 \quad \Longrightarrow & \mathbf{u} \times \mathbf{v} = (\mathbf{u}_T \otimes \mathbf{v}_T)\, \mathbf{e}_3. \quad \text{(ii)} \end{cases} \tag{75}$$

We shall use later the following two elementary results in linear algebra, for which we provide a short proof to make it easy for the reader. The first result is specific to dimension 2.

Lemma 3.1. *For any* 2×2 *matrix* $\mathbf{A}$*, for any* $(\mathbf{u},\mathbf{v}) \in \mathbb{R}^2$*,* $\mathbf{A}\mathbf{u} \otimes \mathbf{A}\mathbf{v} = (\det\mathbf{A})(\mathbf{u} \otimes \mathbf{v})$*.*

Proof. By continuity and density, it suffices to prove the result for diagonalizable $\mathbf{A}$. As the map $(\mathbf{u},\mathbf{v}) \mapsto \mathbf{A}\mathbf{u} \otimes \mathbf{A}\mathbf{v}$ is an alternating bilinear form on $\mathbb{R}^2$, it is proportional to the determinant $\mathbf{u} \otimes \mathbf{v}$, *i.e.* $\mathbf{A}\mathbf{u} \otimes \mathbf{A}\mathbf{v} = \alpha\, \mathbf{u} \otimes \mathbf{v}$. For $\mathbf{u} = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_2$, where $(\mathbf{e}_1, \mathbf{e}_2)$ is an eigenbasis of $\mathbf{A}$ associated to eigenvalues (λ_1, λ_2), $\mathbf{A}\mathbf{e}_1 \otimes \mathbf{A}\mathbf{e}_2 = \lambda_1 \lambda_2 \mathbf{e}_1 \otimes \mathbf{e}_2 = (\det\mathbf{A})\mathbf{e}_1 \otimes \mathbf{e}_2$, which proves $\alpha = \det\mathbf{A}$. □

The second result is known as the Sherman–Morrison formula:

Lemma 3.2. *Let* $\mathbf{A}_0$ *and* $\mathbf{A}$ *be two* $N \times N$ *matrices such that there exist* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ *satisfying*

$$\forall\, \mathbf{u} \in \mathbb{R}^N, \quad \mathbf{A}\mathbf{u} = \mathbf{A}_0 \mathbf{u} + (\mathbf{b} \cdot \mathbf{u})\mathbf{a}.$$

If $\mathbf{A}_0$ *is invertible and* $\mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{a} \neq -1$*,* $\mathbf{A}$ *is invertible and*

$$\forall\, \mathbf{v} \in \mathbb{R}^N, \quad \mathbf{A}^{-1}\mathbf{v} = \mathbf{A}_0^{-1}\mathbf{v} - (1 + \mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{a})^{-1}(\mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{v})\mathbf{A}_0^{-1}\mathbf{a}. \tag{76}$$

Proof. If $\mathbf{v} := \mathbf{A}\mathbf{u}$, $\mathbf{A}_0\mathbf{u} = \mathbf{v} - (\mathbf{b} \cdot \mathbf{u})\mathbf{a}$, then $\mathbf{u} = \mathbf{A}_0^{-1}\mathbf{v} - (\mathbf{b} \cdot \mathbf{u})\mathbf{A}_0^{-1}\mathbf{a}$. Therefore

$$\mathbf{b} \cdot \mathbf{u} = \mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{v} - (\mathbf{b} \cdot \mathbf{u})\left(\mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{a}\right) \text{ which yields } \mathbf{b} \cdot \mathbf{u} = \left(1 + \mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{a}\right)^{-1} \left(\mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{v}\right).$$

This shows that the map $\mathbf{u} \mapsto \mathbf{A}\mathbf{u}$ is injective (if $\mathbf{v} = 0$ then $\mathbf{u} = 0$); hence $\mathbf{A}$ is invertible since $N < +\infty$, which leads to the conclusion since $\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$. □

3.4.2. Transformation of tangential jumps and traces

Obviously, as $\mathcal{F}_T \in C^2(\mathbb{R}^3)$, if $\mathbf{E}$ and $\widehat{\mathbf{E}}$ are related by the anisotropic Piola transform (22), one has

$$\mathbf{E} \in \mathbf{V}^h(\Omega) \iff \widehat{\mathbf{E}} \in \mathbf{V}^h(\widehat{\Omega}), \quad \mathbf{E} \in \mathbf{R}^h(\Omega) \iff \widehat{\mathbf{E}} \in \mathbf{R}^h(\widehat{\Omega}),$$

where $\mathbf{R}_h(\widehat{\Omega})$ and $\mathbf{V}^h(\widehat{\Omega})$ are defined similarly to $\mathbf{R}^h(\Omega)$ and $\mathbf{V}^h(\Omega)$ in Section 3.3, as follows:

$$\begin{aligned}
&\mathbf{V}^h(\widehat{\Omega}) := \{\mathbf{u} \in L^2(\widehat{\Omega})^3 / \forall\, K \in \mathcal{T}_h(\widehat{S}),\ \mathbf{u}_K \in H^1(\widehat{P}_K)^3\} \subset L^2(\widehat{\Omega})^3,\\
&\mathbf{R}^h(\widehat{\Omega}) := \{\mathbf{u} \in \mathbf{V}^h(\widehat{\Omega}) / \forall\, K \in \mathcal{T}_h(\widehat{S}),\ \widehat{\mu}^{-1}(J_T^{-1}\mathbf{D}(\widehat{\boldsymbol{\nabla}} \times \mathbf{u}_K) - \mathbf{P}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\mathbf{u}_K) \in H^1(\widehat{P}_K)^3\}.
\end{aligned} \tag{77}$$

For $\widehat{\mathbf{x}} \in \partial\widehat{P}_K$, $\mathbf{x} = F(\widehat{\mathbf{x}}) \in \partial P_K$, and $\widehat{\mathbf{n}}_K \equiv \widehat{\mathbf{n}}_K(\widehat{\mathbf{x}})$, the outgoing (w.r.t. $\widehat{P}_K$) unit normal vector to $\partial\widehat{P}_K$ at $\widehat{\mathbf{x}}$, is related to $\mathbf{n}_K \equiv \mathbf{n}_K(\mathbf{x})$; the outgoing (w.r.t. P_K) unit normal vector to ∂P_K at $\mathbf{x}$, by the formulas (see for instance [26], page 79)

$$\mathbf{n}_K = \left|D\mathcal{F}(\widehat{\mathbf{x}})^{-*}\widehat{\mathbf{n}}_K\right|^{-1} D\mathcal{F}(\widehat{\mathbf{x}})^{-*}\widehat{\mathbf{n}}_K. \tag{78}$$

As a consequence of the cylindrical nature of the mesh $\mathcal{T}_h(\widehat{\Omega})$, we have the important property

$$\forall\, K \in \mathcal{T}(\widehat{S}), \quad \widehat{\mathbf{n}}_K \cdot \mathbf{e}_3 = 0, \quad i.e.\ \widehat{\mathbf{n}}_K = \left(\widehat{\mathbf{n}}_{K,T}^t, 0\right)^t, \tag{79}$$

so that the tangential jump of $\widehat{\mathbf{E}} \in \mathbf{V}^h$ across $\widehat{\Sigma}_{KL}$, defined similarly to $[\mathbf{n} \times \mathbf{u}]_{KL}$ in (64) up to obvious modifications, satisfies

$$[\widehat{\mathbf{n}} \times \widehat{\mathbf{E}}]_{KL} = [\widehat{E}_3]_{KL}\ (\widehat{\mathbf{n}}_K \times \mathbf{e}_3 + ([\widehat{\mathbf{n}}_{K,T} \otimes \widehat{\mathbf{E}}_T]_{KL})\mathbf{e}_3, \tag{80}$$

where, for any field $\hat{\boldsymbol{f}} : \widehat{\Omega}_h \to \mathbb{R}^d$, $[\hat{\boldsymbol{f}}]_{KL} = \hat{\boldsymbol{f}}_K - \hat{\boldsymbol{f}}_L$. The next result concerns the transformation of the tangential jumps by the anisotropic Piola transform.

Lemma 3.3 (Transformation of tangential jumps)**.** *Given $(\mathbf{E}, \widehat{\mathbf{E}}) \in \mathbf{V}^h(\Omega) \times \mathbf{V}^h(\widehat{\Omega})$ related by (22),*

$$[\mathbf{n} \times \mathbf{E}]_{KL} = [\![\widehat{\mathbf{E}}]\!]_{KL} := J_T^{-1}\mathbf{D}[\widehat{\mathbf{n}} \times \widehat{\mathbf{E}}]_{KL} + [\![\widehat{\mathbf{E}}]\!]_{KL}^\delta, \tag{81}$$

where we have defined

$$[\![\widehat{\mathbf{E}}]\!]_{KL}^\delta := -\mathbf{e}_3 \times \left|D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K\right|^{-1}(\mathbf{b}_T^\delta \cdot \widehat{\mathbf{n}}_K)\mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL}. \tag{82}$$

Remark 3.3 (Abuse of notation)**.** The equality (81) has to be understood up to an abuse of notation: the left-hand side is expressed at $\mathbf{x} \in \Sigma_{KL}$, while the right-hand side is expressed at $\widehat{\mathbf{x}} \in \widehat{\Sigma}_{KL}$, with $\mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}})$. The same observation is applied in the proof of the lemma.

Proof. In what follows, we set $[\mathbf{E}]_{KL} = E_K - E_L$ on Σ_{KL} and $[\widehat{\mathbf{E}}]_{KL} = \widehat{\mathbf{E}}_K - \widehat{\mathbf{E}}_L$ on Σ_{KL} so that, according to the Piola transform (23) and Remark 3.3, $[\mathbf{E}]_{KL} = \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL}$.

Let us compute the tangential jump $[\mathbf{n} \times \mathbf{E}]_{KL}$ using (78). This gives

$$[\mathbf{n} \times \mathbf{E}]_{KL} = \mathbf{n}_K \times [\mathbf{E}]_{KL} = \left|D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K\right|^{-1} D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K \times \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL}. \tag{83}$$

We next compute $D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K$. For this we remark that

$$D\mathcal{F} = \begin{pmatrix} D_T\mathcal{F}_T & \mathbf{v}_T^\delta \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad D\mathcal{F}^* = \begin{pmatrix} D_T\mathcal{F}_T^* & 0 \\ \mathbf{v}_T^{\delta,*} & 1 \end{pmatrix} = \mathbf{D}^* + \begin{pmatrix} 0 & 0 \\ \mathbf{v}_T^{\delta,*} & 0 \end{pmatrix}.$$

We can thus apply Lemma 3.2 with $N = 3$, $\mathbf{A}_0 = \mathbf{D}^*$, $\mathbf{A} = D\mathcal{F}^*$, $\mathbf{a} = \mathbf{e}_3$, and $\mathbf{b} = (\mathbf{v}_T^{\delta,*}, 0)^t$. In particular

$$\mathbf{A}_0^{-1}\mathbf{a} = \mathbf{D}^{-*}\mathbf{e}_3 = \mathbf{e}_3, \text{ which implies } \mathbf{b} \cdot \mathbf{A}_0^{-1}\mathbf{a} = 0.$$

Applying (76), we get $D\mathcal{F}^{-*}\mathbf{w} = \mathbf{D}^{-*}\mathbf{w} - (\mathbf{v}_T^\delta \cdot \mathbf{D}_T^{-*}\mathbf{w}_T)\mathbf{e}_3$ for any $\mathbf{w} = (\mathbf{w}_T, w_3)^t$ in $\mathbb{R}^3$. In particular

$$D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K = \mathbf{D}^{-*}\widehat{\mathbf{n}}_K - (\mathbf{v}_T^\delta \cdot \mathbf{D}_T^{-*}\widehat{\mathbf{n}}_{K,T})\mathbf{e}_3 = \mathbf{D}^{-*}\widehat{\mathbf{n}}_K - (\mathbf{b}_T^\delta \cdot \widehat{\mathbf{n}}_{K,T})\mathbf{e}_3. \tag{84}$$

We deduce from (83) and (84) using the definition (82) of $[\![\widehat{\mathbf{E}}]\!]_{KL}^\delta$ that

$$\left| \begin{aligned} [\mathbf{n}\times\mathbf{E}]_{KL} &= |D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K|^{-1}\left[(\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL}) - (\mathbf{b}_T^\delta \cdot \widehat{\mathbf{n}}_{K,T})\,\mathbf{e}_3 \times \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL}\right] \\ &= |D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K|^{-1}\,(\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL}) + [\![\widehat{\mathbf{E}}]\!]_{KL}^\delta. \end{aligned}\right. \tag{85}$$

Next, writing $\mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL} = (\mathbf{D}_T^{-*}[\widehat{\mathbf{E}}_T]_{KL}, 0)^t + [\widehat{E}_3]_{KL}\,\mathbf{e}_3$, we obtain

$$\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL} = \mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times (\mathbf{D}_T^{-*}[\widehat{\mathbf{E}}_T]_{KL}, 0)^t + [\widehat{E}_3]_{KL}\,\left(\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times \mathbf{e}_3\right). \tag{86}$$

However, as $\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \cdot \mathbf{e}_3 = 0$, using (75)(ii) and Lemma 3.1,

$$\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times (\mathbf{D}_T^{-*}[\widehat{\mathbf{E}}_T]_{KL}, 0)^t = (\mathbf{D}_T^{-*}\widehat{\mathbf{n}}_{K,T} \otimes \mathbf{D}_T^{-*}[\widehat{\mathbf{E}}_T]_{KL})\,\mathbf{e}_3 = J_T^{-1}(\widehat{\mathbf{n}}_{K,T} \otimes [\widehat{\mathbf{E}}_T]_{KL})\mathbf{e}_3.$$

By Lemma 2.3, we have $\mathbf{D}_T^{-*}\mathbf{u}_T \times \mathbf{e}_3 = J_T^{-1}\mathbf{D}_T(\mathbf{u}_T \times \mathbf{e}_3)$, for any $\mathbf{u}_T \in \mathbb{R}^2$; we deduce from (86) that

$$\mathbf{D}^{-*}\widehat{\mathbf{n}}_K \times \mathbf{D}^{-*}[\widehat{\mathbf{E}}]_{KL} = J_T^{-1}(\widehat{\mathbf{n}}_{K,T} \otimes [\widehat{\mathbf{E}}_T]_{KL})\mathbf{e}_3 + J_T^{-1}\mathbf{D}_T(\widehat{\mathbf{n}}_{K,T} \times \mathbf{e}_3)[\widehat{E}_3]_{KL},$$

which, after substitution into (85) and using the block decomposition of $\mathbf{D}$, leads to the conclusion. □

Remark 3.4 (Abuse of notation)**.** From its definition (82), $[\![\widehat{\mathbf{E}}]\!]_{KL}^\delta$ is transverse, *i.e.*, orthogonal to $\mathbf{e}_3$. In the sequel we shall use the same notation for the 3D field and its 2D transverse component.

