Convergence of the uniaxial PML method for time-domain electromagnetic scattering problems

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Abstract

In this paper, we propose and study the uniaxial perfectly matched layer (PML) method for three-dimensional time-domain electromagnetic scattering problems, which has a great advantage over the spherical one in dealing with problems involving anisotropic scatterers. The truncated uniaxial PML problem is proved to be well-posed and stable, based on the Laplace transform technique and the energy method. Moreover, the L^2-norm and L^∞-norm error estimates in time are given between the solutions of the original scattering problem and the truncated PML problem, leading to the exponential convergence of the time-domain uniaxial PML method in terms of the thickness and absorbing parameters of the PML layer. The proof depends on the error analysis between the EtM operators for the original scattering problem and the truncated PML problem, which is different from our previous work (SIAM J. Numer. Anal. 58(3) (2020), 1918-1940).

Keywords: Well-posedness, stability, time-domain electromagnetic scattering, uniaxial PML, exponential convergence

1 Introduction

This paper is concerned with the time-domain electromagnetic scattering by a perfectly conducting obstacle which is modeled by the exterior boundary value problem:

$$\begin{cases} \nabla\times\boldsymbol{E}+\mu\partial_t\boldsymbol{H}=\mathbf{0} & \text{in } (\mathbb{R}^3\backslash\overline{\Omega})\times(0,T), & (1.1a)\\ \nabla\times\boldsymbol{H}-\varepsilon\partial_t\boldsymbol{E}=\boldsymbol{J} & \text{in } (\mathbb{R}^3\backslash\overline{\Omega})\times(0,T), & (1.1b)\\ \boldsymbol{n}\times\boldsymbol{E}=\mathbf{0} & \text{on } \Gamma\times(0,T), & (1.1c)\\ \boldsymbol{E}(x,0)=\boldsymbol{H}(x,0)=\mathbf{0} & \text{in } \mathbb{R}^3\backslash\overline{\Omega}, & (1.1d)\\ \hat{x}\times(\partial_t\boldsymbol{E}\times\hat{x})+\hat{x}\times\partial_t\boldsymbol{H}=o\left(|x|^{-1}\right) & \text{as } |x|\to\infty,\ t\in(0,T). & (1.1e)\end{cases}$$

Here, $\boldsymbol{E}$ and $\boldsymbol{H}$ denote the electric and magnetic fields, respectively, $\Omega\subset\mathbb{R}^3$ is a bounded Lipschitz domain with boundary Γ and $\boldsymbol{n}$ is the unit outer normal vector to Γ. Throughout

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this paper, the electric permittivity ε and the magnetic permeability μ are assumed to be positive constants. Equation (1.1e) is the well-known Silver-Müller radiation condition in the time domain with $\hat{x} := x/|x|$.

Time-domain scattering problems have been widely studied recently due to their capability of capturing wide-band signals and modeling more general materials and nonlinearity, including their mathematical analysis (see, e.g., [1,11,26–28,31,33,39,40] and the references quoted there). The well-posedness and stability of solutions to the problem (1.1a)-(1.1e) have been proved in [16] by employing an exact transparent boundary condition (TBC) on a large sphere. Recently, a spherical PML method has been proposed in [42] to solve the problem (1.1a)-(1.1e) efficiently, based on the real coordinate stretching technique associated with $[\mathrm{Re}(s)]^{-1}$ in the Laplace transform domain with the Laplace transform variable $s \in \mathbb{C}_+ := \{s = s_1 + is_2 \in \mathbb{C} : \ s_1 > 0, \ s_2 \in \mathbb{R}\}$, and its exponential convergence has also been established in terms of the thickness and absorbing parameters of the PML layer.

In this paper, we continue our previous study in [42] and propose and study the uniaxial PML method for the problem (1.1a)-(1.1e), based on the real coordinate stretching technique introduced in [42], which uses a cubic domain to define the PML problem and thus is of great advantage over the spherical one in dealing with problems involving anisotropic scatterers. We first establish the existence, uniqueness and stability estimates of the PML problem by the Laplace transform technique and the energy argument and then prove the exponential convergence in both the L^2-norm and the L^∞-norm in time of the time-domain uniaxial PML method. Our proof for the L^2-norm convergence follows naturally from the error estimate between the EtM operators for the original scattering problem and its truncated PML problem established also in the paper, which is different from [42]. The L^∞-norm convergence is obtained directly from the time-domain variational formulation of the original scattering problem and its truncated PML problem with using special test functions.

The PML method was first introduced in the pioneering work [3] of Bérenger in 1994 for efficiently solving the time-dependent Maxwell's equations. Its idea is to surround the computational domain with a specially designed medium layer of finite thickness in which the scattered waves decay rapidly regardless of the wave incident angle, thereby greatly reducing the computational complexity of the scattering problem. Since then, various PML methods have been developed and studied in the literature (see, e.g., [4, 10, 23–25, 29, 35] and the references quoted there). Convergence analysis of the PML method has also been widely studied for time-harmonic acoustic, electromagnetic, and elastic wave scattering problems. For example, the exponential convergence has been established in terms of the thickness of the PML layer in [2, 4, 8, 13, 15, 21, 30, 32] for the circular or spherical PML method and in [5–7, 14, 17, 19, 20] for the uniaxial (or Cartesian) PML method. Among them, the proof in [2] is based on the error estimate between the electric-to-magnetic (EtM) operators for the original electromagnetic scattering problem and its truncated PML problem, while the key ingredient of the proof in [13] and [14] is the decay property of the PML extensions defined by the series solution and the integral representation solution, respectively. On the other hand, there are also several works on convergence analysis of the time-domain PML method for transient scattering problems. For two-dimensional transient acoustic scattering problems, the exponential convergence was proved in [12] for the circular PML method and in [18] for the uniaxial PML method, based on the complex coordinate stretching technique. For the 3D time-domain electromagnetic scattering problem (1.1a)-(1.1e), the spherical PML method was proposed in [42] based on the real coordinate stretching technique associated with $[\mathrm{Re}(s)]^{-1}$ in the Laplace transform domain with the Laplace transform variable $s \in \mathbb{C}_+$, and its exponential convergence was established by means of the energy argument and the exponential decay estimates of the stretched dyadic Green's function for the Maxwell equations in the free space. In addition, we refer to [1] for the well-posedness and stability estimates of the time-domain PML method for the two-dimensional acoustic-elastic interaction problem, and to [41] for the convergence analysis of the PML method for the fluid-solid interaction problem above an unbounded rough surface.

The remaining part of this paper is as follows. In Section 2, we introduce some basic Sobolev spaces needed in this paper. In Section 3, the well-posedness of the time-domain electromagnetic scattering problem is presented, and some important properties are given for the transparent boundary condition (TBC) in the Cartesian coordinate. In Section 4, we propose the uniaxial PML method in the Cartesian coordinate, study the well-posedness of the truncated PML problem and establish its exponential convergence. Some conclusions are given in Section 5.

2 Functional spaces

We briefly introduce the Sobolev space $H(\mathrm{curl}, \cdot)$ and its related trace spaces which are used in this paper. For a bounded domain $D \subset \mathbb{R}^3$ with Lipschitz continuous boundary Σ, the Sobolev space $H(\mathrm{curl}, D)$ is defined by

$$H(\mathrm{curl}, D) := \{\boldsymbol{u} \in L^2(D)^3 : \ \nabla \times \boldsymbol{u} \in L^2(D)^3\}$$

which is a Hilbert space equipped with the norm

$$\|\boldsymbol{u}\|_{H(\mathrm{curl},D)} = \left(\|\boldsymbol{u}\|^2_{L^2(D)^3} + \|\nabla \times \boldsymbol{u}\|^2_{L^2(D)^3}\right)^{1/2}.$$

Denote by $\boldsymbol{u}_\Sigma = \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n})|_\Sigma$ the tangential component of $\boldsymbol{u}$ on Σ, where $\boldsymbol{n}$ denotes the unit outward normal vector on Σ. By [9] we have the following bounded and surjective trace operators:

$$\begin{aligned}
&\gamma : \ H^1(D) \to H^{1/2}(\Sigma), \quad \gamma\varphi = \varphi \quad \text{on } \ \Sigma,\\
&\gamma_t : \ H(\mathrm{curl}, D) \to H^{-1/2}(\mathrm{Div}, \Sigma), \quad \gamma_t \boldsymbol{u} = \boldsymbol{u} \times \boldsymbol{n} \quad \text{on } \ \Sigma,\\
&\gamma_T : \ H(\mathrm{curl}, D) \to H^{-1/2}(\mathrm{Curl}, \Sigma), \quad \gamma_T \boldsymbol{u} = \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) \quad \text{on } \ \Sigma,
\end{aligned}$$

where γ_t and γ_T are known as the tangential trace and tangential components trace operators, and Div and Curl denote the surface divergence and surface scalar curl operators, respectively (for the detailed definition of $H^{-1/2}(\mathrm{Div}, \Sigma)$ and $H^{-1/2}(\mathrm{Curl}, \Sigma)$, we refer to [9]). By [9] again we know that $H^{-1/2}(\mathrm{Div}, \Sigma)$ and $H^{-1/2}(\mathrm{Curl}, \Sigma)$ form a dual pairing satisfying the integration by parts formula

$$(\boldsymbol{u}, \nabla \times \boldsymbol{v})_D - (\nabla \times \boldsymbol{u}, \boldsymbol{v})_D = \langle \gamma_t \boldsymbol{u}, \gamma_T \boldsymbol{v} \rangle_\Sigma \quad \forall \ \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, D), \tag{2.1}$$

where $(\cdot, \cdot)_D$ and $\langle \cdot, \cdot \rangle_\Sigma$ denote the L^2-inner product on D and the dual product between $H^{-1/2}(\mathrm{Div}, \Sigma)$ and $H^{-1/2}(\mathrm{Curl}, \Sigma)$, respectively.

For any relatively closed and locally Lipschitz continuous subset $S \subset \Sigma$, the subspace with zero tangential trace on S is denoted as

$$H_S(\mathrm{curl}, D) := \{\boldsymbol{u} \in H(\mathrm{curl}, D) : \gamma_t \boldsymbol{u} = 0 \ \text{ on } \ S\}.$$

In particular, if $S = \Sigma$ then we write $H_0(\mathrm{curl}, D) := H_\Sigma(\mathrm{curl}, D)$.

3 The well-posedness of the scattering problem

Let Ω be contained in the interior of the cuboid $B_1 := \{x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 : |x_j| < L_j/2, j = 1, 2, 3\}$ with boundary $\Gamma_1 = \partial B_1$. Denote by $\boldsymbol{n}_1$ the unit outward normal to Γ_1. The computational domain $B_1 \backslash \overline{\Omega}$ is denoted by Ω_1. In this section, we assume that the current density $\boldsymbol{J}$ is compactly supported in B_1 with

$$\boldsymbol{J} \in H^{10}(0, T; L^2(\Omega_1)^3), \quad \partial_t^j \boldsymbol{J}|_{t=0} = 0, \quad j = 0, 1, 2, 3, \ldots 9 \tag{3.1}$$

and that $\boldsymbol{J}$ is extended so that

$$\boldsymbol{J} \in H^{10}(0, \infty; L^2(\Omega_1)^3), \quad \|\boldsymbol{J}\|_{H^{10}(0,\infty;L^2(\Omega_1)^3)} \le C \|\boldsymbol{J}\|_{H^{10}(0,T;L^2(\Omega_1)^3)}. \tag{3.2}$$

Define the following time-domain transparent boundary condition (TBC) on Γ_1:

$$\mathscr{T}[\boldsymbol{E}_{\Gamma_1}] = \boldsymbol{H} \times \boldsymbol{n}_1 \quad \text{on} \quad \Gamma_1 \times (0, T) \tag{3.3}$$

which is essentially an electric-to-magnetic (EtM) Calderón operator, where $\boldsymbol{E}_{\Gamma_1} := \boldsymbol{n}_1 \times (\boldsymbol{E} \times \boldsymbol{n}_1)|_{\Gamma_1}$ is the tangential component of $\boldsymbol{E}$ on Γ_1, $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the exterior Maxwell's equations

$$\begin{cases} \nabla \times \boldsymbol{E} + \mu \partial_t \boldsymbol{H} = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1 \times (0, T), \\ \nabla \times \boldsymbol{H} - \varepsilon \partial_t \boldsymbol{E} = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1 \times (0, T), \\ \boldsymbol{E}(x, 0) = \boldsymbol{H}(x, 0) = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1, \\ \hat{x} \times (\partial_t \boldsymbol{E} \times \hat{x}) + \hat{x} \times \partial_t \boldsymbol{H} = o\left(|x|^{-1}\right) & \text{as} \quad |x| \to \infty, \quad t \in (0, T). \end{cases}$$

Then the original scattering problem (1.1a)-(1.1e) can be equivalently reduced into the initial boundary value problem in a bounded domain $\Omega_1 \times (0, T)$:

$$\begin{cases} \nabla \times \boldsymbol{E} + \mu \partial_t \boldsymbol{H} = \boldsymbol{0} & \text{in} \quad \Omega_1 \times (0, T), \\ \nabla \times \boldsymbol{H} - \varepsilon \partial_t \boldsymbol{E} = \boldsymbol{J} & \text{in} \quad \Omega_1 \times (0, T), \\ \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0} & \text{on} \quad \Gamma \times (0, T), \\ \boldsymbol{E}(x, 0) = \boldsymbol{H}(x, 0) = \boldsymbol{0} & \text{in} \quad \Omega_1, \\ \mathscr{T}[\boldsymbol{E}_{\Gamma_1}] = \boldsymbol{H} \times \boldsymbol{n}_1 & \text{on} \quad \Gamma_1 \times (0, T). \end{cases} \tag{3.4}$$

The well-posedness of the original scattering problem (1.1a)-(1.1e) has been established in [16] by using the transparent boundary condition on a sphere. Thus the problem (3.4) is also well-posed since it is equivalent to the problem (1.1a)-(1.1e). However, for convenience of the subsequent use in the following sections, we study the problem (3.4) directly by studying the property of the EtM operator $\mathscr{T}$, based on the Laplace transform technique [22, 36].

For a Banach space $\mathbb{E}$ we denote by $\mathcal{D}_+(\mathbb{E}) = \{u \in C_0^\infty(\mathbb{R}; \mathbb{E}), u \text{ vanishes on } (-\infty, 0)\}$ the set of smooth and compactly supported $\mathbb{E}$-valued causal functions on the real line. Further, let $\mathcal{D}'_+(\mathbb{E})$ denote the set of $\mathbb{E}$-valued causal distributions on the real line and let $\mathscr{S}'_+(\mathbb{E})$ be the set of the corresponding tempered distributions. Set

$$\mathcal{L}'_+(\mathbb{E}) := \left\{ f \in \mathcal{D}'_+(\mathbb{E}), e^{-\sigma_0 t} f(t) \in \mathscr{S}'_+(\mathbb{E}) \text{ for some } \sigma_0 \in \mathbb{R} \right\}.$$

The Laplace transform of $f \in \mathcal{L}'_+(\mathbb{E})$ is defined by

$$\mathscr{L}[f(t)](s) = \int_0^\infty f(t) e^{-st} dt, \quad s = s_1 + i s_2 \quad \text{for } s_1 > \sigma_0.$$

If $s_1 = 0$, then the Laplace transform coincides with the usual Fourier transform.

