

# A virtual element discretization for the time dependent Navier-Stokes equations in stream-function formulation

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## Abstract

In this work, a new Virtual Element Method (VEM) of arbitrary order  $k \geq 2$  for the time dependent Navier-Stokes equations in stream-function form is proposed and analyzed. Using suitable projection operators, the bilinear and trilinear terms are discretized by only using the proposed degrees of freedom associated with the virtual space. Under certain assumptions on the computational domain, error estimations are derived and shown that the method is optimally convergent in both space and time variables. Finally, to justify the theoretical analysis, four benchmark examples are examined numerically.

**Key words:** virtual element method, Navier–Stokes equations, time dependent problem, stream-function.

**Mathematics subject classifications (2010):** 65N30, 65N12, 76D07, 65N15

## 1 Introduction

In this work, we study a Virtual Element Method (VEM) for a fourth order nonlinear problem arising in the numerical discretization of the Navier-Stokes problem. The VEM, introduced in [6], is a generalization of the Finite Element Method (FEM) which is characterized by the capability of dealing with very general polygonal/polyhedral meshes, and it also permits to construct in a straightforward way highly regular discrete spaces. Indeed, by avoiding the explicit construction of the local basis functions, the VEM can easily handle general polygons/polyhedrons without complex integrations on the element (see [7] for details on the coding aspects of the method). The VEM has been applied successfully for problems in fluid mechanics; see for instance [1, 10, 18, 24, 26, 30, 34, 33, 17, 39, 41], where Stokes, Brinkman, Stokes–Darcy and Navier-Stokes equations have been recently developed.

The Navier-Stokes system is a paradigm of fluid flow problems. Usually, the variables  $\mathbf{u}$  and  $p$  denote the velocity and the pressure field, respectively. It is proved that if the body force  $\mathbf{f}$  and the initial data  $\mathbf{u}_0$  are smooth enough and the boundary of domain  $\Omega$  is locally Lipschitz continuous, then the two dimensional non stationary Navier-Stokes problem has weak solution. In, [38], Temam showed that  $\mathbf{u} \in L^\infty(0, T; [H^2(\Omega)]^2)$  with the assumption that  $\mathbf{f}$  and initial data  $\mathbf{u}_0$  are suitably smooth. Since the model problem consists of nonlinear term, it is not straightforward

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to find analytical solution. Therefore, numerical approximation is the only available option in the majority of the cases found in practice, which involves very difficult initial and boundary conditions. A large amount of articles are contributed to improve the effectiveness of numerical schemes, the computer resolution of Navier-Stokes equations still challenges the applied mathematician and scientist. In this article, we have attempted to develop a VEM scheme for a time dependent stream-function formulation of the Navier-Stokes equations.

In the proposed formulation, the stream-function is the principal unknown of the system [27, 28]. Salient features in formulations of this kind include that: there is only one scalar variable, the incompressible condition is satisfied automatically, the stream-function is one of the most useful tools in flow visualization. On the other hand, we note that the velocity and pressure are not present in the formulation. However, the velocity can be easily obtained by a simple postprocess from stream-function. In 1979 in [27] the standard weak formulation was presented for the first time in terms of the stream-function for the Navier-Stokes equations, in this direction in [20] the authors present conforming finite element method for the steady formulation, including an algorithm for pressure recovery. More recently, in [25] a  $C^1$  finite element method based on the Argyris element has been proposed for the stationary quasi-geostrophic equations, which corresponds to an extension of a stream-function formulation for the Navier-Stokes problem.

It is well-known that conforming finite element methods to solve the fourth order problems require  $C^1$ -continuity. The construction of finite elements with such regularity is not straightforward (see [23]). However, this can be easily achieved by using the virtual element strategy. More precisely, we will follow the VEM approach presented in [16, 21] (see also [12, 8, 35, 36]) to build global discrete spaces of  $H^2(\Omega)$  of arbitrary order to solve the time dependent fourth order nonlinear problem.

There are some works for the approximation by VEM for fluid flow problems using the stream-function formulation. In [4] a  $C^1$  conforming virtual element method has been presented to solve the Stokes problem on general polygonal meshes. More recently, a 2D Stokes complex structure for the VEM was analyzed and a discrete **curl** formulation of the Navier-Stokes problem has been obtained in [13]. The extension to the 3D case of the Stokes complex structure for the VEM has been presented in [9].

The goal of this paper is to propose a conforming  $C^1$  virtual element method to solve continuous weak formulation (2.5) (will be defined in Section 2) and to prove that the method is optimally convergent in both space and time variables. More precisely, we will propose a new VEM discretization to solve the time dependent Navier-Stokes problem written in terms of the stream-function variable. We consider a primal variational formulation of the problem written in  $H^2(\Omega)$ . Then, we propose a direct  $C^1$  global virtual element subspace of arbitrary order  $k \geq 2$  to be used in the semi-discrete and fully-discrete formulations. We construct projection operators in order to write bilinear and trilinear forms that are fully computable. In particular, to discretize the trilinear form, we propose a form which does not need any stabilization. We prove that the fully-discrete problem is well-posed by using fixed point arguments and assuming that the data is in a certain sense small enough. Then, we obtain optimal rate of convergence in  $H^2(\Omega)$  for the proposed discretizations by using an adequate projection operator and under standard assumptions on the computational domain. In addition, the velocity field is then obtained from the discrete stream-function by a postprocess. In a summary, the advantages of the present method are: the  $C^1$  conforming virtual space can be built with a straightforward construction due to the flexibility of the VEM and it provides a very competitive alternative to solve the time dependent Navier-Stokes problem on polygonal meshes.

The rest of the paper is organized as follows: In Section 2, the model problem and the continuous weak formulation are defined. Using the stream-function formulation, we rewrite the continuous weak formulation that is nonlinear time dependent biharmonic problem. Further, the basic settings of the functional analysis and the assumptions required to develop the theory are also highlighted.

In Section 3, we introduce the virtual element subspaces, the polynomial projection operators and the discrete forms which are exploited to construct the discrete schemes. In Section 4, the semi-discrete and fully-discrete schemes are introduced and the well posedness of the schemes are also discussed. A priori error estimates for the semi-discrete and the fully-discrete schemes are investigated in Section 5. The theoretical convergence rates are justified with four numerical examples in Section 6.

## 2 Preliminaries and the model problem

Let  $\Omega \subset \mathbb{R}^2$  be a simply connected polygonal domain with boundary  $\Gamma := \partial\Omega$ . We denote by  $L^2(\Omega)$ , the space of square integrable scalar functions with the standard inner product  $(\psi, \phi)_{0,\Omega} := \int_{\Omega} \psi \phi$ . For each positive integer  $s \in \mathbb{N}$ , we define  $H^s(\Omega)$  [2], the Sobolev space with standard norm

$$\|\phi\|_{s,\Omega} := \left( \sum_{0 \leq \alpha \leq s} \|\partial^\alpha \phi\|_{0,\Omega}^2 \right)^{1/2},$$

where  $\alpha$  is multi index and  $\partial^\alpha \phi$  denotes  $\alpha^{\text{th}}$  partial derivative of  $\phi$ . Let  $t$  denote the time variable taking values in the interval  $I := (0, T]$ , where  $T$  is a given final time. Moreover, the function space  $L^2(0, T; H^s(\Omega))$  consists of scalar functions  $\phi$  such that for almost all  $t \in [0, T]$ ,  $\phi(\cdot, t) \in H^s(\Omega)$  [15] with the norm,

$$\|\phi\|_{L^2(0,T,H^s(\Omega))} := \left( \int_0^T \|\phi(t)\|_{s,\Omega}^2 \right)^{1/2}; \quad \|\phi\|_{L^\infty(0,T,H^s(\Omega))} := \operatorname{ess\,sup}_{0 \leq t \leq T} \|\phi(t)\|_{s,\Omega}.$$

In addition, given any Hilbert space  $V$ , we will denote by  $[V]^2$  the space of vectors functions with entries in  $V$  (see [2]). Further, we define  $\partial_t \phi := \frac{d\phi}{dt}$ ,  $\partial_{tt} \phi := \frac{d^2 \phi}{dt^2}$ ,  $\operatorname{div} \phi := \frac{d\phi}{dx} + \frac{d\phi}{dy}$ ,  $\operatorname{curl} \phi := \left( \frac{d\phi}{dy}, -\frac{d\phi}{dx} \right)$ ,  $\mathbf{D}^2 \phi := (\partial_{ij} \phi)_{1 \leq i, j \leq 2}$  denotes the Hessian matrix of  $\phi$ , and  $\partial_{\mathbf{n}} \phi := \nabla \phi \cdot \mathbf{n}$ , where  $\mathbf{n}$  is outward normal vector. For second order tensor fields  $\boldsymbol{\sigma}, \boldsymbol{\tau} : \Omega \rightarrow \mathbb{R}^{2 \times 2}$ , we define scalar product  $:\ : \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$  by

$$\boldsymbol{\sigma} : \boldsymbol{\tau} := \sum_{1 \leq i, j \leq 2} \sigma_{ij} \tau_{ij},$$

where  $\sigma_{ij}$  and  $\tau_{ij}$  are the entries at  $(i, j)$ -th position of  $\boldsymbol{\sigma}$  and  $\boldsymbol{\tau}$ , respectively.

### 2.1 Model problem

We consider the time-dependent Navier-Stokes problem (for more details, see for instance [28, 37]): given a sufficiently smooth force density  $\mathbf{f} \in [L^2(\Omega)]^2$ , we seek  $(\mathbf{u}(t), p(t))$  such that:

$$\begin{aligned} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} & \text{on } \Gamma, \\ (p, 1)_{0,\Omega} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \tag{2.1}$$

where  $\mathbf{u}, p$  are the velocity and the pressure fields, respectively, and  $\nu > 0$  is the viscosity of the fluid. We introduce the following Hilbert spaces:

$$\mathbf{H} := \{ \mathbf{v} \in [H^1(\Omega)]^2 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma \},$$

and

$$Q := \{q \in L^2(\Omega) : (q, 1)_{0,\Omega} = 0\}.$$

The standard velocity-pressure variational formulation of the Navier-Stokes problem reads as follows: find  $(\mathbf{u}(t), p(t)) \in \mathbf{H} \times Q$ , such that

$$\begin{aligned} \int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{v} + \nu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} - \int_{\Omega} p \operatorname{div} \mathbf{v} &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} & \forall \mathbf{v} \in \mathbf{H}, \\ \int_{\Omega} q \operatorname{div} \mathbf{u} &= 0 & \forall q \in Q. \end{aligned} \quad (2.2)$$

It is well known that (2.2) admits a unique solution (see [28]). Let us introduce the following space of functions in  $\mathbf{H}$  with vanishing divergence

$$\mathbf{Z} := \{\mathbf{v} \in \mathbf{H} : \operatorname{div} \mathbf{v} = 0\}.$$

Then, (2.2) can be rewritten in the following form: find  $\mathbf{u}(t) \in \mathbf{Z}$  such that

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{v} + \nu \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{Z}.$$

Now, we reformulate the above problem as follows: since  $\Omega$  is a simply connected domain, a well known result states that a vector function  $\mathbf{v} \in \mathbf{Z}$  if and only if there exists a scalar function  $\varphi \in H^2(\Omega)$  (called *stream-function*) such that

$$\mathbf{v} = \operatorname{curl} \varphi \in \mathbf{H}.$$

The function  $\varphi$  is defined up to a constant. Thus, we consider the following space

$$H_0^2(\Omega) := \{\varphi \in H^2(\Omega) : \varphi = \partial_n \varphi = 0 \text{ on } \Gamma\}.$$

We endow  $H_0^2(\Omega)$  with the natural norm  $\|\cdot\|_{2,\Omega}$ . Then, (2.2) can be formulated as follows: find  $\psi(t) \in H_0^2(\Omega)$  such that

$$\begin{aligned} \int_{\Omega} \partial_t (\operatorname{curl} \psi) \cdot \operatorname{curl} \phi + \nu \int_{\Omega} \Delta \psi \Delta \phi + \int_{\Omega} \Delta \psi \operatorname{curl} \psi \cdot \nabla \phi &= \int_{\Omega} \mathbf{f} \cdot \operatorname{curl} \phi & \forall \phi \in H_0^2(\Omega), \\ \psi(0) &= \psi_0. \end{aligned} \quad (2.3)$$

Now, we introduce the following trilinear form as follows,

$$\begin{aligned} C(\cdot; \cdot, \cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \times H_0^2(\Omega) &\rightarrow \mathbb{R}, \\ C(v; \psi, \phi) &:= \int_{\Omega} \Delta v \operatorname{curl} \psi \cdot \nabla \phi. \end{aligned} \quad (2.4)$$

It is observed that  $C(v; \phi, \phi) = 0$  and  $C(v; \psi, \phi) = -C(v; \phi, \psi)$ . An application of Hölder inequality and the Sobolev's embedding theorem  $H^1(\Omega) \hookrightarrow L^4(\Omega)$ , we have

$$C(v; \psi, \phi) \leq C_0(\Omega) \|v\|_{2,\Omega} \|\psi\|_{2,\Omega} \|\phi\|_{2,\Omega} \quad \forall v, \psi, \phi \in H^2(\Omega).$$

Further, let  $X'$  be the dual space of  $X$ . For any smooth enough function  $u, v \in H_0^2(\Omega)$ , we define the function  $\mathcal{G}(u, v) \in H^{-2}(\Omega) = [H_0^2(\Omega)]'$  such that

$$\langle \mathcal{G}(u, v), w \rangle_{-2,2,\Omega} := C(u; v, w) \quad \forall w \in H_0^2(\Omega),$$

where  $\langle \cdot, \cdot \rangle_{-2,2,\Omega}$  denotes duality pairing between  $H^{-2}(\Omega)$  and  $H_0^2(\Omega)$ . It can be easily deduced that

$$\|\mathcal{G}(u, u)\|_{-2,\Omega} \leq C_0(\Omega) \|u\|_{2,\Omega}^2 \quad \forall u \in H_0^2(\Omega).$$

Since  $\partial_t(\mathbf{curl} \psi) = \mathbf{curl}(\partial_t \psi)$ , problem (2.3) above can be written as follows: For a given function  $\mathbf{f} \in L^2(0, T; [L^2(\Omega)]^2)$  and  $\psi_0 \in H_0^2(\Omega)$ , the continuous weak formulation is defined as find  $\psi \in L^2(0, T; H_0^2(\Omega))$  such that

$$\begin{aligned} A(\partial_t \psi, \phi) + \nu B(\psi, \phi) + C(\psi; \psi, \phi) &= F(\phi) \quad \forall \phi \in H_0^2(\Omega), \\ \psi(0) &= \psi_0, \end{aligned} \quad (2.5)$$

where  $\psi(t) \in H_0^2(\Omega)$  is the *stream-function* of the velocity field  $\mathbf{u}(t) \in \mathbf{Z}$  ( i.e.,  $\mathbf{u} = \mathbf{curl} \psi$ ) and

$$A(\cdot, \cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}, \quad A(\psi, \phi) := \int_{\Omega} \mathbf{curl} \psi \cdot \mathbf{curl} \phi \quad \forall \psi, \phi \in H_0^2(\Omega), \quad (2.6)$$

$$B(\cdot, \cdot) : H_0^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}, \quad B(\psi, \phi) := \int_{\Omega} \Delta \psi \Delta \phi \quad \forall \psi, \phi \in H_0^2(\Omega), \quad (2.7)$$

$$F(\cdot) : H_0^2(\Omega) \rightarrow \mathbb{R}, \quad F(\phi) := \int_{\Omega} \mathbf{f} \cdot \mathbf{curl} \phi \quad \forall \phi \in H_0^2(\Omega). \quad (2.8)$$

Note that

$$|B(\psi, \phi)| \leq \|\psi\|_{2,\Omega} \|\phi\|_{2,\Omega} \quad \forall \psi, \phi \in H_0^2(\Omega),$$

and

$$B(\phi, \phi) \geq C \|\phi\|_{2,\Omega}^2 \quad \forall \phi \in H_0^2(\Omega),$$

where  $C$  is a positive generic constant. Next, we would like to discuss the well-posedness of continuous weak formulation (2.5).

**Theorem 2.1** *Let  $\mathbf{f} \in L^2(0, T; [L^2(\Omega)]^2)$  and  $\psi_0 \in H_0^2(\Omega)$ . Then, there exists a unique solution  $\psi \in L^2(0, T; H_0^2(\Omega))$  of problem (2.5).*

*Proof.* For detail proof, we refer to [31, Theorem 2.1] (see also [14]). □

### 3 Virtual element method

Let  $\{\mathcal{T}_h\}_h$  be a sequence of decompositions of  $\Omega$  into general polygonal elements  $K$ . Let  $h_K$  denote the diameter of the element  $K$  and  $h$  the maximum of the diameters of all the elements of the mesh, i.e.,  $h := \max_{K \in \mathcal{T}_h} h_K$ . In what follows, we denote by  $N_K$  the number of vertices of  $K$ , by  $e$  a generic edge of  $\mathcal{T}_h$  and for all  $e \in \partial K$ , we define a unit normal vector  $\mathbf{n}_K^e$  that points outside of  $K$  and a unit tangent vector  $\mathbf{t}_K^e$  to  $K$ . Further, we denote by  $h_e$  the length of the edge  $e$ . For each vertex  $V_i$ , we associate a *characteristic length*  $h_{V_i}$  which is the average of the diameter of the elements having  $V_i$  as a vertex.

For the theoretical analysis, we will make the following assumptions: there exists a real number  $C_{\mathcal{T}} > 0$  such that, for every  $h$  and every  $K \in \mathcal{T}_h$ ,

#### Assumption 1

- a) the ratio between the shortest edge and the diameter  $h_K$  of  $K$  is larger than  $C_{\mathcal{T}}$ ;
- b)  $K \in \mathcal{T}_h$  is star-shaped with respect to every point of a ball of radius  $C_{\mathcal{T}} h_K$ .

