FINITE ELEMENT METHODS FOR THE DARCY-FORCHHEIMER PROBLEM COUPLED WITH THE CONVECTION-DIFFUSION-REACTION PROBLEM

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Abstract. In this article, we consider the convection-di usion-reaction problem coupled the Darcy-Forchheimer problem by a non-linear external force depending on the concentration. We establish existence of a solution by using a Galerkin method and we prove uniqueness. We introduce and analyse a numerical scheme based on the nite element method. An optimal *a priori* error estimate is then derived for each numerical scheme. Numerical investigation are performed to con rm the theoretical accuracy of the discretization.

Keywords. Darcy-Forchheimer problem; convection-di usion-reaction equation; nite element method; *a priori* error estimates.

1. Introduction.

This work studies the convection-di usion-reaction equation coupled with Darcy-Forchheimer problem. The system of equations is

$$
(P)\quad \begin{cases}
-K^{-1}u + -|u|u + \nabla p = f(.;C) & \text{in} \ ; \\
\operatorname{div} u = 0 & \text{in} \ ; \\
C + u \ \nabla C + r_0 C = g & \text{in} \ ; \\
u \ n = 0 & \text{on} \ ; \\
C = 0 & \text{on} \ ;
\end{cases}
$$

where $\mathbb{R}^d$, $d = 2;3$, is a bounded simply-connected open domain, having a Lipschitz-continuous boundary with an outer unit normal n. The unknowns are the velocity u, the pressure p and the concentration C of the uid. $|.|$ denotes the Euclidean norm, $|u|^2 = u \ u$. The parameters , and represent the density of the uid, its viscosity and its dynamic viscosity, respectively. is also referred as Forchheimer number when it is a scalar positive constant. The di usion coe cient

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System (P) couples the Darcy-Forchheimer system with the convection-diffusion-reaction equation satisfied by the the concentration of the fluid. The same system can also couple the Darcy-Forchheimer system with the heat equation satisfied by the temperature T of the fluid, it suffices to set $r_0 = 0$ and replace C by T.

Darcy's law (see [31] and [37] for instance for the theoretical derivation) describes the creeping flow of Newtonian fluids in porous media. It is simply the first equation of system (P) without the non-linear term $-|u|u$ and where the function f may depend on the concentration C of the fluid. Forchheimer [19] showed experimentally that when the velocity is higher and the porosity is nonuniform, Darcy's law becomes inadequate. He proposed the Darcy-Forchheimer equation which is the first equation of the system (P). A theoretical derivation of Forchheimer's law can be found in [29]. Multiple theoretical and numerical studies of the Darcy-Forchheimer system were performed and among others we mention [24, 26, 32, 28, 33]. Many numerical investigations are performed and show the importance of the Darcy-Forchheimer equation compared with Darcy equation (see for instance [32] and the references inside).

For the coupled problem of Darcy's law with the heat equation, we can refer to [6] where the coupled problem is treated using the spectral method. The authors in [4] and [14] considered the same stationary system but coupled with a nonlinear viscosity that depends on the temperature. In [15], the authors derived an optimal *a posteriori* error estimate for each of the numerical schemes proposed in [4]. We can also refer to [3] where the authors used a vertex-centred finite volume method to discretize the coupled system. Furthermore, for the time-dependent convection-diffusion-reaction equation coupled with Darcy's equation, we refer to [9, 10] where the authors established the corresponding *a priori* and *a posteriori* errors.

The coupling system (P) has many physical applications for the Darcy-Forchheimer mixed convection case [36]. In this case, the Darcy-Forchheimer system is coupled with the concentration C of the fluid with an external force f. We mention that the work of [6] use the spectral method to treat a coupled system very close to (P) where the Darcy-Forchheimer equation is replaced by the Darcy's one. It turns out that the non-linear term appearing in the first equation of (P) makes the treatment of the coupled system more complex.
We first derive an equivalent variational formulation to (P) and we show the existence of a solution. The uniqueness can be reached under additional constraint on the concentration (see condition (2.27)). Then, we discretize the system by using the finite element method and we show the existence and uniqueness of the corresponding solution. Later, we establish the *a priori* error estimate between the exact and numerical solutions under the condition of smallness of the concentration in the fluid. In order to compute the solution, we introduce an iterative scheme and we study the corresponding convergence. Finally, numerical investigations are performed to validate the theoretical results.
The outline of the paper is as follows:

- Section 2 is devoted to the continuous problem and the analysis of the corresponding variational formulation.
- In section 3, we introduce the discrete problems, recall their main properties, study their *a priori* errors and derive optimal estimates.
- In section 4, we introduce an iterative algorithm and prove its convergence.
- Numerical results validating the convergence analysis are presented in Section 5.

2. Analysis of the model

2.1. Notation. Let $D(\Omega)$ be the space of functions having a compact support in Ω with continuous derivatives of all orders in Ω. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ be a d-uplet of non negative integers, set $|\alpha| =$

$\sum_{i=1}^{d}\alpha_i$, and define the partial derivative ∂^α by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_d^{\alpha_d}}.$$

Then, for any positive integer m and number $p \geq 1$, we recall the classical Sobolev space [2, 30]

$$W^{m,p}(\Omega) = \{ v \in L^p(\Omega);\ \forall |\alpha| \leq m,\ \partial^\alpha v \in L^p(\Omega) \}, \tag{2.1}$$

equipped with the seminorm

$$|v|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^p \, dx \Big)^{\frac{1}{p}} \tag{2.2}$$

and the norm

$$\|v\|_{W^{m,p}(\Omega)} = \Big(\sum_{0\leq k\leq m} |v|^p_{W^{k,p}(\Omega)} \Big)^{\frac{1}{p}}. \tag{2.3}$$

When $p = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let v be a vector valued function; we set

$$\|v\|_{L^p(\Omega)^d} = \Big(\int_\Omega |v|^p \, dx \Big)^{\frac{1}{p}}, \tag{2.4}$$

where $|.|$ denotes the Euclidean vector norm.

For vanishing boundary values, we define

$$\begin{aligned} H^1_0(\Omega) &= \{ v \in H^1(\Omega);\ v_{|\Gamma} = 0 \}, \\ W^{1,q}_0(\Omega) &= \{ v \in W^{1,q}(\Omega);\ v_{|\Gamma} = 0 \}. \end{aligned} \tag{2.5}$$

We shall often use the following Sobolev imbeddings: for any real number $p \geq 1$ when $d = 2$, or $1 \leq p \leq \frac{2d}{d-2}$ when $d \geq 3$, there exist constants S_p and S^0_p such that

$$\forall v \in H^1(\Omega),\ \|v\|_{L^p(\Omega)} \leq S_p \|v\|_{H^1(\Omega)} \tag{2.6}$$

and

$$\forall v \in H^1_0(\Omega),\ \|v\|_{L^p(\Omega)} \leq S^0_p |v|_{H^1(\Omega)}. \tag{2.7}$$

When $p = 2$, (2.7) reduces to Poincaré's inequality.

To deal with the Darcy-Forchheimer, we recall the space

$$L^2_0(\Omega) = \Big\{ v \in L^2(\Omega);\ \int_\Omega v \, dx = 0 \Big\}. \tag{2.8}$$

2.2. Variational formulation.

In this section, we introduce the variational formulation corresponding to the problem (P).

We assume that the volumic and boundary sources verify the following conditions:

Assumption 2.1. *We assume that* f *and* g *verify:*

(1) f *can be written as follows:*

$$\forall x \in \Omega,\ \forall C \in \mathbb{R},\quad f(x,C) = f_0(x) + f_1(C), \tag{2.9}$$

where $f_0 \in L^{\frac{3}{2}}(\Omega)^d$ *and* f_1 *is Lipschitz-continuous with constant* c_{f_1} *and satisfies the inequality*

$$\forall \xi \in \mathbb{R},\ |f_1(\xi)| \leq c_{f_1}|\xi|,$$

where c_{f_1} *is a strictly positive constant.*

(2) $g \in L^2(\Omega)$.

Remark: For the physical interpretation of the choice of the external force f, we refer to [6, 7, 13, 20]. In fact, in [6] page 3, they considered and justified the choice of $f(x,C) = f_1(C)$ (with $f_0 = 0$) with the corresponding properties. For the generalization, we added the function f_0 to get (2.1).

It follows from the nonlinear term in the system (P) that the velocity u and the test function v must belong to $L^3(\Omega)^d$; then, the gradient of the pressure must belong to $L^{\frac{3}{2}}(\Omega)^d$. Furthermore, the concentration C must be in $H^1_0(\Omega)$. Thus, we introduce the spaces

$$X = L^3(\Omega)^d; \quad M = W^{1,\frac{3}{2}}(\Omega) \cap L^2_0(\Omega); \quad Y = H^1_0(\Omega):$$

Furthermore, we recall the following inf-sup condition between X and M [24],

$$\inf_{q \in M} \sup_{v \in X} \frac{\int_\Omega v(x) \cdot \nabla q(x)\,dx}{\|v\|_{L^3(\Omega)^d} \|\nabla q\|_{L^{\frac{3}{2}}(\Omega)}} = 1: \tag{2.10}$$

With these assumptions on the sources, we introduce the following variational formulation associated to problem (P):

$$(V_a)\quad \begin{cases} \text{Find } (u,p,C) \in X \times M \times Y \text{ such that:} \\ \forall v \in X, \ \dfrac{\mu}{\rho}\displaystyle\int_\Omega (K^{-1}u(x)) \cdot v(x)\,dx + \dfrac{\beta}{\rho}\displaystyle\int_\Omega |u(x)|u(x) \cdot v(x)\,dx + \displaystyle\int_\Omega \nabla p(x) \cdot v(x)\,dx \\ \qquad\qquad = \displaystyle\int_\Omega f(x,C(x)) \cdot v(x)\,dx; \\ \forall q \in M, \ \displaystyle\int_\Omega \nabla q(x) \cdot u(x)\,dx = 0; \\ \forall S \in Y, \ \alpha\displaystyle\int_\Omega \nabla C(x) \cdot \nabla S(x)\,dx + \displaystyle\int_\Omega (u \cdot \nabla C)(x)S(x)\,dx + r_0 \displaystyle\int_\Omega C(x)S(x)\,dx = \displaystyle\int_\Omega g(x)S(x)\,dx: \end{cases}$$

Equivalence between (P) and (V_a) in the sense of distribution follows readily from the validity of Green's formula:

$$\forall q \in M, \ \forall v \in H, \quad \int_\Omega \nabla q(x) \cdot v(x)dx = -\int_\Omega q(x)\,\mathrm{div}(v(x))\,dx + \langle q, v \cdot n\rangle_{\partial\Omega}$$

in the space

$$H = \{v \in L^3(\Omega)^d; \mathrm{div}\, v \in L^{\frac{3d}{d+3}}(\Omega)\};$$

and the fact that

$$\begin{aligned} V &= \left\{ v \in H;\ v \cdot n|_{\partial\Omega} = 0; \text{ and } \forall q \in M, \ \int_\Omega \nabla q \cdot v\,dx = 0 \right\}: \\ &= \left\{ v \in H;\ v \cdot n|_{\partial\Omega} = 0;\ \mathrm{div}\, v = 0 \text{ in } \Omega \right\}; \end{aligned}$$

for further details, we refer to [24].

To study problem (V_a), it is convenient to introduce the mapping $v \to \mathcal{A}(v)$ defined by:

$$\begin{aligned} \mathcal{A} : \ L^3(\Omega)^d &\mapsto L^{\frac{3}{2}}(\Omega)^d \\ v &\mapsto \mathcal{A}(v) = \frac{\mu}{\rho}K^{-1}v + \frac{\beta}{\rho}|v|v: \end{aligned}$$

We refer to [24, 17] for the following properties of $\mathcal{A}$.

Property 2.2. $\mathcal{A}$ *satisfies the following properties:*

(1) $\mathcal{A}$ *maps* $L^3(\Omega)^d$ *into* $L^{\frac{3}{2}}(\Omega)^d$ *and we have for all* $v \in L^3(\Omega)^d$*:*

$$\|\mathcal{A}(v)\|_{L^{\frac{3}{2}}(\Omega)^d} \le \frac{\mu}{\rho}\|K^{-1}\|_\infty \|v\|_{L^{\frac{3}{2}}(\Omega)^d} + \frac{\beta}{\rho}\|v\|^2_{L^3(\Omega)^d}:$$

(2) *For all* $(v,w) \in \mathbb{R}^d \times \mathbb{R}^d$*, we have,*

$$|\mathcal{A}(v) - \mathcal{A}(w)| \le \left(\frac{\mu}{\rho}\|K^{-1}\|_\infty + \frac{2\beta}{\rho}(|v| + |w|)\right)|v - w|: \tag{2.11}$$

(3) A *is monotone from* $L^3(\Omega)^d$ *into* $L^{\frac{3}{2}}(\Omega)^d$*, and we have for all* $v, w \in L^3(\Omega)^d$*,*

$$\int_\Omega (A(v(x)) - A(w(x)))\cdot(v(x) - w(x))\,dx \geq \max(c_m\|v - w\|^3_{L^3(\Omega)^d}; -K_m\|v - w\|^2_{L^2(\Omega)^d});$$

where c_m *is a strictly positive constant.*

(4) A *is coercive in* $L^3(\Omega)^d$*:*

$$\lim_{\|u\|_{L^3(\Omega)^d}\to\infty} \frac{\int_\Omega A(u)\cdot u\,dx}{\|u\|_{L^3(\Omega)^d}} = +\infty\,.$$

(5) A *is hemi-continuous in* $L^3(\Omega)^d$*: for fixed* $u, v \in L^3(\Omega)^d$*, the mapping*

$$t \to \int_\Omega A(u + tv)\cdot v\,dx$$

is continuous from $\mathbb{R}$ *into* $\mathbb{R}$*.*

Let us first show that for a given $C \in Y$, the Darcy-Forchheimer problem (first two lines in (V_a)) written as following: Find $(u(C); p(C)) \in X \times M$ such that

$$\begin{cases} \forall v \in X; & \int_\Omega A(u(C)(x))\cdot v(x)\,dx + \int_\Omega \nabla p(C)(x)\cdot v(x)\,dx = \int_\Omega f(x; C(x))\cdot v(x)\,dx;\\ \forall q \in M; & \int_\Omega \nabla q(x)\cdot u(C)(x)\,dx = 0; \end{cases} \tag{2.12}$$

admits a unique solution $(u; p) = (u(C); p(C))$. Problem (2.12) is equivalent to the following: find $(u(C); p(C)) \in X \times M$, such that

$$\forall v \in V; \quad \int_\Omega A(u(C)(x))\cdot v(x)\,dx = \int_\Omega f(x; C(x))\cdot v(x)\,dx. \tag{2.13}$$

Theorem 2.3. *For each* $C \in H^1_0(\Omega)$, $f(\cdot; C) \in L^{\frac{3}{2}}(\Omega)^d$*, the problem* (2.12) *has exactly one solution* $(u(C); p(C)) \in X \times M$*. Furthermore,* $(u; p)$ *satisfies the a priori estimates :*

$$\begin{aligned} &\|u(C)\|_{L^3(\Omega)^d} \leq \left(-\|f(\cdot; C)\|_{L^{\frac{3}{2}}(\Omega)^d}\right)^{\frac{1}{2}};\\ &\|\nabla p(C)\|_{L^{\frac{3}{2}}(\Omega)^d} \leq - \|K^{-1}\|_{L^\infty(\Omega)^{d\times d}}\|u(C)\|_{L^{\frac{3}{2}}(\Omega)^d} + -\|u(C)\|^2_{L^3(\Omega)^d} + \|f(\cdot; C)\|_{L^{\frac{3}{2}}(\Omega)^d}. \end{aligned} \tag{2.14}$$

Proof. Let $C \in H^1_0(\Omega)$. Assumption 2.1 allows us to deduce that $f(\cdot; \cdot C)$ lies in $L^{\frac{3}{2}}(\Omega)^d$. For the proof, we refer to [24] (see Theorem 3 page 172). □

Thus, Problem (V_a) can be rewritten as a function of the single unknown C. Indeed, for a given C, let $(u(C); p(C))$ be the solution of problem (2.12). Then, problem (V_a) is equivalent to the following reduced formulation: Find $C \in Y$ such that

$$\forall S \in Y; \quad \int_\Omega \nabla C(x)\cdot\nabla S(x)\,dx + \int_\Omega (u(C)\cdot\nabla C)(x)S(x)\,dx + r_0\int_\Omega C(x)S(x)dx = \int_\Omega g(x)S(x)\,dx. \tag{2.15}$$

Before proving that problem (2.15) admits a solution, we will show the following intermediate lemma:

Lemma 2.4. *Under Assumption 2.1, let* $(C_k)_{k\geq 1}$ *be a sequence of functions in* $L^2(\Omega)$ *that converges strongly to* C *in* $L^2(\Omega)$*. Then, the sequence* $(u(C_k))_{k\geq 1}$ *converges strongly to* $u(C)$ *in* X *and the sequence* $(p(C_k))_{k\geq 1}$ *converges weakly to* $p(C)$ *in* M*.*

Proof. Assumption 2.1 allows us to deduce that the sequence $(f(\cdot; C_k(\cdot)))_{k\geq 1}$ converges strongly to $f(\cdot; C(\cdot))$ in $L^{\frac{3}{2}}(\Omega)^d$ and then bounded in $L^{\frac{3}{2}}(\Omega)^d$. Bounds (2.14) yield first the weak convergence (up to a subsequence) of $(u(C_k); \nabla p(C_k))$ in $L^3(\Omega)^d \times L^{\frac{3}{2}}(\Omega)$ to some function $(\hat{u}; \hat{h})$. We will show that $(\hat{u}; \hat{h}) = (u(C); \nabla p(C))$. Let us first show that $\hat{u}$ is a solution of Problem (2.13). We show first that $\hat{u} \in V$.