For the sequel, one important consequence of Lemma 3.3 is the corollary below.

Corollary 3.1. *When* $\widehat{\mathbf{E}} \in \mathbf{V}^h(\widehat{\Omega}) \cap H(\mathbf{rot};\widehat{\Omega})$, $[\mathbf{n}\times\mathbf{E}]_{KL} = [\![\widehat{\mathbf{E}}]\!]_{KL}^\delta$.

Finally we state (without proofs) the results that correspond to Lemma 3.3 and Corollary 3.1 for traces instead of jumps.

Lemma 3.4 (Transformation of tangential traces)**.** *Given* $(\mathbf{E}, \widehat{\mathbf{E}}) \in \mathbf{V}^h(\Omega) \times \mathbf{V}^h(\widehat{\Omega})$ *related by* (22)*, one has the formula*

$$(\mathbf{n}\times\mathbf{E})|_{\Gamma_K} = [\![\widehat{\mathbf{E}}]\!]_K := J_T^{-1}\mathbf{D}(\widehat{\mathbf{n}}\times\widehat{\mathbf{E}})|_{\widehat{\Gamma}_K} + \widehat{\mathbf{E}}_K^\delta, \tag{87}$$

where we have defined

$$\widehat{\mathbf{E}}_K^\delta := -\mathbf{e}_3 \times |D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K|^{-1}(\mathbf{b}_T^\delta \cdot \widehat{\mathbf{n}}_K)\mathbf{D}^{-*}\widehat{\mathbf{E}}|_{\widehat{\Gamma}_K}. \tag{88}$$

Corollary 3.2. *When* $\widehat{\mathbf{E}} \in \mathbf{V}^h(\widehat{\Omega}) \cap H(\mathbf{rot};\widehat{\Omega})$ *and* $\widehat{\mathbf{n}}\times\widehat{\mathbf{E}} = 0$ *on* Γ, $(\mathbf{n}\times\mathbf{E})|_{\Gamma_K} = \widehat{\mathbf{E}}_K^\delta$.

We can regroup the results of Lemmas 3.3 and 3.4. For this we use the notation: for any vector field $\hat{\boldsymbol{f}} : \widehat{\Omega}_h \to \mathbb{R}^3$, $[\hat{\boldsymbol{f}}]_h = [\hat{\boldsymbol{f}}]_{KL}$ on Σ_h and $[\hat{\boldsymbol{f}}]_h = \hat{\boldsymbol{f}}|_\Gamma$ on Γ.

Theorem 3.1. *Given* $(\mathbf{E}, \widehat{\mathbf{E}}) \in \mathbf{V}^h(\Omega) \times \mathbf{V}^h(\widehat{\Omega})$ *related by* (22)*, one has, using again Remark 3.3,*

$$[\mathbf{n}\times\mathbf{E}]_h = [\![\widehat{\mathbf{E}}]\!]_h := J_T^{-1}\mathbf{D}[\widehat{\mathbf{n}}\times\widehat{\mathbf{E}}]_h + [\![\widehat{\mathbf{E}}]\!]_h^\delta, \tag{89}$$

where we have defined

$$[\![\widehat{\mathbf{E}}]\!]_h^\delta := -\mathbf{e}_3 \times |D\mathcal{F}^{-*}\widehat{\mathbf{n}}_K|^{-1}(\mathbf{b}_T^\delta \cdot \widehat{\mathbf{n}}_K)\mathbf{D}^{-*}[\widehat{\mathbf{E}}]_h. \tag{90}$$

3.4.3. The DG formulation in the reference domain

Theorem 3.2 (Weak formulation for the problem in $\widehat{\mathbf{E}}$). *If $\mathbf{E}(\cdot,t) \in C^2(\mathbb{R}^+;\mathbf{R}^h(\Omega))$ is a solution of (67), $\widehat{\mathbf{E}}(\cdot,t) \in C^2(\mathbb{R}^+;\mathbf{R}^h(\widehat{\Omega}))$ given by (22) satisfies, for all $\widehat{\mathbf{F}} \in \mathbf{R}^h(\widehat{\Omega})$,*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) + \boldsymbol{k}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) + \boldsymbol{\sigma}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) + \boldsymbol{j}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) = 0, \tag{91}$$

where the various bilinear forms are defined as follows:

- *The bilinear form $\boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}})$ is defined by (48);*
- *Defining the broken operators $\widehat{\boldsymbol{\nabla}}_h\times$ and $\mathbf{P}_h^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)$ by*

$$\forall\, K \in \mathcal{T}_h(\widehat{S}), \quad (\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}})|_{\widehat{P}_K} = \widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}_K, \quad \mathbf{P}_h^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}|_{\widehat{P}_K} = \mathbf{P}^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}_K, \tag{92}$$

  the broken stiffness bilinear form $\boldsymbol{k}_h(\cdot,\cdot)$ is given by

$$\boldsymbol{k}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) := \int_{\widehat{\Omega}} \widehat{\mu}^{-1}|J_T|(J_T^{-1}\mathbf{D}(\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}}) - \mathbf{P}_h^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}) \cdot (J_T^{-1}\mathbf{D}(\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}}) - \mathbf{P}_h^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{F}})\mathrm{d}\widehat{\mathbf{x}}; \tag{93}$$

- *The coupling bilinear form $\boldsymbol{\sigma}_h$ is given by*

$$\begin{aligned}\boldsymbol{\sigma}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}) = &-\int_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]_h \cdot \{\widehat{\mu}^{-1}(J_T^{-1}\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}} - \mathbf{P}_h^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{F}})\}_h \,\mathrm{d}\sigma \\ &- \int_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{F}}]\!]_h \cdot \{\widehat{\mu}^{-1}(J_T^{-1}\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}} - \mathbf{P}_h^\delta(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}})\}_h \,\mathrm{d}\sigma;\end{aligned} \tag{94}$$

- *The penalization bilinear form $\boldsymbol{j}_h$ is given by*

$$\boldsymbol{j}_h\left(\widehat{\mathbf{E}},\widehat{\mathbf{F}}\right) = \int_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} \widehat{\alpha}_h [\![\widehat{\mathbf{E}}]\!]_h \cdot [\![\widehat{\mathbf{F}}]\!]_h \,\mathrm{d}\sigma. \tag{95}$$

In (94) and (95), $[\![\cdot]\!]_h$ is defined as in (89) and (90) and in all the surface integrals $\mathrm{d}\sigma = |J_T||(D\mathcal{F}^*)^{-1}\widehat{\mathbf{n}}|\,\mathrm{d}\widehat{\sigma}$.

Proof. It suffices to rewrite (67) after recomputing $\boldsymbol{m}_\Omega(\mathbf{E},\mathbf{F}), \boldsymbol{k}_{\Omega,h}(\mathbf{E},\mathbf{F})$, and $\boldsymbol{\sigma}_{\Sigma,h}(\mathbf{E},\mathbf{F}), \mathbf{j}_{\Sigma,h}(\mathbf{E},\mathbf{F})$ from their original expressions (68), (70), and (71) in the function of $(\widehat{\mathbf{E}},\widehat{\mathbf{F}})$, provided that $(\mathbf{E},\mathbf{F})$ are deduced from $(\widehat{\mathbf{E}},\widehat{\mathbf{F}})$ via the anisotropic Piola transform (23).

First, using the change of variable $\mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}})$ in integrals over K and Theorem 2.1 for the transformation of the curl operator, we obtain, since $\mathrm{d}\mathbf{x} = |J_T|\,\mathrm{d}\widehat{\mathbf{x}}$,

$$\boldsymbol{m}_\Omega(\mathbf{E},\mathbf{F}) = \boldsymbol{m}(\widehat{\mathbf{E}},\widehat{\mathbf{F}}), \quad \boldsymbol{k}_{\Omega,h}(\mathbf{E},\mathbf{F}) = \boldsymbol{k}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}). \tag{96}$$

We do the same for the integrals over Σ_{KL} and Γ_K for $\boldsymbol{\sigma}_{\Sigma,h}(\mathbf{E},\mathbf{F})$ and $\mathbf{j}_{\Sigma,h}(\mathbf{E},\mathbf{F})$ using Lemma 3.4 and taking into account that the change of variable $\mathbf{x} = \mathcal{F}(\widehat{\mathbf{x}})$ induces a transformation of the surface measure according to $\mathrm{d}\sigma = |J_T||(D\mathcal{F}^*)^{-1}\widehat{\mathbf{n}}|\,\mathrm{d}\widehat{\sigma}$ (see [26], page 80). We show that

$$\boldsymbol{\sigma}_{\Sigma,h}(\mathbf{E},\mathbf{F}) = \boldsymbol{\sigma}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}), \quad \boldsymbol{j}_{\Sigma,h}(\mathbf{E},\mathbf{F}) = \boldsymbol{j}_h(\widehat{\mathbf{E}},\widehat{\mathbf{F}}). \tag{97}$$

This concludes the proof. ☐

4. Space discretization

4.1. DG semi-discretization in the transverse variables

4.1.1. Description of the semi-discrete problem

As usual, it consists in replacing in the weak formulation (91), the space $\mathbf{R}^h(\widehat{\Omega})$ by a finite-element-like subspace (due to "finite dimensional dependence" in the transverse variables), namely

$$\mathbf{V}_h(\widehat{\Omega}) = \{\widehat{\mathbf{E}}_h = (\widehat{\mathbf{E}}_T, \widehat{E}_3) \in H^1(\mathbb{R}, \mathbf{V}_h(\widehat{S})) \times L^2(\mathbb{R}, \mathrm{V}_h(\widehat{S}))\}, \tag{98}$$

where $\mathbf{V}_h(\widehat{\Omega})$ is constructed as a subspace of $H(\widehat{\mathbf{rot}}, \widehat{\Omega})$, *i.e.*

$$\mathbf{V}_h(\widehat{\Omega}) \subset H_0(\widehat{\mathbf{rot}}, \widehat{\Omega}) \cap \mathbf{R}^h(\widehat{\Omega}), \tag{99}$$

by choosing $\mathbf{V}_h(\widehat{S})$ and $V_h(\widehat{S})$ as follows:

$$\begin{cases} \mathbf{V}_h(\widehat{S}) := \{\widehat{\mathbf{E}}_T \in H_0(\widehat{\mathbf{rot}}_T, \widehat{S})^2 / \forall K \in \mathcal{T}_h(\widehat{S}), \widehat{\mathbf{E}}_T|_K \in \mathcal{N}_{2D}\}, \\ \mathrm{V}_h(\widehat{S}) := \{\widehat{E}_3 \in H_0^1(\widehat{S}) / \forall K \in \mathcal{T}_h(\widehat{S}),\ \widehat{E}_3|_K \in \mathbb{P}_1\}, \end{cases} \tag{100}$$

where we recall the definition of the 2D Nedelec space of edge elements (see [26, 28]),

$$\mathcal{N}_{2D} := \left\{a(x_2, -x_1)^t + \mathbf{b}, (a, \mathbf{b}) \in \mathbb{R} \times \mathbb{R}^2\right\} \subset \mathbb{P}_1^2. \tag{101}$$

Contrary to what is more classical with DG methods, the space is not made of fields that are completely discontinuous between elements (see Remark 4.2 for further comment).

Our choice (98) for $\mathbf{V}_h(\widehat{\Omega})$ coincides with the conforming discretization subspace used in our previous article [1] for the case of cylindrical cables, thus keeping the continuity of the tangential traces.

Also, since $\widehat{\mathbf{E}} \times \widehat{\mathbf{n}} = 0$ on $\widehat{\Gamma}$ in $\mathbf{V}_h(\widehat{\Omega})$, the exact boundary condition is taken into account strongly when the cable is cylindrical since $\widehat{\mathbf{n}} = \mathbf{n}$. These two properties will ensure that the method proposed in this paper will be an extension of the method in [1], see Remark 4.1.

Moreover, when writing the weak formulation (91) in $\mathbf{V}_h(\widehat{\Omega})$ instead of $\mathbf{R}^h(\widehat{\Omega})$, the main consequence is a simplification of the bilinear forms $\boldsymbol{\sigma}_h(\cdot,\cdot)$ and $\boldsymbol{j}_h(\cdot,\cdot)$, due to the inclusion (99) which allows us to use Corollaries 3.1 and 3.2.

