Mathematical Modelling and Numerical Analysis Will be set by the publisher
Modélisation Mathématique et Analyse Numérique

A COUNTEREXAMPLE TO ANALYTICITY IN FRICTIONAL DYNAMICS

Christopher R. Dance[1]

Abstract. We consider the motion of a particle acted on by dry friction and a force that is an analytic function of time. We give a counterexample to the claim that such motions are given by analytic functions of time. Several published arguments concerning existence and uniqueness in unilateral dynamics with friction rely on the analyticity of such motions. The counterexample invalidates those arguments for motions in three or more dimensions.

2020 Mathematics Subject Classification. 70F40, 49J52, 34A60.

.

1. Introduction

Contact and impact are typically modelled with *unilateral constraints*: inequalities involving the coordinates of some bodies, which are satisfied when those bodies do not interpenetrate. Such constraints are essential components of models of robots [7, 10] and other systems [6], and it is natural to enquire about the existence and uniqueness of solutions to initial value problems for such models [3]. In the 1960s, it was discovered that the motion of a unilaterally constrained particle acted on by an external force is in general non-unique, even if the force is an infinitely differentiable function of time [1, 4, 14]. However, in the 1990s, it was shown that such motions are unique if there is no friction and the force is an analytic function [1, 13]. (A function f is *analytic* if for every point x_0 of its domain, the Taylor series of f at x_0 converges to $f(x)$ for all points x in some neighbourhood of x_0.)

Several authors have explored how such results might be extended to models involving friction. Ballard and Basseville [2] presented arguments for the existence and uniqueness of solutions to initial value problems for a unilaterally constrained particle acted on by an analytic force and dry friction. Charles and Ballard [5] extended those arguments to finite collections of particles. A key step of those arguments is to derive a local solution given by a power series, and to claim that it corresponds to an analytic function.

In this paper, we present a simple counterexample to that claim. The counterexample invalidates the existence and uniqueness arguments presented in [2, 5] for unilaterally constrained particles in dimension $d > 2$, although those arguments are correct for $d = 2$ to the best of our knowledge. Consequently for $d > 2$, the only general existence result about unilateral dynamics with friction is that of Monteiro Marques [11], which only addresses situations with a single constraint and perfectly inelastic impacts (zero restitution coefficient); and the question of finding sufficient conditions for uniqueness is largely open.

Keywords and phrases: Unilateral dynamics with friction, frictional dynamical contact problems, uniqueness.

[1] NAVER LABS Europe, 6 chemin de Maupertuis, 38240 Meylan, France

This provisional PDF is the accepted version. The article should be cited as: ESAIM: M2AN, doi: 10.1051/m2an/2022033

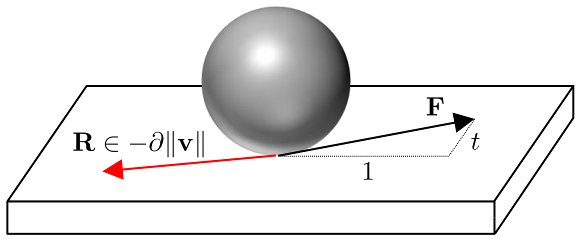

FIGURE 1. The tangential force and reaction acting on the particle in Counterexample 1.

1.1. **Counterexample**

Consider a particle in contact with a flat surface, acted on by a force and friction. The particle's position is an element of $\mathbb{R}^d$, but a normal force holds it in contact with the surface, so we describe its motion in $\mathbb{R}^{d-1}$ using an orthogonal coordinate system tangent to that surface. The particle's tangential velocity $\mathbf{v} : [0, T] \to \mathbb{R}^{d-1}$ satisfies Newton's second law:

$$m\dot{\mathbf{v}} = \mathbf{F} + \mathbf{R},$$

where m is the particle's mass, the dot denotes the time derivative, $\mathbf{F} : [0, T] \to \mathbb{R}^{d-1}$ is the tangential force, and $\mathbf{R} : [0, T] \to \mathbb{R}^{d-1}$ is the tangential reaction. This reaction satisfies Coulomb's law of friction:

$$\mathbf{R}(t)\|\mathbf{v}(t)\| = -\mu N(t)\mathbf{v}(t) \qquad \text{and} \qquad \|\mathbf{R}(t)\| \leq \mu N(t),$$

where $\mu > 0$ is the friction coefficient, and $N(t)$ is the magnitude of the force normal to the surface. As discussed in [3, 12], we may write this compactly as

$$\mathbf{R}(t) \in -\mu N(t)\, \partial\|\mathbf{v}(t)\|,$$

where ∂f denotes the subdifferential of a convex function f. For simplicity, we assume the particle has unit mass and that $\mu N(t) = 1$. The velocity then satisfies the differential inclusion

$$\dot{\mathbf{v}} \in \mathbf{F} - \partial\|\mathbf{v}\|. \tag{1}$$

(This is equivalent to requiring that the differential equation $\dot{\mathbf{v}}(t) = \mathbf{F}(t) - (\mathbf{v}(t)/\|\mathbf{v}(t)\|)$ holds at times when $\mathbf{v}(t) \neq 0$, and that the constraint $\|\dot{\mathbf{v}}(t) - \mathbf{F}(t)\| \leq 1$ holds at times when $\mathbf{v}(t) = 0$.)

