Optimal error estimates to smooth solutions of the central discontinuous Galerkin methods for nonlinear scalar conservation laws

Mengjiao Jiao[1], Yan Jiang[2], Chi-Wang Shu[3] and Mengping Zhang[4]

Abstract

In this paper, we study the error estimates to sufficiently smooth solutions of the nonlinear scalar conservation laws for the semi-discrete central discontinuous Galerkin (DG) finite element methods on uniform Cartesian meshes. A general approach with an explicitly checkable condition is established for the proof of optimal L^2 error estimates of the semi-discrete CDG schemes, and this condition is checked to be valid in one and two dimensions for polynomials of degree up to $k = 8$. Numerical experiments are given to verify the theoretical results.

AMS subject classifications: 65M12, 65M15, 65M60

Key Words: central DG method; nonlinear conservation laws; optimal error estimates

[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China. E-mail: jmj123@mail.ustc.edu.cn

[2]School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China. E-mail: jiangy@ustc.edu.cn. Research supported by NSFC grant 11901555.

[3] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA. E-mail: chi-wang_shu@brown.edu. Research supported by NSF grant DMS-2010107 and AFOSR grant FA9550-20-1-0055.

[4]School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China. E-mail: mpzhang@ustc.edu.cn. Research supported by NSFC grant 11871448.

1 Introduction

In this paper, we study the central discontinuous Galerkin (DG) finite element method for solving scalar conservation laws [10]. The optimal error estimates of the central DG methods have been proved for linear conservation laws in [12]. In this paper, we present the optimal error estimates of central DG approximation based on tensor-product polynomials under suitable assumptions for the general nonlinear scalar conservation laws

$$\begin{cases} u_t + \sum_{i=1}^{d} (f_i(u))_{x_i} = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_d)$ and Ω is a bounded rectangular domain in $\mathbb{R}^d$. Here $u_0(\mathbf{x})$ is a given smooth function. We do not pay attention to boundary conditions in this paper; hence the exact solution is considered to be either periodic or compactly supported. We also assume the flux $f(u)$ is smooth in the variable u; for example, $f \in C^2$ is enough for our proof. The analysis in this paper is for the smooth solutions of (1.1). Discontinuous solutions with shocks are not considered here. We study the cases with $d = 1$ and 2, but the approach is applicable to any d.

The central scheme of Nessyahu and Tadmor [14] computes hyperbolic conservation laws on a staggered mesh and avoids the Riemann solver. In [3], Kurganov and Tadmor introduced a new type of central scheme without the large dissipation error related to the small time step size by using a variable control volume whose size depends on the time step size. To avoid the excessive numerical dissipation for small time steps, Liu [8] uses another coupling technique. The overlapping cell approach evolves two independent cell averages on overlapping cells, which opens up many new possibilities. The advantages of overlapping cells motivate the combination of the central scheme and the DG method, which results in the central DG methods [7, 9, 10]. The central DG method evolves two copies of approximating solutions defined on staggered meshes and avoids using numerical fluxes which can be complicated and costly [4]. Like some previous central schemes, the central DG method also avoids the excessive numerical dissipation for small time steps by a suitable choice of the numerical dissipation term. Besides, the central method carries many features of standard DG methods, such as compact stencil, easy parallel implementation, etc. The central DG method with Runge-Kutta time stepping has a larger CFL number for stability than the standard Runge-Kutta DG method with the same polynomial order k. Also the central DG method has a smaller error than the standard DG method on the same mesh. See [10, 15] for more details. Later in [11], the central local discontinuous Galerkin method was introduced to solve diffusion equations, which is formulated based on the local discontinuous Galerkin scheme on overlapping cells. Recently, the central DG method has been used to solve systems of conservation laws in many applications [6, 5, 22, 18, 17].

In [12], suitable special projections for central DG methods were proposed to yield optimal error estimates for scalar linear conservation laws. The proper local projections were constructed according to the superconvergence property and the duality of overlapping cells, which also required uniform Cartesian meshes. Zhang and Shu firstly presented *a priori* error estimates for the fully discrete second order Runge-Kutta DG methods with smooth solutions for scalar nonlinear conservation laws [19] and symmetrizable systems [20]. The main techniques they used are Taylor expansion and energy estimates. Later these techniques are widely used in error estimates for DG-type methods of nonlinear equations, like the local DG methods for convection-diffusion and KdV equations [16], the ultra weak DG methods for equations with higher order derivatives [1], the third order Runge-Kutta DG methods for scalar conservation laws [21] and for symmetrizable systems [13].

In this paper, we combine the special projections in [12] and the techniques used in [19] to construct new projections to provide the optimal error estimates of the central DG methods on uniform Cartesian meshes for nonlinear scalar conservation laws with smooth solutions. In one dimension, we construct a proper local projection $\mathbb{P}_h^*$ similar to

[12]. The existence and optimal approximation properties of this projection are proved by standard finite element techniques. Moreover, this projection has similar superconvergence property as the projections in [12]. By using this property we develop a general approach with an explicitly checkable condition, and this condition is checked to be valid in one dimension for polynomials of degree up to $k = 8$. This condition could also be checked for larger k with increased algebraic complexity, but it is not carried out in this paper. The optimal convergence results is valid for uniform meshes and for polynomials of degree $k \geq 1$, while for $k = 0$ we need the convection flux to be linear to get the optimal results. For two-dimensional conservation laws, we follow the same arguments as in the one-dimensional case to construct a suitable projection $\mathbb{P}_h^*$ and to analyze its existence and approximation properties. This new projection utilizes Q^k, the space of tensor-product polynomials of degree at most k in each variable. Similarly, the optimal convergence result is valid for uniform meshes and for polynomials of degree $k \geq 2$ in the two-dimensional case, while for $k = 0,\ 1$ we need the convection flux to be linear to get the optimal results. The superconvergence result of $\mathbb{P}_h^*$ on uniform Cartesian meshes will help to yield optimal convergence results under some suitable assumptions. Similar approach with an explicitly checkable condition is established, and here we also check this condition for polynomials of degree up to $k = 8$. Likewise, this condition could also be checked for larger k with increased algebraic complexity, but we will not carried it out. The approach is applicable to higher dimension d, but it will not be discussed in this paper.

The rest of the paper is organized as follows. In section 2, we recall the central DG method for one-dimensional conservation laws. Then we construct a special projection and study its existence, uniqueness and optimal approximation properties. With the help of this projection, we will prove the optimal error estimate for the semi-discrete central DG methods on uniform meshes for the nonlinear conservation laws in one dimension. In section 3, we extend the analysis to two-dimensions. Optimal error estimates are proved by following the same lines of the one dimensional case. We provide numerical examples to show our theoretical results in section 4. In section 5, we give a few concluding remarks and perspectives for future work. Finally, in the appendix we provide proofs for some of the more technical results of the error estimates.

2 The central DG method in one dimension

Here we consider the one-dimensional conservation law given by

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in [a,b] \times (0,T], \\ u(x,0) = u_0(x), & x \in [a,b], \end{cases} \tag{2.1}$$

with periodic boundary condition or compactly supported boundary condition.

2.1 Basic notations

For a given interval $I = [a,b]$, we divide it into N cells as follows:

$$a = x_0 < x_1 < \cdots < x_N = b. \tag{2.2}$$

We denote

$$x_{j+\frac{1}{2}} = \frac{x_j + x_{j+1}}{2},\ I_{j+\frac{1}{2}} = (x_j, x_{j+1}),\ h_{j+\frac{1}{2}} = x_{j+1} - x_j, \quad j = 0, \dots N-1, \tag{2.3}$$

and similarly for the dual mesh

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),\ h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad j = 1, \dots N. \tag{2.4}$$

We let $h = \max_j h_{j+\frac{1}{2}}$ and assume the mesh is regular. Define the approximation space as

$$\begin{aligned} V_h^k &= \{\varphi_h : (\varphi_h)|_{I_j} \in P^k(I_j), j = 1, ..., N\}, \\ W_h^k &= \{\psi_h : (\psi_h)|_{I_{j+\frac{1}{2}}} \in P^k(I_{j+\frac{1}{2}}), j = 0, ..., N-1\}. \end{aligned} \tag{2.5}$$

Here $P^k(I_j)$ denotes the set of all polynomials of degree at most k on I_j. For a function $\varphi_h \in V_h^k$, we use $(\varphi_h)^-_{j+\frac{1}{2}}$ or $(\varphi_h)^+_{j+\frac{1}{2}}$ to refer to the value of φ_h at $x_{j+\frac{1}{2}}$ from the left cell I_j and the right cell I_{j+1}, respectively. For $\psi_h \in W_h^k$, $(\psi_h)^-_j$ and $(\psi_h)^+_j$ have similar meanings. $[\varphi_h]$ or $[\psi_h]$ is used to denote $\varphi_h^+ - \varphi_h^-$ or $\psi_h^+ - \psi_h^-$, i.e. the jump of φ_h or ψ_h at cell interfaces. We denote by C a positive constant independent of h, which may depend on the solution of the problem and other parameters. For our analysis, we need the uniform boundedness of f' and f''. We shall take this as an assumption for simplicity, although such boundedness can be shown *a posteriori* by the eventual boundedness of the numerical solution through the verification of the *a priori* assumptions at the end of section 2 and section 3. Similar to [16, 19], to emphasize the nonlinearity of the flux $f(u)$, we use C_* to denote a non-negative constant depending on the maximum of $|f''|$. We remark $C_* = 0$ for linear fluxes $f(u) = cu$ with a constant c.

2.2 The central DG scheme

We propose the following semi-discrete central DG scheme for periodic boundary condition: find $u_h \in V_h^k$ and $v_h \in W_h^k$, such that for any $\varphi_h \in V_h^k$ and $\psi_h \in W_h^k$,

$$\begin{aligned}\int_{I_j} (u_h)_t \varphi_h dx =& \frac{1}{\tau_{max}} \int_{I_j} (v_h - u_h)\varphi_h dx + \int_{I_j} f(v_h)(\varphi_h)_x dx \\ & - (f(v_h)\varphi_h^-)_{j+\frac{1}{2}} + (f(v_h)\varphi_h^+)_{j-\frac{1}{2}}, \qquad (2.6a)\\ \int_{I_{j+\frac{1}{2}}} (v_h)_t \psi_h dx =& \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h)\psi_h dx + \int_{I_{j+\frac{1}{2}}} f(u_h)(\psi_h)_x dx \\ & - (f(u_h)\psi_h^-)_{j+1} + (f(u_h)\psi_h^+)_j, \qquad (2.6b)\end{aligned}$$

where τ_{max} is an upper bound for the time step size due to the CFL restriction, that is, $\tau_{max} = c\ h$ with a given constant CFL number c dictated by stability. For the initial condition, we simply take $u_h(\cdot, 0) = \mathbb{P}_h u_0(\cdot)$, $v_h(\cdot, 0) = \mathbb{Q}_h u_0(\cdot)$, where $\mathbb{P}_h$ and $\mathbb{Q}_h$ are the L^2 projections into V_h^k and W_h^k, respectively, and we have

$$\begin{aligned}\|u_0 - \mathbb{P}_h u_0\|_{L^2(I_j)} &\le Ch^{k+1}\|u_0\|_{H^{k+1}(I_j)}, \\ \|u_0 - \mathbb{Q}_h u_0\|_{L^2(I_{j+\frac{1}{2}})} &\le Ch^{k+1}\|u_0\|_{H^{k+1}(I_{j+\frac{1}{2}})}.\end{aligned} \qquad (2.7)$$

2.3 L^2 stability for the linear equation

In [10], the following stability result is proved for this scheme if $f(u)$ is linear. Without loss of generality, we take $f(u) = u$. Hence, we have

$$\begin{cases} u_t + u_x = 0, \quad (x,t) \in [a,b] \times (0,T], \\ u(x,0) = u_0(x), \quad x \in [a,b], \end{cases} \tag{2.8}$$

with periodic boundary condition.

Theorem 2.1. *The numerical solutions* u_h *and* v_h *of the CDG scheme* (2.6) *for the equation* (2.8) *have the following* L^2 *stability property*

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_h^2 + v_h^2)dx = -\frac{1}{\tau_{max}}\int_a^b (v_h - u_h)^2 dx \leq 0. \tag{2.9}$$

2.4 Optimal L^2 error estimate

It is worth noting that the L^2 stability for CDG scheme for nonlinear problem is generally not available [10]. But under the assumption of the smoothness of the exact solution, we can still get the error estimate of the nonlinear case. In this subsection, we show *a priori* L^2 error estimate of the scheme (2.6) for the equation (2.1).

Here and below, we use $\|\cdot\|$ to denote the standard L^2 norm. For the proof, we recall the classical inverse and trace inequalities [2]. For any $w_h \in V_h^k$ or $w_h \in W_h^k$, there exists a positive constant C independent of w_h and h, such that

$$\|\partial_x w_h\| \leq Ch^{-1}\|w_h\|, \ \|w_h\|_\Gamma \leq Ch^{-\frac{1}{2}}\|w_h\|, \ \|w_h\|_\infty \leq Ch^{-\frac{1}{2}}\|w_h\|, \tag{2.10}$$

where Γ is the set of boundary points of all elements I_j or $I_{j+\frac{1}{2}}$.

First we introduce some notations. For the numerical solutions u_h and v_h of the CDG scheme (2.6) for equation (2.1), we define

$$\begin{aligned} \tilde{B}_j(u_h, v_h; \varphi_h; f, u) :=& \frac{1}{\tau_{max}}\int_{I_j}(v_h - u_h)\varphi_h dx + \int_{I_j} f'(u(x_j))v_h(\varphi_h)_x \\ &- f'(u(x_j))(v_h\varphi_h^-)_{j+\frac{1}{2}} + f'(u(x_j))(v_h\varphi_h^+)_{j-\frac{1}{2}}, \end{aligned} \tag{2.11}$$

$$\hat{B}_{j+\frac{1}{2}}(u_h, v_h; \psi_h; f, u) := \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h)\psi_h dx + \int_{I_{j+\frac{1}{2}}} f'(u(x_{j+\frac{1}{2}}))u_h(\psi_h)_x - f'(u(x_{j+\frac{1}{2}}))(u_h\psi_h^-)_{j+1} + f'(u(x_{j+\frac{1}{2}}))(u_h\psi_h^+)_j, \tag{2.12}$$

and

$$B_j(u_h, v_h; \varphi_h, \psi_h) := \int_{I_j} (u_h)_t\varphi_h dx + \int_{I_{j+\frac{1}{2}}} (v_h)_t\psi_h dx - \frac{1}{\tau_{max}} \int_{I_j} (v_h - u_h)\varphi_h dx - \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (u_h - v_h)\psi_h dx, \tag{2.13}$$

Obviously, we have

$$B_j(u_h, v_h; \varphi_h, \psi_h) = \int_{I_j} f(v_h)(\varphi_h)_x dx + \int_{I_{j+\frac{1}{2}}} f(u_h)(\psi_h)_x dx - (f(v_h)\varphi_h^-)_{j+\frac{1}{2}} + (f(v_h)\varphi_h^+)_{j-\frac{1}{2}} - (f(u_h)\psi_h^-)_{j+1} + (f(u_h)\psi_h^+)_j, \quad \forall\varphi_h \in V_h^k, \psi_h \in W_h^k. \tag{2.14}$$

It is also clear that the exact solution u of (2.1) satisfies

$$B_j(u, u; \varphi_h, \psi_h) = \int_{I_j} f(u)(\varphi_h)_x dx + \int_{I_{j+\frac{1}{2}}} f(u)(\psi_h)_x dx - (f(u)\varphi_h^-)_{j+\frac{1}{2}} + (f(u)\varphi_h^+)_{j-\frac{1}{2}} - (f(u)\psi_h^-)_{j+1} + (f(u)\psi_h^+)_j, \quad \forall\varphi_h \in V_h^k, \psi_h \in W_h^k. \tag{2.15}$$

Subtracting (2.14) from (2.15), we obtain the error equation

$$\begin{aligned} B_j(u - u_h, u - v_h; \varphi_h, \psi_h) = &\int_{I_j} (f(u) - f(v_h))(\varphi_h)_x dx + \int_{I_{j+\frac{1}{2}}} (f(u) - f(u_h))(\psi_h)_x dx \\ &- ((f(u) - f(v_h))\varphi_h^-)_{j+\frac{1}{2}} + ((f(u) - f(v_h))\varphi_h^+)_{j-\frac{1}{2}} \\ &- ((f(u) - f(u_h))\psi_h^-)_{j+1} + ((f(u) - f(u_h))\psi_h^+)_j \\ :=&H_j(f; u, u_h, v_h; \varphi_h, \psi_h), \ \forall\varphi_h \in V_h^k, \psi_h \in W_h^k. \end{aligned} \tag{2.16}$$

Summing over all j, the error equation becomes

$$\sum_j B_j(u - u_h, u - v_h; \varphi_h, \psi_h) = \sum_j H_j(f; u, u_h, v_h; \varphi_h, \psi_h), \quad \forall\varphi_h \in V_h^k, \psi_h \in W_h^k. \tag{2.17}$$