For instance, for $(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) \in \mathbf{V}_h(\widehat{\Omega})$, by Corollary 3.1, $[\![\widehat{\mathbf{E}}_h]\!]_{KL} = [\![\widehat{\mathbf{E}}_h]\!]^\delta_{KL}$ so that

$$\left|\begin{aligned} &[\![\widehat{\mathbf{E}}_h]\!]_{KL} \cdot \{\widehat{\mu}^{-1}(J_T^{-1}\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{F}}_h - \mathbf{P}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_h)\}_{KL} \\ &\qquad = [\![\widehat{\mathbf{E}}_h]\!]^\delta_{KL} \cdot \{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{F}}_h)_T - \mathbf{P}^\delta_T(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_{h,T})\}_{KL}, \end{aligned}\right.$$

where we also used that $[\![\widehat{\mathbf{E}}_h]\!]^\delta_{KL}$ is transverse (Remark 3.4). As a consequence,

$$\forall\, (\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) \in \mathbf{V}_h(\widehat{\Omega}) \times \mathbf{V}_h(\widehat{\Omega}), \quad \boldsymbol{\sigma}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = \boldsymbol{\sigma}^\delta_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h), \quad \boldsymbol{j}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = \boldsymbol{j}^\delta_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h), \tag{102}$$

where the bilinear forms $\boldsymbol{\sigma}^\delta_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}})$ and $\boldsymbol{j}^\delta_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}})$ are defined in $\mathbf{V}_h(\widehat{\Omega}) \times \mathbf{V}_h(\widehat{\Omega})$ by

$$\left|\begin{aligned} \boldsymbol{\sigma}^\delta_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) \quad &:= -\int_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]^\delta_h \cdot \{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}})_T - \mathbf{P}^\delta_{h,T}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_T)\}_h \,\mathrm{d}\sigma \\ &\quad -\int_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} [\![\widehat{\mathbf{F}}]\!]^\delta_h \cdot \{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}})_T - \mathbf{P}^\delta_{h,T}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_T)\}_h \,\mathrm{d}\sigma, \\ \boldsymbol{j}^\delta_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) &:= \int_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} \widehat{\alpha}_h [\![\widehat{\mathbf{E}}]\!]^\delta_h \cdot [\![\widehat{\mathbf{F}}]\!]^\delta_h \,\mathrm{d}\sigma, \end{aligned}\right. \tag{103}$$

where the notation $(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})_T$ represents the transverse part of $(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})$, and $\mathbf{P}^{\delta}_{h,T}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)$ is defined from $\mathbf{P}^{\delta}_{T}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)$ as $\mathbf{P}^{\delta}_{h}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)$ is defined from $\mathbf{P}^{\delta}_{T}(\widehat{\mathbf{x}}, \widehat{\nabla})$ (see (92)).

The semi-discrete problem is obtained by reducing (91) to the space $\mathbf{V}_h(\widehat{\Omega})$, instead of $\mathbf{R}^h(\widehat{\Omega})$, reads, according to (102): find $\widehat{\mathbf{E}}_h(.,t) : \mathbb{R}^+ \longrightarrow \mathbf{V}_h(\widehat{\Omega})$ such that, for any $\widehat{\mathbf{F}}_h \in \mathbf{V}_h(\widehat{\Omega})$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{m}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{k}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{\sigma}^{\delta}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{j}^{\delta}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = 0. \tag{104}$$

In the sequel, we shall denote by $\boldsymbol{a}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h)$ the total stiffness bilinear form:

$$\boldsymbol{a}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = \boldsymbol{k}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{\sigma}^{\delta}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{j}^{\delta}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h), \tag{105}$$

so that (104) can be written in compact form as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{m}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{a}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = 0, \quad \forall\, \widehat{\mathbf{F}}_h \in \mathbf{V}_h(\widehat{\Omega}). \tag{106}$$

Remark 4.1. The bilinear form $\boldsymbol{k}_h(\cdot,\cdot)$ can obviously be decomposed as

$$\boldsymbol{k}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) = \boldsymbol{k}^0_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \boldsymbol{k}^{\delta}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}), \quad \boldsymbol{k}^0_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) := \sum_K \int_{\widehat{P}_K} \widehat{\mu}^{-1}\mathbf{A}^{-1}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}} \cdot \widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{F}}\, \mathrm{d}\widehat{\mathbf{x}}. \tag{107}$$

According to (93), $\boldsymbol{k}^{\delta}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) \equiv 0$ when $\delta = 0$ (the cylindrical case). In the same way, as $[\![\widehat{\mathbf{E}}]\!]^{\delta}_h = 0$ when $\delta = 0$, cf. (90), $\boldsymbol{\sigma}^{\delta}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) = \boldsymbol{j}^{\delta}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) \equiv 0$ when $\delta = 0$. In that case, (106) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{m}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) + \boldsymbol{k}^0_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = 0, \quad \forall\, \widehat{\mathbf{F}}_h \in \mathbf{V}_h(\widehat{\Omega}), \tag{108}$$

which is nothing but the semi-discrete conforming method of [1].

4.1.2. Discrete coercivity inequality and stability result

Our goal is to obtain a coercivity inequality for the quadratic form

$$\boldsymbol{a}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{E}}_h) = \boldsymbol{k}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{E}}_h) + \boldsymbol{\sigma}^{\delta}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{E}}_h) + \boldsymbol{j}^{\delta}_h(\widehat{\mathbf{E}}_h, \widehat{\mathbf{E}}_h). \tag{109}$$

We follow quite closely what was done by Grote *et al.* in [13], the main difference being that we have to manage variable coefficients and modified curl operators. The proof will rely on the following technical lemma which is a variant of classical discrete trace inequalities in finite element spaces. The additional difficulty with respect to Lemma 4 of [13], for instance, is that we have to deal with infinitely long elements (the prisms P_K) and especially that we need to deal with coefficients which are not necessarily constant inside each element, even when $\widehat{\mu}$ is, because of the presence of the transformation $\mathcal{F}$. For these reasons, usual homogeneity arguments are no longer sufficient and additional technical points need to be dealt with. Although we believe this result holds, a complete proof remains open. This is the reason why we have chosen to state the technical lemma below as a conjecture.

Lemma 4.1 (Conjecture—Discrete trace inequality)**.** *There exists $C_I > 0$ such that, with h_K being defined by (54), for any $K \in \mathcal{T}_h(\widehat{\Omega})$ and $\widehat{\Sigma}_{KL} \subset \partial\widehat{P}_K$, for any $\widehat{\mathbf{F}} \in \mathbf{V}_h(\widehat{\Omega})$,*

$$\left|\begin{aligned} &\int_{\partial\widehat{P}_K} \left|\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{F}}_K)_T - \mathbf{P}^{\delta}_T(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_K)\right|^2 \mathrm{d}\sigma \\ &\qquad \le \frac{C_I^2}{h_K} \int_{\widehat{P}_K} |J_T| \left|\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{F}}_K)_T - \mathbf{P}^{\delta}_T(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_K)\right|^2 \mathrm{d}\widehat{\mathbf{x}}. \end{aligned}\right. \tag{110}$$

As a consequence, we are able to show the following:

Lemma 4.2 (Discrete coercivity inequality)**.** *Let $\nu \in (0,1)$. If the function $\widehat{\alpha}_h$ is chosen large enough so that*

$$\forall K \in \mathcal{T}_h(\widehat{\Omega}), \quad \widehat{\alpha}_h|_{\partial \widehat{P}_K} \geq \frac{C_I^2}{\nu^2 h_K}, \tag{111}$$

one has the discrete coercivity inequality, for all $\widehat{\mathbf{E}} \in \mathbf{V}_h(\widehat{\Omega})$,

$$\boldsymbol{a}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) \geq (1-\nu)(\boldsymbol{k}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) + \boldsymbol{j}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}})). \tag{112}$$

Proof. Since $\boldsymbol{a}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) = \boldsymbol{k}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) + \boldsymbol{j}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) + \boldsymbol{\sigma}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}})$, proving (112) amounts to showing

$$\boldsymbol{\sigma}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) \geq -\nu(\boldsymbol{k}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) + \boldsymbol{j}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}})). \tag{113}$$

However from (103),

$$\boldsymbol{\sigma}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) = -2\int_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}})_T - \mathbf{P}_{h,T}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}})\}_h \,\mathrm{d}\sigma,$$

and using Young's inequality we get,

$$\boldsymbol{\sigma}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) \geq -\nu \boldsymbol{j}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{E}}) - \frac{1}{\nu}\int_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} \widehat{\alpha}_h^{-1} |\{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}})_T - \mathbf{P}_{h,T}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}})\}_h|^2 \,\mathrm{d}\sigma. \tag{114}$$

The second term needs to be bounded from below by $-\nu \boldsymbol{k}_h(\widehat{\mathbf{E}}, \widehat{\mathbf{E}})$ to obtain (113). From the definition of the mean value, we deduce the following local inequality:

$$\left| \begin{aligned} &|\{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}})\}_{KL}|^2 \\ &\quad \leq \tfrac{1}{2}|\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}_K)_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_K)|^2 \\ &\qquad + \tfrac{1}{2}|\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}_L)_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_L)|^2. \end{aligned} \right.$$

After multiplication by $\widehat{\alpha}_h^{-1}$, integration over $\widehat{\Sigma}_{KL}$, and summation on $K \neq L$, we have, denoting by $\mathcal{V}(K)$ the set of elements L in $\mathcal{T}_h$ which are neighbors of K, *i.e.* such that $\widehat{\Sigma}_{KL}$ has a strictly positive measure,

$$\left| \begin{aligned} &\sum_{K \neq L} \int_{\widehat{\Sigma}_{KL}} \widehat{\alpha}_h^{-1} |\{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}})\}_{KL}|^2 \mathrm{d}\sigma \\ &\quad \leq \tfrac{1}{2}\sum_K \sum_{L \in \mathcal{V}(K)} \int_{\widehat{\Sigma}_{KL}} \widehat{\alpha}_h^{-1} |\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}_K)_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_K)|^2 \mathrm{d}\sigma \\ &\qquad + \tfrac{1}{2}\sum_L \sum_{K \in \mathcal{V}(L)} \int_{\widehat{\Sigma}_{KL}} \widehat{\alpha}_h^{-1} |\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}_L)_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_L)|^2 \mathrm{d}\sigma. \end{aligned} \right.$$

By inverting K and L in the second sum, we see that the two terms in the right-hand side of the inequality are the same, and since $\partial \widehat{P}_K = \big(\bigcup_{L \in \mathcal{V}(K)} \widehat{\Sigma}_{KL}\big) \cup \widehat{\Gamma}_K$, we obtain

$$\left| \begin{aligned} &\int_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} \widehat{\alpha}_h^{-1} |\{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}})_T - \mathbf{P}_{h,T}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}})\}_h|^2 \,\mathrm{d}\sigma \\ &\quad \leq \sum_K \int_{\partial \widehat{P}_K} \widehat{\alpha}_h^{-1} |\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}}_K)_T - \mathbf{P}_T^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_K)|^2 \mathrm{d}\sigma. \end{aligned} \right. \tag{115}$$

Then using the trace inequality (110) of Lemma 4.1, we get

$$\left| \begin{aligned} &\int_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} \widehat{\alpha}_h^{-1} \left|\left\{\widehat{\mu}^{-1}\left(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h\times\widehat{\mathbf{E}})_T-\mathbf{P}^{\delta}_{h,T}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}\right)\right\}_h\right|^2 \mathrm{d}\sigma \\ &\quad\le C_I^2\sum_K \frac{\|\widehat{\alpha}_h^{-1}\|_{L^\infty(\partial\widehat{P}_K)}}{h_K}\int_{\widehat{P}_K}|J_T|\left|\widehat{\mu}^{-1}\left(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}\times\widehat{\mathbf{E}}_K)_T-\mathbf{P}^{\delta}_{T}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}}_K\right)\right|^2\mathrm{d}\widehat{\mathbf{x}} \\ &\quad\le C_I^2\left(\sup_K\frac{\|\widehat{\alpha}_h^{-1}\|_{L^\infty(\partial\widehat{P}_K)}}{h_K}\right)\boldsymbol{k}_h(\widehat{\mathbf{E}},\widehat{\mathbf{E}})\le\nu^2\boldsymbol{k}_h(\widehat{\mathbf{E}},\widehat{\mathbf{E}})\quad\text{(thanks to (111))}. \end{aligned}\right. \tag{116}$$

We have obtained

$$-\frac{1}{\nu}\int_{\widehat{\Sigma}_h\cup\widehat{\Gamma}}\widehat{\alpha}_h^{-1}|\{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h\times\widehat{\mathbf{E}})_T-\mathbf{P}^{\delta}_{h,T}(\widehat{\mathbf{x}},\widehat{\nabla}_T)\widehat{\mathbf{E}})\}_h|^2\,\mathrm{d}\sigma\ge-\nu\boldsymbol{k}_h(\widehat{\mathbf{E}},\widehat{\mathbf{E}}),$$

which we can substitute into (114) to conclude the proof. □

To establish the stability of the semi-discrete problem (106), it suffices to recall that the solution of (106) satisfies the discrete energy identity

$$\frac{\mathrm{d}\widehat{\mathcal{E}}_h}{\mathrm{d}t}=0,\quad\text{where } \widehat{\mathcal{E}}_h:=\tfrac{1}{2}\boldsymbol{m}(\widehat{\mathbf{E}}_h,\widehat{\mathbf{E}}_h)+\tfrac{1}{2}\boldsymbol{a}_h(\widehat{\mathbf{E}}_h,\widehat{\mathbf{E}}_h). \tag{117}$$

With the help of Lemma 4.2, the remaining arguments to obtain uniform stability estimates for $\widehat{\mathbf{E}}_h$ under the condition (111) are completely standard and left to the reader.