### 3.1 Local and global virtual spaces

Now, for any subset  $\mathcal{D} \subseteq \mathbb{R}^2$  and non negative integer  $k$ , we will denote by  $\mathcal{P}_k(\mathcal{D})$  the space of polynomials of degree up to  $k$  defined on  $\mathcal{D}$ . Then, for  $(s_1, s_2) \in \mathbb{N} \times \mathbb{N}$ , we define the set of scale monomials as

$$\mathcal{M}_s^*(K) := \left\{ q^* \mid q^* = \left( \frac{x - x_d}{h_K} \right)^{s_1} \left( \frac{y - y_d}{h_K} \right)^{s_2} ; s = s_1 + s_2 \right\},$$

where  $(x_d, y_d)$  denotes centroid of  $K$ . Then, we define  $\mathcal{M}_s(K) := \cup_{j \leq s} \mathcal{M}_j^*(K)$  as a basis of  $\mathcal{P}_s(K)$ . Analogously, we consider the set of the scaled monomials defined on each edge  $e$ :

$$\mathcal{M}_s(e) := \left\{ 1, \frac{\xi - \xi_e}{h_e}, \left( \frac{\xi - \xi_e}{h_e} \right)^2, \dots, \left( \frac{\xi - \xi_e}{h_e} \right)^s \right\},$$

where  $\xi_e$  is the midpoint of  $e$ .

Then, for any  $k \geq 2$  and for every polygon  $K \in \mathcal{T}_h$ , we introduce the following preliminary local virtual space:

$$\begin{aligned} \tilde{V}_h(K) := \{ \phi_h \in H^2(K) : \Delta^2 \phi_h \in \mathcal{P}_{k-2}(K), \phi_h|_{\partial K} \in C^0(\partial K), \phi_h|_e \in \mathcal{P}_r(e) \forall e \in \partial K, \\ \nabla \phi_h|_{\partial K} \in [C^0(\partial K)]^2, \partial_{\mathbf{n}_K^e} \phi_h|_e \in \mathcal{P}_\alpha(e) \forall e \in \partial K \}, \end{aligned}$$

where  $r := \max\{3, k\}$  and  $\alpha := k - 1$ .

Next, for a given  $\phi_h \in \tilde{V}_h(K)$ , we introduce five sets  $\mathbf{D}_1 - \mathbf{D}_5$  of linear operators from the local virtual space  $\tilde{V}_h(K)$  into  $\mathbb{R}$ .

- $\mathbf{D}_1$  : contains linear operators evaluating  $\phi_h$  at the  $N_K$  vertices of  $K$ ;
- $\mathbf{D}_2$  : contains linear operators evaluating  $h_{V_i} \nabla \phi_h(V_i)$  for all vertices  $V_i$  of  $K$ , where  $1 \leq i \leq N_K$ ;
- $\mathbf{D}_3$  : for  $r > 3$ , the moments  $\frac{1}{h_e} \int_e q(\zeta) \phi_h(\zeta) d\zeta \quad \forall q \in \mathcal{M}_{r-4}(e), \quad \forall \text{edge } e$ ;
- $\mathbf{D}_4$  : for  $\alpha > 1$ , the moments  $\int_e q(\zeta) \partial_{\mathbf{n}_K^e} \phi_h(\zeta) d\zeta \quad \forall q \in \mathcal{M}_{\alpha-2}(e), \quad \forall \text{edge } e$ ;
- $\mathbf{D}_5$  : for  $k \geq 4$ , the moments  $\frac{1}{h_K^2} \int_K q(\mathbf{x}) \phi_h(\mathbf{x}) d\mathbf{x} \quad \forall q \in \mathcal{M}_{k-4}(K), \quad \forall \text{polygon } K$ .

In order to construct the discrete scheme, we first observe that

$$\int_{\Omega} \Delta \varphi \Delta \phi = \int_{\Omega} \mathbf{D}^2 \varphi : \mathbf{D}^2 \phi.$$

Then, we decompose the bilinear form (2.7) in the following element by element contribution:

$$B(\varphi, \phi) = \int_{\Omega} \mathbf{D}^2 \varphi : \mathbf{D}^2 \phi = \sum_{K \in \mathcal{T}_h} B_K(\varphi, \phi) = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{D}^2 \varphi : \mathbf{D}^2 \phi, \quad \forall \varphi, \phi \in H_0^2(\Omega).$$

In what follows, we are going to build the discrete version of the local bilinear forms listed above. With this aim, we define the following projector operator  $\Pi_K^{k, \Delta} : \tilde{V}_h(K) \rightarrow \mathcal{P}_k(K) \subseteq \tilde{V}_h(K)$  for each  $\phi_h \in \tilde{V}_h(K)$ , as the solution of the local problems (on each polygon  $K$ ):

$$B_K(\Pi_K^{k, \Delta} \phi_h, q) = B_K(\phi_h, q) \quad \forall q \in \mathcal{P}_k(K), \quad (3.1a)$$

$$\widehat{\Pi_K^{k, \Delta} \phi_h} = \widehat{\phi_h}, \quad \widehat{\nabla \Pi_K^{k, \Delta} \phi_h} = \widehat{\nabla \phi_h}, \quad (3.1b)$$

where  $\widehat{(\cdot)}$  is defined as follows:

$$\widehat{\chi}_h := \frac{1}{N_K} \sum_{i=1}^{N_K} \chi_h(V_i) \quad \forall \chi_h \in C^0(\partial K), \quad (3.2)$$

and  $V_i, 1 \leq i \leq N_K$ , are the vertices of  $K$ .

The following result establishes that the projector  $\Pi_K^{k,\Delta}$  is computable using the output values of the sets  $\mathbf{D}_1 - \mathbf{D}_5$ .

**Lemma 3.1** *The operator  $\Pi_K^{k,\Delta} : \widetilde{V}_h(K) \rightarrow \mathcal{P}_k(K)$  is explicitly computable for every  $\phi_h \in \widetilde{V}_h(K)$ , using only the information of the linear operators  $\mathbf{D}_1 - \mathbf{D}_5$ .*

*Proof.* For detail proof, we refer to [21]. □

For each  $k \geq 2$  and for any  $K \in \mathcal{T}_h$  our local enhanced virtual space is given by:

$$W_h^k(K) := \left\{ \phi_h \in \widetilde{V}_h(K) : \int_K q^* \Pi_K^{k,\Delta} \phi_h = \int_K q^* \phi_h, \quad \forall q^* \in \mathcal{M}_{k-2}^*(K) \cup \mathcal{M}_{k-3}^*(K) \right\}, \quad (3.3)$$

where  $\mathcal{M}_{k-2}^*(K)$  and  $\mathcal{M}_{k-3}^*(K)$  are scaled monomials of degree  $k-2$  and  $k-3$ , respectively, with the convention that  $\mathcal{M}_{-1}^*(K) = \emptyset$ .

By using the linear operators  $\mathbf{D}_1 - \mathbf{D}_5$ , we can evaluate  $\Pi_K^{k,\Delta} \phi_h$  for all  $\phi_h \in W_h^k(K)$ , which is stated explicitly in the next result.

**Lemma 3.2** *The operator  $\Pi_K^{k,\Delta}$  is well defined and computable on the local enhanced virtual space  $W_h^k(K)$ .*

*Proof.* From definition of local enhanced virtual space (3.3), we deduce that  $W_h^k(K) \subseteq \widetilde{V}_h(K)$ . Consequently, for  $\phi_h \in W_h^k(K)$  implies  $\phi_h \in \widetilde{V}_h(K)$  and hence  $\Pi_K^{k,\Delta} \phi_h$  is well-defined and computable. □

On the other hand, we observe that  $\mathcal{P}_k(K) \subseteq W_h^k(K)$  which will guarantee the good approximation properties for the space. Moreover, following similar arguments presented in [16, 3] (see also [21]) we obtain that the sets of linear operators  $\mathbf{D}_1 - \mathbf{D}_5$  constitutes a set of degrees of freedom for  $W_h^k(K)$ .

Now, we introduce the global virtual space by combining the local spaces  $W_h^k(K)$  and incorporating the homogeneous Dirichlet boundary conditions. For every decomposition  $\mathcal{T}_h$  of  $\Omega$  into polygons  $K$ , we define

$$W_h := \{ \phi_h \in H_0^2(\Omega) : \phi_h|_K \in W_h^k(K) \}.$$

### 3.2 Construction of bilinear forms and the force term

In order to build the discrete local and global forms, we observe that the particular condition appearing in the definition of the local virtual space  $W_h^k(K)$  will be useful to construct an  $L^2$ -projection which will be employed to build the discrete bilinear forms. In particular, we consider the  $L^2(K)$ -projection onto  $\mathcal{P}_{k-2}(K)$ . For each  $\phi \in L^2(K)$ ,  $\Pi_K^{k-2} \phi \in \mathcal{P}_{k-2}(K)$  satisfies

$$\int_K (\Pi_K^{k-2} \phi) q = \int_K \phi q \quad \forall q \in \mathcal{P}_{k-2}(K). \quad (3.4)$$

The following lemma establishes that  $\Pi_K^{k-2}$  is computable on  $W_h^k(K)$ . The proof follows from the definition of the local virtual space and the set of degrees of freedom.

**Lemma 3.3** *The operator  $\Pi_K^{k-2} : W_h^k(K) \rightarrow \mathcal{P}_{k-2}(K)$  is explicitly computable for each  $\phi_h \in W_h^k(K)$ , using only the information of the set of degrees freedom  $\mathbf{D}_1 - \mathbf{D}_5$ .*

*Proof.* For a detail proof, we refer to [21].  $\square$

Now, for  $k \geq 2$ , we will introduce some additional projectors which will be used to write the virtual scheme. First, we define  $\Pi_K^{k,\nabla^\perp} : W_h^k(K) \rightarrow \mathcal{P}_k(K) \subseteq W_h^k(K)$  for each  $\phi_h \in W_h^k(K)$  as the solution of the following local problem.

$$\int_K \mathbf{curl} \Pi_K^{k,\nabla^\perp} \phi_h \cdot \mathbf{curl} q = \int_K \mathbf{curl} \phi_h \cdot \mathbf{curl} q \quad \forall q \in \mathcal{P}_k(K), \quad (3.5a)$$

$$\widehat{\Pi_K^{k,\nabla^\perp} \phi_h} = \widehat{\phi_h}, \quad (3.5b)$$

where  $\widehat{(\cdot)}$  has been defined in (3.2). The following result states that this operator is fully computable.

**Lemma 3.4** *The operator  $\Pi_K^{k,\nabla^\perp} : W_h^k(K) \rightarrow \mathcal{P}_k(K) \subseteq W_h^k(K)$  is explicitly computable for each  $\phi_h \in W_h^k(K)$ , using only the information of the set of degrees freedom  $\mathbf{D}_1 - \mathbf{D}_5$ .*

*Proof.* First we note that (3.5b) is computable using the information of the set  $\mathbf{D}_1$ . On the other hand, we integrate by parts on the right hand side of (3.5a) to obtain:

$$\begin{aligned} \int_K \mathbf{curl} \phi_h \cdot \mathbf{curl} q &= - \int_K \phi_h \Delta q + \int_{\partial K} \phi_h \partial_{\mathbf{n}_K} q \quad \forall q \in \mathcal{P}_k(K) \\ &= - \int_K \Pi_K^{k-2} \phi_h \Delta q + \int_{\partial K} \phi_h \partial_{\mathbf{n}_K} q \quad \forall q \in \mathcal{P}_k(K), \end{aligned}$$

where we have used the fact that  $\Delta q \in \mathcal{P}_{k-2}(K)$  and the definition of the projection  $\Pi_K^{k-2}$  (cf. (3.4)). The previous equality allows us to conclude that the polynomial  $\Pi_K^{k,\nabla^\perp} \phi_h$  can be explicitly computed from the degrees of freedom  $\mathbf{D}_1 - \mathbf{D}_5$ .  $\square$

Now, we will introduce an additional projection operator onto the polynomial space  $[\mathcal{P}_{k-1}(K)]^2$ , which will be used to construct a local approximation of  $A(\cdot, \cdot)$  and  $C(\cdot; \cdot, \cdot)$ . For  $K \in \mathcal{T}_h$ , and  $\mathbf{v} \in [L^2(K)]^2$ , we define  $\mathbf{\Pi}_K^{k-1} : [L^2(K)]^2 \rightarrow [\mathcal{P}_{k-1}(K)]^2$  by

$$\int_K \mathbf{\Pi}_K^{k-1} \mathbf{v} \cdot \mathbf{q} = \int_K \mathbf{v} \cdot \mathbf{q} \quad \forall \mathbf{q} \in [\mathcal{P}_{k-1}(K)]^2. \quad (3.6)$$

We observe that for any  $\phi_h \in W_h^k(K)$ , the vector functions  $\mathbf{\Pi}_K^{k-1} \mathbf{curl} \phi_h \in [\mathcal{P}_{k-1}(K)]^2$  and  $\mathbf{\Pi}_K^{k-1} \nabla \phi_h \in [\mathcal{P}_{k-1}(K)]^2$  can be explicitly computed from the degrees of freedom  $\mathbf{D}_1 - \mathbf{D}_5$ . In fact, for all  $K \in \mathcal{T}_h$  and for all  $\phi_h \in W_h^k(K)$ , using integration by parts on the right-hand side of (3.6) (with  $\mathbf{curl} \phi_h$  instead of  $\mathbf{v}$ ), we have (see [28])

$$\begin{aligned} \int_K \mathbf{curl} \phi_h \cdot \mathbf{q} &= \int_K \phi_h \operatorname{rot} \mathbf{q} - \int_{\partial K} \phi_h (\mathbf{q} \cdot \mathbf{t}_K) \quad \forall \mathbf{q} \in [\mathcal{P}_{k-1}(K)]^2 \\ &= \int_K (\Pi_K^{k-2} \phi_h) \operatorname{rot} \mathbf{q} - \int_{\partial K} \phi_h (\mathbf{q} \cdot \mathbf{t}_K) \quad \forall \mathbf{q} \in [\mathcal{P}_{k-1}(K)]^2, \end{aligned}$$

where we have used (3.4). The first term on the right-hand side above depends only on  $\Pi_K^{k-2} \phi_h$  and this depends only on the values of the degrees of freedom (see Lemma 3.3). The second term is an integral on the boundary of the element  $K$ , which is fully computable.



Next, we use the above projection operators to construct computable approximations of the continuous bilinear and trilinear forms, and for the right-hand side. First, let  $s_K^\Delta(\cdot, \cdot)$  and  $s_K^{\text{curl}}(\cdot, \cdot)$  be any symmetric positive definite bilinear forms to be chosen as to satisfy:

$$\begin{aligned} c_0 B_K(\phi_h, \phi_h) &\leq s_K^\Delta(\phi_h, \phi_h) \leq c_1 B_K(\phi_h, \phi_h) & \forall \phi_h \in W_h^k(K), \text{ with } \Pi_K^{k,\Delta} \phi_h = 0, \\ c_2 A_K(\phi_h, \phi_h) &\leq s_K^{\text{curl}}(\phi_h, \phi_h) \leq c_3 A_K(\phi_h, \phi_h) & \forall \phi_h \in W_h^k(K), \text{ with } \Pi_K^{k,\nabla^\perp} \phi_h = 0, \end{aligned} \quad (3.7)$$

with  $c_0, c_1, c_2$  and  $c_3$  are positive constants independent of  $h$  and  $K$ . From (3.7), we deduce that  $s_K^\Delta(\cdot, \cdot)$ , and  $s_K^{\text{curl}}(\cdot, \cdot)$  scale same as  $B_K(\cdot, \cdot)$  and  $A_K(\cdot, \cdot)$ , respectively.

On each element  $K$ , we define the local discrete bilinear forms

$$A_K^h(\cdot, \cdot) : W_h^k(K) \times W_h^k(K) \rightarrow \mathbb{R}, \quad B_K^h(\cdot, \cdot) : W_h^k(K) \times W_h^k(K) \rightarrow \mathbb{R}$$

as follow, for all  $\psi_h, \phi_h \in W_h^k(K)$

$$\begin{aligned} A_K^h(\psi_h, \phi_h) &:= A_K(\Pi_K^{k,\nabla^\perp} \psi_h, \Pi_K^{k,\nabla^\perp} \phi_h) + s_K^{\text{curl}}(\psi_h - \Pi_K^{k,\nabla^\perp} \psi_h, \phi_h - \Pi_K^{k,\nabla^\perp} \phi_h), \\ B_K^h(\psi_h, \phi_h) &:= B_K(\Pi_K^{k,\Delta} \psi_h, \Pi_K^{k,\Delta} \phi_h) + s_K^\Delta(\psi_h - \Pi_K^{k,\Delta} \psi_h, \phi_h - \Pi_K^{k,\Delta} \phi_h). \end{aligned}$$

It can be observed that the forms  $s_K^{\text{curl}}(\cdot, \cdot)$  and  $s_K^\Delta(\cdot, \cdot)$  reduce to zero when one of the two entries  $\phi_h$  or  $\psi_h$  is a polynomial function. Different computable form of the stabilizers are available in the literature [5, 16, 35]. However, we choose the following representation

$$\begin{aligned} s_K^{\text{curl}}(\phi_i - \Pi_K^{k,\nabla^\perp} \phi_i, \phi_j - \Pi_K^{k,\nabla^\perp} \phi_j) &:= \alpha_K^{\text{curl}} \sum_{z=1}^{N_K^{\text{dof}}} \text{dof}_z((I - \Pi_K^{k,\nabla^\perp})\phi_i) \text{dof}_z((I - \Pi_K^{k,\nabla^\perp})\phi_j), \\ s_K^\Delta(\phi_i - \Pi_K^{k,\Delta} \phi_i, \phi_j - \Pi_K^{k,\Delta} \phi_j) &:= \alpha_K^\Delta \sum_{z=1}^{N_K^{\text{dof}}} \text{dof}_z((I - \Pi_K^{k,\Delta})\phi_i) \text{dof}_z((I - \Pi_K^{k,\Delta})\phi_j), \end{aligned}$$

where  $\alpha_K^{\text{curl}}$  and  $\alpha_K^\Delta$  are the suitable constants, and  $N_K^{\text{dof}}$  denotes the number of degrees freedom of  $W_h^k(K)$ . Adding the local contribution, the global forms are defined as

$$A^h(\psi_h, \phi_h) := \sum_{K \in \mathcal{T}_h} A_K^h(\psi_h, \phi_h) \quad \forall \psi_h, \phi_h \in W_h, \quad (3.8)$$

$$B^h(\psi_h, \phi_h) := \sum_{K \in \mathcal{T}_h} B_K^h(\psi_h, \phi_h) \quad \forall \psi_h, \phi_h \in W_h. \quad (3.9)$$

The following result establishes the usual consistency and stability properties for the discrete local bilinear forms.