2.4.1 Projection operators

Similar to [12], we define $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ as the following projections onto V_h^k and W_h^k respectively on uniform meshes. That is, for a given function $w(x)$, we define $\mathbb{P}_h^* w \in V_h^k$, such that $\forall j$,

$$\int_{I_j} \mathbb{P}_h^* w dx = \int_{I_j} w dx, \tag{2.18a}$$

$$\tilde{P}_h(\mathbb{P}_h^* w; \varphi_h; f, u)_j = \tilde{P}_h(w; \varphi_h; f, u)_j, \ \forall \varphi_h \in P^k(I_j), \tag{2.18b}$$

where $\tilde{P}_h(w;\varphi_h)_j$ is defined as follows

$$\begin{aligned}\tilde{P}_h(w;\varphi_h;f,u)_j =& \frac{1}{\tau_{max}}(\int_{x_{j-\frac{1}{2}}}^{x_j} w(x+\frac{h}{2})\varphi_h dx + \int_{x_j}^{x_{j+\frac{1}{2}}} w(x-\frac{h}{2})\varphi_h dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} w(x)\varphi_h dx) \\ &+ \int_{x_{j-\frac{1}{2}}}^{x_j} f'(u(x_j)) w(x+\frac{h}{2})(\varphi_h)_x dx \\ &+ \int_{x_j}^{x_{j+\frac{1}{2}}} f'(u(x_j)) w(x-\frac{h}{2})(\varphi_h)_x dx \\ &- f'(u(x_j))w(x_j)(\varphi_h(x_{j+\frac{1}{2}}^-) - \varphi_h(x_{j-\frac{1}{2}}^+)).\end{aligned} \tag{2.19}$$

Similarly, we define $\mathbb{Q}_h^* w \in W_h^k$, such that $\forall j$,

$$\int_{I_{j+\frac{1}{2}}} \mathbb{Q}_h^* w dx = \int_{I_{j+\frac{1}{2}}} w dx, \tag{2.20a}$$

$$\tilde{Q}_h(\mathbb{Q}_h^* w; \psi_h; f, u)_{j+\frac{1}{2}} = \tilde{Q}_h(w; \psi_h; f, u)_{j+\frac{1}{2}}, \ \forall \psi_h \in P^k(I_{j+\frac{1}{2}}), \tag{2.20b}$$

where $\tilde{Q}_h(w;\psi_h)_{j+\frac{1}{2}}$ is defined as follows

$$\begin{aligned}\tilde{Q}_h(w;\varphi_h;f,u)_{j+\frac{1}{2}} =& \frac{1}{\tau_{max}}(\int_{x_j}^{x_{j+\frac{1}{2}}} w(x+\frac{h}{2})\psi_h dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} w(x-\frac{h}{2})\psi_h dx - \int_{x_j}^{x_{j+1}} w(x)\psi_h dx) \\ &+ \int_{x_j}^{x_{j+\frac{1}{2}}} f'(u(x_{j+\frac{1}{2}})) w(x+\frac{h}{2})(\psi_h)_x dx \\ &+ \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} f'(u(x_{j+\frac{1}{2}})) w(x-\frac{h}{2})(\varphi_h)_x dx \\ &- f'(u(x_{j+\frac{1}{2}}))w(x_{j+\frac{1}{2}})(\varphi_h(x_{j+1}^-) - \varphi_h(x_j^+)).\end{aligned} \tag{2.21}$$

Next, we will discuss the properties of the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$. Without loss of generality we will only consider $\mathbb{P}_h^*$. The equation (2.18a) is required by conservation.

Note that $\tilde{P}_h(w;\varphi_h;f,u)_j = 0$ for $\forall w$ when φ_h is a constant, so (2.18b) alone misses one condition which is provided by (2.18a). The following lemma gives the existence and uniqueness of the special projection $\mathbb{P}_h^*$.

Lemma 2.1. *The projection $\mathbb{P}_h^*$ defined by (2.18) exists and is unique for any smooth function $w(x)$, and the following inequality holds*

$$\|\mathbb{P}_h^* w\| \le C\|w\|_\infty, \tag{2.22}$$

for all k. The positive constant C depends on k, the bound of $f'(u)$, the constant c in the scheme (2.6) and is independent of h and w.

Proof. The proof of this lemma is given in Appendix A.1. □

Since $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ are k-th degree polynomial preserving local projections, standard approximation theory [2] implies, for smooth function w,

$$\begin{aligned} \|\mathbb{P}_h^* w - w\| + h\|\mathbb{P}_h^* w - w\|_\infty + h^{\frac{1}{2}}\|\mathbb{P}_h^* w - w\|_\Gamma \le Ch^{k+1}\|u\|_{H^{k+1}([a,b])}, \\ \|\mathbb{Q}_h^* w - w\| + h\|\mathbb{Q}_h^* w - w\|_\infty + h^{\frac{1}{2}}\|\mathbb{Q}_h^* w - w\|_\Gamma \le Ch^{k+1}\|u\|_{H^{k+1}([a,b])}, \end{aligned} \tag{2.23}$$

Besides the standard approximation results (2.23), the special projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ also have the following superconvergence result.

Proposition 2.1. *For $k = 0, 1..., 8$, assume that u is a $(k+1)$-th degree polynomial function in $P^{k+1}([a,b])$. For a uniform partition on the interval $[a,b]$, set $u_I = \mathbb{P}_h^* u \in V_h^k$ and $v_I = \mathbb{Q}_h^* u \in W_h^k$,. Then we have*

$$\begin{aligned} |\tilde{B}_j(u_I - u, v_I - u; \varphi_h; f, u)| \le Ch^{2k+3} + C\|\varphi_h\|^2_{L^2(I_j)}, \ \forall \varphi_h \in P^k(I_j) \\ |\hat{B}_{j+\frac{1}{2}}(u_I - u, v_I - u; \psi_h; f, u)| \le Ch^{2k+3} + C\|\psi_h\|^2_{L^2(I_{j+\frac{1}{2}})}, \ \forall \psi_h \in P^k(I_{j+\frac{1}{2}}). \end{aligned} \tag{2.24}$$

Proof. The proof of this proposition is given in Appendix A.2. □

2.4.2 *A priori* L^2 error estimates

Theorem 2.2. *For $k = 0, 1..., 8$, let $u(\cdot,t)$ be the exact solution of equation (2.1), which is sufficiently smooth with bounded derivatives, and assume $f \in C^2$ with bounded $f'(u)$*

and $f''(u)$. The numerical solutions u_h and v_h of the CDG scheme (2.6) using uniform meshes satisfies the following L^2 error estimate

$$\|u(\cdot,T)-u_h(\cdot,T)\|^2+\|u(\cdot,T)-v_h(\cdot,T)\|^2 \le Ch^{2k+2}, \tag{2.25}$$

where k is the polynomial degree in the finite element spaces V_h^k and W_h^k, and the constant C depends on k, the final time T, $\|u\|_{H^{k+2}}$ and the bounds on the derivatives $|f^m|$, $m = 1,2$, but is independent of the mesh size h. Here $\|u\|_{H^{k+2}}$ is the maximum $(k+2)$-th order Sobolev norm of u over time in $[0,T]$. For $k=0$ we need $f(u)$ to be linear, i.e. $f(u)=cu$.

Proof. Let $e_u = u-u_h,\ e_v = u-v_h$ be the error between the numerical and exact solutions. To deal with the nonlinearity of $f(u)$, we would like to first make the *a priori* assumption that, for small enough h, we have

$$\|u-u_h\| \le Ch^{\frac{3}{2}},\ \|u-v_h\| \le Ch^{\frac{3}{2}}, \tag{2.26}$$

which also establishes the Lipschitz continuity of the right-hand side of the method of lines semi-discrete ordinary differential equation system, hence the very existence of u_h and v_h. By the interpolation property, we then have

$$\begin{aligned} \|e_u\|_\infty \le Ch \quad &\text{and} \quad \|\mathbb{P}_h^* u - u_h\|_\infty \le Ch, \\ \|e_v\|_\infty \le Ch \quad &\text{and} \quad \|\mathbb{Q}_h^* u - u_h\|_\infty \le Ch. \end{aligned} \tag{2.27}$$

This assumption is not necessary for linear f. We will verify this assumption for $k\ge 1$ later.

By taking

$$\varphi_h = \mathbb{P}_h^* u - u_h,\ \psi_h = \mathbb{Q}_h^* u - v_h, \varphi^e = \mathbb{P}_h^* u - u,\ \psi^e = \mathbb{Q}_h^* u - u, \tag{2.28}$$

we obtain the energy equality

$$\sum_j B_j(\varphi_h-\varphi^e, \psi_h-\psi^e; \varphi_h, \psi_h) = \sum_j H_j(f; u, u_h, v_h; \varphi_h, \psi_h). \tag{2.29}$$

From the definition of B_j, we can obtain

$$\begin{aligned}\sum_j B_j(\varphi_h,\psi_h;\varphi_h,\psi_h) =& \sum_j B_j(\varphi^e,\psi^e;\varphi_h,\psi_h) + \sum_j H_j(f;u,u_h,v_h;\varphi_h,\psi_h)\\ =& \sum_j \int_{I_j} (\psi^e)_t \varphi_h dx + \sum_j \int_{I_{j+\frac{1}{2}}} (\varphi^e)_t \psi_h dx\\ & - \sum_j \frac{1}{\tau_{max}} \int_{I_j} (\psi^e - \varphi^e)\varphi_h dx - \sum_j \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (\varphi^e - \psi^e)\psi_h dx\\ & + \sum_j \int_{I_j} (f(u) - f(v_h))(\varphi_h)_x dx + \sum_j ((f(u) - f(v_h))[\varphi_h])_{j+\frac{1}{2}}\\ & + \sum_j \int_{I_{j+\frac{1}{2}}} (f(u) - f(u_h))(\psi_h)_x dx + \sum_j ((f(u) - f(u_h))[\psi_h])_j.\end{aligned} \tag{2.30}$$

For the left-hand side of (2.30), we follow the L^2 stability proof in Theorem 2.1 for linear case to conclude

$$\sum_j B_j(\varphi_h,\psi_h;\varphi_h,\psi_h) = \frac{1}{2}\frac{d}{dt}\int_a^b (\varphi_h^2 + \psi_h^2)dx + \frac{1}{\tau_{max}}\int_a^b (\varphi_h - \psi_h)^2 dx. \tag{2.31}$$

Similar to [19] and [16], to deal with the nonlinear part of (2.30) we would like to use the following Taylor expansions:

$$\begin{aligned} f(u) - f(u_h) =& f'(u)\varphi_h - f'(u)\varphi^e - \frac{1}{2} f''_u (\varphi_h - \varphi^e)^2,\\ f(u) - f(v_h) =& f'(u)\psi_h - f'(u)\psi^e - \frac{1}{2} f''_v (\psi_h - \psi^e)^2,\end{aligned} \tag{2.32}$$

where f''_u and f''_v are the mean values. These imply the following representation,

$$\begin{aligned}&\sum_j B_j(\varphi^e,\psi^e;\varphi_h,\psi_h) + \sum_j H_j(f;u,u_h,v_h;\varphi_h,\psi_h)\\ =& L + N_1 + N_2 + N_3 + N_4,\end{aligned} \tag{2.33}$$

where

$$\begin{aligned} L =& \sum_j \int_{I_j} (\psi^e)_t \varphi_h dx + \sum_j \int_{I_{j+\frac{1}{2}}} (\varphi^e)_t \psi_h dx,\\ N_1 =& -\sum_j \frac{1}{\tau_{max}} \int_{I_j} (\psi^e - \varphi^e)\varphi_h dx - \sum_j \int_{I_j} f'(u)\psi^e (\varphi_h)_x dx - \sum_j (f'(u)\psi^e[\varphi_h])_{j+\frac{1}{2}},\\ N_2 =& -\sum_j \frac{1}{\tau_{max}} \int_{I_{j+\frac{1}{2}}} (\varphi^e - \psi^e)\psi_h dx - \sum_j \int_{I_{j+\frac{1}{2}}} f'(u)\varphi^e (\psi_h)_x dx - \sum_j (f'(u)\varphi^e[\psi_h])_j,\end{aligned}$$

$$
\begin{aligned}
N_3 =& \sum_j \int_{I_j} f'(u)\psi_h(\varphi_h)_x dx + \sum_j (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} \\
&+ \sum_j \int_{I_{j+\frac{1}{2}}} f'(u)\varphi_h(\psi_h)_x dx + \sum_j (f'(u)\varphi_h[\psi_h])_j, \\
N_4 =& -\frac{1}{2}\Big(\sum_j \int_{I_j} f''_v(\psi_h-\psi^e)^2(\varphi_h)_x dx + \sum_j \int_{I_{j+\frac{1}{2}}} f''_u(\varphi_h-\varphi^e)^2(\psi_h)_x dx \\
&+ \sum_j (f''_v(\psi_h-\psi^e)^2[\varphi_h])_{j+\frac{1}{2}} + \sum_j (f''_u(\varphi_h-\varphi^e)^2[\psi_h])_j \Big).
\end{aligned}
$$

By Young's inequality and (2.23), we have

$$
L \le C(\|\varphi_h\|^2 + \|\psi_h\|^2) + Ch^{2k+2}\|u\|^2_{H^{k+1}([a,b])}. \tag{2.34}
$$

Next we estimate the nonlinear part. First for the N_1 term, we can rewrite it in the form

$$
\begin{aligned}
N_1 =& -\sum_j \frac{1}{\tau_{max}} \int_{I_j} (\psi^e - \varphi^e)\varphi_h dx - \sum_j \int_{I_j} f'(u(x_j))\psi^e(\varphi_h)_x dx \\
& - \sum_j (f'(u(x_j))\psi^e[\varphi_h])_{j+\frac{1}{2}} + \sum_j \int_{I_j} (f'(u(x_j)) - f'(u))\psi^e(\varphi_h)_x dx \\
& - \sum_j (f'(u(x_j)) - f'(u))\psi^e[\varphi_h])_{j+\frac{1}{2}} \\
=& -\sum_j \tilde{B}_j(\varphi^e, \psi^e; \varphi_h) + \sum_j \int_{I_j} (f'(u(x_j)) - f'(u))\psi^e(\varphi_h)_x dx \\
& - \sum_j (f'(u(x_j)) - f'(u))\psi^e[\varphi_h])_{j+\frac{1}{2}}.
\end{aligned}
$$

By the inequality in (2.10), (2.23) and $\|f'(u(x_j)) - f'(u)\|_{L^\infty(I_j)} = O(h)$, we have

$$
N_1 \le -\sum_j \tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) + C_*\|\varphi_h\|^2 + C_* h^{2k+2}\|u\|^2_{H^{k+1}([a,b])}. \tag{2.35}
$$

For $\tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u)$, let $\widehat{u_I}$ be the Taylor polynomial of order $k+1$ of u near x_j i.e. $\widehat{u_I}^j = \sum_{i=0}^{k+1} \frac{1}{i!} u^{(i)}(x_j)(x-x_j)^i, \quad x \in (x_{j-1}, x_{j+1})$. Let r_u denote the residual term i.e. $r_u^j = u - \widehat{u_I}^j$. Recalling the Bramble-Hilbert lemma [2], we have

$$
\|r_u^j\|_{L^\infty(I_j)} \le Ch^{k+\frac{3}{2}}|u|_{H^{k+2}(I_j)}. \tag{2.36}
$$

Then we rewrite φ^e and ψ^e

$$
\varphi^e = \mathbb{P}_h^* u - u = \mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{P}_h^* r_u^j - r_u^j,
$$

$$\psi^e = \mathbb{Q}_h^* u - u = \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{Q}_h^* r_u^j - r_u^j. \tag{2.37}$$

Hence, using Proposition 2.1, we have

$$\begin{aligned}
\tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) =& \tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) \\
=& \tilde{B}_j(\mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j + \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h; f, u) \\
=& \tilde{B}_j(\mathbb{P}_h^* \widehat{u_I}^j - \widehat{u_I}^j, \mathbb{Q}_h^* \widehat{u_I}^j - \widehat{u_I}^j; \varphi_h; f, u) \\
& + \tilde{B}_j(\mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h; f, u) \\
=& \tilde{B}_j(\mathbb{P}_h^* r_u^j - r_u^j, \mathbb{Q}_h^* r_u^j - r_u^j; \varphi_h; f, u) + Ch^{2k+3} + C\|\varphi_h\|^2_{L^2(I_j)}. \quad (2.38)
\end{aligned}$$

Therefore, by using Young's inequality, (2.23), the inequality in (2.10) and (2.36), we have

$$-\sum_j \tilde{B}_j(\varphi^e, \psi^e; \varphi_h; f, u) \leq Ch^{2k+2}|u|_{H^{k+2}([a,b])} + C\|\varphi_h\|^2. \tag{2.39}$$

Hence, for N_1 we have

$$N_1 \leq (C + C_*)\|\varphi_h\|^2 + (C + C_*)h^{2k+2}\|u\|^2_{H^{k+2}([a,b])}. \tag{2.40}$$

Similarly, for N_2 we have

$$N_2 \leq (C + C_*)\|\psi_h\|^2 + (C + C_*)h^{2k+2}\|u\|^2_{H^{k+2}([a,b])}. \tag{2.41}$$

The N_3 term can be rewritten as the following form

$$\begin{aligned}
N_3 =& \sum_j \Big(\int_{x_j}^{x_{j+\frac{1}{2}}} f'(u)(\psi_h\varphi_h)_x dx + \int_{x_{j+\frac{1}{2}}}^{x_{j+1}} f'(u)(\psi_h\varphi_h)_x dx \Big) \\
& + \sum_j (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} + \sum_j (f'(u)\varphi_h[\psi_h])_j \\
=& \sum_j \Big((f'(u)\psi_h\varphi_h^-)_{j+\frac{1}{2}} - (f'(u)\varphi_h\psi_h^+)_j + (f'(u)\varphi_h\psi_h^-)_{j+1} \\
& - (f'(u)\psi_h\varphi_h^+)_{j+\frac{1}{2}} + (f'(u)\psi_h[\varphi_h])_{j+\frac{1}{2}} + (f'(u)\varphi_h[\psi_h])_j \Big) \\
& - \sum_j \int_{x_j}^{x_{j+1}} (f'(u))_x \psi_h\varphi_h dx \\
=& - \sum_j \int_{x_j}^{x_{j+1}} (f'(u))_x \psi_h\varphi_h dx \\
\leq& C\|\psi_h\|\|\varphi_h\| \leq C(\|\psi_h\|^2 + \|\varphi_h\|^2).
\end{aligned} \tag{2.42}$$

N_4 is the high order term in Taylor expansion, it is easy to show that

$$
\begin{aligned}
N_4 \le & C_* h^{-1}(\|e_v\|_\infty \|e_v\| \|\varphi_h\| + \|e_u\|_\infty \|e_u\| \|\psi_h\|) \\
\le & C_* h^{-1}\Big(\|e_v\|_\infty(\|\varphi_h\|\|\psi_h\| + \|\varphi_h\|\|\psi^e\|) + \|e_v\|_\infty(\|\psi_h\|\|\varphi_h\| + \|\psi_h\|\|\varphi^e\|)\Big) \\
\le & C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
& + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty) h^{2k+2}\|u\|^2_{H^{k+1}([a,b])}.
\end{aligned}
\tag{2.43}
$$

Hence, combining (2.34), (2.40), (2.41), (2.42), (2.43), (2.31), we obtain from (2.30)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_a^b (\varphi_h^2 + \psi_h^2)dx \le & (C + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty))(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
& + (C + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty))h^{2k+2}\|u\|^2_{H^{k+2}([a,b])}.
\end{aligned}
\tag{2.44}
$$

When $k \ge 1$, by using *a priori* assumption (2.26) we have

$$
\frac{1}{2}\frac{d}{dt}\int_a^b (\varphi_h^2 + \psi_h^2)dx \le (C + C_*)(\|\varphi_h\|^2 + \|\psi_h\|^2) + (C + C_*)h^{2k+2}\|u\|^2_{H^{k+2}([a,b])}.
\tag{2.45}
$$

Finally, by Gronwall's inequality and the fact that $\|\varphi_h(\cdot,0)\| \le Ch^{k+1}$, $\|\psi_h(\cdot,0)\| \le Ch^{k+1}$ we can get

$$
\int_a^b (\varphi_h^2 + \psi_h^2)dx \le Ch^{2k+2}.
\tag{2.46}
$$

This, together with the approximation result (2.23), implies the desired error estimate.