4.2. Full space discretization

4.2.1. The discrete approximation spaces

We introduce a constant longitudinal space step $\mathrm{h}>0$ in the x_3-direction and decompose the cylindrical cable $\widehat{\Omega}$ into finite cylinders of length h,

$$\widehat{\Omega}=\bigcup_{j\in\mathbb{Z}}\overline{\mathcal{C}_{j+\frac{1}{2}}},\quad\mathcal{C}_{j+\frac{1}{2}}=\{(\widehat{\mathbf{x}}_T,x_3)\in\widehat{\Omega}/j\mathrm{h}\le x_3\le(j+1)\mathrm{h}\}\equiv\widehat{S}\times(j\mathrm{h},(j+1)\mathrm{h}). \tag{118}$$

These cells are separated by transverse cross-section $\widehat{\mathcal{S}}_j$, $j\in\mathbb{Z}$ (see Fig. 4) where, by definition

$$\forall\, j\in\mathbb{Z},\quad\widehat{\mathcal{S}}_j=\{(\mathbf{x}_T,j\mathrm{h}),\mathbf{x}_T\in\widehat{S}\}. \tag{119}$$

According to the transverse mesh $\mathcal{T}_h(\widehat{S})$ introduced in Section 3.2, we thus have

$$\forall\, j\in\mathbb{Z},\quad\overline{\mathcal{C}_{j+\frac{1}{2}}}=\bigcup_{K\in\mathcal{T}_h(\widehat{S})}\overline{P_{K,j+\frac{1}{2}}};\quad\forall\, K\in\mathcal{T}_h(\widehat{S}),\quad P_K=\bigcup_{j\in\mathbb{Z}}\overline{P_{K,j+\frac{1}{2}}},$$

where, by definition, $P_{K,j+\frac{1}{2}}$ is the (finite) prism

$$P_{K,j+\frac{1}{2}}:=K\times(j\mathrm{h},(j+1)\mathrm{h})\,. \tag{120}$$

As in [1], we shall use the letter $\mathbf{h}$ to denote the set of approximation parameters in space, namely

$$\mathbf{h}:=(h,\mathrm{h}). \tag{121}$$

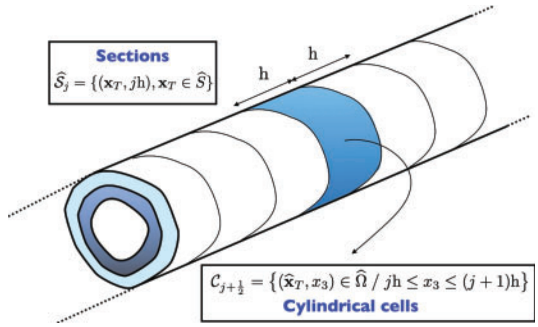

FIGURE 4. Schematic representation of the sections and the cells of the cable $\widehat{\Omega}$ (adapted from [1]).

We thus have built a prismatic mesh $\mathcal{T}_{\mathbf{h}}(\widehat{\Omega})$ of transverse stepsize h and longitudinal stepsize h,

$$\mathcal{T}_{\mathbf{h}}(\widehat{\Omega}) := \bigcup_{j\in\mathbb{Z}} \bigcup_{K\in\mathcal{T}_h(\widehat{S})} P_{K,j+\frac{1}{2}}. \tag{122}$$

The fully discrete space, indexed with $\mathbf{h}$, takes the following form:

$$\mathbf{V}_{\mathbf{h}}(\widehat{\Omega}) = \mathbf{V}_{\mathbf{h},T}(\widehat{\Omega}) \times \mathrm{V}_{\mathbf{h},\ell}(\widehat{\Omega}) \subset \mathbf{V}_h(\widehat{\Omega}),$$

On the one hand, the transverse field $\widehat{\mathbf{E}}_T \in \widehat{\mathbf{V}}_{\mathbf{h},T}$ is searched piecewise linear continuous in x_3 with value in the 2D edge element space $\mathbf{V}_h(\widehat{S})$, see (100),

$$\mathbf{V}_{\mathbf{h},T}(\widehat{\Omega}) := \{\widehat{\mathbf{E}}_T \in C^0(\mathbb{R}; \mathbf{V}_h(\widehat{S}))/\,\forall\, j \in \mathbb{Z},\ \widehat{\mathbf{E}}_T|_{\mathcal{C}_{j+\frac{1}{2}}} \in \mathbb{P}_1(\mathbb{R}; \mathbf{V}_h(\widehat{S}))\}. \tag{123}$$

In other words, for any $\widehat{\mathbf{E}}_T \in \mathbf{V}_{\mathbf{h},T}(\widehat{\Omega})$, denoting $\widehat{\mathbf{E}}_{T,j} = \widehat{\mathbf{E}}_T|_{\widehat{S}_j} \in \mathbf{V}_h(\widehat{S})$,

$$\widehat{\mathbf{E}}_T(\mathbf{x}_T, x_3) = \sum_{j\in\mathbb{Z}} \widehat{\mathbf{E}}_{T,j}(\mathbf{x}_T) w_j(x_3), \tag{124}$$

where w_j is the usual hat function (see Fig. 5). On the other hand, any longitudinal field $E_3 \in \mathrm{V}_{\mathbf{h},\ell}(\widehat{\Omega})$ is searched piecewise constant in x_3 with value in the standard $\mathbb{P}_1$ finite element space.

$$\mathrm{V}_{\mathbf{h},\ell}(\widehat{\Omega}) := \{\widehat{E}_3 : \widehat{\Omega} \to \mathbb{R}/\,\forall j, \forall x_3 \in (j\mathrm{h}, (j+1)\mathrm{h}), \widehat{E}_3(\cdot, x_3) = E_{3,j+\frac{1}{2}} \in \mathrm{V}_h(\widehat{S})\}, \tag{125}$$

with $\mathrm{V}_h(\widehat{S})$ defined in (100). Thus, for any $\widehat{E}_3 \in V_{\mathbf{h},\ell}$, there exist $\widehat{E}_{3,j+\frac{1}{2}} \in V_h(\widehat{S}), j \in \mathbb{Z}$, such that

$$\widehat{E}_3(\mathbf{x}_T, x_3) = \sum_{j\in\mathbb{Z}} \widehat{E}_{3,j+\frac{1}{2}}(\mathbf{x}_T)\chi_{j+\frac{1}{2}}(x_3), \quad \widehat{E}_{3,j+\frac{1}{2}} = \widehat{E}_3|_{\widehat{S}_{j+\frac{1}{2}}}, \tag{126}$$

where $\chi_{j+\frac{1}{2}}$ is the characteristic function of the interval $(j\mathrm{h}, (j+1)\mathrm{h})$ (see Fig. 5).

We have thus two types of degrees of freedom for the discrete electric field (Fig. 6). The first ones, associated to the transverse field, are denoted by $\mathbb{E}_{T,\mathbf{h}} \equiv \{\mathbb{E}_{T,j}\}$, where $\mathbb{E}_{T,j} \in \mathbb{R}^{N_e}$ is the vector of the tangential components of the discrete transverse electric field along the edges in the cross-section S_j.

The second ones, for the longitudinal field, are denoted by $\mathbb{E}_{3,\mathbf{h}} \equiv \{\mathbb{E}_{3,j+\frac{1}{2}}\}$, where $\mathbb{E}_{3,j+\frac{1}{2}} \in \mathbb{R}^N$ is the vector of the values of the discrete longitudinal field at the nodes of the mesh.

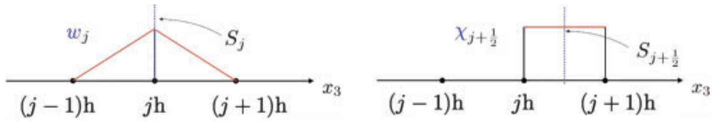

FIGURE 5. The 1D basis functions w_j (left) and $\chi_{j+\frac{1}{2}}$ (right) (picture from [1]).

Transverse field in each section S_j Longitudinal field in each section $S_{j+\frac{1}{2}}$

FIGURE 6. Two types of degrees of freedom (picture from [1]).

Remark 4.2 (About conformity properties)**.** The space $\mathbf{V_h}(\widehat{\Omega})$ is non-conforming with respect to the transverse variable since fields in this space satisfy by construction the jump conditions

$$[\widehat{\mathbf{n}}_K \times \widehat{\mathbf{E}}]_{KL} = 0$$

across the longitudinal interfaces Σ_{KL}, while the correct jump condition is

$$J_T^{-1}\mathbf{D}[\widehat{\mathbf{n}}_K \times \widehat{\mathbf{E}}]_{KL} + [\![\widehat{\mathbf{E}}]\!]_{KL}^{\delta} = 0, \quad \text{with } [\![\widehat{\mathbf{E}}]\!]_{KL}^{\delta} \neq 0 \quad \text{for } \delta > 0, \text{ see (81)}.$$

Conversely, across the transverse interfaces $\widehat{S}_j$, for fields in $\mathbf{V_h}(\widehat{\Omega})$, the only jump condition is the continuity of the transverse field, which precisely corresponds to the physical jump. In this sense, the approximation in space is conforming with respect to the longitudinal variable. Note that our final space discretization will not involve any jump terms across transverse interfaces (see Section 4.2.2).

4.2.2. The fully discrete scheme in space

It consists in rewriting the variational formulation (104) in $\mathbf{V_h}(\widehat{\Omega})$ instead of $\mathbf{V}_h(\widehat{\Omega})$ with an additional approximation corresponding to the ($O(\mathrm{h}^2)$ accurate) trapezoidal quadrature formula in x_3. Setting

$$C^0_{\mathrm{h}}(\mathbb{R}) := \{g : \mathbb{R} \to \mathbb{R} / \forall j,\ g_{j+\frac{1}{2}} := g|_{[j\mathrm{h},(j+1)\mathrm{h}]} \in C^0([j\mathrm{h},(j+1)\mathrm{h}])\}.$$

We define

$$\forall g \in C^0_{\mathrm{h}}(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \oint_{\mathbb{R}} g(x_3)\,\mathrm{d}x_3 := \mathrm{h}\sum_j [g_{j+\frac{1}{2}}((j+1)\mathrm{h}) + g_{j+\frac{1}{2}}(j\mathrm{h})]. \tag{127}$$

From (127) and Fubini's theorem, we build the 3D quadrature formula for integrals over $\widehat{\Omega}$:

$$\forall f \in C^0_{\mathrm{h}}(\widehat{\Omega}) \cap L^1(\widehat{\Omega}), \quad \oint_{\widehat{\Omega}} f\mathrm{d}\widehat{\mathbf{x}} := \int_{\widehat{S}} \left(\oint_{\mathbb{R}} f(\mathbf{x}_T, x_3)\,\mathrm{d}x_3\right)\mathrm{d}\widehat{\mathbf{x}}_T, \tag{128}$$

where by definition $C^0_{\mathrm{h}}(\widehat{\Omega}) := \{f : \widehat{\Omega} \to \mathbb{R} / \forall j, \quad f_{j+\frac{1}{2}} := f|_{\mathcal{C}_{j+\frac{1}{2}}} \in C^0(\widehat{S} \times [j\mathrm{h},(j+1)\mathrm{h}])\}.$

Similarly, we define, writing $\widehat{\Sigma}_{KL} = \widehat{\sigma}_{KL} \times \mathbb{R}$ and $\widehat{\Gamma}_K = \widehat{\gamma}_K \times \mathbb{R}$,

$$\left| \begin{array}{l} \displaystyle\oint_{\widehat{\Sigma}_{KL}} f \,\mathrm{d}\widehat{\sigma} := \int_{\widehat{\sigma}_{KL}} \left(\oint_{\mathbb{R}} f(\widehat{\mathbf{x}}_T, x_3)\,\mathrm{d}x_3 \right) \mathrm{d}\sigma_T, \ f \in C^0_{\mathrm{h}}(\widehat{\Sigma}_{KL}), \\ \displaystyle\oint_{\widehat{\Gamma}_K} f \,\mathrm{d}\widehat{\sigma} := \int_{\widehat{\gamma}_K} \left(\oint_{\mathbb{R}} f(\widehat{\mathbf{x}}_T, x_3)\,\mathrm{d}x_3 \right) \mathrm{d}\sigma_T, \quad f \in C^0_{\mathrm{h}}(\widehat{\Gamma}_K). \end{array} \right.$$

Accordingly we define, with $\widehat{\Sigma}_h = \widehat{\sigma}_h \times \mathbb{R}$ and $\widehat{\Gamma} = \widehat{\gamma} \times \mathbb{R}$,

$$\forall\, f \in C^0_{\mathrm{h}}(\widehat{\Sigma}_h \cup \widehat{\Gamma}) \cap L^1(\widehat{\Sigma}_h \cup \widehat{\Gamma}), \quad \oint_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} f \,\mathrm{d}\widehat{\sigma} := \int_{\widehat{\sigma}_h \cup \widehat{\gamma}} \left(\oint_{\mathbb{R}} f(\mathbf{x}_T, x_3)\,\mathrm{d}x_3 \right) \mathrm{d}\widehat{\mathbf{x}}_T. \tag{129}$$

The above ingredients are of fundamental importance for achieving a numerical scheme which will be explicit in x_3 after time discretization as will be explained in Sections 5.2 and 5.3. According to (104), the fully discrete problem then reads, in variational form:

Find $\widehat{\mathbf{E}}_{\mathbf{h}}(\cdot,t) = (\widehat{\mathbf{E}}_{T,\mathbf{h}}(\cdot,t), \widehat{E}_{3,\mathbf{h}}(\cdot,t)) \in \mathbf{V}_{\mathbf{h}}(\widehat{\Omega}),$ such that $\forall\, \widehat{\mathbf{F}}_{\mathbf{h}} \in \mathbf{V}_{\mathbf{h}}(\widehat{\Omega}),$ we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{m}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) + \boldsymbol{k}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) + \boldsymbol{\sigma}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) + \boldsymbol{j}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) = 0, \quad \forall\, \widehat{\mathbf{F}}_{\mathbf{h}} \in \mathbf{V}_{\mathbf{h}}(\widehat{\Omega}), \tag{130}$$

where a subtle but noticeable difference with (104) is that the bilinear forms $\boldsymbol{m}(\cdot,\cdot)$, $\boldsymbol{k}_h(\cdot,\cdot)$, $\boldsymbol{\sigma}^\delta_h(\cdot,\cdot)$, and $\boldsymbol{j}^\delta_h(\cdot,\cdot)$ defined in (48), (93), and (103), have been replaced by $\boldsymbol{m}_{\mathbf{h}}(\cdot,\cdot)$, $\boldsymbol{k}_{\mathbf{h}}(\cdot,\cdot)$, $\boldsymbol{\sigma}^\delta_{\mathbf{h}}(\cdot,\cdot)$, and $\boldsymbol{j}^\delta_{\mathbf{h}}(\cdot,\cdot)$, defined by replacing the integrals appearing in (48), (93), and (103) by their quadrature approximations (128) or (129) (note that now the bilinear forms depend on $\mathbf{h} := (h, \mathrm{h})$):

$$\left| \begin{array}{lcl} \boldsymbol{m}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) & = & \displaystyle\oint_{\widehat{\Omega}} \widehat{\varepsilon}(\mathbf{A}\widehat{\mathbf{E}} \cdot \widehat{\mathbf{F}})\mathrm{d}\widehat{\mathbf{x}}, \\ \boldsymbol{k}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) & = & \displaystyle\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}|J_T|(\mathbf{D}(J_T^{-1}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}}) - \mathbf{P}^\delta_h(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}) \cdot (J_T^{-1}\mathbf{D}(\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}}) - \mathbf{P}^\delta_h(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}})\,\mathrm{d}\widehat{\mathbf{x}}, \\ \boldsymbol{\sigma}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) & = & \displaystyle -\oint_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]^\delta_h \cdot \{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}})_T - \mathbf{P}^\delta_{h,T}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_T)\}_h\,\mathrm{d}\sigma \\ & & \displaystyle -\oint_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} [\![\widehat{\mathbf{F}}]\!]^\delta_h \cdot \{\widehat{\mu}^{-1}(J_T^{-1}(\mathbf{D}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}})_T - \mathbf{P}^\delta_{h,T}(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_T)\}_h\,\mathrm{d}\sigma, \\ \boldsymbol{j}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) & = & \displaystyle\oint_{\widehat{\Sigma}_h \cup \widehat{\Gamma}} \widehat{\alpha}_h [\![\widehat{\mathbf{E}}]\!]^\delta_h \cdot [\![\widehat{\mathbf{F}}]\!]^\delta_h\,\mathrm{d}\sigma. \end{array} \right. \tag{131}$$

By consistency with the definition (105) of $\boldsymbol{a}_h(\cdot,\cdot)$, we shall define

$$\boldsymbol{a}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) := \boldsymbol{k}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) + \boldsymbol{\sigma}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) + \boldsymbol{j}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}), \tag{132}$$

so that (130) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{m}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) + \boldsymbol{a}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}) = 0, \quad \forall\, \widehat{\mathbf{F}}_{\mathbf{h}} \in \mathbf{V}_{\mathbf{h}}(\widehat{\Omega}). \tag{133}$$

In order to write the problem (133) in algebraic form, we introduce here the (infinite) vectors of degrees of freedom, namely (with obvious notation)

$$\mathbb{E}_{\mathbf{h}} = \begin{pmatrix} \mathbb{E}_{T,\mathbf{h}} \\ \mathbb{E}_{3,\mathbf{h}} \end{pmatrix} \equiv \begin{pmatrix} \mathbb{E}_{T,j} \\ \mathbb{E}_{3,j+\frac{1}{2}} \end{pmatrix} \in \mathbb{V}_{\mathbf{h}} := \mathbb{V}_{\mathbf{h},T} \times \mathbb{V}_{\mathbf{h},3}, \quad \mathbb{E}_{T,j} \in \mathbb{R}^{N_e}, \quad \mathbb{E}_{3,j+\frac{1}{2}} \in \mathbb{R}^N, \tag{134}$$

where $\mathbb{V}_{\mathbf{h},T} = \ell^2(\mathbb{Z}, \mathbb{R}^{N_e})$ and $\mathbb{V}_{\mathbf{h},3} = \ell^2(\mathbb{Z}, \mathbb{R}^{N})$. We shall denote

$$(\mathbb{E}_{\mathbf{h}}, \mathbb{F}_{\mathbf{h}}) := \sum_j \mathbb{E}_{T,j} \cdot \mathbb{F}_{T,j} + \sum_j \mathbb{E}_{3,j+\frac{1}{2}} \cdot \mathbb{F}_{3,j+\frac{1}{2}} \tag{135}$$

and, using Riesz representation theorem, Eq. (130) can be rewritten as

$$\mathbf{M_h} \frac{\mathrm{d}^2\mathbb{E}_{\mathbf{h}}}{\mathrm{d}t^2} + \mathbf{A_h}\mathbb{E}_{\mathbf{h}} = 0, \quad \mathbf{A_h} = \mathbf{K_h} + \mathbf{S}_{\mathbf{h}}^{\delta} + \mathbf{J}_{\mathbf{h}}^{\delta}, \tag{136}$$

where $\mathbf{M_h}$ and $\mathbf{A_h}$ are the (infinite) symmetric mass and stiffness matrices defined by

$$(\mathbf{M_h}\mathbb{E}_{\mathbf{h}}, \mathbb{F}_{\mathbf{h}}) = \boldsymbol{m}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}), \quad (\mathbf{A_h}\mathbb{E}_{\mathbf{h}}, \mathbb{F}_{\mathbf{h}}) = \boldsymbol{a}_{\mathbf{h}}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{F}}_{\mathbf{h}}).$$

Both $\mathbf{M_h}$ and $\mathbf{A_h}$ are positive. According to (132), we have

$$\mathbf{A_h} = \mathbf{K_h} + \mathbf{S}_{\mathbf{h}}^{\delta} + \mathbf{J}_{\mathbf{h}}^{\delta}, \tag{137}$$

where the symmetric matrices $\mathbf{K_h}$, $\mathbf{S}_{\mathbf{h}}^{\delta}$, and $\mathbf{J}_{\mathbf{h}}^{\delta}$ are defined from $\boldsymbol{k}_{\mathbf{h}}(\cdot,\cdot)$, $\boldsymbol{\sigma}_{\mathbf{h}}^{\delta}(\cdot,\cdot)$, and $\boldsymbol{j}_{\mathbf{h}}^{\delta}(\cdot,\cdot)$, respectively.

According to the decomposition of $\mathbb{V}_{\mathbf{h}}$ between transverse and longitudinal fields, the mass matrix $\mathbf{M_h}$ has the following block diagonal form:

$$\begin{cases} \mathbf{M_h} = \begin{pmatrix} \mathbf{M}_{\mathbf{h}}^{T} & 0 \\ 0 & M_{\mathbf{h}}^{3} \end{pmatrix} \quad (\text{according to } \mathbb{V}_{\mathbf{h}} := \mathbb{V}_{\mathbf{h},T} \times \mathbb{V}_{\mathbf{h},3}), \\ \mathbf{M}_{\mathbf{h}}^{T} = \mathrm{diag}\left\{\mathbf{M}_j^T\right\}, \mathbf{M}_j^T \in \mathcal{L}(\mathbb{R}^{N_e}), \quad \mathbf{M}_{\mathbf{h}}^{3} = \mathrm{diag}\{\mathbf{M}^3_{j+\frac{1}{2}}\}, \mathbf{M}^3_{j+\frac{1}{2}} \in \mathcal{L}(\mathbb{R}^N), \quad j \in \mathbb{Z}. \end{cases} \tag{138}$$

In particular, the numerical quadrature used in (131) for $\boldsymbol{m}_{\mathbf{h}}$ allows us to achieve mass lumping in the x_3-direction. The use of the quadrature in the other bilinear forms will also have nice consequences on the structure of $\mathbf{A_h}$, which will be described in Section 5 when considering the time discretization.

4.2.3. Fully discrete coercivity inequality

For any pair of symmetric matrices $\mathbf{X}_h$ and $\mathbf{Y}_h$, we define the operator $\leq$ (and $\geq$) as follows:

$$\mathbf{X}_h \leq \mathbf{Y}_h \quad \Leftrightarrow \quad \forall \mathbb{E}_{\mathbf{h}} \in \mathbb{V}_{\mathbf{h}} \quad (\mathbf{X}_h\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) \leq (\mathbf{Y}_h\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}).$$

The next lemma corresponds to the fully discrete counterpart of the coercivity inequality of Lemma 4.2.

Lemma 4.3 (Discrete coercivity inequality)**.** *Let $\nu \in (0,1)$. If the function $\widehat{\alpha}_h$ is chosen large enough so that*

$$\forall K \in \mathcal{T}_h(\widehat{\Omega}), \quad \widehat{\alpha}_h|_{\partial P_K} \geq \frac{C_I^2}{\nu^2 h_K}, \tag{139}$$

one has

$$-\nu\left(\mathbf{K_h} + \mathbf{J}_{\mathbf{h}}^{\delta}\right) \leq \mathbf{S}_{\mathbf{h}}^{\delta} \leq \nu\left(\mathbf{K_h} + \mathbf{J}_{\mathbf{h}}^{\delta}\right) \quad \textit{and} \quad (1-\nu)\left(\mathbf{K_h} + \mathbf{J}_{\mathbf{h}}^{\delta}\right) \leq \mathbf{A_h}. \tag{140}$$

Proof. We omit the full details of the proof and instead outline the main steps. First, the discrete trace inequality of Lemma 4.1 must be extended to the fully discrete setting. This can be achieved by taking into account the properties of the quadrature rule, in particular the positivity of the weights. With this extension, the argument used in Lemma 4.2 applies in the same way, leading to the desired estimate and discrete coercivity result. □

5. Time discretization

In this section, we are concerned with the time discretization of the semi-discrete problem (136). We shall use a constant time step $\Delta t > 0$, and denote by $\mathbb{E}^n_{\mathbf{h}}, n \in \mathbb{N}$, the corresponding discrete solution such that

$$\mathbb{E}^n_{\mathbf{h}} \simeq \mathbb{E}_{\mathbf{h}}(t^n), \quad t^n := n\Delta t.$$

In Section 5.1, we present the class of finite difference schemes we wish to use as well as the goal we want to attain. In Section 5.2, we recall how we build the splitting to reach this goal in [1], for the case of the cylindrical cable. In Section 5.3, we explain how to extend this construction and the corresponding analysis to the more involved case of the deformed cable.

5.1. Construction of the scheme: basic principles

We wish to use a hybrid explicit–implicit scheme based on an adequate splitting of the stiffness matrix $\mathbf{A}_{\mathbf{h}}$,

$$\mathbf{A}_{\mathbf{h}} = \mathbf{A}^i_{\mathbf{h}} + \mathbf{A}^e_{\mathbf{h}}. \tag{141}$$

The scheme, which belongs to the class of IMEX schemes (for IMplicit/EXplicit schemes), can be seen as an "interpolation" between the explicit leapfrog scheme and the implicit θ-scheme:

$$\mathbf{M}_{\mathbf{h}} \frac{\mathbb{E}^{n+1}_{\mathbf{h}} - 2\mathbb{E}^n_{\mathbf{h}} + \mathbb{E}^{n-1}_{\mathbf{h}}}{\Delta t^2} + \mathbf{A}^i_{\mathbf{h}} \{\mathbb{E}^n_{\mathbf{h}}\}_\theta + \mathbf{A}^e_{\mathbf{h}} \mathbb{E}^n_{\mathbf{h}} = 0, \tag{142}$$

where, $\theta \in \mathbb{R}^+$ being a parameter of the scheme,

$$\{\mathbb{E}^n_{\mathbf{h}}\}_\theta := \theta \mathbb{E}^{n+1}_{\mathbf{h}} + (1 - 2\theta)\mathbb{E}^n_{\mathbf{h}} + \theta \mathbb{E}^{n-1}_{\mathbf{h}}. \tag{143}$$

Obviously, for any θ, (142) is a second-order accurate approximation in Δt of the ODE (136) and at each time step, we have to solve the linear problem

$$\left(\mathbf{M}_{\mathbf{h}} + \theta\ \Delta t^2 \mathbf{A}^i_{\mathbf{h}}\right) \left(\mathbb{E}^{n+1}_{\mathbf{h}} + \mathbb{E}^{n-1}_{\mathbf{h}}\right) = \left(2\mathbf{M}_{\mathbf{h}} - \Delta t^2 \mathbf{A}^e_{\mathbf{h}} - (1-2\theta)\Delta t^2 \mathbf{A}^i_{\mathbf{h}}\right) \mathbb{E}^n_{\mathbf{h}}. \tag{144}$$

The challenge is to build the splitting (141) cleverly enough in order that

(i) The matrix $\mathbf{A}^i_{\mathbf{h}}$ has the same block diagonal structure as the mass matrix $\mathbf{M}_{\mathbf{h}}$;
(ii) The scheme is stable if Δt is only constrained by the longitudinal space step h (and not h).