The following counterexample shows that even if the force $\mathbf{F}$ is an analytic function of time and the final time $T > 0$ is tiny, differential inclusion (1) need not have an analytic solution $\mathbf{v}$. The force appearing in this counterexample is illustrated in Figure 1. The magnitude of this force initially equals the friction limit, but a component of the force orthogonal to its initial direction increases, causing the particle to slip.

Counterexample 1. *Suppose the dimension is $d > 2$, the final time is $T > 0$, and the velocity $\mathbf{v} : [0, T] \to \mathbb{R}^{d-1}$ is an absolutely continuous function with $\mathbf{v}(0) = 0$ that satisfies differential inclusion (1) for the force*

$$\mathbf{F}(t) = (1,\, t,\, 0,\, \ldots,\, 0)^\top \in \mathbb{R}^{d-1},$$

for almost all $t \in [0, T]$. Then $\mathbf{v}$ is not an analytic function.

To prove the velocity $\mathbf{v}(t)$ is not analytic, we derive its Taylor series at $t = 0$ and show that this series diverges for any $t > 0$.

Remark 1.1. It is not uncommon for differential equations to have divergent series solutions [9]. However, the initial condition $\mathbf{v}(0) = 0$ is necessary for a series solution of differential inclusion (1) to diverge for all $t > 0$, when $\mathbf{F}$ is analytic. Indeed, if $\mathbf{v}(0) \neq 0$ then the tangential reaction is $\mathbf{R}(t) = -\mathbf{v}(t)/\|\mathbf{v}(t)\|$ on a neighbourhood of $t = 0$, and this is an analytic function of $\mathbf{v}(t)$ on a neighbourhood of $\mathbf{v}(0)$, so the Cauchy-Kovalevskaya theorem [8] guarantees the existence of a unique analytic solution $\mathbf{v}(t)$ on a neighbourhood of $t = 0$.

Remark 1.2. When working with unilateral constraints, the velocity must in general be formulated as a discontinuous function, so as to allow for impacts [2,5]. Such formulations simplify to ours in the special case that a normal force holds the particle in contact with a surface. In this special case, the unilateral constraint is equivalent to a *bilateral constraint*: an equality constraint on the particle's coordinates. The existence and uniqueness of a solution to Counterexample 1 therefore follow from general results about bilaterally constrained problems with friction — specifically, [2, Proposition 3.3] or [5, Proposition 3.1].

1.2. **Outline**

First, we derive an algorithm to compute the Taylor series of $\mathbf{v}$ (Section 2). We provide numerical evidence that this series diverges (Figure 2) and explain the divergence intuitively. Then we discuss the specific results that our counterexample invalidates, and pinpoint the error in the arguments leading to those results (Section 3). Appendix A gives a proof of divergence, and the supplementary material contains a Python implementation of our algorithm and Maxima code to verify our algebra.

2. Computing the Taylor Series

In this section, we derive an algorithm for computing the Taylor series of the velocity in Counterexample 1, and provide evidence and an intuitive argument for the divergence of that series. We restrict attention to dimension $d = 3$. (We may obtain the velocity for $d > 3$ from that for $d = 3$, simply by setting the $d - 3$ additional components to zero.) We write the Taylor series at $t = 0$ of the velocity $\mathbf{v}$ in the form

$$\mathbf{v}(t) = \begin{pmatrix} t^p u(t) \\ t^q v(t) \end{pmatrix} \qquad \text{where} \qquad u(t) := \sum_{n=0}^{\infty} u_n t^n, \qquad v(t) := \sum_{n=0}^{\infty} v_n t^n \tag{2}$$

for some real coefficients u_n, v_n with $u_0, v_0 \neq 0$, and some non-negative integers p, q.

2.1. **Solving for the Leading Orders**

First we show that $p = 3$ and $q = 4$. It follows from the initial condition $\mathbf{v}(0) = 0$ that $p, q \geq 1$. Also, for $\mathbf{v}(t) \neq 0$ differential inclusion (1) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^p u) = 1 - \frac{t^p u}{\sqrt{t^{2p}u^2 + t^{2q}v^2}}, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^q v) = t - \frac{t^q v}{\sqrt{t^{2p}u^2 + t^{2q}v^2}}. \tag{4}$$

