Convergence of a spectral method for the stochastic incompressible Euler equations

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Abstract

We propose a spectral viscosity method (SVM) to approximate the incompressible Euler equations driven by a *multiplicative* noise. We show that the SVM solution converges to a *dissipative measure-valued martingale* solution of the underlying problem. These solutions are weak in the probabilistic sense i.e. the probability space and the driving Wiener process are an integral part of the solution. We also exhibit a weak (measure-valued)-strong uniqueness principle. Moreover, we establish *strong* convergence of approximate solutions to the regular solution of the limit system at least on the lifespan of the latter, thanks to the weak (measure-valued)–strong uniqueness principle for the underlying system.

Keywords: Euler system; Incompressible fluids; Stochastic forcing; Multiplicative noise; Spectral method; Dissipative measure-valued martingale solution; Weak-strong uniqueness.

1 Introduction

Fluid dynamics is one of the most demanding research areas in mathematics and motivates many questions in stochastic analysis. Since the equations of turbulence are very difficult to examine, many researchers are interested to study the classical models which capture some of the phenomena of turbulence in a more tractable mathematical reference. One typical example is the Euler equations for the motion of an inviscid incompressible fluid which have an intensive role in geophysics; in science; in meteorology; in engineering; in aerospace; in astrophysics and of course, in mathematics where advanced techniques for existence and uniqueness provide important mathematical tool and new theoretical insight. Stochastic partial differential equations (SPDEs) is a subject that has been the focus of much activity during the last decade. Stochastic deformation of classical mechanics is a challenging area in which interactions with stochastic analysis are substantial. To accommodate external influence for which a precise model is missing, it is natural to consider a stochastic version of the Euler equations.

In this article, we consider the stochastic Euler equations governing the time evolution of the velocity $\mathbf{u}$ and the scalar pressure field Π of an inviscid fluid on the three-dimensional torus $\mathbb{T}^3$. The system of equations reads

$$\begin{cases} \mathrm{d}\mathbf{u}(t,x) + \left[\operatorname{div}(\mathbf{u}(t,x)\otimes\mathbf{u}(t,x)) + \nabla_x\Pi(t,x)\right]\,\mathrm{d}t = \sigma(\mathbf{u}(t,x))\,\mathrm{d}W(t), & \text{in } (0,T)\times\mathbb{T}^3, \\ \operatorname{div}\mathbf{u}(t,x) = 0, & \text{in } (0,T)\times\mathbb{T}^3, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & \text{in } \mathbb{T}^3 \end{cases} \tag{1.1}$$

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where $T > 0$ fixed, $\mathbf{u}_0$ is given initial data. Let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ be a stochastic basis, where $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ is a probability space and $(\mathcal{F}_t)_{t\geq 0}$ is a complete filtration with the usual assumptions. We assume that W is a cylindrical Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the coefficient σ is generally nonlinear and satisfies suitable growth assumptions (see Section 2.1 for the complete list of assumptions). In particular, the map $\mathbf{u} \mapsto \sigma(\mathbf{u})$ is a Hilbert space valued function signifying the *multiplicative* nature of the noise.

1.1 Euler equations

The Euler equations are the classical model for the motion of an incompressible, inviscid, homogenous fluid. The addition of stochastic terms to the governing equations is commonly used to account for empirical, numerical, and physical uncertainties in applications ranging from climatology to turbulence theory.

In the deterministic setup, for general initial data, global existence of a smooth solution remains a well-known open problem for the Euler equations and also their dissipative counterpart, the Navier–Stokes equations. Non-uniqueness of solutions for Euler equations was shown for the first time by Scheffer [38] who constructed a nontrivial weak solution of the 2D incompressible Euler equations with compact support in time. Later, De Lellis, Székelyhidi [15, 16] and Chiodaroli et al. [14] established groundbreaking results, that confirms infinitely many weak solutions can be constructed for the Euler equations in three dimensions. In these works, the method of so-called convex integration was used to prove the non-uniqueness of weak solutions to Euler equations. Furthermore, non-uniqueness results were established among weak solutions with dissipating energy, which is one of the well-accepted criteria for the selection of physically relevant solutions. In quest for a global-in-time solution, DiPerna [17] proposed a new concept of solution, known as a measure-valued solution, for the non-linear system of partial differential equations admitting uncontrollable oscillations. Moreover, Brenier et al. [9] proposed a new approach, seeing the measure-valued solutions as possibly the largest class, in which the family of smooth solutions is stable. In particular, they showed the so-called weak (measure-valued)-strong uniqueness principle for the incompressible Euler equations. More specifically, a classical and a measure-valued solution emanating from the same initial data coincide as long as the former exists. Following the philosophy of Brenier et al. [9], we focus on the concept of measure-valued solution in the widest possible sense.

In the stochastic set-up, Glatt-Holtz and Vicol [23] obtained local well-posedness results for strong solutions of the stochastic incompressible Euler equations in two and three dimensions, and global well-posedness results in two dimensions for additive and linear multiplicative noise. Local well-posedness results for the three-dimensional stochastic compressible Euler equations were proved by Breit and Mensah [7]. Moreover, the convex integration method has already been applied in stochastic setting, namely, to the isentropic Euler system by Breit, Feireisl and Hofmanova [6] and to the full Euler system by Chiodaroli, Feireisl and Flandoli [13]. There have been many attempts to define a suitable notion of measure-valued solutions for the stochastic incompressible Euler equations driven by *additive* noise, starting from the work of Kim [28], Breit & Moyo [5], and most recently by Hofmanova et al. [25], where the authors introduced a class of dissipative solutions which allowed them to demonstrate weak-strong uniqueness property and non-uniqueness of solutions in law. However, none of the above-mentioned frameworks can be applied to (1.1), since the driving noise is *multiplicative* in nature. We also mention recent work [11, 12, 24] on the Euler equations driven by a *multiplicative noise*.

1.2 Spectral method

The prototype of spectral methods for the solution of differential equations is the well-known Fourier method which consists of representing the solution as a truncated series expansion, the unknowns being the expansion coefficients. Spectral methods have emerged as a powerful computational technique for the simulation of complex, smooth physical phenomena. Among other applications, they have contributed to our understanding of turbulence by successfully simulating incompressible turbulent flows which have been extensively used in meteorology, geophysics and have been recently applied to time-domain electromagnetic fields (see [37]). The spectral method may be viewed as an extreme development of the class of discretization schemes for non-linear differential equations. We refer to [34, 39, 40, 41, 42] for spectral method related articles.

We also mention the work of Eitan Tadmor [40] in which he discussed behavior and convergence of Fourier methods for scalar nonlinear conservation laws that exhibit spontaneous shock discontinuities. Mishra et al. [33] combined the spectral (viscosity) method and ensemble averaging to propose an algorithm that computes admissible measure-valued solutions of the incompressible Euler equations.

1.3 Aim and scope of this paper

In view of the wide usage of stochastic fluid dynamics, there is an essential need to improve the mathematical foundations of the stochastic partial differential equations of fluid flow, and in particular to study inviscid models such as the stochastic incompressible Euler equations. Spectral methods based on projecting into a finite number of Fourier modes are widely employed particularly in the simulation of flows with periodic boundary conditions, while finite difference and finite element methods are very useful when discretizing the Euler equations in a domain with complex geometry. In that context, we mention the work of Brzézniak [10] where the author studies finite-element-based space-time discretizations of the incompressible Navier–Stokes equations with noise. In the context of compressible flow, we first mention the work of Karper [27], where he has shown the convergence of a mixed finite element-discontinuous Galerkin scheme to compressible Navier–Stokes system. Subsequently, a series of works [18, 19, 20] by Feireisl et al. analyzed convergence issues for several different numerical schemes via the framework of dissipative measure-valued solutions. In [12], the authors proved the existence of measure-valued solutions by showing that weak martingale solutions of the stochastic Navier–Stokes equations converge to a measure-valued solution of (1.1) as the viscosity tends to zero. But in this work, formulation of measure-valued solutions is slightly different from the given formulation in [12] (see Definition 3.1). In fact, in comparison to previous work [12], the main novelty of this work lies in successfully handling the multiplicative noise term. Note that our work bears some similarities with the recent work of Mishra et al. [33] on the deterministic system of the Euler equations. However, our problems need to invoke ideas from spectral methods for deterministic problems and meaningfully fuse them with available approximation methods for SDEs. Indeed, this means that one needs to handle noise-noise interaction terms carefully. In the realm of stochastic conservation laws, noise-noise interaction terms play a fundamental role to establish the well-posedness theory, for details see [2, 3, 4, 29, 30, 31, 32]. The main contributions of this article are as follows:

- We study the convergence of the spectral method for the incompressible Euler equations driven by a *multiplicative noise*. The Cauchy problem for the Euler equations is in general ill-posed in the class of admissible weak solutions. This suggests there might be sequences of approximate solutions that develop fine-scale oscillations. Accordingly, the concept of a measure-valued solution that captures possible oscillations is more suitable for analysis. We show that the sequence of approximate solutions converges to a *dissipative measure-valued martingale* solution to the stochastic Euler equations.

- In view of the new framework based on the theory of measure-valued solutions, we adapt the concept of $\mathcal{K}$-convergence, first developed in the context of Young measures by Balder [1] (see also Feireisl et. al. [20]), to show the pointwise convergence of arithmetic averages (Cesaro means) of approximate solutions to a *dissipative* solution of the limit system (1.1).

- We show that *dissipative measure–valued martingale* solutions satisfy a weak–strong uniqueness principle. More precisely, if for some initial data there is an analytically strong solution (defined up to a stopping time), then it coincides with all dissipative measure–valued martingale solutions having the same initial data.
- When solutions of the stochastic incompressible Euler system possess maximal regularity, by making use of weak (measure-valued)–strong uniqueness principle, we show *unconditional* strong L^1-convergence of approximate solutions to the regular solution of the limit system.

The paper is organized as follows. In Section 2, we first introduce mathematical setting, assumptions, and preliminary results. Then, we introduce the definition of *dissipative measure-valued martingale* solutions for the incompressible Euler system driven by a *multiplicative noise*, keeping in mind that this framework would allow us to establish weak (measure-valued)–strong uniqueness principle, and state the main results of this article in Section 3. In Section 4, we give details of the spectral viscosity method to approximate the stochastic incompressible Euler equations. In Section 5, we prove the convergence of the spectral method in which we present a proof of convergence of approximate solutions to a dissipative measure-valued martingale solution using stochastic compactness method. In Section 6, we use the concept of $\mathcal{K}$-convergence to exhibit the pointwise convergence of approximate solutions. Section 7 is devoted to deriving the weak (measure-valued)-strong uniqueness principle by making use of a suitable relative energy inequality. Finally, in Section 8, we make use of weak (measure-valued)-strong uniqueness property to show the convergence of approximations to the regular solution of the stochastic incompressible Euler system (1.1).

2 Mathematical setting

Function spaces: Let $C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$ be the space of infinitely differentiable 3-dimensional vector fields $\mathbf{u}$ on $\mathbb{T}^3$, satisfying $\nabla\cdot\mathbf{u}=0$.

$$C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)=\{\boldsymbol{\varphi}\in C^\infty(\mathbb{T}^3;\mathbb{R}^3):\nabla\cdot\boldsymbol{\varphi}=0\},$$

$$L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)=\boldsymbol{cl}_{L^2(\mathbb{T}^3)}C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)=\{\boldsymbol{\varphi}\in L^2(\mathbb{T}^3;\mathbb{R}^3):\nabla\cdot\boldsymbol{\varphi}=0\},$$

Helmholtz projection: An important consequence of elliptic theory is the existence of the Helmholtz decomposition. It allows to decompose any vector-valued function in $L^2(\mathbb{T}^3;\mathbb{R}^3)$ into a divergence free part and a gradient part. Set

$$(L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3))^\perp:=\{\mathbf{u}\in L^2(\mathbb{T}^3;\mathbb{R}^3)|\mathbf{u}=\nabla\psi,\ \psi\in H^1(\mathbb{T}^3;\mathbb{R})\}$$

The Helmholtz decomposition is defined by

$$\mathbf{u}=\mathcal{P}_H\mathbf{u}+\mathcal{Q}_H\mathbf{u},\quad\text{for any}\quad\mathbf{u}\in L^2(\mathbb{T}^3;\mathbb{R}^3),$$

where $\mathcal{P}_H$ is the projection from $L^2(\mathbb{T}^3;\mathbb{R}^3)$ to $L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$ and $\mathcal{Q}_H=\mathbb{I}-\mathcal{P}_H$ is also projection from $L^2(\mathbb{T}^3;\mathbb{R}^3)$ to $(L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3))^\perp$. Note that $L^2(\mathbb{T}^3;\mathbb{R}^3)$ admits a decomposition

$$L^2(\mathbb{T}^3;\mathbb{R}^3)=L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)\oplus(L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3))^\perp.$$

This decomposition is orthogonal with respect to $L^2(\mathbb{T}^3;\mathbb{R}^3)$-inner product. By property of projection $\mathcal{P}_H$, we have for $\mathbf{u}\in L^2(\mathbb{T}^3;\mathbb{R}^3)$

$$\langle\mathcal{P}_H\mathbf{u},\psi\rangle=\langle\mathbf{u},\psi\rangle,\qquad\text{for all }\psi\in L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3).\tag{2.1}$$

2.1 Stochastic framework

Here we specify details of the stochastic forcing term.

Brownian motions: Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The stochastic process W is a cylindrical $(\mathcal{F}_t)$-Wiener process in a separable Hilbert space $\mathfrak{U}$. It is formally given by the expansion

$$W(t) = \sum_{k\geq 1} e_k W_k(t),$$

where $\{W_k\}_{k\geq 1}$ is a sequence of mutually independent real-valued Brownian motions relative to $(\mathcal{F}_t)_{t\geq 0}$ and $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $\mathfrak{U}$. Finally, we define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 := \left\{ \mathbf{u} = \sum_{k\geq 1} \beta_k e_k;\ \sum_{k\geq 1} \frac{\beta_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|\mathbf{u}\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{\beta_k^2}{k^2}, \quad \mathbf{u} = \sum_{k\geq 1} \beta_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert–Schmidt. Moreover, $\mathbb{P}$-a.s., trajectories of W are in $C([0,T]; \mathfrak{U}_0)$.

Multiplicative noise: For each $\mathbf{u} \in L^2(\mathbb{T}^3; \mathbb{R}^3)$, we introduce a mapping $\sigma(\mathbf{u}) : \mathfrak{U} \to L^2(\mathbb{T}^3; \mathbb{R}^3)$ given by

$$\sigma(\mathbf{u}) e_k = \sigma_k(\mathbf{u}(\cdot)).$$

In particular, we suppose that the coefficients $\sigma_k : \mathbb{R}^3 \to \mathbb{R}^3$ are C^1-functions that satisfy the following conditions, for every $\xi, \zeta \in \mathbb{R}^3$,

$$\sum_{k\geq 1} |\sigma_k(\xi)|^2 \leq D_0(1 + |\xi|^2), \tag{2.2}$$

$$\sum_{k\geq 1} |\sigma_k(\xi) - \sigma_k(\zeta)|^2 \leq D_1 |\xi - \zeta|^2. \tag{2.3}$$

The assumption (2.2) imposed on σ implies that

$$\sigma : L^2(\mathbb{T}^3; \mathbb{R}^3) \to L_2(\mathfrak{U}; L^2(\mathbb{T}^3; \mathbb{R}^3)),$$

where $L_2(\mathfrak{U}; L^2(\mathbb{T}^3; \mathbb{R}^3))$ denotes the space of Hilbert-Schmidt operators from $\mathfrak{U}$ to $L^2(\mathbb{T}^3; \mathbb{R}^3)$. Thus, given a predictable process $\mathbf{u} \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}^3; \mathbb{R}^3)))$, the stochastic integral

$$\int_0^t \sigma(\mathbf{u})\, \mathrm{d}W = \sum_{k\geq 1} \int_0^t \sigma_k(\mathbf{u})\, \mathrm{d}W_k$$

is a well-defined $(\mathcal{F}_t)$-martingale taking values in $L^2(\mathbb{T}^3; \mathbb{R}^3)$; see [8, Section 2.3] for a detailed construction.