Indeed, achieving (i) and (ii) will result in the following two nice consequences:

(a) The computation, at each time step, of the next iteration is cheap: we have to solve a family of "small" linear systems (indexed by j), each of them consisting in
  - solving a sparse positive definite linear system in dimension N_e for updating $\mathbb{E}_{T,j}$, namely the transverse electric field in the cross-section $x_3 = j\mathrm{h}$,
  - solving a sparse positive definite linear system in dimension N for updating $\mathbb{E}_{3,j+1/2}$, namely the longitudinal electric field in the cross-section $x_3 = (j+1/2)\mathrm{h}$.

  The above systems can be solved in parallel, independently of one another.
(b) The choice of Δt can be made independently of the transverse dimension of the cable, which results in a huge economy of time steps with respect to fully explicit schemes.

For studying stability, the usual trick [7] is to rewrite the scheme as a perturbation of the θ-scheme (142) for $\theta = \frac{1}{4}$. The resulting scheme involves a modified mass matrix. It is rewritten as:

$$\mathbf{M}_{\mathbf{h}}(\Delta t) \frac{\mathbb{E}^{n+1}_{\mathbf{h}} - 2\mathbb{E}^n_{\mathbf{h}} + \mathbb{E}^{n-1}_{\mathbf{h}}}{\Delta t^2} + \mathbf{A}_{\mathbf{h}} \frac{\mathbb{E}^{n+1}_{\mathbf{h}} + 2\mathbb{E}^n_{\mathbf{h}} + \mathbb{E}^{n-1}_{\mathbf{h}}}{4} = 0, \tag{145}$$

where the modified mass matrix $\mathbf{M_h}(\Delta t)$ is given by

$$\mathbf{M_h}(\Delta t) = \mathbf{M_h} + \left(\theta - \tfrac{1}{4}\right) \Delta t^2 \mathbf{A}^i_{\mathbf{h}} - \tfrac{1}{4}\Delta t^2 \mathbf{A}^e_{\mathbf{h}}. \tag{146}$$

Multiplying classically (145) by $\frac{\mathbb{E}^{n+1}_{\mathbf{h}} - \mathbb{E}^{n-1}_{\mathbf{h}}}{2\Delta t}$, we deduce the conservation of the discrete "energy"

$$\mathcal{E}^{n+\frac{1}{2}}_h := \frac{1}{2}\left(\mathbf{M_h}(\Delta t)\frac{\mathbb{E}^{n+1}_{\mathbf{h}} - \mathbb{E}^n_{\mathbf{h}}}{\Delta t}, \frac{\mathbb{E}^{n+1}_{\mathbf{h}} - \mathbb{E}^n_{\mathbf{h}}}{\Delta t} \right) + \frac{1}{2}\left(\mathbf{A_h}\frac{\mathbb{E}^{n+1}_{\mathbf{h}} + \mathbb{E}^n_{\mathbf{h}}}{2}, \frac{\mathbb{E}^{n+1}_{\mathbf{h}} + \mathbb{E}^n_{\mathbf{h}}}{2}\right). \tag{147}$$

The stability analysis is then reduced to showing the positivity of $\mathcal{E}^{n+\frac{1}{2}}_h$ which amounts to proving the positivity of the operator of $\mathbf{M}_h(\Delta t)$. This depends of course on the choice of the matrices $(\mathbf{A}^i_{\mathbf{h}}, \mathbf{A}^e_{\mathbf{h}})$ in the decomposition $\mathbf{A_h} = \mathbf{A}^i_{\mathbf{h}} + \mathbf{A}^e_{\mathbf{h}}$. In Section 5.2, we treat the cylindrical case, which is a recap of what was done in [1]. In Section 5.3, we treat the (difficult) case of the deformed cable.

5.2. Recap on the cylindrical case

Recall the definition (46) of the matrix $\mathbf{A}$,

$$\mathbf{A} := \begin{pmatrix} \mathbf{A}_T & 0 \\ 0 & |J_T| \end{pmatrix} \quad \text{with} \quad \mathbf{A}_T := |J_T| \, \mathbf{D}_T^{-1}\mathbf{D}_T^{-*}. \tag{148}$$

In this case, according to Remark 4.1 and to the longitudinal discretization procedure (see Section 4.2.2), the bilinear form $\boldsymbol{a}_h(\cdot,\cdot)$ reduces to

$$\boldsymbol{a}_{\mathbf{h}}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) = \boldsymbol{k}^0_{\mathbf{h}}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h) := \oint_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}^{-1}\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}}_h \cdot \widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}}_h \, \mathrm{d}\widehat{\mathbf{x}}. \tag{149}$$

The splitting will be based on a particular expression of the bilinear form $\boldsymbol{k}^0_{\mathbf{h}}(\widehat{\mathbf{E}}_h, \widehat{\mathbf{F}}_h)$ which separates the roles of the tangential and longitudinal components of the fields:

$$\boldsymbol{k}^0_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) = \boldsymbol{k}^0_{TT}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \boldsymbol{k}^0_{33}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \mathrm{k}^0_{TT}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \mathrm{k}^0_{3T}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}), \tag{150}$$

where we have set

$$\begin{cases} \mathbf{k}^0_{TT}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) := \displaystyle\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}|J_T|^{-1}\widehat{\mathrm{rot}}_T\widehat{\mathbf{E}}_T\widehat{\mathrm{rot}}_T\widehat{\mathbf{F}}_T \, \mathrm{d}\widehat{\mathbf{x}}, \quad \mathrm{k}^0_{TT}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) := \displaystyle\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}_T^{-1}\widehat{\mathbf{rot}}_T \, \widehat{E}_3 \cdot \widehat{\mathbf{rot}}_T\widehat{F}_3 \, \mathrm{d}\widehat{\mathbf{x}}, \\ \mathbf{k}^0_{33}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) := \displaystyle\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}_T^{-1}(\mathbf{e}_3 \times \partial_3\widehat{\mathbf{E}}_T) \cdot (\mathbf{e}_3 \times \partial_3\widehat{\mathbf{F}}_T) \, \mathrm{d}\widehat{\mathbf{x}}, \\ \mathrm{k}^0_{3T}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) := \displaystyle\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}(\mathbf{A}_T^{-1}\widehat{\mathbf{rot}}_T\widehat{E}_3 \cdot (\mathbf{e}_3 \times \partial_3\widehat{\mathbf{F}}_T) + \mathbf{A}_T^{-1}\widehat{\mathbf{rot}}_T\widehat{F}_3 \cdot (\mathbf{e}_3 \times \partial_3\widehat{\mathbf{E}}_T)) \, \mathrm{d}\widehat{\mathbf{x}}. \end{cases} \tag{151}$$

Remark 5.1 (Remark about the notations)**.** In the definition (151), the indices have been used according to the following rule: The occurrence of T corresponds to the appearance of one transverse derivative (in $\mathbf{x}_T$) of one of the fields in the integrands while the occurrence of 3 corresponds to the appearance of one longitudinal derivative (in x_3). The bilinear form $\mathrm{k}^0_{3T}(\cdot,\cdot)$ couples transverse and longitudinal fields and corresponds to the appearance of both transverse and longitudinal derivatives.

According to (150), we can decompose the matrix $\mathbf{A_h}$ as a sum of four symmetric matrices:

$$\mathbf{A_h} = \mathbf{K}^0_{TT} + \mathrm{K}^0_{TT} + \mathbf{K}^0_{33} + \mathrm{K}^0_{3T}. \tag{152}$$

The important observation is that, in particular thanks to the numerical quadrature,

- The matrices $\mathbf{K}^0_{TT}$ and K^0_{TT} do have the same structure (138) as $\mathbf{M}^T_{\mathbf{h}}$ and $M^3_{\mathbf{h}}$;
- The matrices $\mathbf{K}^0_{33}$ and K^0_{3T} do not since
  - The matrix $\mathbf{K}^0_{33}$ couples $\mathbb{E}_{T,j}$ to $\mathbb{E}_{T,j+1}$ and $\mathbb{E}_{T,j-1}$,
  - The matrix K^0_{3T} couples $\mathbb{E}_{T,j}$ to $\mathbb{E}_{3,j+1/2}$ and $\mathbb{E}_{3,j-1/2}$ and $\mathbb{E}_{3,j+1/2}$ to $\mathbb{E}_{T,j+1}$ and $\mathbb{E}_{T,j}$.

For the above reasons, for the splitting (141) of $\mathbf{A}_h = \mathbf{K}^0_{\mathbf{h}}$, we choose

$$\mathbf{A}^i_{\mathbf{h}} = \mathbf{K}^0_{TT} + \mathrm{K}^0_{TT}, \quad \mathbf{A}^e_{\mathbf{h}} = \mathbf{K}^0_{33} + \mathrm{K}^0_{3T}. \tag{153}$$

By construction, the criterion (i) of Section 5.1 is satisfied. The criterion (ii) can be proved using energy techniques. As explained at the end of Section 5.1, the stability analysis amounts to proving the positivity of the modified mass matrix $\mathbf{M}_h(\Delta t)$ (see (146)) with $(\mathbf{A}^i_h, \mathbf{A}^e_h)$ given by (153), under some upper bound on the time step Δt. This has been done in [1].

The precise stability result involves the function $\beta_{\mathrm{h}}(\widehat{\mathbf{x}}_T) : \widehat{S} \to \mathbb{R}^+$ defined by:

$$\beta_{\mathrm{h}}(\mathbf{x}_T)^2 := \sup_{u_{\mathrm{h}} \in \mathbb{P}_{1,\mathrm{h}}} \frac{\displaystyle\oint_{\mathbb{R}} \mu(\cdot, \mathbf{x}_T)^{-1} |u'_{\mathrm{h}}|^2 \mathrm{d}x_3}{\displaystyle\oint_{\mathbb{R}} \varepsilon(\cdot, \mathbf{x}_T) |u_{\mathrm{h}}|^2 \mathrm{d}x_3} \in L^\infty(\widehat{S}), \tag{154}$$

where $\mathbb{P}_{1,\mathrm{h}}$ denotes the usual 1D finite element space,

$$\mathbb{P}_{1,\mathrm{h}} := \{u_{\mathrm{h}} \in C^0(\mathbb{R}) \cap L^2(\mathbb{R}) / \forall\, j \in \mathbb{Z}, u_{\mathrm{h}}|_{[j\mathrm{h},(j+1)\mathrm{h}]} \in \mathbb{P}_1\}. \tag{155}$$

Proposition 5.1 (Theorem 5.1 of [1]). *In the case $\delta = 0$, the scheme (142) is stable if $\theta > \frac{1}{4}$ and*

$$\frac{\Delta t^2}{4} \|\beta_{\mathrm{h}}\|^2_\infty \leq 1 - \frac{1}{4\theta}. \tag{156}$$

5.3. The decomposition for the deformed cable

5.3.1. The decomposition $\mathbf{A}_{\mathbf{h}} = \mathbf{A}^i_{\mathbf{h}} + \mathbf{A}^e_{\mathbf{h}}$ for the deformed cable

We have to find the equivalent of the decomposition (141). In the spirit of Remark 4.1, we start by rewriting $\boldsymbol{k}_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}})$ as a perturbation of $\boldsymbol{k}^0_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}})$:

$$\boldsymbol{k}_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) = \boldsymbol{k}^0_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \boldsymbol{k}^\delta_{\mathbf{h}}(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}), \tag{157}$$

where, from the definitions of $\boldsymbol{k}_h\left(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}\right)$, see (131), and $\boldsymbol{k}^0_h\left(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}\right)$, see (149), one deduces that

$$\left|\begin{aligned} \boldsymbol{k}^\delta_h(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) \; = \; & \oint_{\widehat{\Omega}} \widehat{\mu}^{-1} |J_T| \mathbf{P}^\delta_h(\widehat{\mathbf{x}}, \widehat{\nabla}_T) \widehat{\mathbf{E}} \cdot \mathbf{P}^\delta_h(\widehat{\mathbf{x}}, \widehat{\nabla}_T) \widehat{\mathbf{F}} \, \mathrm{d}\widehat{\mathbf{x}}, \\ & - \oint_{\widehat{\Omega}} \widehat{\mu}^{-1} |J_T| [J_T^{-1} \mathbf{D}(\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}}) \cdot \mathbf{P}^\delta_h(\widehat{\mathbf{x}}, \widehat{\nabla}_T) \widehat{\mathbf{F}} + J_T^{-1} \mathbf{D}(\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{F}}) \cdot \mathbf{P}^\delta_h(\widehat{\mathbf{x}}, \widehat{\nabla}_T) \widehat{\mathbf{E}}] \, \mathrm{d}\widehat{\mathbf{x}}. \end{aligned}\right.$$

This bilinear form can be decomposed as the sum of three terms using the transverse and longitudinal components of $\widehat{\boldsymbol{\nabla}}_h \times \widehat{\mathbf{E}}$, in particular, using the definition (23) of D,

$$(\mathbf{D} \widehat{\boldsymbol{\nabla}} \times \widehat{\mathbf{E}})_T = \mathbf{D}_T (\widehat{\mathbf{rot}}_T \widehat{E}_3 + \mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T),$$

and using that $\mathbf{P}_h^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}$ is a transverse field (see (34)). We have

$$\boldsymbol{k}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) = \boldsymbol{k}_{TT}^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \mathrm{k}_{3T}^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \mathrm{k}_{TT}^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}), \tag{158}$$

where we have defined, with the same convention of notation as in Remark 5.1,

$$\left|\begin{aligned}
\boldsymbol{k}_{TT}^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) &= \oint_{\widehat{\Omega}} \widehat{\mu}^{-1}|J_T|\mathbf{P}_{h,T}^\delta(\widehat{\nabla}_T)\widehat{\mathbf{E}}_T \cdot \mathbf{P}_{h,T}^\delta(\widehat{\nabla}_T)\widehat{\mathbf{F}}_T \,\mathrm{d}\widehat{\mathbf{x}},\\
\mathrm{k}_{3T}^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) &= -\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}(\mathbf{D}_T(\mathbf{e}_3 \times \partial_3\widehat{\mathbf{E}}_T) \cdot \mathbf{P}_{h,T}^\delta(\widehat{\nabla}_T)\widehat{\mathbf{F}}_T) \,\mathrm{d}\widehat{\mathbf{x}}\\
&\quad -\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}(\mathbf{D}_T(\mathbf{e}_3 \times \partial_3\widehat{\mathbf{F}}_T) \cdot \mathbf{P}_{h,T}^\delta(\widehat{\nabla}_T)\widehat{\mathbf{E}}_T) \,\mathrm{d}\widehat{\mathbf{x}},\\
\mathrm{k}_{TT}^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) &= -\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}(\mathbf{D}_T\widehat{\mathbf{rot}}_T\, \widehat{E}_3 \cdot \mathbf{P}_{h,T}^\delta(\widehat{\nabla}_T)\widehat{\mathbf{F}}_T + \mathbf{D}_T\widehat{\mathbf{rot}}_T\widehat{F}_3 \cdot \mathbf{P}_{h,T}^\delta(\widehat{\nabla}_T)\widehat{\mathbf{E}}_T) \,\mathrm{d}\widehat{\mathbf{x}}.
\end{aligned}\right. \tag{159}$$