Substituting the series expansions (2) and matching the coefficients of t^0 in (4) gives

$$\mathbf{1}_{q=1} v_0 = -\mathbf{1}_{p=q} \frac{v_0}{\sqrt{u_0^2 + v_0^2}} - \mathbf{1}_{p>q} \frac{v_0}{|v_0|},$$

where $\mathbf{1}_{\cdot}$ is the indicator function. If $p = q$, then this expression cannot be satisfied for $v_0 \neq 0$. Thus, we have $p \neq q$. Using this fact, matching the coefficients of t^0 in (3) gives

$$\mathbf{1}_{p=1} u_0 = 1 - \mathbf{1}_{p<q} \frac{u_0}{|u_0|}.$$

If $p > q \geq 1$, then this expression cannot be satisfied. So, we must have $p < q$, in which case this expression reads $\mathbf{1}_{p=1} u_0 = 1 - u_0/|u_0|$. As this cannot be satisfied for $p = 1$, it follows that $p > 1$ and $u_0 > 0$. Matching the coefficients of t^1 in (4), and using $1 < p < q$ (so that $q \neq 2$) gives

$$0 = 1 - \mathbf{1}_{q=p+1} \frac{v_0}{|u_0|},$$

which is only satisfied if $q = p + 1$ and $v_0 = |u_0|$. As $q = p + 1$, Taylor expanding each side of (3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(t^p u) = p t^{p-1} u_0 + O(t^p) \qquad \text{and} \qquad 1 - \frac{u}{\sqrt{u^2 + t^2 v^2}} = \frac{v_0^2}{2u_0^2} t^2 + O(t^3) \qquad \text{as } t \to 0.$$

As the terms involving t^{p-1} and t^2 have non-zero coefficients, they must be equal, and we conclude that

$$p = 3, \qquad q = 4, \qquad \text{and} \qquad u_0 = v_0 = \frac{1}{6}. \tag{5}$$

2.2. **Algorithm**

For the values of p, q in (5), functions u, v should satisfy the differential equations

$$\frac{d}{dt}(t^3 u) = 1 - \frac{u}{\sqrt{u^2 + t^2 v^2}}, \qquad \frac{d}{dt}(t^4 v) = t - \frac{tv}{\sqrt{u^2 + t^2 v^2}}.$$

We write these equations in the form

$$d + e = 1, \qquad f + g = t, \tag{6}$$

in terms of the intermediate functions

$$\begin{gathered} a = u^2 + t^2 v^2, \qquad b = \sqrt{a}, \qquad c = 1/b, \\ d = uc, \qquad e = \frac{\mathrm{d}}{\mathrm{d}t}(t^3 u), \qquad f = tvc, \qquad g = \frac{\mathrm{d}}{\mathrm{d}t}(t^4 v). \end{gathered}$$

By the Cauchy product formula, the coefficients of the formal power series $\sum_{n=0}^{\infty} a_n t^n, \ldots, \sum_{n=0}^{\infty} g_n t^n$ of these intermediate functions satisfy

$$\begin{cases} a_n = \displaystyle\sum_{p+q=n} u_p u_q + \sum_{p+q=n-2} v_p v_q, \\ b_n = \sqrt{a_0} \mathbf{1}_{n=0} + \dfrac{\mathbf{1}_{n\geq 1}}{2b_0} \left(a_n - \displaystyle\sum_{p+q=n,\, p,q>0} b_p b_q \right), \\ c_n = \dfrac{\mathbf{1}_{n=0}}{b_0} - \dfrac{\mathbf{1}_{n\geq 1}}{b_0} \displaystyle\sum_{p+q=n,\, p>0} b_p c_q, \end{cases} \qquad \begin{cases} d_n = \displaystyle\sum_{p+q=n} u_p c_q, \\ e_n = (n+1) u_{n-2}, \\ f_n = \displaystyle\sum_{p+q=n-1} v_p c_q, \\ g_n = (n+1) v_{n-3} \end{cases} \tag{7}$$

for $n \geq 0$. (In our notation, we sum over pairs of non-negative integers p, q, so that $\sum_{p+q=n,\, p>0} b_p c_q = \sum_{p=1}^{n} b_p c_{n-p} = \sum_{q=0}^{n-1} b_{n-q} c_q$, and coefficients with negative indices vanish, so that $a_0 = u_0^2$ and $e_0 = 0$.) Moreover, matching the coefficients of t^n and t^{n-1} in (6) gives

$$d_n + e_n = \mathbf{1}_{n=0}, \qquad f_{n-1} + g_{n-1} = \mathbf{1}_{n=2}. \tag{8}$$

The idea of our algorithm is to treat $u_{n-2}, u_{n-1}, u_n, v_{n-2}$ as unknowns, having already determined values for u_p, v_p for $p < n-2$. Equation (7) then provides expressions for a_p, b_p, c_p for $p = n-2, n-1, n$ and hence for $d_n, e_n, f_{n-1}, g_{n-1}$ in terms of those unknowns. Substituting these expressions in (8) gives a linear equation for u_{n-2}, v_{n-2}, which has a unique solution. (As the intermediate expressions for c_n, c_{n-1} involve u_{n-1}, u_n, it is interesting that the linear equation does not involve those variables.)