For the case of $k = 0$, we assume that the convection term is linear, namely $f(u) = cu$. This is to avoid the need of the *a priori* assumption (2.26) which is no longer justifiable since our L^2 error estimate is only of order $O(h)$ in this case. The proof is similar to that for $k \ge 1$ case given above, and the only difference is $C_* = 0$ in this case. By similar lines of proof, we have

$$
\frac{1}{2}\frac{d}{dt}\int_a^b (\varphi_h^2 + \psi_h^2)dx \le C(\|\varphi_h\|^2 + \|\psi_h\|^2) + Ch^2.
\tag{2.47}
$$

An application of Gronwall's inequality give us that

$$
\int_a^b (\varphi_h^2 + \psi_h^2)dx \le Ch^2.
\tag{2.48}
$$

This, together with the approximation result (2.23), implies the desired error estimate.

Finally, let us justify the *a priori* assumption (2.26) for $k \geq 1$. Similar to [19] and [1], we can verify this by a proof by contradiction. By (2.25), we can consider h small enough so that $Ch^{k+1} < \frac{1}{2}h^{\frac{3}{2}}$, where C is the constant in (2.25) determined by the final time T. Define $t^* = sup\{t : \|u(\cdot,t) - u_h(\cdot,t)\| + \|u(\cdot,t) - v_h(\cdot,t)\| \leq h^{\frac{3}{2}}\}$, then we have $\|u(\cdot,t^*) - u_h(\cdot,t^*)\| + \|u(\cdot,t^*) - v_h(\cdot,t^*)\| = h^{\frac{3}{2}}$ by continuity if t^* is finite. Clearly, (2.25) holds for $t \leq t^*$, in particular, $\|u(\cdot,t^*) - u_h(\cdot,t^*)\| + \|u(\cdot,t^*) - v_h(\cdot,t^*)\| \leq Ch^{k+1} < \frac{1}{2}h^{\frac{3}{2}}$. This is a contradiction if $t^* < T$. Hence, $t^* \geq T$ and our *a priori* assumption is justified. □

3 The central DG method in multi-dimensions

In this section, we consider the semi-discrete central DG method for multidimensional nonlinear conservation laws. Without loss of generality, we will show our central DG scheme and prove the optimal *a priori* error estimates in two dimensions ($d = 2$); all the arguments we present in our analysis depend on the tensor product structure of the mesh and finite element space and can be easily extended to the more general cases $d > 2$. Now we consider the following two-dimensional problem,

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \quad (x,y,t) \in \Omega \times (0,T], \\ u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega, \end{cases} \tag{3.1}$$

with periodic boundary condition or compactly supported boundary condition.

3.1 Basic notations

Let $\{K_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]\}$ be a partition of Ω into uniform square cells, depicted by the solid lines in Fig. 3.1, and tagged by their cell centroid at (x_i, y_j). Define $h = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$. Let $X_h^k := \{v \in L^2(\Omega) : v|_{K_{i,j}} \in Q^k(K_{i,j}), \quad \forall (i,j)\}$, where $Q^k(K_{i,j})$ is the tensor-product polynomials of degrees at most k in each variable defined on $K_{i,j}$ and no continuity is assumed across cell boundaries. Let $K_{i+\frac{1}{2},j+\frac{1}{2}}$ be the dual mesh which consists of a $\frac{h}{2}$ shift of the $K_{i,j}$, depicted by the dashed lines in

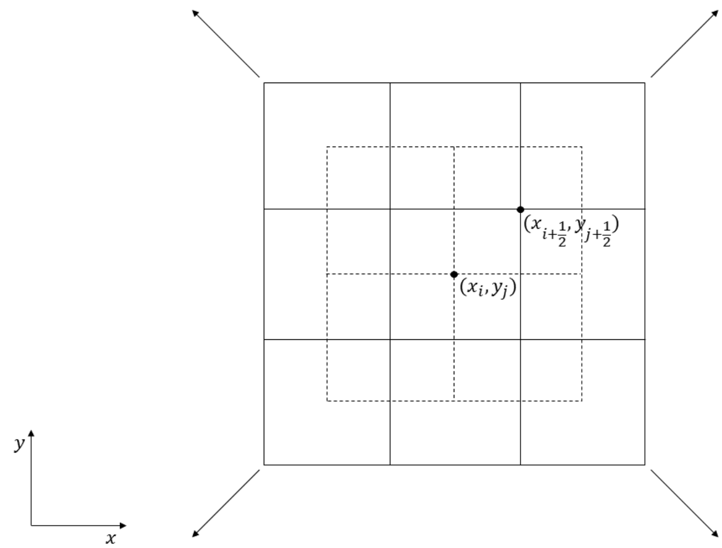

Fig. 3.1. 2D overlapping cells formed by collapsing the staggered dual cells on two adjacent time levels to one time level.

Fig. 3.1. Let $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$ be the cell centroid of the cell $K_{i+\frac{1}{2},j+\frac{1}{2}}$ and let $Y_h^k := \{v \in L^2(\Omega) : v|_{K_{i,j}} \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}), \quad \forall (i,j)\}$ denotes the space of tensor-product polynomials of degrees at most k in each variable defined on $K_{i+\frac{1}{2},j+\frac{1}{2}}$ and no continuity is assumed across the cell boundary. For a function $\varphi_h \in X_h^k$, we use $(\varphi_h)^+_{i+\frac{1}{2},y}$ and $(\varphi_h)^-_{i+\frac{1}{2},y}$ to denote the values of φ_h at $(x_{i+\frac{1}{2}}, y)$ from the right cell $K_{i+1,j}$ and the left cell $K_{i,j}$, respectively, when $y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ on all vertical edges. And for $\psi_h \in Y_h^k$, we use $(\psi_h)^+_{i,y}$ and $(\psi_h)^-_{i,y}$ to denote the values of ψ_h at (x_i, y) from the right cell $K_{i+\frac{1}{2},j+\frac{1}{2}}$ and the left cell $K_{i-\frac{1}{2},j+\frac{1}{2}}$, respectively, when $y \in [y_j, y_{j+1}]$ on all vertical edges. The notation $[\varphi_h]_{i+\frac{1}{2},y}$ or $[\psi_h]_{i+1,y}$ denote $(\varphi_h)^+_{i+\frac{1}{2},y} - (\varphi_h)^-_{i+\frac{1}{2},y}$ or $(\psi_h)^+_{i,y} - (\psi_h)^-_{i,y}$,i.e. the jump of φ_h at $(x_{i+\frac{1}{2}}, y)$ when $y \in [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ or the jump of ψ_h at (x_i, y) when $y \in [y_j, y_{j+1}]$. Similarly, we can define $(\varphi_h)^+_{x,j+\frac{1}{2}}, (\varphi_h)^-_{x,j+\frac{1}{2}}, (\psi_h)^+_{x,j}, (\psi_h)^-_{x,j}, [\varphi_h]_{x,j+\frac{1}{2}}$ and $[\psi_h]_{x,j}$.

3.2 The central DG scheme

We propose the following semi-discrete CDG scheme for periodic boundary condition: find $u_h \in X_h^k$ and $v_h \in Y_h^k$, such that for any $\varphi_h \in X_h^k$ and $\psi_h \in Y_h^k$,

$$\begin{aligned}\int_{K_{i,j}} (u_h)_t \varphi_h dxdy =& \frac{1}{\tau_{max}} \int_{K_{i,j}} (v_h - u_h)\varphi_h dxdy \\ &+ \int_{K_{i,j}} (f(v_h)(\varphi_h)_x + g(v_h)(\varphi_h)_y) dxdy \\ &- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(v_h)\varphi_h^-)_{i+\frac{1}{2},y} - (f(v_h)\varphi_h^+)_{i-\frac{1}{2},y}) dy \\ &- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(v_h)\varphi_h^-)_{x,j+\frac{1}{2}} - (g(v_h)\varphi_h^+)_{x,j+\frac{1}{2}}) dx, \end{aligned} \tag{3.2a}$$

$$\begin{aligned}\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (v_h)_t \psi_h dxdy =& \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (u_h - v_h)\psi_h dxdy \\ &+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u_h)(\psi_h)_x + g(u_h)(\psi_h)_y) dxdy \\ &- \int_{y_j}^{y_{j+1}} ((f(u_h)\psi_h^-)_{i+1,y} - (f(u_h)\psi_h^+)_{i,y}) dy \\ &- \int_{x_i}^{x_{i+1}} ((g(u_h)\psi_h^-)_{x,j+1} - (g(u_h)\psi_h^+)_{x,j}) dx, \end{aligned} \tag{3.2b}$$

where τ_{max} is a max step size, determined by $\tau_{max} = (\text{CFL factor}) \times h/(\text{maximum characteristic speed})$, in which the CFL constant should be less than 1/2. Similarly, for the initial condition we simply take $u_h(\cdot,\cdot,0) = \mathbb{P}_h u_0(\cdot,\cdot)$, $v_h(\cdot,\cdot,0) = \mathbb{Q}_h u_0(\cdot,\cdot)$, where $\mathbb{P}_h$ and $\mathbb{Q}_h$ are the L^2 projections into V_h^k and W_h^k, respectively, and we have

$$\begin{aligned} \|u_0 - \mathbb{P}_h u_0\|_{L^2(K_{i,j})} \le Ch^{k+1} \|u_0\|_{H^{k+1}(K_{i,j})}, \\ \|u_0 - \mathbb{Q}_h u_0\|_{L^2(K_{i+\frac{1}{2},j+\frac{1}{2}})} \le Ch^{k+1} \|u_0\|_{H^{k+1}(K_{i+\frac{1}{2},j+\frac{1}{2}})}. \end{aligned} \tag{3.3}$$

3.3 L^2 Stability for linear equation

The L^2-stability is proved for the CDG scheme (3.2) in [10] if $f(u)$ and $g(u)$ are linear. Without loss of generality, we take $f(u) = g(u) = u$. Hence, we have

$$\begin{cases} u_t + u_x + u_y = 0, & (x,y,t) \in \Omega \times (0,T], \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \end{cases} \tag{3.4}$$

with periodic boundary condition.

Theorem 3.1. *The numerical solutions u_h and v_h of the semi-discrete CDG scheme* (3.2) *for the equation* (3.4) *have the following L^2 stability property*

$$\|u_h(\cdot,\cdot,T)\|^2_{L^2(\Omega)} + \|v_h(\cdot,\cdot,T)\|^2_{L^2(\Omega)} \le \|u_h(\cdot,\cdot,0)\|^2_{L^2(\Omega)} + \|v_h(\cdot,\cdot,0)\|^2_{L^2(\Omega)}. \tag{3.5}$$

3.4 Optimal L^2 error estimate

In this subsection, we show the *a priori* L^2 error estimate of the scheme (3.2) for the equation (3.1).

Here and below, we again use $\|\cdot\|$ to denote the standard L^2 norm. Similar to the one-dimensional case, we recall the classical inverse and trace inequalities [2]. For any $w_h \in X_h^k$ or $w_h \in Y_h^k$, there exists a positive constant C independent of w_h and h, such that

$$\|\partial_x w_h\| \le Ch^{-1}\|w_h\|, \ \|w_h\|_\Gamma \le Ch^{-\frac{1}{2}}\|w_h\|, \ \|w_h\|_\infty \le Ch^{-1}\|w_h\|, \tag{3.6}$$

where Γ is the set of boundaries of all elements $K_{i,j}$ or $K_{i+\frac{1}{2},j+\frac{1}{2}}$.

Similar to the one-dimensional case, we first introduce some notations. Assume u_h and v_h are the numerical solutions of CDG scheme (3.2) for equation (3.1), we define

$$\begin{aligned}\tilde{B}_{i,j}(u_h, v_h; \varphi_h; f, g, u) :=& \frac{1}{\tau_{max}}\int_{K_{i,j}}(v_h - u_h)\varphi_h dxdy \\ &+ \int_{K_{i,j}}(f'(u(x_i,y_j))(\varphi_h)_x + g'(u(x_i,y_j))(\varphi_h)_y)v_h dxdy \\ &- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} f'(u(x_i,y_j))((v_h\varphi_h^-)_{i+\frac{1}{2},y} - (v_h\varphi_h^+)_{i-\frac{1}{2},y})dy \\ &- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} g'(u(x_i,y_j))((v_h\varphi_h^-)_{x,j+\frac{1}{2}} - (v_h\varphi_h^+)_{x,j+\frac{1}{2}})dx,\end{aligned} \tag{3.7a}$$

$$\begin{aligned}\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_h, v_h; \psi_h; f, g, u) :=& \frac{1}{\tau_{max}}\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(u_h - v_h)\psi_h dxdy \\ &+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(f'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))(\psi_h)_x \\ &+ g'(u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}))(\psi_h)_y)u_h dxdy\end{aligned}$$

$$
\begin{aligned}
&-\int_{y_j}^{y_{j+1}} f'(u(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}}))((u_h\psi_h^-)_{i+1,y}-(u_h\psi_h^+)_{i,y})dy \\
&-\int_{x_i}^{x_{i+1}} g'(u(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}}))((u_h\psi_h^-)_{x,j+1}-(u_h\psi_h^+)_{x,j})dx,
\end{aligned} \tag{3.7b}
$$

and

$$
\begin{aligned}
B_{i,j}(u_h,v_h;\varphi_h,\psi_h) = &\int_{K_{i,j}}(u_h)_t\varphi_h dxdy + \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(v_h)_t\psi_h dxdy \\
&-\frac{1}{\tau_{max}}\int_{K_{i,j}}(v_h-u_h)\varphi_h dxdy - \frac{1}{\tau_{max}}\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(u_h-v_h)\psi_h dx,
\end{aligned} \tag{3.8}
$$

Obviously, we have

$$
\begin{aligned}
B_{i,j}(u_h,v_h;\varphi_h,\psi_h) = &\int_{K_{i,j}}(f(v_h)(\varphi_h)_x + g(v_h)(\varphi_h)_y)dxdy \\
&+\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(f(u_h)(\psi_h)_x + g(u_h)(\psi_h)_y)dxdy \\
&-\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}((f(v_h)\varphi_h^-)_{i+\frac{1}{2},y}-(f(v_h)\varphi_h^+)_{i-\frac{1}{2},y})dy \\
&-\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}((g(v_h)\varphi_h^-)_{x,j+\frac{1}{2}}-(g(v_h)\varphi_h^+)_{x,j+\frac{1}{2}})dx \\
&-\int_{y_j}^{y_{j+1}}((f(u_h)\psi_h^-)_{i+1,y}-(f(u_h)\psi_h^+)_{i,y})dy \\
&-\int_{x_i}^{x_{i+1}}((g(u_h)\psi_h^-)_{x,j+1}-(g(u_h)\psi_h^+)_{x,j})dx, \\
&\forall\varphi_h\in Q^k(K_{i,j}), \quad \forall\psi_h\in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.9}
$$

Let u be the exact solution of equation (3.1), clearly we have

$$\begin{aligned}
B_{i,j}(u,u;\varphi_h,\psi_h) = &\int_{K_{i,j}} (f(u)(\varphi_h)_x + g(u)(\varphi_h)_y)dxdy \\
&+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u)(\psi_h)_x + g(u)(\psi_h)_y)dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(u)\varphi_h^-)_{i+\frac{1}{2},y} - (f(u)\varphi_h^+)_{i-\frac{1}{2},y})dy \\
&- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(u)\varphi_h^-)_{x,j+\frac{1}{2}} - (g(u)\varphi_h^+)_{x,j+\frac{1}{2}})dx \\
&- \int_{y_j}^{y_{j+1}} ((f(u)\psi_h^-)_{i+1,y} - (f(u)\psi_h^+)_{i,y})dy \\
&- \int_{x_i}^{x_{i+1}} ((g(u)\psi_h^-)_{x,j+1} - (g(u)\psi_h^+)_{x,j})dx, \\
&\forall \varphi_h \in Q^k(K_{i,j}), \quad \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.10}$$