Indeed, the second equation of problem (2.13) satisfied by $u(C_k)$, and the weak convergence of $u(C_k)$ to $\hat{u}$ led to the relation

$$\forall q \in M, \quad \int_\Omega \hat{u}(x) \cdot \nabla q(x) = 0\,;$$

and then $\hat{u} \in V$. Now, we show that $\hat{u}$ satisfies problem (2.13). The monotonicity of A gives

$$\forall v \in V, \quad \int_\Omega (A(u(C_k)) - A(v))(x) \cdot (u(C_k) - v)(x)\,dx \geq 0. \tag{2.16}$$

The last inequality combined with problem (2.13) satisfied by $u(C_k)$ yields

$$\forall v \in V, \quad \int_\Omega A(v(x)) \cdot (u(C_k) - v)(x)\,dx \leq \int_\Omega f(C_k)(x) \cdot (u(C_k) - v)(x)\,dx. \tag{2.17}$$

We obtain by using the weak convergence of $u(C_k)$ to $\hat{u}$ and the strong convergence of $f(\cdot,C_k)$ to $f(\cdot,C)$, the following relation:

$$\forall v \in V, \quad \int_\Omega A(v(x)) \cdot (\hat{u} - v)(x)\,dx \leq \int_\Omega f(C)(x) \cdot (\hat{u} - v)(x)\,dx. \tag{2.18}$$

By virtue of hemi-continuity of A, a classical argument then yields

$$\forall v \in V, \quad \int_\Omega A(\hat{u}(x)) \cdot v(x)\,dx = \int_\Omega f(C)(x) \cdot v(x)\,dx. \tag{2.19}$$

Hence $\hat{u}$ is a solution of (2.13), and thus $\hat{u} = u(C)$. Furthermore, problem (2.13) gives the relation

$$\forall v \in V, \quad \int_\Omega (A(u(C)(x)) - A(u(C_k)(x))) \cdot v(x)\,dx = \int_\Omega (f(x,C(x)) - f(x,C_k(x))) \cdot v(x)\,dx$$

which allows us by taking $v = u(C_k) - u(C)$, by using the monotonicity of A in $L^3(\Omega)^d$, and the strong convergence of $f(\cdot,C_k)$ to $f(\cdot,C)$, to obtain the following convergence

$$\lim_{k \to \infty} u(C_k) = u(C) \text{ strongly in } L^3(\Omega)^d.$$

Finally, we have to treat the convergence of the pressure. Since $u(C)$ is a solution of problem (2.12), we use the inf-sup condition (2.10) to deduce the existence of $p(C)$ such that $(u(C), p(C))$ is the solution of problem (2.12).
We deduce from problem (2.12) that for all $v \in X$

$$\begin{aligned}\int_\Omega (\nabla p(C) - \nabla p(C_k))(x) \cdot v(x)\,dx &= \int_\Omega (A(u(C)(x)) - A(u(C_k)(x))) \cdot v(x)\,dx \\ &+ \int_\Omega (f(x,C(x)) - f(x,C_k(x))) \cdot v(x)\,dx.\end{aligned}$$

The strong convergence of $u(C_k)$ to $u(C)$ in $L^3(\Omega)^d$, of $f(\cdot,C_k)$ to $f(\cdot,C)$ in $L^{\frac{3}{2}}(\Omega)^d$, and the weak convergence of $\nabla p(C_k)$ to $\hat{h}$ in $L^{\frac{3}{2}}(\Omega)^d$, give

$$\forall v \in X, \quad \int_\Omega (\nabla p(C) - \hat{h})(x) \cdot v(x)\,dx = 0.$$

Thus $\hat{h} = \nabla p(C)$ in $L^{\frac{3}{2}}(\Omega)^d$ which finishes the proof. Finally, uniqueness of the solution of (2.12) implies the convergence of the whole sequence. □

The next theorem shows the existence of at least one solution to the problem (V_a).

Theorem 2.5. *Under assumption 2.1, problem* (V_a) *admits a solution in* $X \times M \times Y$. *Furthermore, each solution* (u, p, C) *of* (V_a) *satisfies the following bounds:*

$$\begin{aligned}
&|C|_{1,\Omega} \leq \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}\,; \\
&\|u\|_{L^3(\Omega)^d} \leq \left(\frac{1}{\alpha}(\|f_0\|_{L^{\frac{3}{2}}(\Omega)^d} + c_{f_1} S^0_{\frac{3}{2}}|C|_{1,\Omega})\right)^{\frac{1}{2}}\,; \\
&\|\nabla p\|_{L^{\frac{3}{2}}(\Omega)^d} \leq \frac{1}{\beta}\left(\|K^{-1}\|_{L^\infty(\Omega)^{d\times d}}\|u\|_{L^{\frac{3}{2}}(\Omega)^d} + \frac{1}{\beta}\|u\|^2_{L^3(\Omega)^d} + \frac{1}{\beta}(\|f_0\|_{L^{\frac{3}{2}}(\Omega)^d} + c_{f_1} S^0_{\frac{3}{2}}|C|_{1,\Omega})\right).
\end{aligned} \tag{2.20}$$

Proof. We propose to construct a solution of (V_a) by Galerkin's method. As $H^1_0(\Omega)$ is separable, it has a countable basis $(\psi_i)_{i\geq 1}$. Let Θ_m be the space spanned by the first m basis functions, $(\psi_i)_{1\leq i\leq m}$. Problem (2.13) is discretized in Θ_m by the square system of nonlinear equations: Find $C_m = \sum_{i=1}^{m} w_i\psi_i \in \Theta_m$ solution of: $\forall 1\leq i\leq m$;

$$\alpha\int_\Omega \nabla C_m(x)\cdot\nabla\psi_i(x)dx + \int_\Omega (u(C_m)\cdot\nabla C_m)(x)\psi_i(x)dx + r_0\int_\Omega C_m(x)\psi_i(x)dx = \int_\Omega g(x)\psi_i(x)\,dx; \quad (2.21)$$

where $(u(C_m);p(C_m))$ solves (2.12) with $C = C_m$. Now, given $C_m \in \Theta_m$, we introduce the auxiliary problem: Find $\Phi(C_m) \in \Theta_m$ such that, for all $S_m \in \Theta_m$, we have

$$\begin{aligned}(\nabla\Phi(C_m);\nabla S_m)_2 = \alpha(\nabla C_m;\nabla S_m)_2 &+ \int_\Omega (u(C_m)\cdot\nabla C_m)(x)S_m(x)\,dx + r_0\int_\Omega C_m(x)S_m(x)\,dx \\ &- \int_\Omega g(x)S_m(x)\,dx. \end{aligned} \quad (2.22)$$

In fact, for each C_m, the left hand side of (2.22) is a bilinear map of the form $(\nabla\cdot;\nabla\cdot)$ and the right hand side is a linear map with respect to S_m. Relation (2.22) defines a continuous mapping from Θ_m into Θ_m, due to the fact that Θ_m is a finite dimension space and Lemma 2.4. By taking $S_m = C_m$, we get

$$\begin{aligned}(\nabla\Phi(C_m);\nabla C_m) &= \alpha|C_m|^2_{1,\Omega} + r_0\|C_m\|^2_{L^2(\Omega)} - \int_\Omega g(x)C_m(x)\,dx \\ &\geq |C_m|_{1,\Omega}\left(\alpha|C_m|_{1,\Omega} - S_2^0\|g\|_{L^2(\Omega)}\right).\end{aligned}$$

In other words, we have

$$(\nabla\Phi(C_m);\nabla C_m) \geq 0$$

for all $C_m \in \Theta_m$ such that

$$|C_m|_{1,\Omega} = \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}.$$

Therefore Brouwer's Fixed-Point Theorem implies immediately the existence of at least one solution to the problem (2.21).

Let C_m be a solution of problem (2.21), satisfying $\forall S_m \in \Theta_m$,

$$\alpha(\nabla C_m;\nabla S_m) + \int_\Omega (u(C_m)\cdot\nabla C_m)(x)S_m(x)\,dx + r_0\int_\Omega C_m(x)S_m(x)\,dx = \int_\Omega g(x)S_m(x)\,dx.$$

By taking $S_m = C_m$ in the last equation, we get immediately the bound

$$|C_m|_{1,\Omega} \leq \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}.$$

The last uniform bound implies that, up to a subsequence, $(C_m)_m$ converges weakly to a function C in $H^1_0(\Omega)$. Therefore, it converges strongly in $L^r(\Omega)$, for any $r < \frac{2d}{d-2}$, and it follows from Lemma 2.4 that $(u(C_m);p(C_m))_m$ converges weakly to $(u(C);p(C))$ in $X\times M$, and $(u(C_m))_m$ converges strongly to $u(C)$ in $L^3(\Omega)^d$. Now, we freeze the index i in (2.21), and let m tends to infinity. The weak convergence of $(C_m)_m$ to C in $H^1_0(\Omega)$, and the strong convergence of $(u(C_m))_m$ to $u(C)$ in $L^3(\Omega)^d$ allow us to deduce that C is a solution of the following problem: Find $C \in H^1_0(\Omega)$ such that

$$\alpha\int_\Omega \nabla C(x)\cdot\nabla\psi_i(x)dx + \int_\Omega (u(C)\cdot\nabla C)(x)\psi_i(x)dx + r_0\int_\Omega C(x)\psi_i(x)dx = \int_\Omega g(x)\psi_i(x)\,dx. \quad (2.23)$$

From this system and the density of the basis in $H^1_0(\Omega)$, we infer that C is a solution of problem (2.15). The first bound in (2.20) can be straightly obtained by taking $S = C$ in the last equation in (2.15). The first and second bounds in (2.20) can be deduced from the inequalities (2.14) and Assumption 2.1. □

Theorem 2.6. *Assume that* Ω *is of class* $C^{1,1}$. *Let* $(u;p;C)$ *be a solution of Problem* (P). *If* $g \in L^\infty(\Omega)$, *then the concentration* C *is in* $L^\infty(\Omega)$ *and satisfies the following bound:*

$$\|C\|_{L^\infty(\Omega)} \leq \frac{1}{r_0}\|g\|_{L^\infty(\Omega)}.$$

Proof. Let (u,p,C) be a solution of Problem (P), then the velocity $u \in V$. Using the fact that V is separable and that the space
$$V_1 = \{ \varphi \in \mathcal{D}(\Omega)^d;\ \operatorname{div}\varphi = 0 \}$$
is dense in V (see for instance [18] Lemma 10.8), there exists a sequence $(u_N)_{N\in\mathbb{N}}$ in V_1 which converges strongly to u in $L^3(\Omega)^d$ when N tends to $+\infty$.
Now, for each $N \in \mathbb{N}$, let $C_N \in H^1_0(\Omega)$ the unique solution to the following problem:
$$\begin{cases} -\Delta C_N + u_N\cdot\nabla C_N + r_0 C_N = g & \text{in } \Omega,\\ C_N = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.24}$$
The elliptic regularity (see [21]) allows us to get that $C_N \in W^{2,p}(\Omega)\cap H^1_0(\Omega)$, for all $p \ge 1$.
Multiplying the first equation of (2.24) by C_N^{2p+1} and integrating by parts give,
$$\frac{(2p+1)}{2(p+1)^2}\int_\Omega |\nabla C_N^{p+1}(x)|^2\,dx + r_0\int_\Omega C_N^{2p+2}(x)\,dx = \int_\Omega g(x)C_N^{2p+1}(x)\,dx.$$
By remarking that the first term of the left hand side of the previous equation is non-negative and by applying Holder's inequality for the right-hand-side with the conjugate exponents $m = \dfrac{2p+2}{2p+1}$ and $n = 2p+2$, we get the following inequality:
$$\|C_N\|_{L^{2p+2}(\Omega)} \le \frac{1}{r_0}\|g\|_{L^{2p+2}(\Omega)}. \tag{2.25}$$
The next step consists to show that C_N converges strongly to $C \in H^1_0(\Omega)$. In order to prove it, we start by subtracting the third equation of problem (P) from first equation of (2.24) to get, for all $S \in H^1_0(\Omega)$,
$$\int_\Omega \nabla(C-C_N)\cdot\nabla S\,dx + r_0\int_\Omega (C-C_N)S\,dx = -\int_\Omega u\cdot\nabla(C-C_N)S\,dx - \int_\Omega (u-u_N)\cdot\nabla C_N S\,dx.$$
By taking $S = C - C_N$ and by using the antisymmetric property and the estimate (2.20), we obtain
$$|C-C_N|_{H^1(\Omega)} \le \frac{S_6^0 S_2^0}{\alpha}\|u-u_N\|_{L^3(\Omega)^d}\|g\|_{L^2(\Omega)}$$
which gives the strong convergence of C_N to C in $H^1_0(\Omega)$.
As C_N is uniformly bounded in $L^{2p+2}(\Omega)$, we can extract a subsequence still denoted by C_N such that C_N converges weakly in $L^{2p+2}(\Omega)$ to some function h satisfying (2.25). The strong convergence of C_N to C in $H^1(\Omega)$ and the uniqueness of the limit allows us to deduce that $h = C$ in $L^{\frac{2p+1}{2p+2}}(\Omega)$ and we get
$$\|C\|_{L^{2p+2}(\Omega)} \le \frac{1}{r_0}\|g\|_{L^{2p+2}(\Omega)} \quad \forall p \ge 1. \tag{2.26}$$
Thus $h = C \in L^{2p+2}(\Omega)$ and finally, taking the limit in (2.26) as $p \to \infty$, we get the desired result.