Of course, (157) and (158) correspond, with the obvious notation, to the matrix decompositions

$$\mathbf{K_h} = \mathbf{K_h^0} + \mathbf{K_h^\delta}, \qquad \mathbf{K_h^\delta} = \mathbf{K}_{TT}^\delta + \mathrm{K}_{3T}^\delta + \mathrm{K}_{TT}^\delta. \tag{160}$$

Similarly, one can deduce from formula (131), the following decomposition of $\boldsymbol{\sigma}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}})$:

$$\boldsymbol{\sigma}_h^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) = \boldsymbol{\sigma}_T^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \sigma_3^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) + \sigma_T^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}), \tag{161}$$

where we have set, again with the convention of Remark 5.1,

$$\left|\begin{aligned}
\boldsymbol{\sigma}_T^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) &= \oint_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}\mathbf{P}_{h,T}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{F}}_T\}_h \,\mathrm{d}\sigma + \oint_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{F}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}\mathbf{P}_{h,T}^\delta(\widehat{\mathbf{x}}, \widehat{\nabla}_T)\widehat{\mathbf{E}}_T\}_h \,\mathrm{d}\sigma,\\
\sigma_3^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) &= -\oint_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}J_T^{-1}\mathbf{D}_T(\mathbf{e}_3 \times \partial_3\widehat{\mathbf{F}}_T)\}_h \,\mathrm{d}\sigma\\
&\quad -\oint_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{F}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}J_T^{-1}\mathbf{D}_T(\mathbf{e}_3 \times \partial_3\widehat{\mathbf{E}}_T)\}_h \,\mathrm{d}\sigma,\\
\sigma_T^\delta(\widehat{\mathbf{E}}, \widehat{\mathbf{F}}) &= -\oint_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{E}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}J_T^{-1}\mathbf{D}_T\widehat{\mathbf{rot}}_T\widehat{F}_3\}_h \,\mathrm{d}\sigma\\
&\quad -\oint_{\widehat{\Sigma}_h\cup\widehat{\Gamma}} [\![\widehat{\mathbf{F}}]\!]_h^\delta \cdot \{\widehat{\mu}^{-1}J_T^{-1}\mathbf{D}_T\widehat{\mathbf{rot}}_T\widehat{E}_3\}_h \,\mathrm{d}\sigma.
\end{aligned}\right. \tag{162}$$

According to (161), we have the matrix decomposition,

$$\mathbf{S_h^\delta} = \mathbf{S}_T^\delta + \mathrm{S}_3^\delta + \mathrm{S}_T^\delta, \tag{163}$$

where the matrices $\mathbf{S}_T^\delta$, S_3^δ, and S_T^δ are defined from $\boldsymbol{\sigma}_T^\delta(\cdot,\cdot)$, $\sigma_3^\delta(\cdot,\cdot)$, and $\sigma_T^\delta(\cdot,\cdot)$, respectively.

The interest in the decompositions of $\mathbf{K_h}$ and $\mathbf{S_h^\delta}$ given by (160) and (163) respectively is explained below.

Among the matrices appearing in the right-hand side of (160) and (163), the ones that make the index 3 appear, namely $\mathbf{K}_{33}^0, \mathrm{K}_{3T}^0, \mathrm{K}_{3T}^\delta$, and S_3^δ couple the degrees of freedom of S_j to those of S_{j+1} (or those of $S_{j-\frac{1}{2}}$ to those of $S_{j+\frac{1}{2}}$). These matrices must then be incorporated in "the explicit part" $\mathbf{A}_h^e$.

This is also the case of the matrices K_{TT}^δ and S_T^δ since they couple the degrees of freedom of S_j to those of $S_{j+\frac{1}{2}}$ because they couple the transverse and longitudinal fields. Even though the matrix $\mathbf{S}_T^\delta$ does not couple different cross-sections, we shall make the choice to incorporate it in "the explicit part" $\mathbf{A}_h^e$ so that the whole matrix $\mathbf{S_h^\delta}$ is "treated explicitly" (see also Remark 5.2).

The matrices $\mathbf{K}^0_{TT}$, K^0_{TT}, and $\mathbf{K}^\delta_{TT}$, as well as the matrix $\mathbf{J}^\delta_{\mathbf{h}}$ do not imply any coupling between different cross-sections (and thus have the same structure (138) as $\mathbf{M}^T_{\mathbf{h}}$) and will be incorporated in "the implicit part" $\mathbf{A}^i_h$.

Therefore, according to the criterion (i) of Section 5.1, we need to adopt the following splitting of $\mathbf{A}_{\mathbf{h}}$:

$$\left\{ \begin{aligned} \mathbf{A}^i_{\mathbf{h}} &= \mathbf{K}^0_{TT} + \mathrm{K}^0_{TT} + \mathbf{K}^\delta_{TT} + \mathbf{J}^\delta_{\mathbf{h}}, \\ \mathbf{A}^e_{\mathbf{h}} &= \mathbf{K}^0_{33} + \mathrm{K}^0_{3T} + \mathrm{K}^\delta_{3T} + \mathrm{K}^\delta_{TT} + \mathbf{S}^\delta_{\mathbf{h}}. \end{aligned} \right. \tag{164}$$

Remark 5.2. An alternative choice would be to move the matrix $\mathbf{S}^\delta_T$ into $\mathbf{A}^i_h$. The corresponding alternative scheme would have similar stability and complexity properties to the one corresponding to the choice (164). However, its stability analysis would be slightly more involved than the one for the choice (164).

5.3.2. Stability analysis

According to the choice (164) for $\mathbf{A}^i_h$ and $\mathbf{A}^e_h$, we choose to decompose $\mathbf{M}_h(\Delta t)$ (see (146)) as follows:

$$\left| \begin{array}{l} \mathbf{M}_h(\Delta t) = \mathbf{M}_h \ + (\theta - \tfrac{1}{4})\Delta t^2(\mathbf{K}^0_{TT} + \mathrm{K}^0_{TT} + \mathbf{K}^\delta_{TT} + \mathbf{J}^\delta_{\mathbf{h}}) \\ \qquad\qquad\qquad\quad - \tfrac{1}{4}\Delta t^2(\mathbf{K}^0_{33} + \mathrm{K}^0_{3T} + \mathrm{K}^\delta_{3T} + \mathrm{K}^\delta_{TT} + \mathbf{S}^\delta_{\mathbf{h}}). \end{array} \right.$$

We use (140), more precisely $\mathbf{S}^\delta_{\mathbf{h}} \le \nu\left(\mathbf{K}_{\mathbf{h}} + \mathbf{J}^\delta_{\mathbf{h}}\right)$, to deduce that

$$\left| \begin{array}{l} \mathbf{M}_h(\Delta t) \ge \mathbf{M}_h \ + (\theta - \tfrac{1}{4})\Delta t^2(\mathbf{K}^0_{TT} + \mathrm{K}^0_{TT} + \mathbf{K}^\delta_{TT} + \mathbf{J}^\delta_{\mathbf{h}}) \\ \qquad\qquad\qquad\quad - \tfrac{1}{4}\Delta t^2(\mathbf{K}^0_{33} + \mathrm{K}^0_{3T} + \mathrm{K}^\delta_{3T} + \mathrm{K}^\delta_{TT}) \\ \qquad\qquad\qquad\quad - \tfrac{\nu}{4}\Delta t^2(\mathbf{K}_{\mathbf{h}} + \mathbf{J}^\delta_{\mathbf{h}}). \end{array} \right.$$

Using the decomposition $\mathbf{K}_{\mathbf{h}} = \mathbf{K}^0_{TT} + \mathrm{K}^0_{TT} + \mathbf{K}^0_{33} + \mathrm{K}^0_{3T} + \mathbf{K}^\delta_{TT} + \mathrm{K}^\delta_{3T} + \mathrm{K}^\delta_{TT}$ (see (152) and (160)), one can rewrite the right-hand side of the previous inequality to obtain

$$\left| \begin{array}{l} \mathbf{M}_h(\Delta t) \ge \mathbf{M}_h \ + (\theta - \tfrac{1+\nu}{4})\Delta t^2(\mathbf{K}^0_{TT} + \mathrm{K}^0_{TT} + \mathbf{K}^\delta_{TT} + \mathbf{J}^\delta_{\mathbf{h}}) \\ \qquad\qquad\qquad\quad - \tfrac{1+\nu}{4}\Delta t^2(\mathbf{K}^0_{33} + \mathrm{K}^0_{3T} + \mathrm{K}^\delta_{3T} + \mathrm{K}^\delta_{TT}). \end{array} \right. \tag{165}$$

We have the following lemma.

Lemma 5.1. *For any $\eta > 0$,*

$$\left| \begin{array}{l} \mathrm{K}^\delta_{TT} \le \tfrac{1}{\eta}\mathrm{K}^0_{TT} + \eta\mathbf{K}^\delta_{TT}, \\ \mathrm{K}^0_{3T} \le \tfrac{1}{\eta}\mathbf{K}^0_{33} + \eta\mathrm{K}^0_{TT}, \\ \mathrm{K}^\delta_{3T} \le \tfrac{1}{\eta}\mathbf{K}^0_{33} + \eta\mathbf{K}^\delta_{TT}. \end{array} \right.$$

Proof. We show only the first inequality. The two others are proven similarly. We recall that, for a vector of degrees of freedom $\mathbb{E}_{\mathbf{h}}$ corresponding to $\mathbf{E}_{\mathbf{h}}$ an element of $\mathbf{V}_{\mathbf{h}}$, we have

$$\mathrm{K}^\delta_{TT} \le \frac{1}{\eta}\mathrm{K}^0_{TT} + \eta\mathbf{K}^\delta_{TT} \quad \Leftrightarrow \quad (\mathrm{K}^\delta_{TT}\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) \le \frac{1}{\eta}(\mathrm{K}^0_{TT}\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) + \eta(\mathbf{K}^\delta_{TT}\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}), \tag{166}$$

and

$$(\mathrm{K}^\delta_{TT}\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) = \mathrm{k}^\delta_{TT}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{E}}_{\mathbf{h}}), \quad (\mathrm{K}^0_{TT}\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) = k^0_{TT}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{E}}_{\mathbf{h}}), \quad (\mathbf{K}^\delta_{TT}\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) = \boldsymbol{k}^\delta_{TT}(\widehat{\mathbf{E}}_{\mathbf{h}}, \widehat{\mathbf{E}}_{\mathbf{h}}),$$

the inequality (166) is then a trivial consequence of Young's inequality: indeed, using $\mathbf{A}_T^{-1} = |J_T|^{-1}\mathbf{D}_T^*\mathbf{D}_T$ (see (46)),

$$\begin{aligned}
\mathrm{k}_{TT}^{\delta}(\widehat{\mathbf{E}},\widehat{\mathbf{E}}) &= -2\oint_{\mathbb{R}}\left(\int_{\widehat{S}}\widehat{\mu}^{-1}\mathbf{D}_T\widehat{\mathbf{rot}}_T\ \widehat{E}_3\cdot\mathbf{P}_{h,T}^{\delta}(\widehat{\nabla}_T)\widehat{\mathbf{E}}_T\mathrm{d}\widehat{\mathbf{x}}_T\right)\mathrm{d}x_3\\
&\le \frac{1}{\eta}\oint_{\mathbb{R}}\left(\int_{\widehat{S}}\widehat{\mu}^{-1}\mathbf{A}_T^{-1}\widehat{\mathbf{rot}}_T\ \widehat{E}_3\cdot\widehat{\mathbf{rot}}_T\ \widehat{E}_3\mathrm{d}\widehat{\mathbf{x}}_T\right)\mathrm{d}x_3\\
&\quad+\eta\oint_{\mathbb{R}}\left(\int_{\widehat{S}}\widehat{\mu}^{-1}|J_T||\mathbf{P}_{h,T}^{\delta}(\widehat{\nabla}_T)\widehat{\mathbf{E}}_T|^2\mathrm{d}\widehat{\mathbf{x}}_T\right)\mathrm{d}x_3\\
&= \frac{1}{\eta}k_{TT}^{0}(\widehat{\mathbf{E}}_{\mathbf{h}},\widehat{\mathbf{E}}_{\mathbf{h}})+\eta\boldsymbol{k}_{TT}^{\delta}(\widehat{\mathbf{E}}_{\mathbf{h}},\widehat{\mathbf{E}}_{\mathbf{h}}).
\end{aligned}$$