We now apply the idea of the previous paragraph, treating $n = 3, 4$ and $n \geq 5$ separately, as the formulas have "special cases" for small n. (For instance, the sum $\sum_{p+q=n} u_p u_q$ involves the term u_{n-2}^2 for $n = 4$ but the corresponding term is $2u_2 u_{n-2}$ for $n \geq 5$.) Full details of the following derivations are given in `algebra.txt` in the supplementary material. For $n = 3$ and 4, we get the linear equations

$$\begin{pmatrix} d_3 + e_3 \\ f_2 + g_2 \end{pmatrix} = \begin{pmatrix} 10 & -6 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} d_4 + e_4 \\ f_3 + g_3 \end{pmatrix} = \begin{pmatrix} 11 & -6 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} + \begin{pmatrix} 3/8 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with the unique solutions

$$u_1 = 0, \qquad v_1 = 0, \qquad u_2 = -13/120, \qquad v_2 = -49/360. \tag{9}$$

For $n \geq 5$, we get

$$\begin{pmatrix} d_n + e_n \\ f_{n-1} + g_{n-1} \end{pmatrix} = \begin{pmatrix} n+7 & -6 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} u_{n-2} \\ v_{n-2} \end{pmatrix} - \begin{pmatrix} m_n^u \\ m_n^v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{10}$$

where

$$\begin{cases} m_n^u := -R^n_{d_n} - (R^n_{c_{n-2}}/2) + R^n_{c_n} + (18/5)(R^n_{b_{n-2}} - R^n_{a_{n-2}}) + 18(R^n_{a_n} - R^n_{b_n}), \\ m_n^v := -n v_{n-4} - R^n_{f_{n-1}} + R^n_{c_{n-2}} + 18(R^n_{a_{n-2}} - R^n_{b_{n-2}}), \end{cases} \tag{11}$$

in which the terms $R^n_\cdot$ are the parts of the sums in (7) that depend neither on the unknowns $u_{n-2}, u_{n-1}, u_n, v_{n-2}$, nor on the quantities b_p, c_p for $p = n-2, n-1, n$ that involve those unknowns:

$$\begin{cases} R^n_{a_{n-2}} = \sum\limits_{\substack{p+q=n-2 \\ p,q>0}} u_p u_q + \sum\limits_{p+q=n-4} v_p v_q, \quad R^n_{b_{n-2}} = \sum\limits_{\substack{p+q=n-2 \\ p,q>0}} b_p b_q, \quad R^n_{c_{n-2}} = \sum\limits_{\substack{p+q=n-2 \\ p,q>0}} b_p c_q, \\ R^n_{a_n} = \sum\limits_{\substack{p+q=n \\ p,q>2}} u_p u_q + \sum\limits_{\substack{p+q=n-2 \\ p,q>0}} v_p v_q, \quad R^n_{b_n} = \sum\limits_{\substack{p+q=n \\ p,q>2}} b_p b_q, \quad R^n_{c_n} = \sum\limits_{\substack{p+q=n \\ p,q>2}} b_p c_q, \\ R^n_{d_n} = \sum\limits_{\substack{p+q=n \\ p,q>2}} u_p c_q, \quad R^n_{f_{n-1}} = \sum\limits_{\substack{p+q=n-2 \\ p,q>0}} v_p c_q. \end{cases} \tag{12}$$

It turns out that u_n, v_n, a_n, b_n, c_n vanish for all odd n. We show this by induction. In the base case, equation (9) gives $u_1 = v_1 = 0$, so (7) gives $a_1 = b_1 = c_1 = 0$. Otherwise, suppose $n-2$ is odd and that u_p, v_p, a_p, b_p, c_p vanish for all all odd $p < n-2$. Consider any term of any of the sums $R^n_\cdot$ in (12). This term is of the form $x_p y_q$ for appropriate sequences $x, y \in \{u, v, b, c\}$, where one of p, q is odd and less than $n-2$. It follows from the induction hypothesis that one of the factors x_p or y_q vanishes. Thus all the sums $R^n_\cdot$ vanish. It also follows from the induction hypothesis that $v_{n-4} = 0$. Thus, (11) gives $m_n^u = m_n^v = 0$, so the linear equation (10) has the unique solution $u_{n-2} = v_{n-2} = 0$. Examining the sums giving $a_{n-2}, b_{n-2}, c_{n-2}$ in (7), we see that their terms are also of the form $x_p y_q$, where one of the factors x_p or y_q vanishes. Therefore $a_{n-2}, b_{n-2}, c_{n-2}$ also vanish. This completes the induction.

In the light of the above discussion, Algorithm 1 computes the coefficients of the Taylor series of the velocity appearing in Counterexample 1.