Theorem 2.7. *Under Assumption 2.1, We suppose that the problem* (V_a) *admits a solution* $(u,p,C) \in X \times M \times Y$ *such that*
$$\|C\|_{L^\infty(\Omega)} \le \frac{K_m}{c_{f_1} S_2^0}, \tag{2.27}$$
then, the solution of the problem (V_a) *is unique.*

Proof. Let (u_1,p_1,C_1) and (u_2,p_2,C_2) be two solutions of Problem (V_a), and let $u = u_1 - u_2$, $p = p_1 - p_2$ and $C = C_1 - C_2$. Then, (u,p,C) satisfies for all $(v,S) \in X \times Y$,
$$\begin{aligned} &\int_\Omega (A(u_1) - A(u_2))(x)\cdot v(x)\,dx = \int_\Omega (f(C_1) - f(C_2))(x)\cdot v(x)\,dx - \int_\Omega \nabla(p_1 - p_2)(x)\cdot v(x)dx;\\ &\int_\Omega \nabla C(x)\cdot\nabla S(x)\,dx + \int_\Omega (u\cdot\nabla C_1 + u_2\cdot\nabla C)(x)S(x)\,dx + r_0\int_\Omega C(x)S(x)\,dx = 0. \end{aligned} \tag{2.28}$$
By taking $S = C$ in the second equation of (2.28), we get by using the Green formula,
$$|C|^2_{1,\Omega} + r_0\|C\|^2_{L^2(\Omega)} = -\int_\Omega (u\cdot\nabla C)(x)C_1(x)\,dx;$$

and then

$$|C|_{1,\Omega}^2 + r_0 \|C\|^2_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)^d} \|\nabla C\|_{L^2(\Omega)^d} \|C_1\|_{L^\infty(\Omega)}.$$

Finally, we get

$$|C|_{1,\Omega} \le \frac{1}{\alpha} \|u\|_{L^2(\Omega)^d} \|C_1\|_{L^\infty(\Omega)}. \tag{2.29}$$

Substituting v by u in the first equation of (2.28), we get

$$\int_\Omega (A(u_1) - A(u_2))(x) \cdot (u_1 - u_2)(x)\,dx = \int_\Omega (f(C_1) - f(C_2))(x) \cdot (u_1 - u_2)(x)\,dx.$$

By using the monotonicity of A, Assumption 2.1, and the fact that f_1 is c_{f_1}-Lipschitz, we obtain

$$\begin{aligned} -K_m \|u\|_{L^2(\Omega)^d} &\le \|f_1(C_1) - f_1(C_2)\|_{L^2(\Omega)^d} \\ &\le c_{f_1} S_2^0 |C|_{1,\Omega}. \end{aligned} \tag{2.30}$$

Thus, relations (2.29) and (2.30), give

$$-K_m \|u\|_{L^2(\Omega)^d} \le \frac{c_{f_1}}{\alpha} S_2^0 \|u\|_{L^2(\Omega)^d} \|C_1\|_{L^\infty(\Omega)}.$$

Relation (2.27) allows us to deduce that $\|u\|_{L^2(\Omega)} = 0$ and then $u_1 = u_2$. Relation (2.29) implies $C_1 = C_2$. Finally, the first equation of system (2.28) and the inf-sup condition provide $p_1 = p_2$, which yields the uniqueness of the solution. □

Corollary 2.8. *Under Assumption 2.1 and Theorems 2.6 and 2.7, if the data* g *satisfies the following smallness condition*

$$\|g\|_{L^\infty(\Omega)} < \frac{r_0 K_m}{c_{f_1} S_2^0};$$

then the solution $(u; p; C)$ *of Problem* (P) *is unique in* $L^3(\Omega)^d \times \big(W^{1,\frac{3}{2}}(\Omega) \cap L^2_0(\Omega)\big) \times H^1_0(\Omega)$.

3. Discretization

From now on, we assume that Ω is a polygon when $d = 2$ or polyhedron when $d = 3$, so it can be completely meshed. For the space discretization, we consider a regular (see Ciarlet [11]) family of triangulations $(\mathcal{T}_h)_h$ of Ω which is a set of closed non degenerate triangles for $d = 2$ or tetrahedra for $d = 3$, called elements, satisfying

- for each h, $\bar\Omega$ is the union of all elements of $\mathcal{T}_h$;
- the intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge (or face when $d = 3$);
- the ratio of the diameter h_κ of an element $\kappa \in \mathcal{T}_h$ to the diameter ρ_κ of its inscribed circle when $d = 2$ or ball when $d = 3$ is bounded by a constant independent of h, that is, there exists a strictly positive constant σ independent of h such that,

$$\max_{\kappa \in \mathcal{T}_h} \frac{h_\kappa}{\rho_\kappa} \le \sigma. \tag{3.1}$$

As usual, h denotes the maximal diameter of all elements of $\mathcal{T}_h$. To define the finite element functions, let r be a non negative integer. For each κ in $\mathcal{T}_h$, we denote by $\mathbb{P}_r(\kappa)$ the space of restrictions to κ of polynomials in d variables and total degree at most r, with a similar notation on the faces or edges of κ. For every edge (when $d = 2$) or face (when $d = 3$) e of the mesh $\mathcal{T}_h$, we denote by h_e the diameter of e.

We shall use the following inverse inequality [15]: for any dimension d, there exists a constant C_I such that for any polynomial function v_h of degree r on K,

$$\|v_h\|_{L^3(\kappa)} \le C_I h_\kappa^{-\frac{d}{6}} \|v_h\|_{L^2(\kappa)}. \tag{3.2}$$

The constant C_I depends on the regularity parameter σ of (3.1), but for the sake of simplicity this is not indicated.

Let $X_h \subset X$, $M_h \subset M$ and $Y_h \subset Y$ be the discrete spaces corresponding to the velocity, the pressure and the concentration. We assume that X_h and M_h satisfy the following inf-sup condition:

$$\forall q_h \in M_h; \ \sup_{v_h \in X_h} \frac{\int_\Omega \nabla q_h \cdot v_h\, dx}{\|v_h\|_{X_h}} \geq \beta_2 \|q_h\|_{M_h}; \tag{3.3}$$

where β_2 is a strictly positive constant independent of h.

Problem (V_a) can be discretized as following: Find $(u_h; p_h; C_h) \in X_h \times M_h \times Y_h$ such that

$$(V_{ah})\left\{\begin{array}{l} \forall v_h \in X_h; \quad \int_\Omega A(u_h)\cdot v_h\, dx + \int_\Omega \nabla p_h \cdot v_h\, dx = \int_\Omega f(C_h)\cdot v_h\, dx;\\ \forall q_h \in M_h; \quad \int_\Omega \nabla q_h \cdot u_h\, dx = 0;\\ \forall S_h \in Y_h; \quad \int_\Omega \nabla C_h \cdot \nabla S_h\, dx + \int_\Omega (u_h \cdot \nabla C_h) S_h\, dx + \frac{1}{2}\int_\Omega \operatorname{div}(u_h)\, C_h S_h\, dx\\ \qquad\qquad + r_0 \int_\Omega C_h S_h\, dx = \int_\Omega g S_h\, dx. \end{array}\right. \tag{3.4}$$

In the following, we will introduce the finite dimension spaces $X_h; M_h$ and Y_h. Let κ be an element of T_h with vertices a_i, $1 \leq i \leq d+1$, and corresponding barycentric coordinates λ_i. We denote by $b_\kappa \in \mathbb{P}_{d+1}(\kappa)$ the basic bubble function :

$$b_\kappa(x) = \lambda_1(x) \ldots \lambda_{d+1}(x). \tag{3.5}$$

We observe that $b_\kappa(x) = 0$ on $\partial\kappa$ and that $b_\kappa(x) > 0$ in the interior of κ.
We introduce the following discrete spaces:

$$\begin{aligned} X_h &= \{v_h \in (C^0(\bar\Omega))^d;\ \forall \kappa \in T_h;\ v_h|_\kappa \in P(\kappa)^d\};\\ M_h &= \{q_h \in C^0(\bar\Omega);\ \forall \kappa \in T_h;\ q_h|_\kappa \in \mathbb{P}_1(\kappa)\} \cap L^2_0(\Omega);\\ Y_h &= \{q_h \in C^0(\bar\Omega);\ \forall \kappa \in T_h;\ q_h|_\kappa \in \mathbb{P}_1(\kappa)\} \cap H^1_0(\Omega);\\ V_h &= \{v_h \in X_h; \forall q_h \in M_h;\ \int_\Omega \nabla q_h \cdot v_h\, dx = 0\}; \end{aligned} \tag{3.6}$$

where

$$P(\kappa) = \mathbb{P}_1(\kappa) \oplus \operatorname{Vect}\{b_\kappa\}.$$

In this case, for the inf-sup condition (3.3), we refer to [23].

We shall use the following results:

(1) For the concentration: there exists an approximation operator (when $d = 2$, see Bernardi and Girault [5] or Clément [12]; when $d = 2$ or $d = 3$, see Scott and Zhang [35]), R_h in $\mathcal{L}(W^{1,p}(\Omega); Y_h)$ such that for all κ in T_h, $m = 0, 1$, $l = 0, 1$, and all $p \geq 1$,

$$\forall S \in W^{l+1,p}(\Omega); \ |S - R_h(S)|_{W^{m,p}(\kappa)} \leq c(p, m, l)\, h^{l+1-m}_\kappa |S|_{W^{l+1,p}(\Delta_\kappa)}; \tag{3.7}$$

where Δ_κ is the macro element containing the values of S used in defining $R_h(S)$.

(2) For the velocity: We introduce a variant of R_h denoted by F_h (see [4] and [22]) which is stable in $L^3(\Omega)^d$:

$$\forall v \in L^3(\Omega)^d; \ \|F_h(v)\|_{L^3(\Omega)^d} \leq C_s \|v\|_{L^3(\Omega)^d}; \tag{3.8}$$

such that $F_h(v) \in V_h$ when $\operatorname{div} v = 0$, and satisfies (3.7).

(3) For the pressure: As M_h contains all constants, an easy modification of R_h yields an operator $r_h \in \mathcal{L}(W^{1,p}(\Omega) \cap L^2_0(\Omega); M_h)$ (see for instance Abboud, Girault and Sayah [1]), satisfying (3.7). Indeed, r_h can be constructed as follows:

$$\forall q \in M; \quad r_h q = R_h q - \frac{1}{|\Omega|}\int_\Omega (R_h q)(x)\, dx.$$

Existence of a solution of (V_{ah}) is derived by duplicating the steps of the previous section concerning the existence of a solution of Problem (V_a). First (V_{ah}) is split as in the previous section, i.e., find $C_h \in Y_h$ such that: $\forall S_h \in Y_h$,

$$\int_\Omega \nabla C_h \cdot \nabla S_h\,dx + \int_\Omega (u_h(C_h)\cdot\nabla C_h)S_h\,dx + \frac{1}{2}\int_\Omega \operatorname{div}(u_h(C_h))\,C_h S_h\,dx + r_0\int_\Omega C_h S_h\,dx = \int_\Omega g S_h\,dx; \tag{3.9}$$

where $u_h(C_h)$ is the velocity solution of: Find $(u_h(C_h); p_h(C_h)) \in X_h \times M_h$, such that

$$\begin{cases} \forall v_h \in X_h; & \int_\Omega A(u_h(C_h))\cdot v_h\,dx + \int_\Omega \nabla p_h(C_h)\cdot v_h\,dx = \int_\Omega f(\cdot;C_h)\cdot v_h\,dx; \\ \forall q_h \in M_h; & \int_\Omega \nabla q_h \cdot u_h\,dx = 0. \end{cases} \tag{3.10}$$

For each $C_h \in X_h$, an easy finite-dimensional variant of the argument of Theorem 2.3 allows one to prove that the scheme (3.10) has a unique solution $(u_h(C_h); p_h(C_h)) \in X_h \times M_h$, and this solution satisfies the a priori estimates similar to (2.14):

$$\|u_h(C_h)\|_{L^3(\Omega)^d} \le \left(\frac{\rho}{\beta}\|f(\cdot;C_h)\|_{L^{\frac{3}{2}}(\Omega)^d}\right)^{\frac{1}{2}};$$

$$\beta_2\|\nabla p_h(C_h)\|_{L^{\frac{3}{2}}(\Omega)^d} \le \frac{\mu}{\rho}\|K^{-1}\|_{L^\infty(\Omega)^{d\times d}}\|u_h(C_h)\|_{L^{\frac{3}{2}}(\Omega)^d} + \frac{\beta}{\rho}\|u_h(C_h)\|^2_{L^3(\Omega)^d} + \|f(\cdot;C_h)\|_{L^{\frac{3}{2}}(\Omega)^d}. \tag{3.11}$$

We address now the existence of at least one solution of the problem (3.9) written with the only variable C_h. For this purpose, we apply Brouwer's Fixed-Point Theorem. Indeed, we introduce the following map: For a given $C_h \in Y_h$, find $\Phi(C_h) \in Y_h$ such that: $\forall S_h \in Y_h$,

$$\begin{aligned}(\nabla\Phi(C_h);\nabla S_h) = {} & \int_\Omega \nabla C_h\cdot\nabla S_h\,dx + \int_\Omega (u_h(C_h)\cdot\nabla C_h)S_h\,dx \\ & + \frac{1}{2}\int_\Omega \operatorname{div}(u_h(C_h))\,C_h S_h\,dx + r_0\int_\Omega C_h S_h\,dx - \int_\Omega g S_h\,dx.\end{aligned}$$

This last relation defines a mapping from Y_h into itself, and we easily derive its continuity. By taking $S_h = C_h$, we get

$$\begin{aligned}(\nabla\Phi(C_h);\nabla C_h) = {} & |C_h|^2_{1,\Omega} + r_0\|C_h\|^2_{L^2(\Omega)} - \int_\Omega g(x)C_h(x)\,dx; \\ \ge {} & |C_h|_{1,\Omega}\left(|C_h|_{1,\Omega} - S_2^0\|g\|_{L^2(\Omega)}\right).\end{aligned}$$

In other words, we have

$$(\nabla\Phi(C_h);\nabla C_h) \ge 0,$$

for all $C_h \in Y_h$ such that

$$|C_h|_{1,\Omega} = S_2^0\|g\|_{L^2(\Omega)}.$$

The Brouwer's Fixed-Point Theorem implies immediately the existence of at least one solution of the problem (3.9). Hence, problem (V_{ah}) admits at least one solution $(u_h; p_h; C_h) \in X_h \times M_h \times Y_h$. Furthermore, by taking $S_h = C_h$ in the last equation of (V_{ah}) gives, in addition to inequality (3.11), the following bound:

$$|C_h|_{1,\Omega} \le S_2^0\|g\|_{L^2(\Omega)}. \tag{3.12}$$

Finally, uniqueness follows easily since C_h belongs to $L^\infty(\Omega)$. This is summed up in the following existence and uniqueness theorems.

Theorem 3.1. *Under assumption 2.1,* (V_{ah}) *has at least a solution* $(u_h; p_h; C_h) \in X_h \times M_h \times Y_h$. *Moreover, every solution of* (V_{ah}) *satisfies bounds similar to* (2.20).

Theorem 3.2. *We assume that the data* f *and* g *satisfies assumption 2.1. Suppose that problem* (V_{ah}) *has a solution* $(u_h; p_h; C_h) \in X_h \times M_h \times Y_h$ *such that*

$$\|C_h\|_{L^\infty(\Omega)} + S_6^0|C_h|_{W^{1,3}(\Omega)} < \frac{2K_m}{c_{f_1}S_2^0}. \tag{3.13}$$

Then problem (V_{ah}) *has no other solution in* $X_h \times M_h \times Y_h$.