□

We now use Lemma 5.1. We set $\eta = 1$ in the first inequality of this lemma, and then sum the three inequalities of the lemma to deduce that for any $\eta > 0$,

$$\mathrm{K}_{3T}^{0}+\mathrm{K}_{3T}^{\delta}+\mathrm{K}_{TT}^{\delta}\le(1+\eta)\left(\mathrm{K}_{TT}^{0}+\mathbf{K}_{TT}^{\delta}\right)+\tfrac{2}{\eta}\mathbf{K}_{33}^{0}.$$

With this inequality, we deduce from (165) that

$$\left|\begin{aligned}\mathbf{M}_h(\Delta t)\ge\mathbf{M}_h\ &+\left(\theta-\tfrac{1+\nu}{4}(2+\eta)\right)\Delta t^2\left(\mathrm{K}_{TT}^{0}+\mathbf{K}_{TT}^{\delta}\right)\\ &+\left(\theta-\tfrac{1+\nu}{4}\right)\Delta t^2\left(\mathbf{K}_{TT}^{0}+\mathbf{J}_{\mathbf{h}}^{\delta}\right)\\ &-(1+\tfrac{2}{\eta})\tfrac{1+\nu}{4}\Delta t^2\mathbf{K}_{33}^{0}\end{aligned}\right. \tag{167}$$

Accordingly, the quadratic form associated to $\mathbf{M}_h(\Delta t)$ satisfies, for

$$\theta\ge\frac{1+\nu}{4}\,(2+\eta), \tag{168}$$

the following estimate:

$$(\mathbf{M}_h(\Delta t)\mathbb{E}_{\mathbf{h}},\mathbb{E}_{\mathbf{h}})\ge\oint_{\widehat{\Omega}}\widehat{\varepsilon}\mathbf{A}\mathbf{E}_{\mathbf{h}}\cdot\mathbf{E}_{\mathbf{h}}\,\mathrm{d}\widehat{\mathbf{x}}-(1+\tfrac{2}{\eta})\tfrac{1+\nu}{4}\Delta t^2\oint_{\widehat{\Omega}}\widehat{\mu}^{-1}\mathbf{A}_T^{-1}(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\cdot(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\,\mathrm{d}\widehat{\mathbf{x}}, \tag{169}$$

where $\mathbf{E_h}$ denotes the element of $\mathbf{V_h}$ associated to the vector of degrees of freedom $\mathbb{E}_{\mathbf{h}}$.

Theorem 5.1. *Assume that $\theta > (1+\nu)/2$, where ν is the parameter in Lemma 4.3. Then the scheme is stable under the sufficient CFL condition*

$$\frac{\Delta t^2}{4}\|\widehat{\boldsymbol{\beta}}_{\mathrm{h}}\|_\infty^2\le 1-\frac{(1+\nu)^2}{2\theta+\nu+\nu^2}, \tag{170}$$

where the function $\widehat{\boldsymbol{\beta}}_{\mathrm{h}}(\mathbf{x}_T)\in L^\infty(\widehat{S})$ is defined by

$$\widehat{\boldsymbol{\beta}}_{\mathrm{h}}(\mathbf{x}_T):=\sup_{\mathbf{u}_h\in\mathbb{P}_{1,\mathrm{h}}^2}\frac{\displaystyle\oint_{\mathbb{R}}\widehat{\mu}(\cdot,x_T)^{-1}\left(\mathbf{A}_T(\cdot,x_T)\mathbf{u}_h'\cdot\mathbf{u}_h'\right)\mathrm{d}x_3}{\displaystyle\oint_{\mathbb{R}}\widehat{\varepsilon}(\cdot,x_T)\left(\mathbf{A}_T(\cdot,x_T)\mathbf{u}_h\cdot\mathbf{u}_h\right)\mathrm{d}x_3},\quad \textit{with } \mathbb{P}_{1,\mathrm{h}} \textit{ defined in } (155). \tag{171}$$

Proof. We can use the formula (128) for $f=\mathbf{A}_T^{-1}(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\cdot(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)$, which gives

$$\oint_{\widehat{\Omega}}\widehat{\mu}^{-1}\mathbf{A}_T^{-1}(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\cdot(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\,\mathrm{d}\widehat{\mathbf{x}}=\int_{\widehat{S}}\left(\oint_{\mathbb{R}}\widehat{\mu}^{-1}\mathbf{A}_T^{-1}(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T)\cdot\left(\mathbf{e}_3\times\partial_3\widehat{\mathbf{E}}_T\right)\mathrm{d}x_3\right)\mathrm{d}\widehat{\mathbf{x}}_T.$$

Using the definition (46) of $\mathbf{A}_T$ and Lemma 2.3, we leave it to the reader to verify that

$$\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}_T^{-1}(\mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T)\cdot(\mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T)\,\mathrm{d}\widehat{\mathbf{x}} = \oint_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}_T \partial_3 \widehat{\mathbf{E}}_T \cdot \partial_3 \widehat{\mathbf{E}}_T\,\mathrm{d}\widehat{\mathbf{x}}. \tag{172}$$

As a consequence, introducing $\widehat{\boldsymbol{\beta}}_{\mathrm{h}}$ as given by (171) (by analogy with (154)), we have

$$\oint_{\widehat{\Omega}} \widehat{\mu}^{-1}\mathbf{A}_T^{-1}(\mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T)\cdot(\mathbf{e}_3 \times \partial_3 \widehat{\mathbf{E}}_T)\,\mathrm{d}\widehat{\mathbf{x}} \le \|\widehat{\boldsymbol{\beta}}_{\mathrm{h}}\|_\infty^2 \oint_{\widehat{\Omega}} \widehat{\varepsilon}\mathbf{A}_T \widehat{\mathbf{E}}_T \cdot \widehat{\mathbf{E}}_T\,\mathrm{d}\widehat{\mathbf{x}}. \tag{173}$$

Now, from (168), we choose the largest value of η (θ being fixed) such that (169) holds, *i.e.*,

$$\eta = \left(\frac{4\theta}{1+\nu} - 2\right),$$

which is positive since $\theta > (1+\nu)/2$. Now, (169), together with (173) and the chosen value of η, reads

$$(\mathbf{M}_h(\Delta t)\mathbb{E}_{\mathbf{h}}, \mathbb{E}_{\mathbf{h}}) \ge \left(1 - \chi \tfrac{\Delta t^2}{4}\|\widehat{\boldsymbol{\beta}}_{\mathrm{h}}\|_\infty^2\right) \oint_{\widehat{\Omega}} \widehat{\varepsilon}\mathbf{A}_T \widehat{\mathbf{E}}_T \cdot \widehat{\mathbf{E}}_T, \tag{174}$$

with

$$\chi = 1 + 2\,(1+\nu)\left(\tfrac{4\theta}{1+\nu} - 2\right)^{-1} \quad \Rightarrow \quad \chi^{-1} = \frac{2\theta - 1 - \nu}{2\theta + \nu + \nu^2},$$

which proves that the stability of the scheme will be obtained under the CFL condition (170). □

From the theorem above, one can deduce a CFL condition similar to the one of Proposition 5.1 as soon as $\widehat{\alpha}$ is chosen sufficiently large.

Corollary 5.1. *Assume that $\theta > 1/2$ and that the function $\widehat{\alpha}_h$ is chosen large enough. Then the scheme is stable under the sufficient CFL condition*

$$\frac{\Delta t^2}{4}\|\widehat{\boldsymbol{\beta}}_{\mathrm{h}}\|_\infty^2 < 1 - \frac{1}{2\theta}\cdot \tag{175}$$

Proof. First note that it can be shown, since $\theta > 1/2$, that the function

$$\nu \mapsto f_\theta(\nu) := 1 - \frac{(1+\nu)^2}{2\theta + \nu + \nu^2}$$

is smooth, monotonically decreasing on $[0,1]$ and $f_\theta(0) = 1 - 1/(2\theta)$. Therefore since (175) is verified, there exists $\nu_* \in (0,1)$ such that

$$\frac{\Delta t^2}{4}\|\widehat{\boldsymbol{\beta}}_{\mathrm{h}}\|_\infty^2 \le f_\theta(\nu_*).$$

To conclude the proof, it is sufficient to assume that $\widehat{\alpha}_h$ is sufficiently large so that

$$\forall K \in \mathcal{T}_h(\widehat{\Omega}), \quad \widehat{\alpha}_h|_{\partial \widehat{P}_K} \ge \frac{C_I^2}{\nu_*^2 h_K}\cdot$$

□

References

[1] A. Beni Hamad, G. Beck, S. Imperiale and P. Joly, An efficient numerical method for time domain electromagnetic wave propagation in co-axial cables. *Comput. Methods Appl. Math.* **22** (2022) 861–888.

[2] D.N. Arnold, F. Brezzi, B. Cockburn and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* **39** (2002) 1749–1779.

[3] G. Beck, *Modëlisation et ëtude mathëmatique de rëseaux de câbles ëlectriques.* Thëse de doctorat. Universitë Paris-Saclay (2016).

[4] G. Beck, S. Imperiale and P. Joly, Mathematical modelling of multi conductor cables. *Discrete Contin. Dyn. Syst. Ser. S* **8** (2015) 521–546.

[5] M. Bergot and M. Duruflé, High-order optimal edge elements for pyramids, prisms and hexahedra. *J. Comput. Phys.* **232** (2013) 189–213.

[6] A. Burel, S. Imperiale and P. Joly, Solving the homogeneous isotropic linear elastodynamics equations using potentials and finite elements. The case of the rigid boundary condition. *Numer. Anal. Appl.* **5** (2012) 136–143.

[7] J. Chabassier and S. Imperiale, Space/time convergence analysis of a class of conservative schemes for linear wave equations. *C. R. Math.* **355** (2017) 282–289.

[8] J. Chabassier and S. Imperiale, Fourth-order energy-preserving locally implicit time discretization for linear wave equations. *Int. J. Numer. Methods Eng.* **106** (2015) 593–622.

[9] J.-L. Coulomb, F.-X. Zgainski and Y. Marechal, A pyramidal element to link hexahedral, prismatic and tetrahedral edge finite elements. *Institute of Electrical and Electronics Engineers* **33** (1997) 1362–1365.

[10] R. Dautray and J.L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 1. Springer-Verlag (1990).

[11] S. Descombes, S. Lanteri and L. Moya, Locally implicit time integration strategies in a discontinuous Galerkin method for Maxwell's equations. *J. Sci. Comput.* **56** (2013) 190–218.

[12] M. Grote, A. Schneebeli and D. Schötzau, Discontinuous Galerkin finite element method for the wave equation. *SIAM J. Numer. Anal.* **44** (2006) 2408–2431.

[13] M. Grote, A. Schneebeli and D. Schötzau, Interior penalty discontinuous Galerkin method for Maxwell's equations: energy norm error estimates. *J. Comput. Appl. Math.* **204** (2007) 375–386.

[14] M. Grote, A. Schneebeli and D. Schötzau, Interior penalty discontinuous Galerkin method for Maxwell's equations: optimal L 2-norm error estimates. *IMA J. Numer. Anal.* **28** (2008) 440–468.

[15] M. Hochbruck, T. Jahnke and R. Schnaubelt, Convergence of an ADI splitting for Maxwell's equations. *Numer. Math.* **129** (2014) 535–561.

[16] M. Hochbruck and J. Leibold, An implicit-explicit time discretization scheme for second-order semilinear wave equations with application to dynamic boundary conditions. *Numer. Math.* **147** (2021) 869–899.

[17] M. Hochbruck and A. Sturm, Error analysis of a second-order locally implicit method for linear Maxwell's equations. *SIAM J. Numer. Anal.* **54** (2016) 3167–3191.

[18] P. Houston, I. Perugia, A. Schneebeli and D. Schötzau, Interior penalty method for the indefinite time-harmonic Maxwell equations. *Numer. Math.* **100** (2005) 485–518.

[19] S. Imperiale and P. Joly, Error estimates for 1D asymptotic models in coaxial cables with non-homogeneous cross-section. *Adv. Appl. Math. Mech.* **4** (2012) 647–664.

[20] S. Imperiale and P. Joly, Mathematical modeling of electromagnetic wave propagation in heterogeneous lossy coaxial cables with variable cross-section. *Appl. Numer. Math.* **79** (2014) 42–61.

[21] S. Imperiale, *Modëlisation mathëmatique et numërique de capteurs piëzoëlectriques.* Thëse de doctorat, Paris IX (2012).

[22] A. Kameni, F. Loete, S. Ziani, K. Kahalerras and L. Pichon, Time domain modeling of soft faults in wiring system by a nodal Discontinuous Galerkin Method with high-order hexahedral meshes. in *Proc. of the IEEE International Conference on the Computation of Electromagnetic Fields* (2015).

[23] J. Lee and B. Fornberg, A split step approach for the 3-D Maxwell's equations. *J. Comput. Appl. Math.* **158** (2003) 485–505.

[24] J. Lee and B. Fornberg, Some unconditionally stable time stepping methods for the 3D Maxwell's equations. *J. Comput. Appl. Math.* **166** (2004) 497–523.

[25] J. Li, E. Machorro and S. Shields, Numerical study of signal propagation in corrugated coaxial cables. *J. Comput. Appl. Math.* **309** (2017) 230–243.

[26] P. Monk, Finite Element Methods for Maxwell's Equations. Oxford University Press, Oxford (2003).

[27] P. Monk and G.R. Richter, A discontinuous Galerkin method for linear symmetric hyper-bolic systems in inhomogeneous media. *J. Sci. Comput.* **22–23** (2005) 443–477.

[28] J.C. Nědělec, Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **35** (1980) 315–341.

[29] T. Rylander and A. Bondeson, Stability of explicit-implicit hybrid time-stepping schemes for Maxwell's equations. *J. Comput. Phys.* **179** (2002) 426–438.