2.2 Preliminary results

Modified version of Jakubowski-Skorokhod theorem: Note that, strong convergence of approximate solutions in ω variable plays a pivotal role in the upcoming analysis. In that context, we need Jakubowski-Skorokhod theorem, delivering a new probability space and new random variables, with the same laws as the original ones, converging almost surely. However, for technical reasons, we have to use a modified version of Jakubowski-Skorokhod theorem [35, Corollary 7.3] which is stated below.

Theorem 2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and S_1 be separable metric space and S_2 be a quasi-polish space (there is a sequence of continuous functions $h_n : S_2 \to [-1, 1]$ that separates points of S_2). $\mathcal{B}(S_1) \otimes \mathcal{S}_2$ is sigma algebra associated with product space $S_1 \times S_2$, where $\mathcal{S}_2$ is the sigma algebra generated by the sequence of h_n. Let $U_n : \Omega \to S_1 \times S_2$, $n \in \mathbb{N}$, be a family of random variables, such that the sequence $\{\mathcal{L}aw(U_n) : n \in \mathbb{N}\}$ is weakly convergent on $S_1 \times S_2$. For $k = 1, 2$, let $\pi_i : S_1 \times S_2$ be the projection onto S_i, i.e.*

$$U = (U_1, U_2) \in S_1 \times S_2 \mapsto \pi_i(U) = U_i \in S_i.$$

Finally let us assume that there exists a random variable $X : \Omega \to S_1$ such that $\mathcal{L}aw(\pi(U_n)) = \mathcal{L}aw(X)$, $\forall\, n \in \mathbb{N}$. Then, there exist a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, a family of $S_1 \times S_2$-valued random variables $\{\widetilde{U}_n : n \in \mathbb{N}\}$, on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and a random variable $\widetilde{U} : \widetilde{\Omega} \to S_1 \times S_2$ such that

1. $\mathcal{L}aw(\widetilde{U}_n) = \mathcal{L}aw(U_n)\, \forall\, n \in \mathbb{N}$;
2. $\widetilde{U}_n \to \widetilde{U}$ *in* $S_1 \times S_2$, $\mathbb{P}-$ *a.s.*
3. $\pi_1(\widetilde{U}_n)(\widetilde{w}) = \pi_1(\widetilde{U})(\widetilde{w})$, $\forall\, \widetilde{w} \in \widetilde{\Omega}$.

3 Definitions and main results

3.1 Dissipative measure-valued martingale solutions

We are ready to introduce the concept of *dissipative measure–valued martingale solution* for the stochastic incompressible Euler system. In what follows, let $\mathcal{M} = \mathbb{R}^3$ be the phase space associated to the incompressible Euler system.

Definition 3.1 (Dissipative measure-valued martingale solution)**.** Let Λ be a Borel probability measure on $L^2_{\mathrm{div}}(\mathbb{T}^3)$. Then $\big[(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}); \mathcal{V}^\omega_{t,x}, W, \lambda_{\mathcal{C}}, \lambda_{\mathcal{D}}, \mathcal{H}\big]$ is a dissipative measure-valued martingale solution of (1.1), with initial condition $\mathcal{V}^\omega_{0,x}$; if

(a) $\mathcal{V}^\omega$ is a random variable taking values in the space of Young measures on $L^\infty_{w^*}\big([0,T] \times \mathbb{T}^3; \mathcal{P}(\mathcal{M})\big)$. In other words, $\mathbb{P}$-a.s. $\mathcal{V}^\omega_{t,x} : (t, x) \in [0, T] \times \mathbb{T}^3 \to \mathcal{P}(\mathcal{M})$ is a parametrized family of probability measures on $\mathcal{M}$,

(b) $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ is a stochastic basis with a complete right-continuous filtration,

(c) W is a $(\mathcal{F}_t)$-cylindrical Wiener process in $\mathfrak{U}$,

(d) the average velocity $\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle$ [1] satisfies, for any $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3; \mathbb{R}^3)$, $t \mapsto \langle\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle(t, \cdot), \boldsymbol{\varphi}\rangle \in C([0, T]; \mathbb{R}^3)$, $\mathbb{P}$-a.s., the function $t \mapsto \langle\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle(t, \cdot), \boldsymbol{\phi}\rangle$ is progressively measurable, and for any $\boldsymbol{\varphi} \in C^1(\mathbb{T}^3)$,

$$\int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle \cdot \nabla_x \boldsymbol{\varphi}\, dx = 0$$

for all $t \in [0, T]$, $\mathbb{P}-$a.s., and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \|\langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\rangle(t, \cdot)\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)}\right] < \infty$$

for all $1 \leq p < \infty$,

[1] Here $\langle \mathcal{V}^\omega_{t,x}; f(\mathbf{u})\rangle := \int_{\mathcal{M}} f(\mathbf{u}) d\mathcal{V}^\omega_{t,x}(\mathbf{u})$, for any measurable function f.

(e) $\Lambda = \mathcal{L}[\langle \mathcal{V}^\omega_{0,x}; \mathbf{u}\rangle]$,

(f) the integral identity

$$\int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{\tau,x}; \mathbf{u}\rangle \cdot \boldsymbol{\varphi}\, \mathrm{d}x - \int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{0,x}; \mathbf{u}\rangle \cdot \boldsymbol{\varphi}\, \mathrm{d}x = \int_0^\tau \int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{t,x}; \mathbf{u}\otimes\mathbf{u}\rangle : \nabla_x \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t + \int_0^\tau \int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{\tau,x}; \sigma(\mathbf{u})\rangle \cdot \boldsymbol{\varphi}\, \mathrm{d}W(t)\, \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^3} \nabla_x \boldsymbol{\varphi} : d\lambda_{\mathcal{C}}, \tag{3.1}$$

holds $\mathbb{P}$-a.s., for all $\tau \in [0,T)$, and for all $\boldsymbol{\varphi} \in C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$, where $\lambda_{\mathcal{C}} : \Omega \to L^\infty_{w*}([0,T];\mathcal{M}_b(\mathbb{T}^3;\mathbb{R}^3\times\mathbb{R}^3))$ is a random variable[2]; $\lambda_{\mathcal{C}}$ is called tensor-valued random concentration defect measures;

(g) there exists a real-valued square integrable continuous martingale $\mathcal{M}^2_E$, such that the following inequality

$$\begin{aligned}\mathcal{E}(t+) \leq &\mathcal{E}(s-) + \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{\tau,x}; |\sigma_k(\mathbf{u})|^2\right\rangle \,\mathrm{d}x\,\mathrm{d}\tau \\ &- \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \Big(\mathcal{Q}_H \left\langle \mathcal{V}^\omega_{\tau,x}; |\sigma_k(\mathbf{u})|\right\rangle\Big)^2 \mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_s^t\int_{\mathbb{T}^3} d\lambda_{\mathcal{D}} + \int_s^t d\mathcal{M}^2_E,\end{aligned} \tag{3.2}$$

holds $\mathbb{P}$-a.s., for all $0 \leq s < t \in (0,T)$ with

$$\mathcal{E}(t-) := \liminf_{r\to 0^+} \frac{1}{r}\int_{t-r}^t \left(\int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{s,x}; \frac{|\mathbf{u}|^2}{2}\right\rangle \mathrm{d}x + \mathcal{H}(s)\right) \mathrm{d}s$$

$$\mathcal{E}(t+) := \liminf_{r\to 0^+} \frac{1}{r}\int_t^{t+r} \left(\int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{s,x}; \frac{|\mathbf{u}|^2}{2}\right\rangle \mathrm{d}x + \mathcal{H}(s)\right) \mathrm{d}s$$

Here $\lambda_{\mathcal{D}} : \Omega \to L^\infty_{w*}([0,T];\mathcal{M}_b(\mathbb{T}^3))$ is a random variable[3], $\mathcal{H} \in L^\infty(0,T)$, $\mathcal{H} \geq 0$, $\mathbb{P}$-almost surely, and

$$\mathbb{E}\big[\sup_{t\in(0,T)} \mathcal{H}(t)\big] < \infty,$$

with initial energy

$$\mathcal{E}(0-) = \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}_0|^2\,\mathrm{d}x,$$

(h) there exists a constant $C > 0$ such that

$$\int_0^\tau \int_{\mathbb{T}^3} d|\lambda_{\mathcal{C}}| + \int_0^\tau \int_{\mathbb{T}^3} d|\lambda_{\mathcal{D}}| \leq C \int_0^\tau \mathcal{H}(t)dt, \tag{3.3}$$

$\mathbb{P}$-a.s., for every $\tau \in (0,T)$.

Remark 3.2. Notice that, a standard Lebesgue point argument applied to (3.2) reveals that the energy inequality holds for a.e. $0 \leq s < t$ in $(0,T)$:

$$\int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{t,x}; \frac{|\mathbf{u}|^2}{2}\right\rangle \mathrm{d}x + \mathcal{H}(t) \leq \int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{s,x}; \frac{|\mathbf{u}|^2}{2}\right\rangle \mathrm{d}x + \mathcal{H}(s) + \frac{1}{2}\sum_{k\geq 1}\int_s^t \int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{\tau,x}; |\sigma_k(\mathbf{u})|^2\right\rangle \,\mathrm{d}x\,\mathrm{d}\tau$$

[2]For any $\psi \in L^1\big([0,T];C(\mathbb{T}^3;\mathbb{R}^3\times\mathbb{R}^3)\big)$, $\langle \lambda_{\mathcal{C}}, \psi\rangle : \Omega \to \mathbb{R}$ is a random variable.
[3]For any $\phi \in L^1\big([0,T];C(\mathbb{T}^3;\mathbb{R})\big)$, $\langle \lambda_{\mathcal{D}}, \phi\rangle : \Omega \to \mathbb{R}$ is a random variable.

$$- \frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\Big(\mathcal{Q}_H\left\langle \mathcal{V}^{\omega}_{\tau,x}; |\sigma_k(\mathbf{u})|\right\rangle\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_s^t\int_{\mathbb{T}^3} d\lambda_D + \int_s^t d\mathcal{M}_E^2,\ \mathbb{P}-a.s. \tag{3.4}$$

However, as it is evident from Section 7, we require energy inequality to hold for *all* $s,t \in (0,T)$ to demonstrate weak-strong uniqueness principle.

Remark 3.3. Note that the above solution concept differs from the dissipative martingale solution concept [25, Definition 3.1] introduced by Hofmanová et. al. Indeed, the main difference lies in the successful identification of the martingale term present in (3.1), thanks to the weak continuity of Itô integral. Energy inequality (3.2) also differs from that of [25, Definition 3.1(M3)].

3.2 Strong pathwise solutions

We are also interested in establishing weak (measure-valued)–strong uniqueness principle for dissipative measure-valued solutions to (1.1). Since such an argument requires the existence of a strong solution, therefore, we first recall the notion of a local strong pathwise solution for the stochastic incompressible Euler equations. We remark that such a solution can be constructed on any given stochastic basis, that is, solutions are probabilistically strong, and satisfies the underlying equation (1.1) pointwise (not only in the sense of distributions), that is, solutions are srtong from the PDE standpoint. Existence of such a solution was first established by Glatt-Holtz & Vicol in [23].

Definition 3.4. (Local strong pathwise solution). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration, and W be an $(\mathcal{F}_t)$-cylindrical Wiener process in $\mathfrak{U}$. Suppose that $p \geq 2$, $m > \frac{3}{p} + 1$. Let $\mathbf{u}_0$ be a $W^{m,p}_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$-valued $\mathcal{F}_0$-measurable random variable. Then $(\mathbf{u}, \mathfrak{t})$ is said to be a local strong pathwise solution to the system (1.1) provided

(a) $\mathfrak{t}$ is an a.s. strictly positive $(\mathcal{F}_t)$-stopping time;

(b) the velocity $\mathbf{u}$ is a $W^{m,p}_{\mathrm{div}}(\mathbb{T}^3)$-valued $(\mathcal{F}_t)$-predictable measurable process satisfying

$$\mathbf{u}(\cdot\wedge\mathfrak{t}) \in C([0,T]; W^{m,p}_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)) \quad \mathbb{P}\text{-a.s.};$$

(c) for all $t \geq 0$,

$$\mathbf{u}(t\wedge\mathfrak{t}) = \mathbf{u}_0 - \int_0^{t\wedge\mathfrak{t}} \mathcal{P}_H(\mathbf{u}\cdot\nabla\mathbf{u})\mathrm{d}s + \int_0^{t\wedge\mathfrak{t}} \mathcal{P}_H\sigma(\mathbf{u})\,\mathrm{d}W. \tag{3.5}$$

It is evident that classical solutions require spatial derivatives of the velocity field $\mathbf{u}$ to be continuous $\mathbb{P}$-a.s. This motivates the following definition.

Definition 3.5. (Maximal strong pathwise solution). Fix an initial condition, and a complete stochastic basis with a cylindrical Wiener process as in Definition 3.4. Then a triplet

$$(\mathbf{u}, (\tau_R)_{R\in\mathbb{N}}, \mathfrak{t})$$

is said to be a maximal strong pathwise solution to system (1.1) provided

(a) $\mathfrak{t}$ is an a.s. strictly positive $(\mathbb{F}_t)$-stopping time;

(b) $(\tau_R)_{R\in\mathbb{N}}$ is an increasing sequence of $(\mathbb{F}_t)$-stopping times such that $\lim_{R\to\infty}\tau_R = \mathfrak{t}$ a.s. and

$$\sup_{t\in[0,\tau_R]} \|\mathbf{u}(t)\|_{W^{1,\infty}(\mathbb{T}^3;\mathbb{R}^3)} \geq R \quad \text{on} \quad [\mathfrak{t} < T]; \tag{3.6}$$

(c) each pair $(\mathbf{u}, \tau_R)$, $R \in \mathbb{N}$, is a local strong pathwise solution in the sense of Definition 3.4.

3.3 Statements of main results

We now state the main results of this paper. To begin with, regarding the existence of dissipative measure-valued martingale solutions, we have the following result.

Theorem 3.6 (**Existence of measure-valued solutions**). *Let $\mathbf{u}_0 \in L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$. Then approximating solutions $\mathbf{u}_n$ resulted by the spectral viscosity method (4.4) (semi-discrete scheme) generate a dissipative measure-valued martingale solution $\big[(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}});\widetilde{\mathcal{V}}^{\omega}_{t,x},\widetilde{W},\lambda_{\mathcal{C}},\lambda_{\mathcal{D}},\mathcal{H}\big]$ in the sense of Definition 3.1 to the incompressible Euler system (1.1).*

Next, we make use of the $\mathcal{K}$-convergence in the context of Young measures to conclude the following pointwise convergence of averages of approximate solutions to a dissipative martingale solution to (1.1).