**Proposition 3.1** *For  $k \geq 2$ , the local bilinear forms  $A_K^h(\cdot, \cdot)$  and  $B_K^h(\cdot, \cdot)$  on each element  $K$  satisfy*

- *Consistency: for all  $h > 0$  and for all  $K \in \mathcal{T}_h$ , we have that*

$$A_K^h(q, \phi_h) = A_K(q, \phi_h) \quad \forall q \in \mathcal{P}_k(K), \quad \forall \phi_h \in W_h^k(K), \quad (3.10)$$

$$B_K^h(q, \phi_h) = B_K(q, \phi_h) \quad \forall q \in \mathcal{P}_k(K), \quad \forall \phi_h \in W_h^k(K). \quad (3.11)$$

- *Stability and boundedness: There exist positive constants  $\alpha_i, i = 1, \dots, 4$ , independent of  $K$ , such that:*

$$\alpha_1 A_K(\phi_h, \phi_h) \leq A_K^h(\phi_h, \phi_h) \leq \alpha_2 A_K(\phi_h, \phi_h) \quad \forall \phi_h \in W_h^k(K), \quad (3.12)$$

$$\alpha_3 B_K(\phi_h, \phi_h) \leq B_K^h(\phi_h, \phi_h) \leq \alpha_4 B_K(\phi_h, \phi_h) \quad \forall \phi_h \in W_h^k(K). \quad (3.13)$$

*Proof.* For further details, we refer to [21, 19, 36].  $\square$

We observe that from the symmetry of  $A^h(\cdot, \cdot)$  and  $B^h(\cdot, \cdot)$  and the stability conditions stated before imply the continuity of  $A^h$  and  $B^h$ . In fact, for all  $\psi_h, \phi_h \in W_h$ :

$$\begin{aligned} |A^h(\psi_h, \phi_h)| &\leq C_A \|\psi_h\|_{1,\Omega} \|\phi_h\|_{1,\Omega}, \\ |B^h(\psi_h, \phi_h)| &\leq C_B \|\psi_h\|_{2,\Omega} \|\phi_h\|_{2,\Omega}. \end{aligned} \quad (3.14)$$

The following result establishes that by virtue of (3.13), bilinear form  $B^h(\cdot, \cdot)$  is uniformly elliptic.

**Lemma 3.5** *There exists a constant  $\alpha > 0$ , independent of  $h$ , such that*

$$B^h(v_h, v_h) \geq \alpha \|v_h\|_{2,\Omega}^2 \quad \forall v_h \in W_h.$$

Now, we proceed to discretize the force function as follows

$$F_K^h(\cdot) : W_h^k(K) \rightarrow \mathbb{R}$$

such that

$$F_K^h(\phi_h) := \int_K \mathbf{\Pi}_K^{k-1} \mathbf{f} \cdot \mathbf{curl} \phi_h = \int_K \mathbf{f} \cdot \mathbf{\Pi}_K^{k-1} \mathbf{curl} \phi_h \quad \forall \phi_h \in W_h^k(K). \quad (3.15)$$

Globally, the force function  $F_K^h$  is defined as follows

$$F^h(\phi_h) := \sum_{K \in \mathcal{T}_h} F_K^h(\phi_h) \quad \forall \phi_h \in W_h. \quad (3.16)$$

### 3.3 Discretization of the nonlinear term

In this section, we would like to discretize the trilinear term associated with the problem (2.3). Using the projection operators  $\mathbf{\Pi}_K^{k,\Delta}$  and  $\mathbf{\Pi}_K^{k-1}$ , we discretize the trilinear term.

$$C_K^h(\cdot; \cdot, \cdot) : W_h^k(K) \times W_h^k(K) \times W_h^k(K) \rightarrow \mathbb{R}$$

such that

$$C_K^h(v_h; \psi_h, \phi_h) := \int_K (\Delta \mathbf{\Pi}_K^{k,\Delta} v_h) \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h \cdot \mathbf{\Pi}_K^{k-1} \nabla \phi_h \quad \forall v_h, \psi_h, \phi_h \in W_h^k(K).$$

The term is fully computable from the degrees of freedom. Globally, the trilinear term is defined as

$$C^h(v_h; \psi_h, \phi_h) := \sum_{K \in \mathcal{T}_h} C_K^h(v_h; \psi_h, \phi_h) \quad \forall v_h, \psi_h, \phi_h \in W_h. \quad (3.17)$$

Moreover, it can be shown that the discrete trilinear form  $C^h(\cdot; \cdot, \cdot)$  is uniformly bounded on  $W_h$ .

**Lemma 3.6** *Let  $C^h(v_h; \psi_h, \phi_h)$  be the trilinear form defined in (3.17). Then there exists a positive constant  $\hat{C}$  such that*

$$|C^h(v_h; \psi_h, \phi_h)| \leq \hat{C} \|v_h\|_{2,\Omega} \|\psi_h\|_{2,\Omega} \|\phi_h\|_{2,\Omega},$$

where  $\hat{C}$  is independent of mesh size  $h$ .

*Proof.* An application of boundedness of the projection operators  $\Pi_K^{k,\Delta}$  and  $\Pi_K^{k-1}$ , Hölder inequality, and Sobolev's embedding theorem yields the proof.  $\square$

Moreover, we have the following properties of  $C^h(\cdot; \cdot, \cdot)$

$$C^h(v_h; \phi_h, \phi_h) = 0 \quad \forall v_h, \phi_h \in W_h. \quad (3.18)$$

In addition, we observe that  $C^h(\cdot; \cdot, \cdot)$  can be extended to  $H_0^2(\Omega)$  only taking the projections of the continuous  $v, \psi, \phi \in H_0^2(\Omega)$ .

**Remark 3.1** *The discrete trilinear form  $C^h(\cdot; \cdot, \cdot)$  does not contain non-polynomial part or stabilizer. It is defined using the projection operators  $\Pi_K^{k,\Delta}$  and  $\Pi_K^{k-1}$  which are computable from the information provided by the degrees of freedom. With this definition, we will show that the semi-discrete and fully-discrete schemes are well-posed and we will obtain the corresponding error estimates.*

## 4 Discrete schemes and their well posedness

In this section we will introduce the semi-discrete and fully-discrete virtual element schemes for problem (2.5), by using the discrete forms introduced in Sections 3.2 and 3.3. We will also prove that under some assumptions on  $\nu$ , the fully-discrete scheme is well posed.

### 4.1 Semi-discrete formulation

The semi-discrete VEM formulation for the time dependent Navier-Stokes problem reads as follows. For all  $t > 0$ , find  $\psi_h \in L^2(0, T; W_h)$  such that

$$A^h(\partial_t \psi_h(t), \phi_h) + \nu B^h(\psi_h(t), \phi_h) + C^h(\psi_h(t); \psi_h(t), \phi_h) = F^h(\phi_h) \quad \forall \phi_h \in W_h. \quad (4.1)$$

Additionally, we set  $\psi_h(0) = \psi_I(0)$ , where  $\psi_I(0)$  is a suitable interpolation of  $\psi_0$  (see Proposition 5.2). In this section, we will discuss the well-posedness of semi-discrete scheme (4.1).

First, we observe that the matrix representation corresponding to the discrete bilinear form  $A^h(\cdot, \cdot)$  is positive definite and hence inverse exists. Further, let us assume that  $\mathbf{A}, \mathbf{B}$  be the matrix representation corresponding to the discrete forms  $A^h(\cdot, \cdot), B^h(\cdot, \cdot)$ , respectively. Therefore, problem (4.1) reduces to a system of nonlinear differential equations as follows

$$\mathbf{A} \frac{d\boldsymbol{\psi}_h}{dt} + \nu \mathbf{B} \boldsymbol{\psi}_h + \mathbf{C}(\boldsymbol{\psi}_h) = \mathbf{F} \quad (4.2)$$

$$\boldsymbol{\psi}_h(0) = \boldsymbol{\psi}_0, \quad (4.3)$$

where  $\boldsymbol{\psi}_h$  denotes the vector whose entries are the components in the basis of  $\psi_h$ . Moreover,  $\mathbf{C}(\boldsymbol{\psi}_h)$  is the matrix corresponding to the nonlinear term and  $\mathbf{F}$  be the right hand side load vector corresponding to the basis  $\phi_h$ . Before going into further details, we would like to prove that the nonlinear term, i.e.,  $C^h(\psi_h; \psi_h, \phi_h)$  satisfies Lipschitz's continuity condition. In this direction, let

$\psi_h^1, \psi_h^2$  be two elements in  $W_h$ . Then, we can write as

$$\begin{aligned} \left| C^h(\psi_h^1; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^2, \phi_h) \right| &= \left| C^h(\psi_h^1; \psi_h^1, \phi_h) - C(\psi_h^2; \psi_h^1, \phi_h) + C(\psi_h^2; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^2, \phi_h) \right| \\ &\leq \sum_{K \in \mathcal{T}_h} \left( \underbrace{\left| \int_K \Delta \Pi_K^{k,\Delta} (\psi_h^1 - \psi_h^2) \Pi_K^{k-1} \mathbf{curl} \psi_h^1 \cdot \Pi_K^{k-1} \nabla \phi_h \right|}_{:=L_1} \right. \\ &\quad \left. + \underbrace{\left| \int_K \Delta \Pi_K^{k,\Delta} \psi_h^2 \Pi_K^{k-1} \mathbf{curl} (\psi_h^1 - \psi_h^2) \cdot \Pi_K^{k-1} \nabla \phi_h \right|}_{:=L_2} \right). \end{aligned} \quad (4.4)$$

An application of Hölder inequality and using the continuity of  $\Pi_K^{k-1}$  with respect to  $L^4$ -norm and stability of  $\Pi_K^{k,\Delta}$ , we obtain

$$\sum_{K \in \mathcal{T}_h} L_1 \leq C \|\psi_h^1 - \psi_h^2\|_{2,\Omega} \|\mathbf{curl} \psi_h^1\|_{L^4(\Omega)} \|\nabla \phi_h\|_{L^4(\Omega)},$$

using Sobolev's embedding theorem, we obtain

$$\sum_{K \in \mathcal{T}_h} L_1 \leq C \|\psi_h^1 - \psi_h^2\|_{2,\Omega} \|\mathbf{curl} \psi_h^1\|_{1,\Omega} \|\nabla \phi_h\|_{1,\Omega}. \quad (4.5)$$

Using analogous arguments, we derive that

$$\sum_{K \in \mathcal{T}_h} L_2 \leq C \|\psi_h^2\|_{2,\Omega} \|\mathbf{curl} (\psi_h^1 - \psi_h^2)\|_{1,\Omega} \|\nabla \phi_h\|_{1,\Omega}. \quad (4.6)$$

Inserting (4.5) and (4.6) into (4.4), we can claim that the nonlinear term  $C^h(\psi_h; \psi_h, \phi_h)$  is Lipschitz continuous. Therefore, from Picard's Theorem of existence and uniqueness of system of differential equation, we can deduce that (4.2)-(4.3) has a unique solution.

## 4.2 Fully-discrete formulation

A classical backward Euler integration method is employed for the time discretization of (4.1) with time step  $\Delta t = T/N$ , where  $N$  is a positive integer. In addition, we introduce  $\psi_h^n := \psi_h(t_n)$  for  $n = 0, 1, 2, \dots, N$ . This results in the following fully discrete method: find  $\psi_h^n \in W_h$  such that

$$\begin{aligned} A^h \left( \frac{\psi_h^n - \psi_h^{n-1}}{\Delta t}, \phi_h \right) + \nu B^h(\psi_h^n, \phi_h) + C^h(\psi_h^n; \psi_h^n, \phi_h) &= F^h(\phi_h) \quad \forall \phi_h \in W_h, \\ \psi_h^0 &= \psi_I(0), \end{aligned} \quad (4.7)$$

where  $\psi_h^0 \in W_h$  is an initial approximation of  $\psi$  at  $t = 0$ . Next, we prove the well posedness of the fully-discrete scheme (4.7). In this direction, we first recollect Brouwer's fixed point theorem. Then, under certain assumption, we will show that the fully-discrete scheme has unique solution  $\psi_h^n$  and the solution is bounded, i.e,  $\|\psi_h^n\|_{2,\Omega} \leq \mathcal{R}$ , where  $\mathcal{R}$  is a positive constant which will be specified later.

**Theorem 4.1** (*Brouwer's Theorem*) *Let  $\mathcal{K}$  be a Banach space and let  $\mathcal{B} \subset \mathcal{K}$  be a compact and convex subset. If  $\mathcal{L} : \mathcal{B} \rightarrow \mathcal{B}$  is continuous, then  $\mathcal{L}$  has a fixed point.*

**Theorem 4.2** *Assume that*

$$\max \left\{ \frac{2\hat{C}^2}{\alpha_3^2\nu^2}, \frac{2\hat{C}^2 C_B^2}{\alpha_3^4\nu^2} \right\} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 < 1. \quad (4.8)$$

*Then, for  $1 \leq n \leq N$  and for sufficiently small values of  $\Delta t$ , there exists a unique solution of the fully-discrete problem (4.7) and the solution  $\psi_h^n$  satisfies the condition  $\|\psi_h^n\|_{2,\Omega} \leq \mathcal{R}$ , with*

$$\mathcal{R} := \frac{\left( \frac{\Delta t}{\alpha_1\alpha_3\nu} \|\mathbf{f}\|_{L^\infty(0,t_{n-1};L^2(\Omega))}^2 + \frac{2C_B^2}{\alpha_3^2} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 \right)^{1/2}}{\left( 1 - \frac{2\hat{C}^2}{\alpha_3^2\nu^2} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 \right)^{1/2}}.$$

*Proof.* Let  $\psi_h^{n-1} \in W_h$ . Define a mapping  $\mathcal{F} : W_h \rightarrow W_h$  such that  $\psi_h^n = \mathcal{F}(\xi_h)$  for all  $\xi_h \in W_h$ , where  $\psi_h$  is defined by

$$\frac{1}{\Delta t} A^h(\psi_h^n, \phi_h) + \nu B^h(\psi_h^n, \phi_h) + C^h(\xi_h; \psi_h^n, \phi_h) = F^h(\phi_h) + \frac{1}{\Delta t} A^h(\psi_h^{n-1}, \phi_h). \quad (4.9)$$

The proof of the result will be divided in three steps. We first define a mapping  $\mathcal{F}$  from  $W_h$  to  $W_h$  and prove that the mapping is well-defined and maps a ball  $\mathcal{B}_{\mathcal{R}}$  to a ball  $\mathcal{B}_{\mathcal{R}}$ . In second stage, we prove that the mapping is continuous. Then from *Brouwer's Theorem*, we deduce that  $\mathcal{F}$  has a fixed point inside the ball  $\mathcal{B}_{\mathcal{R}}$  which is the solution of the fully-discrete scheme (4.7). Finally, using assumption (4.8), we prove that the solution is unique.

**Well posedness of  $\mathcal{F}$ :** Since the bilinear form  $B^h(\phi_h, \phi_h)$  is elliptic (cf. Lemma 3.5), (3.18) and the fact that  $A^h(\phi_h, \phi_h) > 0$ , we have that problem (4.9) is well-posed, which follows from the Lax-Milgram Theorem. Further, to present the analysis we denote by  $\mathcal{D}_t \psi_h^n := \frac{\psi_h^n - \psi_h^{n-1}}{\Delta t}$ .