In what follows, we aim to find solutions of the problem (1.1a)-(1.1e) in the Sobolev space (see [16, Theorem 3.1]):

$$\boldsymbol{E}(x,t) \in L^2(0,T;H(\mathrm{curl},\Omega_1)) \cap H^1\left(0,T;L^2(\Omega_1)^3\right),$$
$$\boldsymbol{H}(x,t) \in L^2(0,T;H(\mathrm{curl},\Omega_1)) \cap H^1\left(0,T;L^2(\Omega_1)^3\right)$$

which are L^2-integrable in the time variable. It can be easily seen that $e^{-\sigma_0 t}\boldsymbol{E}(\cdot,t) \in \mathscr{S}'_+(L^2(\Omega_1)^3)$ for $\sigma_0 > 0$. Thus $\boldsymbol{E}(\cdot,t)$ and $\boldsymbol{H}(\cdot,t)$ are Laplace-transformable in time. Define

$$\check{\boldsymbol{E}}(x,s) = \mathscr{L}(\boldsymbol{E})(x,s) = \int_0^\infty e^{-st}\boldsymbol{E}(x,t)dt,$$
$$\check{\boldsymbol{H}}(x,s) = \mathscr{L}(\boldsymbol{H})(x,s) = \int_0^\infty e^{-st}\boldsymbol{H}(x,t)dt$$

for any $s \in \mathbb{C}_+ := \{s = s_1 + is_2 \in \mathbb{C} : \ s_1 > 0, \ s_2 \in \mathbb{R}\}$, which is usually adopted in the time domain scattering problems (see, e.g., [16,42]). Let $\mathscr{B} : H^{-1/2}(\mathrm{Curl},\Gamma_1) \to H^{-1/2}(\mathrm{Div},\Gamma_1)$ be the EtM operator

$$\mathscr{B}[\check{\boldsymbol{E}}_{\Gamma_1}] = \check{\boldsymbol{H}} \times \boldsymbol{n}_1 \quad \text{on} \quad \Gamma_1, \tag{3.5}$$

where $\check{\boldsymbol{E}}$ and $\check{\boldsymbol{H}}$ satisfy the exterior Maxwell's equation in the Laplace domain

$$\begin{cases} \nabla \times \check{\boldsymbol{E}} + \mu s \check{\boldsymbol{H}} = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1, \\ \nabla \times \check{\boldsymbol{H}} - \varepsilon s \check{\boldsymbol{E}} = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1, \\ \hat{x} \times (\check{\boldsymbol{E}} \times \hat{x}) + \hat{x} \times \check{\boldsymbol{H}} = o\left(\frac{1}{|x|}\right) & \text{as} \quad |x| \to \infty. \end{cases} \tag{3.6}$$

It is obvious that $\mathscr{T} = \mathscr{L}^{-1} \circ \mathscr{B} \circ \mathscr{L}$, where $\mathscr{L}^{-1}$ is the inverse Laplace transform given by

$$f = \mathscr{F}^{-1}(e^{s_1 t}\mathscr{L}(f)(s)) = \frac{1}{2\pi i}\int_\gamma e^{st}\mathscr{L}(f)(s)ds$$

where γ is the vertical line $\mathrm{Re}(s) = s_1 > \sigma_0$ in the complex plane.

For each $s \in \mathbb{C}_+$ it is known that, by the Lax-Milgram theorem the problem (3.6) has a unique solution $(\check{\boldsymbol{E}}, \check{\boldsymbol{H}}) \in H(\mathrm{curl}, \mathbb{R}^3 \backslash \overline{B}_1)$. Thus the operator $\mathscr{B}$ is a well-defined, continuous linear operator.

Lemma 3.1. *For each $s \in \mathbb{C}_+$, $\mathscr{B} : H^{-1/2}(\mathrm{Curl},\Gamma_1) \to H^{-1/2}(\mathrm{Div},\Gamma_1)$ is bounded with the estimate*

$$\|\mathscr{B}\|_{L(H^{-1/2}(\mathrm{Curl},\Gamma_1),H^{-1/2}(\mathrm{Div},\Gamma_1))} \lesssim |s|^{-1} + |s|, \tag{3.7}$$

where $L(X,Y)$ denotes the standard space of bounded linear operators from the Hilbert space X to the Hilbert space Y. Further, we have

$$\mathrm{Re}\langle \mathscr{B}\boldsymbol{\omega}, \boldsymbol{\omega}\rangle_{\Gamma_1} \geq 0 \quad \textit{for any } \boldsymbol{\omega} \in H^{-1/2}(\mathrm{Curl},\Gamma_1), \tag{3.8}$$

where $\langle \cdot \rangle_{\Gamma_1}$ denotes the dual product between $H^{-1/2}(\mathrm{Div},\Gamma_1)$ and $H^{-1/2}(\mathrm{Curl},\Gamma_1)$.

Proof. First, eliminating $\check{\boldsymbol{H}}$ from (3.6) and multiplying both sides of the resulting equation with $\overline{\boldsymbol{V}} \in H(\mathrm{curl}, \mathbb{R}^3\backslash\overline{B}_1)$ yield

$$\begin{aligned}\left|\langle \mathscr{B}[\check{\boldsymbol{E}}_{\Gamma_1}], \gamma_T \boldsymbol{V}\rangle_{\Gamma_1}\right| &= \left|\int_{\mathbb{R}^3\backslash\overline{B}_1} \left[(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}\cdot\nabla\times\overline{\boldsymbol{V}} + \varepsilon s\check{\boldsymbol{E}}\cdot\overline{\boldsymbol{V}}dx\right]\right| \\ &\lesssim (|s|^{-1}+|s|)\|\check{\boldsymbol{E}}\|_{H(\mathrm{curl},\mathbb{R}^3\backslash\overline{B}_1)}\|\boldsymbol{V}\|_{H(\mathrm{curl},\mathbb{R}^3\backslash\overline{B}_1)},\end{aligned}$$

which implies (3.7).

Now, for any $\boldsymbol{\omega} \in H^{-1/2}(\mathrm{Curl}, \Gamma_1)$ suppose $(\check{\boldsymbol{E}}, \check{\boldsymbol{H}})$ is the solution to the problem (3.6) satisfying the boundary condition $\gamma_T\check{\boldsymbol{E}} = \boldsymbol{\omega}$ on Γ_1. Let $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ contain the domain B_1. Eliminating $\check{\boldsymbol{H}}$ from (3.6) and integrating by parts the resulting equation multiplied with $\overline{\check{\boldsymbol{E}}}$ over $B_R\backslash\overline{B}_1$, we obtain that

$$\begin{aligned}&\int_{B_R\backslash\overline{B}_1} \left((\mu s)^{-1}|\nabla\times\check{\boldsymbol{E}}|^2 + \varepsilon s|\check{\boldsymbol{E}}|^2\right)dx - \langle\mathscr{B}\boldsymbol{\omega},\boldsymbol{\omega}\rangle_{\Gamma_1} \\ &\qquad + \int_{\partial B_R} \hat{x}\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}\cdot\overline{\check{\boldsymbol{E}}}d\gamma = 0.\end{aligned} \tag{3.9}$$

Taking the real part of (3.9) and noting that

$$\begin{aligned}&\left|\hat{x}\times(\check{\boldsymbol{E}}\times\hat{x}) - \hat{x}\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}\right|^2 \\ &\quad = |\hat{x}\times(\check{\boldsymbol{E}}\times\hat{x})|^2 + |\hat{x}\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}|^2 - 2\mathrm{Re}(\hat{x}\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}})\cdot\overline{\check{\boldsymbol{E}}},\end{aligned}$$

we have

$$\begin{aligned}&\frac{s_1}{\mu|s|^2}\|\nabla\times\check{\boldsymbol{E}}\|^2_{L^2(B_R\backslash\overline{B}_1)^3} + \varepsilon s_1\|\check{\boldsymbol{E}}\|^2_{L^2(B_R\backslash\overline{B}_1)^3} - \mathrm{Re}\langle\mathscr{B}\boldsymbol{\omega},\boldsymbol{\omega}\rangle_{\Gamma_1} \\ &\qquad + \frac{1}{2}\left\|\hat{x}\times(\check{\boldsymbol{E}}\times\hat{x})\right\|^2_{L^2(\partial B_R)^3} + \frac{1}{2}\left\|\hat{x}\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}\right\|^2_{L^2(\partial B_R)^3} \\ &\quad = \frac{1}{2}\left\|\hat{x}\times(\check{\boldsymbol{E}}\times\hat{x}) - \hat{x}\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}\right\|^2_{L^2(\partial B_R)^3}.\end{aligned} \tag{3.10}$$

By the Silver-Müller radiation condition (1.1e) in the s-domain, it is known that the right-hand side of (3.10) tends to zero as $R \to \infty$. This implies that $\mathrm{Re}\langle\mathscr{B}\boldsymbol{\omega},\boldsymbol{\omega}\rangle_{\Gamma_1} \ge 0$. The proof is thus complete. ☐

By using Lemma 3.1 and [40, Lemmas 4.5-4.6], the time-domain EtM operator $\mathscr{T}$ has the following positive properties which will be used in the error analysis of the time-domain PML solution.

Lemma 3.2. *Given $\xi \ge 0$ and $\boldsymbol{\omega}(\cdot, t) \in L^2(0, \xi; H^{-1/2}(\mathrm{Curl}, \Gamma_1))$ it holds that*

$$\mathrm{Re}\int_{\Gamma_1}\int_0^{\xi}\left(\int_0^t \mathscr{C}[\boldsymbol{\omega}](x,\tau)d\tau\right)\overline{\boldsymbol{\omega}}(x,t)dtd\gamma \ge 0,$$

where $\mathscr{C} = \mathscr{L}^{-1}\circ s\mathscr{B}\circ\mathscr{L}$.

Proof. We extend $\boldsymbol{\omega}$ by $\mathbf{0}$ with respect to t outside the interval $[0, \xi]$ which is denoted again by $\boldsymbol{\omega}$. Recall the Parseval identity for the Laplace transform (see [22, (2.46)])

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\check{\boldsymbol{u}}(s)\cdot\check{\boldsymbol{v}}(s)ds_2 = \int_0^{\infty} e^{-2s_1t}\boldsymbol{u}(t)\cdot\boldsymbol{v}(t)dt \tag{3.11}$$

for all $s_1 > \lambda$, where λ is the abscissa of convergence for the Laplace transform of $\boldsymbol{u}$ and $\boldsymbol{v}$. This, together with Lemma 3.1 and the Laplace transform property $\mathscr{L}\left(\int_0^t \boldsymbol{u}(\tau)d\tau\right)(s) = s^{-1}\mathscr{L}(\boldsymbol{u})(s)$, implies that

$$
\begin{aligned}
&\text{Re}\int_{\Gamma_1}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{C}[\boldsymbol{\omega}](x,\tau)d\tau\right)\overline{\boldsymbol{\omega}}(x,t)dtd\gamma \\
&= \text{Re}\int_{\Gamma_1}\int_0^\infty e^{-2s_1 t}\left(\int_0^t \mathscr{L}^{-1}\circ s\mathscr{B}\circ\mathscr{L}[\boldsymbol{\omega}](x,\tau)d\tau\right)\overline{\boldsymbol{\omega}}(x,t)dtd\gamma \\
&= \frac{1}{2\pi}\text{Re}\int_{-\infty}^\infty\int_{\Gamma_1}\mathscr{B}\circ\mathscr{L}(\boldsymbol{\omega})\cdot\mathscr{L}(\overline{\boldsymbol{\omega}})(s)d\gamma ds_2 \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \text{Re}\langle\mathscr{B}[\check{\boldsymbol{\omega}}],\check{\boldsymbol{\omega}}\rangle_{\Gamma_1}ds_2 \\
&\geq 0.
\end{aligned}
$$

This completes the proof after taking $s_1 \to 0$. □

Lemma 3.3. *Given $\xi \geq 0$ and $\boldsymbol{\omega}(\cdot,t) \in H^1(0,\xi;H^{-1/2}(\text{Curl},\Gamma_1))$ with $\boldsymbol{\omega}(\cdot,0) = \boldsymbol{0}$, it holds that*

$$
\text{Re}\int_{\Gamma_1}\int_0^\xi\left(\int_0^t \mathscr{C}[\partial_\tau\boldsymbol{\omega}](x,\tau)d\tau\right)\partial_\tau\overline{\boldsymbol{\omega}}(x,t)dtd\gamma \geq 0.
$$

Proof. The proof is similar to that for Lemma 3.2 with $\boldsymbol{\omega}$ replaced by $\partial_\tau\boldsymbol{\omega}$. So we omit the detailed proof. □

We now introduce the equivalent variational formulation in the Laplace transform domain to the problem (3.4). To this end, eliminate the magnetic field $\boldsymbol{H}$ and take the Laplace transform of (3.4) to get

$$
\begin{cases}
\nabla\times[(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}] + \varepsilon s\check{\boldsymbol{E}} = -\check{\boldsymbol{J}} & \text{in} \quad \Omega_1, \\
\boldsymbol{n}\times\check{\boldsymbol{E}} = \boldsymbol{0} & \text{on} \quad \Gamma, \\
\mathscr{B}[\check{\boldsymbol{E}}_{\Gamma_1}] = -(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}\times\boldsymbol{n}_1 & \text{on} \quad \Gamma_1.
\end{cases} \tag{3.12}
$$

The variational formulation of (3.12) is then as follows: find a solution $\check{\boldsymbol{E}} \in H_\Gamma(\text{curl},\Omega_1)$ such that

$$
a(\check{\boldsymbol{E}},\boldsymbol{V}) = -\int_{\Omega_1}\check{\boldsymbol{J}}\cdot\overline{\boldsymbol{V}}dx, \quad \forall\,\boldsymbol{V}\in H_\Gamma(\text{curl},\Omega_1), \tag{3.13}
$$

where the sesquilinear form $a(\cdot,\cdot)$ is defined as

$$
a(\check{\boldsymbol{E}},\boldsymbol{V}) = \int_{\Omega_1}\left[(s\mu)^{-1}(\nabla\times\check{\boldsymbol{E}})\cdot(\nabla\times\overline{\boldsymbol{V}}) + \varepsilon s\check{\boldsymbol{E}}\cdot\overline{\boldsymbol{V}}\right]dx + \langle\mathscr{B}[\check{\boldsymbol{E}}_{\Gamma_1}],\boldsymbol{V}_{\Gamma_1}\rangle_{\Gamma_1}. \tag{3.14}
$$

By Lemma 3.1 it is easy to see that $a(\cdot,\cdot)$ is uniformly coercive, that is,

$$
\begin{aligned}
\text{Re}[a(\check{\boldsymbol{E}},\check{\boldsymbol{E}})] &\gtrsim \frac{s_1}{|s|^2}(\|\nabla\times\check{\boldsymbol{E}}\|^2_{L^2(\Omega_1)^3} + \|s\check{\boldsymbol{E}}\|^2_{L^2(\Omega_1)^3}) \\
&\geq s_1\min\{|s|^{-2},1\}\|\check{\boldsymbol{E}}\|^2_{H(\text{curl},\Omega_1)}.
\end{aligned} \tag{3.15}
$$

Then, by the Lax-Milgram theorem the problem (3.12) is well-posed for each $s \in \mathbb{C}_+$. Thus, and by the energy argument in conjunction with the inversion theorem of the Laplace transform (cf. [16]) the well-posedness of the problem (3.4) follows. In particular, $\mathscr{T}[\boldsymbol{E}_{\Gamma_1}] \in L^2\left(0,T;H^{-1/2}(\text{Div},\Gamma_1)\right)$.