Subtracting (3.9) from (3.10), we get the error equation for two-dimensional case,

$$\begin{aligned}
&B_{i,j}(u-u_h,u-v_h;\varphi_h,\psi_h) = \\
&\int_{K_{i,j}} (f(u)-f(v_h))(\varphi_h)_x + (g(u)-g(v_h))(\varphi_h)_y dxdy \\
&+ \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u)-f(u_h))(\psi_h)_x + (g(u)-g(u_h))(\psi_h)_y dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} [((f(u)-f(v_h))\varphi_h^-)_{i+\frac{1}{2},y} - ((f(u)-f(v_h))\varphi_h^+)_{i-\frac{1}{2},y}]dy \\
&- \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} [((g(u)-g(v_h))\varphi_h^-)_{x,j+\frac{1}{2}} - ((g(u)-g(v_h))\varphi_h^+)_{x,j+\frac{1}{2}}]dx \\
&- \int_{y_j}^{y_{j+1}} [((f(u)-f(u_h))\psi_h^-)_{i+1,y} - ((f(u)-f(u_h))\psi_h^+)_{i,y}]dy \\
&- \int_{x_i}^{x_{i+1}} [((g(u)-g(u_h))\psi_h^-)_{x,j+1} - ((g(u)-g(u_h))\psi_h^+)_{x,j}]dx \\
&:= H_{i,j}(f;u,u_h,v_h;\varphi_h,\psi_h),\ \forall \varphi_h \in Q^k(K_{i,j}),\ \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.11}$$

Summing over all i and j, the error equation becomes

$$\begin{aligned}
&\sum_{i,j} B_{i,j}(u-u_h,u-v_h;\varphi_h,\psi_h) = \sum_{i,j} H_{i,j}(f;u,u_h,v_h;\varphi_h,\psi_h), \\
&\quad \forall \varphi_h \in Q^k(K_{i,j}), \quad \forall \psi_h \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}}).
\end{aligned} \tag{3.12}$$

3.4.1 Projection operators

To prove the error estimates for two-dimensional problems in uniform Cartesian meshes, we need two suitable projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ similar to the one-dimensional case. By applying the shifting technique in the two-dimensional case, for x and y variables respectively, for a given function $w(x)$ we define $\mathbb{P}_h^* w \in Q^k(K_{i,j})$ over $K_{i,j}$ satisfying the following two equations,

$$\int_{K_{i,j}} \mathbb{P}_h^* w dxdy = \int_{K_{i,j}} w dxdy, \tag{3.13a}$$

$$\tilde{P}_h(\mathbb{P}_h^* w; \varphi_h; f, g, u)_{i,j} = \tilde{P}_h(w; \varphi_h; f, g, u)_{i,j}, \ \forall \varphi_h \in Q^k(K_{i,j}) \tag{3.13b}$$

where $\tilde{P}_h(w; \varphi_h; f, g, u)_{i,j}$ is defined as follows,

$$\begin{aligned}
\tilde{P}_h(w; \varphi_h; f, g, u)_{i,j} =& \frac{1}{\tau_{max}} \Bigg(\int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_{i-\frac{1}{2}}}^{x_i} w(x + \frac{h}{2}, y + \frac{h}{2}) \varphi_h dxdy \\
&+ \int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_i}^{x_{i+\frac{1}{2}}} w(x - \frac{h}{2}, y + \frac{h}{2}) \varphi_h dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} w(x + \frac{h}{2}, y - \frac{h}{2}) \varphi_h dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_i}^{x_{i+\frac{1}{2}}} w(x - \frac{h}{2}, y - \frac{h}{2}) \varphi_h dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, y) \varphi_h dxdy \Bigg) \\
&+ \int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_{i-\frac{1}{2}}}^{x_i} w(x + \frac{h}{2}, y + \frac{h}{2}) (f'(u(x_i, y_j)) \partial_x \varphi_h + g'(u(x_i, y_j)) \partial_y \varphi_h) dxdy \\
&+ \int_{y_{j-\frac{1}{2}}}^{y_j} \int_{x_i}^{x_{i+\frac{1}{2}}} w(x - \frac{h}{2}, y + \frac{h}{2}) (f'(u(x_i, y_j)) \partial_x \varphi_h + g'(u(x_i, y_j)) \partial_y \varphi_h) dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_i} w(x + \frac{h}{2}, y - \frac{h}{2}) (f'(u(x_i, y_j)) \partial_x \varphi_h + g'(u(x_i, y_j)) \partial_y \varphi_h) dxdy \\
&+ \int_{y_j}^{y_{j+\frac{1}{2}}} \int_{x_i}^{x_{i+\frac{1}{2}}} w(x - \frac{h}{2}, y - \frac{h}{2}) (f'(u(x_i, y_j)) \partial_x \varphi_h + g'(u(x_i, y_j)) \partial_y \varphi_h) dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_j} f'(u(x_i, y_j)) w(x_i, y + \frac{h}{2}) \Big(\varphi_h(x_{i+\frac{1}{2}}^-, y) - \varphi_h(x_{i-\frac{1}{2}}^+, y) \Big) dy \\
&- \int_{y_j}^{y_{j+\frac{1}{2}}} f'(u(x_i, y_j)) w(x_i, y - \frac{h}{2}) \Big(\varphi_h(x_{i+\frac{1}{2}}^-, y) - \varphi_h(x_{i-\frac{1}{2}}^+, y) \Big) dy
\end{aligned}$$

$$
\begin{aligned}
&- \int_{x_{i-\frac{1}{2}}}^{x_i} g'(u(x_i, y_j)) w(x + \frac{h}{2}, y_j) \Big(\varphi_h(x, y^-_{j+\frac{1}{2}}) - \varphi_h(x, y^+_{j-\frac{1}{2}}) \Big) dx \\
&- \int_{x_i}^{x_{i+\frac{1}{2}}} g'(u(x_i, y_j)) w(x - \frac{h}{2}, y_j) \Big(\varphi_h(x, y^-_{j+\frac{1}{2}}) - \varphi_h(x, y^+_{j-\frac{1}{2}}) \Big) dx,
\end{aligned} \tag{3.14}
$$

Similarly, we can define the projection $\mathbb{Q}_h^*$ from $w \in L^\infty(K_{i+\frac{1}{2},j+\frac{1}{2}})$ into $\mathbb{Q}_h^* w \in Q^k(K_{i+\frac{1}{2},j+\frac{1}{2}})$ over $K_{i+\frac{1}{2},j+\frac{1}{2}}$. Next we will discuss the properties of these two special projections. Without loss of generality we will only consider $\mathbb{P}_h^*$. The equation (3.13a) is required by conservation. Note that $\tilde{P}_h(w; \varphi_h)_{i,j} = 0$ for $\forall w$ when φ_h is a constant, so (3.13b) alone misses one condition which is provided by (3.13a), just like the one-dimensional case. Existence and optimal approximate property of the projection $\mathbb{P}_h^*$ are established in the following lemma.

Lemma 3.1. *The projection $\mathbb{P}_h^*$ defined by (3.13) exists and is unique for any smooth function $w(x)$, and the following inequality holds*

$$
\|\mathbb{P}_h^* w - w\| + h \|\mathbb{P}_h^* w - w\|_\infty + h^{\frac{1}{2}} \|\mathbb{P}_h^* w - w\|_\Gamma \leq C h^{k+1} \|w\|_{H^{k+1}(\Omega)}, \tag{3.15}
$$

for all k. The positive constant C depends on k, the bound of $f'(u)$, $g'(u)$, the constant c and is independent of h and w.

Proof. The proof of this lemma is given in Appendix A.3. □

Similarly, for $\mathbb{Q}_h^*$ we have

$$
\|\mathbb{Q}_h^* w - w\| + h \|\mathbb{Q}_h^* w - w\|_\infty + h^{\frac{1}{2}} \|\mathbb{Q}_h^* w - w\|_\Gamma \leq C h^{k+1} \|w\|_{H^{k+1}(\Omega)}, \tag{3.16}
$$

if w is a smooth function.

Again, the projections $\mathbb{P}_h^*$ and $\mathbb{Q}_h^*$ satisfy the following superconvergence result.

Lemma 3.2. *For $m = 0, 1..., 8$, assume that $u = x^{k+1}$ or y^{k+1}, let $u_I = \mathbb{P}_h^* u$ and $v_I = \mathbb{Q}_h^* u$ then*

$$
|\tilde{B}_{i,j}(u_I - u, v_I - u; \varphi_h; f, g, u)| \leq C h^{2k+4} + C \|\varphi_h\|^2_{L^2(K_{i,j})}, \tag{3.17}
$$

$$
|\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_I - u, v_I - u; \psi_h; f, g, u)| \leq C h^{2k+4} + C \|\psi_h\|^2_{L^2(K_{i+\frac{1}{2},j+\frac{1}{2}})}. \tag{3.18}
$$

Proof. The proof of this lemma is given in Appendix A.4. □

3.5 *A priori* L^2 error estimates

Now let us give the *a priori* error estimate for the two-dimensional case.

Theorem 3.2. *For $k = 0, 1..., 8$, let $u(\cdot,\cdot,t)$ be the exact solution of equation (3.1), which is sufficiently smooth with bounded derivatives, and assume $f \in C^2$ with bounded $f'(u)$ and $f''(u)$. The numerical solutions u_h and v_h of the CDG scheme (3.2) using uniform meshes satisfies the following L^2 error estimate*

$$\|u(\cdot,\cdot,T) - u_h(\cdot,\cdot,T)\|^2 + \|u(\cdot,\cdot,T) - v_h(\cdot,\cdot,T)\|^2 \le Ch^{2k+2}, \tag{3.19}$$

where k is the polynomial degree in the finite element spaces X_h^k and Y_h^k, and the constant C depends on k, the final time T, $\|u\|_{H^{k+2}}$ and the bounds on the derivatives $|f^{(m)}|$, $|g^{(m)}|$, $m = 1, 2$, but is independent of the mesh size h. Here $\|u\|_{H^{k+2}}$ is the maximum $(k+2)$-th order Sobolev norm of u over time in $[0,T]$. For $k = 0$ and 1 we need $f(u)$ and $g(u)$ to be linear, i.e. $f(u) = c_1 u$ and $g(u) = c_2 u$ with constants c_1 and c_2.

Proof. Let $e_u = u - u_h$, $e_v = u - v_h$ be the error between the numerical and exact solutions. Similar to the one-dimensional case, to deal with the nonlinearity of $f(u)$ and $g(u)$, we would like first make *a priori* assumption that, for small enough h, we have

$$\|u - u_h\| \le Ch^2, \ \|u - v_h\| \le Ch^2, \tag{3.20}$$

which also establishes the Lipschitz continuity of the right-hand side of the method of lines semi-discrete ordinary differential equation system, hence the very existence of u_h and v_h. By the interpolation property, we then have

$$\begin{aligned} \|e_u\|_\infty \le Ch \quad &\text{and} \quad \|\mathbb{P}_h^* u - u_h\|_\infty \le Ch, \\ \|e_v\|_\infty \le Ch \quad &\text{and} \quad \|\mathbb{Q}_h^* u - u_h\|_\infty \le Ch. \end{aligned} \tag{3.21}$$

This assumption is not necessary for linear f and g. We will verify this assumption for $k \ge 2$ later.

By taking

$$\varphi_h = \mathbb{P}_h^* u - u_h,\ \ \psi_h = \mathbb{Q}_h^* u - v_h, \varphi^e = \mathbb{P}_h^* u - u,\ \ \psi^e = \mathbb{Q}_h^* u - u, \tag{3.22}$$

we obtain the energy equality

$$\sum_{i,j} B_{i,j}(\varphi_h - \varphi^e, \psi_h - \psi^e; \varphi_h, \psi_h) = \sum_{i,j} H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h). \tag{3.23}$$

From the definition of $B_{i,j}$, we can obtain

$$\begin{aligned}
&\sum_{i,j} B_{i,j}(\varphi_h, \psi_h; \varphi_h, \psi_h) \\
=&\sum_{i,j} B_{i,j}(\varphi^e, \psi^e; \varphi_h, \psi_h) + \sum_{i,j} H_{i,j}(f; u, u_h, v_h; \varphi_h, \psi_h) \\
=&\sum_{i,j} \int_{K_{i,j}} (\psi^e)_t \varphi_h dxdy + \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (\varphi^e)_t \psi_h dxdy \\
&- \sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i,j}} (\psi^e - \varphi^e)\varphi_h dxdy - \sum_{i,j} \frac{1}{\tau_{max}} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (\varphi^e - \psi^e)\psi_h dxdy \\
&+ \sum_{i,j} \int_{K_{i,j}} (f(u) - f(v_h))(\varphi_h)_x + (g(u) - g(v_h))(\varphi_h)_y dxdy \\
&+ \sum_{i,j} \int_{K_{i+\frac{1}{2},j+\frac{1}{2}}} (f(u) - f(u_h))(\psi_h)_x + (g(u) - g(u_h))(\psi_h)_y dxdy \\
&+ \sum_{i,j} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} ((f(u) - f(v_h))[\varphi_h])_{i+\frac{1}{2},y} dy + \sum_{i,j} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} ((g(u) - g(v_h))[\varphi_h])_{x,j+\frac{1}{2}} dx \\
&+ \sum_{i,j} \int_{y_j}^{y_{j+1}} ((f(u) - f(u_h))[\psi_h])_{i,y} dy + \sum_{i,j} \int_{x_i}^{x_{i+1}} ((g(u) - g(u_h))[\psi_h])_{x,j} dx.
\end{aligned} \tag{3.24}$$

For the left-hand side of (3.24), we follow the L^2 stability proof in Theorem 3.1 for linear case to conclude

$$\sum_{i,j} B_{i,j}(\varphi_h, \psi_h; \varphi_h, \psi_h) = \frac{1}{2}\frac{d}{dt}\int_\Omega (\varphi_h^2 + \psi_h^2) dx + \frac{1}{\tau_{max}} \int_\Omega (\varphi_h - \psi_h)^2 dx. \tag{3.25}$$

Similar to the proof in [19] and [16], to deal with the nonlinear part of (3.24) we would like to use the following Taylor expansions:

$$\begin{aligned} f(u)-f(u_h) =& f'(u)\varphi_h - f'(u)\varphi^e - \frac{1}{2}f''_u(\varphi_h-\varphi^e)^2, \\ f(u)-f(v_h) =& f'(u)\psi_h - f'(u)\psi^e - \frac{1}{2}f''_v(\psi_h-\psi^e)^2, \\ g(u)-g(u_h) =& g'(u)\varphi_h - g'(u)\varphi^e - \frac{1}{2}g''_u(\varphi_h-\varphi^e)^2, \\ g(u)-g(v_h) =& g'(u)\psi_h - g'(u)\psi^e - \frac{1}{2}g''_v(\psi_h-\psi^e)^2, \end{aligned} \tag{3.26}$$

where f''_u, f''_v and g''_u, g''_v are the mean values. These imply the following representation,

$$\begin{aligned} &\sum_{i,j} B_{i,j}(\varphi^e,\psi^e;\varphi_h,\psi_h) + \sum_{i,j} H_{i,j}(f;u,u_h,v_h;\varphi_h,\psi_h) \\ =& \mathcal{L} + \mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N}_4, \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} \mathcal{L} =& \sum_{i,j}\int_{K_{i,j}} (\psi^e)_t\varphi_h dxdy + \sum_{i,j}\int_{K_{i+\frac12,j+\frac12}} (\varphi^e)_t\psi_h dxdy, \\ \mathcal{N}_1 =& -\sum_{i,j}\frac{1}{\tau_{max}}\int_{K_{i,j}}(\psi^e-\varphi^e)\varphi_h dxdy \\ &-\sum_{i,j}\int_{K_{i,j}}(f'(u)\psi^e(\varphi_h)_x + g'(u)\psi^e(\varphi_h)_y)dxdy \\ &-\sum_{i,j}\int_{y_{j-\frac12}}^{y_{j+\frac12}}(f'(u)\psi^e[\varphi_h])_{i+\frac12,y}dy - \sum_{i,j}\int_{x_{i-\frac12}}^{x_{i+\frac12}}(g'(u)\psi^e[\varphi_h])_{x,j+\frac12}dx, \\ \mathcal{N}_2 =& -\sum_{i,j}\frac{1}{\tau_{max}}\int_{K_{i+\frac12,j+\frac12}}(\varphi^e-\psi^e)\psi_h dxdy \\ &-\sum_{i,j}\int_{K_{i+\frac12,j+\frac12}}(f'(u)\varphi^e(\psi_h)_x + g'(u)\varphi^e(\psi_h)_y)dxdy \\ &-\sum_{i,j}\int_{y_j}^{y_{j+1}}(f'(u)\varphi^e[\psi_h])_{i,y}dy - \sum_{i,j}\int_{x_i}^{x_{i+1}}(g'(u)\varphi^e[\psi_h])_{x,j}dx, \\ \mathcal{N}_3 =& \sum_{i,j}\int_{K_{i,j}}(f'(u)\psi_h(\varphi_h)_x + g'(u)\psi_h(\varphi_h)_y)dxdy \\ &+\sum_{i,j}\int_{K_{i+\frac12,j+\frac12}}(f'(u)\varphi_h(\psi_h)_x + g'(u)\varphi_h(\psi_h)_y)dxdy \\ &+\sum_{i,j}\int_{y_{j-\frac12}}^{y_{j+\frac12}}(f'(u)\psi_h[\varphi_h])_{i+\frac12,y}dy + \sum_{i,j}\int_{x_{i-\frac12}}^{x_{i+\frac12}}(g'(u)\psi_h[\varphi_h])_{x,j+\frac12}dx \\ &+\sum_{i,j}\int_{y_j}^{y_{j+1}}(f'(u)\varphi_h[\psi_h])_{i,y}dy + \sum_{i,j}\int_{x_i}^{x_{i+1}}(g'(u)\varphi_h[\psi_h])_{x,j}dx, \end{aligned}$$

$$\begin{aligned}
\mathcal{N}_4 = &-\frac{1}{2}\Big(\sum_{i,j}\int_{K_{i,j}}(f_v''(\psi_h-\psi^e)^2(\varphi_h)_x+g_v''(\psi_h-\psi^e)^2(\varphi_h)_y)dxdy\\
&+\sum_{i,j}\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(f_u''(\varphi_h-\varphi^e)^2(\psi_h)_x+g_u''(\varphi_h-\varphi^e)^2(\psi_h)_y)dxdy\\
&+\sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}(f_v''(\psi_h-\psi^e)^2[\varphi_h])_{i+\frac{1}{2},y}dy+\sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(g_v''(\psi_h-\psi^e)^2[\varphi_h])_{x,j+\frac{1}{2}}dx\\
&+\sum_{i,j}\int_{y_j}^{y_{j+1}}(f_u''(\varphi_h-\varphi^e)^2[\psi_h])_{i,y}dy+\sum_{i,j}\int_{x_i}^{x_{i+1}}(g_u''(\varphi_h-\varphi^e)^2[\psi_h])_{x,j}dx\Big).
\end{aligned}$$