Now, we will construct a ball with radius  $\mathcal{R}$  say  $\mathcal{B}_{\mathcal{R}}$  such that  $\mathcal{F} : \mathcal{B}_{\mathcal{R}} \rightarrow \mathcal{B}_{\mathcal{R}}$ . We consider  $\phi_h = \mathcal{D}_t \psi_h^n$  in (4.9) and obtain

$$A^h(\mathcal{D}_t \psi_h^n, \mathcal{D}_t \psi_h^n) + \nu B^h(\psi_h^n, \mathcal{D}_t \psi_h^n) + C^h(\xi_h; \psi_h^n, \mathcal{D}_t \psi_h^n) = F^h(\mathcal{D}_t \psi_h^n).$$

An application of the stability property of the discrete bilinear forms (3.12)-(3.13), and (3.18) yields

$$\alpha_1 \|\mathbf{curl} \mathcal{D}_t \psi_h^n\|_{0,\Omega}^2 + \frac{\alpha_3\nu}{\Delta t} \|\psi_h^n\|_{2,\Omega}^2 \leq |F^h(\mathcal{D}_t \psi_h^n)| + \frac{\nu}{\Delta t} B^h(\psi_h^n, \psi_h^{n-1}) + \frac{1}{\Delta t} C^h(\xi_h; \psi_h^n, \psi_h^{n-1}). \quad (4.10)$$

By exploiting Cauchy-Schwarz inequality and boundedness of the  $L^2$ -projection operator  $\mathbf{\Pi}_K^{k-1}$ , we obtain

$$|F^h(\mathcal{D}_t \psi_h^n)| = \left| \sum_{K \in \mathcal{T}_h} \int_K \mathbf{\Pi}_K^{k-1} \mathbf{f} \cdot \mathbf{curl} \mathcal{D}_t \psi_h^n \right| \leq \|\mathbf{f}\|_{0,\Omega} \|\mathbf{curl} \mathcal{D}_t \psi_h^n\|_{0,\Omega}. \quad (4.11)$$

Using the continuity property of  $B^h(\cdot, \cdot)$  (cf. (3.14)), we get

$$|B^h(\psi_h^n, \psi_h^{n-1})| \leq C_B \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega}. \quad (4.12)$$

Inserting (4.11)-(4.12) in (4.10), and using boundedness of the trilinear term  $C^h(\cdot; \cdot, \cdot)$  (cf. Lemma 3.6), we obtain

$$\begin{aligned} \alpha_1 \|\mathbf{curl} \mathcal{D}_t \psi_h^n\|_{0,\Omega}^2 + \frac{\alpha_3\nu}{\Delta t} \|\psi_h^n\|_{2,\Omega}^2 &\leq \left( \|\mathbf{f}\|_{0,\Omega} \|\mathbf{curl} \mathcal{D}_t \psi_h^n\|_{0,\Omega} + \frac{C_B\nu}{\Delta t} \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega} \right. \\ &\quad \left. + \frac{\hat{C}}{\Delta t} \|\xi_h\|_{2,\Omega} \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega} \right). \end{aligned} \quad (4.13)$$

Upon employing Young's inequality, we obtain

$$\begin{aligned} \frac{\alpha_1}{2} \|\mathbf{curl} \mathcal{D}_t \psi_h^n\|_{0,\Omega}^2 + \frac{\alpha_3 \nu}{\Delta t} \|\psi_h^n\|_{2,\Omega}^2 &\leq \frac{1}{2\alpha_1} \|\mathbf{f}\|_{0,\Omega}^2 + \frac{C_{B\nu}}{\Delta t} \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega} \\ &+ \frac{\hat{C}}{\Delta t} \|\xi_h\|_{2,\Omega} \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega}. \end{aligned}$$

Since the term  $\frac{\alpha_1}{2} \|\mathbf{curl} \mathcal{D}_t \psi_h^n\|_{0,\Omega}^2$  is positive, we can neglect the term and obtain

$$\alpha_3 \nu \|\psi_h^n\|_{2,\Omega}^2 \leq \frac{\Delta t}{2\alpha_1} \|\mathbf{f}\|_{0,\Omega}^2 + C_{B\nu} \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega} + \hat{C} \|\xi_h\|_{2,\Omega} \|\psi_h^n\|_{2,\Omega} \|\psi_h^{n-1}\|_{2,\Omega}.$$

Exploiting Young's inequality and kick-back argument, we obtain

$$\begin{aligned} \frac{\alpha_3 \nu}{2} \|\psi_h^n\|_{2,\Omega}^2 &\leq \frac{\Delta t}{2\alpha_1} \|\mathbf{f}\|_{0,\Omega}^2 + \frac{C_{B\nu}^2}{\alpha_3} \|\psi_h^{n-1}\|_{2,\Omega}^2 + \frac{\hat{C}^2}{\alpha_3 \nu} \|\xi_h\|_{2,\Omega}^2 \|\psi_h^{n-1}\|_{2,\Omega}^2 \\ &\leq \frac{\Delta t}{2\alpha_1} \|\mathbf{f}\|_{L^\infty(0,t_{n-1};L^2(\Omega))}^2 + \left( \frac{C_{B\nu}^2}{\alpha_3} + \frac{\hat{C}^2}{\alpha_3 \nu} \|\xi_h\|_{2,\Omega}^2 \right) \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2. \end{aligned} \quad (4.14)$$

Moreover, we adopt that  $\|\xi_h\|_{2,\Omega} \leq \mathcal{R}$ . Then from (4.14), we derive as

$$\frac{\Delta t}{\alpha_1 \alpha_3 \nu} \|\mathbf{f}\|_{L^\infty(0,t_{n-1};L^2(\Omega))}^2 + \left( \frac{2C_{B\nu}^2}{\alpha_3^2} + \frac{2\hat{C}^2}{\alpha_3^2 \nu^2} \mathcal{R}^2 \right) \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 \leq \mathcal{R}^2. \quad (4.15)$$

Upon writing explicitly (4.15), we obtain

$$\mathcal{R}^2 = \frac{\left( \frac{\Delta t}{\alpha_1 \alpha_3 \nu} \|\mathbf{f}\|_{L^\infty(0,t_{n-1};L^2(\Omega))}^2 + \frac{2C_{B\nu}^2}{\alpha_3^2} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 \right)}{\left( 1 - \frac{2\hat{C}^2}{\alpha_3^2 \nu^2} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 \right)}.$$

Now, according to assumption (4.8), the term  $\left( 1 - \frac{2\hat{C}^2}{\alpha_3^2 \nu^2} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 \right)$  is positive and consequently, we define  $\mathcal{B}_{\mathcal{R}} := \{v_h \in W_h : \|v_h\|_{2,\Omega} \leq \mathcal{R}\}$ . Therefore, we deduce that  $\mathcal{F} : \mathcal{B}_{\mathcal{R}} \rightarrow \mathcal{B}_{\mathcal{R}}$  is well defined.

**Continuity of  $\mathcal{F}$ :** Let  $\varepsilon > 0$  be a small number and let  $\xi_h^*$ ,  $\xi_h$  be two elements in  $\mathcal{B}_{\mathcal{R}}$  such that  $\psi_h^* := \mathcal{F}(\xi_h^*)$  and  $\psi_h := \mathcal{F}(\xi_h)$  and  $\|\Delta(\xi_h^* - \xi_h)\|_{0,\Omega} < \delta$ . Then, from (4.9), we have

$$A^h(\psi_h - \psi_h^*, \phi_h) + \Delta t \nu B^h(\psi_h - \psi_h^*, \phi_h) + \Delta t \left( C^h(\xi_h; \psi_h, \phi_h) - C^h(\xi_h^*; \psi_h^*, \phi_h) \right) = 0. \quad (4.16)$$

Putting  $\phi_h = (\psi_h - \psi_h^*) \in W_h$  in (4.16), we obtain

$$A^h(\psi_h - \psi_h^*, \psi_h - \psi_h^*) + \Delta t \nu B^h(\psi_h - \psi_h^*, \psi_h - \psi_h^*) + \Delta t \left( C^h(\xi_h; \psi_h, \psi_h - \psi_h^*) - C^h(\xi_h^*; \psi_h^*, \psi_h - \psi_h^*) \right) = 0.$$

An application of stability property of  $A^h(\cdot, \cdot)$  and  $B^h(\cdot, \cdot)$  yields

$$\alpha_1 \|\mathbf{curl}(\psi_h - \psi_h^*)\|_{0,\Omega}^2 + \Delta t \nu \alpha_3 \|\Delta(\psi_h - \psi_h^*)\|_{0,\Omega}^2 \leq \Delta t |C^h(\xi_h^*; \psi_h^*, \psi_h - \psi_h^*) - C^h(\xi_h; \psi_h, \psi_h - \psi_h^*)|. \quad (4.17)$$

The difference on the right hand side above can be bound as follows:

$$\begin{aligned}
& \left| \sum_{K \in \mathcal{T}_h} \left( \int_K \Delta \Pi_K^{k,\Delta} \xi_h \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h \cdot \mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*) - \int_K \Delta \Pi_K^{k,\Delta} \xi_h^* \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h^* \cdot \mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*) \right) \right| \\
& \leq \sum_{K \in \mathcal{T}_h} \underbrace{\left| \left( \int_K \Delta \Pi_K^{k,\Delta} \xi_h \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h \cdot \mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*) - \int_K \Delta \Pi_K^{k,\Delta} \xi_h^* \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h \cdot \mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*) \right) \right|}_{=:T_1} \\
& \quad + \underbrace{\left| \left( \int_K \Delta \Pi_K^{k,\Delta} \xi_h^* \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h \cdot \mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*) - \int_K \Delta \Pi_K^{k,\Delta} \xi_h^* \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h^* \cdot \mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*) \right) \right|}_{=:T_2}.
\end{aligned}$$

Using Hölder's inequality and the boundedness of the projection operator  $\Pi_K^{k,\Delta}$ , the term  $T_1$  can be bounded as follows:

$$\begin{aligned}
|T_1| & \leq \|\Delta \Pi_K^{k,\Delta} \xi_h - \Delta \Pi_K^{k,\Delta} \xi_h^*\|_{0,K} \|\mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h\|_{L^4(K)} \|\mathbf{\Pi}_K^{k-1} \nabla(\psi_h - \psi_h^*)\|_{L^4(K)} \\
& \leq \|\Delta(\xi_h - \xi_h^*)\|_{0,K} \|\mathbf{curl} \psi_h\|_{L^4(K)} \|\nabla(\psi_h - \psi_h^*)\|_{L^4(K)}.
\end{aligned}$$

Summing the above inequality for all elements  $K$  and using the Sobolev's embedding theorem,

$$\sum_{K \in \mathcal{T}_h} |T_1| \leq \widehat{C} \|\Delta(\xi_h - \xi_h^*)\|_{0,\Omega} \|\mathbf{curl} \psi_h\|_{1,\Omega} \|\nabla(\psi_h - \psi_h^*)\|_{1,\Omega}. \quad (4.18)$$

Now, using (3.18), we deduce that the term  $T_2 = 0$ .

Inserting (4.18) into (4.17), we obtain

$$\alpha_1 \|\mathbf{curl}(\psi_h - \psi_h^*)\|_{0,\Omega}^2 + \Delta t \alpha_3 \nu \|\psi_h - \psi_h^*\|_{2,\Omega}^2 \leq \widehat{C} \Delta t \delta \|\mathbf{curl} \psi_h\|_{1,\Omega} \|\nabla(\psi_h - \psi_h^*)\|_{1,\Omega}.$$

Since the term  $\alpha_1 \|\mathbf{curl}(\psi_h - \psi_h^*)\|_{0,\Omega}^2$  is positive, we deduce that

$$\|\psi_h - \psi_h^*\|_{2,\Omega} \leq \frac{\widehat{C} \delta}{\alpha_3 \nu} \|\mathbf{curl} \psi_h\|_{1,\Omega}.$$

Hence, using that  $\psi_h \in \mathcal{B}_{\mathcal{R}}$ , we conclude that  $\|\psi_h - \psi_h^*\|_{2,\Omega} < \varepsilon$  if

$$\delta < \frac{\alpha_3 \nu \varepsilon}{\widehat{C} \mathcal{R}},$$

which implies that the function  $\mathcal{F}$  is continuous.

Therefore, we claim that the function  $\mathcal{F}$  has a fixed point  $\psi_h \in \mathcal{B}_{\mathcal{R}}$  from Brouwer's fixed point theorem (cf. Theorem 4.1) such that  $\psi_h = \mathcal{F}(\psi_h)$ . Hence the fully-discrete scheme (4.7) has a solution. Now we proceed to show that the solution is unique.

**Uniqueness of solution:** Let  $\psi_h^1, \psi_h^2 \in \mathcal{B}_{\mathcal{R}}$  be two solutions of (4.7). Then from (4.9), we have

$$A^h(\psi_h^1 - \psi_h^2, \phi_h) + \Delta t \nu B^h(\psi_h^1 - \psi_h^2, \phi_h) + \Delta t \left( C^h(\psi_h^1; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^2, \phi_h) \right) = 0. \quad (4.19)$$

To bound the nonlinear term in (4.19), we add and subtract  $C^h(\psi_h^2; \psi_h^1, \phi_h)$  in the above equality, to get

$$\begin{aligned}
C^h(\psi_h^1; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^2, \phi_h) & = C^h(\psi_h^1; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^1, \phi_h) \\
& \quad + C^h(\psi_h^2; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^2, \phi_h).
\end{aligned} \quad (4.20)$$

Using Hölder's inequality and Sobolev embedding theorem, the first two terms in (4.20) can be bounded as follows

$$\begin{aligned} |C^h(\psi_h^1; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^1, \phi_h)| &= \left| \sum_{K \in \mathcal{T}_h} \int_K \left( \Delta \Pi_K^{k, \Delta}(\psi_h^1 - \psi_h^2) \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi_h^1 \cdot \mathbf{\Pi}_K^{k-1} \nabla \phi_h \right) \right| \\ &\leq \widehat{C} \|\psi_h^1 - \psi_h^2\|_{2, \Omega} \|\mathbf{curl} \psi_h^1\|_{1, \Omega} \|\nabla \phi_h\|_{1, \Omega}. \end{aligned}$$

The last two terms on the right hand side of (4.20) can be written as follows,

$$C^h(\psi_h^2; \psi_h^1, \phi_h) - C^h(\psi_h^2; \psi_h^2, \phi_h) = C^h(\psi_h^2; \psi_h^1 - \psi_h^2, \phi_h).$$

Taking  $\phi_h = (\psi_h^1 - \psi_h^2) \in W_h$  in (4.19), we have that the right hand side above vanish (cf. (3.18)), and using the stability property of the discrete bilinear forms  $A^h(\cdot, \cdot)$  and  $B^h(\cdot, \cdot)$ , and Young's inequality, we obtain

$$\alpha_1 \|\mathbf{curl}(\psi_h^1 - \psi_h^2)\|_{0, \Omega}^2 + \alpha_3 \Delta t \nu \|\psi_h^1 - \psi_h^2\|_{2, \Omega}^2 \leq \Delta t \widehat{C} \|\psi_h^1 - \psi_h^2\|_{2, \Omega} \|\mathbf{curl} \psi_h^1\|_{1, \Omega} \|\nabla(\psi_h^1 - \psi_h^2)\|_{1, \Omega}.$$

An application of kick back arguments, we obtain

$$\alpha_1 \|\mathbf{curl}(\psi_h^1 - \psi_h^2)\|_{0, \Omega}^2 + \Delta t \left( \alpha_3 \nu - \widehat{C} \mathcal{R} \right) \|\psi_h^1 - \psi_h^2\|_{2, \Omega}^2 \leq 0.$$

Now, according to assumption (4.8), and for sufficiently small values of  $\Delta t$ ,  $\left( \alpha_3 \nu - \widehat{C} \mathcal{R} \right) > 0$ . Hence, we have  $\|\psi_h^1 - \psi_h^2\|_{2, \Omega} = 0$ ; therefore  $\psi_h^1 = \psi_h^2$ , and we conclude the proof.  $\square$

**Remark 4.1** In Theorem 4.2, we have proved the fully-discrete scheme (4.7) has unique solution based on certain feasible assumption on the viscosity  $\nu$  (cf. (4.8)) and for sufficiently small values of time-step  $\Delta t$ . In particular, for sufficiently large values of  $\nu$ , we have

$$\left( \alpha_3^2 \nu^2 - \frac{2 \widehat{C}^2 C_B^2}{\alpha_3^2} \|\psi_h\|_{L^\infty(0, t_{n-1}; H^2(\Omega))}^2 \right) > 0.$$

Then for sufficiently small values of  $\Delta t$ , we have

$$\nu \left( \alpha_3^2 \nu^2 - \frac{2 \widehat{C}^2 C_B^2}{\alpha_3^2} \|\psi_h\|_{L^\infty(0, t_{n-1}; H^2(\Omega))}^2 \right) > \frac{\widehat{C}^2}{\alpha_1 \alpha_3} \Delta t \|\mathbf{f}\|_{L^\infty(0, t_{n-1}; L^2(\Omega))}^2,$$

which implies

$$\left( \alpha_3 \nu - \widehat{C} \mathcal{R} \right) > 0.$$

## 5 Convergence analysis

In this section, we will derive *a priori* error estimation for the virtual element semi-discrete and fully-discrete schemes. With this aim, first we introduce a discrete energy projection operator  $\mathcal{S}_h : H_0^2(\Omega) \rightarrow W_h$ , which is defined as follows:

$$B^h(\mathcal{S}_h u, w_h) = B(u, w_h) \quad \forall w_h \in W_h. \quad (5.1)$$

Upon exploiting the energy projection operator  $\mathcal{S}_h$ , we will split the error of the stream-function as

$$\psi - \psi_h = \psi - \mathcal{S}_h \psi + \mathcal{S}_h \psi - \psi_h.$$



Next, we define

$$\begin{aligned}\rho_h &:= \psi - \mathcal{S}_h\psi, \\ \theta_h &:= \psi_h - \mathcal{S}_h\psi.\end{aligned}\tag{5.2}$$

In what follows, we will prove approximations properties for  $\mathcal{S}_h$ , thus the first term  $\rho_h$  will be easily bounded. Then, using the continuous problem (2.3) and the polynomial approximation properties, we bound the term  $\theta_h$ . In this regard, we introduce the polynomial approximation property and the interpolation operator  $\psi_I$  on the virtual element space  $W_h$ . Further, to derive the *a priori* error estimations of the semi-discrete and fully-discrete schemes, some additional results are needed which will be presented in the next subsection.

## 5.1 Preliminary results

We start with the following approximation result, on star-shaped polygons, which is derived by interpolation between Sobolev spaces (see for instance [28, Theorem I.1.4] from the analogous result for integer values of  $s$ ). We mention that this result has been stated in [6, Proposition 4.2] for integer values and follows from the classical Scott-Dupont theory (see [15] and [5, Proposition 3.1]):

**Proposition 5.1** *There exists a constant  $C > 0$ , independent of mesh size  $h$  but depends on mesh regularity parameter  $C_{\mathcal{T}}$  (Assumption 1) such that for every  $v \in H^\delta(K)$  there exists  $v_\pi \in \mathcal{P}_k(K)$ ,  $k \geq 0$  such that*

$$\|v - v_\pi\|_{\ell,K} \leq Ch_K^{\delta-\ell} \|v\|_{\delta,K} \quad 0 \leq \delta \leq k+1, \ell = 0, \dots, [\delta],$$

with  $[\delta]$  denoting largest integer equal or smaller than  $\delta \in \mathbb{R}$ .

Now, we present an interpolation result in the virtual space  $W_h$  (see [5, 12]).

**Proposition 5.2** *Assume A1–A2 are satisfied, then for all  $v \in H^\delta(K)$  there exist  $v_I \in W_h$  and  $C > 0$  independent of  $h$  such that*

$$\|v - v_I\|_{\ell,K} \leq Ch_K^{\delta-\ell} \|v\|_{\delta,K}, \quad \ell = 0, 1, 2, \quad 2 \leq \delta \leq k+1,$$

where  $C$  is independent of mesh size  $h$  but depends on mesh regularity parameter  $C_{\mathcal{T}}$  (Assumption 1).

Next, in order to prove the convergence of our method, we introduce the following broken  $H^\ell$ -seminorm ( $\ell = 1, 2$ ):

$$|v|_{\ell,h} := \left( \sum_{K \in \mathcal{T}_h} |v|_{\ell,K}^2 \right)^{1/2},$$

which is well defined for every  $v \in L^2(\Omega)$  such that  $v|_K \in H^\ell(K)$  for all polygon  $K \in \mathcal{T}_h$ .