Algorithm 1 Determining the coefficients of the Taylor series defined in (2).

function TAYLORCOEFFICIENTS($n_{\max}$) ▷ Assume $n_{\max} \geq 2$ is even
  $u_0, v_0, b_0, c_0 \leftarrow 1/6, 1/6, 1/6, 6$ ▷ By (5) and (7)
  $u_2, v_2 \leftarrow -13/120, -49/360$ ▷ By (9)
  $u_n, v_n, b_n, c_n \leftarrow 0$ for odd $n < n_{\max}$
  for $n = 6, 8, \ldots, n_{\max} + 2$ **do**
    $a_{n-4} \leftarrow \sum_{p+q=n-4} u_p u_q + \sum_{p+q=n-6} v_p v_q$ ▷ By (7)
    $b_{n-4} \leftarrow 3\big(a_{n-4} - \sum_{p+q=n-4,\, p,q>0} b_p b_q\big)$
    $c_{n-4} \leftarrow -6 \sum_{p+q=n-4,\, p>0} b_p c_q$
    Compute the sums $R^n_{\cdot}$ defined in (12)
    Compute m^u_n, m^v_n as defined in (11)
    $u_{n-2} \leftarrow (m^u_n + m^v_n)/(n+1)$, $v_{n-2} \leftarrow u_{n-2} + (m^v_n/6)$ ▷ This solves (10)
  end for
  return $u_0, v_0, \ldots, u_{n_{\max}}, v_{n_{\max}}$
end function

2.3. Intuitive Argument and Numerical Evidence for Divergence

If v_n does not vanish for all large enough n, and each of the sums in (12) satisfies $R^n_{\cdot}/v_{n-4} = \text{constant} + O(1/n)$ for even n as $n \to \infty$, then (10) gives

$$\frac{v_{n-2}}{v_{n-4}} = -\frac{n}{6} + \text{constant} + O\left(\frac{1}{n}\right). \tag{13}$$

For such a sequence, there is no $t > 0$ such that $\sum_{n=0}^{\infty} v_n t^n$ converges.

We implemented the above algorithm using Python's `fractions` module, which provides exact arithmetic for rational numbers, and provide the resulting implementation as `series.py` in the supplementary material. We find the following initial terms of the series:

$$u(t) = \frac{1}{6}t^0 - \frac{13}{120}t^2 + \frac{79}{720}t^4 - \frac{7439}{51840}t^6 + \frac{1289987}{5702400}t^8 - \frac{370576091}{889574400}t^{10} + O(t^{12})$$
$$v(t) = \frac{1}{6}t^0 - \frac{49}{360}t^2 + \frac{23}{144}t^4 - \frac{61297}{259200}t^6 + \frac{7176649}{17107200}t^8 - \frac{772992989}{889574400}t^{10} + O(t^{12})$$

as $t \to 0$. Figure 2 plots the ratio v_n/v_{n-2}, which does indeed decrease nearly linearly, as suggested by (13).

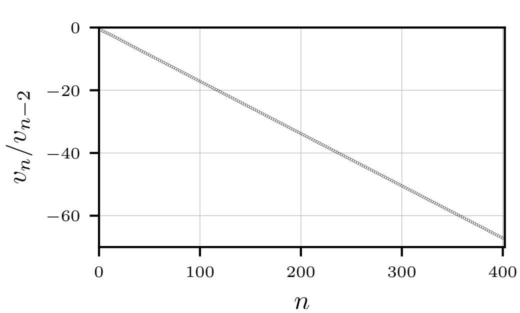

FIGURE 2. The ratio v_n/v_{n-2} for even n.

3. Relation to Previous Work

Ballard and Basseville [2] argued for the analyticity of the motion of a single particle, under the action of an analytic force and Coulomb friction, and Charles and Ballard [5] extended those arguments to multiple particles. Both papers apply Lemma 3.4 of [2] to draw this conclusion. This lemma involves a function G that is hypothesized to be analytic in a neighbourhood of the origin. While both papers claim that the function to which they apply this lemma ($\tilde{G}$ in [2, p. 70] and $\tilde{\mathbf{g}}^i$ in [5, p. 13]) is analytic in a neighbourhood of the origin, this claim is false in dimension $d > 2$, as illustrated in the example below. Moreover, Counterexample 1 shows that it is not possible to circumvent this error: the counterexample directly contradicts the results about the existence of a local analytic solution presented in [2, Proposition 3.5 and Theorem 4.1] and [5, Proposition 4.1]. As these existence results are central to the uniqueness arguments presented in [2, Theorem 4.2] and [5, Theorem 4.2], the counterexample also invalidates those arguments in dimension $d > 2$.

Example 3.1. For a single particle i that is initially at rest but slides immediately thereafter, which is acted on by a tangential force $\mathbf{F}(t) \in \mathbb{R}^{d-1}$ and normal force $N(t)$, the function in [5, p. 13] is

$$\tilde{\mathbf{g}}^i(t, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) := \frac{1}{t^{p_0+1}} \left[\mathbf{F}(t) - \ddot{\mathbf{S}}_{p_0+2}(t) - \mu |N(t)| \frac{\dot{\mathbf{S}}_{p_0+2}(t) + t^{p_0+1}\tilde{\mathbf{v}}}{\|\dot{\mathbf{S}}_{p_0+2}(t) + t^{p_0+1}\tilde{\mathbf{v}}\|} \right], \qquad t \in [0, T],\ \tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in \mathbb{R}^{d-1},$$