Proof. We consider two solutions $(u_{h,1}, p_{h,1}, C_{h,1})$ and $(u_{h,2}, p_{h,2}, C_{h,2})$ of problem (V_{ah}) and we denote by $u_h = u_{h,1} - u_{h,2}$, $p_h = p_{h,1} - p_{h,2}$ and $C_h = C_{h,1} - C_{h,2}$. By following the same steps of the proof of Theorem 2.7, u_h satisfies the analogue of (2.30),

$$-K_m \|u_h\|_{L^2(\Omega)^3} \le c_{f_1} S_2^0 |C_h|_{1,\Omega}. \tag{3.14}$$

The treatment of the concentration is slightly different. By using the Green's formula, the difference of the equations satisfied by the concentrations reads with $S_h = T_h$,

$$|C_h|^2_{1,\Omega} \le \frac{1}{2}\left(\int_\Omega (u_h \cdot \nabla C_h) C_{h,1}\,dx - \int_\Omega (u_h \cdot \nabla C_{h,1}) C_h\,dx\right). \tag{3.15}$$

Therefore, using Hölder's inequality, we obtain

$$|C_h|_{1,\Omega} \le \frac{\|u_h\|_{L^2(\Omega)^d}}{2}\left(\|C_{h,1}\|_{L^\infty(\Omega)} + S_6^0 |C_{h,1}|_{W^{1,3}(\Omega)}\right). \tag{3.16}$$

Thus, inequalities (3.14) and (3.16) give,

$$-K_m \|u_h\|_{L^2(\Omega)^d} \le \frac{c_{f_1} S_2^0 \|u_h\|_{L^2(\Omega)^d}}{2}\left(\|C_{h,1}\|_{L^\infty(\Omega)} + S_6^0 |C_{h,1}|_{W^{1,3}(\Omega)}\right). \tag{3.17}$$

Condition (3.13) allows us to deduce that $\|u_h\|_{L^2(\Omega)^d} = 0$ and hence $u_{h,1} = u_{h,2}$. Inequality (3.16) gives $C_{h,1} = C_{h,2}$. Finally, the inf-sup condition provides $p_{h,1} = p_{h_2}$. □

Now, we address the convergence of the subsequence of the numerical solution to the exact one. Bounds (3.12) and (3.11), and the compactness of the embedding of $H^1(\Omega)$ into $L^p(\Omega)$ (($p \ge 1$ if $d = 2$; $1 \le p < 6$ if $d = 3$), allow us to get the following lemma:

Lemma 3.3. *Let* f *and* g *satisfy Assumption 2.1 and let* (u_h, p_h, C_h) *be any solution of the discrete problem* (V_{ah}). *We can extract a subsequence, still denoted* (u_h, p_h, C_h) *verifying*

$$\begin{aligned}
&\lim_{h \to 0} C_h = C \quad \textit{weakly in } H_0^1(\Omega);\\
&\lim_{h \to 0} C_h = C \quad \textit{strongly in } L^p(\Omega); \quad (p \ge 1 \textit{ if } d = 2;\ 1 \le p < 6 \textit{ if } d = 3);\\
&\lim_{h \to 0} u_h = u \quad \textit{weakly in } L^3(\Omega)^d;\\
&\lim_{h \to 0} \nabla p_h = h \quad \textit{weakly in } L^{\frac{3}{2}}(\Omega)^d;
\end{aligned} \tag{3.18}$$

where $u \in L^3(\Omega)^d$, $h \in L^{\frac{3}{2}}(\Omega)^d$ *and* $C \in H_0^1(\Omega)$.

Proposition 3.4. *Let* (u_h, p_h, C_h) *be any solution of the discrete problem* (V_{ah}). *Under assumption of Lemma 3.3, we have* $h = \nabla p$, *where* (u, p) *solves the first two equations of* (V_a) *with* $C = C$. *Furthermore, we have the following strong convergence :*

$$\lim_{h \to 0} u_h = u \quad \textit{strongly in } L^3(\Omega)^d. \tag{3.19}$$

Proof. First, we shall show that u is a solution of problem (2.12) for $C = C$.
The monotonicity of A gives

$$\forall v_h \in V_h, \quad \int_\Omega (A(u_h) - A(v_h)) \cdot (u_h - v_h)\,dx \ge 0. \tag{3.20}$$

As u_h is a solution of problem (V_{ah}), we get

$$\forall v_h \in V_h, \quad \int_\Omega A(u_h) \cdot (u_h - v_h)\,dx = \int_\Omega f(\cdot, C_h) \cdot (u_h - v_h)\,dx. \tag{3.21}$$

Therefore,

$$\forall v_h \in V_h, \quad \int_\Omega A(v_h) \cdot (u_h - v_h)\,dx \le \int_\Omega f(\cdot, C_h) \cdot (u_h - v_h)\,dx. \tag{3.22}$$

We now choose $v_h = \mathcal{F}_h(v)$ where v is an arbitrary element of V. The strong covergence of $\mathcal{F}_h(v)$ to v in $L^3(\Omega)^d$ and (2.11) allow us to get

$$A(\mathcal{F}_h(v)) \xrightarrow[h\to 0]{} A(v) \text{ strongly in } L^{\frac{3}{2}}(\Omega)^d. \tag{3.23}$$

Furthermore, since $u_h - \mathcal{F}_h(v)$ converges weakly to $u - v$ in $L^3(\Omega)^d$ and $f(\cdot;C_h)$ converges strongly to $f(\cdot;C)$ in $L^{\frac{3}{2}}(\Omega)^d$, we get by passing to the limit in (3.22),

$$\forall v \in V, \quad \int_\Omega A(v)\cdot(u-v)\,dx \le \int_\Omega f(\cdot;C)\cdot(u-v)\,dx. \tag{3.24}$$

In particular, for $v = u + tw$, where $t \in \mathbb{R}$ and $w \in V$, we get

$$\forall t \in \mathbb{R}; \quad t\int_\Omega A(u+tw)\cdot w\,dx \ge t\int_\Omega f(C)\cdot w\,dx.$$

By taking $t > 0$ (resp. $t < 0$), simplifying by t, tending t to 0 and using the hemi-continuity of A, we get:

$$\forall v \in V; \quad \int_\Omega A(u)\cdot v\,dx \ge \int_\Omega f(\cdot;C)\cdot v\,dx \quad \left(\text{resp.} \int_\Omega A(u)\cdot v\,dx \le \int_\Omega f(\cdot;C)\cdot v\,dx\right);$$

We obtain finally:

$$\forall v \in V; \quad \int_\Omega A(u)\cdot v\,dx = \int_\Omega f(\cdot;C)\cdot v\,dx. \tag{3.25}$$

Hence u is the solution of (2.12), and we construct by using the inf-sup condition (2.10) the corresponding pressure p.

The next step consists to show that u_h converges strongly to u in $L^3(\Omega)^d$. In order to prove it, we start by taking $v = v_h$ in (2.12) and subtracting from (3.10), we have

$$\begin{aligned}\forall v_h \in V_h; \quad \int_\Omega (A(u_h) - A(u))\cdot v_h\,dx &= \int_\Omega (f(\cdot;C_h) - f(\cdot;C))\cdot v_h\,dx + \int_\Omega \nabla(p - p_h)\cdot v_h\,dx \\ &= \int_\Omega (f(\cdot;C_h) - f(\cdot;C))\cdot v_h\,dx + \int_\Omega \nabla p\cdot v_h\,dx.\end{aligned} \tag{3.26}$$

By inserting $A(\mathcal{F}_h(u))$ and taking $v_h = u_h - \mathcal{F}_h(u)$, we get

$$\begin{aligned}\int_\Omega (A(u_h) - A(\mathcal{F}_h(u)))\cdot(u_h - \mathcal{F}_h(u))\,dx &= \int_\Omega (A(\mathcal{F}_h(u)) - A(u))\cdot(u_h - \mathcal{F}_h(u))\,dx \\ &+ \int_\Omega \nabla p\cdot(u_h - \mathcal{F}_h(u))\,dx + \int_\Omega (f(\cdot;C_h) - f(\cdot;C))\cdot(u_h - \mathcal{F}_h(u))\,dx.\end{aligned} \tag{3.27}$$

The monotonicity of A allows us to obtain,

$$\begin{aligned}c_m\|u_h - \mathcal{F}_h(u)\|^3_{L^3(\Omega)^d} &\le \int_\Omega (A(\mathcal{F}_h(u)) - A(u))\cdot(u_h - \mathcal{F}_h(u))\,dx \\ &+ \int_\Omega \nabla p\cdot(u_h - \mathcal{F}_h(u))\,dx + \int_\Omega (f(\cdot;C_h) - f(\cdot;C))\cdot(u_h - \mathcal{F}_h(u))\,dx.\end{aligned} \tag{3.28}$$

We pass to the limit in the previous equation. We deduce from the strong convergence of $A(\mathcal{F}_h(u))$ to $A(u)$ and of $f(\cdot;C_h)$ to $f(\cdot;C)$ that the first and last terms of the right hand side of the previous inequality tend to 0. Furthermore, the weak convergence of u_h to u, and the strong convergence of $\mathcal{F}_h(u)$ to u imply the convergence of the second term of the right hand side of the previous inequality to 0. Thus, u_h converges strongly to u in $L^3(\Omega)^d$.
To finish the proof, it remains to show that $h = \nabla p$, which can be easily obtained by passing to the limit in (3.26), and by using the strong convergence of u_h to u in $L^3(\Omega)^d$, and the uniqueness of the weak limit.

□

Theorem 3.5. *Let f and g satisfy Assumption 2.1, the limit $(u;p;C)$ defined in Proposition 3.4 is a solution of problem (V_a).*

Proof. We have proved in Proposition 3.4 that (u,p,C) solves the first two equations of problem (V_a). It remains to show that (u,C) solves the third equation of problem (V_a). We consider the third equation of problem (V_{ah}). By taking $S_h = R_h S$ for a regular $S \in \mathcal{D}(\Omega)$ (taking into account the density of $\mathcal{D}(\Omega)$ in $H^1_0(\Omega)$), we can show easily the convergence of the linear terms except the non-linear ones which can be written as:

$$\int_\Omega (u_h \cdot \nabla C_h) S_h \, dx + \frac{1}{2}\int_\Omega \mathrm{div}(u_h)\, C_h S_h \, dx = \frac{1}{2}\int_\Omega (u_h \cdot \nabla C_h) S_h \, dx - \frac{1}{2}\int_\Omega (u_h \cdot \nabla S_h) C_h \, dx. \quad (3.29)$$

The strong convergence of S_h to S in $H^1_0(\Omega)$, and in $L^6(\Omega)$, the strong convergence of u_h to u in $L^3(\Omega)^d$, and the weak convergence of C_h to C in $H^1_0(\Omega)$ lead to the convergence of (3.29). □

After showing the convergence of the discrete solution (u_h, p_h, C_h) of problem (V_{ah}) to a solution (u,p,C) of problem (V_a), we next derive the corresponding *a priori* error estimate.

Theorem 3.6. *Under Assumption 2.1, let* (u_h, p_h, C_h) *be a solution of problem* (V_{ah})*, and* (u,p,C) *be a solution of problem* (V_{ah})*. If* (u,p,C) *are such that* $C \in W^{1,3}(\Omega) \cap L^\infty(\Omega)$*,* $u \in L^\infty(\Omega)^d$ *and* $p \in H^1(\Omega)$*, and satisfies the following condition:*

$$S_6^0 |C|_{W^{1,3}(\Omega)} + \|C\|_{L^\infty(\Omega)} \le \frac{K_m}{2\sqrt{2}\, C_{f_1} S_2^0}; \quad (3.30)$$

then, we have the following a priori error estimates:

$$|C - C_h|_{H^1(\Omega)} \le \frac{1}{1 - c_{2r}\sqrt{c_{2u}}}\Big((1 + c_{1r})|C - R_h(C)|_{H^1(\Omega)} + c_{2r}\sqrt{c_{1u}}\,\|\nabla(r_h(p) - p)\|_{L^2(\Omega)^d} + c_{2r}\sqrt{c_{3u}}\,\|F_h(u) - u\|^{\frac{3}{2}}_{L^3(\Omega)^d} + c_{2r}\sqrt{c_{4u}}\,\|F_h(u) - u\|_{L^2(\Omega)^d}\Big); \quad (3.31)$$

$$\|u - u_h\|^2_{L^2(\Omega)^d} \le c_{1u}\|\nabla(p - r_h(p))\|^2_{L^2(\Omega)^d} + c_{2u}|C - C_h|^2_{H^1(\Omega)} + c_{3u}\|F_h(u) - u\|^3_{L^3(\Omega)^d} + c_{4u}\|F_h(u) - u\|^2_{L^2(\Omega)^d}; \quad (3.32)$$

$$\|u - u_h\|_{L^3(\Omega)^d} \le c^0_{1u}\|\nabla(p - r_h(p))\|^{2/3}_{L^2(\Omega)^d} + c^0_{2u}|C - C_h|^{2/3}_{H^1(\Omega)} + c^0_{3u}\|F_h(u) - u\|_{L^3(\Omega)^d} + c^0_{4u}\|F_h(u) - u\|^{2/3}_{L^2(\Omega)^d}; \quad (3.33)$$

and

$$\|\nabla(p - p_h)\|_{L^{\frac{3}{2}}(\Omega)^d} \le c_{1p}|C - C_h|_{1,\Omega} + c_{2p}\|u - u_h\|_{L^3(\Omega)^d} + c_{3p}\|\nabla(r_h(p) - p)\|_{L^{\frac{3}{2}}(\Omega)^d}; \quad (3.34)$$

where c_{1r}, c_{2r} *are constants given in relation* (3.52), $c_{1u}, c_{2u}, c_{3u}, c_{4u}$ *are given in relation* (3.42), $c^0_{1u}, c^0_{2u}, c^0_{3u}, c^0_{4u}$ *are given in relation* (3.42), *and* c_{1p}, c_{2p}, c_{3p} *are given in relation* (3.49).

Proof. We shall proof the result by proceeding by steps.

1) Let us estimate the velocity error in terms of the temperature error. By taking the difference between the first equations of (V) and $(V_{h,1})$ and testing with $v = v_h \in V_h$, we obtain

$$\int_\Omega (A(u) - A(u_h)) \cdot v_h \, dx = \int_\Omega (f_1(C) - f_1(C_h)) \cdot v_h \, dx - \int_\Omega \nabla(p - r_h(p)) \cdot v_h \, dx. \quad (3.35)$$

Then by inserting $F_h(u)$, and testing with $v_h = F_h(u) - u_h$ that belongs indeed to V_h, we easily derive

$$\int_\Omega (A(F_h(u)) - A(u_h)) \cdot v_h dx = \int_\Omega (f_1(C) - f_1(C_h)) \cdot v_h \, dx + \int_\Omega (A(F_h(u)) - A(u)) \cdot v_h dx - \int_\Omega \nabla(p - r_h(p)) \cdot v_h \, dx. \quad (3.36)$$

Let us bound the second term in the right hand side of (3.36). We have

$$\int_\Omega (A(F_h(u)) - A(u))\cdot v_h\,dx = \frac{}{}\int_\Omega K^{-1}(F_h(u)-u)\cdot v_h\,dx + \frac{}{}\int_\Omega (|F_h(u)|-|u|)(F_h(u)-u)\cdot v_h\,dx + \frac{}{}\int_\Omega |u|(F_h(u)-u)\cdot v_h\,dx + \frac{}{}\int_\Omega (|F_h(u)|-|u|)u\cdot v_h\,dx. \tag{3.37}$$

Then,

$$\int_\Omega (A(F_h(u)) - A(u))\cdot v_h\,dx \le \frac{K_M}{}\|F_h(u)-u\|_{L^2(\Omega)^d}\|v_h\|_{L^2(\Omega)^d} + \frac{}{}\|F_h(u)-u\|^2_{L^3(\Omega)^d}\|v_h\|_{L^3(\Omega)^d} + \frac{2}{}\|u\|_{L^\infty(\Omega)^d}\|F_h(u)-u\|_{L^2(\Omega)^d}\|v_h\|_{L^2(\Omega)^d}. \tag{3.38}$$

Thus, the monoticity of A and the fact that f_1 is c_{f_1}-Lipschitz with values in $\mathbb{R}^d$ allow us to obtain,

$$\frac{c_s}{2}\|F_h(u)-u_h\|^3_{L^3(\Omega)^d} + \frac{K_m}{2}\|F_h(u)-u_h\|^2_{L^2(\Omega)^d} \le \|\nabla(r_h(p)-p)\|_{L^2(\Omega)^d}\|v_h\|_{L^2(\Omega)^d} + c_{f_1}S_2^0|C-C_h|_{H^1(\Omega)}\|v_h\|_{L^2(\Omega)^d} + \frac{K_M}{}\|F_h(u)-u)\|_{L^2(\Omega)^d}\|v_h\|_{L^2(\Omega)^d} + \frac{}{}\|F_h(u)-u\|^2_{L^3(\Omega)^d}\|v_h\|_{L^3(\Omega)^d} + \frac{2}{}\|u\|_{L^\infty(\Omega)^d}\|F_h(u)-u\|_{L^2(\Omega)^d}\|v_h\|_{L^2(\Omega)^d}. \tag{3.39}$$