In order to obtain the error estimates, we prove the following approximation properties of the discrete projection operator  $\mathcal{S}_h$  (cf. (5.1)).

**Lemma 5.1** *For each  $u \in H_0^2(\Omega) \cap H^{2+s}(\Omega)$ , with  $1/2 < s \leq k-1$ , there exists a unique function  $\mathcal{S}_h(u) \in W_h$ , such that the following approximation properties hold:*

1) *There exists a positive constant  $C$ , independent of  $h$ , such that*

$$\|u - \mathcal{S}_h(u)\|_{2,\Omega} \leq Ch^s \|u\|_{2+s,\Omega}.\tag{5.3}$$

2) There exist a positive constant  $C$  and  $\tilde{s} \in (1/2, 1]$ , independent of  $h$ , such that

$$\|u - \mathcal{S}_h(u)\|_{1,\Omega} \leq Ch^{\tilde{s}+s} \|u\|_{2+s,\Omega}. \quad (5.4)$$

*Proof.* We begin by proving 1). Since the bilinear form  $B^h(\cdot, \cdot)$  is bounded and coercive in  $W_h$  and the function  $B(u, \cdot)$  is continuous on  $W_h$  hence there exists a unique solution of the variational formulation (5.1).

Now, we proceed to prove the approximation results. In this direction, we rewrite the term  $u - \mathcal{S}_h(u)$  as follows.

$$u - \mathcal{S}_h(u) = u - u_I + u_I - \mathcal{S}_h(u),$$

where  $u_I \in W_h$  is the interpolation operator introduced in Proposition 5.2.

The estimation of  $u - u_I$  is known, hence we proceed to bound the term  $\xi_h := (\mathcal{S}_h(u) - u_I) \in W_h$ . Using coercivity (cf. Lemma 3.5), we have

$$\begin{aligned} \alpha |\xi_h|_{2,\Omega}^2 &\leq B^h(\xi_h, \xi_h) \\ &= B^h(\mathcal{S}_h(u), \xi_h) - B^h(u_I, \xi_h) \\ &= B(u, \xi_h) - \sum_{K \in \mathcal{T}_h} \left( B_K^h(u_I - u_\pi, \xi_h) + B_K^h(u_\pi, \xi_h) \right) \\ &= B(u, \xi_h) - \sum_{K \in \mathcal{T}_h} \left( B_K^h(u_I - u_\pi, \xi_h) + B_K(u_\pi, \xi_h) \right) \\ &= \sum_{K \in \mathcal{T}_h} \left( B_K^h(u_\pi - u_I, \xi_h) + B_K(u - u_\pi, \xi_h) \right), \end{aligned}$$

where we have added and subtracted  $u_\pi \in \mathcal{P}_k(K)$ ,  $k \geq 2$  and then we have used (3.11). Using the continuity property of the bilinear form  $B^h(\cdot, \cdot)$  and Cauchy-Schwartz inequality, we have

$$\alpha |\xi_h|_{2,\Omega}^2 \leq C (|u_\pi - u|_{2,h} + |u - u_I|_{2,\Omega}) |\xi_h|_{2,\Omega}.$$

Now, an application of the approximation properties of the interpolation operator and the projection operator  $u_I$  and  $u_\pi$ , respectively, we obtain

$$\alpha |\xi_h|_{2,\Omega} \leq Ch^s \|u\|_{2+s,\Omega}. \quad (5.5)$$

Thus, we have that

$$|u - \mathcal{S}_h(u)|_{2,\Omega} \leq Ch^s \|u\|_{2+s,\Omega}. \quad (5.6)$$

Now, we proceed to prove 2). Let  $\phi \in H_0^2(\Omega)$  be the solution of the auxiliary variational problem: find  $\phi$  such that

$$B(\phi, v) = \int_{\Omega} \nabla(u - \mathcal{S}_h(u)) \cdot \nabla v \quad \forall v \in H_0^2(\Omega), \quad (5.7)$$

where  $B(\cdot, \cdot)$  is the bilinear form defined in (2.6).

As a consequence of a classical regularity result for the biharmonic problem with its right hand side in  $H^{-1}(\Omega) := [H_0^1(\Omega)]'$  (cf. [29]), there exists  $\tilde{s} \in (1/2, 1]$  such that  $\phi \in H^{2+\tilde{s}}(\Omega)$  and

$$\|\phi\|_{2+\tilde{s},\Omega} \leq C \|u - \mathcal{S}_h(u)\|_{1,\Omega}. \quad (5.8)$$

Now, let  $\phi_I \in W_h$  be such that Proposition 5.2 holds true. Taking  $v := (u - \mathcal{S}_h(u)) \in H_0^2(\Omega)$  as test function in (5.7), using the symmetry of the bilinear form and adding and subtracting  $\phi_I$ ,

we obtain

$$\begin{aligned}
|u - \mathcal{S}_h(u)|_{1,\Omega}^2 &\leq B(u - \mathcal{S}_h(u), \phi) \\
&= B(u - \mathcal{S}_h(u), \phi - \phi_I) + B(u - \mathcal{S}_h(u), \phi_I) \\
&= B(u - \mathcal{S}_h(u), \phi - \phi_I) + B(u, \phi_I) - B(\mathcal{S}_h(u), \phi_I) \\
&= B(u - \mathcal{S}_h(u), \phi - \phi_I) + B^h(\mathcal{S}_h(u), \phi_I) - B(\mathcal{S}_h(u), \phi_I) \\
&=: T_1 + T_2,
\end{aligned} \tag{5.9}$$

where we have used the definition of the continuous (see (2.5)) and discrete problems (see (5.1)).

Now, we bound the terms  $T_1$  and  $T_2$  separately. We start with  $T_1$  as follows.

$$\begin{aligned}
T_1 &= B(u - \mathcal{S}_h(u), \phi - \phi_I) \leq \|u - \mathcal{S}_h(u)\|_{2,\Omega} \|\phi - \phi_I\|_{2,\Omega} \\
&\leq Ch^s (\|u\|_{2+s,\Omega}) h^{\bar{s}} \|\phi\|_{2+\bar{s},\Omega} \\
&\leq Ch^{\bar{s}+s} (\|u\|_{2+s,\Omega}) |u - \mathcal{S}_h(u)|_{1,\Omega},
\end{aligned} \tag{5.10}$$

where we have used the continuity of bilinear form  $B(\cdot, \cdot)$ , (5.6) and Proposition 5.2.

Now, we continue with the term  $T_2$  in (5.9). Let  $u_\pi \in \mathcal{P}_k(K)$ ,  $\phi_\pi \in \mathcal{P}_2(K)$  such that Proposition 5.1 holds true with respect to  $u$  and  $\phi$ , respectively. Using (3.11), we have

$$\begin{aligned}
T_2 &= \sum_{K \in \mathcal{T}_h} [B_K^h(\mathcal{S}_h(u) - u_\pi, \phi_I - \phi_\pi) + B_K(u_\pi - \mathcal{S}_h(u), \phi_I - \phi_\pi)] \\
&\leq \sum_{K \in \mathcal{T}_h} C \|\mathcal{S}_h(u) - u_\pi\|_{2,K} \|\phi_I - \phi_\pi\|_{2,K} \\
&\leq \sum_{K \in \mathcal{T}_h} C (\|\mathcal{S}_h(u) - u\|_{2,K} + \|u - u_\pi\|_{2,K}) (\|\phi_I - \phi\|_{2,K} + \|\phi - \phi_\pi\|_{2,K}) \\
&\leq \sum_{K \in \mathcal{T}_h} C (\|\mathcal{S}_h(u) - u\|_{2,K} + h_K^s \|u\|_{2+s,K}) (\|\phi_I - \phi\|_{2,K} + Ch_K^{\bar{s}} \|\phi\|_{2+\bar{s},K}), \\
&\leq C (\|\mathcal{S}_h(u) - u\|_{2,\Omega} + Ch^s \|u\|_{2+s,\Omega}) (Ch^{\bar{s}} \|\phi\|_{2+\bar{s},\Omega} + \|\phi_I - \phi\|_{2,\Omega}) \\
&\leq C (h^s \|u\|_{2+s,\Omega}) Ch^{\bar{s}} \|\phi\|_{2+\bar{s},\Omega} \\
&\leq Ch^{\bar{s}+s} (\|u\|_{2+s,\Omega}) |u - \mathcal{S}_h(u)|_{1,\Omega},
\end{aligned} \tag{5.11}$$

where we have used continuity of local bilinear forms  $B_K(\cdot, \cdot)$  and  $B_K^h(\cdot, \cdot)$ , Proposition 5.1, (5.6) and (5.8). Thus, (5.3) follows from (5.9), (5.10) and (5.11). The proof is complete.  $\square$

## 5.2 Error estimation for semi-discrete scheme

In this section, we will derive the error estimation for the semi-discrete scheme (cf. (4.1)). With this end, we state the following lemma.

**Lemma 5.2** *Let  $\psi(t) \in W$  be the solution of problem (2.5). Assume that  $\psi(t) \in H^{2+r}(\Omega)$ , for  $\frac{1}{2} < r \leq k-1$ , for almost all  $t \in [0, T]$ . Then there exists a positive generic constant  $C$ , which could be depended on mesh regularity Assumption 1, Sobolev regularity of the solution  $\psi$ , but independent of mesh size  $h$  such that*

$$|C(\psi; \psi, \phi_h) - C^h(\psi; \psi, \phi_h)| \leq Ch^r \left( |\psi|_{1+r,\Omega} + \|\psi\|_{2,\Omega} \right) |\psi|_{2+r,\Omega} \|\phi_h\|_{2,\Omega} \quad \forall \phi_h \in W_h. \tag{5.12}$$

*Proof.* Using the definition of the continuous nonlinear term  $C(\cdot; \cdot, \cdot)$  (see (2.4)) and discrete nonlinear term  $C^h(\cdot; \cdot, \cdot)$  (see (3.17)), we have that

$$\begin{aligned}
& C(\psi; \psi, \phi_h) - C^h(\psi; \psi, \phi_h) \\
&= \sum_{K \in \mathcal{T}_h} \left( \int_K \Delta \psi \mathbf{curl} \psi \cdot \nabla \phi_h - \int_K (\Delta \Pi_K^{k, \Delta} \psi) \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \nabla \phi_h \right), \\
&= \sum_{K \in \mathcal{T}_h} \left( \int_K \Delta \psi \mathbf{curl} \psi \cdot (I - \Pi_K^{k-1})(\nabla \phi_h) + \int_K \Delta \psi (I - \Pi_K^{k-1}) \mathbf{curl} \psi \cdot \Pi_K^{k-1} \nabla \phi_h \right. \\
&\quad \left. + \int_K \Delta (I - \Pi_K^{k, \Delta}) \psi \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \nabla \phi_h \right) \quad (5.13) \\
&=: D_1 + D_2 + D_3.
\end{aligned}$$

Next, we bound the term  $D_1$ , and we split it in two cases. First, we consider the case  $1/2 < r \leq 1$ .

$$\begin{aligned}
D_1 &= \sum_{K \in \mathcal{T}_h} \int_K \Delta \psi \mathbf{curl} \psi \cdot (I - \Pi_K^{k-1}) \nabla \phi_h \\
&\leq \sum_{K \in \mathcal{T}_h} \|\Delta \psi\|_{L^4(K)} \|\mathbf{curl} \psi\|_{L^4(K)} \|(I - \Pi_K^{k-1}) \nabla \phi_h\|_{0,K} \\
&\leq Ch \left( \sum_{K \in \mathcal{T}_h} \|\Delta \psi\|_{L^4(K)}^4 \right)^{1/4} \left( \sum_{K \in \mathcal{T}_h} \|\mathbf{curl} \psi\|_{L^4(K)}^4 \right)^{1/4} \left( \sum_{K \in \mathcal{T}_h} |\phi_h|_{2,K}^2 \right)^{1/2} \\
&\leq Ch \|\Delta \psi\|_{L^4(\Omega)} \|\mathbf{curl} \psi\|_{L^4(\Omega)} |\phi_h|_{2,\Omega} \\
&\leq Ch \|\psi\|_{2+r,\Omega} \|\mathbf{curl} \psi\|_{r,\Omega} |\phi_h|_{2,\Omega} \\
&\leq Ch \|\psi\|_{2+r,\Omega} \|\psi\|_{1+r,\Omega} |\phi_h|_{2,\Omega},
\end{aligned}$$

where we have used the Sobolev embedding  $H^r(\Omega) \hookrightarrow L^4(\Omega)$  for  $r \in (1/2, 1]$ . On the other hand, for the case  $1 \leq r \leq k-1$ , we proceed as follows. Using the orthogonality property of the projection operator  $\Pi_K^{k-1}$  on the polynomial function of degree  $k-1$  and Cauchy-Schwarz inequality, we bound the first term on the right hand side of (5.13).

$$\begin{aligned}
D_1 &= \sum_{K \in \mathcal{T}_h} \int_K \Delta \psi \mathbf{curl} \psi \cdot (I - \Pi_K^{k-1}) \nabla \phi_h = \sum_{K \in \mathcal{T}_h} \int_K (I - \Pi_K^{k-1})(\Delta \psi \mathbf{curl} \psi) (I - \Pi_K^{k-1}) \nabla \phi_h \\
&\leq \sum_{K \in \mathcal{T}_h} \|(I - \Pi_K^{k-1})(\Delta \psi \mathbf{curl} \psi)\|_{0,K} \|(I - \Pi_K^{k-1}) \nabla \phi_h\|_{0,K} \\
&\leq Ch^{r-1} |(\Delta \psi \mathbf{curl} \psi)|_{r-1,\Omega} \tilde{C} h |\phi_h|_{2,\Omega}.
\end{aligned}$$

An application of Hölder's inequality and Sobolev embedding theorem [11, Lemma 4.2] yields,

$$|(\Delta \psi \mathbf{curl} \psi)|_{r-1,\Omega} \leq C \|\Delta \psi\|_{W^{r-1,4}(\Omega)} \|\mathbf{curl} \psi\|_{W^{r-1,4}(\Omega)} \leq C \|\psi\|_{2+r,\Omega} \|\psi\|_{1+r,\Omega}.$$

Collecting the above inequalities, we obtain for  $r > 1/2$  that

$$D_1 = \sum_{K \in \mathcal{T}_h} \int_K \Delta \psi \mathbf{curl} \psi \cdot ((I - \Pi_K^{k-1}) \nabla \phi_h) \leq Ch^r \|\psi\|_{1+r,\Omega} \|\psi\|_{2+r,\Omega} \|\phi_h\|_{2,\Omega}. \quad (5.14)$$

Using Hölder's inequality, the term  $D_2$  in (5.13) can be bounded as follows

$$\sum_{K \in \mathcal{T}_h} \int_K \Delta \psi ((I - \Pi_K^{k-1}) \mathbf{curl} \psi) \cdot \Pi_K^{k-1} \nabla \phi_h \leq \sum_{K \in \mathcal{T}_h} C \|\Delta \psi\|_{0,K} \|(I - \Pi_K^{k-1}) \mathbf{curl} \psi\|_{L^4(K)} \|\Pi_K^{k-1} \nabla \phi_h\|_{L^4(K)}.$$

Using the continuity of  $\mathbf{\Pi}_K^{k-1}$  on the space  $L^4(K)$  and optimal approximation property of the polynomial projection operator, we have

$$\begin{aligned} \|(I - \mathbf{\Pi}_K^{k-1})\mathbf{curl} \psi\|_{L^4(K)} &\leq \|\mathbf{curl}(\psi - \psi_\pi)\|_{L^4(K)} + \|\mathbf{\Pi}_K^{k-1}(\mathbf{curl} \psi - \mathbf{curl} \psi_\pi)\|_{L^4(K)} \\ &\leq Ch^r |\psi|_{W^{r,4}(K)}. \end{aligned}$$

Also, the term can be bounded as

$$\|\mathbf{\Pi}_K^{k-1} \nabla \phi_h\|_{L^4(\Omega)} \leq \|\nabla \phi_h\|_{L^4(\Omega)} \leq C |\phi_h|_{2,\Omega},$$

where we have used that  $H^1(\Omega) \hookrightarrow L^4(\Omega)$ . Hence, by the Sobolev embedding  $H^{1+r}(\Omega) \hookrightarrow W^{r,4}(\Omega)$ , we can write

$$D_2 \leq \sum_{K \in \mathcal{T}_h} \left| \int_K \Delta \psi ((I - \mathbf{\Pi}_K^{k-1})\mathbf{curl} \psi) \cdot \mathbf{\Pi}_K^{k-1} \nabla \phi_h \right| \leq Ch^r \|\Delta \psi\|_{0,\Omega} |\psi|_{2+r,\Omega} |\phi_h|_{2,\Omega}. \quad (5.15)$$

Using the Hölder's inequality and suming over each element  $K$ , the third term of (5.13) can be bounded as follows

$$\begin{aligned} D_3 &= \sum_{K \in \mathcal{T}_h} \int_K \Delta (I - \mathbf{\Pi}_K^{k,\Delta}) \psi \mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi \cdot \mathbf{\Pi}_K^{k-1} \nabla \phi_h \\ &\leq \sum_{K \in \mathcal{T}_h} \int_K |(I - \mathbf{\Pi}_K^{k,\Delta}) \psi|_{2,K} \|\mathbf{\Pi}_K^{k-1} \mathbf{curl} \psi\|_{L^4(K)} \|\mathbf{\Pi}_K^{k-1} \nabla \phi_h\|_{L^4(K)}. \end{aligned}$$

Therefore, from the fact that  $\mathbf{\Pi}_K^{k,\Delta}$  is the projector defined by (3.1a), the continuity of  $\mathbf{\Pi}_K^{k-1}$  on the space  $L^4(K)$  and Sobolev's embedding theorem, we obtain

$$D_3 \leq Ch^r |\psi|_{2+r,\Omega} |\psi|_{2,\Omega} |\phi_h|_{2,\Omega}. \quad (5.16)$$

Finally, the proof follows by collecting all the estimations (5.14), (5.15), (5.16) and inserting into (5.13).  $\square$

The following theorem provides the error estimates for the semi-discrete virtual element scheme.