Theorem 3.7 (**Point-wise convergence to a dissipative solution**). *Suppose that the approximate solutions $\mathbf{u}_n$ to (4.4) for the stochastic Euler system generate a dissipative measure-valued martingale solution $\big[(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}});\widetilde{\mathcal{V}}^{\omega}_{t,x},\widetilde{W},\lambda_{\mathcal{C}},\lambda_{\mathcal{D}},\mathcal{H}\big]$ in the sense of Definition 3.1. Then there exists a sequence of approximate solutions $\widetilde{\mathbf{u}}_n$ to (4.4) on probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}})$ for which following holds true,*

1. *$\widetilde{\mathbb{P}}$-a.s.*
$$\widetilde{\mathbf{u}}_n \to \langle \widetilde{\mathcal{V}}^{\omega}_{t,x};\widetilde{\mathbf{u}}\rangle \text{ in } C_w([0,T],L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)),$$
2. *$\widetilde{\mathbb{P}}$-a.s., there exists subsequence $\widetilde{\mathbf{u}}_{n_k}$ such that*
$$\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{u}}_{n_k} \to \langle \widetilde{\mathcal{V}}^{\omega}_{t,x};\widetilde{\mathbf{u}}\rangle, \text{ as } N\to\infty \text{ a.e. in } (0,T)\times\mathbb{T}^3.$$

Theorem 3.8 (**Weak-strong uniqueness**). *Let $\big[(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P});\mathcal{V}^{\omega}_{t,x},W,\lambda_{\mathcal{C}},\lambda_{\mathcal{D}},\mathcal{H}\big]$ be a dissipative measure-valued martingale solution to the system (1.1). On the same stochastic basis $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}\big)$, let us consider the unique maximal strong pathwise solution in sense of Definition 3.5 to the Euler system (1.1) given by $(\bar{\mathbf{u}},(\mathfrak{t}_R)_{R\in\mathbb{N}},\mathfrak{t})$ driven by the same cylindrical Wiener process W with the initial data $\bar{\mathbf{u}}(0)$ satisfying*
$$\mathcal{V}^{\omega}_{0,x} = \delta_{\bar{\mathbf{u}}(0,x)},\ \mathbb{P}-a.s., \text{for a.e. } x\in\mathbb{T}^3.$$
Then, $\mathbb{P}$-a.s. a.e. $t\in[0,T]$, $\mathcal{H}(t\wedge\mathfrak{t}_R)=0$, and $\mathbb{P}-a.s.$,
$$\mathcal{V}^{\omega}_{t\wedge\mathfrak{t}_R,x} = \delta_{\bar{\mathbf{u}}(t\wedge\mathfrak{t}_R,x)}, \text{ for a.e. } (t,x)\in(0,T)\times\mathbb{T}^3.$$

Finally, making use of the weak (measure-valued)–strong uniqueness principle (cf. Theorem 3.8), we prove the following result justifying the strong convergence to the regular solution.

Theorem 3.9 (**Strong convergence to regular solution**). *Let $u_0 \in L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$. Suppose that the approximate solutions $\mathbf{u}_n$ to (4.4) for the stochastic Euler system generate a dissipative measure-valued martingale solution $\big[(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}});\widetilde{\mathcal{V}}^{\omega}_{t,x},\widetilde{W},\lambda_{\mathcal{C}},\lambda_{\mathcal{D}},\mathcal{H}\big]$ in the sense of Definition 3.1. In addition, let the Euler equations (1.1) possess the unique strong (continuously differentiable) solution $(\bar{\mathbf{u}},(\mathfrak{t}_R)_{R\in\mathbb{N}},\mathfrak{t}) = (\bar{\mathbf{u}},(\mathfrak{t}_R)_{R\in\mathbb{N}},\mathfrak{t})$, emanating form the initial data (1.1). Then there exists a sequence of approximate solutions $\widetilde{\mathbf{u}}_n$ to (4.4) on probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}})$ such that $\widetilde{\mathbb{P}}$-a.s.*

$$\widetilde{\mathbf{u}}_n(\cdot\wedge\mathfrak{t}_R) \to \bar{\mathbf{u}}(\cdot\wedge\mathfrak{t}_R) \text{ weakly-(*) in } L^\infty(0,T;L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)) \text{ and strongly in } L^1((0,T)\times\mathbb{T}^3;\mathbb{R}^3).$$

4 Fourier method for the incompressible Euler system

We demonstrate the abstract theory applying the results to the approximate solutions resulting from the Fourier approximation of the Euler system.

- **Existence of approximate solutions.** First, we recall the existence of the Fourier approximate solutions $\mathbf{u}_n$ of the semi-discrete scheme in Fourier mode for any discretization $n \in \mathbb{N}$. Here we introduce a kind of spectrally accurate vanishing viscosity to augment the Fourier approximation of such nonlinear equations.
- **Stability and a priori bounds.** We assure that the scheme is energy dissipative. We recover required energy bounds from energy inequality.
- **Consistency.** We provide a consistent formulation and establish suitable bounds on the error terms.
- **Convergence of spectral method.** Using the stochastic compactness technique we show that approximate solutions generate a *dissipative measure-valued martingale* solution. The proof relies on a compactness argument combined with Jakubowski–Skorokhod's representation theorem. Due to the limited compactness of the Euler system, it is necessary to work with dissipative rather than analytically weak solutions.

4.1 Preliminaries for spectral method

We begin by reviewing some basic tools associated with the spectral method.
Fourier Coefficient: Consider the spatial Fourier expansion $\mathbf{u}(x,t) = \sum_k \widehat{\mathbf{u}}_k(t) e^{ik\cdot x}$ with coefficients $\widehat{\mathbf{u}}_k$ given by

$$\widehat{\mathbf{u}}_k(t) = \int_{\mathbb{T}^3} \mathbf{u}(x,t) e^{-ik\cdot x}\, \mathrm{d}x,$$

Truncation Operator: Truncation Operator $\mathcal{T}_n$ project vector field of the form $\mathbf{u} = \sum_k \widehat{\mathbf{u}}_k(t) e^{ik\cdot x}$ to $\sum_{|k|\,\le n} \widehat{\mathbf{u}}_k(t) e^{ik\cdot x}$ (only Fourier modes below threshold n). That is

$$\mathcal{T}_n(\mathbf{u}) = \sum_{|k|\,\le n} \hat{\mathbf{u}}_k(t) e^{ik\cdot x}$$

Projection Operator: We know Helmholtz projection project vector field $\mathbf{u} = \sum_k \widehat{\mathbf{u}}_k(t) e^{ik\cdot x}$ to divergence-free vector field given by

$$\mathcal{P}_H(\mathbf{u}) = \sum_k \big(\widehat{\mathbf{u}}_k - \frac{\widehat{\mathbf{u}}_k \cdot k}{|k|^2} k\big) e^{ik\cdot x}$$

Here we consider finite trunction of Holmoltz Projection as $\mathcal{P}_N$ given by

$$\mathcal{P}_N(\mathbf{u}) = \mathcal{T}_n\big(\mathcal{P}_H(\mathbf{u})\big) = \sum_{|k|\le N} \big(\widehat{\mathbf{u}}_k - \frac{\widehat{\mathbf{u}}_k \cdot k}{|k|^2} k\big) e^{ik\cdot x}$$

yielding a divergene-free vector field with Fourier modes $|k| \le N$. We also define

$$\mathcal{Q}_{n,m}(\mathbf{u}) = \mathcal{T}_n(\mathbf{u}) - \mathcal{P}_m(\mathbf{u})$$

where $m<n$ and $\mathcal{Q}_n$ shows the projection onto upper modes.

4.2 Semi-discrete scheme for spectral method

We propose a spectral viscosity method (SVM) to approximate the stochastic incompressible Euler equations and prove that SVM solution converges to a dissipative measure-valued martingale solution.
Motivation: To motivate the semi-discrete scheme, let $(\mathbf{u}, \Pi)$ be solutions to (1.1) with periodic boundary conditions. We focus on the spectral method based on the Fourier expansion and at the heart of a spectral method lies the assumption that the solutions $\mathbf{u}(x,t)$ can be expressed by a series of smooth basis functions. So we consider the spatial Fourier expansion $\mathbf{u}(x,t) = \sum_k \widehat{\mathbf{u}}_k(t)e^{ik\cdot x}$, $B(\mathbf{u}(x,t)) = \mathbf{u}(x,t)\cdot\nabla_x\mathbf{u}(x,t) = \sum_k \widehat{B}_k(\mathbf{u})(t)e^{ik\cdot x}$ and $\sigma_j(\mathbf{u}(x,t)) = \sum_k \widehat{\sigma}_{j,k}(\mathbf{u})(t)e^{ik\cdot x}$. Therefore divergence free condition gives that

$$i\sum_k(\widehat{\mathbf{u}}_k(t)\cdot k)e^{ik\cdot x} = 0 \iff \widehat{\mathbf{u}}_k(t)\cdot k = 0 \;\; \forall\, k \tag{4.1}$$

In terms of Fourier coefficients we have equation (1.1) in this form

$$\mathrm{d}\widehat{\mathbf{u}}_k(t) + \widehat{B}_k(\mathbf{u})(t)\,\mathrm{d}t + ik\widehat{\Pi}_k(t)\,\mathrm{d}t = \sum_{j\geq 1}\widehat{\sigma}_{j,k}(\mathbf{u})(t)\mathrm{d}W_j \tag{4.2}$$

Take dot product of (4.2) with k and used (4.1), therefore

$$\widehat{B}_k(\mathbf{u})\cdot k\,\mathrm{d}t + i|k|^2\widehat{\Pi}_k\,\mathrm{d}t = \sum_{j\geq 1}\widehat{\sigma}_{j,k}(\mathbf{u})\cdot k\mathrm{d}W_j$$

Eliminate pressure term from (4.2) using above expression,

$$\mathrm{d}\widehat{\mathbf{u}}_k + (\widehat{B}_k(\mathbf{u}) - \frac{\widehat{B}_k(\mathbf{u})\cdot k}{|k|^2}k)\,\mathrm{d}t = \sum_{j\geq 1}(\widehat{\sigma}_{j,k}(\mathbf{u}) - \frac{\widehat{\sigma}_{j,k}(\mathbf{u})\cdot k}{|k|^2}k)\mathrm{d}W_j \tag{4.3}$$

For the coefficient $\widehat{\mathbf{u}}_k$ with k=0, we can assume that $\int_{\mathbb{T}^3}\mathbf{u}_0\,\mathrm{d}x = 0$.
Semi-discrete scheme: To obtain a semi-discretized approximation to system (1.1), we restrict our attention to only the Fourier modes below some threshold n. In fact, we consider velocity field of the form $\mathbf{u}_n = \sum_{|k|\leq n}\widehat{\mathbf{u}}_k\, e^{ik\cdot x}$. In what follows, we will consider the following spectral vanishing viscosity scheme for the stochastic incompressible Euler equations:

$$\begin{cases}\mathrm{d}\mathbf{u}_n + \mathcal{P}_n(\mathbf{u}_n\cdot\nabla\mathbf{u}_n)\,\mathrm{d}t = \varepsilon\,\mathrm{div}(\mathcal{Q}_{\mathrm{n,m}}\nabla\mathbf{u}_{\mathrm{n}})\,\mathrm{d}t + \mathcal{P}_{\mathrm{n}}\sigma(\mathbf{u}_{\mathrm{n}})\mathrm{dW}\\ \mathbf{u}_n(0) = \mathcal{T}_n(\mathbf{u}_0)\end{cases} \tag{4.4}$$

In this scheme, we adopt a small $\varepsilon := \varepsilon(n)$ ($\varepsilon(n)\to 0$ as $n\to\infty$) and an integer $m<n$. Here the integer m handles as a threshold between small and large Fourier modes. We have also added a small amount of numerical viscosity to ensure the stability of the resulting scheme. The idea behind the SVM is that dissipation is only applied on the upper part of the spectrum ($m<n$). Above system includes following Navier–Stokes system (for example $m=0$).

$$\begin{cases}\mathrm{d}\mathbf{u}_n + \mathcal{P}_n(\mathbf{u}_n\cdot\nabla\mathbf{u}_n)\,\mathrm{d}t = \varepsilon\,\Delta\mathbf{u}_n\,\mathrm{d}t + \mathcal{P}_n\sigma(\mathbf{u}_n)\mathrm{d}W\\ \mathbf{u}_n(0) = \mathcal{T}_n(\mathbf{u}_0)\end{cases} \tag{4.5}$$

Existence of approximate solutions $\mathbf{u}_n$: The existence of solutions $\mathbf{u}_n$ to (4.4) is classical and relies on a priori bounds that are established using the cancellation property. For a proof, one can follow the similar approach as proposed in [22].

4.3 Stability and energy bounds

Energy inequality for $\mathbf{u}_n$: We derive the energy inequality from the scheme. In fact, the energy inequality is a direct consequence of the Itô formula.

Lemma 4.1. *Let $\mathbf{u}_n$ be the solution of the semi-discrete scheme (4.4). Then, $\mathbb{P}$-a.s., for all $s<t$,*

$$\frac{1}{2}\|\mathbf{u}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \varepsilon\int_s^t \|\mathcal{Q}_{n,m}(\nabla\mathbf{u}_n(\tau))\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)}\mathrm{d}\tau$$
$$= \frac{1}{2}\|\mathbf{u}_n(s)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \int_s^t\int_{\mathbb{T}^3}\mathbf{u}_n(\tau)\cdot\mathcal{P}_n(\sigma(\mathbf{u}_n(\tau)))\mathrm{d}W(\tau) + \frac{1}{2}\int_s^t \|\mathcal{P}_n(\sigma(\mathbf{u}_n(\tau)))\|^2_{L_2(\mathfrak{U},L^2(\mathbb{T}^3;\mathbb{R}^3))}\,\mathrm{d}x\mathrm{d}\tau. \tag{4.6}$$

In particular, $\mathbb{P}$-a.s., for all $t\in[0,T]$

$$\frac{1}{2}\|\mathbf{u}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \varepsilon\int_0^t \|\mathcal{Q}_{n,m}(\nabla\mathbf{u}_n(\tau))\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)}\mathrm{d}\tau$$
$$\leq \frac{1}{2}\|\mathbf{u}_0(s)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \int_0^t\int_{\mathbb{T}^3}\mathbf{u}_n(\tau)\cdot\mathcal{P}_n(\sigma(\mathbf{u}_n(\tau)))\mathrm{d}W(\tau) + \frac{1}{2}\int_0^t \|\mathcal{P}_n(\sigma(\mathbf{u}_n(\tau)))\|^2_{L_2(\mathfrak{U},L^2(\mathbb{T}^3;\mathbb{R}^3))}\,\mathrm{d}x\mathrm{d}\tau. \tag{4.7}$$

Proof. Apply Itô formula to $F(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)}$, we get $\mathbb{P}$-a.s., for all $s<t\in[0,T]$

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} =& \frac{1}{2}\|\mathbf{u}_n(s)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} - \int_s^t\int_{\mathbb{T}^3}\big(\mathbf{u}_n\cdot\mathcal{P}_n(\mathbf{u}_n\cdot\nabla\mathbf{u}_n) - \varepsilon\mathbf{u}_n\cdot\mathrm{div}(\mathcal{Q}_{\mathrm{n,m}}\nabla\mathbf{u}_n)\big)\,\mathrm{d}x\mathrm{d}\tau\\ &+ \int_s^t\int_{\mathbb{T}^3}\mathbf{u}_n\cdot\mathcal{P}_n(\sigma(\mathbf{u}_n))\mathrm{d}x\mathrm{d}W(\tau) + \frac{1}{2}\int_s^t\|\mathcal{P}_n(\sigma(\mathbf{u}_n))\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^3;\mathbb{R}^3))}d\tau,\end{aligned} \tag{4.8}$$

$$\int_{\mathbb{T}^3}\mathbf{u}_n\cdot\mathcal{P}_n(\mathbf{u}_n\cdot\nabla\mathbf{u}_n)\,\mathrm{d}x = \int_{\mathbb{T}^3}\mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla\mathbf{u}_n)\,\mathrm{d}x = \int_{\mathbb{T}^3}\mathrm{div}(\frac{1}{2}|\mathbf{u}_\mathrm{n}|^2\mathbf{u}_\mathrm{n})\,\mathrm{d}x = 0, \tag{4.9}$$

$$\begin{aligned}\varepsilon\int_{\mathbb{T}^3}\mathbf{u}_n\cdot\mathrm{div}(\mathcal{Q}_{\mathrm{n,m}}\nabla\mathbf{u}_\mathrm{n})\,\mathrm{d}x =& -\varepsilon\int_{\mathbb{T}^3}\nabla\mathbf{u}_n : \mathcal{Q}_{n,m}\nabla\mathbf{u}_n\,\mathrm{d}x = -\varepsilon\int_{\mathbb{T}^3}\mathcal{Q}_{n,m}\nabla\mathbf{u}_n : \mathcal{Q}_{n,m}\nabla\mathbf{u}_n\,\mathrm{d}x\\ &= -\varepsilon\int_{\mathbb{T}^3}|\mathcal{Q}_{n,m}(\nabla\mathbf{u}_n)|^2\,\mathrm{d}x.\end{aligned} \tag{4.10}$$

By using (4.8)-(4.10), we get (4.6).