By Young's inequality and (3.15), (3.16) we have

$$\mathcal{L}\le C(\|\varphi_h\|^2+\|\psi_h\|^2)+Ch^{2k+2}\|u\|^2_{H^{k+1}(\Omega)}. \tag{3.28}$$

Next we estimate the nonlinear part. First for the $\mathcal{N}_1$ term, we can rewrite it as

$$\begin{aligned}
\mathcal{N}_1 = &-\sum_{i,j}\frac{1}{\tau_{max}}\int_{K_{i,j}}(\psi^e-\varphi^e)\varphi_h dxdy\\
&-\sum_{i,j}\int_{K_{i,j}}(f'(u(x_i,y_j))\psi^e(\varphi_h)_x+g'(u(x_i,y_j))\psi^e(\varphi_h)_y)dxdy\\
&-\sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}(f'(u(x_i,y_j))\psi^e[\varphi_h])_{i+\frac{1}{2},y}dy-\sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}(g'(u(x_i,y_j))\psi^e[\varphi_h])_{x,j+\frac{1}{2}}dx\\
&+\sum_{i,j}\int_{K_{i,j}}((f'(u(x_i,y_j))-f'(u))\psi^e(\varphi_h)_x+(g'(u(x_i,y_j))-g'(u))\psi^e(\varphi_h)_y)dxdy\\
&+\sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}((f'(u(x_i,y_j))-f'(u))\psi^e[\varphi_h])_{i+\frac{1}{2},y}dy\\
&+\sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}((g'(u(x_i,y_j))-g'(u))\psi^e[\varphi_h])_{x,j+\frac{1}{2}}dx\\
= &-\sum_{i,j}\tilde{B}_{i,j}(\varphi^e,\psi^e;\varphi_h)\\
&+\sum_{i,j}\int_{K_{i,j}}((f'(u(x_i,y_j))-f'(u))\psi^e(\varphi_h)_x+(g'(u(x_i,y_j))-g'(u))\psi^e(\varphi_h)_y)dxdy\\
&+\sum_{i,j}\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}((f'(u(x_i,y_j))-f'(u))\psi^e[\varphi_h])_{i+\frac{1}{2},y}dy\\
&+\sum_{i,j}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}((g'(u(x_i,y_j))-g'(u))\psi^e[\varphi_h])_{x,j+\frac{1}{2}}dx.
\end{aligned}$$

By using the inequality in (3.6), (3.15), (3.16) and $\|f'(u(x_i,y_j))-f'(u)\|_{L^\infty(K_{i,j})}=$

$O(h), \|g'(u(x_i, y_j)) - g'(u)\|_{L^\infty(K_{i,j})} = O(h)$, we have

$$N_1 \leq -\sum_j \tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h) + C_* \|\varphi_h\|^2 + C_* h^{2k+2} \|u\|^2_{H^{k+1}([a,b])}. \tag{3.29}$$

For $\tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h)$, we know that for an arbitrary element $K_{i,j}$, we can obtain the following results from Lemma 3.2, for $\forall u \in P^{k+1}([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]),\ \forall \varphi_h \in Q^k(K_{i,j})$

$$|\tilde{B}_{i,j}(\mathbb{P}_h^* u - u, \mathbb{Q}_h^* u - u; \varphi_h; f, g, u)| \leq C h^{2k+4} + C \|\varphi_h\|^2_{L^2(K_{i,j})}, \tag{3.30}$$

On each element $K_{i,j}$ we consider the following Taylor expansion of u around (x_i, y_j),

$$u = Tu + Ru, \tag{3.31}$$

where

$$Tu = \sum_{l=0}^{k+1} \sum_{m=0}^{l} \frac{1}{m!(l-m)!} \frac{\partial^l u(x_i, y_j)}{\partial x^{l-m} \partial y^m} (x - x_i)^{l-m} (y - y_j)^m, \tag{3.32}$$

$$Ru = \sum_{m=0}^{k+2} \frac{(k+2)(x - x_i)^{k+2-m} (y - y_j)^m}{m!(k+2-m)!} \int_0^1 (1-s)^{k+1} \frac{\partial^{k+2} u(x_i^{(s)}, y_j^{(s)})}{\partial x^{k+2-m} \partial y^m} ds. \tag{3.33}$$

with $x_i^{(s)} = x_i + s(x - x_i),\ y_j^{(s)} = y_j + s(y - y_j)$. It is obvious that $Tu \in P^k([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}])$. Note that the operator $\mathbb{P}_h^*$ is a linear operator and $\mathbb{P}_h^* u = \mathbb{P}_h^* Tu + \mathbb{P}_h^* Ru$, we obtain from (3.30) that

$$\begin{aligned}
\tilde{B}_{i,j}(\varphi^e, \psi^e; \varphi_h; f, g, u) =& \tilde{B}_{i,j}(\mathbb{P}_h^* Tu - Tu + \mathbb{P}_h^* Ru - Ru, \\
& \mathbb{Q}_h^* Tu - Tu + \mathbb{Q}_h^* Ru - Ru; \varphi_h; f, g, u) \\
=& \tilde{B}_{i,j}(\mathbb{P}_h^* Tu - Tu, \mathbb{Q}_h^* Tu - Tu; \varphi_h; f, g, u) \\
& + \tilde{B}_{i,j}(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru; \varphi_h; f, g, u) \\
=& \tilde{B}_{i,j}(\mathbb{P}_h^* Ru - Ru, \mathbb{Q}_h^* Ru - Ru; \varphi_h; f, g, u) \\
& + C h^{2k+4} + C \|\varphi_h\|^2_{L^2(K_{i,j})}.
\end{aligned} \tag{3.34}$$

Recalling the Bramble-Hilbert lemma [2], we have

$$\|Ru\|_{L^\infty(K_{i,j})} \leq C h^{k+1} |u|_{H^{k+2}(K_{i,j})}. \tag{3.35}$$

Therefore, by using Young's inequality, (3.15), (3.16), (3.6) and (3.35), we have

$$-\sum_{i,j}\tilde{B}_{i,j}(\varphi^e,\psi^e;\varphi_h;f,g,u)\le Ch^{2k+2}\|u\|_{H^{k+2}(\Omega)}+C\|\varphi_h\|^2. \tag{3.36}$$

Hence, for N_1 we have

$$\mathcal{N}_1\le (C+C_*)\|\varphi_h\|^2+(C+C_*)h^{2k+2}\|u\|^2_{H^{k+2}(\Omega)}. \tag{3.37}$$

Similarly, for N_2 we have

$$\mathcal{N}_2\le (C+C_*)\|\psi_h\|^2+(C+C_*)h^{2k+2}\|u\|^2_{H^{k+2}(\Omega)}. \tag{3.38}$$

Similar to the one-dimensional case, the $\mathcal{N}_3$ term can be rewritten as

$$\begin{aligned}
\mathcal{N}_3 =&\sum_{i,j}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{x_{i-\frac12}}^{x_i} f'(u)(\psi_h\varphi_h)_x dxdy+\sum_{i,j}\int_{y_{j-\frac12}}^{y_{j+\frac12}}\int_{x_i}^{x_{i+\frac12}} f'(u)(\psi_h\varphi_h)_x dxdy\\
&+\sum_{i,j}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{y_{j-\frac12}}^{y_j} g'(u)(\psi_h\varphi_h)_y dydx+\sum_{i,j}\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{y_j}^{y_{j+\frac12}} g'(u)(\psi_h\varphi_h)_y dydx\\
&+\sum_{i,j}\int_{y_{j-\frac12}}^{y_{j+\frac12}}(f'(u)\psi_h[\varphi_h])_{i+\frac12,y}dy+\sum_{i,j}\int_{x_{i-\frac12}}^{x_{i+\frac12}}(g'(u)\psi_h[\varphi_h])_{x,j+\frac12}dx\\
&+\sum_{i,j}\int_{y_j}^{y_{j+1}}(f'(u)\varphi_h[\psi_h])_{i,y}dy+\sum_{i,j}\int_{x_i}^{x_{i+1}}(g'(u)\varphi_h[\psi_h])_{x,j}dx\\
=&\sum_{i,j}\Big(\int_{y_{j-\frac12}}^{y_{j+\frac12}}((f'(u)\psi_h\varphi_h^-)_{i+\frac12,y}-(f'(u)\psi_h\varphi_h^+)_{i-\frac12,y}\\
&+(f'(u)\varphi_h\psi_h^-)_{i,y}-(f'(u)\varphi_h\psi_h^+)_{i,y}+(f'(u)\psi_h[\varphi_h])_{i+\frac12,y})dy\\
&+\int_{x_{i-\frac12}}^{x_{i+\frac12}}((g'(u)\psi_h\varphi_h^-)_{x,j+\frac12}-(g'(u)\psi_h\varphi_h^+)_{x,j-\frac12}\\
&+(g'(u)\varphi_h\psi_h^-)_{x,j}-(g'(u)\varphi_h\psi_h^+)_{x,j}+(g'(u)\psi_h[\varphi_h])_{x,j+\frac12})dx\\
&+\int_{y_j}^{y_{j+1}}(f'(u)\varphi_h[\psi_h])_{i,y}dy+\int_{x_i}^{x_{i+1}}(g'(u)\varphi_h[\psi_h])_{x,j}dx\\
&-\int_{K_{i,j}}((f'(u))_x+(g'(u))_y)\varphi_h\psi_h dxdy\Big)\\
=&-\sum_{i,j}\int_{K_{i,j}}((f'(u))_x+(g'(u))_y)\varphi_h\psi_h dxdy\\
\le& C_*\|\varphi_h\|\|\psi_h\|\le C_*(\|\varphi_h\|^2+\|\psi_h\|^2).
\end{aligned} \tag{3.39}$$

$\mathcal{N}_4$ is the high order term in Taylor expansion, its easy to show that

$$
\begin{aligned}
\mathcal{N}_4 \leq & C_* h^{-1}(\|e_v\|_\infty \|e_v\| \|\varphi_h\| + \|e_u\|_\infty \|e_u\| \|\psi_h\|) \\
\leq & C_* h^{-1}\Big(\|e_v\|_\infty (\|\varphi_h\| \|\psi_h\| + \|\varphi_h\| \|\psi^e\|) + \|e_v\|_\infty (\|\psi_h\| \|\varphi_h\| + \|\psi_h\| \|\varphi^e\|)\Big) \\
\leq & C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
& + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty) h^{2k+2} \|u\|^2_{H^{k+1}(\Omega)}.
\end{aligned} \tag{3.40}
$$

Then by combining (3.28), (3.37), (3.38), (3.39), (3.40), (3.25), we obtain from (3.24)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (\varphi_h^2 + \psi_h^2) dxdy \leq & (C + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty))(\|\varphi_h\|^2 + \|\psi_h\|^2) \\
& + (C + C_*(h^{-1}\|e_v\|_\infty + h^{-1}\|e_u\|_\infty)) h^{2k+2} \|u\|^2_{H^{k+2}(\Omega)}.
\end{aligned} \tag{3.41}
$$

When $k \geq 2$, by using *a priori* assumption (3.21) we have

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega (\varphi_h^2 + \psi_h^2) dxdy \leq (C + C_*)(\|\varphi_h\|^2 + \|\psi_h\|^2) + (C + C_*) h^{2k+2} \|u\|^2_{H^{k+2}(\Omega)}. \tag{3.42}
$$

Finally, by Gronwall's inequality and the fact that $\|\varphi_h(\cdot,\cdot,0)\| \leq Ch^{k+1}$, $\|\psi_h(\cdot,\cdot,0)\| \leq Ch^{k+1}$ we can get

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega (\varphi_h^2 + \psi_h^2) dxdy \leq Ch^{2k+2}. \tag{3.43}
$$

This, together with the approximation result (3.15), (3.16) implies the desired error estimate.

For the case of $k = 0$ or 1, we assume that $f(u)$ and $g(u)$ are linear fluxes, namely $f(u) = c_1 u,\ g(u) = c_2 u$ with constants $c_1,\ c_2$. This is to avoid the need of the *a priori* assumption (3.20) which is no longer justifiable in this case. By similar lines of proof and noting that $C_* = 0$ in this case, we can obtain

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega (\varphi_h^2 + \psi_h^2) dxdy \leq C(\|\varphi_h\|^2 + \|\psi_h\|^2) + Ch^{2k+2},\ k = 0,\ 1. \tag{3.44}
$$

By using the Gronwall's inequality we have

$$
\int_\Omega (\varphi_h^2 + \psi_h^2) dxdy \leq Ch^{2k+2},\ k = 0,\ 1. \tag{3.45}
$$

This, together with the approximation result (3.15), (3.16), implies the desired error estimate for $k = 0,\ 1$ with linear fluxes.

Just like the one-dimensional case, let us justify the *a priori* assumption (3.20) with $k \geq 2$. Similar to [19] and [1], we can verify this by a proof by contradiction. By (3.19), we can consider h small enough so that $Ch^{k+1} < \frac{1}{2}h^2$, where C is the constant in (3.19) determined by the final time T. Define $t^* = sup\{t : \|u(\cdot,\cdot,t) - u_h(\cdot,\cdot,t)\| + \|u(\cdot,\cdot,t) - v_h(\cdot,\cdot,t)\| \leq h^2\}$, then we have $\|u(\cdot,\cdot,t^*) - u_h(\cdot,\cdot,t^*)\| + \|u(\cdot,\cdot,t^*) - v_h(\cdot,\cdot,t^*)\| = h^2$ by continuity if t^* is finite. Clearly, (3.19) holds for $t \leq t^*$, in particular, $\|u(\cdot,\cdot,t^*) - u_h(\cdot,\cdot,t^*)\| + \|u(\cdot,\cdot,t^*) - v_h(\cdot,\cdot,t^*)\| \leq Ch^{k+1} < \frac{1}{2}h^2$. This is a contradiction if $t^* < T$. Hence, $t^* \geq T$ and our *a priori* assumption is justified. □

4 Numerical examples

In this section, we present numerical examples to verify our theoretical findings. Uniform meshes are used in all examples. The schemes are integrated in time with the third order SSP Runge-Kutta method. We would like to compute on elements of degree $k = 0, 1, 2, 3$. We set the CFL number to be 0.05. For $k = 0, 1, 2$ we let $\Delta t = CFL \cdot h$ and $\Delta t = CFL \cdot h^{\frac{4}{3}}$ for $k = 3$ where h is the characteristic length of the mesh, so that the time error will be dominated by the spatial error.

Example 4.1. *We solve the one-dimensional Burgers equation given by*

$$\begin{cases} u_t + (\dfrac{u^2}{2})_x = 0, \ \ x \in [-\pi, \pi], \\ u(x,0) = \sin(x), \ \ x \in [-\pi, \pi], \\ u(-\pi, t) = u(\pi, t). \end{cases} \tag{4.1}$$

The exact solution is obtained by Newton iteration. In this example, we use $\tau_{max} = \frac{h}{2k+1}$, $h = \frac{2\pi}{N}$ to test the numerical schemes. The errors and numerical order of accuracy at $T = 0.5$ with $0 \leq k \leq 3$ are listed in Tables 4.1.

Table 4.1 shows that the order of convergence of the error achieves the expected $(k+1)$-th order of accuracy.

k	N	L^1 error	order	L^2 error	order	L^∞ error	order
0	10	6.73E-001	-	3.65E-001	-	5.60E-001	-
	20	3.34E-001	1.01	1.83E-001	0.99	3.04E-001	0.88
	40	1.66E-001	1.00	9.19E-002	1.00	1.56E-001	0.97
	80	8.31E-002	1.00	4.60E-002	1.00	7.90E-002	0.98
	160	4.15E-002	1.00	2.30E-002	1.00	3.97E-002	0.99
1	10	6.90E-002	-	4.40E-002	-	8.69E-002	-
	20	1.86E-002	1.89	1.25E-002	1.81	2.58E-002	1.75
	40	4.73E-003	1.98	3.21E-003	1.97	7.34E-003	1.81
	80	1.19E-003	1.99	8.11E-004	1.98	1.95E-003	1.92
	160	2.98E-004	2.00	2.04E-004	1.99	4.94E-004	1.98
2	10	9.68E-003	-	8.58E-003	-	2.53E-002	-
	20	8.97E-004	3.43	9.29E-004	3.21	4.24E-003	2.58
	40	1.13E-004	2.99	1.14E-004	3.02	6.03E-004	2.82
	80	1.42E-005	2.99	1.44E-005	2.98	7.87E-005	2.94
	160	1.78E-006	3.00	1.81E-006	2.99	9.99E-006	2.98
3	10	6.06E-04	-	6.47E-04	-	3.26E-03	-
	20	6.17E-05	3.30	6.91E-05	3.23	2.73E-04	3.58
	40	4.54E-06	3.77	5.54E-06	3.64	3.21E-05	3.09
	80	2.86E-07	3.99	3.49E-07	3.99	2.06E-06	3.96
	160	1.79E-08	4.00	2.19E-08	4.00	1.30E-07	3.99

Table 4.1. Errors and numerical orders of accuracy for Example 4.1 on a uniform mesh of N cells. Here $\tau_{max} = \frac{h}{2k+1}$ and final time $T = 0.5$.