**Theorem 5.1** *Let  $\psi(t) \in W$  be the solution of problem (2.5) and let  $\psi_h(t) \in W_h$  be the solution of problem (4.1). Assume that  $\psi(t) \in H^{2+r}(\Omega)$ ,  $\partial_t \psi(t) \in H^{1+r}(\Omega)$ , for  $\frac{1}{2} < r \leq k-1$  and for almost all  $t \in [0, T]$ . In addition, assume that  $\psi, \psi_h \in \mathcal{K} := \left\{ v \in W : \frac{2\|v\|_{2,\Omega}}{\alpha_3 \nu} < 1 \right\}$ . Then there exists a positive generic constant  $C$ , which could be depended on mesh regularity Assumption 1, Sobolev regularity of the solution  $\psi$ ,  $\partial_t \psi$ ,  $\partial_{tt} \psi$  but independents of mesh size  $h$  such that the following estimation holds*

$$\begin{aligned} \|\psi - \psi_h\|_{L^\infty(0,t,H^1(\Omega))} + C \|\psi - \psi_h\|_{L^2(0,t,H^2(\Omega))} &\leq \|\psi(0) - \psi_h(0)\|_{1,\Omega} + C(C_A)h^r \left( \|\psi(0)\|_{1+r,\Omega} \right. \\ &\quad \left. + \|\mathbf{f}\|_{L^2(0,t,H^{r-1}(\Omega))} + \|\partial_t \psi\|_{L^2(0,t,H^{1+r}(\Omega))} + \|\psi\|_{L^2(0,t,H^{2+r}(\Omega))} \right). \end{aligned}$$

*Proof.* Upon applying the projection operator  $\mathcal{S}_h$ , we split the error as (see (5.2)):

$$\begin{aligned} \psi_h - \psi &:= \psi_h - \mathcal{S}_h \psi + \mathcal{S}_h \psi - \psi \\ &=: \theta_h + \rho_h. \end{aligned}$$

Since the estimation for  $\rho_h$  is known from the Lemma 5.1, we attempt to bound  $\theta_h$ .

Using the semi-discrete scheme (4.1), definition of the projection operator  $\mathcal{S}_h$  (5.1) and the continuous weak formulation (2.3), we obtain

$$\begin{aligned} A^h(\partial_t \theta_h(t), \phi_h) + \nu B^h(\theta_h(t), \phi_h) &= F^h(\phi_h) - C^h(\psi_h; \psi_h, \phi_h) - A^h(\partial_t \mathcal{S}_h \psi, \phi_h) - \nu B^h(\mathcal{S}_h \psi, \phi_h) \\ &= F^h(\phi_h) - F(\phi_h) - C^h(\psi_h; \psi_h, \phi_h) + C(\psi; \psi, \phi_h) \\ &\quad - A^h(\partial_t \mathcal{S}_h \psi, \phi_h) + A(\partial_t \psi, \phi_h) - \nu B^h(\mathcal{S}_h \psi, \phi_h) + \nu B(\psi, \phi_h) \end{aligned} \quad (5.17)$$

Now, we will bound all the terms on the right hand side above. We start with the nonlinear terms  $C(\psi; \psi, \phi_h) - C^h(\psi_h; \psi_h, \phi_h)$ . We rewrite the term as follows:

$$C(\psi; \psi, \phi_h) - C^h(\psi_h; \psi_h, \phi_h) = \underbrace{C(\psi; \psi, \phi_h) - C^h(\psi; \psi, \phi_h)}_{=: A_1} + \underbrace{C^h(\psi; \psi, \phi_h) - C^h(\psi_h; \psi_h, \phi_h)}_{=: A_2}.$$

The term  $A_1$  has been bounded in Lemma 5.2. Thus, we will bound the term  $A_2$ .

We rewrite the term  $A_2$  as follows

$$\begin{aligned} &C^h(\psi; \psi, \phi_h) - C^h(\psi_h; \psi_h, \phi_h) \\ &= \sum_{K \in \mathcal{T}_h} \left( \int_K \Delta \Pi_K^{k, \Delta} \psi \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h - \int_K \Delta \Pi_K^{k, \Delta} \psi_h \Pi_K^{k-1} \mathbf{curl} \psi_h \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h \right), \\ &= \sum_{K \in \mathcal{T}_h} \left( \int_K \Delta \Pi_K^{k, \Delta} \psi \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h - \int_K \Delta \Pi_K^{k, \Delta} \psi_h \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h \right. \\ &\quad \left. + \int_K \Delta \Pi_K^{k, \Delta} \psi_h \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h - \int_K \Delta \Pi_K^{k, \Delta} \psi_h \Pi_K^{k-1} \mathbf{curl} \psi_h \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h \right), \\ &= \sum_{K \in \mathcal{T}_h} \left( \int_K \Delta (\Pi_K^{k, \Delta} \psi - \Pi_K^{k, \Delta} \psi_h) \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h \right. \\ &\quad \left. + \int_K \Delta \Pi_K^{k, \Delta} \psi_h \Pi_K^{k-1} (\mathbf{curl} \psi - \mathbf{curl} \psi_h) \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h \right). \end{aligned} \quad (5.18)$$

Using Hölder's inequality, the first term on the right hand side of (5.18) can be bounded as follows,

$$\begin{aligned} &\sum_{K \in \mathcal{T}_h} \int_K \Delta (\Pi_K^{k, \Delta} \psi - \Pi_K^{k, \Delta} \psi_h) \Pi_K^{k-1} \mathbf{curl} \psi \cdot \Pi_K^{k-1} \mathbf{curl} \phi_h \\ &\leq C \sum_{K \in \mathcal{T}_h} \|\Delta \Pi_K^{k, \Delta} (\psi - \psi_h)\|_{0, K} \|\Pi_K^{k-1} \mathbf{curl} \psi\|_{L^4(K)} \|\Pi_K^{k-1} \mathbf{curl} \phi_h\|_{L^4(K)} \\ &\leq C \|\Delta \Pi_K^{k, \Delta} (\psi - \psi_h)\|_{0, \Omega} \|\Pi_K^{k-1} \mathbf{curl} \psi\|_{L^4(\Omega)} \|\Pi_K^{k-1} \mathbf{curl} \phi_h\|_{L^4(\Omega)} \\ &\leq C |\psi - \psi_h|_{2, \Omega} \|\mathbf{curl} \psi\|_{1, \Omega} \|\mathbf{curl} \phi_h\|_{1, \Omega} \\ &\leq C (|\psi - \mathcal{S}_h \psi|_{2, \Omega} + |\mathcal{S}_h \psi - \psi_h|_{2, \Omega}) \|\mathbf{curl} \psi\|_{1, \Omega} \|\mathbf{curl} \phi_h\|_{1, \Omega} \\ &\leq C (h^r |\psi|_{2+r, \Omega} + |\theta_h(t)|_{2, \Omega}) \|\psi\|_{2, \Omega} \|\phi_h\|_{2, \Omega}, \end{aligned} \quad (5.19)$$

where we have used the boundedness of  $\Pi_K^{k, \Delta}$ , Sobolev's embedding theorem and the approximation property of the operator  $\mathcal{S}_h$  (cf. (5.3)).

Using Hölder's inequality, we can bound the second term on the right hand side of (5.18) as

follows,

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} \int_K \Delta(\Pi_K^{k,\Delta} \psi_h) \mathbf{\Pi}_K^{k-1}(\mathbf{curl} \psi - \mathbf{curl} \psi_h) \mathbf{\Pi}_K^{k-1} \nabla \phi_h \\
& \leq \sum_{K \in \mathcal{T}_h} \|\Delta \Pi_K^{k,\Delta} \psi_h\|_{L^2(K)} \|\mathbf{\Pi}_K^{k-1}(\mathbf{curl} \psi - \mathbf{curl} \psi_h)\|_{L^4(K)} \|\mathbf{\Pi}_K^{k-1} \nabla \phi_h\|_{L^4(K)} \\
& \leq C |\psi_h|_{2,\Omega} \|\mathbf{curl} \psi - \mathbf{curl} \psi_h\|_{L^4(\Omega)} \|\nabla \phi_h\|_{L^4(\Omega)} \\
& \leq C |\psi_h|_{2,\Omega} \|\mathbf{curl} \psi - \mathbf{curl} \psi_h\|_{1,\Omega} \|\nabla \phi_h\|_{1,\Omega} \\
& \leq C |\psi_h|_{2,\Omega} (h^r |\psi|_{2+r,\Omega} + |\theta_h(t)|_{2,\Omega}) \|\phi_h\|_{2,\Omega},
\end{aligned} \tag{5.20}$$

where we have exploited Sobolev inequality and approximation property of  $\mathcal{S}_h$  (cf. (5.3)).

Using the polynomial consistency property of discrete bilinear form  $A^h(\cdot, \cdot)$  (cf. (3.10)), we can represent the time derivative term in (5.17) as

$$\begin{aligned}
-A^h(\partial_t \mathcal{S}_h \psi, \phi_h) + A(\partial_t \psi, \phi_h) &= A^h(-\mathcal{S}_h \partial_t \psi, \phi_h) + A(\partial_t \psi, \phi_h) \\
&+ \sum_{K \in \mathcal{T}_h} \left( A_K^h(\Pi_K^{k,\nabla^\perp} \partial_t \psi, \phi_h) - A_K(\Pi_K^{k,\nabla^\perp} \partial_t \psi, \phi_h) \right) \\
&= \sum_{K \in \mathcal{T}_h} A_K^h(\Pi_K^{k,\nabla^\perp} \partial_t \psi - \mathcal{S}_h \partial_t \psi, \phi_h) + A_K(\partial_t \psi - \Pi_K^{k,\nabla^\perp} \partial_t \psi, \phi_h) \\
&\leq C \sum_{K \in \mathcal{T}_h} |\Pi_K^{k,\nabla^\perp} \partial_t \psi - \mathcal{S}_h \partial_t \psi|_{1,K} |\phi_h|_{1,K} + |\partial_t \psi - \Pi_K^{k,\nabla^\perp} \partial_t \psi|_{1,K} |\phi_h|_{1,K} \\
&\leq C(C_A) h^r |\partial_t \psi|_{1+r,\Omega} |\phi_h|_{2,\Omega},
\end{aligned} \tag{5.21}$$

where we have used the Cauchy-Schwarz inequality, the approximation property of the projection operator  $\Pi_K^{k,\nabla^\perp}$  and Lemma 5.1.

Now, we move to bound the load term in (5.17). To bound this term (cf. (3.15)), we will exploit the approximation property of the projection operator  $\mathbf{\Pi}_K^{k-1}$ , which implies

$$\begin{aligned}
|F^h(\phi_h) - F(\phi_h)| &= \sum_{K \in \mathcal{T}_h} \left| \int_K (\mathbf{\Pi}_K^{k-1} \mathbf{f} - \mathbf{f}) \cdot \mathbf{curl} \phi_h \right| \\
&\leq \sum_{K \in \mathcal{T}_h} \|\mathbf{\Pi}_K^{k-1} \mathbf{f} - \mathbf{f}\|_{0,K} \|\mathbf{\Pi}_K^{k-1} \mathbf{curl} \phi_h - \mathbf{curl} \phi_h\|_{0,K} \\
&\leq C h^r |\mathbf{f}|_{r-1,\Omega} |\phi_h|_{2,\Omega}.
\end{aligned} \tag{5.22}$$

Now, inserting the estimations (5.19), (5.12), (5.20), (5.21), (5.22) into (5.17), choosing test function  $\phi_h = \theta_h(t)$ , using stability properties of the bilinear forms  $A^h(\cdot, \cdot)$  and  $B^h(\cdot, \cdot)$ , we derive

$$\begin{aligned}
\alpha_1 \frac{1}{2} \frac{d}{dt} \|\mathbf{curl} \theta_h(t)\|_{0,\Omega}^2 + \alpha_3 \nu \|\theta_h(t)\|_{2,\Omega}^2 &\leq C h^r |\mathbf{f}|_{r-1,\Omega} \|\theta_h(t)\|_{2,\Omega} + C(C_A) h^r |\partial_t \psi|_{1+r,\Omega} \|\theta_h(t)\|_{2,\Omega} \\
&+ C \left( h^r |\psi|_{2+r,\Omega} + \|\theta_h(t)\|_{2,\Omega} \right) \|\psi\|_{2,\Omega} \|\theta_h(t)\|_{2,\Omega} \\
&+ \|\psi_h\|_{2,\Omega} \left( h^r |\psi|_{2+r,\Omega} + \|\theta_h(t)\|_{2,\Omega} \right) \|\theta_h(t)\|_{2,\Omega}.
\end{aligned}$$

Upon applying kick-back argument, we obtain

$$\begin{aligned}
\alpha_1 \frac{1}{2} \frac{d}{dt} \|\mathbf{curl} \theta_h(t)\|_{0,\Omega}^2 + \left( \alpha_3 \nu - \|\psi\|_{2,\Omega} - \|\psi_h\|_{2,\Omega} \right) \|\theta_h(t)\|_{2,\Omega}^2 \\
\leq C(C_A) h^r \left( |\mathbf{f}|_{r-1,\Omega} + |\partial_t \psi|_{1+r,\Omega} + |\psi|_{2+r,\Omega} \right) \|\theta_h(t)\|_{2,\Omega}.
\end{aligned} \tag{5.23}$$

Since  $\psi, \psi_h \in \mathcal{K}$ , from the assumption of the theorem, and applying Young's inequality and taking integration on both sides of (5.23), we derive

$$\begin{aligned} \|\mathbf{curl} \theta_h(t)\|_{0,\Omega} + C(\alpha_3, \nu, \mathcal{R}) \|\theta_h(t)\|_{L^2(0,t,H^2(\Omega))} &\leq \|\mathbf{curl} \theta_h(0)\|_{0,\Omega} + C(C_A) h^r \left( |\mathbf{f}|_{L^2(0,t,H^{r-1}(\Omega))} \right. \\ &\quad \left. + |\partial_t \psi|_{L^2(0,t,H^{1+r}(\Omega))} + |\psi|_{L^2(0,t,H^{2+r}(\Omega))} \right), \end{aligned}$$

for almost all  $t \in (0, T]$ . Using the approximation properties of  $\mathcal{S}_h$  (Lemma 5.1), we derive

$$\begin{aligned} \|\psi - \psi_h\|_{L^\infty(0,t,H^1(\Omega))} + C(\alpha_3, \nu, \mathcal{R}) \|\psi - \psi_h\|_{L^2(0,t,H^2(\Omega))} &\leq \|\psi(0) - \psi_h(0)\|_{1,\Omega} + C(C_A) h^r \left( |\psi(0)|_{r+1,\Omega} \right. \\ &\quad \left. + |\mathbf{f}|_{L^2(0,t,H^{r-1}(\Omega))} + |\partial_t \psi|_{L^2(0,t,H^{1+r}(\Omega))} + |\psi|_{L^2(0,t,H^{2+r}(\Omega))} \right). \end{aligned}$$

The proof is complete.  $\square$

### 5.3 Error estimation for fully-discrete scheme

In this section, we would like to derive the error estimation for the fully-discrete scheme (cf. (4.7)). To derive the estimates, we first split the error as follows:

$$\psi_h^n - \psi(t_n) = \psi_h^n - \mathcal{S}_h \psi(t_n) + \mathcal{S}_h \psi(t_n) - \psi(t_n) = \theta_h^n + \rho_h^n,$$

where  $\theta_h^n := \psi_h^n - \mathcal{S}_h \psi(t_n)$  and  $\rho_h^n := \mathcal{S}_h \psi(t_n) - \psi(t_n)$ . Since the estimation for  $\rho_h^n$  is known from the Lemma 5.1, we will focus in bounding the term  $\theta_h^n$ .

**Theorem 5.2** *Let  $\psi_h^n \in W_h$  be the virtual element solution generated by (4.7), and  $\psi(t_n) \in W$  be the analytical solution of the problem (2.3) at time  $t = t_n$ . Assume that  $\psi(t_n), \psi_h^n \in \tilde{\mathcal{K}} := \{v \in \mathcal{B}_{\mathcal{R}} : \frac{2\mathcal{R}}{\alpha_3 \nu} < 1\}$ . Also, we assume that the Assumption 1 is satisfied on mesh regularity. Then, under the condition of the Theorem 5.1, there exists a positive generic constant  $C$  that depends on mesh regularity parameter, and Sobolev regularity of the exact solution  $\psi$ ,  $\partial_t \psi$ ,  $\partial_{tt} \psi$  and force function  $\mathbf{f}$  but independent of mesh size  $h$  and time steps  $\Delta t$  such that the following estimation holds*

$$\|\psi_h^n - \psi(t_n)\|_{1,\Omega} + \left( \Delta t \sum_{j=1}^n \|\psi_h^j - \psi(t_j)\|_{2,\Omega}^2 \right)^{1/2} \leq C(C_A, \alpha_3, \mathcal{R}, \psi, \partial_t \psi, \partial_{tt} \psi, \mathbf{f}) (h^r + \Delta t),$$

for  $\frac{1}{2} < r \leq k - 1$ .