4 The uniaxial PML method

In practical applications, the scattering problems may involve anisotropic scatterers. In this case, the uniaxial PML method has a big advantage over the circular or spherical PML method as it provides greater flexibility and efficiency in solving such problems. Thus, in this section, we propose and study the uniaxial PML method for solving the time-domain electromagnetic scattering problem (1.1a)-(1.1e).

4.1 The PML equation in the Cartesian coordinates

In this subsection, we derive the PML equation in the Cartesian coordinates. To this end, define $B_2 := \{x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 : |x_j| < L_j/2 + d_j,\ j = 1, 2, 3\}$ with boundary $\Gamma_2 = \partial B_2$

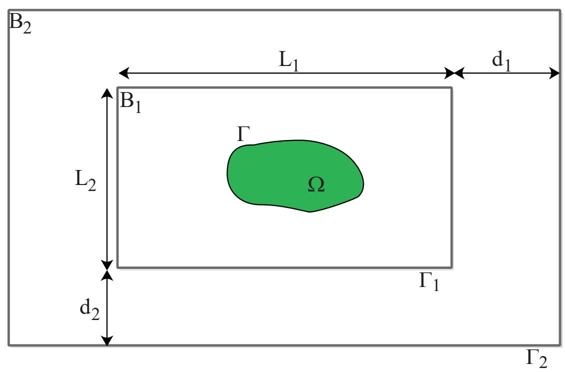

Figure 1: Geometric configuration of the uniaxial PML

which is a cubic domain surrounding B_1. Denote by $\boldsymbol{n}_2$ the unit outward normal to Γ_2. Let $\Omega^{\rm PML} = B_2 \backslash \overline{B}_1$ be the PML layer and let $\Omega_2 = B_2 \backslash \overline{\Omega}$ be the truncated PML domain. See Figure 1 for the uniaxial PML geometry.

For $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, let $s_1 > 0$ be an arbitrarily fixed parameter and let us define the PML medium property as

$$\alpha_j(x_j) = 1 + s_1^{-1}\sigma_j(x_j), \quad j = 1, 2, 3,$$

where

$$\sigma_j(x_j) = \begin{cases} 0, & |x_j| \le L_j/2, \\ \widetilde{\sigma}_j \left(\dfrac{|x_j| - L_j/2}{d_j} \right)^m, & L_j/2 < |x_j| \le L_j/2 + d_j, \\ \widetilde{\sigma}_j, & L_j/2 + d_j < |x_j| < \infty \end{cases} \tag{4.1}$$

with positive constants $\widetilde{\sigma}_j$, $j = 1, 2, 3$, and integer $m \ge 1$. In what follows, we will take the real part of the Laplace transform variable $s \in \mathbb{C}_+$ to be s_1, that is, $\mathrm{Re}(s) = s_1$.

In the rest of this paper, we always make the following assumptions on the thickness of the PML layer and the parameters $\widetilde{\sigma}_j$, which are reasonable in our model:

$$d_1 = d_2 = d_3 := d, \qquad L = \max\{L_1, L_2, L_3\} \le C_0 d, \tag{4.2}$$

$$\widetilde{\sigma}_1 = \widetilde{\sigma}_2 = \widetilde{\sigma}_3 := \sigma_0 > 0 \tag{4.3}$$

for a fixed generic constant C_0. Under the assumptions (4.2) and (4.3) we have

$$\int_0^{L_j/2+d_j} \sigma_j(\tau)d\tau = \frac{\sigma_0 d}{m+1}, \quad j=1,2,3. \tag{4.4}$$

We remark that the constant assumption on d_j and $\widetilde{\sigma}_j$ in (4.2)-(4.3) is only to simplify the convergence analysis but not mandatory. We now introduce the real stretched Cartesian coordinates $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)^\top$ with

$$\widetilde{x}_j = \int_0^{x_j} \alpha_j(\tau)d\tau, \quad j=1,2,3. \tag{4.5}$$

We now derive the PML extension under the stretched coordinates $\widetilde{x}$ by following [42]. By [34, Theorem 12.2] the solution of the exterior problem (3.6) in $\mathbb{R}^3\backslash\overline{B}_1$ can be given by the integral representation

$$\check{\boldsymbol{E}}(x) = -\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{q})(x) - \boldsymbol{\Psi}_{\rm DL}(\boldsymbol{p})(x), \quad \check{\boldsymbol{H}}(x) = -(\mu s)^{-1}{\rm curl}\, \check{\boldsymbol{E}}(x), \tag{4.6}$$

where

$$\boldsymbol{\Psi}_{\rm SL}(\boldsymbol{q}) = \int_{\Gamma_1} \mathbb{G}^T(s,x,y)\boldsymbol{q}(y)d\gamma(y), \quad \boldsymbol{\Psi}_{\rm DL}(\boldsymbol{p}) = \int_{\Gamma_1} ({\rm curl}_y \mathbb{G})^T(s,x,y)\boldsymbol{p}(y)d\gamma(y),$$

denote the Maxwell single- and double-layer potentials, respectively, $\boldsymbol{p} = \gamma_t(\check{\boldsymbol{E}})$ and $\boldsymbol{q} = \gamma_t({\rm curl}\ \check{\boldsymbol{E}})$ are the Dirichlet trace and Neumann trace of the solution on Γ_1, and $\mathbb{G}$ is the dyadic Green's function for Maxwell's equations in the free space defined as a matrix function (see [34, (12.1)]):

$$\mathbb{G}(s,x,y) = \Phi_s(x,y)\mathbb{I} + \frac{1}{k^2}\nabla_y\nabla_y\Phi_s(x,y), \quad x \neq y.$$

Hereafter, $s \in \mathbb{C}_+$ with ${\rm Re}(s) = s_1$, $\mathbb{I}$ is the 3×3 identity matrix, $\Phi_s(x,y)$ is the fundamental solution of the Helmholtz equation with complex wave number $k = i\sqrt{\varepsilon\mu}s$ defined by

$$\Phi_s(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} = \frac{e^{-\sqrt{\varepsilon\mu}s|x-y|}}{4\pi|x-y|}, \tag{4.7}$$

and $\nabla_y\nabla_y\Phi_s(x,y)$ is the Hessian matrix of $\Phi_s(x,y)$ with its (l,m)th element

$$(\nabla_y\nabla_y\Phi_s(x,y))_{l,m} = \frac{\partial^2\Phi_s(x,y)}{\partial_{y_l}\partial_{y_m}}, \quad 1 \le l,m \le 3. \tag{4.8}$$

Now, for $x \in \mathbb{R}^3 \setminus \overline{B}_1$ define the stretched single- and double-layer potentials

$$\widetilde{\boldsymbol{\Psi}}_{\rm SL}(\boldsymbol{q}) = \int_{\Gamma_1} \widetilde{\mathbb{G}}^T(s,x,y)\boldsymbol{q}(y)d\gamma(y), \quad \widetilde{\boldsymbol{\Psi}}_{\rm DL}(\boldsymbol{p}) = \int_{\Gamma_1} ({\rm curl}_y \widetilde{\mathbb{G}})^T(s,x,y)\boldsymbol{p}(y)d\gamma(y),$$

where the stretched dyadic Green's function

$$\widetilde{\mathbb{G}}(s,x,y) = \widetilde{\Phi}_s(x,y)\mathbb{I} + \frac{1}{k^2}\nabla_y\nabla_y\widetilde{\Phi}_s(x,y), \quad x \neq y, \quad k = i\sqrt{\varepsilon\mu}s \tag{4.9}$$

with the stretched fundamental solution and the complex distance

$$\widetilde{\Phi}_s(x,y) = \frac{e^{-\sqrt{\varepsilon\mu}\rho_s(\widetilde{x},y)}}{4\pi\rho_s(\widetilde{x},y)s^{-1}}, \qquad \rho_s(\widetilde{x},y) = s|\widetilde{x}-y|. \tag{4.10}$$

For any $\boldsymbol{p} \in H^{-1/2}(\mathrm{Div}, \Gamma_1)$ and $\boldsymbol{q} \in H^{-1/2}(\mathrm{Div}, \Gamma_1)$, define

$$\mathbb{E}(\boldsymbol{p}, \boldsymbol{q})(x) := -\widetilde{\boldsymbol{\Psi}}_{\mathrm{SL}}(\boldsymbol{q})(x) - \widetilde{\boldsymbol{\Psi}}_{\mathrm{DL}}(\boldsymbol{p})(x), \quad x \in \mathbb{R}^3 \backslash \overline{B}_1 \tag{4.11}$$

to be the PML extensions in the s-domain of $\boldsymbol{p}$ and $\boldsymbol{q}$. Introduce the stretched curl operator acting on vector $\boldsymbol{u} = (u_1, u_2, u_3)^\top$:

$$\widetilde{\mathrm{curl}}\ \boldsymbol{u} = \widetilde{\nabla} \times \boldsymbol{u} := \left(\frac{\partial u_3}{\partial \widetilde{x}_2} - \frac{\partial u_2}{\partial \widetilde{x}_3}, \frac{\partial u_1}{\partial \widetilde{x}_3} - \frac{\partial u_3}{\partial \widetilde{x}_1}, \frac{\partial u_2}{\partial \widetilde{x}_1} - \frac{\partial u_1}{\partial \widetilde{x}_2} \right)^\top = \mathbb{A} \nabla \times \mathbb{B} \boldsymbol{u}$$

with the diagonal matrices

$$\mathbb{A} = \mathrm{diag}\left\{ \frac{1}{\alpha_2 \alpha_3}, \frac{1}{\alpha_1 \alpha_3}, \frac{1}{\alpha_1 \alpha_2} \right\} \quad \text{and} \quad \mathbb{B} = \mathrm{diag}\{\alpha_1, \alpha_2, \alpha_3\}. \tag{4.12}$$

Then the PML extension in the s-domain in $\mathbb{R}^3 \backslash \overline{B}_1$ of $\gamma_t(\check{\boldsymbol{E}})|_{\Gamma_1}$ and $\gamma_t(\mathrm{curl}\ \check{\boldsymbol{E}})|_{\Gamma_1}$ is defined as

$$\check{\tilde{\boldsymbol{E}}}(x) = \mathbb{E}(\gamma_t(\check{\boldsymbol{E}}), \gamma_t(\mathrm{curl}\ \check{\boldsymbol{E}})), \quad x \in \mathbb{R}^3 \backslash \overline{B}_1. \tag{4.13}$$

Define $\check{\widetilde{\boldsymbol{H}}}(x) := -(\mu s)^{-1} \widetilde{\mathrm{curl}}\ \check{\tilde{\boldsymbol{E}}}(x)$ for $x \in \mathbb{R}^3 \backslash \overline{B}_1$. Then it is easy to see that $(\check{\tilde{\boldsymbol{E}}}, \check{\widetilde{\boldsymbol{H}}})$ satisfies the Maxwell equation in the s-domain:

$$\widetilde{\nabla} \times \check{\tilde{\boldsymbol{E}}} + \mu s \check{\widetilde{\boldsymbol{H}}} = \boldsymbol{0}, \quad \widetilde{\nabla} \times \check{\widetilde{\boldsymbol{H}}} - \varepsilon s \check{\tilde{\boldsymbol{E}}} = \boldsymbol{0} \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{B}_1. \tag{4.14}$$

Define

$$(\boldsymbol{E}^{\mathrm{PML}}, \boldsymbol{H}^{\mathrm{PML}}) := \mathbb{B}(\mathscr{L}^{-1}(\check{\tilde{\boldsymbol{E}}}), \mathscr{L}^{-1}(\check{\widetilde{\boldsymbol{H}}})).$$

Then $(\boldsymbol{E}^{\mathrm{PML}}, \boldsymbol{H}^{\mathrm{PML}})$ can be viewed as the extension in the region $\mathbb{R}^3 \backslash \overline{B}_1$ of the solution of the problem (1.1a)-(1.1e) since, by the fact that $\alpha_j = 1$ on Γ_1 for $j = 1, 2, 3$ we have $\boldsymbol{E}^{\mathrm{PML}} = \boldsymbol{E}$, $\boldsymbol{H}^{\mathrm{PML}} = \boldsymbol{H}$ on Γ_1. If we set $\boldsymbol{E}^{\mathrm{PML}} = \boldsymbol{E}$ and $\boldsymbol{H}^{\mathrm{PML}} = \boldsymbol{H}$ in $\Omega_1 \times (0, T)$, then $(\boldsymbol{E}^{\mathrm{PML}}, \boldsymbol{H}^{\mathrm{PML}})$ satisfies the PML problem:

$$\begin{cases} \nabla \times \boldsymbol{E}^{\mathrm{PML}} + \mu (\mathbb{B}\mathbb{A})^{-1} \partial_t \boldsymbol{H}^{\mathrm{PML}} = \boldsymbol{0} & \text{in} \quad (\mathbb{R}^3 \backslash \overline{\Omega}) \times (0, T), \\ \nabla \times \boldsymbol{H}^{\mathrm{PML}} - \varepsilon (\mathbb{B}\mathbb{A})^{-1} \partial_t \boldsymbol{E}^{\mathrm{PML}} = \boldsymbol{J} & \text{in} \quad (\mathbb{R}^3 \backslash \overline{\Omega}) \times (0, T), \\ \boldsymbol{n} \times \boldsymbol{E}^{\mathrm{PML}} = \boldsymbol{0} & \text{on} \quad \Gamma \times (0, T), \\ \boldsymbol{E}^{\mathrm{PML}}(x, 0) = \boldsymbol{H}^{\mathrm{PML}}(x, 0) = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}. \end{cases} \tag{4.15}$$