Example 4.2. *We solve the two-dimensional Burgers equation given by*

$$\begin{cases} u_t + (\frac{u^2}{2})_x + (\frac{u^2}{2})_y = 0, \ (x,y) \in [-\pi,\pi]^2, \\ u(x,y,0) = \sin(x+y), \ (x,y) \in [-\pi,\pi]^2, \end{cases} \tag{4.2}$$

with periodic boundary condition. The exact solution follows from the solution of one-dimensional Burgers equation with $\xi = x + y$. In this example, we use $\tau_{max} = \frac{h}{2k+1}$, $h = \frac{2\pi}{N}$ to test the numerical schemes. The central DG scheme is evolved up to $T = 0.2$ when the solution is still smooth. The errors and numerical order of accuracy with $0 \le k \le 3$ are listed in Tables 4.2.

Table 4.2 shows that the order of convergence of the error achieves the expected $(k+1)$-th order of accuracy.

k	N× N	L^1 error	order	L^2 error	order	L^∞ error	order
0	10×10	5.57E+00	-	1.22E+00	-	8.16E-01	-
	20×20	2.76E+00	1.01	6.17E-01	0.98	4.87E-01	0.74
	40×40	1.37E+00	1.01	3.09E-01	1.00	2.57E-01	0.92
	80×80	6.81E-01	1.01	1.54E-01	1.00	1.30E-01	0.98
	160×160	3.40E-01	1.00	7.72E-02	1.00	6.54E-02	0.99
1	10×10	9.12E-01	-	2.34E-01	-	2.60E-01	-
	20×20	2.37E-01	1.94	6.25E-02	1.90	8.19E-02	1.67
	40×40	5.99E-02	1.99	1.60E-02	1.97	2.19E-02	1.90
	80×80	1.50E-02	2.00	4.02E-03	1.99	5.71E-03	1.94
	160×160	3.75E-03	2.00	1.01E-03	2.00	1.45E-03	1.98
2	10×10	1.49E-01	-	5.03E-02	-	1.22E-01	-
	20×20	1.91E-02	2.97	6.44E-03	2.97	2.14E-02	2.52
	40×40	2.38E-03	3.00	8.33E-04	2.95	3.00E-03	2.83
	80×80	3.00E-04	2.99	1.05E-04	2.98	3.87E-04	2.96
	160×160	3.77E-05	2.99	1.33E-05	2.99	4.87E-05	2.99
3	10×10	2.06E-02	-	7.45E-03	-	2.20E-02	-
	20×20	2.04E-03	3.33	8.72E-04	3.09	3.30E-03	2.74
	40×40	1.48E-04	3.79	6.09E-05	3.84	2.50E-04	3.72
	80×80	9.70E-06	3.93	4.02E-06	3.92	1.78E-05	3.81
	160×160	6.19E-07	3.97	2.62E-07	3.94	1.17E-06	3.92

Table 4.2. Errors and numerical orders of accuracy for Example 4.2 on a uniform mesh of $N \times N$ cells. Here $\tau_{max} = \frac{h}{2k+1}$ and final time $T = 0.2$.

5 Concluding remarks

In this paper, *a priori* optimal L^2 error estimates to central DG methods on uniform meshes applied to nonlinear conservation laws with smooth solutions are proved with polynomial degrees of $k \leq 8$. The main techniques used in this paper are special projections and Taylor expansions. Our analysis is carried out both in one dimension and in two-dimensions for uniform Cartesian meshes and tensor-product polynomial spaces. We also give some numerical examples to verify the results of our theoretical analysis. The error estimates for nonlinear conservation laws in this paper were obtained using stability for the linear case and the smoothness of the exact solution. It is not clear whether stability holds for the scalar nonlinear conservation laws with general non-smooth solutions. Such a stability proof for the central DG schemes and the extension of this work to non-uniform meshes and unstructured triangular meshes are interesting and challenging, and constitutes our ongoing work.

A Appendix: Collection of technical proofs

In this appendix, we collect the proofs of some technical lemmas and propositions.

A.1 Proof of Lemma 2.1

Proof. We only consider $\mathbb{P}_h^*$, while the proof for $\mathbb{Q}_h^*$ follows similar lines. For $\forall j$, we let $\xi = \frac{2(x-x_j)}{h}$ on I_j, for a smooth function $\omega(x)$ and a k-th order polynomial $\varphi_h(x)$ on I_j, and define

$$\begin{aligned}\tilde{\omega}(\xi) &= \omega(\frac{h}{2}\xi + x_j) = \omega(x),\\ \phi_h(\xi) &= \varphi_h(\frac{h}{2}\xi + x_j) = \varphi_h(x).\end{aligned} \tag{A.1}$$

Note that the procedure to find the $\mathbb{P}_h^*\tilde{\omega} \in \mathbb{P}^k([-1,1])$ is to solve for a linear system, so existence of the projection can be proved by proving its uniqueness. Thus, we only need to prove the uniqueness of the projection $\mathbb{P}_h^*$. We set $\omega_I(\xi) = \mathbb{P}_h^*\tilde{\omega}(\xi) = \mathbb{P}_h^*\omega(x)$ with $\tilde{\omega}(\xi) = \omega(x) = 0$, and would like to prove $\omega_I(\xi) = 0$. Then by the definition of the projection $\mathbb{P}_h^*$, we have:

$$\begin{aligned}\tilde{P}_h(\omega_I;\phi_h;f,u)_j =& \frac{h}{2\tau_{max}}(\int_{-1}^0 \omega_I(\xi+1)\phi_h(\xi)d\xi + \int_0^1 \omega_I(\xi-1)\phi_h(\xi)d\xi\\ &- \int_{-1}^1 \omega_I(\xi)\phi_h(\xi)d\xi) + \int_{-1}^0 f'(u(x_j))\omega_I(\xi+1)(\phi_h(\xi))_\xi d\xi\\ &+ \int_0^1 f'(u(x_j))\omega_I(\xi-1)(\phi_h(\xi))_\xi d\xi\\ &- f'(u(x_j))\omega_I(0)(\phi_h(1)-\phi_h(-1))\\ =&0,\end{aligned} \tag{A.2a}$$

$$\frac{h}{2}\int_{-1}^1 \omega_I(\xi)d\xi = 0. \tag{A.2b}$$

Let $\phi_h(\xi) = \omega_I(\xi) \in \mathbb{P}^k([-1,1])$, we get

$$\begin{aligned}\tilde{P}_h(\omega_I;\omega_I;f,u)_j =& \frac{h}{2\tau_{max}}\left(\int_{-1}^0 \omega_I(\xi+1)\omega_I(\xi)d\xi + \int_0^1 \omega_I(\xi-1)\omega_I(\xi)d\xi - \int_{-1}^1 \omega_I(\xi)^2 d\xi\right)\\ &+ \int_{-1}^0 f'(u(x_j))\omega_I(\xi+1)(\omega_I(\xi))_\xi d\xi + \int_0^1 f'(u(x_j))\omega_I(\xi-1)(\omega_I(\xi))_\xi d\xi\end{aligned}$$

$$- f'(u(x_j))\omega_I(0)\left(\omega_I(1) - \omega_I(-1)\right) = 0. \tag{A.3}$$

We rewrite $\tilde{P}_h(\omega_I; \omega_I; f, u)_j$ by a change of variable $\xi \to \xi + 1$ for the integrations over $[-1, 0]$ to get

$$\begin{aligned}
\tilde{P}_h(\omega_I; \omega_I; f, u)_j =& \frac{h}{2\tau_{max}}\left(2\int_0^1 \omega_I(\xi-1)\omega_I(\xi)d\xi - \int_0^1 \omega_I(\xi-1)^2 d\xi - \int_0^1 \omega_I(\xi)^2 d\xi\right) \\
&+ \int_0^1 f'(u(x_j))\omega_I(\xi)(\omega_I(\xi-1))_\xi d\xi + \int_0^1 f'(u(x_j))\omega_I(\xi-1)(\omega_I(\xi))_\xi d\xi \\
&- f'(u(x_j))\omega_I(0)(\omega_I(1) - \omega_I(-1)) \\
=& - \frac{h}{2\tau_{max}}\int_0^1 (\omega_I(\xi) - \omega_I(\xi-1))^2 d\xi = 0.
\end{aligned} \tag{A.4}$$

Thus,

$$\omega_I(\xi) = \omega_I(\xi - 1), \quad \forall \xi \in (0, 1). \tag{A.5}$$

Next we will show that $\omega_I(\xi)$ is a constant on $[-1, 1]$. Let $\omega_I(\xi) = \sum_{i=0}^k a_i\xi^i,\ \xi \in [-1, 1]$. For $k = 0$ it clearly holds. For $k \geq 1$, now from (A.5) we have

$$G(\xi) := \omega_I(\xi) - \omega_I(\xi - 1) = \sum_{i=1}^k a_i(\xi^i - (\xi - 1)^i) = 0, \quad \forall \xi \in (0, 1). \tag{A.6}$$

Assume $a_i,\ 1 \leq i \leq k$ are not all zeros, then $G(\xi)$ is a non-zero polynomial of degree at most $k-1$, thus it has at most $k-1$ roots, which is a contradiction to (A.6). Hence, we have $a_i = 0,\ \forall 1 \leq i \leq k$, which indicates that $\omega_I(\xi)$ is a constant on $[-1, 1]$. Hence, by (A.2b), we have

$$\frac{h}{2}\int_{-1}^1 \omega_I(\xi)d\xi = h\omega_I(\xi) = 0, \tag{A.7}$$

which implies $\omega_I(\xi) \equiv 0$ on $[-1, 1]$.

We have now finished the proof of uniqueness. Next we move to prove the boundedness. Let $\omega_I(x) = \mathbb{P}_h^*\omega(x) = \sum_{i=0}^k a_i x^i$ and set the test functions $\varphi_h = x, x^2, ..., x^k$. Then we have

$$\tilde{P}_h(\omega_I; x^l; f, u)_j = \sum_{i-0}^k \alpha_{il}a_i,\ 1 \leq l \leq k, \tag{A.8}$$

$$\int_{-1}^1 w_I(x)dx = \sum_{i=0}^k \frac{1^{i+1} - (-1)^{i+1}}{i+1}a_i = \sum_{i=0}^k \alpha_{i0}a_i. \tag{A.9}$$

By calculation, for $1 \leq l \leq k$ we have

$$\begin{aligned}\alpha_{il} =& \frac{h}{2\tau_{max}}[\frac{i!l!\left((-1)^i + (-1)^l\right)}{(i+l+1)!} + \frac{(-1)^{i+l}+1}{i+l+1}] \\ &+ f'(u(x_j))\frac{li!(l-1)!\left((-1)^i + (-1)^{l+1}\right)}{(i+l)!} \\ =& \frac{h}{2\tau_{max}}\mu_{il} + f'(u(x_j))\eta_{il},\end{aligned} \tag{A.10}$$

where

$$\begin{aligned}\mu_{il} =& \frac{i!l!\left((-1)^i + (-1)^l\right)}{(i+l+1)!} + \frac{(-1)^{i+l}+1}{i+l+1}, \\ \eta_{il} =& \frac{li!(l-1)!\left((-1)^i + (-1)^{l+1}\right)}{(i+l)!}.\end{aligned} \tag{A.11}$$

We denote $\beta = (a_0, ..., a_k)^T,\ A(i,l) = \alpha_{il},\ 0 \leq i \leq k,\ 0 \leq l \leq k$ and $b_0 = \int_{-1}^{1} w(x)dx,\ b_l = \tilde{P}_h(w; x^l; f, u),\ 1 \leq l \leq k,\ B = (b_0, ..., b_k)^T$. We will solve the following linear system to get the coefficients β,

$$A^T\beta = B. \tag{A.12}$$

We can rewrite A as the following form,

$$A = \frac{h}{2\tau_{max}}\mathcal{M} + f'(u(x_j))\mathcal{H} + \mathcal{C}, \tag{A.13}$$

where

$$\mathcal{M}(i,l) = \begin{cases} \mu_{il},\ 0 \leq i \leq k,\ 1 \leq l \leq k, \\ 0,\ 0 \leq i \leq k,\ l = 0, \end{cases} \tag{A.14}$$

$$\mathcal{H}(i,l) = \begin{cases} \eta_{il},\ 0 \leq i \leq k,\ 1 \leq l \leq k, \\ 0,\ 0 \leq i \leq k,\ l = 0, \end{cases} \tag{A.15}$$

$$\mathcal{C}(i,l) = \begin{cases} 0,\ 0 \leq i \leq k,\ 1 \leq l \leq k, \\ \alpha_{i0},\ 0 \leq i \leq k,\ l = 0. \end{cases} \tag{A.16}$$

From the formulation of the scheme (2.6) we have $\tau_{max} = c\ h$, here c is a constant dictated by stability. Then we have

$$A^T = \frac{1}{2c}\mathcal{M}^T + f'(u(x_j))\mathcal{H}^T + \mathcal{C}^T. \tag{A.17}$$

From (A.11) we know that

$$\mu_{il} = \begin{cases} \dfrac{2((i+l)!+i!l!)}{(i+l+1)!}, \, if \ i \ and \ l \ are \ even, \\ \dfrac{2((i+l)!-i!l!)}{(i+l+1)!}, \, if \ i \ and \ l \ are \ odd, \\ 0, \, if \ (i+l) \ is \ odd, \end{cases} \tag{A.18}$$

$$\eta_{il} = \begin{cases} \dfrac{-2li!(l-1)!}{(i+l)!}, \, if \ i \ is \ odd \ and \ l \ is \ even, \\ \dfrac{2li!(l-1)!}{(i+l)!}, \, if \ i \ is \ even \ and \ l \ is \ odd, \\ 0, \, if \ (i+l) \ is \ even, \end{cases} \tag{A.19}$$

and from (A.9) we have

$$\alpha_{i0} = \begin{cases} \dfrac{2}{i+1}, \, if \ i \ is \ even, \\ 0, \, if \ i \ is \ odd. \end{cases} \tag{A.20}$$

Hence, we can estimate the infinity norm of A^T,

$$\begin{aligned} \|A^T\|_\infty =& \|\frac{1}{2c}\mathcal{M}^T + f'(u(x_j))\mathcal{H}^T + \mathcal{C}^T\|_\infty \\ =& \max\{\sum_{i=0}^{k} |\alpha_{i0}|, \max_{1\le l\le k} \sum_{i=0}^{k} (\frac{1}{2c}|\mu_{il}| + |f'(u(x_j))\eta_{il}|)\}. \end{aligned} \tag{A.21}$$

Since $\mu_{il} > 0$ for $(i+l)$ is even and $f'(u(x_j))$ is bounded, then we have

$$\|A^T\|_\infty \le \mathcal{E}, \tag{A.22}$$

where $\mathcal{E}$ is a constant which depends on polynomial degree k, the bound of $f'(u(x_j))$ and constant c. Since the first row of the matrix A^T are constants α_{i0} which only depends on degree k and the other elements of A^T either only contain $\frac{1}{2c}$ or only $f'(u(x_j))$, the by the definition of determinant we have

$$\det(A^T) = \sum_{i=0}^{k} \mathcal{D}_i(k)(\frac{1}{2c})^i (f'(u(x_j)))^{k-i}, \tag{A.23}$$

where $\mathcal{D}_i(k)$ is a constant which only depends on degree k. Notice that if $f'(u(x_j)) = 0$ in (A.23), then $\det(A^T) = \mathcal{D}_k(k)(\frac{1}{2c})^k$. From the previous proof of the existence and uniqueness of the projection, we know that A^T is always invertible which means

$\det(A^T) \neq 0$ holds for any value of $f'(u(x_j))$. Hence, here we have $\mathcal{D}_k(k) \neq 0$. Therefore, we can take c small enough so that

$$|\sum_{i=0}^{k-1} \mathcal{D}_i(k)(\frac{1}{2c})^i (f'(u(x_j)))^{k-i}| \leq \frac{|\mathcal{D}_k(k)|}{2}(\frac{1}{2c})^k. \tag{A.24}$$

We emphasize that this choice of c is only a sufficient condition for our proof, in numerical computation c should be chosen as the largest CFL number for linear stability to avoid excessive numerical dissipation. We now have

$$|\det(A^T)| \geq \frac{|\mathcal{D}_k(k)|}{2}(\frac{1}{2c})^k > 0, \tag{A.25}$$

holds for all $f'(u(x_j))$. Next let $\sigma_i(A^T)$ denotes the $i-$th singular value of A^T which are in descending order from 0 to k, $\sigma_{\max}(A^T)$ and $\sigma_{\min}(A^T)$ represent the largest and smallest singular value of matrix A^T. Then we have

$$\begin{aligned}
\|A^{-T}\|_2 =& \frac{1}{\sigma_{\min}(A^T)} \\
\leq& \frac{1}{\sigma_{\min}(A^T)} \cdot (\prod_{i=0}^{k-1} \frac{\sigma_{\max}(A^T)}{\sigma_i(A^T)}) \\
=& \frac{(\sigma_{\max}(A^T))^k}{\prod_{i=0}^{k} \sigma_i(A^T)} \\
=& \frac{\|A^T\|_2^k}{|\det(A^T)|} \\
\leq& \frac{2(2c)^k}{\mathcal{D}_k(k)} \|A^T\|_2^k.
\end{aligned} \tag{A.26}$$