*Proof.* Using the fully discrete scheme (4.7), weak formulation (2.3), and the biharmonic projection operator  $\mathcal{S}_h$ , we obtain

$$\begin{aligned} A^h \left( \frac{\theta_h^n - \theta_h^{n-1}}{\Delta t}, \phi_h \right) + \nu B^h(\theta_h^n, \phi_h) &= A^h \left( \frac{\psi_h^n - \psi_h^{n-1}}{\Delta t}, \phi_h \right) + \nu B^h(\psi_h^n, \phi_h) \\ &\quad - A^h \left( \frac{\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})}{\Delta t}, \phi_h \right) - \nu B^h(\mathcal{S}_h \psi(t_n), \phi_h) \\ &= F^h(\phi_h) - F(\phi_h) - C^h(\psi_h^n, \psi_h^n, \phi_h) + C(\psi(t_n); \psi(t_n), \phi_h) \\ &\quad + A(\partial_t \psi(t_n), \phi_h) - A^h \left( \frac{\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})}{\Delta t}, \phi_h \right). \end{aligned} \tag{5.24}$$



In order to derive the desired estimation, we will put  $\phi_h = \theta_h^n$  into (5.24).

$$\begin{aligned} A^h \left( \frac{\theta_h^n - \theta_h^{n-1}}{\Delta t}, \theta_h^n \right) + \nu B^h(\theta_h^n, \theta_h^n) &= F^h(\theta_h^n) - F(\theta_h^n) - C^h(\psi_h^n; \psi_h^n, \theta_h^n) + C(\psi(t_n); \psi(t_n), \theta_h^n) \\ &\quad + A(\partial_t \psi(t_n), \theta_h^n) - A^h \left( \frac{\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})}{\Delta t}, \theta_h^n \right). \end{aligned} \quad (5.25)$$

The nonlinear term can be rewritten as follows:

$$\begin{aligned} C(\psi(t_n); \psi(t_n), \theta_h^n) - C^h(\psi_h^n; \psi_h^n, \theta_h^n) &= C(\psi(t_n); \psi(t_n), \theta_h^n) - C^h(\psi(t_n); \psi(t_n), \theta_h^n) \\ &\quad + C^h(\psi(t_n); \psi(t_n), \theta_h^n) - C^h(\psi_h^n; \psi_h^n, \theta_h^n). \end{aligned} \quad (5.26)$$

Using analogous arguments as in the proof of the Theorem 5.1, we have

$$|C(\psi(t_n); \psi(t_n), \theta_h^n) - C^h(\psi(t_n); \psi(t_n), \theta_h^n)| \leq Ch^r |\psi(t_n)|_{2+r, \Omega} \|\psi(t_n)\|_{2, \Omega} \|\theta_h^n\|_{2, \Omega}. \quad (5.27)$$

Proceeding same as semi-discrete case, another term of (5.26) can be estimated as follows

$$\begin{aligned} |C^h(\psi(t_n); \psi(t_n), \theta_h^n) - C^h(\psi_h^n; \psi_h^n, \theta_h^n)| &\leq C \left( h^r |\psi(t_n)|_{2+r, \Omega} + \|\Delta \theta_h^n\|_{0, \Omega} \right) \|\mathbf{curl} \psi(t_n)\|_{1, \Omega} \|\mathbf{curl} \theta_h^n\|_{1, \Omega} \\ &\quad + \|\Delta \psi_h^n\|_{0, \Omega} \left( h^r |\psi(t_n)|_{2+r, \Omega} + \|\theta_h^n\|_{0, \Omega} \right) \|\nabla \theta_h^n\|_{1, \Omega}. \end{aligned} \quad (5.28)$$

Using the projection operator  $\Pi_K^{k, \Delta}$ , and polynomial consistency property of the discrete bilinear form  $A^h(\cdot, \cdot)$ , we obtain

$$\begin{aligned} A(\psi_t(t_n), \theta_h^n) - A^h \left( \frac{\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})}{\Delta t}, \theta_h^n \right) &= A \left( \psi_t(t_n) - \frac{\psi(t_n) - \psi(t_{n-1})}{\Delta t}, \theta_h^n \right) \\ &\quad + \sum_{K \in \mathcal{T}_h} A_K \left( \frac{\psi(t_n) - \psi(t_{n-1})}{\Delta t} - \left( \frac{\Pi_K^{k, \Delta}(\psi(t_n) - \psi(t_{n-1}))}{\Delta t} \right), \theta_h^n \right) \\ &\quad + \sum_{K \in \mathcal{T}_h} A_K^h \left( \left( \frac{\Pi_K^{k, \Delta}(\psi(t_n) - \psi(t_{n-1}))}{\Delta t} \right) - \frac{\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})}{\Delta t}, \theta_h^n \right) \\ &:= \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \quad (5.29)$$

Following the analogous arguments as [40, Theorem 3.3], we can write the term  $\Sigma_1$  as follows,

$$\begin{aligned} \Sigma_1 &\leq \frac{1}{\Delta t} |\psi(t_n) - \psi(t_{n-1}) - \Delta t \psi_t(t_n)|_{1, \Omega} |\theta_h^n|_{1, \Omega} \leq \frac{1}{\Delta t} \left| \int_{t_{n-1}}^{t_n} (s - t_{j-1}) \psi_{tt}(s) \right|_{1, \Omega} |\theta_h^n|_{1, \Omega} \\ &\leq \|\partial_{tt} \psi\|_{L^1(t_{n-1}, t_n; L^2(\Omega))} |\theta_h^n|_{1, \Omega}. \end{aligned}$$

Further, in order to bound  $\Sigma_2$ , we derive the following estimate

$$\begin{aligned} \Sigma_2 &\leq \frac{1}{\Delta t} \sum_{K \in \mathcal{T}_h} \|\mathbf{curl}(\psi(t_n) - \psi(t_{n-1})) - \mathbf{curl} \Pi_K^{k, \Delta}(\psi(t_n) - \psi(t_{n-1}))\|_{0, K} |\theta_h^n|_{1, K} \\ &\leq \frac{1}{\Delta t} Ch^r \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; H^{r+1}(\Omega))} |\theta_h^n|_{1, \Omega}. \end{aligned}$$

Adding and subtracting  $\mathbf{curl}(\psi(t_n) - \psi(t_{n-1}))$ , the term  $\Sigma_3$  (cf. (5.29)) can be written as

$$\begin{aligned}\Sigma_3 &\leq \frac{1}{\Delta t} C_A \sum_{K \in \mathcal{T}_h} \|\mathbf{curl}(\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})) - \mathbf{curl} \Pi_K^{k,\Delta}(\psi(t_n) - \psi(t_{n-1}))\|_{0,K} |\theta_h^n|_{1,K} \\ &\leq \frac{1}{\Delta t} C_A \sum_{K \in \mathcal{T}_h} \|\mathbf{curl}(\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})) - \mathbf{curl}(\psi(t_n) - \psi(t_{n-1}))\|_{0,K} |\theta_h^n|_{1,K} \\ &\quad + \|\mathbf{curl}(\psi(t_n) - \psi(t_{n-1})) - \mathbf{curl} \Pi_K^{k,\Delta}(\psi(t_n) - \psi(t_{n-1}))\|_{0,K} |\theta_h^n|_{1,K}.\end{aligned}$$

Using the approximation properties of the operator  $\mathcal{S}_h$ , we obtain

$$\begin{aligned}\sum_{K \in \mathcal{T}_h} \|\mathbf{curl}(\mathcal{S}_h \psi(t_n) - \mathcal{S}_h \psi(t_{n-1})) - \mathbf{curl}(\psi(t_n) - \psi(t_{n-1}))\|_{0,K} |\theta_h^n|_{1,K} \\ \leq Ch^r \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; H^{1+r}(\Omega))} |\theta_h^n|_{1,\Omega}.\end{aligned}$$

Analogously, we have

$$\begin{aligned}\sum_{K \in \mathcal{T}_h} \|\mathbf{curl}(\psi(t_n) - \psi(t_{n-1})) - \mathbf{curl} \Pi_K^{k,\Delta}(\psi(t_n) - \psi(t_{n-1}))\|_{0,K} |\theta_h^n|_{1,K} \\ \leq Ch^r \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; H^{1+r}(\Omega))} |\theta_h^n|_{1,\Omega}.\end{aligned}$$

Inserting the above estimates in (5.29), and then using it together with (5.27)-(5.28) into (5.25), and multiplying by  $\Delta t$ , we obtain that

$$\begin{aligned}\alpha_1 \|\mathbf{curl} \theta_h^n\|_{0,\Omega}^2 + \nu \Delta t \alpha_3 \|\theta_h^n\|_{2,\Omega}^2 &\leq C \Delta t h^r \|\mathbf{f}\|_{r-1,\Omega} \|\theta_h^n\|_{2,\Omega} + C \Delta t \left( h^r |\psi(t_n)|_{r+2,\Omega} + \|\theta_h^n\|_{2,\Omega} \right) \\ &\quad \times \|\psi(t_n)\|_{2,\Omega} \|\theta_h^n\|_{2,\Omega} + C \Delta t \|\psi_h^n\|_{2,\Omega} \left( h^r |\psi(t_n)|_{r+2,\Omega} + \|\theta_h^n\|_{2,\Omega} \right) \|\theta_h^n\|_{2,\Omega} \\ &\quad + C(C_A) \Delta t h^r \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; H^{r+1}(\Omega))} \|\mathbf{curl} \theta_h^n\|_{0,\Omega} + C \Delta t \|\partial_{tt} \psi\|_{L^1(t_{n-1}, t_n; L^2(\Omega))} \|\mathbf{curl} \theta_h^n\|_{0,\Omega} \\ &\quad + \alpha_1 \|\mathbf{curl} \theta_h^n\|_{0,\Omega} \|\mathbf{curl} \theta_h^{n-1}\|_{0,\Omega}.\end{aligned}$$

An application of the Young's inequality and kick-back argument yields

$$\begin{aligned}\frac{\alpha_1}{2} \|\mathbf{curl} \theta_h^n\|_{0,\Omega}^2 - \frac{\alpha_1}{2} \|\mathbf{curl} \theta_h^{n-1}\|_{0,\Omega}^2 + \Delta t (\nu \alpha_3 - \|\psi(t_n)\|_{2,\Omega} - \|\psi_h^n\|_{2,\Omega}) \|\theta_h^n\|_{2,\Omega}^2 \\ \leq Ch^r \left( \|\mathbf{f}\|_{r-1,\Omega} + |\psi(t_n)|_{r+2,\Omega} \right) \|\theta_h^n\|_{2,\Omega} + C(C_A) h^r \|\partial_t \psi\|_{L^1(t_{n-1}, t_n; H^{r+1}(\Omega))} \|\mathbf{curl} \theta_h^n\|_{0,\Omega} \\ + C \Delta t \|\partial_{tt} \psi\|_{L^1(t_{n-1}, t_n; L^2(\Omega))} \|\mathbf{curl} \theta_h^n\|_{0,\Omega}.\end{aligned}$$

Since  $\psi(t_n), \psi_h^n \in \tilde{\mathcal{K}}$ , from the assumption of the theorem, and using Young's inequality and iterating  $j = 1, \dots, n$ , we have

$$\begin{aligned}\frac{\alpha_1}{2} \|\mathbf{curl} \theta_h^n\|_{0,\Omega}^2 + \Delta t C(\nu, \alpha_3, \mathcal{R}) \sum_{j=1}^n \|\theta_h^j\|_{2,\Omega}^2 &\leq \frac{\alpha_1}{2} \|\mathbf{curl} \theta_h^0\|_{0,\Omega}^2 + C \Delta t^2 \|\partial_{tt} \psi\|_{L^1(0, t_n; L^2(\Omega))}^2 \\ &\quad + C(C_A) h^{2r} \left( \|\mathbf{f}\|_{L^\infty(0, t_n; H^{r-1}(\Omega))}^2 + \|\psi\|_{L^\infty(0, t_n; H^{r+2}(\Omega))}^2 + \|\partial_t \psi\|_{L^\infty(0, t_n; H^{r+1}(\Omega))}^2 \right) \\ &\leq C \|\psi_h^0 - \mathcal{S}_h(\psi(0))\|_{1,\Omega}^2 + C(C_A) h^{2r} \left( \|\mathbf{f}\|_{L^\infty(0, t_n; H^{r-1}(\Omega))}^2 + \|\psi\|_{L^\infty(0, t_n; H^{r+2}(\Omega))}^2 \right. \\ &\quad \left. + \|\partial_t \psi\|_{L^\infty(0, t_n; H^{r+1}(\Omega))}^2 \right) + C \Delta t^2 \|\partial_{tt} \psi\|_{L^1(0, t_n; L^2(\Omega))}^2 \\ &\leq C (\|\psi_I(0) - \psi(0)\|_{1,\Omega}^2 + \|\psi(0) - \mathcal{S}_h(\psi(0))\|_{1,\Omega}^2) + C \Delta t^2 \|\partial_{tt} \psi\|_{L^1(0, t_n; L^2(\Omega))}^2 \\ &\quad + C(C_A) h^{2r} \left( \|\mathbf{f}\|_{L^\infty(0, t_n; H^{r-1}(\Omega))}^2 + \|\psi\|_{L^\infty(0, t_n; H^{r+2}(\Omega))}^2 + \|\partial_t \psi\|_{L^\infty(0, t_n; H^{r+1}(\Omega))}^2 \right).\end{aligned}$$

Using the approximation properties for  $\mathcal{S}_h$ , and Proposition 5.2, we get

$$\begin{aligned} \frac{\alpha_1}{2} \|\mathbf{curl} \theta_h^n\|_{0,\Omega}^2 + \Delta t C(\nu, \alpha_3, \mathcal{S}) \sum_{j=1}^n \|\theta_h^j\|_{2,\Omega}^2 &\leq C \Delta t^2 \|\partial_{tt}\psi\|_{L^1(0,t_n;L^2(\Omega))}^2 \\ &+ C(C_A)h^{2r} \left( \|\psi(0)\|_{2+r,\Omega}^2 + \|\mathbf{f}\|_{L^\infty(0,t_n;H^{r-1}(\Omega))}^2 + \|\psi\|_{L^\infty(0,t_n;H^{r+2}(\Omega))}^2 \right. \\ &\left. + \|\partial_t\psi\|_{L^\infty(0,t_n;H^{r+1}(\Omega))}^2 \right). \end{aligned}$$

Finally, using the fact that  $\psi_h^n - \psi(t_n) = \theta_h^n + \rho_h^n$ , from the above estimation and approximation properties for  $\mathcal{S}_h$  (see Lemma 5.1), we have the desired thesis.  $\square$

**Remark 5.1** In Theorem 5.1, and Theorem 5.2, we have chosen that the analytical solution  $\psi$  at time  $t = t_n$  and fully discrete solution  $\psi_h^n$  belong to a bounded subset of  $\mathcal{B}_{\mathcal{R}}$ . To satisfy this condition, an additional condition has to be imposed on viscosity  $\nu$ , which leads

$$\max \left\{ \frac{2\hat{C}^2}{\alpha_3^2\nu^2}, \frac{2C_B^2(4 + \hat{C}^2)}{\alpha_3^4\nu^2} \right\} \|\psi_h\|_{L^\infty(0,t_{n-1};H^2(\Omega))}^2 < 1.$$

For sufficiently small values of  $\Delta t$  and  $\nu$  satisfying the above mentioned condition, we advocate a numerical approximation of (2.3) that converges optimally in both space and time variable. The primary advantage of this scheme is that the condition imposed on  $\nu$  is independent of  $1/\Delta t$  which ensures the robustness of the scheme for very small values of  $\Delta t$ .

## 6 Numerical experiments

In this section, we report the results of four numerical tests carried out with the fully-discrete virtual element scheme proposed in Section 4, in order to validate the theoretical results presented in Section 5. We have developed a MATLAB code that implements the fully-discrete scheme for  $k = 2$  and  $k = 3$ . For the time discretization, we employ a backward Euler scheme and for each time step, we use the Newton-Raphson method to solve the resulting nonlinear system, with maximum 10 iterations, a user specified tolerance  $\text{tol} := 10^{-8}$ , and taking  $\psi_h^{\text{in}} = 0$  as initial guess.

For our numerical tests, we have used different families of polygonal meshes (see Figure 1):

- $\mathcal{T}_h^1$ : Uniform triangular meshes;
- $\mathcal{T}_h^2$ : Trapezoidal meshes;
- $\mathcal{T}_h^3$ : Sequence of Centroidal Voronoi Tessellation;
- $\mathcal{T}_h^4$ : Distorted concave rhombic quadrilaterals.

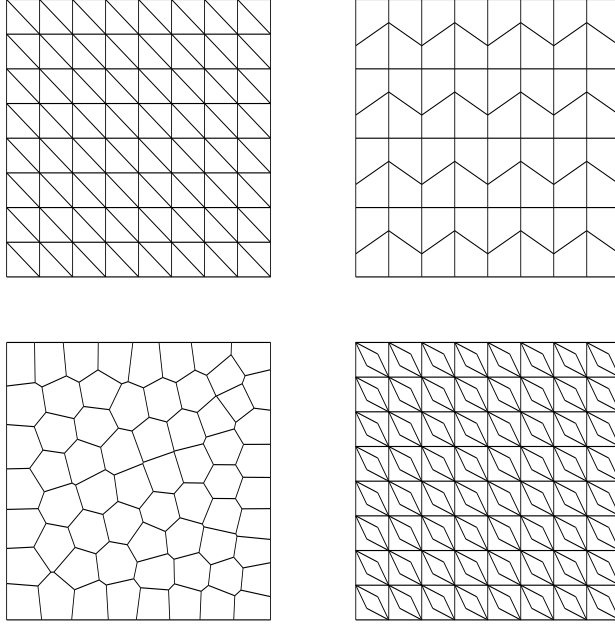


Figure 1: Sample meshes:  $\mathcal{T}_h^1$  (top left),  $\mathcal{T}_h^2$  (top right),  $\mathcal{T}_h^3$  (bottom left) and  $\mathcal{T}_h^4$  (bottom right).

In order to test the convergence properties of the VEM method, we measure the errors as the difference between the exact solution  $\psi$  and the suitable projections of the numerical solution  $\psi_h$ . More precisely, for the norm  $L^2(0, T; H^2(\Omega))$ , we consider the following quantity:

$$\mathbf{E}_2(\psi) := \mathbf{error}(\psi, L^2, H^2) = \left( \Delta t \sum_{n=1}^N \|\psi(t_n) - \Pi_K^{k, \Delta} \psi_h^n\|_{2, h}^2 \right)^{1/2}.$$

## 6.1 Test 1. Homogeneous Dirichlet boundary conditions and initial data.

In this numerical test, we solve the Navier-Stokes problem (2.1) on the square domain  $\Omega := (0, 1)^2$ . We take the load term  $\mathbf{f}$ , boundary and initial conditions in such a way that the analytical solution is given by:

$$\mathbf{u}(x, y, t) = 0.1 \begin{pmatrix} \sin(t) x^2(1-x)^2(2y-6y^2+4y^3) \\ -\sin(t) y^2(1-y)^2(2x-6x^2+4x^3) \end{pmatrix}, \quad p(x, y, t) = t^2 \left( x^3 y^3 - \frac{1}{16} \right),$$

$$\psi(x, y, t) = 0.1 \sin(t) x^2(1-x)^2 y^2(1-y)^2.$$

We consider the time interval  $[0, 1]$ , for the viscosity we consider the values  $\nu = 1$  and  $\nu = 10^{-3}$  and we start the process with  $h_0 = 1/8$  and  $\Delta t_0 = 1/16$ .