where $\mathbf{S}_{p_0+2}(t) = \sum_{k=0}^{p_0+2} \mathfrak{u}_k t^k$ is the truncated Taylor series for the tangential position of the particle for suitable coefficients $\mathfrak{u}_k \in \mathbb{R}^{d-1}$, and p_0 is the integer such that $\dot{\mathbf{S}}_{p_0+2}(t)$ has a single non-zero term. Now, if the force satisfies $\mathbf{F}(t) = (2, 0)^\top$ and $\mu |N(t)| = 1$, then the solution for the tangential velocity is $\mathbf{v}(t) = (t, 0)^\top$, so the truncated Taylor series satisfies $\dot{\mathbf{S}}_{p_0+2}(t) = (t, 0)^\top$ with $p_0 = 0$. It follows that

$$\tilde{\mathbf{g}}^i(t, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \frac{1}{t} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{\mathbf{v}}}{\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{\mathbf{v}} \right\|} \right],$$

which has a simple pole at $t = 0$ whenever the second component of $\tilde{\mathbf{v}}$ does not vanish. Thus $\tilde{\mathbf{g}}^i$ is not analytic on any neighbourhood of the origin.

Appendix A. Proof of Counterexample 1

The proof is based on Algorithm 1. It is straightforward, but laborious as the algorithm involves many intermediate sums, each of which must be bounded. The file `algebra.txt` in the supplementary material contains computer algebra code for all calculations involved.

Let $B_p := |b_p|, C_p := |c_p|, U_p := |u_p|$ and $V_p := |v_p|$. The proof proceeds by induction with the hypothesis:

$$H_p : \begin{cases} |b_{p-2}/v_{p-6} + 1/6| \leq k_b/(p-2), & \text{where } k_b := 20, \text{ if } p \geq 6, \\ |c_{p-2}/v_{p-6} - 6| \leq k_c/(p-2), & \text{where } k_c := 746, \text{ if } p \geq 6, \\ B_{p-2} \leq V_{p-2}/4, & \text{if } p \geq 6, \\ C_{p-2} \leq 8V_{p-2}, & \text{if } p \geq 6, \\ |u_p/v_{p-2} + 1| \leq K/p^2, & \text{where } K := 60, \text{ if } p \geq 2, \\ |v_p/v_{p-2} + (p+2.9)/6| \leq K_v/p, & \text{where } K_v := 5, \text{ if } p \geq 2, \\ U_p \leq V_p, & \text{and} \\ V_{p-2}/V_{p-4} \leq V_p/V_{p-2} & \text{if } p \geq 4. \end{cases}$$

We refer to the four lines concerning $b_{p-2}, c_{p-2}, B_{p-2}, C_{p-2}$ as H_p^{bc}, and we define the hypothesis

$$H_{\leq n}: \qquad n \text{ is even and } H_p \text{ holds for all even } p \text{ with } 0 \leq p \leq n.$$

Before giving the proof itself, we present three lemmas: Lemmas A.1 and A.2 are simple bounds on ratios involving V_n; and Lemma A.3 gives upper bounds of the form $\frac{\text{constant}}{n}$ on the sums defining a_n, b_n, c_n and $R^n_\cdot$.

Lemma A.1. *Suppose $k \geq 6$ and $n \geq k+10$ are even integers and that $H_{\leq n-4}$ holds. Then*

$$\frac{V_{n-k}}{V_{n-4}} \leq \prod_{q=4,\, q \text{ even}}^{k-2} \frac{6}{n-q} =: \rho(n,k).$$

Proof. As $H_{\leq n-4}$ holds, for even p with $2 \leq p \leq n-4$, we have $\frac{V_p}{V_{p-2}} \geq \frac{p+2.9}{6} - \frac{K_v}{p}$. For $p \geq 12$, the right-hand side is greater than $p/6$. As $12 \leq n-k+2 \leq n-4$, it follows that

$$\frac{V_{n-k}}{V_{n-4}} = \frac{V_{n-k}}{V_{n-k+2}} \frac{V_{n-k+2}}{V_{n-k+4}} \cdots \frac{V_{n-6}}{V_{n-4}} \leq \frac{6}{n-k+2} \frac{6}{n-k+4} \cdots \frac{6}{n-4}$$

as claimed. □

Lemma A.2. *Suppose $p \geq 182$ is even and H_p holds. Then $V_p/V_{p-2} \leq p/5.9$.*

Proof. As H_p holds, we have $\frac{V_p}{V_{p-2}} \leq \frac{p+2.9}{6} + \frac{K_v}{p}$. The result follows as $\left(\frac{1}{5.9} - \frac{1}{6}\right) p \geq \frac{2.9}{6} + \frac{5}{p}$ for $p \geq 182$. □

The following lemma involves the function

$$S_{\alpha,\beta}^{X,x,Y,y}(n) := \left(\frac{xy}{2} (n_{\min} - \beta - 18) V_{10} \rho(n_{\min}, \beta+10) + \sum_{p=\alpha+2}^{8} (xY_p + yX_p) \rho(n_{\min}, \beta+p) \right) \frac{n_{\min}}{n},$$

in which X, Y are real sequences, x, y are real numbers, α, β, n are even integers, and $\rho(n,k)$ is as in Lemma A.1. We used $n_{\min} = 200$ to obtain numerical values.