The truncated PML problem in the time domain is to find $(\boldsymbol{E}^p, \boldsymbol{H}^p)$, which is an approximation to $(\boldsymbol{E}, \boldsymbol{H})$ in Ω_1, such that

$$\begin{cases} \nabla \times \boldsymbol{E}^p + \mu (\mathbb{B}\mathbb{A})^{-1} \partial_t \boldsymbol{H}^p = \boldsymbol{0} & \text{in} \quad \Omega_2 \times (0, T), \\ \nabla \times \boldsymbol{H}^p - \varepsilon (\mathbb{B}\mathbb{A})^{-1} \partial_t \boldsymbol{E}^p = \boldsymbol{J} & \text{in} \quad \Omega_2 \times (0, T), \\ \boldsymbol{n} \times \boldsymbol{E}^p = \boldsymbol{0} & \text{on} \quad \Gamma \times (0, T), \\ \boldsymbol{n}_2 \times \boldsymbol{E}^p = \boldsymbol{0} & \text{on} \quad \Gamma_2 \times (0, T), \\ \boldsymbol{E}^p(x, 0) = \boldsymbol{H}^p(x, 0) = \boldsymbol{0} & \text{in} \quad \Omega_2. \end{cases} \tag{4.16}$$

4.2 Well-posedness of the truncated PML problem

We now study the well-posedness of the truncated PML problem (4.16), employing the Laplace transform technique and a variational method. Eliminate $\boldsymbol{H}^p$ and take the Laplace transform of (4.16) to obtain that

$$\begin{cases} \nabla\times\left[(\mu s)^{-1}\mathbb{B}\mathbb{A}\nabla\times\check{\boldsymbol{E}}^p\right]+\varepsilon s(\mathbb{B}\mathbb{A})^{-1}\check{\boldsymbol{E}}^p=-\check{\boldsymbol{J}} \quad \text{in} \quad \Omega_2, \\ \boldsymbol{n}\times\check{\boldsymbol{E}}^p=\boldsymbol{0} \quad \text{on} \quad \Gamma, \\ \boldsymbol{n}_2\times\check{\boldsymbol{E}}^p=\boldsymbol{0} \quad \text{on} \quad \Gamma_2. \end{cases} \tag{4.17}$$

The variational formulation of (4.17) can be derived as follows: find a solution $\check{\boldsymbol{E}}^p\in H_0(\mathrm{curl},\Omega_2)$ such that

$$a_p\left(\check{\boldsymbol{E}}^p,\boldsymbol{V}\right)=-\int_{\Omega_1}\check{\boldsymbol{J}}\cdot\overline{\boldsymbol{V}}dx, \qquad \forall\,\boldsymbol{V}\in H_0(\mathrm{curl},\Omega_2), \tag{4.18}$$

where the sesquilinear form $a_p(\cdot,\cdot)$ is defined as

$$a_p\left(\check{\boldsymbol{E}}^p,\boldsymbol{V}\right)=\int_{\Omega_2}\left[(\mu s)^{-1}\mathbb{B}\mathbb{A}(\nabla\times\check{\boldsymbol{E}}^p)\cdot(\nabla\times\overline{\boldsymbol{V}})dx+\varepsilon s(\mathbb{B}\mathbb{A})^{-1}\check{\boldsymbol{E}}^p\cdot\overline{\boldsymbol{V}}\right]dx. \tag{4.19}$$

We have the following result on the well-posedness of the variational problem (4.18).

Lemma 4.1. *For each $s\in\mathbb{C}_+$ with $\mathrm{Re}(s)=s_1>0$ the variational problem (4.18) has a unique solution $\check{\boldsymbol{E}}^p\in H_0(\mathrm{curl},\Omega_2)$. Further, it holds that*

$$\|\nabla\times\check{\boldsymbol{E}}^p\|_{L^2(\Omega_2)^3}+\|s\check{\boldsymbol{E}}^p\|_{L^2(\Omega_2)^3}\lesssim s_1^{-1}(1+s_1^{-1}\sigma_0)^2\|s\check{\boldsymbol{J}}\|_{L^2(\Omega_1)^3}. \tag{4.20}$$

Proof. By the definition of the diagonal matrix $\mathbb{B}\mathbb{A}$ (see (4.12)) and a direct calculation it easily follows that for $i=1,2,3$

$$(1+s_1^{-1}\sigma_0)^{-2}\le|(\mathbb{B}\mathbb{A})_{ii}|\le(1+s_1^{-1}\sigma_0) \quad \text{in} \;\; \Omega^{\mathrm{PML}}, \tag{4.21}$$

$$(1+s_1^{-1}\sigma_0)^{-1}\le|(\mathbb{B}\mathbb{A})_{ii}^{-1}|\le(1+s_1^{-1}\sigma_0)^2, \quad \text{in} \;\; \Omega^{\mathrm{PML}}. \tag{4.22}$$

Thus, it is derived that

$$\mathrm{Re}[a_p(\check{\boldsymbol{E}}^p,\check{\boldsymbol{E}}^p)]\gtrsim\frac{1}{(1+s_1^{-1}\sigma_0)^2}\frac{s_1}{|s|^2}\left(\|\nabla\times\check{\boldsymbol{E}}^p\|_{L^2(\Omega_2)^3}+\|s\check{\boldsymbol{E}}^p\|_{L^2(\Omega_2)^3}\right). \tag{4.23}$$

The existence and uniqueness of solutions to the problem (4.18) then follow from the Lax-Milgram theorem. The estimate (4.20) can be obtained by combining (4.18), (4.23) and the Cauchy-Schwartz inequality. The proof is thus complete. □

To show the well-posedness of the truncated PML problem (4.16) in the time domain, we need the following lemma which is the analog of the Paley-Wiener-Schwartz theorem for the Fourier transform of the distributions with compact support in the case of Laplace transform [36, Theorem 43.1].

Lemma 4.2. [36, Theorem 43.1]. *Let $\check{\boldsymbol{\omega}}(s)$ denote a holomorphic function in the half complex plane $s_1=\mathrm{Re}(s)>\sigma_0$ for some $\sigma_0\in\mathbb{R}$, valued in the Banach space $\mathbb{E}$. Then the following statements are equivalent:*

1) there is a distribution $\boldsymbol{\omega} \in \mathcal{D}_+'(\mathbb{E})$ whose Laplace transform is equal to $\breve{\boldsymbol{\omega}}(s)$, where $\mathcal{D}_+'(\mathbb{E})$ is the space of distributions on the real line which vanish identically in the open negative half-line;

2) there is a σ_1 with $\sigma_0 \le \sigma_1 < \infty$ and an integer $m \ge 0$ such that for all complex numbers s with $s_1 = \mathrm{Re}(s) > \sigma_1$ it holds that $\|\breve{\boldsymbol{\omega}}(s)\|_{\mathbb{E}} \lesssim (1+|s|)^m$.

The well-posedness and stability of the truncated PML problem (4.16) can be proved by using Lemmas 4.1 and 4.2 and the energy method (cf. [16, Theorem 3.1]).

Theorem 4.3. *Let $s_1 = 1/T$. Then the truncated PML problem* (4.16) *in the time domain has a unique solution $(\boldsymbol{E}^p(x,t), \boldsymbol{H}^p(x,t))$ with*

$$\begin{aligned} \boldsymbol{E}^p &\in L^2\big(0,T; H_0(\mathrm{curl},\Omega_2)\big) \cap H^1\left(0,T; L^2(\Omega_2)^3\right), \\ \boldsymbol{H}^p &\in L^2\big(0,T; H_0(\mathrm{curl},\Omega_2)\big) \cap H^1\left(0,T; L^2(\Omega_2)^3\right) \end{aligned}$$

and satisfying the stability estimate

$$\begin{aligned} &\max_{t\in[0,T]} \left[\|\partial_t \boldsymbol{E}^p\|_{L^2(\Omega_2)^3} + \|\nabla\times\boldsymbol{E}^p\|_{L^2(\Omega_2)^3} + \|\partial_t \boldsymbol{H}^p\|_{L^2(\Omega_2)^3} + \|\nabla\times\boldsymbol{H}^p\|_{L^2(\Omega_2)^3}\right] \\ &\quad \lesssim (1+\sigma_0 T)^3 \|\boldsymbol{J}\|_{H^1(0,T;L^2(\Omega_1)^3)}. \end{aligned} \tag{4.24}$$

Proof. Existence and uniqueness of solutions of the truncated PML problem (4.16) follows directly from [16, Theorem 3.1] and Lemmas 4.1 and 4.2. We now establish the stability estimate (4.24). Define the energy function

$$e(t) = \|\varepsilon^{1/2}(\mathbb{BA})^{-1/2}\boldsymbol{E}^p(\cdot,t)\|^2_{L^2(\Omega_2)^3} + \|\mu^{1/2}(\mathbb{BA})^{-1/2}\boldsymbol{H}^p(\cdot,t)\|^2_{L^2(\Omega_2)^3}, \quad t\in(0,T).$$

From the zero initial conditions of $\boldsymbol{E}^p$ and $\boldsymbol{H}^p$, we know that $e(\cdot)$ can be equivalently written as

$$e(t) = e(t) - e(0) = \int_0^t e'(\tau)d\tau.$$

By a simple calculation with using the system (4.16) and integration by parts, we have

$$\begin{aligned} \int_0^t e'(\tau)d\tau &= 2\mathrm{Re}\int_0^t\int_{\Omega_2}\left(\varepsilon(\mathbb{BA})^{-1}\partial_\tau\boldsymbol{E}^p\cdot\overline{\boldsymbol{E}^p} + \mu(\mathbb{BA})^{-1}\partial_\tau\boldsymbol{H}^p\cdot\overline{\boldsymbol{H}^p}\right)dxd\tau \\ &= 2\mathrm{Re}\int_0^t\int_{\Omega_2}\left(\nabla\times\boldsymbol{H}^p\cdot\overline{\boldsymbol{E}^p} + \nabla\times\boldsymbol{E}^p\cdot\overline{\boldsymbol{H}^p}\right)dxd\tau - 2\mathrm{Re}\int_0^t\int_{\Omega_2}\boldsymbol{J}\cdot\overline{\boldsymbol{E}^p}dxd\tau \\ &= 2\mathrm{Re}\int_0^t\int_{\Omega_2}\left((\nabla\times\overline{\boldsymbol{E}^p})\cdot\boldsymbol{H}^p - (\nabla\times\boldsymbol{E}^p)\cdot\overline{\boldsymbol{H}^p}\right)dxd\tau - 2\mathrm{Re}\int_0^t\int_{\Omega_2}\boldsymbol{J}\cdot\overline{\boldsymbol{E}^p}dxd\tau \\ &= -2\mathrm{Re}\int_0^t\int_{\Omega_2}\boldsymbol{J}\cdot\overline{\boldsymbol{E}^p}dxd\tau \le 2\max_{t\in[0,T]}\|\boldsymbol{E}^p(\cdot,t)\|_{L^2(\Omega_2)^3}\|\boldsymbol{J}\|_{L^1(0,T;L^2(\Omega_1)^3)}. \end{aligned}$$

This, combined with the definition of $e(t)$, the estimate for $(\mathbb{BA})^{-1}$ (see (4.22)) and the Cauchy-Schwartz inequality, yields

$$\max_{t\in[0,T]}\left(\|\boldsymbol{E}^p(\cdot,t)\|_{L^2(\Omega_2)^3} + \|\boldsymbol{H}^p(\cdot,t)\|_{L^2(\Omega_2)^3}\right) \lesssim (1+\sigma_0 T)\|\boldsymbol{J}\|_{L^1(0,T;L^2(\Omega_1)^3)}. \tag{4.25}$$

Taking the derivative of (4.16) with respect to t, we know that $(\partial_t \boldsymbol{E}^p, \partial_t \boldsymbol{H}^p)$ satisfy the same set of equations with the source $\boldsymbol{J}$ replaced by $\partial_t \boldsymbol{J}$, and the initial conditions replaced by $\partial_t \boldsymbol{E}^p|_{t=0} = \varepsilon^{-1}\mathbb{B}\mathbb{A}\nabla \times \boldsymbol{H}^p|_{t=0} = \mathbf{0}$, $\partial_t \boldsymbol{H}^p|_{t=0} = -\mu^{-1}\mathbb{B}\mathbb{A}\nabla \times \boldsymbol{E}^p|_{t=0} = \mathbf{0}$. Hence, following the similar steps as in deriving (4.25) for $(\partial_t \boldsymbol{E}^p, \partial_t \boldsymbol{H}^p)$ we have

$$\max_{t\in[0,T]} \left(\|\partial_t \boldsymbol{E}^p(\cdot,t)\|_{L^2(\Omega_2)^3} + \|\partial_t \boldsymbol{H}^p(\cdot,t)\|_{L^2(\Omega_2)^3}\right) \lesssim (1+\sigma_0 T)\|\partial_t \boldsymbol{J}\|_{L^1(0,T;L^2(\Omega_1)^3)}. \tag{4.26}$$

Combining (4.25)-(4.26) and the Maxwell system (4.16) yields the desired estimate (4.24). □

To study the convergence of the uniaxial PML method, we introduce the EtM operator $\widehat{\mathscr{B}} : H^{-1/2}(\mathrm{Curl}, \Gamma_1) \to H^{-1/2}(\mathrm{Div}, \Gamma_1)$ associated with the truncated PML problem (4.17) in the s-domain. Given $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div}, \Gamma_1)$, define

$$\widehat{\mathscr{B}}(\boldsymbol{\lambda} \times \boldsymbol{n}_1) := \boldsymbol{n}_1 \times (\mu s)^{-1}\nabla \times \boldsymbol{u} \qquad \text{on } \Gamma_1, \tag{4.27}$$

where $\boldsymbol{u}$ satisfies the following problem in the PML layer:

$$\begin{cases} \nabla \times \left[(\mu s)^{-1}\mathbb{B}\mathbb{A}\nabla \times \boldsymbol{u}\right] + \varepsilon s(\mathbb{B}\mathbb{A})^{-1}\boldsymbol{u} = 0 \qquad \text{in } \Omega^{\mathrm{PML}}, \\ \boldsymbol{n}_1 \times \boldsymbol{u} = \boldsymbol{\lambda} \qquad \text{on } \Gamma_1, \qquad \boldsymbol{n}_2 \times \boldsymbol{u} = \mathbf{0} \qquad \text{on } \Gamma_2. \end{cases} \tag{4.28}$$