To prove the second part (4.7), we use the fact that $\|\mathcal{T}_n(\mathbf{u}_0)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \leq \|\mathbf{u}_0\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)}$. This finishes the proof. □

A priori estimates: In what follows, we can now derive a priori bounds from the above energy inequality. Indeed, after taking $p-$th power and expectation of both sides (4.7), making use of Gronwall's and BDG inequality, we immediately get the following uniform bounds in n, for all $p\geq 1$,

$$\mathbb{E}\big[\sup_{t\in[0,T]}\|\mathbf{u}_n(t)\|^p_{L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)}\big] \leq \|\mathbf{u}_0\|^p_{L^2(\mathbb{T}^3;\mathbb{R}^3)}. \tag{4.11}$$

4.4 Consistency formulation

In this section, our aim is to prove the consistency of the momentum equation. Indeed, we demonstrate a consistency formulation and derive suitable bounds on the error terms. The consistency formulation of semi-discrete scheme for the incompressible Euler equations reads, for all $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\langle \mathbf{u}_n(t), \boldsymbol{\varphi} \rangle = \langle \mathcal{T}_n(\mathbf{u}_0), \boldsymbol{\varphi} \rangle + \int_0^t \langle \mathbf{u}_n \otimes \mathbf{u}_n, \nabla \boldsymbol{\varphi} \rangle ds + \int_0^t \langle \sigma(\mathbf{u}_n), \boldsymbol{\varphi} \rangle \mathrm{d}W(s) + \mathcal{R}_1(n,t,\boldsymbol{\varphi}) + \mathcal{N}(n,m,t,\boldsymbol{\varphi}), \tag{4.12}$$

where $\mathcal{R}_1(n,t,\boldsymbol{\varphi})$, $\mathcal{N}(n,t,m,\boldsymbol{\varphi})$ satiesfies $\mathbb{P}$-a.s

$$\mathcal{R}_1(n,t,\boldsymbol{\varphi}) := -\int_0^t \int_{\mathbb{T}^3} \nabla(\mathbb{I} - \mathcal{P}_n)\boldsymbol{\varphi} : (\mathbf{u}_n \otimes \mathbf{u}_n)\, \mathrm{d}x ds,$$

$$\mathcal{N}(n,m,t,\boldsymbol{\varphi}) := \varepsilon \int_0^t \int_{\mathbb{T}^3} (\mathbb{I} - \mathcal{P}_m)\Delta\boldsymbol{\varphi} \cdot \mathbf{u}_n \, \mathrm{d}x ds,$$

$$|\mathcal{R}_1(n,t,\boldsymbol{\varphi})| \le C_T \sup_{t\in[0,T]} \|\mathbf{u}_n\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \|(\mathbb{I} - \mathcal{P}_n)\boldsymbol{\varphi}\|_{H^{3/2}(\mathbb{T}^3;\mathbb{R}^3)},$$

$$|\mathcal{N}(n,m,t,\boldsymbol{\varphi})| \le C_T \,\varepsilon \sup_{t\in[0,T]} \|\mathbf{u}_n\|_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \|(\mathbb{I} - \mathcal{P}_m)\boldsymbol{\varphi}\|_{H^2(\mathbb{T}^3;\mathbb{R}^3)}.$$

To establish this, we procced with each term step by step and estimate the consistency errors. For that, let $\boldsymbol{\varphi} \in C^\infty_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$ be a divergence free test function. Then
Convective term:

$$\int_{\mathbb{T}^3} \boldsymbol{\varphi} \cdot \mathcal{P}_n(\mathbf{u}_n \nabla \mathbf{u}_n)\, \mathrm{d}x = \int_{\mathbb{T}^3} \mathrm{div}(\mathbf{u}_\mathrm{n} \otimes \mathbf{u}_\mathrm{n}) \cdot \boldsymbol{\varphi}\, \mathrm{dx} + \mathcal{R}_1(\mathrm{n}, \mathrm{t}, \boldsymbol{\varphi})$$

where $\mathcal{R}_1$ estimated as follows

$$\begin{aligned}
\mathcal{R}_1(n,t,\boldsymbol{\varphi}) &= \int_0^t \int_{\mathbb{T}^3} \boldsymbol{\varphi} \mathcal{P}_n(\mathbf{u}_n \cdot \nabla \mathbf{u}_n)\, \mathrm{d}x ds - \int_0^t \int_{\mathbb{T}^3} \mathrm{div}(\mathbf{u}_\mathrm{n} \otimes \mathbf{u}_\mathrm{n}) \cdot \boldsymbol{\varphi}\, \mathrm{dxds} \\
&= \int_0^t \int_{\mathbb{T}^3} \boldsymbol{\varphi} \cdot \mathcal{P}_n(\mathrm{div}(\mathbf{u}_n \otimes \mathbf{u}_n))\, \mathrm{d}x ds - \int_0^t \int_{\mathbb{T}^3} \mathrm{div}(\mathbf{u}_\mathrm{n} \otimes \mathbf{u}_\mathrm{n}) \cdot \boldsymbol{\varphi}\, \mathrm{dxds} \\
&= -\int_0^t \int_{\mathbb{T}^3} \boldsymbol{\varphi} \cdot \mathrm{div}\big((\mathbb{I} - \mathcal{P}_\mathrm{n})(\mathbf{u}_\mathrm{n} \otimes \mathbf{u}_\mathrm{n})\big)\, \mathrm{dxds} \\
&= \int_0^t \int_{\mathbb{T}^3} \nabla\boldsymbol{\varphi} : (\mathbb{I} - \mathcal{P}_n)(\mathbf{u}_n \otimes \mathbf{u}_n)\, \mathrm{d}x ds \\
&= \int_0^t \int_{\mathbb{T}^3} \nabla(\mathbb{I} - \mathcal{P}_n)\boldsymbol{\varphi} : (\mathbf{u}_n \otimes \mathbf{u}_n)\, \mathrm{d}x ds
\end{aligned}$$

It implies that

$$\begin{aligned}
|\mathcal{R}_1(n,t,\boldsymbol{\varphi})| &\le \int_0^t \|\mathbf{u}_n\|_{L^\infty(\mathbb{T}^3)} \|\mathbf{u}_n\|_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \|\nabla(\mathbb{I} - \mathcal{P}_n)\boldsymbol{\varphi}\|_{L^2(\mathbb{T}^3;\mathbb{R}^3)} ds \\
&\le C_T\, n^{1/2} \sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \|\nabla(\mathbb{I} - \mathcal{P}_n)\boldsymbol{\varphi}\|_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \\
&\le C_T \sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \|(\mathbb{I} - \mathcal{P}_n)\boldsymbol{\varphi}\|_{H^{3/2}(\mathbb{T}^3;\mathbb{R}^3)}
\end{aligned}$$

where $\|(\mathbb{I}-\mathcal{P}_n)\boldsymbol{\varphi}\|_{H^{3/2}(\mathbb{T}^3;\mathbb{R}^3)} \to 0$ as $n \to \infty$.
Diffusion term:

$$\mathcal{N}(n,m,t,\boldsymbol{\varphi}) := \varepsilon \int_0^t \int_{\mathbb{T}^3} \mathrm{div}(\mathcal{Q}_{\mathrm{n,m}}\nabla \mathbf{u}_{\mathrm{n}}) \cdot \boldsymbol{\varphi} \,\mathrm{dxds} = \varepsilon \int_0^{\mathrm{t}} \int_{\mathbb{T}^3} \mathrm{div}(\mathbb{I}-\mathcal{P}_{\mathrm{m}})\nabla \boldsymbol{\varphi} \cdot \mathbf{u}_{\mathrm{n}} \,\mathrm{dxds}$$
$$= \varepsilon \int_0^t \int_{\mathbb{T}^3} (\mathbb{I}-\mathcal{P}_m)\Delta\boldsymbol{\varphi} \cdot \mathbf{u}_n \,\mathrm{d}x ds$$

It implies that

$$|\mathcal{N}(n,m,t,\boldsymbol{\varphi})| \le C_T\, \varepsilon \sup_{t\in[0,T]} \|\mathbf{u}_n(t)\|_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \|(\mathbb{I}-\mathcal{P}_m)\boldsymbol{\varphi}\|_{H^2(\mathbb{T}^3;\mathbb{R}^3)}.$$

Stochastic term:

$$\int_0^t \int_{\mathbb{T}^3} \boldsymbol{\varphi} \cdot \mathcal{P}_n \sigma(\mathbf{u}_n)\,\mathrm{d}x\,\mathrm{d}W(s) = \int_0^t \int_{\mathbb{T}^3} \boldsymbol{\varphi} \cdot\, \sigma(\mathbf{u}_n)\,\mathrm{d}x\,\mathrm{d}W(s).$$

5 Convergence of spectral method

In this section, we discuss the tools required for the proof of convergence of the scheme. In fact, our aim is to verify the passage to the limit which in turn gives the existence of a *dissipative measure-valued martingale* solution to the original equation. Nevertheless, the limit argument is quite technical and has to be done in several steps. It is based on the compactness method: the uniform energy estimates yield tightness of sequence of approximate solutions and thus, on another probability space, this sequence converges almost surely, thanks to the Jakubowski-Skorokhod representation theorem. Let us now prepare the setup for our compactness method. To establish the tightness of the laws generated by the approximations, let us define the path space $\mathcal{K}$ to be the product of the following spaces:

$$\mathcal{K}_{\mathbf{u}} = C_w([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)), \qquad \mathcal{K}_W = C([0,T];\mathfrak{U}_0),$$
$$\mathcal{K}_{\mathcal{C}} = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3;\mathbb{R}^3\times\mathbb{R}^3)), w^*\big), \qquad \mathcal{K}_{\mathcal{E}} = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big),$$
$$\mathcal{K}_{\mathcal{D}} = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big) \qquad \mathcal{K}_{\mathcal{V}} = \big(L^\infty((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), w^*\big),$$
$$\mathcal{K}_{\mathcal{G}} = \big(L^\infty(0,T;\mathcal{M}_b(\mathbb{T}^3)), w^*\big), \qquad \mathcal{K}_{\mathcal{N}} = C([0,T];\mathbb{R}).$$

Let us denote by $\lambda_{\mathbf{u}_n}$, and λ_{W_n} respectively, the law of $\mathbf{u}_n$ and W_n on the corresponding path space. Moreover, let $\lambda_{\mathcal{N}_n}$ denotes the law of martingales $\mathcal{N}_n(t) := \sum_{k\ge1}\int_0^t\int_{\mathbb{T}^3} \mathbf{u}_n \cdot \mathcal{P}_n\sigma_k(\mathbf{u}_n)\,\mathrm{d}W$ on the corresponding path space. Furthermore, let $\lambda_{\mathcal{C}_n}$, $\lambda_{\mathcal{D}_n}$, $\lambda_{\mathcal{E}_n}$, and $\lambda_{\mathcal{V}_n}$ denote the law of

$$\mathcal{C}_n := \mathbf{u}_n\otimes\mathbf{u}_n, \quad \mathcal{D}_n := \frac{1}{2}\sum_{k\ge1}|\sigma_k(\mathbf{u}_n)|^2, \quad \mathcal{E}_n := \frac{1}{2}|\mathbf{u}_n|^2, \qquad \mathcal{V}_n := \delta_{\mathbf{u}_n}, \quad \mathcal{G}_n := \frac{1}{2}\sum_{k\ge1}|\mathcal{Q}_H\sigma_k(\mathbf{u}_n)|^2,$$

respectively, on the corresponding path spaces. Finally, let λ^n denotes the joint law of all the variables on $\mathcal{K}$. To proceed further, it is necessary to establish tightness of $\{\lambda^n;\, n\in\mathbb{N}\}$. To this end, we observe that the tightness of λ_W is immediate. So we show the tightness of other variables. We can easily prove the following uniform estimate which helps to conclude that laws given by approximate solutions are tight.

Lemma 5.1 (Compactness in time). *Let $\mathbf{u}_n$ be the solution of the semi-discrete scheme (4.4). Then there exists $0<\alpha<\frac{1}{2}$, $C>0$ such that, for $\gamma>\frac{5}{2}$, $r>2$*

$$\mathbb{E}\big[\|\mathbf{u}_n\|_{C^\alpha([0,T],W^{-\gamma,2}(\mathbb{T}^3;\mathbb{R}^3))}\big] \le C, \tag{5.1}$$

and

$$\mathbb{E}\left[\Big\|\int_0^{\cdot}\int_{\mathbb{T}^3}\mathbf{u}_n\cdot\mathcal{P}_n(\sigma(\mathbf{u}_n))\,\mathrm{d}x\mathrm{d}W\Big\|^r_{L_2(\mathfrak{U};C^\alpha([0,T];\mathbb{R}))}\right]\leq C. \tag{5.2}$$

Proof. For a proof, we refer to [12, Propositions 3.1 & 3.5]. □

5.1 Stochastic compactness

Tightness of law: To proceed, it is necessary to establish tightness of $\{\lambda^n; n\in\mathbb{N}\}$. In fact, we have all in hand to conclude our compactness argument by showing the tightness of a certain collection of laws.

Lemma 5.2. *$\{\lambda^n, n\in\mathbb{N}\}$ is tight on $\mathcal{K}$.*

Proof. Compact embeddings give tightness of laws. For a proof, we refer to [12, Propositions 3.1-3.5, Corollary 3.6]. □

Since the path space $\mathcal{K}$ is not a Polish space, our compactness argument is based on the modified version of Jakubowski-Skorokhod representation theorem, instead of the classical Skorokhod representation theorem. To be more precise, passing to a weakly convergent subsequence λ^n and denoting by λ the limit law, we infer the following result.