We report in Table 1 and Table 2 the errors  $\mathbf{E}_2(\psi)$  for different refinement levels and time steps and using the family of meshes  $\mathcal{T}_h^2$  for  $k = 2$  and the family  $\mathcal{T}_h^4$  for  $k = 3$ . Moreover, the maximum number of iterations that are required for the Newton method in this example is 4 for all the meshes and for  $k = 2, 3$ . It can be seen along the diagonal of Table 1, that the error  $\mathbf{E}_2(\psi)$  reduces linearly with respect to  $h$  and along the diagonal of Table 2 can be observed that the error  $\mathbf{E}_2(\psi)$

reduces quadratically with respect to  $h$ , which are the expected order of convergence for  $k = 2$  and  $k = 3$ , respectively. In addition, we have highlighted the errors which are dominated by space in Table 1 and Table 2 for small values of time-step  $\Delta t$ . We also observed that for big values of  $h$ , the error  $E_2(\psi)$  is almost constant with respect to  $\Delta t$ . Further, to examine the rate of convergence for the space variable, we have included convergence graph using all family of meshes in Figure 2, for  $k = 2$  and  $k = 3$ .

dof	$h$	$\Delta t_0$	$\Delta t_0/2$	$\Delta t_0/4$	$\Delta t_0/8$	$\Delta t_0/16$
147	$h_0$	1.076690e-3	1.055725e-3	1.045194e-3	1.039916e-3	1.037273e-3
675	$h_0/2$	5.450550e-4	5.344520e-4	5.291258e-4	5.264563e-4	5.251199e-4
2883	$h_0/4$	2.723653e-4	2.670638e-4	2.644015e-4	2.630672e-4	2.623994e-4
11907	$h_0/8$	1.361231e-4	1.334675e-4	1.321353e-4	1.314681e-4	1.311342e-4
48387	$h_0/16$	6.807611e-5	6.673602e-5	6.606687e-5	6.573250e-5	6.556536e-5

Table 1: Test 1. Errors for the stream-function  $\psi_h$  in the discrete  $L^2(0, T; H^2(\Omega))$  norm obtained with  $k = 2$ ,  $\nu = 1$  and  $\mathcal{T}_h^2$ .

dof	$h$	$\Delta t_0$	$\Delta t_0/2$	$\Delta t_0/4$	$\Delta t_0/8$	$\Delta t_0/16$
211	$h_0$	9.546949e-4	9.395878e-4	9.320480e-4	9.282763e-4	9.263893e-4
899	$h_0/2$	2.760235e-4	2.701636e-4	2.678057e-4	2.667518e-4	2.662540e-4
3715	$h_0/4$	8.270615e-5	7.376976e-5	7.135559e-5	7.064923e-5	7.041604e-5
15107	$h_0/8$	4.423838e-5	2.603643e-5	1.969686e-5	1.785950e-5	1.736430e-5
60931	$h_0/16$	4.077910e-5	1.976831e-5	1.021248e-5	6.047226e-6	4.554491e-6

Table 2: Test 1. Errors for the stream function  $\psi_h$  in the discrete  $L^2(0, T; H^2(\Omega))$  norm obtained with  $k = 3$ ,  $\nu = 10^{-3}$  and  $\mathcal{T}_h^4$ .

## 6.2 Test 2. Non-homogeneous Dirichlet boundary conditions and initial conditions.

In this numerical test, we solve the Navier-Stokes problem (2.1) on the square domain  $\Omega := (0, 1)^2$ . We take the load term  $\mathbf{f}$ , non-homogeneous Dirichlet boundary and initial conditions in such a way that the analytical solution is given by:

$$\mathbf{u}(x, y, t) = 0.1\pi \begin{pmatrix} -\exp(0.1t)(1 + \cos(\pi x)) \sin(\pi y) \\ \exp(0.1t)(1 + \cos(\pi y)) \sin(\pi x) \end{pmatrix}, \quad p(x, y, t) = \exp(-t) \left( x^2 + y^2 - \frac{2}{3} \right),$$

$$\psi(x, y, t) = 0.1 \exp(0.1t)(1 + \cos(\pi x))(1 + \cos(\pi y)).$$

In this test, we consider the time interval to  $[0, 1]$ , the viscosity  $\nu = 10^{-3}$  and we start with  $h_0 = 1/4$  and  $\Delta t_0 = 1/16$ .

We report in Table 3 the errors  $E_2(\psi)$  for the family of meshes  $\mathcal{T}_h^4$  and different refinement levels and time steps. In this example, the maximum number of iterations that are required for the Newton method is 5. Once again, it can be seen along the diagonal of Table 3 that the error  $E_2(\psi)$  reduces linearly with respect to  $h$ , which is the expected order of convergence for  $k = 2$ . In addition, we have highlighted the errors which are dominated by space in Table 3 for small values of time-step  $\Delta t$ .

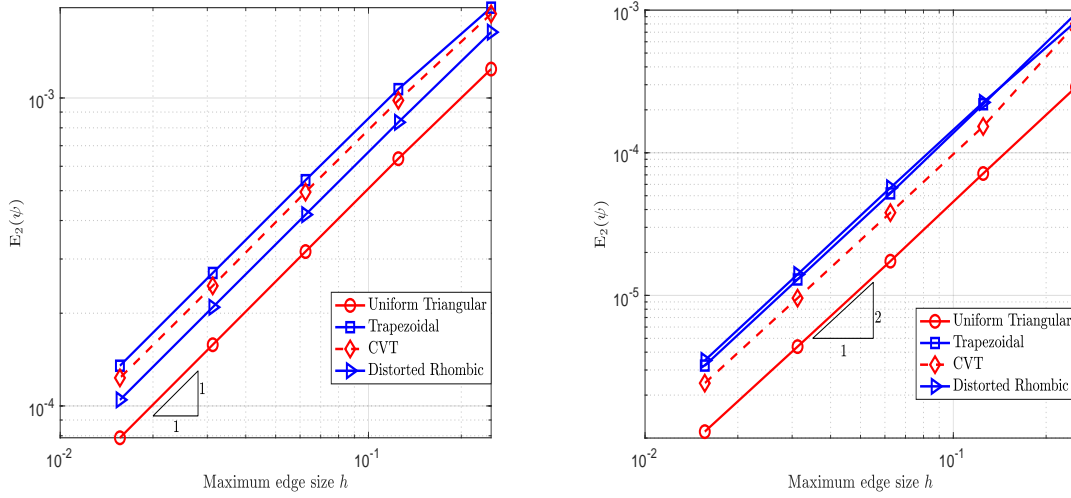


Figure 2: Test 1: Rate of convergence for space variable for different meshes with meshsize  $h = 1/4, \dots, 1/64$ ,  $\nu = 1$  and for small values of time-step  $\Delta t$ . For  $k = 2$  and  $\Delta t = 0.05$  (left panel) and  $k = 3$  and  $\Delta t = 0.0025$  (right panel).

dof	$h$	$\Delta t_0$	$\Delta t_0/2$	$\Delta t_0/4$	$\Delta t_0/8$	$\Delta t_0/16$
123	$h_0$	3.312118e-1	3.307043e-1	3.304508e-1	3.303242e-1	3.302609e-1
531	$h_0/2$	1.674332e-1	1.672076e-1	1.670951e-1	1.670389e-1	1.670108e-1
2211	$h_0/4$	8.358238e-2	8.345864e-2	8.339822e-2	8.336843e-2	8.335366e-2
9027	$h_0/8$	4.229862e-2	4.222367e-2	4.218726e-2	4.216934e-2	4.216047e-2
36483	$h_0/16$	2.163267e-2	2.160239e-2	2.158950e-2	2.158367e-2	2.158092e-2

Table 3: Test 2. Errors for the stream-function  $\psi_h$  in the discrete  $L^2(0, T; H^2(\Omega))$  norm obtained with  $\mathcal{T}_h^4$  and  $\nu = 10^{-3}$ .

### 6.3 Test 3. Example with dominating time error

In the present numerical test, we study the order of convergence of the virtual scheme in time. We solve the Navier-Stokes problem (2.1) on the square domain  $\Omega := (0, 1)^2$ . We take the load term  $\mathbf{f}$ , boundary and initial conditions in such a way that the analytical solution is given by:

$$\mathbf{u}(x, y, t) = \begin{pmatrix} \sin(10\pi t) x^2(1-x)^2(2y-6y^2+4y^3) \\ -\sin(10\pi t) y^2(1-y)^2(2x-6x^2+4x^3) \end{pmatrix},$$

$$p(x, y, t) = -\left(x^3 + y^3 - \frac{1}{2}\right) \left(\frac{3}{2} + \frac{1}{2} \sin(10\pi t)\right), \quad \psi(x, y, t) = \sin(10\pi t) x^2(1-x)^2 y^2(1-y)^2.$$

We consider the time interval  $[0, 1]$ , the viscosity  $\nu = 1$  and we start the process with  $h_0 = 1/8$  and  $\Delta t_0 = 1/8$ .

We report in Table 4 the errors  $E_2(\psi)$  obtained for  $k = 2$  and using the family of meshes  $\mathcal{T}_h^1$  with different refinement levels and time steps. In this experiment, differently to the first and second tests, we can observe that, for small values of  $h$ , the error  $E_2(\psi)$  reduces linearly with respect to  $\Delta t$  (see the last row of Table 4), which is the expected order of convergence in time according to

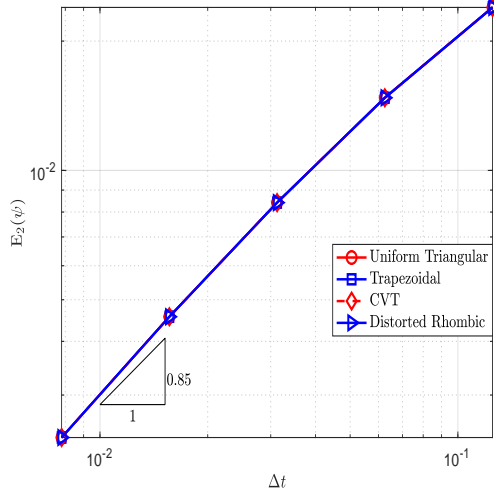


Figure 3: Rate of convergence for time variable for different meshes with time-steps  $\Delta t = 1/8, \dots, 1/128$  and meshsize  $h = 1/64$ ,  $\nu = 1$  and  $k = 3$ .

Theorem 5.2. We also observe that for big values of  $\Delta t$  the error  $E_2(\psi)$  is almost constant with respect to  $h$ . In this numerical test, the maximum number of iterations that are required for the Newton method is 3.

dof	$h$	$\Delta t_0$	$\Delta t_0/2$	$\Delta t_0/4$	$\Delta t_0/8$	$\Delta t_0/16$
147	$h_0$	2.556643e-2	1.851436e-2	1.439994e-2	1.256422e-2	1.186603e-2
675	$h_0/2$	2.437394e-2	1.579445e-2	1.025426e-2	7.432721e-3	6.296528e-3
2883	$h_0/4$	2.406341e-2	1.502749e-2	8.911963e-3	5.428423e-3	3.772418e-3
11907	$h_0/8$	2.398373e-2	1.482783e-2	8.541759e-3	4.795144e-3	2.800171e-3
48387	$h_0/16$	2.396363e-2	1.477733e-2	8.446522e-3	4.623010e-3	2.498059e-3

Table 4: Test 3. Errors and experimental rates for the stream-function  $\psi_h$  in norm  $L^2(0, T; H^2(\Omega))$  obtained with the meshes  $\mathcal{T}_h^1$ ,  $k = 2$  and  $\nu = 1$ .

Further, in Figure 3, we have posted a convergence graph where errors  $E_2(\psi)$  are dominated by time and virtual element space of order  $k = 3$  is chosen. We have used all family of meshes. We deduce that the rate of convergence is closer to 1 (*expected order of convergence for time variable*) for very small values of  $h$ .

#### 6.4 Test 4. Chorin problem

In this example, we consider the well-known Chorin problem for incompressible Navier-Stokes equations [22]. For this test the load term is  $\mathbf{f} = \mathbf{0}$ , and the initial and boundary conditions correspond to the analytical solution:

$$\mathbf{u}(x, y, t) = \begin{pmatrix} -\cos(2\pi x) \sin(2\pi y) e^{-8\pi^2 \nu t} \\ \sin(2\pi x) \cos(2\pi y) e^{-8\pi^2 \nu t} \end{pmatrix}, \quad p(x, y, t) = -\frac{1}{4} (\cos(4\pi x) + \cos(4\pi y)) e^{-8\pi^2 \nu t},$$

$$\psi(x, y, t) = \frac{1}{2\pi} \cos(2\pi x) \cos(2\pi y) e^{-8\pi^2 \nu t}.$$

It has been observed that some finite element methods for a velocity-pressure formulation, the  $L^2$  error of the velocity converge suboptimally or even lock for small values of the viscosity (see [32, Section 4.2]). It can be seen that the  $L^2$  error of the velocity is related with the  $H^1$  error for the stream-function. Thus, in order to assess the performance of the virtual scheme for this numerical example, we introduce the following discrete quantity:

$$\mathbf{E}_1(\psi) := \mathbf{error}(\psi, L^2, H^1) = \left( \Delta t \sum_{n=1}^N |\psi(t_n) - \Pi_K^{k,\Delta} \psi_h^n|_{1,h}^2 \right)^{1/2}.$$

We observe that an additional order of convergence is expected in this discrete error. To show this fact, we will compute experimental rates of convergence for each individual error as follows:

$$\mathbf{r}_i(\psi) := \frac{\log(\mathbf{E}_i(\psi)/\mathbf{E}'_i(\psi))}{\log(h/h')}, \quad i = 1, 2,$$

where  $h, h'$  denote two consecutive mesh sizes with their respective errors  $\mathbf{E}_i$  and  $\mathbf{E}'_i$ .

We report in Table 5 the discrete errors  $\mathbf{E}_1(\psi)$  and  $\mathbf{E}_2(\psi)$ , for the family of meshes  $\mathcal{T}_h^3$ . The results were obtained by considering the final time  $T = 0.01$  and time stepping  $\Delta t = 0.001$ . For the viscosity  $\nu$ , we take two values:  $\nu = 10^{-3}$  and  $\nu = 10^{-6}$ . The maximum number of iterations that are required for the Newton method in this example is 4 for  $\nu = 10^{-3}$  and 5 for  $\nu = 10^{-6}$ .

$\nu$	dof	$h$	$\mathbf{E}_1(\psi)$	$\mathbf{r}_1(\psi)$	$\mathbf{E}_2(\psi)$	$\mathbf{r}_2(\psi)$
$10^{-3}$	54	1/4	2.589181e-2	—	3.689706e-1	—
	294	1/8	6.856293e-3	1.916	1.979657e-1	0.898
	1371	1/16	1.796994e-3	1.931	1.012519e-1	0.967
	5796	1/32	4.332627e-4	2.052	5.059189e-2	1.000
	23874	1/64	1.078390e-4	2.006	2.519974e-2	1.005
$10^{-6}$	54	1/4	2.590922e-2	—	3.691334e-1	—
	294	1/8	6.858939e-3	1.917	1.980514e-1	0.898
	1371	1/16	1.796887e-3	1.932	1.012955e-1	0.967
	5796	1/32	4.328584e-4	2.053	5.061318e-2	1.000
	23874	1/64	1.074008e-4	2.010	2.520898e-2	1.005

Table 5: Test 4. Errors and experimental rates for the stream-function  $\psi_h$ , using the meshes  $\mathcal{T}_h^3$ , and with different values of  $\nu$ .

It can be clearly observed from Table 5 a linear order of convergence in the norm  $\mathbf{E}_2(\psi)$  and a quadratic order in the norm  $\mathbf{E}_1(\psi)$  (which has not been proved). Thus, we conclude that our virtual scheme does not suffer from a suboptimal order of convergence or locking phenomenon.

Exact and approximate solutions (including a postprocessed velocity field) are illustrated in Figure 4.



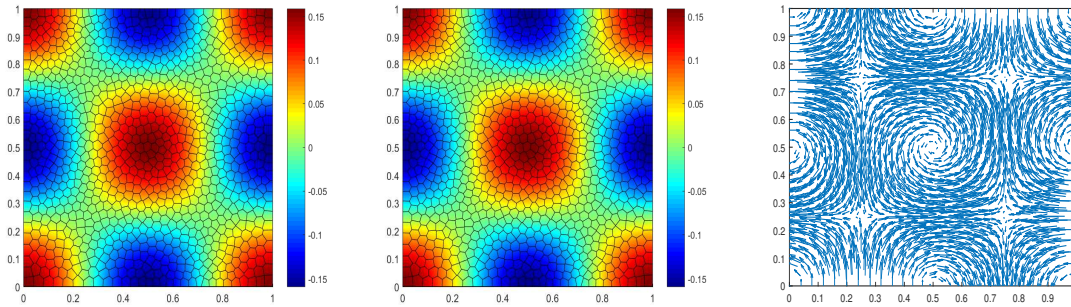


Figure 4: Test 4. Exact and approximate solutions  $\psi$ ,  $\psi_h$  and the postprocessed velocity field  $\mathbf{u}_h := \text{curl } \psi_h$ , using  $\Delta t = 0.001$  for  $T = 0.01$ , using the mesh  $\mathcal{T}_h^3$  with  $h = 1/32$  and  $\nu = 10^{-6}$ .

## Conclusion

In this article, we have proposed a  $C^1$  VEM to discretize the time-dependent Navier-Stokes problem in the stream-function form. Exploiting enhanced virtual element spaces, we have approximated the spatial variables and we have discretized the time variable using the backward Euler scheme. We have derived *a priori* error estimations for semi-discrete and fully-discrete schemes and the theoretical results are verified by four numerical experiments. Moreover, the fourth numerical experiment allows us to conclude that our virtual scheme does not suffer from a suboptimal order of convergence when the diffusion coefficient  $\nu$  is small, in contrast to some finite element methods in velocity-pressure formulation, where a suboptimal convergence is observed in  $L^2$ -norm of the velocity for the Chorin problem with small values of  $\nu$  (see [32]).

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