By the equivalence of norms

$$\|A^T\|_2 \leq \sqrt{k+1}\|A^T\|_\infty, \tag{A.27}$$

$$\|A^{-T}\|_\infty \leq \sqrt{k+1}\|A^{-T}\|_2, \tag{A.28}$$

we have

$$\|A^{-T}\|_\infty \leq \frac{2(2c)^k (k+1)^{\frac{k+1}{2}}}{\mathcal{D}_k(k)} \mathcal{E}^k. \tag{A.29}$$

It is obvious that $\|B\|_\infty \leq \tilde{C}\|w\|_\infty$ due to the boundedness of $f'(u(x_j))$. Here $\tilde{C}$ is a constant which depends on degree k and the bound of $f'(u(x_j))$. Hence, for the coefficients β we have

$$\|\beta\|_\infty \leq \|A^{-T}\|_\infty \|B\|_\infty \leq \frac{2(2c)^k (k+1)^{\frac{k+1}{2}}}{\mathcal{D}_k(k)} \mathcal{E}^k \tilde{C} \|w\|_\infty. \tag{A.30}$$

which immediately results in the boundedness of $\mathbb{P}_h^* w$. □

A.2 Proof of Proposition 2.1

Let $u_I = \mathbb{P}_h^* u \in V_h^k,\ v_I = \mathbb{Q}_h^* u \in W_h^k,\ a_j = f'(u(x_j)),\ a_{j+\frac{1}{2}} = f'(u(x_{j+\frac{1}{2}}))$, by the definition of $\tilde{B}_j$ and $\hat{B}_{j+\frac{1}{2}}$, we have

$$\begin{aligned}
&\tilde{B}_j(u_I, v_I; \varphi_h; f, u) - \tilde{B}_j(u, u; \varphi_h; f, u) \\
=&\frac{1}{\tau_{max}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (v_I - u_I)\varphi_h dx + a_j[\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (v_I - u)(\varphi_h)_x \\
&- (v_I(x_{j+\frac{1}{2}}) - u(x_{j+\frac{1}{2}}))\varphi_h(x^-_{j+\frac{1}{2}}) + (v_I(x_{j-\frac{1}{2}}) - u(x_{j-\frac{1}{2}}))\varphi_h(x^+_{j-\frac{1}{2}})] \\
=&\tilde{P}_h(u_I - u; \varphi_h; f, u)_j + \frac{1}{\tau_{max}}[\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - u - u_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))\varphi_h dx \\
&+ \int_{x_{j-\frac{1}{2}}}^{x_j} (v_I - u - u_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))\varphi_h dx] \\
&+ a_j[\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - u - u_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))(\varphi_h)_x dx \\
&+ \int_{x_{j-\frac{1}{2}}}^{x_j} (v_I - u - u_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))(\varphi_h)_x dx \\
&- (v_I(x_{j+\frac{1}{2}}) - u(x_{j+\frac{1}{2}}) - u_I(x_j) + u(x_j))\varphi_h(x^-_{j+\frac{1}{2}}) \\
&+ (v_I(x_{j-\frac{1}{2}}) - u(x_{j-\frac{1}{2}}) - u_I(x_j) + u(x_j))\varphi_h(x^+_{j-\frac{1}{2}})],
\end{aligned} \tag{A.31}$$

and

$$\begin{aligned}
&\hat{B}_{j+\frac{1}{2}}(u_I, v_I; \psi_h; f, u) - \hat{B}_{j+\frac{1}{2}}(u, u; \psi_h; f, u) \\
=&\frac{1}{\tau_{max}} \int_{x_j}^{x_{j+1}} (u_I - v_I)\psi_h dx + a_{j+\frac{1}{2}}[\int_{x_j}^{x_{j+1}} (u_I - u)(\psi_h)_x \\
&- (u_I(x_{j+1}) - u(x_{j+1}))\psi_h(x^-_{j+1}) + (u_I(x_j) - u(x_j))\psi_h(x^+_j)] \\
=&\tilde{Q}_h(v_I - u; \psi_h; f, u)_{j+\frac{1}{2}} + \frac{1}{\tau_{max}}[\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} (u_I - u - v_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))\psi_h dx \\
&+ \int_{x_j}^{x_{j+\frac{1}{2}}} (u_I - u - v_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))\psi_h dx]
\end{aligned}$$

$$
\begin{aligned}
&+ a_{j+\frac{1}{2}}[\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} (u_I - u - v_I(x - \frac{h}{2}) + u(x - \frac{h}{2}))(\psi_h)_x dx \\
&+ \int_{x_j}^{x_{j+\frac{1}{2}}} (u_I - u - v_I(x + \frac{h}{2}) + u(x + \frac{h}{2}))(\psi_h)_x dx \\
&- (u_I(x_{j+1}) - u(x_{j+1}) - v_I(x_{j+\frac{1}{2}}) + u(x_{j+\frac{1}{2}}))\psi_h(x_{j+1}^-) \\
&+ (u_I(x_j) - u(x_j) - v_I(x_{j+\frac{1}{2}}) + u(x_{j+\frac{1}{2}}))\psi_h(x_j^+)].
\end{aligned} \tag{A.32}
$$

For $u = x^{k+1}$, to get the desired result we need to estimate $\|v_I - x^{k+1} - u_I(x + \frac{h}{2}) + (x + \frac{h}{2})^{k+1}\|_{L^2(x_{j-\frac{1}{2}}, x_j)}$, $\|v_I - x^{k+1} - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|_{L^2(x_j, x_{j+\frac{1}{2}})}$ and $\|u_I - x^{k+1} - v_I(x+\frac{h}{2}) + (x+\frac{h}{2})^{k+1}\|_{L^2(x_j, x_{j+\frac{1}{2}})}$, $\|u_I - x^{k+1} - v_I(x - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|_{L^2(x_{j+\frac{1}{2}}, x_{j+1})}$. We will only show that $\|v_I - x^{k+1} - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|_{L^2(x_j, x_{j+\frac{1}{2}})} \leq Ch^{2k+5}$ with $k = 0, 1..., 8$, as the other cases are similar.

For $k = 0, 1..., 8$, by using the definition of the projection and the property that $\|a_j - a_{j+\frac{1}{2}}\|_{L^\infty(I_j)} = \|a_j - a_{j-\frac{1}{2}}\|_{L^\infty(I_j)} = O(h)$ we have the following results. For $u = x^{k+1}$, by the definition (for $k = 0$ we only have the first equation in the definition),

$$
\begin{aligned}
\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_I dx &= \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} x^{k+1} dx, \\
\tilde{P}_h(u_I; x^l; f, u)_j &= \tilde{P}_h(x^{k+1}; x^l; f, u)_j, \ \ l = 1, ..., k, \\
\int_{x_j}^{x_{j+1}} v_I dx &= \int_{x_j}^{x_{j+1}} x^{k+1} dx, \\
\tilde{Q}_h(v_I; x^l; f, u)_{j+\frac{1}{2}} &= \tilde{Q}_h(x^{k+1}; x^l; f, u)_{j+\frac{1}{2}}, \ \ l = 1, ..., k,
\end{aligned} \tag{A.33}
$$

then we have

$$
\begin{aligned}
u_I &= \sum_{l=0}^{k} \alpha_l x^l, \ \ \forall x \in I_j, \\
v_I &= \sum_{l=0}^{k} \beta_l x^l, \ \ \forall x \in I_{j+\frac{1}{2}}.
\end{aligned} \tag{A.34}
$$

Here α_l and β_l are the coefficients obtained by solving the local linear system (A.33). We leave the detailed calculations and formulas for k up to 8 in a separate file, as a supplement to this paper, since they are too lengthy. We then have, $k = 0, 1.,,, 8$, that

$$
\int_{x_j}^{x_{j+\frac{1}{2}}} (v_I - x^2 - u_I(x - \frac{h}{2}) + (x - \frac{h}{2})^2)^2 dx = O(h^{2k+5}), \tag{A.35}
$$

and therefore we can prove that

$$\|v_I - x^{k+1} - u_I(x-\frac{h}{2}) + (x-\frac{h}{2})^{k+1}\|^2_{L^2(x_j,x_{j+\frac{1}{2}})} \le Ch^{2k+5}. \tag{A.36}$$

Then by using Holder's inequality and Young's inequality, we obtain from (A.31)

$$|\tilde{B}_j(u_I,v_I;\varphi_h;f,u) - \tilde{B}_j(u,u;\varphi_h;f,u)| \le Ch^{2k+3} + C\|\varphi_h\|^2_{L^2(I_j)}. \tag{A.37}$$

Similarly, for $\hat{B}_{j+\frac{1}{2}}$ we have

$$|\hat{B}_{j+\frac{1}{2}}(u_I,v_I;\psi_h;f,u) - \hat{B}_{j+\frac{1}{2}}(u,u;\psi_h;f,u)| \le Ch^{2k+3} + C\|\psi_h\|^2_{L^2(I_{j+\frac{1}{2}})}. \tag{A.38}$$

A.3 Proof of Lemma 3.1

Proof. Let u_I denote $\mathbb{P}_h^* u$. Assume that $u \equiv 0$. Take $\varphi_h = u_I$ in (3.14), we get

$$\begin{aligned}
0 = \tilde{P}_h(u_I,u_I)_{i,j} =& \frac{1}{\tau_{max}}\Big(\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} 2u_I(x+\frac{h}{2},y+\frac{h}{2})u_I(x,y) \\
&+ 2u_I(x+\frac{h}{2},y)u_I(x,y+\frac{h}{2})\,dxdy \\
&- \int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} u_I(x,y)^2 + u_I(x,y+\frac{h}{2})^2 \\
&+ u_I(x+\frac{h}{2},y)^2 + u_I(x+\frac{h}{2},y+\frac{h}{2})^2\,dxdy\Big) \\
=& -\frac{1}{\tau_{max}}\Big(\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} (u_I(x+\frac{h}{2},y+\frac{h}{2}) - u_I(x,y))^2\,dxdy \\
&+ \int_{y_{j-\frac{1}{2}}}^{y_j}\int_{x_{i-\frac{1}{2}}}^{x_i} (u_I(x+\frac{h}{2},y) - u_I(x,y+\frac{h}{2}))^2\,dxdy\Big),
\end{aligned} \tag{A.39}$$

where we have again used change of variable to shift all the integration regions to the same subcell $(x_{i-\frac{1}{2}},x_i)\times(y_{j-\frac{1}{2}},y_j)$ to simplify the formulation. Then

$$u_I(x,y) = u_I(x+\frac{h}{2},y+\frac{h}{2}), \quad u_I(x+\frac{h}{2},y) = u_I(x,y+\frac{h}{2}), \forall (x,y)\in(x_{i-\frac{1}{2}},x_i)\times(y_{j-\frac{1}{2}},y_j).$$

Thus $u_I(x,y)\equiv c_0$ on $K_{i,j}$, c_0 is a constant. By (3.13) we immediately get $u_I\equiv 0$, and we have finished the proof of uniqueness, hence also existence. We note that this projection is a local projection, hence we can make a change of variables to the reference element

$[-1,1]\times[-1,1]$ by taking $\xi=\frac{2(x-x_i)}{h}$ and $\eta=\frac{2(y-y_j)}{h}$. Taking a similar derivation as in the proof of (A.1), we obtain

$$\|u_I\|_{L^\infty(K_{i,j})}\le C(k)\|u\|_{L^\infty(K_{i,j})}. \tag{A.40}$$

Again standard approximation theory [2] implies the optimal approximating estimates.

□

A.4 Proof of Lemma 3.2

Proof. Let $u_I=\mathbb{P}_h^*u\in X_h^k,\ v_I=\mathbb{Q}_h^*u\in Y_h^k$, and $a_{i,j}=f'(u(x_i,y_j)),\ b_{i,j}=g'(u(x_i,y_j))$, $a_{i+\frac12,j+\frac12}=f'(u(x_{i+\frac12},y_{j+\frac12})),\ b_{i+\frac12,j+\frac12}=g'(u(x_{i+\frac12},y_{j+\frac12}))$, then by the definition of $\tilde{B}_{i,j}$ and $\hat{B}_{i+\frac12,j+\frac12}$, we have

$$\begin{aligned}
&\tilde{B}_{i,j}(u_I,v_I;\varphi_h;f,g,u)-\tilde{B}_{i,j}(u,u;\varphi_h;f,g,u)\\
=&\frac{1}{\tau_{max}}\int_{K_{i,j}}(v_I-u_I)\varphi_h dxdy+a_{i,j}\Bigg[\int_{K_{i,j}}(v_I-u)(\varphi_h)_x dxdy\\
&-\int_{y_{j-\frac12}}^{y_{j+\frac12}}\Big((v_I-u)(x_{i+\frac12},y)\varphi_h(x^-_{i+\frac12},y)-(v_I-u)(x_{i-\frac12},y)\varphi_h(x^+_{i-\frac12},y)\Big)\Bigg]\\
&+b_{i,j}\Bigg[\int_{K_{i,j}}(v_I-u)(\varphi_h)_y dxdy-\int_{x_{i-\frac12}}^{x_{i+\frac12}}\Big((v_I-u)(x,y_{j+\frac12})\varphi_h(x,y^-_{j+\frac12})\\
&-(v_I-u)(x,y_{j-\frac12})\varphi_h(x,y^+_{j-\frac12})\Big)\Bigg]\\
=&\tilde{P}_h(u_I-u;\varphi_h;f,g,u)_{i,j}\\
&+\frac{1}{\tau_{max}}\Bigg[\int_{x_i}^{x_{i+\frac12}}\int_{y_j}^{y_{j+\frac12}}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)\varphi_h dxdy\\
&+\int_{x_{i-\frac12}}^{x_i}\int_{y_j}^{y_{j+\frac12}}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)\varphi_h dxdy\\
&+\int_{x_i}^{x_{i+\frac12}}\int_{y_{j-\frac12}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)\varphi_h dxdy\\
&+\int_{x_{i-\frac12}}^{x_i}\int_{y_{j-\frac12}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)\varphi_h dxdy\Bigg]\\
&+a_{i,j}\Bigg[\int_{x_i}^{x_{i+\frac12}}\int_{y_j}^{y_{j+\frac12}}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_x dxdy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_x dxdy\\
&-\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x_{i+\frac{1}{2}},y)-u(x_{i+\frac{1}{2}},y)-u_I(x_i,y-\frac{h}{2})+u(x_i,y-\frac{h}{2})\Big)\varphi_h(x^-_{i+\frac{1}{2}},y)dy\\
&-\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x_{i+\frac{1}{2}},y)-u(x_{i+\frac{1}{2}},y)-u_I(x_i,y+\frac{h}{2})+u(x_i,y+\frac{h}{2})\Big)\varphi_h(x^-_{i+\frac{1}{2}},y)dy\\
&+\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x_{i-\frac{1}{2}},y)-u(x_{i-\frac{1}{2}},y)-u_I(x_i,y-\frac{h}{2})+u(x_i,y-\frac{h}{2})\Big)\varphi_h(x^+_{i-\frac{1}{2}},y)dy\\
&+\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x_{i-\frac{1}{2}},y)-u(x_{i-\frac{1}{2}},y)-u_I(x_j,y+\frac{h}{2})+u(x_j,y+\frac{h}{2})\Big)\varphi_h(x^+_{i-\frac{1}{2}},y)dy\Bigg]\\
&+b_{i,j}\Bigg[\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y-\frac{h}{2})+u(x-\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_y dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_j}^{y_{j+\frac{1}{2}}}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\varphi_h)_y dxdy\\
&+\int_{x_i}^{x_{i+\frac{1}{2}}}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_y dxdy\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\int_{y_{j-\frac{1}{2}}}^{y_j}\Big(v_I(x,y)-u(x,y)-u_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\varphi_h)_y dxdy\\
&-\int_{x_i}^{x_{i+\frac{1}{2}}}\Big(v_I(x,y_{j+\frac{1}{2}})-u(x,y_{j+\frac{1}{2}})-u_I(x-\frac{h}{2},y_j)+u(x-\frac{h}{2},y_j)\Big)\varphi_h(x,y^-_{j+\frac{1}{2}})dx\\
&-\int_{x_{i-\frac{1}{2}}}^{x_i}\Big(v_I(x,y_{j+\frac{1}{2}})-u(x,y_{j+\frac{1}{2}})-u_I(x+\frac{h}{2},y_j)+u(x+\frac{h}{2},y_j)\Big)\varphi_h(x,y^-_{j+\frac{1}{2}})dx\\
&+\int_{x_i}^{x_{i+\frac{1}{2}}}\Big(v_I(x,y_{j-\frac{1}{2}})-u(x,y_{j-\frac{1}{2}})-u_I(x-\frac{h}{2},y_j)+u(x-\frac{h}{2},y_j)\Big)\varphi_h(x,y^+_{j-\frac{1}{2}})dx\\
&+\int_{x_{i-\frac{1}{2}}}^{x_i}\Big(v_I(x,y_{j-\frac{1}{2}})-u(x,y_{j-\frac{1}{2}})-u_I(x+\frac{h}{2},y_j)+u(x+\frac{h}{2},y_j)\Big)\varphi_h(x,y^+_{j-\frac{1}{2}})dx\Bigg],
\end{aligned}
\tag{A.41}
$$