Proposition 5.3. *There exist a subsequence λ^n (not relabelled), a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ with $\mathcal{K}$-valued Borel measurable random variables $(\widetilde{\boldsymbol{u}}_n,\widetilde{W}_n\widetilde{\mathcal{C}}_n,\widetilde{\mathcal{D}}_n,\widetilde{\mathcal{E}}_n,\widetilde{\mathcal{N}}_n,\widetilde{\mathcal{G}}_n,\widetilde{\mathcal{V}}_n)$, $n\in\mathbb{N}$, and $(\widetilde{\boldsymbol{u}},\widetilde{W},\widetilde{\mathcal{C}},\widetilde{\mathcal{D}},\widetilde{\mathcal{E}},\widetilde{\mathcal{N}},\widetilde{\mathcal{G}},\widetilde{\mathcal{V}})$ such that*

(1) the law of $(\widetilde{\boldsymbol{u}}_n,\widetilde{W}_n\widetilde{\mathcal{C}}_n,\widetilde{\mathcal{D}}_n,\widetilde{\mathcal{E}}_n,\widetilde{\mathcal{N}}_n,\widetilde{\mathcal{G}}_n,\widetilde{\mathcal{V}}_n)$ is given by λ^n, $n\in\mathbb{N}$,

(2) the law of $(\widetilde{\boldsymbol{u}},\widetilde{W},\widetilde{\mathcal{C}},\widetilde{\mathcal{D}},\widetilde{\mathcal{E}},\widetilde{\mathcal{N}},\widetilde{\mathcal{G}},\widetilde{\mathcal{V}})$, denoted by λ, is a Radon measure,

(3) $(\widetilde{\boldsymbol{u}}_n,\widetilde{W}_n\widetilde{\mathcal{C}}_n,\widetilde{\mathcal{D}}_n,\widetilde{\mathcal{E}}_n,\widetilde{\mathcal{N}}_n,\widetilde{\mathcal{G}}_n,\widetilde{\mathcal{V}}_n)$ converges $\widetilde{\mathbb{P}}$-almost surely to $(\widetilde{\boldsymbol{u}},\widetilde{W},\widetilde{\mathcal{C}},\widetilde{\mathcal{D}},\widetilde{\mathcal{E}},\widetilde{\mathcal{N}},\widetilde{\mathcal{G}},\widetilde{\mathcal{V}})$ in the topology of $\mathcal{K}$, i.e.,

$$\begin{aligned}
&\widetilde{\boldsymbol{u}}_n\to\bar{\boldsymbol{u}} \text{ in } C_w([0,T];L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)), && \widetilde{W}_n\to\widetilde{W} \text{ in } C([0,T];\mathcal{U}_0)),\\
&\widetilde{\mathcal{C}}_n\to\widetilde{\mathcal{C}} \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3;\mathbb{R}^3\times\mathbb{R}^3)), && \widetilde{\mathcal{D}}_n\to\widetilde{\mathcal{D}} \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),\\
&\widetilde{\mathcal{N}}_n\to\widetilde{N} \text{ in } C([0,T];\mathbb{R}), && \widetilde{\mathcal{E}}_n\to\widetilde{\mathcal{E}} \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),\\
&\widetilde{\mathcal{V}}_n\to\widetilde{\mathcal{V}} \text{ weak-}* \text{ in } L^\infty_{w*}((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^3)), && \widetilde{\mathcal{G}}_n\to\widetilde{\mathcal{G}} \text{ weak-}* \text{ in } L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),
\end{aligned}$$

(4) For any $n\in\mathbb{N}$, $\widetilde{W}_n=\widetilde{W}$.

Proof. Proof of the items (1), (2), and (3) directly follow from Jakubowski-Skorokhod representation theorem. For the proof of the item (4), we refer to Theorem 2.1, and [26]. □

Martingale solution: In the following result, we will show that $\widetilde{\mathbf{u}}_n$ is also a solution of the approximate scheme (4.4) in another probability space.

Proposition 5.4. *For every $n\in\mathbb{N}$, $\big((\widetilde{\Omega},\widetilde{\mathcal{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}}),\widetilde{\mathbf{u}}_n,\widetilde{W}\big)$ is a finite energy martingale solution to (4.4) with the initial data $\mathcal{T}_n(\mathbf{u}_0)$.*

Proof. Proof of the above proposition directly follows form the Theorem 2.9.1 of the monograph by Breit et. al. [8]. □

We note that the above proposition implies that the new random variables satisfy the following equations and the energy inequality on the new probability space,

- for all $\boldsymbol{\varphi} \in C^\infty_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$ we have

$$\begin{aligned}\langle \widetilde{\mathbf{u}}_n(t), \boldsymbol{\varphi}\rangle &= \langle \widetilde{\mathbf{u}}_n(0), \boldsymbol{\varphi}\rangle - \int_0^t \langle \widetilde{\mathbf{u}}_n(s) \otimes \widetilde{\mathbf{u}}_n(s), \nabla_x \boldsymbol{\varphi}\rangle \,\mathrm{d}s + \varepsilon \int_0^t \langle (\mathbb{I} - \mathcal{P}_m)\nabla_x \widetilde{\mathbf{u}}_n(s)\,, \nabla_x \boldsymbol{\varphi}\rangle \mathrm{d}s \\ &\quad + \int_0^t \langle \sigma(\widetilde{\mathbf{u}}_n(s)), \boldsymbol{\varphi}\rangle \,\mathrm{d}W + \widetilde{\mathcal{R}}(n,t,\boldsymbol{\varphi}) + \widetilde{\mathcal{N}}(n,m,t,\boldsymbol{\varphi})\end{aligned} \tag{5.3}$$

  $\widetilde{\mathbb{P}}$-a.s. for all $t \in [0,T]$, where $\widetilde{\mathcal{R}}(n,t,\varphi)$, and $\widetilde{\mathcal{N}}(n,m,t,\boldsymbol{\varphi})$ are defined similarly as in (4.12), in the new probability space.

- the energy inequality, $\widetilde{\mathbb{P}}$-a.s., for all $0 \le s < t \le T$,

$$\begin{aligned}&\frac{1}{2}\|\widetilde{\mathbf{u}}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \varepsilon \int_s^t \|\mathcal{Q}_{n,m}(\nabla \widetilde{\mathbf{u}}_n(s))\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \mathrm{d}s \\ &\le \frac{1}{2}\|\widetilde{\mathbf{u}}_n(s)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \int_s^t \int_{\mathbb{T}^3} \widetilde{\mathbf{u}}_n(s)\cdot \mathcal{P}_n(\sigma(\widetilde{\mathbf{u}}_n(s)))\mathrm{d}\widetilde{W}(s) + \frac{1}{2}\int_s^t \|\mathcal{P}_n(\sigma(\widetilde{\mathbf{u}}_n(s)))\|^2_{L_2(\mathfrak{U}, L^2(\mathbb{T}^3;\mathbb{R}^3))}\mathrm{d}s\end{aligned} \tag{5.4}$$

Note that we can easily prove the energy inequality (5.4) in the new probability space from equation (5.3) as proved in Lemma 4.1. It is a direct consequence of the Itô formula.
Filtration: Note that, since $(\mathbf{u}_n, \mathcal{N}_n)$ are random variables with values in $C([0,T]; L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)) \times C([0,T];\mathbb{R})$. By [43, Lemma A.3] and [36, Corollary A.2], $(\widetilde{\mathbf{u}}_n, \widetilde{\mathcal{N}}_n)$ are also random variables with values in $C([0,T], L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)) \times C([0,T];\mathbb{R})$. Let $(\widetilde{\mathcal{F}}^n_t)$ be the $\widetilde{\mathbb{P}}$-augmented canonical filtration of the process $(\widetilde{\mathbf{u}}_n, \widetilde{W}, \widetilde{\mathcal{N}}_n)$, that is

$$\widetilde{\mathcal{F}}^n_t = \sigma\big(\sigma\big(\mathbf{r}_t\widetilde{\mathbf{u}}_n,\ \mathbf{r}_t\widetilde{W}, \mathbf{r}_t\widetilde{\mathcal{N}}_n\big) \cup \big\{N \in \widetilde{\mathcal{F}};\ \widetilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0,T],$$

where we denote by $\mathbf{r}_t$ the operator of restriction to the interval $[0,t]$ acting on various path spaces. Let us remark that by assuming that the initial filtration $(\mathcal{F}_t)_{t\ge 0}$ is the one generated by W, by [43, Lemma A.6], one can consider $(\widetilde{\mathcal{F}}^h_t) = (\widetilde{\mathcal{F}_t})$ is the filtration generated by $\widetilde{W}$. By [8, Theorem 2.1.34], $\widetilde{W}$ is a $(\widetilde{\mathcal{F}}_t)$-cylindrical Wiener process in $\mathfrak{U}$.
Almost surely limit: The lack of strong convergence of sequence $\widetilde{\mathbf{u}}_n$ does not allow us to identify the limit of the terms where the dependence on $\widetilde{\mathbf{u}}_n$ is nonlinear, namely, the convective term in momentum equation and nonlinear terms in energy inequality. Next we want to pass the limit $n \to \infty$ in (5.3) and (5.4). To complete this, first, we recall that a-priori bounds (4.11) which remains hold for the new random variables. Young measure capture the weak limit [8, section 2.8]. Thus, by the implementation of [8, Theorem 2.8], we conclude that $\widetilde{\mathbb{P}}$-a.s.,

$$\widetilde{\mathbf{u}}_n \rightharpoonup \langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\mathbf{u}}\rangle, \text{ weakly in } L^2((0,T); L^2_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)). \tag{5.5}$$

We first introduce the following random concentration defect measures to pass limit in the nonlinear terms present in the equations.

$$\widetilde{\lambda}_{\mathcal{C}} = \widetilde{\mathcal{C}} - \left\langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; \widetilde{\mathbf{u}} \otimes \widetilde{\mathbf{u}}\right\rangle dxdt,\ \widetilde{\lambda}_{\mathcal{E}} = \widetilde{\mathcal{E}} - \left\langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; \frac{1}{2}|\widetilde{\mathbf{u}}|^2\right\rangle dx,\ \widetilde{\lambda}_{\mathcal{D}} = \widetilde{\mathcal{D}} - \sum_{k\ge 1}\left\langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; \frac{1}{2}|\sigma_k(\widetilde{\mathbf{u}})|^2\right\rangle dxdt.$$

$$\widetilde{\lambda}_{\mathcal{G}} = \widetilde{\mathcal{G}} - \frac{1}{2}\sum_{k\geq 1}\left(\mathcal{Q}_H\left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; |\sigma_k(\widetilde{\mathbf{u}})|\right\rangle\right)^2$$

Make use of these random concentration defect measures, we conclude that $\widetilde{\mathbb{P}}$-a.s.

$$\begin{aligned}
&\widetilde{\mathcal{C}}_n \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \widetilde{\mathbf{u}}\otimes\widetilde{\mathbf{u}}\right\rangle dxdt + \widetilde{\lambda}_{\mathcal{C}}, \quad \text{weak-}* \text{ in } L^{\infty}_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3;\mathbb{R}^3\times\mathbb{R}^3)),\\
&\widetilde{\mathcal{D}}_n \rightharpoonup \sum_{k\geq 1}\left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{1}{2}|\sigma_k(\widetilde{\mathbf{u}})|^2\right\rangle dxdt + \widetilde{\lambda}_{\mathcal{D}}, \text{ weak-}* \text{ in } L^{\infty}_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),\\
&\widetilde{\mathcal{E}}_n \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{1}{2}|\widetilde{\mathbf{u}}|^2\right\rangle dxdt + \widetilde{\lambda}_{\mathcal{E}}, \text{ weak-}* \text{ in } L^{\infty}_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3)),\\
&\widetilde{\mathcal{G}}_n \rightharpoonup \frac{1}{2}\sum_{k\geq 1}\left(\mathcal{Q}_H\left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; |\sigma_k(\widetilde{\mathbf{u}})|\right\rangle\right)^2 + \widetilde{\lambda}_{\mathcal{G}} \text{ weak-}* \text{ in } L^{\infty}_{w*}(0,T;\mathcal{M}^+_b(\mathbb{T}^3)).
\end{aligned}$$

These concentration defect measures are $\widetilde{\mathbb{P}}$-almost surely, limits of sequences of random variables (consult the proof of [21, Lemma 2.1]), so these concentration defect measures are random variables.

Pass to limit in approximation of the momentum equation: By making use of [8, Theorem 2.8], we have $\widetilde{\mathbb{P}}$-a.s.,

$$\sigma_k(\widetilde{\mathbf{u}}_n) \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \sigma_k(\widetilde{\mathbf{u}})\right\rangle \text{ weakly in } L^2([0,T];L^2(\mathbb{T}^3;\mathbb{R}^3)). \tag{5.6}$$

Note that the Itô integral

$$I_t : \varphi \to \int_0^t \varphi(s)\mathrm{d}\widetilde{W}_k(s)$$

is a linear and continuous (hence weakly continuous) map from $L^2(\Omega\times[0,T];L^2(\mathbb{T}^3))$ to $L^2(\Omega;L^2(\mathbb{T}^3))$. Therefore, we can make use of weak continuity of Itô integral, and item (4) of Proposition 5.3, to conclude $I_t(\sigma_k(\widetilde{\mathbf{u}}_n))$ converges weakly to $I_t\big(\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\sigma_k(\widetilde{\mathbf{u}})\rangle\big)$ in $L^2(\Omega;L^2(\mathbb{T}^3;\mathbb{R}^3))$. Make use of above information and energy bounds we can conclude that

$$\begin{aligned}
\int_{\widetilde{\Omega}} \big\langle\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\widetilde{\mathbf{u}}\rangle,\boldsymbol{\varphi}\big\rangle\alpha(\omega)\mathrm{d}\widetilde{\mathbb{P}}(\omega) = \int_{\widetilde{\Omega}}\Big[\big\langle\langle\widetilde{\mathcal{V}}^{\omega}_{0,x};\widetilde{\mathbf{u}}\rangle,\boldsymbol{\varphi}\big\rangle + \int_0^t \big\langle\langle\widetilde{\mathcal{V}}_{s,x};\widetilde{\mathbf{u}}\otimes\widetilde{\mathbf{u}}\rangle,\nabla\boldsymbol{\varphi}\big\rangle\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\nabla\boldsymbol{\varphi}\mathrm{d}\widetilde{\lambda}_{\mathcal{C}}\\
+ \int_0^t \big\langle\langle\widetilde{\mathcal{V}}^{\omega}_{s,x};\sigma(\widetilde{\mathbf{u}})\rangle,\boldsymbol{\varphi}\big\rangle\mathrm{d}\widetilde{W}(s)\Big]\alpha(\omega)\mathrm{d}\widetilde{\mathbb{P}}(\omega)
\end{aligned} \tag{5.7}$$

holds for all $t\in[0,T)$, for all $\alpha\in L^2(\widetilde{\Omega})$ and for all $\boldsymbol{\varphi}\in C^{\infty}_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$. Since $C^{\infty}_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$ is separable space with sup norm, above equation (5.7) implies that

$$\begin{aligned}
\big\langle\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\widetilde{\mathbf{u}}\rangle,\boldsymbol{\varphi}\big\rangle = \big\langle\langle\widetilde{\mathcal{V}}^{\omega}_{0,x};\widetilde{\mathbf{u}}\rangle,\boldsymbol{\varphi}\big\rangle + \int_0^t \big\langle\langle\widetilde{\mathcal{V}}_{s,x};\widetilde{\mathbf{u}}\otimes\widetilde{\mathbf{u}}\rangle,\nabla\boldsymbol{\varphi}\big\rangle\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3}\nabla\boldsymbol{\varphi}\mathrm{d}\widetilde{\lambda}_{\mathcal{C}}\\
+ \int_0^t \big\langle\langle\widetilde{\mathcal{V}}^{\omega}_{s,x};\sigma(\widetilde{\mathbf{u}})\rangle,\boldsymbol{\varphi}\big\rangle\mathrm{d}\widetilde{W}(s)
\end{aligned}$$

for all $t\in[0,T]$, $\widetilde{\mathbb{P}}$-a.s., for all $\boldsymbol{\varphi}\in C^{\infty}_{\text{div}}(\mathbb{T}^3;\mathbb{R}^3)$.

$\widetilde{\lambda}_{\mathcal{G}}$ **is a nonnegative measure:** It is clear from (5.6) that $\widetilde{\mathbb{P}}$-a.s.,

$$\mathcal{Q}_H(\sigma_k(\widetilde{\mathbf{u}}_n)) \rightharpoonup \mathcal{Q}_H\big(\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\sigma_k(\widetilde{\mathbf{u}})\rangle\big) \text{ weakly in } L^2([0,T];(L^2(\mathbb{T}^3;\mathbb{R}^3))^{\perp}).$$

Let $\psi \in C([0,T]\times\mathbb{T}^3)$ with $\psi \ge 0$. Making use of weakly lower semi-continuity of norm, then we have $\widetilde{\mathbb{P}}$-a.s.,

$$\Big\langle \psi\ ,\ |\mathcal{Q}_H\big(\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\sigma_k(\widetilde{\mathbf{u}})\rangle\big)|^2\Big\rangle \le \liminf_{n\to\infty}\Big\langle \psi\ ,\ |\mathcal{Q}_H\big(\sigma_k(\widetilde{\mathbf{u}}_n)\big)|^2\Big\rangle$$

It shows that $\tilde{\lambda}_{\mathcal{G}}$ is non-negative measure.