To treat the last inequality, we bound all the terms of the right hand side containing $b=\|v_h\|_{L^2(\Omega)^d}$ using the formula

$$ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2; \quad \text{with } \varepsilon = \frac{K_m}{8};$$

and, the term containing $b=\|v_h\|_{L^3(\Omega)^d}$, using the formula

$$a^2b \le \frac{1}{3}\big(2\,{}^{\frac{3}{2}}a^3 + (\frac{1}{})^3b^3\big); \quad \text{with } = (\frac{4}{3c_s})^{1/3}.$$

Then we infer the following bound:

$$\frac{c_s}{4}\|F_h(u)-u_h\|^3_{L^3(\Omega)^d} + \frac{K_m}{4}\|F_h(u)-u_h\|^2_{L^2(\Omega)^d} \le \frac{4}{K_m}\|\nabla(r_h(p)-p)\|^2_{L^2(\Omega)^d} + \frac{4}{K_m}(c_{f_1}S_2^0)^2|C-C_h|^2_{H^1(\Omega)} + \frac{4K_M^2}{K_m}\|F_h(u)-u)\|^2_{L^2(\Omega)^d} + \frac{4\,{}^{\frac{3}{2}}}{3\,{}^{\frac{3}{2}}\sqrt{3c_s}}\|F_h(u)-u\|^3_{L^3(\Omega)^d} + \frac{16\,^2}{K_m}\|u\|^2_{L^\infty(\Omega)^d}\|F_h(u)-u\|^2_{L^2(\Omega)^d}. \tag{3.40}$$

By using the following triangle inequality

$$\frac{1}{2}\|u-u_h\|^2_{L^2(\Omega)^d} \le \|F_h(u)-u_h\|^2_{L^2(\Omega)^d} + \|F_h(u)-u\|^2_{L^2(\Omega)^d};$$

we get

$$\|u-u_h\|^2_{L^2(\Omega)^d} \le c_{1u}\|\nabla(r_h(p)-p)\|^2_{L^2(\Omega)^d} + c_{2u}|C-C_h|^2_{H^1(\Omega)} + c_{3u}\|F_h(u)-u\|^3_{L^3(\Omega)^d} + c_{4u}\|F_h(u)-u)\|^2_{L^2(\Omega)^d}; \tag{3.41}$$

where

$$c_{1u} = \frac{32\,^2}{^2K_m^2}; \quad c_{2u} = c_{1u}(c_{f_1}S_2^0)^2; \quad c_{3u} = \frac{32\,{}^{\frac{3}{2}}}{3K_m\sqrt{3c_s}} \text{ and } c_{4u} = \frac{32K_M^2}{K_m^2} + \frac{128\,^2}{^2K_m^2}\|u\|^2_{L^\infty(\Omega)^d} + 2. \tag{3.42}$$

Furthermore, relation (3.40) gives

$$\|F_h(u)-u_h\|^3_{L^3(\Omega)^d} \le \frac{K_m}{2c_s}\Big(c_{1u}\|\nabla(r_h(p)-p)\|^2_{L^2(\Omega)^d} + c_{2u}|C-C_h|^2_{H^1(\Omega)} + c_{3u}\|F_h(u)-u\|^3_{L^3(\Omega)^d} + c_{4u}\|F_h(u)-u\|^2_{L^2(\Omega)^d}\Big).$$

Thus, a triangle inequality allows us to get

$$\|u - u_h\|_{L^3(\Omega)^d} \le c^0_{1u}\|\nabla(r_h(p) - p)\|^{2/3}_{L^2(\Omega)^d} + c^0_{2u}|C - C_h|^{2/3}_{H^1(\Omega)} + c^0_{3u}\|F_h(u) - u\|_{L^3(\Omega)^d} + c^0_{4u}\|F_h(u) - u\|^{2/3}_{L^2(\Omega)^d};$$

where

$$c^0_u = \sqrt[3]{\frac{\rho K_m}{2c_s}};\quad c^0_{1u} = c^0_u\sqrt[3]{c_{1u}};\quad c^0_{2u} = c^0_u\sqrt[3]{c_{2u}};\quad c^0_{3u} = c^0_u(1 + \sqrt[3]{c_{3u}}) \text{ and } c^0_{4u} = c^0_u\sqrt[3]{c_{4u}}. \tag{3.43}$$

Hence relations (3.32) and (3.33) are proved.

2) The proof of the error estimate for the pressure follows the same lines of the previous step. By taking the difference between the first equations of (V_a) and (V_{ah}), inserting $r_h(p)$ and testing with $v_h \in X_h$, we obtain

$$\int_\Omega \nabla(r_h(p) - p_h)\cdot v_h\,dx = \int_\Omega (f_1(C) - f_1(C_h))\cdot v_h\,dx - \int_\Omega (A(u) - A(u_h))\cdot v_h\,dx - \int_\Omega \nabla(p - r_h(p))\cdot v_h\,dx. \tag{3.44}$$

In order to estimate the second term in the right hand side of (3.44), we will proceed as follows

$$\begin{aligned}\int_\Omega (A(u) - A(u_h))\cdot v_h\,dx &= \frac{\mu}{\rho}\int_\Omega K^{-1}(u - u_h)\cdot v_h\,dx + \frac{\beta}{\rho}\int_\Omega |u|(u - u_h)\cdot v_h\,dx\\ &\quad + \frac{\beta}{\rho}\int_\Omega (|u| - |u_h|)u_h\cdot v_h\,dx\\ &\le \frac{\mu K_M}{\rho}|\Omega|^{1/3}\|u - u_h\|_{L^3(\Omega)^d}\|v_h\|_{L^3(\Omega)^d} + \frac{\beta}{\rho}\|u - u_h\|_{L^3(\Omega)^d}\|v_h\|_{L^3(\Omega)^d}(\|u\|_{L^3(\Omega)^d} + \|u_h\|_{L^3(\Omega)^d}).\end{aligned} \tag{3.45}$$

Applying (2.20) and (3.11), we get

$$\begin{aligned}\int_\Omega (A(u) - A(u_h))\cdot v_h\,dx \le{}& \frac{\mu K_M}{\rho}|\Omega|^{1/3}\|u - u_h\|_{L^3(\Omega)^d}\|v_h\|_{L^3(\Omega)^d} +\\ & \frac{2\beta}{\rho}\left(\frac{1}{\nu}\|f_0\|_{L^{\frac{3}{2}}(\Omega)^d} + \frac{c_{f_1}S^0_{\frac{3}{2}}S^0_2}{\nu}\|g\|_{L^2(\Omega)^d}\right)^{\frac{1}{2}}\|u - u_h\|_{L^3(\Omega)^d}\|v_h\|_{L^3(\Omega)^d}.\end{aligned} \tag{3.46}$$

We denote $\gamma = \left(\frac{1}{\nu}\|f_0\|_{L^{\frac{3}{2}}(\Omega)^d} + \frac{c_{f_1}S^0_{\frac{3}{2}}S^0_2}{\nu}\|g\|_{L^2(\Omega)^d}\right)^{\frac{1}{2}}$. By following the same steps of the previous part, and the inf-sup condition (3.3), we get

$$\begin{aligned}\beta_2\|\nabla(r_h(p) - p_h)\|_{L^{\frac{3}{2}}(\Omega)^d} \le{}& c_{f_1}|\Omega|^{1/6}S^0_2|C - C_h|_{1,\Omega} + \left(\frac{\mu K_M}{\rho}|\Omega|^{1/3} + \frac{2\beta}{\rho}\gamma\right)\|u - u_h\|_{L^3(\Omega)^d} + \|\nabla(p - r_h(p))\|_{L^{\frac{3}{2}}(\Omega)^d}.\end{aligned} \tag{3.47}$$

The following triangle inequality

$$\|\nabla(p - p_h)\|_{L^{\frac{3}{2}}(\Omega)^d} \le \|\nabla(r_h(p) - p_h)\|_{L^{\frac{3}{2}}(\Omega)^d} + \|\nabla(r_h(p) - p)\|_{L^{\frac{3}{2}}(\Omega)^d}$$

allows us to get

$$\|\nabla(p - p_h)\|_{L^{\frac{3}{2}}(\Omega)^d} \le c_{1p}|C - C_h|_{1,\Omega} + c_{2p}\|u - u_h\|_{L^3(\Omega)^d} + c_{3p}\|\nabla(r_h(p) - p)\|_{L^{\frac{3}{2}}(\Omega)^d}; \tag{3.48}$$

where

$$c_{1p} = \frac{1}{\beta_2}c_{f_1}S^0_2|\Omega|^{1/6};\quad c_{2p} = \frac{1}{\beta_2}\left(\frac{\mu K_M}{\rho}|\Omega|^{1/3} + \frac{2\beta}{\rho}\gamma\right) \text{ and } c_{3p} = \left(\frac{1}{\beta_2} + 1\right). \tag{3.49}$$

Hence relation (3.34) is proved.

3) Next we estimate the error of the concentration in terms of the velocity error. We take the difference between the third equations of systems (V_a) and (V_{ah}), insert $S_h(C)$, and use the Green formula to get for all $S_h \in X_h$

$$\begin{aligned}\int_\Omega \nabla(R_h(C)-C_h)\cdot\nabla S_h\,dx + r_0\int_\Omega (R_h(C)-C_h)\,S_h\,dx = {}& \int_\Omega \nabla(R_h(C)-C)\cdot\nabla S_h\,dx + r_0\int_\Omega (R_h(C)-C)\,S_h\,dx\\ &+\frac{1}{2}\int_\Omega (u_h\cdot\nabla(R_h(C)-C))S_h\,dx - \frac{1}{2}\int_\Omega (u_h\cdot\nabla S_h)(R_h(C)-C)\,dx\\ &+\frac{1}{2}\int_\Omega ((u_h-u)\cdot\nabla C)S_h\,dx - \frac{1}{2}\int_\Omega ((u_h-u)\cdot\nabla S_h)C\,dx.\end{aligned} \tag{3.50}$$

The terms in the last two lines of the right-hand side are bounded by

$$\begin{aligned}&\|u_h\|_{L^3(\Omega)^d}\big(|C-R_h(C)|_{H^1(\Omega)}\|S_h\|_{L^6(\Omega)} + \|C-R_h(C)\|_{L^6(\Omega)}|S_h|_{H^1(\Omega)}\big)\\ &+\|u_h-u\|_{L^2(\Omega)^d}\big(|C|_{W^{1,3}(\Omega)}\|S_h\|_{L^6(\Omega)} + \|C\|_{L^\infty(\Omega)}|S_h|_{H^1(\Omega)}\big).\end{aligned} \tag{3.51}$$

Then the choice $S_h = R_h(C) - C_h$, the antisymmetric property of the transport term, the fact that u_h is bounded in $L^3(\Omega)^d$ as

$$\|u_h\|_{L^3(\Omega)^d} \le \Big(\frac{\;}{\;}(\|f_0\|_{L^{\frac{3}{2}}(\Omega)^d} + c_{f_1}S^0_{\frac{3}{2}}|C|_{1,\Omega})\Big)^{\frac{1}{2}};$$

and Sobolev's imbedding yield

$$|R_h(C)-C_h|_{H^1(\Omega)} \le c_{1r}|C-R_h(C)|_{H^1(\Omega)} + c_{2r}\|u_h-u\|_{L^2(\Omega)^d};$$

where

$$c_{1r} = \Big(1+\frac{r_0(S_2^0)^2}{\;} + \frac{S_6^0}{\;}\Big(\frac{\;}{\;}(\|f_0\|_{L^{\frac{3}{2}}(\Omega)^d} + c_{f_1}S^0_{\frac{3}{2}}|C|_{1,\Omega})\Big)^{\frac{1}{2}}\Big) \text{ and } c_{2r} = \frac{1}{2}(S_6^0|C|_{W^{1,3}(\Omega)} + \|C\|_{L^\infty(\Omega)}). \tag{3.52}$$

By using the following triangle inequality:

$$|C-C_h|_{H^1(\Omega)} \le |R_h(C)-C|_{H^1(\Omega)} + |R_h(C)-C_h|_{H^1(\Omega)};$$

we get

$$|C-C_h|_{H^1(\Omega)} \le (1+c_{1r})|C-R_h(C)|_{H^1(\Omega)} + c_{2r}\|u_h-u\|_{L^2(\Omega)^d}. \tag{3.53}$$

4) Finally, by combining relations (3.41) and (3.53), and using relation (3.30), we obtain relation (3.31).□

By using the properties of the operators R_h, F_h and r_h, we get the following result:

Theorem 3.7. *Under the assumptions of Theorem 3.6 and if the solution* (u,p,C) *of problem* (V_a) *satisfies* $C \in H^2(\Omega)$, $u \in W^{1,3}(\Omega)^d$ *and* $p \in H^2(\Omega)$, *then we have the following a priori error estimates:*

$$|C-C_h|_{H^1(\Omega)} + \|u-u_h\|_{L^2(\Omega)^d} + \|\nabla(p-p_h)\|_{L^{\frac{3}{2}}(\Omega)^d} \le C_1 h \tag{3.54}$$

and

$$\|u-u_h\|_{L^3(\Omega)^d} \le C_2 h^{2/3}; \tag{3.55}$$

where C_1 *and* C_2 *are strictly positive constants independent of* h.

4. Iterative algorithm

In order to solve the discrete system, we propose in this section an iterative algorithm which converges to the exact solution under additional conditions on the exact solution.

The algorithm proceeds as follows: Let $u_h^0 \in X_h$ and $C_h^0 \in Y_0$ the initial guesses. Having $(u_h^i, C_h^i) \in X_h \times Y_h$ at each iteration i, we compute $(u_h^{i+1}, p_h^{i+1}, C_h^{i+1}) \in X_h \times M_h \times Y_h$, such that

$$(V_{ahi})\begin{cases} \forall v_h \in X_h; \quad \gamma\int_\Omega (u_h^{i+1} - u_h^i)\cdot v_h\,dx + \frac{\mu}{\rho}\int_\Omega (K^{-1}u_h^{i+1})\cdot v_h\,dx + \frac{\beta}{\rho}\int_\Omega |u_h^i|u_h^{i+1}\cdot v_h\,dx \\ \qquad\qquad + \int_\Omega \nabla p_h^{i+1}\cdot v_h\,dx = \int_\Omega f(C_h^i)\cdot v_h\,dx; \\ \forall q_h \in M_h; \quad \int_\Omega \nabla q_h\cdot u_h^{i+1}\,dx = 0; \\ \forall S_h \in Y_h; \quad \alpha\int_\Omega \nabla C_h^{i+1}\cdot\nabla S_h\,dx + \int_\Omega (u_h^{i+1}\cdot\nabla C_h^{i+1})S_h\,dx + \frac{1}{2}\int_\Omega \operatorname{div}(u_h^{i+1})C_h^{i+1}S_h\,dx \\ \qquad\qquad + r_0\int_\Omega C_h^{i+1}S_h\,dx = \int_\Omega gS_h\,dx; \end{cases} \tag{4.1}$$

where γ is a real strictly positive parameter. Later on, the parameter γ will be chosen to ensure the convergence of algorithm (V_{ahi}). At each iteration i, having u_h^i and C_h^i, the first two lines of (V_{ahi}) computes (u_h^{i+1}, p_h^{i+1}). Next, we substitute u_h^{i+1} by its value in the third equation of (V_{ahi}) to compute C_h^{i+1}.

In the following, we study Scheme (V_{ahi}), and we begin by proving the existence and uniqueness of the corresponding solution.