We need to show that (4.28) has a unique solution, so $\widehat{\mathscr{B}}$ is well-defined. To this end, we consider the following general problem with the tangential trace $\boldsymbol{\xi}$ on Γ_2, which is needed for the convergence analysis of the PML method:

$$\begin{cases} \nabla \times \left[(\mu s)^{-1}\mathbb{B}\mathbb{A}\nabla \times \boldsymbol{u}\right] + \varepsilon s(\mathbb{B}\mathbb{A})^{-1}\boldsymbol{u} = 0 \qquad \text{in } \Omega^{\mathrm{PML}}, \\ \boldsymbol{n}_1 \times \boldsymbol{u} = \boldsymbol{\lambda} \qquad \text{on } \Gamma_1, \qquad \boldsymbol{n}_2 \times \boldsymbol{u} = \boldsymbol{\xi} \qquad \text{on } \Gamma_2. \end{cases} \tag{4.29}$$

Define the sesquilinear form $a^{\mathrm{PML}} : H(\mathrm{curl}, \Omega^{\mathrm{PML}}) \times H(\mathrm{curl}, \Omega^{\mathrm{PML}}) \to \mathbb{C}$ as

$$a^{\mathrm{PML}}(\boldsymbol{u}, \boldsymbol{V}) := \int_{\Omega^{\mathrm{PML}}} (\mu s)^{-1}\mathbb{B}\mathbb{A}(\nabla \times \boldsymbol{u}) \cdot (\nabla \times \overline{\boldsymbol{V}})dx + \int_{\Omega^{\mathrm{PML}}} \varepsilon s(\mathbb{B}\mathbb{A})^{-1}\boldsymbol{u} \cdot \overline{\boldsymbol{V}}dx. \tag{4.30}$$

Then the variational formulation of (4.29) is as follows: Given $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div}, \Gamma_1)$ and $\boldsymbol{\xi} \in H^{-1/2}(\mathrm{Div}, \Gamma_2)$, find $\boldsymbol{u} \in H(\mathrm{curl}, \Omega^{\mathrm{PML}})$ such that $\boldsymbol{n}_1 \times \boldsymbol{u} = \boldsymbol{\lambda}$ on Γ_1, $\boldsymbol{n}_2 \times \boldsymbol{u} = \boldsymbol{\xi}$ on Γ_2 and

$$a^{\mathrm{PML}}(\boldsymbol{u}, \boldsymbol{V}) = 0, \quad \forall\, \boldsymbol{V} \in H_0(\mathrm{curl}, \Omega^{\mathrm{PML}}). \tag{4.31}$$

Arguing similarly as in proving (4.23), we obtain that for any $\boldsymbol{V} \in H_0(\mathrm{curl}, \Omega^{\mathrm{PML}})$,

$$\mathrm{Re}\left[a^{\mathrm{PML}}(\boldsymbol{V}, \boldsymbol{V}\right] \gtrsim \frac{1}{(1+s_1^{-1}\sigma_0)^2}\frac{s_1}{|s|^2}\left[\|\nabla \times \boldsymbol{V}\|^2_{L^2(\Omega^{\mathrm{PML}})^3} + \|s\boldsymbol{V}\|^2_{L^2(\Omega^{\mathrm{PML}})^3}\right]. \tag{4.32}$$

By (4.32) and the Lax-Milgram theorem it follows that the variational problem (4.31) has a unique solution. We have the following stability result for the solution to the problem (4.29).

Lemma 4.4. *For any $\boldsymbol{\lambda} \in H^{-1/2}(\mathrm{Div}, \Gamma_1)$ and $\boldsymbol{\xi} \in H^{-1/2}(\mathrm{Div}, \Gamma_2)$, let $\boldsymbol{u}$ be the solution to the problem* (4.29). *Then*

$$\begin{aligned} &\|\nabla \times \boldsymbol{u}\|_{L^2(\Omega^{\mathrm{PML}})^3} + \|s\boldsymbol{u}\|_{L^2(\Omega^{\mathrm{PML}})^3} \\ &\qquad \lesssim s_1^{-1}(1+s_1^{-1}\sigma_0)^4|s|(1+|s|)(\|\boldsymbol{\lambda}\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)} + \|\boldsymbol{\xi}\|_{H^{-1/2}(\mathrm{Div},\Gamma_2)}). \end{aligned} \tag{4.33}$$

Proof. Let $\boldsymbol{u}_0 \in H(\mathrm{curl}, \Omega^{\mathrm{PML}})$ be such that $\boldsymbol{n}_1 \times \boldsymbol{u}_0 = \boldsymbol{\lambda}$, $\boldsymbol{n}_2 \times \boldsymbol{u}_0 = \boldsymbol{\xi}$ on Γ_2. Then, by (4.31) we have $\boldsymbol{\omega} := \boldsymbol{u} - \boldsymbol{u}_0 \in H_0(\mathrm{curl}, \Omega^{\mathrm{PML}})$ and

$$a^{\mathrm{PML}}(\boldsymbol{\omega}, \boldsymbol{V}) = -a^{\mathrm{PML}}(\boldsymbol{u}_0, \boldsymbol{V}), \quad \forall\, \boldsymbol{V} \in H_0(\mathrm{curl}, \Omega^{\mathrm{PML}}). \tag{4.34}$$

This, combined with (4.30)-(4.32) and the Cauchy-Schwartz inequality, gives

$$\begin{aligned}
&\frac{1}{(1+s_1^{-1}\sigma_0)^2}\frac{s_1}{|s|^2}\left(\|\nabla\times\boldsymbol{\omega}\|^2_{L^2(\Omega^{\mathrm{PML}})^3} + \|s\boldsymbol{\omega}\|^2_{L^2(\Omega^{\mathrm{PML}})^3}\right)\\
&\quad\lesssim \mathrm{Re}\left[a^{\mathrm{PML}}(\boldsymbol{\omega},\boldsymbol{\omega})\right]\\
&\quad\lesssim \frac{(1+s_1^{-1}\sigma_0)^2}{|s|}\sqrt{1+|s|^2}\left(\|\nabla\times\boldsymbol{\omega}\|^2_{L^2(\Omega^{\mathrm{PML}})^3} + \|s\boldsymbol{\omega}\|^2_{L^2(\Omega^{\mathrm{PML}})^3}\right)^{1/2}\|\boldsymbol{u}_0\|_{H(\mathrm{curl},\Omega^{\mathrm{PML}})},
\end{aligned}$$

yielding

$$\left(\|\nabla\times\boldsymbol{\omega}\|^2_{L^2(\Omega^{\mathrm{PML}})^3} + \|s\boldsymbol{\omega}\|^2_{L^2(\Omega^{\mathrm{PML}})^3}\right)^{1/2} \lesssim \frac{(1+s_1^{-1}\sigma_0)^4|s|\sqrt{1+|s|^2}}{s_1}\|\boldsymbol{u}_0\|^2_{H(\mathrm{curl},\Omega^{\mathrm{PML}})}.$$

This, together with the definition of $\boldsymbol{\omega}$ and the Cauchy-Schwartz inequality, implies that

$$\|\nabla\times\check{\boldsymbol{u}}\|_{L^2(\Omega^{\mathrm{PML}})^3} + \|s\check{\boldsymbol{u}}\|_{L^2(\Omega^{\mathrm{PML}})^3} \lesssim \frac{(1+s_1^{-1}\sigma_0)^4|s|(1+|s|)}{s_1}\|\boldsymbol{u}_0\|_{H(\mathrm{curl},\Omega^{\mathrm{PML}})}.$$

The desired estimate (4.33) then follows from the trace theorem. □

Now, by using $\widehat{\mathscr{B}}$ the truncated PML problem (4.17) for the electric field $\check{\boldsymbol{E}}^p$ can be equivalently reduced to the boundary value problem in Ω_1:

$$\begin{cases} \nabla\times\left[(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}^p\right] + \varepsilon s\check{\boldsymbol{E}}^p = -\check{\boldsymbol{J}} \ \text{ in } \ \Omega_1,\\ \boldsymbol{n}\times\check{\boldsymbol{E}}^p = \boldsymbol{0} \quad \text{on} \ \ \Gamma, \qquad \widehat{\mathscr{B}}[\check{\boldsymbol{E}}^p_{\Gamma_1}] = \boldsymbol{n}_1\times(\mu s)^{-1}\nabla\times\check{\boldsymbol{E}}^p \quad \text{on} \ \ \Gamma_1. \end{cases} \tag{4.35}$$

Similarly, for the problem (4.35) we can derive its equivalent variational formulation: find $\check{\boldsymbol{E}}^p \in H_{\Gamma_1}(\mathrm{curl}, \Omega_1)$ such that

$$\widehat{a}\left(\check{\boldsymbol{E}}^p, \boldsymbol{V})\right) = -\int_{\Omega_1}\check{\boldsymbol{J}}\cdot\overline{\boldsymbol{V}}dx, \quad \forall\, \boldsymbol{V} \in H_{\Gamma_1}(\mathrm{curl}, \Omega_1), \tag{4.36}$$

where the sesquilinear form $\widehat{a}(\cdot,\cdot)$ is defined as

$$\widehat{a}\left(\check{\boldsymbol{E}}^p, \boldsymbol{V}\right) := \int_{\Omega_1}\left[(\mu s)^{-1}(\nabla\times\check{\boldsymbol{E}}^p)\cdot(\nabla\times\overline{\boldsymbol{V}})dx + \varepsilon s\check{\boldsymbol{E}}^p\cdot\overline{\boldsymbol{V}}\right]dx + \langle\widehat{\mathscr{B}}[\check{\boldsymbol{E}}_{\Gamma_1}], \boldsymbol{V}_{\Gamma_1}\rangle_{\Gamma_1}. \tag{4.37}$$

By using $\widehat{\mathscr{B}}$ and the Laplace and inverse Laplace transform it can be shown that the truncated PML problem (4.16) is equivalent to the initial boundary value problem in Ω_1:

$$\begin{cases} \nabla\times\boldsymbol{E}^p + \mu\partial_t\boldsymbol{H}^p = \boldsymbol{0} & \text{in} \ \ \Omega_1\times(0,T),\\ \nabla\times\boldsymbol{H}^p - \varepsilon\partial_t\boldsymbol{E}^p = \boldsymbol{J} & \text{in} \ \ \Omega_1\times(0,T),\\ \boldsymbol{n}\times\boldsymbol{E}^p = \boldsymbol{0} & \text{on} \ \ \Gamma\times(0,T),\\ \boldsymbol{E}^p(x,0) = \boldsymbol{H}^p(x,0) = \boldsymbol{0} & \text{in} \ \ \Omega_1,\\ \hat{\mathscr{T}}[\boldsymbol{E}^p_{\Gamma_1}] = \boldsymbol{H}^p\times\boldsymbol{n}_1 & \text{on} \ \ \Gamma_1\times(0,T), \end{cases} \tag{4.38}$$

where $\hat{\mathscr{T}} = \mathscr{L}^{-1} \circ \widehat{\mathscr{B}} \circ \mathscr{L}$ is the time-domain EtM operator for the PML problem. In fact, it is easy to see that any solution $(\boldsymbol{E}^p, \boldsymbol{H}^p)$ of the truncated PML problem (4.16) restricted to Ω_1 is a solution of the problem (4.38). Conversely, let $(\boldsymbol{E}^p, \boldsymbol{H}^p)$ be a solution of the problem (4.38). Let $\check{\boldsymbol{E}}^{\rm PML}$ be the solution of (4.28) with $\boldsymbol{\lambda} = \boldsymbol{n}_1 \times \check{\boldsymbol{E}}^p|_{\Gamma_1}$, where $\check{\boldsymbol{E}}^p|_{\Gamma_1}$ is the Laplace transform of $\boldsymbol{E}^p$ on Γ_1. Then $\check{\boldsymbol{E}}^{\rm PML}$ is actually an extension in $\Omega^{\rm PML}$ of $\check{\boldsymbol{E}}^p$ on Γ_1 and $\boldsymbol{n}_2 \times \check{\boldsymbol{E}}^{\rm PML} = 0$ on Γ_2. Define $\check{\boldsymbol{H}}^{\rm PML} := -(\mu s)^{-1} \nabla \times \check{\boldsymbol{E}}^{\rm PML}$ in $\Omega^{\rm PML}$. Then, by the definition of the transparent operators $\hat{\mathscr{T}}$ and $\widehat{\mathscr{B}}$ (see (4.27) above) we know that $\hat{\mathscr{T}}[\boldsymbol{E}^p_{\Gamma_1}] = \boldsymbol{H}^{\rm PML} \times \boldsymbol{n}_1$ on $\Gamma_1 \times (0, T)$. This, together with the last equation in (4.38), gives that $\boldsymbol{H}^{\rm PML} \times \boldsymbol{n}_1 = \boldsymbol{H}^p \times \boldsymbol{n}_1$ on $\Gamma_1 \times (0, T)$, and thus $\check{\boldsymbol{H}}^{\rm PML}$ is actually an extension in $\Omega^{\rm PML}$ of $\check{\boldsymbol{H}}^p$ on Γ_1, where $\check{\boldsymbol{H}}^p|_{\Gamma_1}$ is the Laplace transform of $\boldsymbol{H}^p$ on Γ_1. Define $(\boldsymbol{E}^p, \boldsymbol{H}^p) := (\boldsymbol{E}^{\rm PML}, \boldsymbol{H}^{\rm PML})$ in $\Omega^{\rm PML}$. Then $(\boldsymbol{E}^p, \boldsymbol{H}^p)$ is a solution of the truncated PML problem (4.16).

4.3 Exponential convergence of the uniaxial PML method

In this subsection, we prove the exponential convergence of the uniaxial PML method. The proof depends on the error analysis between the EtM operators for the original scattering problem and the truncated PML problem, which is concluded as a boundary value problem in the PML layer with the PML extension as outer boundary value. The convergence in then obtained by combining the stability result of the PML system in Lemma 4.4 and the exponential decay of the PML extension. We begin with the following lemma which is useful in the proof of the exponential decay property of the stretched fundamental solution $\widetilde{\Phi}_s(x, y)$.