Lemma A.3. *Suppose that $(X,x), (Y,y) \in \{(U,1), (V,1), (B,1/4), (C,8)\}$, that $\alpha, \beta \in \{0,2,4,6\}$ with $\alpha+\beta \geq 4$, that $n-4 \geq 200$, and that $H_{\leq n-4}$ holds. Then*

$$\sum_{p+q=n-\beta,\, p,q>\alpha} \frac{X_p Y_q}{V_{n-4}} \leq S_{\alpha,\beta}^{X,x,Y,y}(n).$$

Proof. As $H_{\leq n-4}$ holds, we have $B_{n-6} \leq V_{n-6}/4$, and more generally for $4 \leq p \leq n-6$, we have

$$X_p \leq xV_p, \qquad Y_p \leq yV_p.$$

For the hypothesized α, β, n, it follows that

$$\sum_{p+q=n-\beta,\, p,q>\alpha} X_p Y_q \leq \sum_{p+q=n-\beta,\, p,q>\alpha} (X_p \mathbf{1}_{p\leq 8} + xV_p \mathbf{1}_{p>8})(Y_q \mathbf{1}_{q\leq 8} + yV_q \mathbf{1}_{q>8}).$$

As $H_{\leq n-4}$ holds, we have $V_p/V_{p-2} \leq V_{q+2}/V_q$ for any even integers $2 \leq p \leq q+2 \leq n-4$. So for any even integers $8 < p \leq q+2$ appearing in this sum, we have

$$V_p V_q \leq V_{p-2} V_{q+2} \leq \cdots \leq V_{10} V_{n-\beta-10}.$$

As the sum $\sum_{p+q=n-\beta,\, p,q>8} \cdots$ involves $(n-\beta-18)/2$ terms with even p, it follows that

$$\sum_{p+q=n-\beta,\, p,q>\alpha} \frac{X_p Y_q}{V_{n-4}} \le \frac{xy}{2}(n-\beta-18)\frac{V_{10}V_{n-\beta-10}}{V_{n-4}} + \sum_{p=\alpha+2}^{8} (X_p y + xY_p)\frac{V_{n-\beta-p}}{V_{n-4}}. \tag{14}$$

By Lemma A.1, we have

$$\frac{V_{n-\beta-p}}{V_{n-4}} \le \rho(n, \beta+p) \qquad \text{for even } \beta+p \text{ with } 6 \le \beta+p \le n-10.$$

As $\alpha+\beta \ge 4$, the condition $6 \le \beta+p$ applies to all appearances of $V_{n-\beta-p}$ in (14), giving

$$\sum_{p+q=n-\beta,\, p,q>\alpha} \frac{X_p Y_q}{V_{n-4}} \le \frac{xy}{2}(n-\beta-18)V_{10}\rho(n,\beta+10) + \sum_{p=\alpha+2}^{8} (X_p y + xY_p)\rho(n,\beta+p). \tag{15}$$

Let f be any of the functions $n \mapsto \rho(n,\beta+p)$ for $\alpha+2 \le p \le 8$, or $n \mapsto (n-\beta-18)\rho(n,\beta+10)$. By the definition of ρ and the hypotheses on α, β, the function $nf(n)$ is decreasing for $n \ge n_{\min}$. Therefore, $f(n) \le n_{\min} f(n_{\min})/n$ for $n \ge n_{\min}$. Applying this to (15) completes the proof. □

Proof of Counterexample 1. First, we consider the base case, $H_{\le 200}$ (in Step 1). Next, we show that if $n-4 \ge 200$ and $H_{\le n-4}$ holds, then H^{bc}_{n-2} and H_{n-2} hold (Steps 2 and 3). Finally, we conclude (Step 4).

Step 1: Base Case ($H_{\le 200}$). Our implementation of Algorithm 1, which uses exact arithmetic for rational numbers from Python's `fractions` module, gives

$$\begin{aligned}
&\max_{6\le p\le 200} (p-2)|b_{p-2}/v_{p-6} + 1/6| = 1.466\cdots \le k_b, && \max_{2\le p\le 200} p^2|u_p/v_{p-2}+1| = 10.913\cdots \le K,\\
&\max_{6\le p\le 200} (p-2)|c_{p-2}/v_{p-6} - 6| = 37.089\cdots \le k_c, && \max_{2\le p\le 200} p|v_p/v_{p-2} + (p+2.9)/6| = 1.881\cdots \le K_v,\\
&\max_{6\le p\le 200} (p-2)B_{p-2}/V_{p-2} = 0.043\cdots \le 1/4, && \max_{0\le p\le 200} U_p/V_p = 1,\\
&\max_{6\le p\le 200} (p-2)C_{p-2}/V_{p-2} = 1.073\cdots \le 8, && \max_{4\le p\le 200} V_{p-2}^2/(V_pV_{p-4}) = 0.990\cdots \le 1,
\end{aligned}$$

in which each maximum is taken only over even values of p. Therefore $H_{\le 200}$ holds. (While one might find a proof involving a "smaller" base case, for instance $H_{\le 50}$, we do not pursue this as we believe it would still require computer algebra.)