Theorem 4.1. *In addition to assumption 2.1, we suppose that* $f_0 \in L^2(\Omega)^d$. *For each* $(u_h^i, C_h^i) \in X_h \times Y_h$, *problem* (V_{ahi}) *admits a unique solution* $(u_h^{i+1}, p_h^{i+1}, C_h^{i+1}) \in X_h \times M_h \times Y_h$. *Moreover, we have the following bound*

$$|C_h^{i+1}|_{1,\Omega} \le \frac{S_2^0}{\alpha}\|g\|_{L^2(\Omega)}. \tag{4.2}$$

Furthermore, if the initial value u_h^0 *satisfies the condition*

$$\|u_h^0\|^2_{L^2(\Omega)^d} \le L_1(f, g); \tag{4.3}$$

where

$$L_1(f, g) = \frac{1}{4}\Big(\|f_0\|_{L^2(\Omega)^d} + c_{f_1}\frac{(S_2^0)^2}{\alpha}\|g\|_{L^2(\Omega)}\Big)^2;$$

and if γ *satisfies the condition*

$$\gamma > \frac{32}{27}C_I^9 L_2(f, g, L_1(f, g))\,h^{-3d/2}; \tag{4.4}$$

where

$$L_2(f, g, \gamma) = \frac{1}{\sqrt{\gamma}}\Big(\frac{3}{2K_m}\Big(\|f_0\|_{L^2(\Omega)^d} + c_{f_1}\frac{(S_2^0)^2}{\alpha}\|g\|_{L^2(\Omega)}\Big)^2 + \frac{3K_M^2}{2K_m} + \frac{3\gamma^2}{2K_m}C_I^6 h^{-d-2}\Big)^{\frac{1}{2}}; \tag{4.5}$$

then, the following inequalities hold

$$\|u_h^{i+1}\|^2_{L^2(\Omega)^d} \le L_1(f, g); \tag{4.6}$$

and

$$\|u_h^{i+1}\|^3_{L^3(\Omega)^d} \le \frac{2}{\beta}\Big(\frac{2}{K_m} + \frac{\gamma}{2}\Big)L_1(f, g). \tag{4.7}$$

Proof. To prove the existence and uniqueness of the solution of Problem (V_{ahi}) which is a square finite dimension linear system, it suffices to show the uniqueness which is readily checked for each $(u_h^i, C_h^i) \in X_h \times Y_h$. In fact, let $(u_{h,1}^{i+1}, p_{h,1}^{i+1}, C_{h,1}^{i+1})$ and $(u_{h,2}^{i+1}, p_{h,2}^{i+1}, C_{h,2}^{i+1})$ be two solutions of problem (V_{ahi}). Denote $w_h = u_{h,1}^{i+1} - u_{h,2}^{i+1}$ and $\lambda_h = p_{h,1}^{i+1} - p_{h,2}^{i+1}$. We deduce from the problem (V_{ahi}) that (w_h, λ_h) is the solution of the following problem

$$\begin{cases} \forall v_h \in X_h, & \alpha \int_\Omega w_h \cdot v_h \, dx + \frac{\mu}{\rho} \int_\Omega K^{-1} w_h \cdot v_h \, dx + \frac{\beta}{\rho} \int_\Omega |u_h^i| w_h \cdot v_h \, dx + \int_\Omega \nabla \lambda_h \cdot v_h \, dx = 0, \\ \forall q_h \in M_h, & \int_\Omega \nabla q_h \cdot w_h \, dx = 0. \end{cases}$$

Taking $(v_h, q_h) = (w_h, \lambda_h)$ and remarking that $\int_\Omega |u_h^i||w_h|^2 dx$ is non negative, we obtain by using the properties of K^{-1}, the following bound

$$\left(\alpha + \frac{\mu K_m}{\rho}\right) \|w_h\|^2_{L^2(\Omega)^d} \le 0.$$

Thus, we deduce that $w_h = 0$ $(u_{h,1}^{i+1} = u_{h,2}^{i+1})$ and the discrete inf-sup condition (3.3) implies that $\lambda_h = 0$ $(p_{h,1}^{i+1} = p_{h,2}^{i+1})$. This gives the uniqueness of the velocity and the pressure for each iteration.

Let us now prove the uniqueness of the concentration. We denote by $C_h^{i+1} = C_{h,1}^{i+1} - C_{h,2}^{i+1}$. Then, the third equation of problem (V_{ahi}) gives: Find $C_h^{i+1} \in Y_h$ such that for all $S_h \in Y_h$

$$\alpha \int_\Omega \nabla C_h^{i+1} \cdot \nabla S_h \, dx + \int_\Omega (u_h^{i+1} \cdot \nabla C_h^{i+1}) S_h \, dx + \frac{1}{2} \int_\Omega \mathrm{div}(u_h^{i+1}) C_h^{i+1} S_h \, dx + r_0 \int_\Omega C_h^{i+1} S_h \, dx = 0, \quad (4.8)$$

where (u_h^{i+1}, p_h^{i+1}) is the unique solution of the first two equations of problem (V_{ahi}). By taking $S_h = C_h^{i+1}$ and using the antisymmetric property we get the uniqueness of the concentration.

The bound (4.2) can be deduced immediately by taking $S_h = C_h^{i+1}$ in the third equation of problem (V_{ahi}), and by using the Cauchy-Schwartz inequality.

To prove the bound (4.6), we need first to estimate the error $\|u_h^{i+1} - u_h^i\|_{L^2(\Omega)^d}$ in terms of the previous value u_h^i. Taking the first equation of problem (V_{ahi}) with $v_h = u_h^{i+1} - u_h^i$ yields

$$\alpha \|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} + \frac{\mu}{\rho} \int_\Omega K^{-1} u_h^{i+1} \cdot (u_h^{i+1} - u_h^i) dx + \frac{\beta}{\rho} \int_\Omega |u_h^i| u_h^{i+1} \cdot (u_h^{i+1} - u_h^i) dx = \int_\Omega f(C_h^i) \cdot (u_h^{i+1} - u_h^i) dx.$$

By inserting u_h^i in the second and third terms of the last equation, we get,

$$\begin{aligned} \alpha \|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} &+ \frac{\mu}{\rho} \int_\Omega K^{-1}(u_h^{i+1} - u_h^i) \cdot (u_h^{i+1} - u_h^i) \, dx + \frac{\beta}{\rho} \int_\Omega |u_h^i||u_h^{i+1} - u_h^i|^2 \, dx \\ &= \int_\Omega f(C_h^i) \cdot (u_h^{i+1} - u_h^i) \, dx - \frac{\mu}{\rho} \int_\Omega K^{-1} u_h^i \cdot (u_h^{i+1} - u_h^i) \, dx - \frac{\beta}{\rho} \int_\Omega |u_h^i| u_h^i \cdot (u_h^{i+1} - u_h^i) \, dx. \end{aligned} \quad (4.9)$$

Using the properties of K^{-1}, the Cauchy-Schwartz inequality and relation (3.2) give the following

$$\begin{aligned} \alpha \|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} + \frac{\mu K_m}{\rho} \|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} &\le \|f(C_h^i)\|_{L^2(\Omega)^d} \|u_h^{i+1} - u_h^i\|_{L^2(\Omega)^d} \\ &+ \frac{\mu K_M}{\rho} \|u_h^{i+1} - u_h^i\|_{L^2(\Omega)^d} \|u_h^i\|_{L^2(\Omega)^d} + \frac{\beta}{\rho} C_I^3 h^{-d/2} \|u_h^i\|^2_{L^2(\Omega)^d} \|u_h^{i+1} - u_h^i\|_{L^2(\Omega)^d}. \end{aligned} \quad (4.10)$$

We apply the relation $ab \le \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2$ with $\varepsilon = \frac{\mu K_m}{3\rho}$ to each term on the right-hand side of the previous inequality, and we obtain

$$\begin{aligned} \alpha \|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} + \frac{\mu K_m}{2\rho} \|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} &\le \frac{3\rho}{2\mu K_m} \|f(C_h^i)\|^2_{L^2(\Omega)^d} + \frac{3\mu K_M^2}{2\rho K_m} \|u_h^i\|^2_{L^2(\Omega)^d} \\ &+ \frac{3\beta^2}{2\rho\mu K_m} C_I^6 h^{-d} \|u_h^i\|^4_{L^2(\Omega)^d}. \end{aligned} \quad (4.11)$$

Therefore, Assumption 2.1 and relation (4.2) allow us to get

$$\|u_h^{i+1} - u_h^i\|_{L^2(\Omega)^d} \le L_2(f, g, \|u_h^i\|^2_{L^2(\Omega)^d}); \tag{4.12}$$

where

$$L_2(f, g, \xi) = \frac{1}{\sqrt{\gamma}}\Big(\frac{3}{2K_m}(\|f_0\|_{L^2(\Omega)^d} + c_{f_1}\frac{(S_2^0)^2}{\nu}\|g\|_{L^2(\Gamma)})^2 + \frac{3K_M^2}{2K_m}\xi + \frac{3\beta^2}{2K_m}C_I^6 h^{-d}\xi^2\Big)^{\frac{1}{2}}. \tag{4.13}$$

Then, we are now in position to show relation (4.6). We consider the first equation of problem (V_{ahi}) with $v_h = u_h^{i+1}$, and we obtain

$$\frac{1}{\delta t}\int_\Omega (u_h^{i+1} - u_h^i)\cdot u_h^{i+1}\,dx + \frac{\mu}{\rho}\int_\Omega K^{-1}u_h^{i+1}\cdot u_h^{i+1}\,dx + \frac{\beta}{\rho}\|u_h^{i+1}\|^3_{L^3(\Omega)^d} = \int_\Omega f(C_h^i)\cdot u_h^{i+1}\,dx + \frac{\beta}{\rho}\int_\Omega (|u_h^{i+1}| - |u_h^i|)|u_h^{i+1}|^2\,dx. \tag{4.14}$$

Using the properties of K^{-1}, the Cauchy-Shwartz inequality and the relations $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ and $a^2b \le \frac{1}{3}(\frac{1}{\eta^3}b^3 + 2\eta^{\frac{3}{2}}a^3)$ wih $\varepsilon = \frac{K_m}{\mu}$ and $\eta = (\frac{3\beta}{4})^{2/3}$, we get

$$\begin{aligned}\frac{\gamma}{2}\|u_h^{i+1}\|^2_{L^2(\Omega)^d} - \frac{\gamma}{2}\|u_h^i\|^2_{L^2(\Omega)^d} + \frac{\gamma}{2}\|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} + \frac{K_m}{2}\|u_h^{i+1}\|^2_{L^2(\Omega)^d} + \frac{\beta}{2}\|u_h^{i+1}\|^3_{L^3(\Omega)^d} \\ \le \frac{1}{2K_m}\|f(C_h^i)\|^2_{L^2(\Omega)^d} + \frac{16}{27}C_I^9 h^{-3d/2}\|u_h^{i+1} - u_h^i\|^3_{L^2(\Omega)^d}.\end{aligned} \tag{4.15}$$

We denote by

$$C_1(\|u_h^i\|_{L^2(\Omega)^d}) = \frac{\gamma}{2} - \frac{16}{27}C_I^9 h^{-3d/2}L_2(f, g, \|u_h^i\|^2_{L^2(\Omega)^d})$$

and we get by using (4.12) that

$$C_1(\|u_h^i\|_{L^2(\Omega)^d}) \le \frac{\gamma}{2} - \frac{16}{27}C_I^9 h^{-3d/2}\|u_h^{i+1} - u_h^i\|_{L^2(\Omega)^d}.$$

Therefore, we obtain the following bound:

$$\begin{aligned}\frac{\gamma}{2}\|u_h^{i+1}\|^2_{L^2(\Omega)^d} - \frac{\gamma}{2}\|u_h^i\|^2_{L^2(\Omega)^d} + C_1(\|u_h^i\|_{L^2(\Omega)^d})\|u_h^{i+1} - u_h^i\|^2_{L^2(\Omega)^d} + \frac{K_m}{2}\|u_h^{i+1}\|^2_{L^2(\Omega)^d} \\ + \frac{\beta}{2}\|u_h^{i+1}\|^3_{L^3(\Omega)^d} \le \frac{2}{K_m}L_1(f, g).\end{aligned} \tag{4.16}$$

We now prove estimate (4.6) by induction on $i \ge 1$ under some condition on γ. Starting with relation (4.3), we suppose that we have

$$\|u_h^i\|^2_{L^2(\Omega)^d} \le L_1(f, g). \tag{4.17}$$

We have two situations:

- $\|u_h^{i+1}\|_{L^2(\Omega)^d} \le \|u_h^i\|_{L^2(\Omega)^d}$, which immediately leads to
$$\|u_h^{i+1}\|^2_{L^2(\Omega)^d} \le L_1(f, g).$$
- $\|u_h^{i+1}\|_{L^2(\Omega)^d} \ge \|u_h^i\|_{L^2(\Omega)^d}$. By using the induction condition (4.17), taking
$$\begin{aligned}\frac{\gamma}{2} &> \frac{16}{27}C_I^9 h^{-3d/2}L_2(f, L_1(f, g)) \\ &> \frac{16}{27}C_I^9 h^{-3d/2}L_2(f, \|u_h^i\|^2_{L^2(\Omega)^d});\end{aligned} \tag{4.18}$$
we get $C_1(\|u_h^i\|_{L^2(\Omega)^d}) > 0$, and deduce from relation (4.16) that
$$\|u_h^{i+1}\|^2_{L^2(\Omega)^d} \le L_1(f, g).$$

Then relation (4.6) holds. The bound (4.7) is a simple consequence of (4.16) and (4.6). □

The next theorem shows the convergence of the solution (u_h^i, p_h^i, C_h^i) of problem (V_{ahi}) to the solution of problem (V_{ah}).

Theorem 4.2. *In addition to assumption 2.1, we assume that the concentration solution of the problem* (V_a) *satisfies*

$$S_6^0|C|_{W^{1,3}(\Omega)} + \|C\|_{L^\infty(\Omega)} \le \frac{K_m}{2\, c_{f_1} S_2^0}. \tag{4.19}$$

Under the assumptions of Theorem 4.1, and if α *satisfies the condition*

$$\alpha > \frac{2\, C_2^2}{K_m} h^{-d}; \tag{4.20}$$

where $C_2 = -C_I^3(L_1(f,g))^{1/2}$ *and if*

$$h \le \left(\frac{1}{2C_I\, C_1}\Big(|C|_{W^{1,3}(\Omega)} + \frac{\|C\|_{L^\infty(\Omega)}}{S_6^0}\Big)\right)^{6/(6-d)}; \tag{4.21}$$

where C_1 *is the constant in* (3.54)*, then the solution* (u_h^i, p_h^i, C_h^i) *of problem* (V_{ahi}) *converges in* $L^2(\Omega)^d \times L^2(\Omega) \times H^1(\Omega)$ *to the solution of problem* (V_{ah}).