Lemma 4.5. *Let $s = s_1 + is_2$ with $s_1 > 0,\ s_2 \in \mathbb{R}$. Then, for any $x \in \Gamma_2$ and $y \in \Gamma_1$ the complex distance ρ_s defined by* (4.10) *satisfies*

$$|\rho_s(\widetilde{x}, y)/s| \ge d, \quad {\rm Re}[\rho_s(\widetilde{x}, y)] \ge \frac{\sigma_0 d}{m+1}.$$

Proof. For $x \in \Gamma_2$ and $y \in \Gamma_1$, $\widetilde{x}_j - y_j = (x_j - y_j) + s_1^{-1} x_j \hat{\sigma}_j(x_j)$, where

$$\hat{\sigma}_j(x_j) = \frac{1}{x_j} \int_0^{x_j} \sigma_j(\tau) d\tau.$$

Then, by the definition of the complex distance $\rho_s(\widetilde{x}, y)$ (see (4.10)) we have

$$\begin{aligned} |\rho_s(\widetilde{x}, y)/s| = |\widetilde{x} - y| &= \sqrt{(\widetilde{x}_1 - y_1)^2 + (\widetilde{x}_2 - y_2)^2 + (\widetilde{x}_3 - y_3)^2} \\ &= \left(\sum_{j=1}^{3} \left[(x_j - y_j)^2 + 2 s_1^{-1} x_j \hat{\sigma}_j(x_j)(x_j - y_j) + s_1^{-2} x_j^2 \hat{\sigma}_j^2(x_j) \right] \right)^{1/2} \\ &\ge |x - y| \ge d, \end{aligned}$$

where we have used the fact that $x_j \hat{\sigma}_j(x_j)(x_j - y_j) \ge 0$ for $x \in \Gamma_2$ and $y \in \Gamma_1$. In addition,

$$\begin{aligned} {\rm Re}\left[\rho_s(\widetilde{x}, y)\right] &= {\rm Re}\left[s^2 \left((\widetilde{x}_1 - y_1)^2 + (\widetilde{x}_2 - y_2)^2 + (\widetilde{x}_3 - y_3)^2 \right) \right]^{1/2} \\ &= s_1 \sqrt{(\widetilde{x}_1 - y_1)^2 + (\widetilde{x}_2 - y_2)^2 + (\widetilde{x}_3 - y_3)^2} \\ &\ge \left(\sum_{j=1}^{3} x_j^2 \hat{\sigma}_j^2(x_j) \right)^{1/2}. \end{aligned}$$

If $x_j = \pm(L_j/2+d_j) \in \Gamma_2$, then, by (4.4) we have $|x_j\hat{\sigma}_j(x_j)| = \sigma_0 d/(m+1)$. Thus, $\mathrm{Re}\,[\rho_s(\widetilde{x}, y)] \geq \sigma_0 d/(m+1)$. The proof is thus complete. □

By Lemma 4.5, and arguing similarly as in the proof of [42, Lemma 5.3], we have similar estimates as in [42, Lemma 5.3] for the stretched dyadic Green's function $\widetilde{\mathbb{G}}$ in the PML layer.

Lemma 4.6. *Assume that the conditions in* (4.2) *and* (4.3) *are satisfied. Then we have that for* $x \in \Gamma_2, y \in \Gamma_1$,

$$
\begin{aligned}
&|\widetilde{\mathbb{G}}(s,x,y)| \lesssim s_1^{-2} d^{-1} \left(1 + s_1^{-1}\sigma_0\right)^2 e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}},\\
&\left|\mathrm{curl}_{\widetilde{x}}\, \widetilde{\mathbb{G}}(s,x,y)\right|, \left|\mathrm{curl}_y\, \widetilde{\mathbb{G}}(s,x,y)\right| \lesssim d^{-1}(1+|s|)\left(1 + s_1^{-1}\sigma_0\right) e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}},\\
&\left|\mathrm{curl}_{\widetilde{x}}\,\mathrm{curl}_y\, \widetilde{\mathbb{G}}(s,x,y)\right|, \left|\mathrm{curl}_y\,\mathrm{curl}_y\, \widetilde{\mathbb{G}}(s,x,y)\right|\\
&\qquad \lesssim \left(1+|s|^2\right)\left(1 + s_1^{-1}\sigma_0\right)^2 d^{-1} e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}},\\
&\left|\mathrm{curl}_{\widetilde{x}}\,\mathrm{curl}_y\,\mathrm{curl}_y\, \widetilde{\mathbb{G}}(s,x,y)\right| \lesssim \left(1+|s|^3\right) d^{-1}\left(1 + s_1^{-1}\sigma_0\right)^3 e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}},
\end{aligned}
$$

where $\widetilde{\mathbb{G}}$ *is the stretched dyadic Green's function and* $s = s_1 + is_2 \in \mathbb{C}_+$.

By Lemma 4.6, the trace theorem for $H(\mathrm{curl}, \cdot)$ and the PML extension in the s-domain defined in terms of the integral representation (4.11), the following lemma on the decay property of the PML extension can be easily proved by following the proof of [42, Theorem 5.4] with Γ_R replaced by Γ_1 (cf. [42, Theorem 5.4]).

Lemma 4.7. *For any* $\boldsymbol{p}$, $\boldsymbol{q} \in H^{-1/2}(\mathrm{Div}, \Gamma_1)$ *let* $\mathbb{E}(\boldsymbol{p}, \boldsymbol{q})$ *be the PML extension in the s-domain defined in* (4.11)*. Then we have that for any* $x \in \Omega^{\mathrm{PML}}$,

$$
\begin{aligned}
&|\mathbb{E}(\boldsymbol{p},\boldsymbol{q})(x)| \qquad (4.39)\\
&\lesssim s_1^{-2} d^{1/2}(1+s_1^{-1}\sigma_0)^2 e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}} \left[(1+|s|)\|\boldsymbol{q}\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)} + (1+|s|^2)\|\boldsymbol{p}\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&|\mathrm{curl}_{\widetilde{x}}\, \mathbb{E}(\boldsymbol{p},\boldsymbol{q})(x)| \qquad (4.40)\\
&\lesssim d^{1/2}(1+s_1^{-1}\sigma_0)^3 e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}} \left[(1+|s|^2)\|\boldsymbol{q}\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)} + (1+|s|^3)\|\boldsymbol{p}\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}\right].
\end{aligned}
$$

We now establish the L^2-norm and L^∞-norm error estimates in time between solutions to the original scattering problem and the truncated PML problem (4.16) in the computational domain Ω_1.

Theorem 4.8. *Let* $(\boldsymbol{E}, \boldsymbol{H})$ *and* $(\boldsymbol{E}^p, \boldsymbol{H}^p)$ *be the solutions of the problems* (1.1a)-(1.1e) *and* (4.16) *with* $s_1 = 1/T$*, respectively. If the assumptions* (3.1) *and* (3.2) *are satisfied, then we have the error estimates*

$$
\begin{aligned}
&\|\boldsymbol{E}-\boldsymbol{E}^p\|_{L^2(0,T;H(\mathrm{curl},\Omega_1))} + \|\boldsymbol{H}-\boldsymbol{H}^p\|_{L^2(0,T;H(\mathrm{curl},\Omega_1))}\\
&\quad \lesssim T^5 d^2(1+\sigma_0 T)^{15} e^{-\sigma_0 d\sqrt{\varepsilon\mu}/2}\|\boldsymbol{J}\|_{H^{10}(0,T;L^2(\Omega_1)^3)}.
\end{aligned} \tag{4.41}
$$

and

$$
\begin{aligned}
&\|\boldsymbol{E}-\boldsymbol{E}^p\|_{L^\infty(0,T;H(\mathrm{curl},\Omega_1))} + \|\boldsymbol{H}-\boldsymbol{H}^p\|_{L^\infty(0,T;H(\mathrm{curl},\Omega_1))}\\
&\quad \lesssim T^{11/2} d^2(1+\sigma_0 T)^{15} e^{-\sigma_0 d\sqrt{\varepsilon\mu}/2}\|\boldsymbol{J}\|_{H^{9}(0,T;L^2(\Omega_1)^3)}.
\end{aligned} \tag{4.42}
$$

Proof. We first prove (4.41). Let $\boldsymbol{U} = \boldsymbol{E} - \boldsymbol{E}^p$ and $\boldsymbol{V} = \boldsymbol{H} - \boldsymbol{H}^p$ and let $\check{\boldsymbol{E}}$ and $\check{\boldsymbol{E}}^p$ be the solutions to the variational problems (3.13) and (4.36), respectively. Then, by (3.13) and (4.36) we get

$$a(\check{\boldsymbol{U}}, \check{\boldsymbol{U}}) = \widehat{a}(\check{\boldsymbol{E}}, \check{\boldsymbol{U}}) - a(\check{\boldsymbol{E}}^p, \check{\boldsymbol{U}}) = \langle (\widehat{\mathscr{B}} - \mathscr{B})[\check{\boldsymbol{E}}^p_{\Gamma_1}], \check{\boldsymbol{U}}_{\Gamma_1} \rangle_{\Gamma_1}. \tag{4.43}$$

This, together with the uniform coercivity (3.15) of $a(\cdot, \cdot)$, implies that

$$\|\check{\boldsymbol{U}}\|_{H(\mathrm{curl},\Omega_1)} \lesssim s_1^{-1} \left(1 + |s|^2\right) \|(\widehat{\mathscr{B}} - \mathscr{B})[\check{\boldsymbol{E}}^p_{\Gamma_1}]\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}. \tag{4.44}$$

From the Maxwell equations in Ω_1 obtained by taking the Laplace transform of the problems (1.1a)-(1.1e) and (4.16), it follows that

$$\|\check{\boldsymbol{V}}\|_{H(\mathrm{curl},\Omega_1)} \lesssim (|s| + |s|^{-1}) \|\check{\boldsymbol{U}}\|_{H(\mathrm{curl},\Omega_1)}.$$

This, combined with (4.44), leads to the result

$$\begin{aligned} &\|\check{\boldsymbol{U}}\|_{H(\mathrm{curl},\Omega_1)} + \|\check{\boldsymbol{V}}\|_{H(\mathrm{curl},\Omega_1)} \\ &\qquad \lesssim s_1^{-1}(|s|^{-1} + |s|^3) \|(\widehat{\mathscr{B}} - \mathscr{B})[\check{\boldsymbol{E}}^p_{\Gamma_1}]\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}. \end{aligned} \tag{4.45}$$

We now estimate the norm $\|(\widehat{\mathscr{B}} - \mathscr{B})[\check{\boldsymbol{E}}^p_{\Gamma_1}]\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}$. For $\check{\boldsymbol{E}}^p|_{\Gamma_1}$ define its PML extension $\tilde{\check{\boldsymbol{E}}}^p$ in the s-domain to be the solution of the exterior problem

$$\begin{cases} \widetilde{\nabla} \times [(\mu s)^{-1} \widetilde{\nabla} \times \boldsymbol{v}] + \varepsilon s \boldsymbol{v} = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1, \\ \boldsymbol{n}_1 \times \boldsymbol{v} = \boldsymbol{n}_1 \times \check{\boldsymbol{E}}^p & \text{on} \quad \Gamma_1, \\ \hat{x} \times (\mu s \boldsymbol{v} \times \hat{x}) - \hat{x} \times (\widetilde{\nabla} \times \boldsymbol{v}) = o\left(|\widetilde{x}|^{-1}\right) & \text{as} \quad |\widetilde{x}| \to \infty. \end{cases}$$

By [34, Theorem 12.2] it is easy to see that $\tilde{\check{\boldsymbol{E}}}^p$ has the integral representation

$$\tilde{\check{\boldsymbol{E}}}^p = \mathbb{E}(\gamma_t(\check{\boldsymbol{E}}^p), \gamma_t(\widetilde{\mathrm{curl}}\ \tilde{\check{\boldsymbol{E}}}^p)).$$

Define $\widetilde{\check{\boldsymbol{H}}}^p := -(\mu s)^{-1} \widetilde{\mathrm{curl}} \tilde{\check{\boldsymbol{E}}}^p$. Then $(\tilde{\check{\boldsymbol{E}}}^p, \widetilde{\check{\boldsymbol{H}}}^p)$ satisfies the stretched Maxwell equations in (4.14) in $\mathbb{R}^3 \backslash \overline{B}_1$. It is worth noting that $\widetilde{\check{\boldsymbol{H}}}^p$ is not the extension of $\check{\boldsymbol{H}}^p|_{\Gamma_1}$.

Noting that $\widetilde{\nabla} \times \boldsymbol{v} = \mathbb{A} \nabla \times \mathbb{B} \boldsymbol{v}$, we know that $\mathbb{B} \tilde{\check{\boldsymbol{E}}}^p$ satisfies the problem

$$\begin{cases} \nabla \times \left[(\mu s)^{-1} \mathbb{B} \mathbb{A} \nabla \times \boldsymbol{v}\right] + \varepsilon s (\mathbb{B} \mathbb{A})^{-1} \boldsymbol{v} = \boldsymbol{0} & \text{in} \quad \mathbb{R}^3 \backslash \overline{B}_1, \\ \boldsymbol{n}_1 \times \boldsymbol{v} = \boldsymbol{n}_1 \times \check{\boldsymbol{E}}^p & \text{on} \quad \Gamma_1, \\ \hat{x} \times (\mu s \mathbb{B}^{-1} \boldsymbol{v} \times \hat{x}) - \hat{x} \times (\mathbb{A} \nabla \times \boldsymbol{v}) = o\left(|\widetilde{x}|^{-1}\right) & \text{as} \quad |\widetilde{x}| \to \infty, \end{cases}$$

where we have used the fact that $\tilde{\check{\boldsymbol{E}}}^p$ is the extension of $\check{\boldsymbol{E}}^p|_{\Gamma_1}$ and $\mathbb{B} = \mathrm{diag}\{1, 1, 1\}$ on Γ_1. By the definition of $\mathscr{B}$, and since $\mathbb{A} = \mathrm{diag}\{1, 1, 1\}$ on Γ_1, it is easy to see that

$$\mathscr{B}[\check{\boldsymbol{E}}^p_{\Gamma_1}] = \boldsymbol{n}_1 \times (\mu s)^{-1} \widetilde{\nabla} \times \tilde{\check{\boldsymbol{E}}}^p = \boldsymbol{n}_1 \times (\mu s)^{-1} \nabla \times \mathbb{B} \tilde{\check{\boldsymbol{E}}}^p.$$

By the definition of $\widehat{\mathscr{B}}$ in (4.27), we obtain that

$$(\widehat{\mathscr{B}} - \mathscr{B})[\check{\boldsymbol{E}}^p_{\Gamma_1}] = \boldsymbol{n}_1 \times (\mu s)^{-1} \nabla \times \boldsymbol{\omega} \tag{4.46}$$

where $\boldsymbol{\omega}$ satisfies

$$\nabla \times \left[(\mu s)^{-1}\mathbb{B}\mathbb{A}\nabla \times \boldsymbol{\omega}\right] + \varepsilon s(\mathbb{B}\mathbb{A})^{-1}\boldsymbol{\omega} = 0 \quad \text{in } \ \Omega^{\rm PML},$$
$$\boldsymbol{n}_1 \times \boldsymbol{\omega} = \boldsymbol{0} \quad \text{on } \ \Gamma_1, \qquad \boldsymbol{n}_2 \times \boldsymbol{\omega} = \gamma_t(\mathbb{B}\check{\tilde{\boldsymbol{E}}}^p) \quad \text{on } \ \Gamma_2.$$