Energy inequality and concentration defect: In this subsection, we show that an appropriate form of energy inequality also holds for dissipative measure-valued martingale solutions in the following steps.

Step-1. Martingale term.

Proposition 5.5. *$\widetilde{\mathbb{P}}$-a.s., $\widetilde{\mathcal{N}}_n \to \widetilde{\mathcal{N}}$ in $C([0,T];\mathbb{R})$, and $\widetilde{\mathcal{N}}$ is a real valued square-integrable martingale.*

Proof. Note that, thanks to Proposition 5.3, we have $\widetilde{\mathbb{P}}$-a.s., $\widetilde{\mathcal{N}}_n \to \widetilde{\mathcal{N}}$, in $C([0,T];\mathbb{R})$. To conclude that $(\widetilde{\mathcal{N}}(t))_{t\in[0,T]}$ is a martingale, it is enough to show that

$$\widetilde{\mathbb{E}}[\widetilde{\mathcal{N}}(t)|\widetilde{\mathcal{F}}_s] = \widetilde{\mathcal{N}}(s),$$

for all $t,s\in[0,T]$ with $s\le t$. To prove this, we have to show that, for $A\in\widetilde{\mathcal{F}}_s$

$$\widetilde{\mathbb{E}}\Big[\mathcal{I}_A\big(\widetilde{\mathcal{N}}(t)-\widetilde{\mathcal{N}}(s)\big)\Big] = 0,$$

By using the fact that $\widetilde{\mathcal{N}}_n$ is a martingale, we know that

$$\widetilde{\mathbb{E}}\Big[\mathcal{I}_A\big(\widetilde{\mathcal{N}}_n(t)-\widetilde{\mathcal{N}}_n(s)\big)\Big] = 0,$$

for all $A\in\widetilde{\mathcal{F}}_s$. For each t, $\widetilde{\mathcal{N}}_n(t)$ is uniformly bounded in $L^2(\widetilde{\Omega})$, with the help of Vitali's convergence theorem, we can pass to the limit in n to conclude that $\widetilde{\mathcal{N}}$ is a martingale. In this manner, we cannot secure the structure of the martingale $\widetilde{\mathcal{N}}$, which is expected due of lack of sufficient regularity. □

Step 2. Control on concentration defect measures:

Lemma 5.6. *The concentration defect $0\le\widetilde{\mathcal{H}}(r) := \widetilde{\lambda}_{\mathcal{E}}(r)(\mathbb{T}^3)$ dominates defect measures $\widetilde{\lambda}_{\mathcal{D}}$ & $\widetilde{\lambda}_{\mathcal{C}}$. More precisely, there exists a constant $C>0$ such that*

$$\int_0^r\int_{\mathbb{T}^3} d|\widetilde{\lambda}_{\mathcal{C}}| + \int_0^r\int_{\mathbb{T}^3} d|\widetilde{\lambda}_{\mathcal{D}}| \le C\int_0^r \widetilde{\mathcal{H}}(t)\,dt,$$

$\widetilde{\mathbb{P}}$-a.s., for all $r\in(0,T)$.

Proof. With the help of [24, Lemma 2.3], it is clear that $\widetilde{\lambda}_{\mathcal{E}}$ dominates defect measures $\widetilde{\lambda}_{\mathcal{C}}$. To show the dominance of $\widetilde{\lambda}_{\mathcal{E}}$ over $\widetilde{\lambda}_{\mathcal{D}}$, observe that, by virtue of hypotheses (2.2), (2.3), the function

$$[\mathbf{u}] \mapsto |\sigma_k(\mathbf{u})|^2 \text{ is continuous,}$$

and as such dominated by the total energy

$$\sum_{k\ge1}|\sigma_k(\mathbf{u})|^2 \le c\left(1+|\mathbf{u}|^2\right)$$

Hence, a consequence of [24, Lemma 2.3] completes the proof of the lemma. □

Step 3. Energy inequality. Use the fact that $\|\mathcal{P}_n(\sigma(\widetilde{\mathbf{u}}_n))\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^3;\mathbb{R}^3))} \le \|\mathcal{P}_H(\sigma(\widetilde{\mathbf{u}}_n))\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^3))}$ then from (4.6), we have $\widetilde{\mathbb{P}}$-a.s, for all $0 \le s < t \le T$,

$$\frac{1}{2}\|\widetilde{\mathbf{u}}_n(t)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \le \frac{1}{2}\|\widetilde{\mathbf{u}}_n(s)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} + \int_s^t \int_{\mathbb{T}^3} \widetilde{\mathbf{u}}_n(s)\cdot \mathcal{P}_n(\sigma(\widetilde{\mathbf{u}}_n(s)))\mathrm{d}W(s) + \frac{1}{2}\int_s^t \|\mathcal{P}_H(\sigma(\widetilde{\mathbf{u}}_n))\|^2_{L_2(\mathfrak{U};L^2(\mathbb{T}^3;\mathbb{R}^3))}\mathrm{d}s \tag{5.8}$$

Case 1: Suppose that $0<s<t \le T$. Now we would like to pass limit $n \to \infty$ in energy inequality. Let r, δ are small enough positive real numbers. Then we have $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}\frac{1}{2r}\int_t^{t+r}\int_{\mathbb{T}^3} |\widetilde{\mathbf{u}}_n(\tau)|^2\,\mathrm{d}x\mathrm{d}\tau \le \frac{1}{2\delta}\int_{s-\delta}^s\int_{\mathbb{T}^3}|\widetilde{\mathbf{u}}_n(a)|^2\,\mathrm{d}x\mathrm{d}a \\ + \frac{1}{r}\int_t^{t+r}\Big(\frac{1}{\delta}\int_{s-\delta}^s\int_a^b\int_{\mathbb{T}^3}\widetilde{\mathbf{u}}_n(s)\cdot\mathcal{P}_n(\sigma(\widetilde{\mathbf{u}}_n(\tau)))\mathrm{d}W(\tau)\mathrm{d}a\Big)\mathrm{d}b \\ + \sum_{k\ge 1}\frac{1}{2r}\int_t^{t+r}\Big(\frac{1}{\delta}\int_{s-\delta}^s\int_a^b\int_{\mathbb{T}^3}|\sigma_k(\widetilde{\mathbf{u}}_n(\tau))|^2 dx\mathrm{d}\tau\mathrm{d}a\Big)\mathrm{d}b \\ - \sum_{k\ge 1}\frac{1}{2r}\int_t^{t+r}\Big(\frac{1}{\delta}\int_{s-\delta}^s\int_a^b\int_{\mathbb{T}^3}|\mathcal{Q}_H(\sigma_k(\widetilde{\mathbf{u}}_n(\tau)))|^2\mathrm{d}x\mathrm{d}\tau\mathrm{d}a\Big)\mathrm{d}b\end{aligned}$$

We pass to limit $n \to \infty$ in above equation and make use of information of limits to conclude that $\widetilde{\mathbb{P}}$-a.s.,

$$\begin{aligned}\frac{1}{r}\int_t^{t+r}\int_{\mathbb{T}^3}\Big\langle \widetilde{\mathcal{V}}^{\omega}_{b,x};\frac{1}{2}|\widetilde{\mathbf{u}}|^2\Big\rangle\,\mathrm{d}x\mathrm{d}b + \frac{1}{r}\int_t^{t+r}\widetilde{\mathcal{H}}(b)\mathrm{d}b \le \frac{1}{\delta}\int_{s-\delta}^s\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^{\omega}_{a,x};\frac{1}{2}|\widetilde{\mathbf{u}}|^2\Big\rangle\,\mathrm{d}x\mathrm{d}a + \frac{1}{\delta}\int_{s-\delta}^s\widetilde{\mathcal{H}}(a)\mathrm{d}a \\ + \frac{1}{r}\int_t^{t+r}\Big(\frac{1}{\delta}\int_{s-\delta}^s(\widetilde{\mathcal{N}}(b)-\widetilde{\mathcal{N}}(a))\mathrm{d}a\Big)\mathrm{d}b \\ + \sum_{k\ge 1}\frac{1}{r}\int_t^{t+r}\Big(\frac{1}{\delta}\int_{s-\delta}^s\int_a^b\int_{\mathbb{T}^3}\Big(\Big\langle\widetilde{\mathcal{V}}^{\omega}_{(a,x)};\frac{1}{2}|\sigma_k(\widetilde{\mathbf{u}})|^2\Big\rangle + \mathrm{d}\widetilde{\lambda}_{\mathcal{D}}(x,a)\Big)\,\mathrm{d}x\mathrm{d}a\Big)\mathrm{d}b \\ - \sum_{k\ge 1}\frac{1}{2r}\int_t^{t+r}\Big(\frac{1}{\delta}\int_{s-\delta}^s\int_a^b\int_{\mathbb{T}^3}\Big(\Big(\mathcal{Q}_H\Big\langle\widetilde{\mathcal{V}}^{\omega}_{(a,x)};|\sigma_k(\widetilde{\mathbf{u}})|\Big\rangle\Big)^2 - \mathrm{d}\widetilde{\lambda}_{\mathcal{G}}(x,a)\Big)\,\mathrm{d}x\mathrm{d}a\Big)\mathrm{d}b\end{aligned}$$

Letting liminf both sides as $r,\delta \to 0$ and use that $\widetilde{\lambda}_{\mathcal{G}}$ is non-negative measure, then we have $\widetilde{\mathbb{P}}$-a.s., for all $0<s<t\le T$,

$$\begin{aligned}\liminf_{r\to 0}\frac{1}{r}\int_t^{t+r}\Big[\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^{\omega}_{b,x};\frac{1}{2}|\widetilde{\mathbf{u}}|^2\Big\rangle\,\mathrm{d}x\mathrm{d}b + \widetilde{\mathcal{H}}(b)\Big]\mathrm{d}b \le \liminf_{\delta\to 0}\frac{1}{\delta}\int_{s-\delta}^s\Big[\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^{\omega}_{a,x};\frac{1}{2}|\widetilde{\mathbf{u}}|^2\Big\rangle\,\mathrm{d}x + \widetilde{\mathcal{H}}(a)\Big]\mathrm{d}a \\ + \widetilde{\mathcal{N}}(t) - \widetilde{\mathcal{N}}(s) + \sum_{k\ge 1}\int_s^t\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^{\omega}_{(b,x)};\frac{1}{2}|\sigma_k(\widetilde{\mathbf{u}})|^2\Big\rangle\,\mathrm{d}x\mathrm{d}b + \int_s^t\int_{\mathbb{T}^3}\mathrm{d}\widetilde{\lambda}_{\mathcal{D}}(x,s)\mathrm{d}b \\ - \frac{1}{2}\sum_{k\ge 1}\int_s^t\int_{\mathbb{T}^3}\Big(\mathcal{Q}_H\Big\langle\widetilde{\mathcal{V}}^{\omega}_{(b,x)};|\sigma_k(\widetilde{\mathbf{u}})|\Big\rangle\Big)^2\,\mathrm{d}x\mathrm{d}b\end{aligned} \tag{5.9}$$

Case 2: When $s = 0$ and $t \in (0,T]$ in (5.8). From energy inequality (5.8), we have $\widetilde{\mathbb{P}}$-a.s.

$$\frac{1}{2r}\int_t^{t+r}\int_{\mathbb{T}^3}|\widetilde{\mathbf{u}}_n(s)|^2\,\mathrm{d}x\mathrm{d}s \le \frac{1}{2}\int_{\mathbb{T}^3}|\widetilde{\mathbf{u}}_n(0)|^2\,\mathrm{d}x + \frac{1}{r}\int_t^{t+r}\Big(\int_0^s\int_{\mathbb{T}^3}\widetilde{\mathbf{u}}_n(b)\mathcal{P}_n(\sigma(\widetilde{\mathbf{u}}_n(b)))\mathrm{d}W(b)\Big)\mathrm{d}s$$

$$
\begin{aligned}
&+\sum_{k\geq 1}\frac{1}{2r}\int_t^{t+r}\Big(\int_0^s\int_{\mathbb{T}^3}|\sigma_k(\widetilde{\mathbf{u}}_n(b))|^2dx\mathrm{d}b\Big)\mathrm{d}s\\
&-\sum_{k\geq 1}\frac{1}{2r}\int_t^{t+r}\Big(\int_0^s\int_{\mathbb{T}^3}|\mathcal{Q}_H(\sigma_k(\widetilde{\mathbf{u}}_n(b)))|^2\mathrm{d}x\mathrm{d}b\Big)\mathrm{d}s
\end{aligned}
$$

As in the previous case, we can conclude that $\widetilde{\mathbb{P}}$-a.s., for all $t\in(0,T]$,

$$
\begin{aligned}
\liminf_{r\to 0}\frac{1}{r}\int_t^{t+r}\Big[\int_{\mathbb{T}^3}\left\langle\widetilde{\mathcal{V}}^{\omega}_{b,x};\frac{1}{2}|\widetilde{\mathbf{u}}|^2\right\rangle\mathrm{d}x\mathrm{d}b+\widetilde{\mathcal{H}}(b)\Big]\mathrm{d}b \leq \int_{\mathbb{T}^3}\left\langle\widetilde{\mathcal{V}}^{\omega}_{0,x};\frac{1}{2}|\widetilde{\mathbf{u}}|^2\right\rangle\mathrm{d}x\\
+\widetilde{\mathcal{N}}(t)-\widetilde{\mathcal{N}}(0)+\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3}\left\langle\widetilde{\mathcal{V}}^{\omega}_{(b,x)};\frac{1}{2}|\sigma_k(\widetilde{\mathbf{u}})|^2\right\rangle\mathrm{d}x\mathrm{d}b+\int_0^t\int_{\mathbb{T}^3}\mathrm{d}\widetilde{\lambda}_{\mathcal{D}}(x,b)\mathrm{d}b\\
-\frac{1}{2}\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}^3}\left(\mathcal{Q}_H\left\langle\widetilde{\mathcal{V}}^{\omega}_{(b,x)};|\sigma_k(\widetilde{\mathbf{u}})|\right\rangle\right)^2\mathrm{d}x\mathrm{d}b
\end{aligned}
\tag{5.10}
$$

Last both cases show that there exists a real-valued square-integrable continuous martingale $\widetilde{\mathcal{M}}^2_E=\widetilde{\mathcal{N}}$, such that the following inequality

$$
\begin{aligned}
\widetilde{\mathcal{E}}(t+)\leq&\,\widetilde{\mathcal{E}}(s-)+\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\left\langle\widetilde{\mathcal{V}}^{\omega}_{\tau,x};|\sigma_k(\widetilde{\mathbf{u}})|^2\right\rangle\mathrm{d}x\,\mathrm{d}\tau\\
&-\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\left(\mathcal{Q}_H\left\langle\widetilde{\mathcal{V}}^{\omega}_{\tau,x};|\sigma_k(\widetilde{\mathbf{u}})|\right\rangle\right)^2\mathrm{d}x\,\mathrm{d}\tau+\frac{1}{2}\int_s^t\int_{\mathbb{T}^3}d\widetilde{\lambda}_{\mathcal{D}}+\int_s^t d\widetilde{\mathcal{M}}^2_E,
\end{aligned}
\tag{5.11}
$$

holds $\mathbb{P}$-a.s., for all $0\leq s<t\in(0,T)$ with

$$
\widetilde{\mathcal{E}}(t-):=\liminf_{r\to 0^+}\frac{1}{r}\int_{t-r}^t\left(\int_{\mathbb{T}^3}\left\langle\mathcal{V}^{\omega}_{s,x};\frac{|\widetilde{\mathbf{u}}|^2}{2}\right\rangle\mathrm{d}x+\widetilde{\mathcal{H}}(s)\right)\mathrm{d}s
$$

$$
\widetilde{\mathcal{E}}(t+):=\liminf_{r\to 0^+}\frac{1}{r}\int_t^{t+r}\left(\int_{\mathbb{T}^3}\left\langle\mathcal{V}^{\omega}_{s,x};\frac{|\widetilde{\mathbf{u}}|^2}{2}\right\rangle\mathrm{d}x+\widetilde{\mathcal{H}}(s)\right)\mathrm{d}s
$$

and initial energy

$$
\mathcal{E}(0-)=\int_{\mathbb{T}^3}\frac{1}{2}|\mathbf{u}_0|^2\,\mathrm{d}x.
$$

6 Proof of Theorem 3.7: Convergence to a dissipative solution

With the help of the Proposition 5.3, and convergence results given by (5.5), we conclude that there exists a subsequence $\widetilde{\mathbf{u}}_{n_k}$ such that $\widetilde{\mathbb{P}}$-a.s,

$$
\widetilde{\mathbf{u}}_{n_k}\to\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\widetilde{\mathbf{u}}\rangle\ \text{in}\ C_w([0,T],L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)).
$$

For the pointwise converegnce of approximations, we can make use of [11, Proposition 2.4]. Indeed, we obtain $\widetilde{\mathbb{P}}$-a.s., there exists a subsequece $\widetilde{\mathbf{u}}_{n_k}$ such that

$$
\frac{1}{N}\sum_{k=1}^N\widetilde{\mathbf{u}}_{n_k}\to\langle\widetilde{\mathcal{V}}^{\omega}_{t,x};\widetilde{\mathbf{u}}\rangle,\ \text{as}\ N\to\infty\ \text{a.e. in}\ (0,T)\times\mathbb{T}^3.
$$

7 Weak-strong uniqueness

Weak Itô formula: Here we give some outlines of proof of weak strong uniqueness. In this section, we prove Theorem 3.8 through some auxiliary results. We start with the following lemma, which is a variant of [5, Lemma 2.4]. We give the proof for the convenience of the reader.