Proof. We start by substracting the third equation of problem (V_{ahi}) from the one of problem (V_{ah}) to get

$$\begin{aligned}\int_\Omega \nabla(C_h - C_h^{i+1})\cdot\nabla S_h\,dx &+ \int_\Omega u_h\cdot\nabla C_h S_h\,dx - \int_\Omega u_h^{i+1}\cdot\nabla C_h^{i+1} S_h\,dx + r_0\int_\Omega (C_h - C_h^{i+1})S_h\,dx\\ &= \frac{1}{2}\int_\Omega \operatorname{div} u_h^{i+1}\, C_h^{i+1} S_h\,dx - \frac{1}{2}\int_\Omega \operatorname{div} u_h C_h S_h\,dx.\end{aligned} \tag{4.22}$$

Inserting ∇C_h in the last term of the left-hand side and C_h in the first term of the right-hand side of the previous relation lead to

$$\begin{aligned}\int_\Omega \nabla(C_h - C_h^{i+1})\cdot\nabla S_h\,dx &+ r_0\int_\Omega (C_h - C_h^{i+1})S_h\,dx - \int_\Omega u_h^{i+1}\cdot\nabla(C_h^{i+1} - C_h)S_h\,dx\\ = \frac{1}{2}\int_\Omega \operatorname{div} u_h^{i+1}(C_h^{i+1} - C_h)S_h\,dx &+ \int_\Omega (u_h^{i+1} - u_h)\cdot\nabla C_h S_h\,dx + \frac{1}{2}\int_\Omega \operatorname{div}(u_h^{i+1} - u_h)C_h S_h\,dx.\end{aligned} \tag{4.23}$$

Finally, by inserting ∇C in the second term of right-hand side and using the Green's formula and the antisymmetric property, we get

$$\begin{aligned}\int_\Omega \nabla(C_h - C_h^{i+1})\cdot\nabla S_h\,dx + r_0\int_\Omega (C_h - C_h^{i+1})S_h\,dx =\\ \frac{1}{2}\int_\Omega (u_h^{i+1} - u_h)\cdot\nabla(C_h - C)S_h\,dx + \frac{1}{2}\int_\Omega (u_h^{i+1} - u_h)\cdot\nabla C S_h\,dx\\ -\frac{1}{2}\int_\Omega (u_h^{i+1} - u_h)\cdot\nabla S_h(C_h - C)\,dx - \frac{1}{2}\int_\Omega (u_h^{i+1} - u_h)\cdot\nabla S_h C\,dx.\end{aligned} \tag{4.24}$$

By taking $S_h = C_h - C_h^{i+1}$, we obtain

$$\begin{aligned}|C_h - C_h^{i+1}|_{1,\Omega} \le{}& S_6^0\|u_h^{i+1} - u_h\|_{L^3(\Omega)^d}|C_h - C|_{1,\Omega}\\ &+ \frac{S_6^0}{2}\|u_h^{i+1} - u_h\|_{L^2(\Omega)^d}|C|_{W^{1,3}(\Omega)} + \frac{1}{2}\|C\|_{L^\infty(\Omega)}\|u_h^{i+1} - u_h\|_{L^2(\Omega)^d}.\end{aligned} \tag{4.25}$$

Finally, we get by using relation (3.2):

$$|C_h - C_h^{i+1}|_{1,\Omega} \le \frac{S_6^0}{2}\left[2C_I\, h^{-d/6}|C - C_h|_{1,\Omega} + |C|_{W^{1,3}(\Omega)} + \frac{\|C\|_{L^\infty(\Omega)}}{S_6^0}\right]\|u_h^{i+1} - u_h\|_{L^2(\Omega)^d}. \tag{4.26}$$

Furthermore, by taking the difference between the first equations of problems (V_{ah}) and (V_{ahi}) with $v_h = u_h^{i+1} - u_h$, we get

$$\begin{aligned}\frac{\rho}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} - \frac{\rho}{2}\|u_h^{i}-u_h\|^2_{L^2(\Omega)^d} + \frac{\rho}{2}\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} + \nu\int_\Omega K^{-1}|u_h^{i+1}-u_h|^2\,dx \\ + \nu\int_\Omega (|u_h^i| - |u_h^{i+1}|)u_h^{i+1}\cdot(u_h^{i+1}-u_h)\,dx + \nu\int_\Omega (|u_h^{i+1}|u_h^{i+1} - |u_h|u_h)\cdot(u_h^{i+1}-u_h)\,dx \\ = \int_\Omega (f(C_h^i) - f(C_h))\cdot(u_h^{i+1}-u_h)\,dx. \end{aligned} \tag{4.27}$$

By using the monotonicity property of the operator A we obtain,

$$\begin{aligned}\frac{\rho}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} - \frac{\rho}{2}\|u_h^{i}-u_h\|^2_{L^2(\Omega)^d} + \frac{\rho}{2}\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} + K_m\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} \\ \le C_I^3 h^{-d/2}\|u_h^{i+1}-u_h^i\|_{L^2(\Omega)^d}\|u_h^{i+1}\|_{L^2(\Omega)^d}\|u_h^{i+1}-u_h\|_{L^2(\Omega)^d} + c_{f_1}S_2^0|C_h^i - C_h|_{1,\Omega}\|u_h^{i+1}-u_h\|_{L^2(\Omega)^d} \\ \le C_I^3 h^{-d/2}(L_1(f,g))^{1/2}\|u_h^{i+1}-u_h^i\|_{L^2(\Omega)^d}\|u_h^{i+1}-u_h\|_{L^2(\Omega)^d} + c_{f_1}S_2^0|C_h^i - C_h|_{1,\Omega}\|u_h^{i+1}-u_h\|_{L^2(\Omega)^d}. \end{aligned} \tag{4.28}$$

We denote by $C_2 = C_I^3(L_1(f,g))^{1/2}$, and we use the relation $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ with $\varepsilon = \frac{K_m}{2}$, we get

$$\begin{aligned}\frac{\rho}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} - \frac{\rho}{2}\|u_h^{i}-u_h\|^2_{L^2(\Omega)^d} + \frac{\rho}{2}\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} + \frac{K_m}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} \\ \le \frac{C_2^2}{K_m}h^{-d}\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} + \frac{(c_{f_1}S_2^0)^2}{K_m}|C_h^i - C_h|^2_{1,\Omega}. \end{aligned} \tag{4.29}$$

We choose $\frac{\rho}{2} > \frac{C_2^2}{K_m}h^{-d}$, and we denote by $C_3 = \frac{\rho}{2} - \frac{C_2^2}{K_m}h^{-d} > 0$ to conclude that

$$\begin{aligned}\frac{\rho}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} - \frac{\rho}{2}\|u_h^{i}-u_h\|^2_{L^2(\Omega)^d} + C_3\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} + \frac{K_m}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} \\ \le \frac{(c_{f_1}S_2^0)^2}{K_m}|C_h^i - C_h|^2_{1,\Omega}. \end{aligned} \tag{4.30}$$

Combining (4.30) with (4.26) and using the priori estimate (3.54), we get

$$\begin{aligned}\frac{\rho}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} - \frac{\rho}{2}\|u_h^{i}-u_h\|^2_{L^2(\Omega)^d} + C_3\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} + \frac{K_m}{2}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} \\ \le \frac{1}{K_m}\left(\frac{c_{f_1}S_2^0S_6^0}{2}\right)^2\left[2C_I C_1 h^{(6-d)/6} + |C|_{W^{1,3}(\Omega)} + \|C\|_{L^\infty(\Omega)}\right]^2\|u_h^i - u_h\|^2_{L^2(\Omega)^d}. \end{aligned} \tag{4.31}$$

Finally, Assumptions (4.19) and (4.21) allow us to get

$$\begin{aligned}\left(\frac{\rho}{2} + \frac{K_m}{4}\right)\left(\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} - \|u_h^{i}-u_h\|^2_{L^2(\Omega)^d}\right) + C_3\|u_h^{i+1}-u_h^i\|^2_{L^2(\Omega)^d} \\ + \frac{K_m}{4}\|u_h^{i+1}-u_h\|^2_{L^2(\Omega)^d} \le 0. \end{aligned} \tag{4.32}$$

We deduce that, for all $i \ge 1$, we have (if $\|u_h^i - u_h\|_{L^2(\Omega)^d} \neq 0$)

$$\|u_h^{i+1}-u_h\|_{L^2(\Omega)^d} < \|u_h^{i}-u_h\|_{L^2(\Omega)^d};$$

and then we deduce the convergence of the sequence $(u_h^{i+1} - u_h)$ in $L^2(\Omega)^d$, and then the convergence of the sequence u_h^i in $L^2(\Omega)^d$. By taking the limit of (4.32), we get that u_h^{i+1} converges to u_h in $L^2(\Omega)^d$. Relation (4.26) allows us to deduce that C_h^{i+1} converges to C_h in $H_0^1(\Omega)$.

Next, we study the convergence of the pressure, taking the difference between the first equations of systems (V_{ah}) and (V_{ahi}), we obtain for all $v_h \in X_h$ the equation

$$\begin{aligned}\int_\Omega \nabla(p_h^{i+1} - p_h)\cdot v_h\,dx =& \int_\Omega (f(C_h) - f(C_h^i))\cdot v_h\,dx - \int_\Omega (u_h^{i+1} - u_h^i)\cdot v_h\,dx\\ &+ \frac{\mu}{\rho}\int_\Omega K^{-1}(u_h - u_h^{i+1})\cdot v_h\,dx + \frac{\beta}{\rho}\int_\Omega (|u_h| - |u_h^i|)u_h\cdot v_h\,dx\\ &+ \frac{\beta}{\rho}\int_\Omega |u_h^i|(u_h - u_h^{i+1})\cdot v_h\,dx.\end{aligned}$$

We get by using the inverse inequality (3.2) the following:

$$\begin{aligned}\frac{\int_\Omega \nabla(p_h^{i+1} - p_h)\cdot v_h\,dx}{\|v_h\|_{L^3(\Omega)^d}} \le\;& c_{f_1}S_2^0|C_h - C_h^i|_{1,\Omega}\frac{\|v_h\|_{L^2(\Omega)^d}}{\|v_h\|_{L^3(\Omega)^d}}\\ &+\Big(\|u_h^i - u_h^{i+1}\|_{L^2(\Omega)^d} + \frac{\mu K_M}{\rho}\|u_h - u_h^{i+1}\|_{L^2(\Omega)^d}\Big)\frac{\|v_h\|_{L^2(\Omega)^d}}{\|v_h\|_{L^3(\Omega)^d}}\\ &+ \cdot\, C_I h^{-\frac{d}{6}}\|u_h - u_h^i\|_{L^2(\Omega)^d}(\|u_h\|_{L^3(\Omega)^d} + \|u_h^i\|_{L^3(\Omega)^d}).\end{aligned}$$

For a given mesh, owning the inf-sup condition (3.3), and using the strong convergence of u_h^i to u_h in $L^2(\Omega)^d$, and of C_h^i to C_h in $H_0^1(\Omega)$, we deduce the strong convergence of ∇p_h^i to ∇p_h in $L^{\frac{3}{2}}(\Omega)$. Furthermore, the fact that p_h^i and p_h are in the discrete space of P_1 finite elements $M_h \subset L_0^2(\Omega)$ which is defined in (3.6), allows us to deduce the strong convergence of p_h^i to p_h in $L^2(\Omega)$. □

Remark 4.3. *If* h *is not small enough, we can replace the conditions* (4.19) *and* (4.21) *in Theorem 4.2 by the following condition*

$$\left[\frac{\rho}{K_m}\Big(\frac{c_{f_1}S_2^0S_6^0}{2}\Big)^2 \cdot 2C_I C_1(\mathrm{diam}(\Omega))^{(6-d)/6} + |C|_{W^{1,3}(\Omega)} + \frac{\|C\|_{L^\infty(\Omega)}}{S_6^0}\right]^2 < \frac{K_m}{4}.$$

In fact, this condition can be used in the relation (4.31) *in the proof of the previous theorem.*

Remark 4.4. *The convergence of the iterative solution* $(u_h^i; p_h^i; C_h^i)$ *of problem* (V_{ahi}) *to the exact solution* $(u; p; C)$ *of problem* (V_a) *is a simple consequence of Theorems 3.6 and 4.2. In fact, conditions* (4.4) *and* (4.20) *satisfied by* γ *to ensure the convergence of the iterative solution* $(u_h^i; p_h^i; C_h^i)$ *to the discrete solution* $(u_h; p_h; C_h)$ *lead us to consider* γ *as a function of* h *(*$\gamma(h)$*) and satisfying these two relations. Thanks to the triangle inequality, we have:*

$$\begin{aligned}\|(u; p; C) - (u_h^i; p_h^i; C_h^i)\|_{X\times M\times Y} \le\;& \|(u; p; C) - (u_h; p_h; C_h)\|_{X\times M\times Y}\\ &+ \|(u_h; p_h; C_h) - (u_h^i; p_h^i; C_h^i)\|_{X\times M\times Y}.\end{aligned}$$

Under the assumptions of Theorem 4.2 with $\gamma(h)$ *satisfying conditions* (4.4) *and* (4.20), $(u_h^i; p_h^i; C_h^i)$ *converges to* $(u_h; p_h; C_h)$ *in* $X_h \times M_h \times Y_h$*, and, there exists an integer* $i_0(h)$ *depending on* h *such that for all* $i \ge i_0(h)$ *we have*

$$\|(u_h; p_h; C_h) - (u_h^i; p_h^i; C_h^i)\|_{X\times M\times Y} \le \|(u; p; C) - (u_h; p_h; C_h)\|_{X\times M\times Y}.$$

Consequently, for all $i \ge i_0(h)$*, we get*

$$\|(u; p; C) - (u_h^i; p_h^i; C_h^i)\|_{X\times M\times Y} \le 2\|(u; p; C) - (u_h; p_h; C_h)\|_{X\times M\times Y}.$$

Thus we obtain the convergence of the iterative solution to the exact one.

5. Numerical results

In this section, we present numerical experiments corresponding to our coupled problem for $d = 2$. These simulations have been performed using the code FreeFem++ due to F. Hecht and O. Pironneau, see [25].

We will show in this section numerical investigations corresponding to problems (V_{ahi}) by using for the convergence the stopping criterion $Err_L \leq \varepsilon$ where ε is a given tolerance considered in this work equal to 10^{-5} and Err_L is defined by

$$Err_L = \frac{\|u_h^{i+1} - u_h^i\|_{L^3(\Omega)^2} + \|\nabla(p_h^{i+1} - p_h^i)\|_{L^{\frac{3}{2}}(\Omega)^2} + |C_h^{i+1} - C_h^i|_{1,\Omega}}{\|u_h^{i+1}\|_{L^3(\Omega)^2} + \|\nabla p_h^{i+1}\|_{L^{\frac{3}{2}}(\Omega)^2} + |C_h^{i+1}|_{1,\Omega}}.$$

The initial guesses u_h^0 and C_h^0 are considered in one of these two situations:

(1) $C_h^0 = 0$ and $u_h^0 = 0$.
(2) $C_h^0 = 0$ and $u_h^0 = u_{hd}^0$ are calculated by using Darcy's problem which corresponds to $= \ = 0$.

We will see later that the second case where u_h^0 is the solution of Darcy's problem improve the convergence of the algorithms.

We also consider the errors

$$Err = \frac{\|u_h^i - u\|_{L^2(\Omega)^2} + \|\nabla(p_h^i - p)\|_{L^{\frac{3}{2}}(\Omega)^2} + |C_h^i - C|_{1,\Omega}}{\|u\|_{L^2(\Omega)^2} + \|\nabla p\|_{L^{\frac{3}{2}}(\Omega)^2} + |C|_{1,\Omega}};$$

and

$$ErrL3 = \frac{\|u_h^i - u\|_{L^3(\Omega)^2}}{\|u\|_{L^3(\Omega)^2}};$$

which describe the rate of the *a priori* error estimation for a large values of the iteration index i.

5.1. First numerical test: analytical solution. In this section, we will show numerical results corresponding to the problem where we know the exact solution. Let $\Omega =]0,1[^2 \subset \mathbb{R}^2$ where each edge is divided into N equal segments so that Ω is divided into N^2 equal squares and finally into $2N^2$ equal triangles. For simplicity, we take $= \ = 1$.
We consider the following exact solution with a parameter :

$$\begin{cases} p(x,y) = \cos(\ x)\cos(\ y); \\ u(x,y) = (\ \sin(\ x)\cos(\ y); \cos(\ x)\sin(\ y))^T; \\ C(x,y) = x^2(x-1)^2y^2(y-1)^2; \end{cases} \tag{5.1}$$

where $\operatorname{div} u = 0$ in Ω, $u \cdot n = 0$ and $C = 0$ on $\partial\Omega$. Furthermore, we take $f_1(C) = (4C; 3\sin(C))$. Thus, we compute f and g by using their expressions in problem (P).

5.1.1. *Case where* $K = I$.
In this part, we take $K = I$. To study the dependency of the convergence with the parameter , we consider $N = 60$, $= 20$, $= 20$, $= r_0 = 1$, and for each , we stop the algorithm (V_{ahi}) when the error $Err_L < 1e-5$. Tables 1 shows the error Err and the number of iterations Nbr for $u_h^0 = 0$ and $C_h^0 = 0$, while Table 2 shows similar results for $u_0 = u_{hd}^0$ and $C_h^0 = 0$. These two tables describe the convergence of algorithm (V_{ahi}) with respect to . We remark that the number of iterations is relatively small when is large. In both cases, the best convergence is obtained for $= 100$. The main advantage is for the case where $u_h^0 = u_{hd}^0$ are computed with Darcy's problem is that the number of iterations Nbr is less than the one obtained with the case $u_h^0 = 0$.