By Lemma 4.4 and the estimate for $\mathbb{B}\mathbb{A}$ and $(\mathbb{B}\mathbb{A})^{-1}$ in (4.21)-(4.22), we have

$$\begin{aligned}
&\|\boldsymbol{n}_1 \times (\mu s)^{-1}\nabla \times \boldsymbol{\omega}\|_{H^{-1/2}({\rm Div},\Gamma_1)} \\
&\lesssim (1+s_1^{-1}\sigma_0)^2\|(\mu s)^{-1}\mathbb{B}\mathbb{A}\nabla \times \boldsymbol{\omega}\|_{H({\rm curl},\Omega^{\rm PML})} \\
&\lesssim (1+s_1^{-1}\sigma_0)^2\left(\frac{(1+s_1^{-1}\sigma_0)^2}{|s|^2}\|\nabla \times \boldsymbol{\omega}\|^2_{L^2(\Omega^{\rm PML})^3} + (1+s_1^{-1}\sigma_0)^4\|s\boldsymbol{\omega}\|^2_{L^2(\Omega^{\rm PML})^3}\right)^{1/2} \\
&\lesssim s_1^{-1}(1+s_1^{-1}\sigma_0)^8(1+|s|)^2\|\gamma_t(\mathbb{B}\check{\tilde{\boldsymbol{E}}}^p)\|_{H^{-1/2}({\rm Div},\Gamma_2)}. \qquad (4.47)
\end{aligned}$$

Since $\widetilde{\nabla} \times \boldsymbol{v} = \mathbb{A}\nabla \times \mathbb{B}\boldsymbol{v}$ and $|\mathbb{A}^{-1}| \le (1+\sigma_0)^2$ in $\Omega^{\rm PML}$, we have by the boundedness of the trace operator γ_t that

$$\|\gamma_t(\mathbb{B}\check{\tilde{\boldsymbol{E}}}^p)\|_{H^{-1/2}({\rm Div},\Gamma_2)} \lesssim \|\mathbb{B}\check{\tilde{\boldsymbol{E}}}^p\|_{H({\rm curl},\Omega^{\rm PML})} \lesssim (1+s_1^{-1}\sigma_0)^2\|\check{\tilde{\boldsymbol{E}}}^p\|_{H(\widetilde{\rm curl},\Omega^{\rm PML})}. \qquad (4.48)$$

By Lemma 4.7 and the boundedness of γ_T and γ_t it is derived that

$$\begin{aligned}
\|\check{\tilde{\boldsymbol{E}}}^p\|^2_{H(\widetilde{\rm curl},\Omega^{\rm PML})} &\le (\|\check{\tilde{\boldsymbol{E}}}^p\|^2_{L^\infty(\Omega^{\rm PML})} + \|\widetilde{\rm curl}\check{\tilde{\boldsymbol{E}}}^p\|^2_{L^\infty(\Omega^{\rm PML})})|\Omega^{\rm PML}| \\
&\lesssim s_1^{-4}d^4(1+s_1^{-1}\sigma_0)^6 e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\Big[(1+|s|^4)\|\gamma_t(\widetilde{\rm curl}\,\check{\tilde{\boldsymbol{E}}}^p)\|^2_{H^{-1/2}({\rm Div},\Gamma_1)} \\
&\quad + (1+|s|^6)\|\gamma_t\check{\boldsymbol{E}}^p\|^2_{H^{-1/2}({\rm Div},\Gamma_1)}\Big] \\
&\lesssim s_1^{-4}d^4(1+s_1^{-1}\sigma_0)^6 e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\Big[(1+|s|^4)^2\|\gamma_T\check{\tilde{\boldsymbol{E}}}^p\|^2_{H^{-1/2}({\rm Curl},\Gamma_1)} \\
&\qquad + (1+|s|^6)\|\gamma_t\check{\boldsymbol{E}}^p\|^2_{H^{-1/2}({\rm Div},\Gamma_1)}\Big] \\
&\lesssim s_1^{-4}d^4(1+s_1^{-1}\sigma_0)^6 e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\sum_{l=0}^{4}\|s^l\check{\boldsymbol{E}}^p\|^2_{H({\rm curl},\,\Omega_1)} \\
&\lesssim s_1^{-6}d^4(1+s_1^{-1}\sigma_0)^{10} e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\sum_{l=0}^{5}\|s^l\check{\boldsymbol{J}}\|^2_{L^2(\Omega_1)^3}, \qquad (4.49)
\end{aligned}$$

where we have used Lemma 4.1 and the upper bound estimate (3.7) of the EtM operator $\mathscr{B}$. Combining (4.45)-(4.49) yields that

$$\begin{aligned}
&\|\check{\boldsymbol{U}}\|^2_{H({\rm curl},\Omega_1)} + \|\check{\boldsymbol{V}}\|^2_{H({\rm curl},\Omega_1)} \\
&\quad \lesssim s_1^{-10}d^4(1+s_1^{-1}\sigma_0)^{30} e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\sum_{l=0}^{10}\|s^l\check{\boldsymbol{J}}\|^2_{L^2(\Omega_1)^3}. \qquad (4.50)
\end{aligned}$$

This, together with the Parseval identity for the Laplace transform (3.11), gives

$$\begin{aligned}
&\|\boldsymbol{U}\|^2_{L^2(0,T;H(\mathrm{curl},\Omega_1))} + \|\boldsymbol{V}\|^2_{L^2(0,T;H(\mathrm{curl},\Omega_1))} \\
&= \int_0^T \left(\|\boldsymbol{U}\|^2_{H(\mathrm{curl},\Omega_1)} + \|\boldsymbol{V}\|^2_{H(\mathrm{curl},\Omega_1)}\right) dt \\
&\leq e^{2s_1T} \int_0^\infty e^{-2s_1t} \left(\|\boldsymbol{U}\|^2_{H(\mathrm{curl},\Omega_1)} + \|\boldsymbol{V}\|^2_{H(\mathrm{curl},\Omega_1)}\right) dt \\
&\lesssim e^{2s_1T} \int_0^\infty s_1^{-10} d^4 (1+s_1^{-1}\sigma_0)^{30} e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}} \sum_{l=0}^{10} \|s^l \check{\boldsymbol{J}}\|^2_{L^2(\Omega_1)^3} ds_2 \\
&\lesssim e^{2s_1T} s_1^{-10} d^4 (1+s_1^{-1}\sigma_0)^{30} e^{-2\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}} \|\boldsymbol{J}\|^2_{H^{10}(0,T;L^2(\Omega_1)^3)},
\end{aligned} \tag{4.51}$$

where we have used the assumptions (3.1) and (3.2) to get the last inequality. It is obvious that m should be chosen small enough to ensure rapid convergence (thus we need to take $m = 1$). Since $s_1^{-1} = T$ in (4.51), we obtain the required estimate (4.41) by using the Cauchy-Schwartz inequality.

We now prove (4.42). Since $(\boldsymbol{E}, \boldsymbol{H})$ and $(\boldsymbol{E}^p, \boldsymbol{H}^p)$ satisfy the equations (3.4) and (4.38), respectively, it is easy to verify that $(\boldsymbol{U}, \boldsymbol{V})$ satisfies the problem

$$\begin{cases}
\nabla \times \boldsymbol{U} + \mu \partial_t \boldsymbol{V} = \boldsymbol{0} & \text{in } \ \Omega_1 \times (0,T), \\
\nabla \times \boldsymbol{V} - \varepsilon \partial_t \boldsymbol{U} = \boldsymbol{0} & \text{in } \ \Omega_1 \times (0,T), \\
\boldsymbol{n} \times \boldsymbol{U} = \boldsymbol{0} & \text{on } \ \Gamma \times (0,T), \\
\boldsymbol{U}(x,0) = \boldsymbol{V}(x,0) = \boldsymbol{0} & \text{in } \ \Omega_1, \\
\boldsymbol{V} \times \boldsymbol{n}_1 = (\mathscr{T} - \hat{\mathscr{T}})[\boldsymbol{E}^p_{\Gamma_1}] + \mathscr{T}[\boldsymbol{U}_{\Gamma_1}] & \text{on } \ \Gamma_1 \times (0,T).
\end{cases} \tag{4.52}$$

Eliminating $\boldsymbol{V}$ yields that

$$\begin{cases}
\nabla \times (\mu^{-1} \nabla \times \boldsymbol{U}) + \varepsilon \partial_t^2 \boldsymbol{U} = \boldsymbol{0} & \text{in } \ \Omega_1 \times (0,T), \\
\boldsymbol{n} \times \boldsymbol{U} = \boldsymbol{0} & \text{on } \ \Gamma \times (0,T), \\
\boldsymbol{U}(x,0) = \partial_t \boldsymbol{U}(x,0) = \boldsymbol{0} & \text{in } \ \Omega_1, \\
\mu^{-1}(\nabla \times \boldsymbol{U}) \times \boldsymbol{n}_1 + \mathscr{C}[\boldsymbol{U}_{\Gamma_1}] = (\hat{\mathscr{T}} - \mathscr{T})[\partial_t \boldsymbol{E}^p_{\Gamma_1}] & \text{on } \ \Gamma_1 \times (0,T),
\end{cases} \tag{4.53}$$

where $\mathscr{C} = \mathscr{L}^{-1} \circ s\mathscr{B} \circ \mathscr{L}$. The variational problem of (4.53) is to find $\boldsymbol{U} \in H_\Gamma(\mathrm{curl}, \Omega_1)$ for all $t > 0$ such that

$$\begin{aligned}
\int_{\Omega_1} \varepsilon \partial_t^2 \boldsymbol{U} \cdot \overline{\boldsymbol{\omega}} dx = &- \int_{\Omega_1} \mu^{-1} (\nabla \times \boldsymbol{U})(\nabla \times \overline{\boldsymbol{\omega}}) dx \\
&+ \int_{\Gamma_1} (\hat{\mathscr{T}} - \mathscr{T})[\partial_t \boldsymbol{E}^p_{\Gamma_1}] \cdot \overline{\boldsymbol{\omega}}_{\Gamma_1} d\gamma - \int_{\Gamma_1} \mathscr{C}[\boldsymbol{U}_{\Gamma_1}] \cdot \overline{\boldsymbol{\omega}}_{\Gamma_1} d\gamma, \quad \forall \boldsymbol{\omega} \in H_\Gamma(\mathrm{curl}, \Omega_1).
\end{aligned} \tag{4.54}$$

For $0 < \xi < T$, introduce the auxiliary function

$$\boldsymbol{\Psi}_1(x,t) = \int_t^\xi \boldsymbol{U}(x,\tau) d\tau, \quad x \in \Omega_1, 0 \leq t \leq \xi.$$

Then it is easy to verify that

$$\boldsymbol{\Psi}_1(x,\xi) = 0, \quad \partial_t \boldsymbol{\Psi}_1(x,t) = -\boldsymbol{U}(x,t). \tag{4.55}$$

For any $\boldsymbol{\phi}(x,t) \in L^2\left(0,\xi; L^2(\Omega_1)^3\right)$, using integration by parts and condition (4.55), we have

$$\int_0^\xi \boldsymbol{\phi}(x,t)\cdot\overline{\boldsymbol{\Psi}}_1(x,t)dt = \int_0^\xi\int_0^t \boldsymbol{\phi}(x,\tau)d\tau\cdot\overline{\boldsymbol{U}}(x,t)dt. \tag{4.56}$$

Taking the test function $\boldsymbol{\omega} = \boldsymbol{\Psi}_1$ in (4.54) and using (4.55) give

$$\begin{aligned}\mathrm{Re}\int_0^\xi\int_{\Omega_1}\varepsilon\partial_t^2\boldsymbol{U}\cdot\overline{\boldsymbol{\Psi}}_1 dxdt =& \mathrm{Re}\int_{\Omega_1}\int_0^\xi \varepsilon\Big(\partial_t(\partial_t\boldsymbol{U}\cdot\overline{\boldsymbol{\Psi}}_1) + \partial_t\boldsymbol{U}\cdot\overline{\boldsymbol{U}}\Big)dtdx\\ =&\frac{1}{2}\|\sqrt{\varepsilon}\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3}.\end{aligned} \tag{4.57}$$

By (4.56) we have the estimate

$$\begin{aligned}&\mathrm{Re}\int_0^\xi\int_{\Omega_1}\mu^{-1}(\nabla\times\boldsymbol{U})\cdot(\nabla\times\overline{\boldsymbol{\Psi}}_1)dxdt\\ &= \mathrm{Re}\int_{\Omega_1}\int_0^\xi\mu^{-1}(\nabla\times\boldsymbol{U})\cdot\int_t^\xi(\nabla\times\overline{\boldsymbol{U}}(x,\tau))d\tau dtdx\\ &= \int_{\Omega_1}\mu^{-1}\Big|\int_0^\xi(\nabla\times\boldsymbol{U})(x,t)dt\Big|^2dx - \mathrm{Re}\int_0^\xi\int_{\Omega_1}\mu^{-1}(\nabla\times\boldsymbol{U})\cdot(\nabla\times\overline{\boldsymbol{\Psi}}_1)dxdt,\end{aligned}$$

which implies that

$$\mathrm{Re}\int_0^\xi\int_{\Omega_1}\mu^{-1}(\nabla\times\boldsymbol{U})\cdot(\nabla\times\overline{\boldsymbol{\Psi}}_1)dxdt = \frac{1}{2}\int_{\Omega_1}\mu^{-1}\Big|\int_0^\xi\nabla\times\boldsymbol{U}(x,t)dt\Big|^2dx. \tag{4.58}$$

Integrating (4.54) from $t=0$ to $t=\xi$ and taking the real parts yield

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\varepsilon}\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3} + \frac{1}{2}\int_{\Omega_1}\mu^{-1}\Big|\int_0^\xi\nabla\times\boldsymbol{U}(x,t)dt\Big|^2\\ &= \mathrm{Re}\int_0^\xi\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{1\Gamma_1}d\gamma dt - \mathrm{Re}\int_0^\xi\int_{\Gamma_1}\mathscr{C}[\boldsymbol{U}_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{1\Gamma_1}d\gamma dt.\end{aligned} \tag{4.59}$$

First, using (4.56) and Lemma 3.2, we have

$$\mathrm{Re}\int_0^\xi\int_{\Gamma_1}\mathscr{C}[\boldsymbol{U}_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{1\Gamma_1}d\gamma dt = \mathrm{Re}\int_{\Gamma_1}\int_0^\xi\left(\int_0^t\mathscr{C}[\boldsymbol{U}_{\Gamma_1}](x,\tau)d\tau\right)\cdot\overline{\boldsymbol{U}}_{\Gamma_1}(x,t)dt\,d\gamma \ge 0. \tag{4.60}$$