$$
\begin{aligned}
&\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_I,v_I;\psi_h;f,u)-\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u,u;\psi_h;f,u)\\
=&\frac{1}{\tau_{max}}\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(u_I-v_I)\psi_h dxdy+a_{i+\frac{1}{2},j+\frac{1}{2}}\Bigg[\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}(v_I-u)(\psi_h)_x dxdy
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \int_{y_j}^{y_{j+1}} \Big((u_I - u)(x_{i+\frac{1}{2}}, y)\psi_h(x^-_{i+\frac{1}{2}}, y) - (u_I - u)(x_{i-\frac{1}{2}}, y)\psi_h(x^+_{i-\frac{1}{2}}, y) \Big) \Bigg] \\
&\quad + b_{i,j} \Bigg[\int_{K_{i,j}} (u_I - u)(\psi_h)_y dxdy - \int_{x_i}^{x_{i+1}} \Big((u_I - u)(x, y_{j+\frac{1}{2}})\psi_h(x, y^-_{j+\frac{1}{2}}) \\
&\quad - (u_I - u)(x, y_{j-\frac{1}{2}})\psi_h(x, y^+_{j-\frac{1}{2}}) \Big) \Bigg] \\
&= \tilde{Q}_h(v_I - u; \psi_h; f, g, u)_{i+\frac{1}{2}, j+\frac{1}{2}} \\
&\quad + \frac{1}{\tau_{max}} \Bigg[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \Big(u_I(x,y) - u(x,y) - v_I(x - \frac{h}{2}, y - \frac{h}{2}) + u(x - \frac{h}{2}, y - \frac{h}{2}) \Big) \psi_h dxdy \\
&\quad + \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \Big(u_I(x,y) - u(x,y) - v_I(x + \frac{h}{2}, y - \frac{h}{2}) + u(x + \frac{h}{2}, y - \frac{h}{2}) \Big) \psi_h dxdy \\
&\quad + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{y_j}^{y_{j+\frac{1}{2}}} \Big(u_I(x,y) - u(x,y) - v_I(x - \frac{h}{2}, y + \frac{h}{2}) + u(x - \frac{h}{2}, y + \frac{h}{2}) \Big) \psi_h dxdy \\
&\quad + \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} \Big(u_I(x,y) - u(x,y) - v_I(x + \frac{h}{2}, y + \frac{h}{2}) + u(x + \frac{h}{2}, y + \frac{h}{2}) \Big) \psi_h dxdy \Bigg] \\
&\quad + a_{i+\frac{1}{2}, j+\frac{1}{2}} \Bigg[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \Big(u_I(x,y) - u(x,y) - v_I(x - \frac{h}{2}, y - \frac{h}{2}) + u(x - \frac{h}{2}, y - \frac{h}{2}) \Big) (\psi_h)_x dxdy \\
&\quad + \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \Big(u_I(x,y) - u(x,y) - v_I(x + \frac{h}{2}, y - \frac{h}{2}) + u(x + \frac{h}{2}, y - \frac{h}{2}) \Big) (\psi_h)_x dxdy \\
&\quad + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{y_j}^{y_{j+\frac{1}{2}}} \Big(u_I(x,y) - u(x,y) - v_I(x - \frac{h}{2}, y + \frac{h}{2}) + u(x - \frac{h}{2}, y + \frac{h}{2}) \Big) (\psi_h)_x dxdy \\
&\quad + \int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} \Big(u_I(x,y) - u(x,y) - v_I(x + \frac{h}{2}, y + \frac{h}{2}) + u(x + \frac{h}{2}, y + \frac{h}{2}) \Big) (\psi_h)_x dxdy \\
&\quad - \int_{y_j}^{y_{j+\frac{1}{2}}} \Big(u_I(x_{i+1}, y) - u(x_{i+1}, y) - v_I(x_{i+\frac{1}{2}}, y - \frac{h}{2}) + u(x_{i+\frac{1}{2}}, y - \frac{h}{2}) \Big) \psi_h(x^-_{i+1}, y) dy \\
&\quad - \int_{y_{j-\frac{1}{2}}}^{y_j} \Big(u_I(x_{i+1}, y) - u(x_{i+1}, y) - v_I(x_{i+\frac{1}{2}}, y + \frac{h}{2}) + u(x_{i+\frac{1}{2}}, y + \frac{h}{2}) \Big) \psi_h(x^-_{i+1}, y) dy \\
&\quad + \int_{y_j}^{y_{j+\frac{1}{2}}} \Big(u_I(x_i, y) - u(x_i, y) - v_I(x_{i+\frac{1}{2}}, y - \frac{h}{2}) + u(x_{i+\frac{1}{2}}, y - \frac{h}{2}) \Big) \psi_h(x^+_i, y) dy \\
&\quad + \int_{y_{j-\frac{1}{2}}}^{y_j} \Big(u_I(x_i, y) - u(x_i, y) - v_I(x_{i+\frac{1}{2}}, y + \frac{h}{2}) + u(x_{i+\frac{1}{2}}, y + \frac{h}{2}) \Big) \psi_h(x^+_i, y) dy \Bigg] \\
&\quad + b_{i+\frac{1}{2}, j+\frac{1}{2}} \Bigg[\int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \int_{y_{j+\frac{1}{2}}}^{y_{j+1}} \Big(u_I(x,y) - u(x,y) - v_I(x - \frac{h}{2}, y - \frac{h}{2}) + u(x - \frac{h}{2}, y - \frac{h}{2}) \Big) (\psi_h)_y dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{x_i}^{x_{i+\frac12}}\int_{y_{j+\frac12}}^{y_{j+1}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y-\frac{h}{2})+u(x+\frac{h}{2},y-\frac{h}{2})\Big)(\psi_h)_y dxdy\\
&+\int_{x_{i+\frac12}}^{x_{i+1}}\int_{y_j}^{y_{j+\frac12}}\Big(u_I(x,y)-u(x,y)-v_I(x-\frac{h}{2},y+\frac{h}{2})+u(x-\frac{h}{2},y+\frac{h}{2})\Big)(\psi_h)_y dxdy\\
&+\int_{x_i}^{x_{i+\frac12}}\int_{y_j}^{y_{j+\frac12}}\Big(u_I(x,y)-u(x,y)-v_I(x+\frac{h}{2},y+\frac{h}{2})+u(x+\frac{h}{2},y+\frac{h}{2})\Big)(\psi_h)_y dxdy\\
&-\int_{x_{i+\frac12}}^{x_{i+1}}\Big(u_I(x,y_{j+1})-u(x,y_{j+1})-v_I(x-\frac{h}{2},y_{j+\frac12})+u(x-\frac{h}{2},y_{j+\frac12})\Big)\psi_h(x,y_{j+1}^-)dx\\
&-\int_{x_i}^{x_{i+\frac12}}\Big(u_I(x,y_{j+1})-u(x,y_{j+1})-v_I(x+\frac{h}{2},y_{j+\frac12})+u(x+\frac{h}{2},y_{j+\frac12})\Big)\psi_h(x,y_{j+1}^-)dx\\
&+\int_{x_{i+\frac12}}^{x_{i+1}}\Big(u_I(x,y_j)-u(x,y_j)-v_I(x-\frac{h}{2},y_{j+\frac12})+u(x-\frac{h}{2},y_{j+\frac12})\Big)\psi_h(x,y_j^+)dx\\
&+\int_{x_i}^{x_{i+\frac12}}\Big(u_I(x,y_j)-u(x,y_j)-v_I(x+\frac{h}{2},y_{j+\frac12})+u(x+\frac{h}{2},y_{j+\frac12})\Big)\psi_h(x,y_j^+)dx\Bigg].
\end{aligned}
\tag{A.42}
$$

For $u(x,y)=x^{k+1}$ *or* y^{k+1}, we only need to estimate $\|v_I(x,y)-x^{k+1}-u_I(x-\frac{h}{2},y-\frac{h}{2})+(x-\frac{h}{2})^{k+1}\|_{L^2((x_i,x_{i+\frac12})\times(y_j,y_{j+\frac12}))}$ and $\|v_I(x,y)-y^{k+1}-u_I(x-\frac{h}{2},y-\frac{h}{2})+(y-\frac{h}{2})^{k+1}\|_{L^2((x_i,x_{i+\frac12})\times(y_j,y_{j+\frac12}))}$ as the other cases are similar.

For $k=0,1...,8$, by using the definition of the projection and the property that $\|a_{i,j}-a_{i+\frac12,j+\frac12}\|_{L^\infty(K_{i,j})}=O(h)$, $\|b_{i,j}-b_{i+\frac12,j+\frac12}\|_{L^\infty(K_{i,j})}=O(h)$ we have the following results:

1) $u=x^{k+1}$, by the definition of the projection (for $k=0$ we only have the first equation in the definition),

$$
\begin{aligned}
\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{y_{j-\frac12}}^{y_{j+\frac12}}u_I dxdy &=\int_{x_{i-\frac12}}^{x_{i+\frac12}}\int_{y_{j-\frac12}}^{y_{j+\frac12}}x^{k+1}dxdy,\\
\tilde{P}_h(u_I;x^my^n;f,g,u)_{i,j} &=\tilde{P}_h(x^{k+1};x^my^n;f,g,u)_{i,j},\ m,n=0,...,k,\\
\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}v_I dxdy &=\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}x^{k+1}dxdy,\\
\tilde{Q}_h(v_I;x^my^n;f,g,u)_{i+\frac12,j+\frac12} &=\tilde{Q}_h(x^{k+1};x^my^n;f,g,u)_{i+\frac12,j+\frac12},\ m,n=0,...,k,
\end{aligned}
\tag{A.43}
$$

then we have

$$
u_I=\sum_{m=0}^{k}\sum_{n=0}^{k}\alpha_{m,n}x^my^n,\ \forall(x,y)\in K_{i,j}, \tag{A.44}
$$

$$v_I = \sum_{m=0}^{k}\sum_{n=0}^{k} \beta_{m,n} x^m y^n, \ \forall (x,y) \in K_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{A.45}$$

Here $\alpha_{m,n}$ and $\beta_{m,n}$ are the coefficients obtained by solving the local linear system (A.43). We leave the detailed calculations and formulas for k up to 8 in a separate file, as a supplement to this paper, since they are too lengthy. We then have, for $k = 0, 1., , , 8$, that

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x,y) - x^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^{k+1})^2 dxdy = O(h^{2k+6}). \tag{A.46}$$

2) $u = y^{k+1}$, by the definition of the projection (for $k = 0$ we only have the first equation in the definition),

$$\begin{aligned}
\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u_I dxdy &= \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} y^{k+1} dxdy, \\
\tilde{P}_h(u_I; x^m y^n; f, g, u)_{i,j} &= \tilde{P}_h(x^{k+1}; x^m y^n; f, g, u)_{i,j}, \ m, n = 0, ..., k, \\
\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} v_I dxdy &= \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} y^{k+1} dxdy, \\
\tilde{Q}_h(v_I; x^m y^n; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}} &= \tilde{Q}_h(y^{k+1}; x^m y^n; f, g, u)_{i+\frac{1}{2},j+\frac{1}{2}}, \ m, n = 0, ..., k,
\end{aligned} \tag{A.47}$$

then we have

$$u_I = \sum_{m=0}^{k}\sum_{n=0}^{k} \alpha_{m,n} x^m y^n, \ \forall (x,y) \in K_{i,j}, \tag{A.48}$$

$$v_I = \sum_{m=0}^{k}\sum_{n=0}^{k} \beta_{m,n} x^m y^n, \ \forall (x,y) \in K_{i+\frac{1}{2},j+\frac{1}{2}}. \tag{A.49}$$

Here $\alpha_{m,n}$, $\beta_{m,n}$ are the coefficients obtained by solving the local linear system (A.47). We do not give detailed calculations here since for $u = y^{k+1}$ in two-dimensional case the formulas are symmetric to those of $u = x^{k+1}$ by switching x and y (i and j). Hence, by some calculation we have

$$\int_{x_i}^{x_{i+\frac{1}{2}}} \int_{y_j}^{y_{j+\frac{1}{2}}} (v_I(x,y) - y^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (y - \frac{h}{2})^{k+1})^2 dxdy = O(h^{2k+6}). \tag{A.50}$$

Hence, for $k = 0, 1., , , 8$ we have proved that

$$\|v_I(x,y) - x^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (x - \frac{h}{2})^{k+1}\|^2_{L^2((x_i, x_{i+\frac{1}{2}})\times(y_j, y_{j+\frac{1}{2}}))} \le Ch^{2k+6}, \tag{A.51}$$

$$\|v_I(x,y) - y^{k+1} - u_I(x - \frac{h}{2}, y - \frac{h}{2}) + (y - \frac{h}{2})^{k+1}\|^2_{L^2((x_i, x_{i+\frac{1}{2}})\times(y_j, y_{j+\frac{1}{2}}))} \le Ch^{2k+6}. \tag{A.52}$$

Then by using Holder's inequality and Young's inequality, we obtain from (A.41)

$$|\tilde{B}_{i,j}(u_I, v_I; \varphi_h; f, g, u) - \tilde{B}_{i,j}(u, u; \varphi_h; f, g, u)| \le Ch^{2k+4} + C\|\varphi_h\|^2_{L^2(K_{i,j})}. \tag{A.53}$$

Similarly, for $\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}$ we have

$$|\hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u_I, v_I; \psi_h; f, g, u) - \hat{B}_{i+\frac{1}{2},j+\frac{1}{2}}(u, u; \psi_h; f, g, u)| \le Ch^{2k+4} + C\|\psi_h\|^2_{L^2(K_{i+\frac{1}{2},j+\frac{1}{2}})}. \tag{A.54}$$

□

References

[1] Y. Cheng and C.-W. Shu. A discontinuous Galerkin finite element method for time dependent partial differential equations with higher order derivatives. *Mathematics of Computation*, 77(262):699–730, 2008.

[2] P. G Ciarlet. The Finite Element Method for Elliptic Problems, North Holland Publishing Company, 1978.

[3] A. Kurganov and E. Tadmor. New high-resolution central schemes for nonlinear conservation laws and convection–diffusion equations. *Journal of Computational Physics*, 160(1):241–282, 2000.

[4] R. J. LeVeque. *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002.

[5] F. Li and L. Xu. Arbitrary order exactly divergence-free central discontinuous Galerkin methods for ideal MHD equations. *Journal of Computational Physics*, 231(6):2655–2675, 2012.

[6] F. Li, L. Xu, and S. Yakovlev. Central discontinuous Galerkin methods for ideal MHD equations with the exactly divergence-free magnetic field. *Journal of Computational Physics*, 230(12):4828–4847, 2011.

[7] Y. Liu. Central schemes and central discontinuous Galerkin methods on overlapping cells. *Conference on Analysis, Modeling and Computation of PDE and Multiphase Flow, Stony Brook, NY, 2004*.

[8] Y. Liu. Central schemes on overlapping cells. *Journal of Computational Physics*, 209(1):82–104, 2005.

[9] Y. Liu, C.-W. Shu, E. Tadmor, and M. Zhang. Central discontinuous Galerkin methods on overlapping cells with a nonoscillatory hierarchical reconstruction. *SIAM Journal on Numerical Analysis*, 45(6):2442–2467, 2007.

[10] Y. Liu, C.-W. Shu, E. Tadmor, and M. Zhang. L^2 stability analysis of the central discontinuous Galerkin method and a comparison between the central and regular discontinuous Galerkin methods. *ESAIM: Mathematical Modelling and Numerical Analysis*, 42(4):593–607, 2008.

[11] Y. Liu, C.-W. Shu, E. Tadmor, and M. Zhang. Central local discontinuous Galerkin methods on overlapping cells for diffusion equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 45(6):1009–1032, 2011.

[12] Y. Liu, C.-W. Shu, and M. Zhang. Optimal Error estimates of the semidiscrete central discontinuous Galerkin methods for linear hyperbolic equations. *SIAM Journal on Numerical Analysis*, 56(1):520–541, 2018.

[13] J. Luo, C.-W. Shu, and Q. Zhang. A priori error estimates to smooth solutions of the third order Runge-Kutta discontinuous Galerkin method for symmetrizable systems of conservation laws. *ESAIM: Mathematical Modelling and Numerical Analysis*, 49(4):991–1018, 2015.

[14] H. Nessyahu and E. Tadmor. Non-oscillatory central differencing for hyperbolic conservation laws. *Journal of Computational Physics*, 87(2):408–463, 1990.

[15] M.A. Reyna and F. Li. Operator bounds and time step conditions for DG and central DG methods. *Journal of Scientific Computing*, 62:532–554, 2015.

[16] Y. Xu and C.-W. Shu. Error estimates of the semi-discrete local discontinuous Galerkin method for nonlinear convection–diffusion and KdV equations. *Computer Methods in Applied Mechanics and Engineering*, 196(37-40):3805–3822, 2007.

[17] Z. Xu and Y. Liu. New central and central discontinuous Galerkin schemes on overlapping cells of unstructured grids for solving ideal magnetohydrodynamic equations with globally divergence-free magnetic field. *Journal of Computational Physics*, 327:203–224, 2016.

[18] S. Yakovlev, L. Xu, and F. Li. Locally divergence-free central discontinuous Galerkin methods for ideal MHD equations. *Journal of Computational Science*, 4(1-2):80–91, 2013.

[19] Q. Zhang and C.-W. Shu. Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin methods for scalar conservation laws. *SIAM Journal on Numerical Analysis*, 42(2):641–666, 2004.

[20] Q. Zhang and C.-W. Shu. Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin method for symmetrizable systems of conservation laws. *SIAM Journal on Numerical Analysis*, 44(4):1703–1720, 2006.

[21] Q. Zhang and C.-W. Shu. Stability analysis and a priori error estimates of the third order explicit Runge-Kutta discontinuous Galerkin method for scalar conservation laws. *SIAM Journal on Numerical Analysis*, 48(3):1038–1063, 2010.

[22] J. Zhao and H. Tang. Runge-Kutta central discontinuous Galerkin methods for the special relativistic hydrodynamics. *Communications in Computational Physics*, 22(3):643–682, 2017.