	0.001	.01	.1	1	10	100	1000
Nbr	4078	4009	3432	1440	234	29	115
Err	0.068	0.068	0.068	0.068	0.068	0.068	0.068

Table 1. Error Err (in logarithmic scale) and number of iterations Nbr for each associated to Example (5.1) with algorithm (V_{ahi}) for $u_h^0 = 0$. ($= 20$, $= 20$).

	0.001	.01	.1	1	10	100	1000
Nbr	2027	1993	1710	714	110	18	76
Err	0.068	0.068	0.068	0.068	0.068	0.068	0.068

Table 2. Error Err (in logarithmic scale) and number of iterations Nbr for each associated to Example (5.1) with algorithm (V_{ahi}) for $u_h^0 = u_{hd}^0$. (= 20, = 20).

For further studies, we consider = 10, = 20, = 100 and the initial guesses $u_h^0 = u_{hd}^0$ and $C_h^0 = 0$. Tables 3 and 4 show the obtained rate of convergence which seems in agreement with the theoretical ndings. We notice that the theoretical rate of convergence of the velocity in norm $L^2( )^2$, the pressure in norm $W^{1;\frac{3}{2}}( )$ and the concentration in norm $H_0^1( )$ are equal to 1; the rate of convergence of the velocity in norm $L^3( )^3$ is $2=3$.

h	$\log_{10}$ k u u_h k_{L^2} = k u k_{L^2}	Rate	$\log_{10}$ k u u_h k_{L^3} = k u k_{L^3}	Rate
1/120	-4.2812		-4.1598	
1/140	-4.4047	1.84	-4.2431	1.23
1/160	-4.5111	1.83	-4.3154	1.24
1/180	-4.6029	1.79	-4.3711	1.09
1/200	-4.6830	1.74	-4.4183	1.03

Table 3. Rate of convergence of the velocity in norms $L^2( )^2$ and $L^3( )^3$. Example (5.1) with algorithm (V_{ahi}). (= 20, = 20).

h	$\log_{10}$ k p p_h $k_{W^{1;\frac{3}{2}}}$	Rate	$\log_{10}$ k C C_h $k_{H_0^1}$	Rate
1/120	-1.8902		-1.7090	
1/140	-1.9611	1.05	-1.7759	0.99
1/160	-2.0215	1.04	-1.8339	1.00
1/180	-2.0742	1.03	-1.8851	1.00
1/200	-2.1210	1.02	-1.93	0.98

Table 4. Rate of convergence of the pressure in norm $W^{1;\frac{3}{2}}( )$ and the concentration in norm $H_0^1( )$. Example (5.1) with algorithm (V_{ahi}). (= 20, = 20).

5.1.2. *Case where* $K \neq I$.
In this part, we take K such that K^{-1} is equal to:

$$K^{-1} = \begin{pmatrix} 3 + \sin(x)\sin(y) & x^2y^2 \\ x^2y^2 & 3 + \sin(x)\sin(y) \end{pmatrix} :$$

We follow the same numerical tests performed above and we consider $N = 60$, = 20, = 20, = $r_0 = 1$. Based on the previous test case, we will perform here numerical simulations with $u_h^0 = u_{hd}^0$ and $C_h^0 = 0$. Table 5 describe the convergence of algorithm (V_{ahi}) with respect to . In this case, the best convergence is also obtained for = 100.

	.1	1	10	100	1000
Nbr	518	346	84	13	45
Err	0.068	0.068	0.068	0.068	0.068

Table 5. Error Err (in logarithmic scale) and number of iterations Nbr for each associated to Example (5.1) with algorithm (V_{ahi}) for $u_h^0 = u_{hd}^0$. (= 20, = 20).

Let us now study the rate of convergence of the errors. we consider = 10, = 20, = 100 and the initial guesses $u_h^0 = u_{hd}^0$ and $C_h^0 = 0$. Tables 6 and 7 show that the numerical rate of convergence seems in agreement with the theoretical ndings.

h	$\log_{10} \|u - u_h\|_{L^2} / \|u\|_{L^2}$	Rate	$\log_{10} \|u - u_h\|_{L^3} / \|u\|_{L^3}$	Rate
1/120	-4.2927		-4.2127	
1/140	-4.4245	1.96	-4.3432	1.94
1/160	-4.5379	1.95	-4.4539	1.90
1/180	-4.6368	1.93	-4.5480	1.84
1/200	-4.7242	1.90	-4.6274	1.73

Table 6. Rate of convergence of the velocity in norms $L^2(\ )^2$ and $L^3(\ )^3$. Example (5.1) with algorithm (V_{ahi}). (= 20, = 20).

h	$\log_{10} \|p - p_h\|_{W^{1;\frac{3}{2}}}$	Rate	$\log_{10} \|C - C_h\|_{H_0^1}$	Rate
1/120	-1.8902		-1.7090	
1/140	-1.9612	1.05	-1.7759	0.99
1/160	-2.0216	1.04	-1.8339	1.00
1/180	-2.0743	1.03	-1.8851	1.00
1/200	-2.1211	1.02	-1.9308	0.99

Table 7. Rate of convergence of the pressure in norm $W^{1;\frac{3}{2}}(\ )$ and the concentration in norm $H_0^1(\ )$. Example (5.1) with algorithm (V_{ahi}). (= 20, = 20).

5.2. Comparison between Darcy and Darcy-Forchheimer. In this section, we treat numerical test (taken from [32]) showing the difference between Darcy and Darcy-Forchheimer systems. We take $N = 60$, $= 10$,

$$f_0(x;y) = (100(1-x)^2(1-y)^2;0)^T; \quad g(x;y) = 10x^2y^2;$$

$= \ = 1$, $= r_0 = 1$, $K = I$ and $f_1 = 0$.

In the following, we will compare the numerical velocity, pressure and concentration corresponding to Darcy-Forchheimer Problem (for $= 10$) and Darcy Problem (for $= 0$). Figures 1-6 show that there are differences between the distributions and values of the numerical velocities, pressures and concentration. The biggest difference between the Darcy and the Darcy-Forchheimer is in the values of the velocity (which was expected).

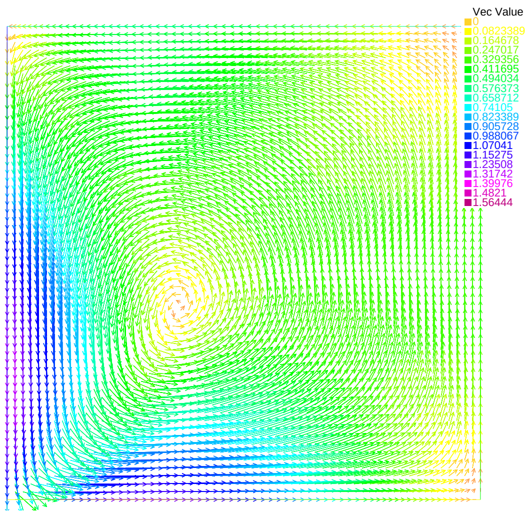

Figure 1. Numerical Darcy-Forchheimer velocity (= 10).

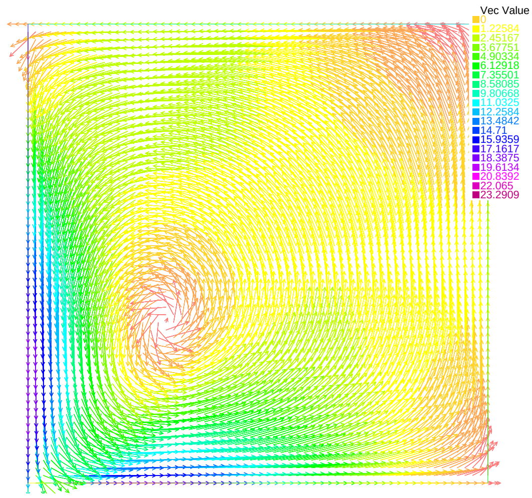

Figure 2. Numerical Darcy velocity.

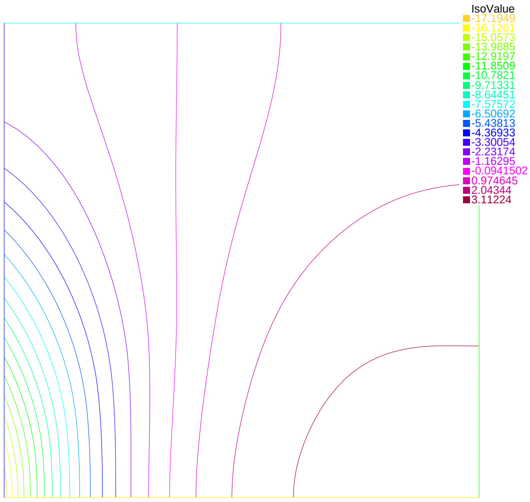

Figure 3. Numerical Darcy-Forchheimer pressure (= 10)

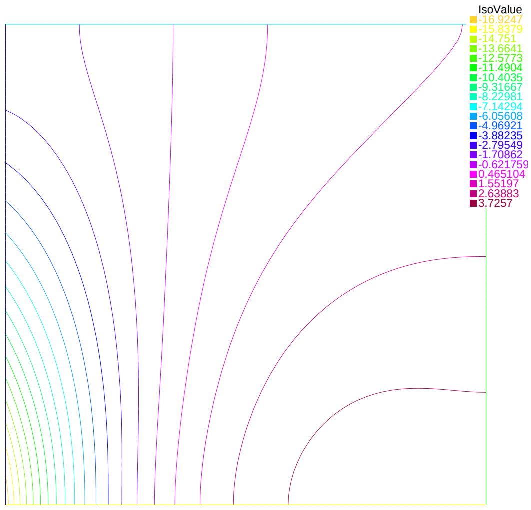

Figure 4. Numerical Darcy pressure

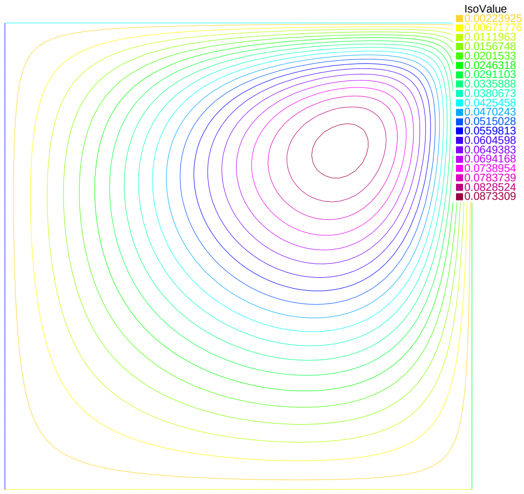

Figure 5. Numerical Darcy-Forchheimer concentration (= 10)

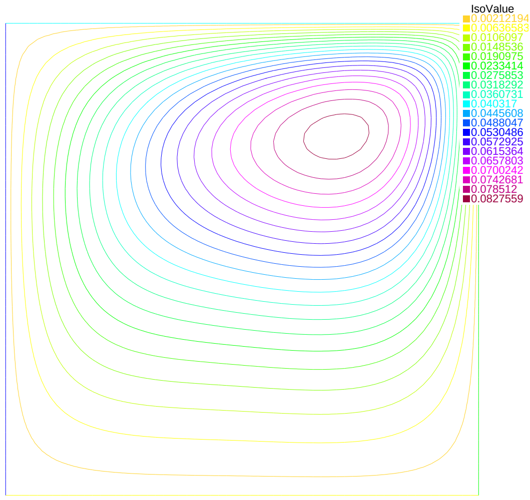

Figure 6. Numerical Darcy concentration

5.3. Second numerical test: Driven cavity. The driven cavity is a standard benchmark for testing the performance of algorithms in uid problems. It is treated in several works ([27], [8], [34] and [16]). In this section, we show numerical simulation corresponding to the Lid Driven Cavity in order to study the dependency of the convergence with respect to and the data.

Let =]0 ;1[^{2}, K = I , = 1, $r_0 = 0$, = 20, f_0 = 0, $f_1(C) = (10C; 10C)$, and g = 0. We complete the Darcy-Forchheimer equations with the boundary conditions $u{:}n = 0$ on @, and the concentration equation with the boundary condition C = (is a parameter) on $_1$ = [0;1] f 1g (top of), and C = 0 on @ n $_1$. In this section, the initial guesses of algorithm (V_{ahi}) are $C_h^0 = 1$ and $u_h^0 = u_{hg}^0$.

We begin rst by testing the convergence of the algorithm with respect of for a given = 20. We consider N = 20 and we test the algorithm for multiple values of . We consider that the algorithm doesn't converge if the condition $Err_L < 1e^{5}$ is not reached after 5000 iterations. Table 8 shows for

= 20, the dependency of the convergence of the algorithm with respect to and the better convergence corresponds to the value = 10. This result shows clearly that the convergence depends on as announced in relation (4.20) of Theorem 4.2.

	0.001	.01	.1	.5	.55	.6	.7	.8	1	10	100	1000
Nbr	{	{	{	{	2440	1463	732	504	313	27	49	283
cov/div	div	div	div	div	conv	conv	conv	conv	conv	conv	conv	conv

Table 8. Test of the convergence for the driven cavity with respect to for = 20.

Figures 7, 8 and 9 show for = 10 and = 20, the velocity, the pressure and the concentration in .

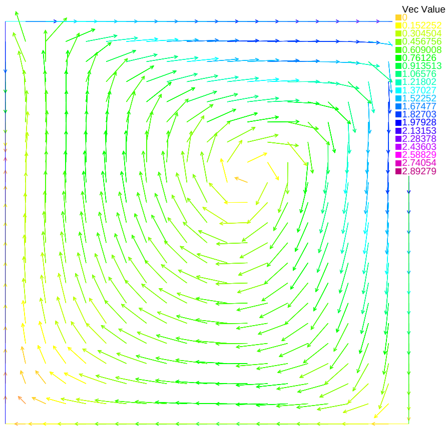

Figure 7. Numerical velocity (Driven cavity), = 10; = 20.

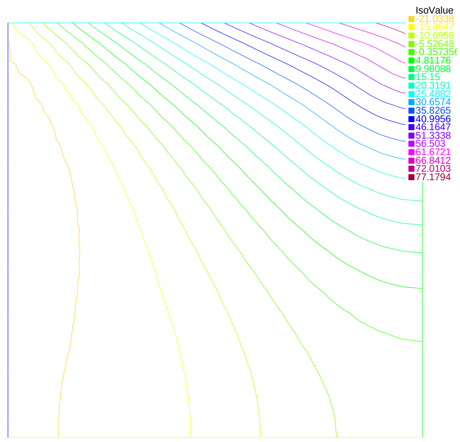

Figure 8. Numerical pressure (Driven cavity), = 10; = 20.

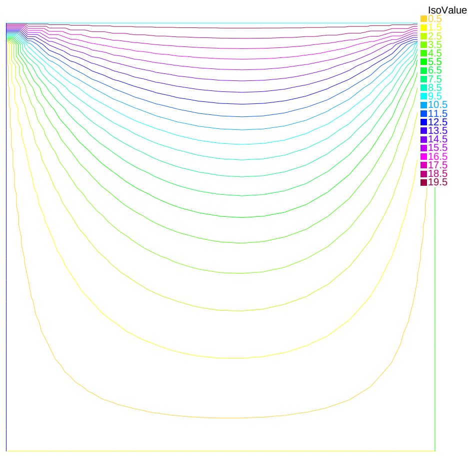

Figure 9. Numerical concentration (Driven cavity), = 10; = 20.

Let us now test the convergence with respect to for = 10. Table 9 shows the dependency of the convergence of the algorithm with respect to (i.e. with respect to the concentration C). This result shows clearly that the convergence depends on the concentration as announced in relation (4.19) of Theorem 4.2.

	1	20	100	150	170	175	180	200
Nbr	24	27	77	235	728	1493	{	{
conv/div	conv	conv	conv	conv	cov	conv	div	div

Table 9. Test of the convergence for the driven cavity with respect to for = 10.

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