Then, and by (4.56) we deduce the estimate

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\varepsilon}\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3} + \frac{1}{2}\int_{\Omega_1}\mu^{-1}\Big|\int_0^\xi\nabla\times\boldsymbol{U}(x,t)dt\Big|^2dx\\ &\le \mathrm{Re}\int_0^\xi\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{1\Gamma_1}d\gamma dt\\ &= \mathrm{Re}\int_0^\xi\int_{\Gamma_1}\left(\int_0^t(\hat{\mathscr{T}}-\mathscr{T})[\partial_\tau\boldsymbol{E}^p_{\Gamma_1}]d\tau\right)\overline{\boldsymbol{U}}_{\Gamma_1}(x,t)d\gamma dt\\ &\lesssim \left(\int_0^\xi\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}](\cdot,t)\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}dt\right)\left(\int_0^\xi\|\boldsymbol{U}(\cdot,t)\|_{H(\mathrm{curl},\Omega_1)}dt\right),\end{aligned} \tag{4.61}$$

where we have used the trace theorem to get the last inequality. The right-hand of (4.61) contains the term

$$\int_0^\xi \|\boldsymbol{U}(\cdot,t)\|_{H(\mathrm{curl},\Omega_1)}dt = \int_0^\xi \left(\|\boldsymbol{U}(\cdot,t)\|^2_{L^2(\Omega_1)^3} + \|\nabla\times\boldsymbol{U}(\cdot,t)\|^2_{L^2(\Omega_1)^3}\right)^{\frac{1}{2}} dt$$

which cannot be controlled by the left-hand of (4.61). To address this issue, we consider the new problem

$$\begin{cases} \nabla\times(\mu^{-1}\nabla\times(\partial_t\boldsymbol{U})) + \varepsilon\partial_t^2(\partial_t\boldsymbol{U}) = \boldsymbol{0} & \text{in} \quad \Omega_1\times(0,T), \\ \boldsymbol{n}\times\partial_t\boldsymbol{U} = \boldsymbol{0} & \text{on} \quad \Gamma\times(0,T), \\ \partial_t\boldsymbol{U}(x,0) = \partial_t^2\boldsymbol{U}(x,0) = \boldsymbol{0} & \text{in} \quad \Omega_1, \\ \mu^{-1}(\nabla\times(\partial_t\boldsymbol{U}))\times\boldsymbol{n}_1 + \mathscr{C}[\partial_t\boldsymbol{U}_{\Gamma_1}] = (\hat{\mathscr{T}} - \mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}] & \text{on} \quad \Gamma_1\times(0,T), \end{cases} \tag{4.62}$$

which is obtained by differentiating each equation of (4.53) with respect to t. By a similar argument as in deriving (4.54), we obtain the variational formulation of (4.62): find u such that for all $\boldsymbol{\omega}\in H_\Gamma(\mathrm{curl},\Omega_1)$,

$$\begin{aligned}\int_{\Omega_1}\varepsilon\partial_t^2(\partial_t\boldsymbol{U})\cdot\overline{\boldsymbol{\omega}}dx = &-\int_{\Omega_1}\mu^{-1}(\nabla\times(\partial_t\boldsymbol{U}))(\nabla\times\overline{\boldsymbol{\omega}})dx \\ &+\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\cdot\overline{\boldsymbol{\omega}}_{\Gamma_1}d\gamma - \int_{\Gamma_1}\mathscr{C}[\partial_t\boldsymbol{U}_{\Gamma_1}]\cdot\overline{\boldsymbol{\omega}}_{\Gamma_1}d\gamma. \end{aligned}\tag{4.63}$$

Define the auxiliary function

$$\boldsymbol{\Psi}_2(x,t) = \int_t^\xi \partial_\tau\boldsymbol{U}(x,\tau)d\tau, \quad x\in\Omega_1, 0\le t\le\xi.$$

Similarly as in the derivation of (4.57)-(4.58), we conclude by integration by parts that

$$\mathrm{Re}\int_0^\xi\int_{\Omega_1}\varepsilon\partial_t^2(\partial_t\boldsymbol{U})\cdot\overline{\boldsymbol{\Psi}}_2dxdt = \frac{1}{2}\|\sqrt{\varepsilon}\partial_t\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3}, \tag{4.64}$$

$$\mathrm{Re}\int_0^\xi\int_{\Omega_R}\mu_e^{-1}(\nabla\times(\partial_t\boldsymbol{U}))\cdot(\nabla\times\overline{\boldsymbol{\Psi}}_2)dxdt = \frac{1}{2}\left\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{U}(\cdot,\xi)\right\|^2_{L^2(\Omega_1)^3}. \tag{4.65}$$

Choosing the test function $\boldsymbol{\omega}=\boldsymbol{\Psi}_2$ in (4.63), integrating the resulting equation with respect to t from $t=0$ to $t=\xi$ and taking the real parts yield

$$\begin{aligned}&\frac{1}{2}\|\sqrt{\varepsilon}\partial_t\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3} + \frac{1}{2}\left\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{U}(\cdot,\xi)\right\|^2_{L^2(\Omega_1)^3} \\ &= \mathrm{Re}\int_0^\xi\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{2\Gamma_1}d\gamma dt - \mathrm{Re}\int_0^\xi\int_{\Gamma_1}\mathscr{C}[\partial_t\boldsymbol{U}_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{2\Gamma_1}d\gamma dt. \end{aligned}\tag{4.66}$$

Similarly to (4.60), it follows from (4.56) and Lemma 3.3 that

$$\mathrm{Re}\int_0^\xi\int_{\Gamma_1}\mathscr{C}[\partial_t\boldsymbol{U}_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{2\Gamma_1}d\gamma dt \ge 0.$$

Thus, and by (4.66) we have

$$\begin{aligned}
&\frac{1}{2}\|\sqrt{\varepsilon}\partial_t \boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3} + \frac{1}{2}\Big\|\frac{1}{\sqrt{\mu}}\nabla\times\boldsymbol{U}(\cdot,\xi)\Big\|^2_{L^2(\Omega_1)^3}\\
&\le \mathrm{Re}\int_0^\xi\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\cdot\overline{\boldsymbol{\Psi}}_{2\Gamma_1}d\gamma dt\\
&= \mathrm{Re}\int_0^\xi\int_{\Gamma_1}\left(\int_0^t(\hat{\mathscr{T}}-\mathscr{T})[\partial_\tau^2\boldsymbol{E}^p_{\Gamma_1}]d\tau\right)\partial_t\overline{\boldsymbol{U}}_{\Gamma_1}(x,t)d\gamma dt\\
&= \mathrm{Re}\int_0^\xi\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\cdot\overline{\boldsymbol{U}}_{\Gamma_1}(x,\xi)d\gamma dt - \mathrm{Re}\int_0^\xi\int_{\Gamma_1}(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\overline{\boldsymbol{U}}_{\Gamma_1}(x,t)d\gamma dt\\
&\le \int_0^\xi\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}\cdot\left(\|\boldsymbol{U}(\cdot,\xi)\|_{H(\mathrm{curl},\Omega_1)}+\|\boldsymbol{U}(\cdot,t)\|_{H(\mathrm{curl},\Omega_1)}\right)dt \qquad (4.67)
\end{aligned}$$

Combining (4.61) and (4.67) gives

$$\begin{aligned}
&\|\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3}+\|\partial_t\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3}+\|\nabla\times\boldsymbol{U}(\cdot,\xi)\|^2_{L^2(\Omega_1)^3}\\
&\lesssim\left(\int_0^\xi\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}](\cdot,t)\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}dt\right)\left(\int_0^\xi\|\boldsymbol{U}(\cdot,t)\|_{H(\mathrm{curl},\Omega_1)}dt\right)\\
&\quad+\int_0^\xi\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\|_{H^{-1/2}(\mathrm{Div},\Gamma_1)}\cdot\left(\|\boldsymbol{U}(\cdot,\xi)\|_{H(\mathrm{curl},\Omega_1)}+\|\boldsymbol{U}(\cdot,t)\|_{H(\mathrm{curl},\Omega_1)}\right)dt. \qquad (4.68)
\end{aligned}$$

Taking the L^∞-norm of both sides of (4.68) with respect to ξ and using the Young inequality yield

$$\begin{aligned}
&\|\boldsymbol{U}\|^2_{L^\infty(0,T;L^2(\Omega_1)^3)}+\|\partial_t\boldsymbol{U}\|^2_{L^\infty(0,T;L^2(\Omega_1)^3)}+\|\nabla\times\boldsymbol{U}\|^2_{L^\infty(0,T;L^2(\Omega_1)^3)}\\
&\lesssim T^2\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}]\|^2_{L^1(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}+\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\|^2_{L^1(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))},
\end{aligned}$$

which, together with the Cauchy-Schwartz inequality, implies that

$$\begin{aligned}
&\|\boldsymbol{U}\|_{L^\infty(0,T;L^2(\Omega_1)^3)}+\|\partial_t\boldsymbol{U}\|_{L^\infty(0,T;L^2(\Omega_1)^3)}+\|\nabla\times\boldsymbol{U}\|_{L^\infty(0,T;L^2(\Omega_1)^3)} \qquad (4.69)\\
&\lesssim T^{3/2}\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}]\|_{L^2(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}+T^{1/2}\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2\boldsymbol{E}^p_{\Gamma_1}]\|_{L^2(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}.
\end{aligned}$$

We now only need to estimate the right-hand term of (4.69). By (4.46) and the definition of $\hat{\mathscr{T}}$ (see (4.38)) we know that $(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}]=\boldsymbol{n}_1\times\mu^{-1}\nabla\times\boldsymbol{v}$, where $\boldsymbol{v}$ satisfies the problem

$$\begin{cases}
\nabla\times(\mu^{-1}\mathbb{B}\mathbb{A}\nabla\times\boldsymbol{v})+\varepsilon(\mathbb{B}\mathbb{A})^{-1}\partial_t^2\boldsymbol{v}=\boldsymbol{0} & \text{in} \quad \Omega^{\mathrm{PML}}\times(0,T),\\
\boldsymbol{n}_1\times\boldsymbol{v}=\boldsymbol{0} & \text{on} \quad \Gamma_1\times(0,T),\\
\boldsymbol{n}_2\times\boldsymbol{v}=\gamma_t(\mathbb{B}\widetilde{\boldsymbol{E}}^p) & \text{on} \quad \Gamma_2\times(0,T),\\
\boldsymbol{v}(x,0)=\partial_t\boldsymbol{v}(x,0)=\boldsymbol{0} & \text{in} \quad \Omega^{\mathrm{PML}}.
\end{cases} \qquad (4.70)$$

Thus we deduce that

$$\begin{aligned}
\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t\boldsymbol{E}^p_{\Gamma_1}]\|^2_{L^2(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}&=\|\boldsymbol{n}_1\times\mu^{-1}\nabla\times\boldsymbol{v}\|^2_{L^2(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}\\
&\le e^{2s_1T}\int_0^\infty e^{-2s_1t}\|\mu^{-1}\nabla\times\boldsymbol{v}\|^2_{H(\mathrm{curl},\Omega^{\mathrm{PML}})}dt\\
&\lesssim e^{2s_1T}\int_{-\infty}^\infty\|\mu^{-1}\nabla\times\check{\boldsymbol{v}}\|^2_{H(\mathrm{curl},\Omega^{\mathrm{PML}})}ds_2.
\end{aligned}$$

Repeating (4.47)-(4.49) yields

$$\begin{aligned}
&\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t \boldsymbol{E}^p_{\Gamma_1}]\|_{L^2(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}\\
&\lesssim e^{s_1T}s_1^{-4}d^2(1+s_1^{-1}\sigma_0)^{15}e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\left[\int_{-\infty}^{\infty}\sum_{l=0}^{8}\|s^l\check{\boldsymbol{J}}\|^2_{L^2(\Omega_1)^3}ds_2\right]^{1/2}\\
&\lesssim e^{s_1T}s_1^{-4}d^2(1+s_1^{-1}\sigma_0)^{15}e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\|\boldsymbol{J}\|_{H^8(0,T;L^2(\Omega_1)^3)}.
\end{aligned}$$

Similarly, we have

$$\|(\hat{\mathscr{T}}-\mathscr{T})[\partial_t^2 \boldsymbol{E}^p_{\Gamma_1}]\|_{L^2(0,T;H^{-1/2}(\mathrm{Div},\Gamma_1))}\lesssim e^{s_1T}s_1^{-4}d^2(1+s_1^{-1}\sigma_0)^{15}e^{-\frac{\sqrt{\varepsilon\mu}\sigma_0 d}{m+1}}\|\boldsymbol{J}\|_{H^9(0,T;L^2(\Omega_1)^3)}.$$

By (4.69) and the above two estimates it follows on setting $s_1=1/T$ and $m=1$ that

$$\begin{aligned}
&\|\boldsymbol{U}\|_{L^\infty(0,T;L^2(\Omega_1)^3)}+\|\partial_t\boldsymbol{U}\|_{L^\infty(0,T;L^2(\Omega_1)^3)}+\|\nabla\times\boldsymbol{U}\|_{L^\infty(0,T;L^2(\Omega_1)^3)}\\
&\lesssim T^{11/2}d^2(1+\sigma_0 T)^{15}e^{-\sqrt{\varepsilon\mu}\sigma_0 d/2}\|\boldsymbol{J}\|_{H^9(0,T;L^2(\Omega_1)^3)}.
\end{aligned}$$

From this, the definition of $\boldsymbol{U}$ and Maxwell's system (4.52) the required estimate (4.42) then follows. The proof is thus complete. □

Remark 4.9. The L^2-norm error estimate (4.41) can also be obtained by integrating (4.68) with respect to ξ from 0 to T. The idea of using the uniform coercivity of the variational form in our proof of the L^2-norm error estimate (4.41) is also known for the time-harmonic PML method. This builds a connection between our proposed time-domain PML method with the real coordinate stretching technique and the time-harmonic PML method in some sense.

5 Conclusions

In this paper, by using the real coordinate stretching technique we proposed a uniaxial PML method in the Cartesian coordinates for 3D time-domain electromagnetic scattering problems, which is of advantage over the spherical one in dealing with scattering problems involving anisotropic scatterers. The well-posedness and stability estimates of the truncated uniaxial PML problem in the time domain were established by employing the Laplace transform technique and the energy argument. The exponential convergence of the uniaxial PML method was also proved in terms of the thickness and absorbing parameters of the PML layer, based on the error estimate between the EtM operators for the original scattering problem and the truncated PML problem established in this paper via the decay estimate of the dyadic Green's function.

Our method can be extended to other electromagnetic scattering problems such as scattering by inhomogeneous media or bounded elastic bodies as well as scattering in a two-layered medium. It is also interesting to study the spherical and Cartesian PML methods for time-domain elastic scattering problems, which is more challenging due to the existence of shear and compressional waves with different wave speeds. We hope to report such results in the near future.

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