Lemma 7.1 (Weak Itô formula). *Let $\mathbf{V}$ be a stochastic process on $\big(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\big)$ such that*

$$\mathbf{V} \in C_w([0,T]; L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)) \cap L^\infty((0,T); L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)) \qquad \mathbb{P} - a.s.$$

$$\mathbb{E}\big[\sup_{t\in[0,T]} \|\mathbf{V}\|^2_{L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)}\big] < \infty,$$

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{V}(t)\cdot\boldsymbol{\varphi}\,dx = \int_{\mathbb{T}^3} \mathbf{V}(0)\cdot\boldsymbol{\varphi}\,dx + \int_0^t\int_{\mathbb{T}^3} \mathbf{V}_1 : \nabla\boldsymbol{\varphi} dx ds + \int_0^t\int_{\mathbb{T}^3} \nabla\boldsymbol{\varphi} :\, d\lambda(x,s)\,ds \\ + \int_0^t \langle \boldsymbol{\varphi}, \sigma(\mathbf{V})\rangle dW(s)\end{aligned} \tag{7.1}$$

for all $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$, for all $t\in[0,T]$, $\mathbb{P}$-a.s. Here $\mathbf{V}_1$, λ satisfy with

$$\mathbf{V}_1 \in L^2(\Omega; L^1(0,T; L^1(\mathbb{T}^3))), \qquad \lambda \in L^1(\Omega; L^\infty_{w*}(0,T;\mathcal{M}_b(\mathbb{T}^3))).$$

Let $\mathbf{U}$ be a stochastic process on $\big(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P}\big)$ satisfying

$$\mathbf{U} \in C([0,T]; C^1(\mathbb{T}^3;\mathbb{R}^3)),\ \mathbb{P} - a.s. \text{ and } \ \mathbb{E}\big[\sup_{t\in[0,T]} \|\mathbf{U}\|^2_{L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)\cap C(\mathbb{T}^3;\mathbb{R}^3)}\big] < \infty,$$

$$\mathrm{d}\mathbf{U} = \mathbf{U}_1 \mathrm{d}t + \mathbf{U}_2\,\mathrm{d}W \tag{7.2}$$

Here $\mathbf{U}_1, \mathbf{U}_2$ are progressively measurable with

$$\mathbf{U}_1 \in L^2(\Omega; L^1((0,T); L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3))) \qquad \mathbf{U}_2 \in L^2(\Omega; L^2((0,T); L_2(\mathfrak{U}; L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3))))$$

.

$$\sum_k^\infty \int_0^T \|\mathcal{P}_H \mathbf{U}_2(e_k)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)} \in L^1(\Omega).$$

Then, for all $t\in[0,T]$, $\mathbb{P}$-a.s

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{V}(t)\cdot\mathbf{U}(t)\,\mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{V}(0)\cdot\mathbf{U}(0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \mathbf{V}_1 : \nabla\mathbf{U}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \nabla\mathbf{U} :\, \mathrm{d}\lambda\,\mathrm{d}s \\ + \int_0^t\int_{\mathbb{T}^3} \mathbf{U}\cdot\sigma(V)\,\mathrm{d}x\mathrm{d}W + \int_0^t\int_{\mathbb{T}^3} \mathbf{U}_1\cdot\mathbf{V}\ \mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathbf{V}\cdot\mathbf{U}_2\,\mathrm{d}W\ \mathrm{d}x \\ + \int_0^t\int_{\mathbb{T}^3} \mathcal{P}_H(\sigma(\mathbf{V}))\mathcal{P}_H(\sigma(\mathbf{U}))\,\mathrm{d}x\mathrm{d}s\end{aligned} \tag{7.3}$$

Proof. Let $\boldsymbol{\varphi} \in L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$, then $\boldsymbol{\varphi}_r = \boldsymbol{\varphi} * \boldsymbol{\rho}_r \in C^\infty_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)$, we have $\mathbb{P}$-a.s., for all $t\in[0,T]$

$$\begin{aligned}\int_{\mathbb{T}^3} \mathbf{V}(t)\cdot\boldsymbol{\varphi}_r\,dx = \int_{\mathbb{T}^3} \mathbf{V}(0)\cdot\boldsymbol{\varphi}_r\,dx + \int_0^t\int_{\mathbb{T}^3} \mathbf{V}_1 : \nabla\boldsymbol{\varphi}_r dx ds + \int_0^t\int_{\mathbb{T}^3} \nabla\boldsymbol{\varphi}_r :\, d\lambda(x,s)\,ds \\ + \int_{\mathbb{T}^3} \boldsymbol{\varphi}_r \cdot \int_0^t \sigma(\mathbf{V})\mathrm{d}W(s)\,\mathrm{d}x\end{aligned}$$

After shiftting molification on other variable, we have $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} \mathbf{V}_r(t)\cdot\boldsymbol{\varphi}\,dx = \int_{\mathbb{T}^3} \mathbf{V}_r(0)\cdot\boldsymbol{\varphi}\,dx + \int_0^t\int_{\mathbb{T}^3} (\mathbf{V}_1)_r : \nabla\boldsymbol{\varphi} dx ds + \int_0^t\int_{\mathbb{T}^3} \nabla\boldsymbol{\varphi} : d\lambda_r(x,s)\,ds$$
$$+ \int_{\mathbb{T}^3} \boldsymbol{\varphi}\cdot\int_0^t \sigma(V)_r \mathrm{d}W(s)\ \mathrm{d}x,$$

It implies that $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$\mathbf{V}_r(t) = \mathbf{V}_r(0) - \int_0^t \mathcal{P}_H\big(\mathrm{div}(\mathbf{V}_1)_r\big)\mathrm{d}s - \int_0^t \mathcal{P}_H\big(\mathrm{div}\lambda_r\big)ds + \int_0^t \mathcal{P}_H(\sigma(V)_r)\mathrm{d}W(s) \tag{7.4}$$

Let $(e_i)_{i\geq 1}$ is countable orthonormal basis of $L^2(\mathbb{T}^3;\mathbb{R}^3)$, and from equations (7.2)-(7.4), we have $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$\langle \mathbf{U}(t), e_i\rangle = \langle \mathbf{U}(0), e_i\rangle + \int_0^T \langle \mathbf{U}_1(s), e_i\rangle \mathrm{d}s + \int_0^t \langle \mathcal{P}_H(\sigma(\mathbf{U})), e_i\rangle \mathrm{d}W(s)$$

and

$$\langle \mathbf{V}_r(t), e_i\rangle = \langle \mathbf{V}_r(0), e_i\rangle - \int_0^t \langle \mathcal{P}_H\big(\mathrm{div}(\mathbf{V}_1)_r\big), e_i\rangle \mathrm{d}s - \int_0^t \langle \mathcal{P}_H\big(\mathrm{div}\lambda_r\big), e_i\rangle ds$$
$$+ \int_0^t \langle \mathcal{P}_H(\sigma(V))_r, e_i\rangle \mathrm{d}W(s)$$

Now, we apply Itô product rule to $t \mapsto \langle \mathbf{U}(t), e_i\rangle\cdot\langle \mathbf{V}_r(t), e_i\rangle$, we have $\mathbb{P}$-a.s, for all $t \in [0,T]$

$$\langle \mathbf{U}(t), e_i\rangle\cdot\langle \mathbf{V}_r(t), e_i\rangle = \langle \mathbf{U}(0), e_i\rangle\cdot\langle \mathbf{V}_r(0), e_i\rangle + \int_0^t \langle \mathcal{P}_H\big(\mathrm{div}(\mathbf{V}_1)_r\big), e_i\rangle\cdot\langle \mathbf{U}(s), e_i\rangle \mathrm{d}s$$
$$+ \int_0^t \langle \mathcal{P}_H\big(\mathrm{div}\lambda_r\big), e_i\rangle\cdot\langle \mathbf{U}(s), e_i\rangle \mathrm{d}s + \int_0^t \langle \mathcal{P}_H(\sigma(V))_r, e_i\rangle\cdot\langle \mathbf{U}(s), e_i\rangle \mathrm{d}W(s)$$
$$+ \int_0^t \langle \mathbf{V}_r(t), e_i\rangle\cdot\langle \mathbf{U}_1(s), e_i\rangle \mathrm{d}s + \int_0^t \langle \mathbf{V}_r(t), e_i\rangle\cdot\langle \mathcal{P}_H(\sigma(\mathbf{U})), e_i\rangle \mathrm{d}W(s)$$
$$+ \int_0^t \langle \mathcal{P}_H(\sigma(\mathbf{U})), e_i\rangle\cdot\langle \mathcal{P}_H(\sigma(\mathbf{V}))_r, e_i\rangle \mathrm{d}s$$

We use the fact that $\int_{\mathbb{T}^3} \mathbf{U}\cdot\mathbf{V}\,\mathrm{d}x = \sum_{i\geq 1}\langle \mathbf{U}, e_i\rangle\cdot\langle \mathbf{V}, e_i\rangle$, then we have, $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$\int_{\mathbb{T}^3} \mathbf{U}(t)\cdot\mathbf{V}_r(t)\,\mathrm{d}x = \int_{\mathbb{T}^3} \mathbf{U}(0)\cdot\mathbf{V}_r(0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3} \mathcal{P}_H\big(\mathrm{div}(\mathbf{V}_1)\big)_r\cdot\mathbf{U}(s)\rangle\,\mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t\int_{\mathbb{T}^3} \mathcal{P}_H\big(\mathrm{div}\lambda_r\big)\cdot\mathbf{U}(s)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathcal{P}_H(\sigma(V))_r\cdot\mathbf{U}(s)\,\mathrm{d}x\mathrm{d}W(s)$$
$$+ \int_0^t\int_{\mathbb{T}^3} \mathbf{V}_r(t)\cdot\mathbf{U}_1(s)\,\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbb{T}^3} \mathbf{V}_r(t)\cdot\mathcal{P}_H(\sigma(\mathbf{U}))\,\mathrm{d}x\mathrm{d}W(s)$$
$$+ \int_0^t\int_{\mathbb{T}^3} \mathcal{P}_H(\sigma(\mathbf{U}))\cdot\mathcal{P}_H(\sigma(\mathbf{V}))_r\,\mathrm{d}x\mathrm{d}s.$$

Now we are able to perform the limit $r \to 0$ in the above relation by using the hypotheses of Lemma 7.1, completing the proof. □

7.1 Relative energy inequality (the Euler system)

Relative energy functional: We proceed further and introduce the *relative energy (entropy)* functional. The commonly used form of the *relative energy* functional in the context of measure-valued solutions to the incompressible Euler system reads, $\mathbb{P}$-a.s, for all $t \in [0,T]$

$$\mathfrak{E}_{\rm mv}\left(\mathbf{u}\,\Big|\mathbf{U}\right)(t) := \liminf_{r\to 0}\frac{1}{r}\int_t^{t+r}\left[\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{s,x};\frac{1}{2}|\mathbf{u}|^2\right\rangle \mathrm{d}x + \mathcal{H}(s)\right]\mathrm{d}s - \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathbf{U}(t)\,\mathrm{d}x$$
$$+\frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{U}(t)|^2\,\mathrm{d}x.$$

We note that by Lebesgue differentiation theorem and energy inequality (3.4), we have $\mathbb{P}$-a.s, almost every $t \in [0,T]$

$$\mathfrak{E}_{\rm mv}\left(\mathbf{u}\,\Big|\mathbf{U}\right)(t) := \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\frac{1}{2}|\mathbf{u}|^2\right\rangle \mathrm{d}x - \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathbf{U}(t)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}|\mathbf{U}(t)|^2\,\mathrm{d}x + \mathcal{H}(t).$$

Relative energy inequality:

Proposition 7.2 (Relative Energy)**.** *Let* $\left[(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\ge 0},\mathbb{P});\mathcal{V}^\omega_{t,x},W\right]$ *be a dissipative measure-valued martingale solution to the system* (1.1)*. Suppose* $\mathbf{U}$ *be stochastic processes which is adapted to the filtration* $(\mathfrak{F}_t)_{t\ge0}$ *and satisfies*

$$\mathrm{d}\mathbf{U} = \mathbf{U}_1\,\mathrm{d}t + \mathcal{P}_H\mathbf{U}_2\,\mathrm{d}W,$$

with

$$\mathbf{U}\in C([0,T];C^1_{\rm div}(\mathbb{T}^3;\mathbb{R}^3)),\qquad \mathbb{P}\text{-}a.s.,\qquad \mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{U}\|^2_{L^2_{\rm div}(\mathbb{T}^3;\mathbb{R}^3)}\right]<\infty, \tag{7.5}$$

Moreover, $\mathbf{U}$ *satisfies*

$$\mathbf{U}_1\in L^2(\Omega;L^2(0,T;L^2(\mathbb{T}^3;\mathbb{R}^3))),\quad \mathbf{U}_2\in L^2(\Omega;L^2(0,T;L_2(\mathfrak{U};L^2(\mathbb{T}^3;\mathbb{R}^3)))), \tag{7.6}$$
$$\int_0^T\sum_{k\ge1}\|\mathcal{P}_H\mathbf{U}_2(e_k)\|^2_{L^2(\mathbb{T}^3;\mathbb{R}^3)}\in L^1(\Omega).$$

Then the following relative energy inequality *holds:*

$$\mathcal{E}_{\rm mv}\left(\boldsymbol{u}\,\Big|\mathbf{U}\right)(t)\le \mathcal{E}_{\rm mv}\left(\boldsymbol{u}\,\Big|\mathbf{U}\right)(0) + M_{RE}(t) + \int_0^t \mathcal{R}_{\rm mv}\big(\boldsymbol{u}\,|\,\mathbf{U}\big)(s)\,\mathrm{d}s \tag{7.7}$$

$\mathbb{P}$*-a.s., where*

$$\mathcal{R}_{\rm mv}\big(\mathbf{u}\,|\,\mathbf{U}\big) = \int_{\mathbb{T}^3}\left\langle\mathcal{V}^\omega_{t,x};\mathbf{u}\otimes\mathbf{u}\right\rangle : \nabla_x\boldsymbol{U}\,\mathrm{d}x + \int_{\mathbb{T}^3}\left\langle\mathcal{V}^\omega_{t,x};\mathbf{u}\right\rangle\cdot\mathbf{U}_1 dxdt - \int_{\mathbb{T}^3}\nabla_x\mathbf{U}:d\lambda_{\mathcal{C}} + \frac{1}{2}\int_{\mathbb{T}^3}d\lambda_{\mathcal{D}}$$
$$+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{T}^3}\left\langle\mathcal{V}^\omega_{t,x};\left|\sigma_k(\mathbf{u})-\mathbf{U}_2(e_k)\right|^2\right\rangle\mathrm{d}x \tag{7.8}$$

Here M_{RE} *is a real valued square integrable martingale, and the norm of this martingale depends only on the norms of* $\mathbf{U}$ *in the aforementioned spaces.*

Proof. Here we shall complete proof of mention result in following several steps.

Step 1: In order to compute $\mathrm{d}\int_{\mathbb{T}^3} \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{u} \rangle \cdot \mathbf{U} \mathrm{d}x$, we first recall that $\mathbf{V} = \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{u} \rangle$ satisfies the hypotheses of Lemma 7.1. Therefore we can apply the Lemma 7.1 to conclude that $\mathbb{P}$-almost surely, for all $\tau \in [0, T]$

$$
\begin{aligned}
\int_{\mathbb{T}^3} \langle \mathcal{V}^{\omega}_{\tau,x}; \mathbf{u} \rangle \cdot \mathbf{U}(\tau) \,\mathrm{d}x \,\mathrm{d}t &= \int_{\mathbb{T}^3} \langle \mathcal{V}^{\omega}_{0,x}; \mathbf{u} \rangle \cdot \mathbf{U}(0) \,\mathrm{d}x \,\mathrm{d}t + \int_0^{\tau} \int_{\mathbb{T}^3} \nabla_x \mathbf{U} : d\mu_C \, dt \\
&+ \int_0^{\tau} \int_{\mathbb{T}^3} \left[\langle \mathcal{V}^{\omega}_{t,x}; \mathbf{u} \rangle \cdot \mathbf{U}_1(t) + \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{u} \otimes \mathbf{u} \rangle : \nabla_x \mathbf{u} \right] \mathrm{d}x \mathrm{d}t \\
&+ \sum_{k \geq 1} \int_0^{\tau} \int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{U}_2(e_k) \cdot \mathcal{P}_H \langle \mathcal{V}^{\omega}_{t,x}; \sigma_k(\mathbf{u}) \rangle \,\mathrm{d}x \,\mathrm{d}t + \mathrm{d}\mathcal{M}_1,
\end{aligned}
\tag{7.9}
$$

where the square integrable martingale $\mathcal{M}_1(t)$ is given by

$$
\mathcal{M}_1(t) = \int_{\mathbb{T}^3} \int_0^t \mathbf{U} \cdot \langle \mathcal{V}^{\omega}_{t,x}; \sigma_k(\mathbf{u}) \rangle \,\mathrm{d}W \, dx + \int_0^t \int_{\mathbb{T}^3} \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{u} \rangle \cdot \mathcal{P}_H \mathbf{U}_2 \,\mathrm{d}W
$$

Step 2: Next, we see that $\mathbb{P}$-a.s., for all $\tau \in [0, T]$

$$
\int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{U}(\tau)|^2 \,\mathrm{d}x = \int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{U}(0)|^2 \,\mathrm{d}x + \frac{1}{2} \int_0^{\tau} \sum_{k \geq 1} \int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{U}_2(e_k)|^2 \,\mathrm{d}x \ \mathrm{d}t + \mathrm{d}\mathcal{M}_2,
\tag{7.10}
$$

where

$$
\mathcal{M}_2(t) = \int_0^t \int_{\mathbb{T}^3} \mathbf{U} \cdot \mathcal{P}_H \mathbf{U}_2 \,\mathrm{d}W.
$$

Step 3: We have from energy inequality, $\mathbb{P}$-a.s., for all $\tau \in [0, T]$,

$$
\begin{aligned}
\mathfrak{E}(\tau+) \leq \mathfrak{E}(0) &+ \frac{1}{2} \sum_{k \geq 1} \int_0^{\tau} \int_{\mathbb{T}^3} \langle \mathcal{V}^{\omega}_{s,x}; |\sigma_k(\mathbf{u})|^2 \rangle \,\mathrm{d}x\mathrm{d}s - \frac{1}{2} \sum_{k \geq 1} \int_0^{\tau} \int_{\mathbb{T}^3} \Big(\mathcal{Q}_H \langle \mathcal{V}^{\omega}_{s,x}; |\sigma_k(\mathbf{u})| \rangle \Big)^2 \mathrm{d}x\mathrm{d}s \\
&+ \frac{1}{2} \int_0^{\tau} \int_{\mathbb{T}^3} d\mu_D + \int_0^{\tau} d\mathcal{M}_E^2.
\end{aligned}
\tag{7.11}
$$

We manipulate the product term in the equality (7.9) using properties of projections $\mathcal{P}_H$ and $\mathcal{Q}_H$. Indeed, note that

$$
\begin{aligned}
&\int_{\mathbb{T}^3} \mathcal{P}_H \mathbf{U}_2(e_k) \cdot \mathcal{P}_H \langle \mathcal{V}^{\omega}_{t,x}; \sigma_k(\mathbf{u}) \rangle \\
&\qquad = \int_{\mathbb{T}^3} \mathbf{U}_2(e_k) \cdot \langle \mathcal{V}^{\omega}_{t,x}; \sigma_k(\mathbf{u}) \rangle - \int_{\mathbb{T}^3} \mathcal{Q}_H \mathbf{U}_2(e_k) \cdot \mathcal{Q}_H \langle \mathcal{V}^{\omega}_{t,x}; \sigma_k(\mathbf{u}) \rangle,
\end{aligned}
$$

and

$$
\int_{\mathbb{T}^3} |\mathcal{P}_H \mathbf{U}_2(e_k)|^2 = \int_{\mathbb{T}^3} |\mathbf{U}_2(e_k)|^2 - \int_{\mathbb{T}^3} |\mathcal{Q}_H \mathbf{U}_2(e_k)|^2.
$$

These properties of projections imply that

$$\begin{aligned}
&\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}|\mathcal{P}_H\mathbf{U}_2(e_k)|^2-\sum_{k\geq 1}\int_{\mathbb{T}^3}\mathcal{P}_H\mathbf{U}_2(e_k)\cdot\mathcal{P}_H\big\langle\mathcal{V}^{\omega}_{t,x};\sigma_k(\mathbf{u})\big\rangle\\
&\qquad+\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\big\langle\mathcal{V}^{\omega}_{s,x};|\sigma_k(\mathbf{u})|^2\big\rangle\,\mathrm{d}x\,\mathrm{d}\tau-\frac{1}{2}\sum_{k\geq 1}\int_s^t\int_{\mathbb{T}^3}\Big(\mathcal{Q}_H\big\langle\mathcal{V}^{\omega}_{s,x};\sigma_k(\mathbf{u})\big\rangle\Big)^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&=\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\Big\langle\mathcal{V}^{\omega}_{t,x};\big|\sigma_k(\mathbf{u})-\mathbf{U}_2(e_k)\big|^2\Big\rangle\,\mathrm{d}x-\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\Big|\mathcal{Q}_H\mathbf{U}_2(e_k)-\mathcal{Q}_H\big\langle\mathcal{V}^{\omega}_{t,x};\sigma_k(\mathbf{u})\big\rangle\Big|^2\,\mathrm{d}x\\
&\leq\frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\Big\langle\mathcal{V}^{\omega}_{t,x};\big|\sigma_k(\mathbf{u})-\mathbf{U}_2(e_k)\big|^2\Big\rangle\,\mathrm{d}x.
\end{aligned}\tag{7.12}$$

Finally, in view of the above observations given by (7.9)-(7.12), we can now add the resulting expressions to establish (7.7). Note that the square integrable martingale $\mathcal{M}_{RE}(t)$ is given by $\mathcal{M}_{RE}(t):=\mathcal{M}_1(t)+\mathcal{M}_2(t)+\mathcal{M}_E^2(t)$. □

7.2 Proof of weak-strong principle 3.8

Since $\bar{\mathbf{u}}$ is the strong pathwise solution to stystem (1.1), so taking $\mathbf{U}=\bar{\mathbf{u}}$ in the relative energy inequality (7.7). Then we get $\mathbb{P}$-a.s., for all $t\in[0,T]$,

$$\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(t\wedge\mathfrak{t}_R)\leq\ \mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(0)+\mathcal{M}_{RE}(t\wedge\mathfrak{t}_R)+\int_0^{t\wedge\mathfrak{t}_R}\mathfrak{R}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(\tau)\mathrm{d}\tau,\tag{7.13}$$

where $\mathfrak{R}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)$ is given by

$$\begin{aligned}
\mathfrak{R}_{\mathrm{mv}}\left(\mathbf{u}|\bar{\mathbf{u}}\right)&=\int_{\mathbb{T}^3}\big\langle\mathcal{V}^{\omega}_{t,x};|(\mathbf{u}-\bar{\mathbf{u}})\otimes(\bar{\mathbf{u}}-\mathbf{u})|\big\rangle\,|\nabla_x\bar{\mathbf{u}}|\,dx+\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{T}^3}\big\langle\mathcal{V}^{\omega}_{t,x};\big|\sigma_k(\mathbf{u})-\sigma_k(\bar{\mathbf{u}})\big|^2\big\rangle\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{T}^3}|\nabla_x\bar{\mathbf{u}}|\cdot d|\lambda_{\mathcal{C}}|+\int_{\mathbb{T}^3}d|\lambda_{\mathcal{D}}|.
\end{aligned}$$

Now we use the following facts

$$\|\bar{\mathbf{u}}\|_{W^{1,\infty}(\mathbb{T}^3)}\leq\ c(R)\ \text{for all}\ t\ \leq\ \tau_R\ \&\ \ |(\mathbf{u}-\bar{\mathbf{u}})\otimes(\bar{\mathbf{u}}-\mathbf{u})|\leq|\mathbf{u}-\bar{\mathbf{u}}|^2,$$

$$\sum_{k\geq 1}\big|\sigma_k(\mathbf{u})-\sigma_k(\bar{\mathbf{u}})\big|^2\leq D_1|\mathbf{u}-\bar{\mathbf{u}}|^2,$$

to conclude that

$$\frac{1}{2}\sum_{k\in\mathbb{N}}\int_{\mathbb{T}^3}\big\langle\mathcal{V}^{\omega}_{t,x};\big|\sigma_k(\mathbf{u})-\sigma_k(\bar{\mathbf{u}})\big|^2\big\rangle\,\mathrm{d}x\leq c(L)\,\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,|\,\bar{\mathbf{u}}\big),$$

and

$$\int_0^{t\wedge\mathfrak{t}_R}\mathfrak{R}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(s)\,\mathrm{d}s\leq\ c(R)\int_0^{t\wedge\mathfrak{t}_R}\big(\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(s)\,\mathrm{d}s.\tag{7.14}$$

In view of (7.13) and (7.14), a simple consequence of Gronwall's lemma gives, for all $t\in[0,T]$

$$\mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(t\wedge\mathfrak{t}_R)\big]\leq c(R)\,\mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(0)\big].$$

Since, initial data are same for both solutions, right hand side of above inequality equals to zero. Therefore it implies that for all $t \in [0,T]$

$$\mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\bar{\mathbf{u}}\big)(t\wedge \mathfrak{t}_R)\big] = 0.$$

This also implies that

$$\int_0^T \mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\bar{\mathbf{u}}\big)(s\wedge \mathfrak{t}_R)\big]\,\mathrm{d}s = 0.$$

In view of a priori estimates, a usual Lebesgue point argument, and application of Fubini's theorem reveals that $\mathbb{P}$-a.s.,

$$\int_0^T \mathfrak{E}_{\mathrm{mv}}\big(\mathbf{u}\,\big|\,\bar{\mathbf{u}}\big)(t\wedge \mathfrak{t}_R)\,\mathrm{d}t = 0.$$

Since, the defect measure $\mathcal{H} \geq 0$, we have $\mathbb{P}$-a.s., for a.e. $t \in [0,T]$, $\mathcal{H}(t\wedge \mathfrak{t}_R) = 0$, Moreover, $\mathbb{P}-$ a.s.

$$\mathcal{V}^{\omega}_{t\wedge \mathfrak{t}_R, x} = \delta_{\bar{\mathbf{u}}(t\wedge \mathfrak{t}_R, x)}, \text{ for a.e. } (t,x) \in (0,T)\times \mathbb{T}^3.$$

This proves our claim.

8 Proof of Theorem 3.9: Convergence to regular solution

We have proven that the approximate solutions $\widetilde{\mathbf{u}}_n$ to (4.4) for the stochastic incompressible Euler system converges to a *dissipative measure-valued martingale* solution, in the sense of Definition 3.1. Using the corresponding weak (measure-valued)–strong uniqueness results (cf. Theorem 3.8), we can prove the strong convergence of approximate solutions to the strong solution of the system on its lifespan.

First observe that, Proposition 5.3 and Theorem 3.8 give the required weak-$*$ convergence. Indeed, from Proposition 5.3, we have $\widetilde{\mathbb{P}}$-a.s.,

$$\widetilde{\mathbf{u}}_n(\cdot\wedge \mathfrak{t}_R) \to \langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\mathbf{u}}\rangle(\cdot\wedge \mathfrak{t}_R) \text{ in } C_w([0,T], L^2_{\mathrm{div}}(\mathbb{T}^3;\mathbb{R}^3)),$$

Information of above convergence and Theorem 3.8 give the required weak-* convergence. In the proof of strong convergence of $\widetilde{\mathbf{u}}_n$ in $L^1(\mathbb{T}^3)$, we use the results of Proposition 5.3 & Theorem 3.8, energy bounds (4.11) and the fact limit Young measure of any subsequence $\delta_{\widetilde{\mathbf{u}}_{n_k}(\cdot\wedge \mathfrak{t}_R)}$ is $\delta_{\bar{\mathbf{u}}(\cdot\wedge \mathfrak{t}_R)}$ to conclude that $\widetilde{\mathbb{P}}$-a.s., sequence of young measure converges to Dirac Young measure, i.e. $\widetilde{\mathbb{P}}$-a.s.

$$\delta_{\widetilde{\mathbf{u}}_n(\cdot\wedge \mathfrak{t}_R)} \to \delta_{\bar{\mathbf{u}}(\cdot\wedge \mathfrak{t}_R)}, \text{ weak-}* \text{ in } L^{\infty}((0,T)\times \mathbb{T}^3; \mathcal{P}(\mathbb{R}^3))$$

By theory of Young measure [1, Proposition 4.16], it implies that, $\widetilde{\mathbb{P}}$-a.s. $\widetilde{\mathbf{u}}_n(\cdot\wedge \mathfrak{t}_R)$ converges to $\bar{\mathbf{u}}(\cdot\wedge \mathfrak{t}_R)$ in measure respectively. Note that, $\widetilde{\mathbb{P}}$-a.s. sequence $\widetilde{\mathbf{u}}_n(\cdot\wedge \mathfrak{t}_R)$ is uniformly integrable and converges in measure, therefore Vitali's convergence theorem implies that $\widetilde{\mathbb{P}}$-a.s,

$$\widetilde{\mathbf{u}}_n(\cdot\wedge \mathfrak{t}_R) \to \bar{\mathbf{u}}(\cdot\wedge \mathfrak{t}_R) \text{ strongly in } L^1((0,T)\times \mathbb{T}^3;\mathbb{R}^3),$$

This finishes the proof of the Theorem 3.9.

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