Step 2: Showing that $H_{\le n-4} \Rightarrow H^{bc}_{n-2}$. Suppose that $H_{\le m}$ holds where $m := n-4 \ge 200$. As H_m holds, we have $|u_m/v_{m-2}+1| \le K/m^2$, so Lemma A.2 gives

$$\frac{|u_0u_m + v_0v_{m-2}|}{V_{m-4}} \le \left|u_0\left(\frac{u_m}{v_{m-2}}+1\right) - u_0 + v_0\right|\frac{V_{m-2}}{V_{m-4}} \le \left(\frac{U_0K}{m^2} + |v_0-u_0|\right)\frac{m-2}{5.9} \le \frac{U_0K}{5.9m} =: r_{u_m}.$$

Similarly, we have

$$\left|u_2\frac{u_{m-2}}{v_{m-4}} + v_2 + \frac{1}{36}\right| \le \frac{U_2K}{(m-2)^2} + \left|v_2 + \frac{1}{36} - u_2\right| = \frac{U_2K}{(m-2)^2} =: r_{u_{m-2}}$$

and Lemma A.1 gives

$$\frac{|u_4u_{m-4} + v_4v_{m-6}|}{V_{m-4}} \le \left(\frac{U_4K}{(m-4)^2} + |v_4-u_4|\right)\rho(m,6) =: r_{u_{m-4}}.$$

So the definition of a_m in (7), Lemma A.3, and the fact that $m \mapsto m(r_{u_m} + r_{u_{m-2}} + r_{u_{m-4}})$ is decreasing give

$$\begin{aligned}\left|\frac{a_m}{v_{m-4}} + \frac{1}{18}\right| &\leq 2\frac{|u_0 u_m + v_0 v_{m-2}|}{V_{m-4}} + \left|2u_2 \frac{u_{m-2}}{v_{m-4}} + 2v_2 + \frac{1}{18}\right| + 2\frac{|u_4 u_{m-4} + v_4 v_{m-6}|}{V_{m-4}} \\ &\quad + \sum_{p+q=m,\, p,q>4} \frac{U_p U_q}{V_{m-4}} + \sum_{p+q=m-2,\, p,q>4} \frac{V_p V_q}{V_{m-4}} \\ &\leq \frac{2}{m}\left[m\left(r_{u_m} + r_{u_{m-2}} + r_{u_{m-4}}\right)\right]_{m=200} + S_{4,0}^{U,1,U,1}(m) + S_{4,2}^{V,1,V,1}(m) \leq \frac{6.5}{m}. \qquad (16)\end{aligned}$$

As H_m and H_{m-2} hold, Lemma A.1 gives

$$\frac{B_{m-2}}{V_{m-4}} \leq \left(\frac{1}{6} + \frac{k_b}{m-2}\right)\rho(m,6) =: r_{B_{m-2}}, \qquad \frac{B_{m-4}}{V_{m-4}} \leq \left(\frac{1}{6} + \frac{k_b}{m-4}\right)\rho(m,8) =: r_{B_{m-4}}.$$

So the definition of b_m, inequality (16), and Lemma A.3 give

$$\begin{aligned}\left|\frac{b_m}{v_{m-4}} + \frac{1}{6}\right| &\leq 3\left(\left|\frac{a_m}{v_{m-4}} + \frac{1}{18}\right| + 2B_2 \frac{B_{m-2}}{V_{m-4}} + 2B_4 \frac{B_{m-4}}{V_{m-4}} + \sum_{p+q=m,\, p,q>4} \frac{B_p B_q}{V_{m-4}}\right) \\ &\leq 3\left(\frac{6.5}{m} + \frac{1}{m}\left[m\left(2B_2 r_{B_{m-2}} + 2B_4 r_{B_{m-4}}\right)\right]_{m=200} + S_{4,0}^{B,\frac{1}{4},B,\frac{1}{4}}(m)\right) = \frac{19.8\cdots}{m} \leq \frac{k_b}{m}. \qquad (17)\end{aligned}$$

As H_m and H_{m-2} hold, Lemma A.1 gives

$$\frac{C_{m-2}}{V_{m-4}} \leq \left(6 + \frac{k_c}{m-2}\right)\rho(m,6) =: r_{C_{m-2}}, \qquad \frac{C_{m-4}}{V_{m-4}} \leq \left(6 + \frac{k_c}{m-4}\right)\rho(m,8) =: r_{C_{m-4}},$$

so that

$$\begin{aligned}\left|\frac{c_m}{v_{m-4}} - 6\right| &\leq 6\left(\left|\frac{c_0 b_m}{v_{m-4}} - 1\right| + B_2 \frac{C_{m-2}}{V_{m-4}} + C_2 \frac{B_{m-2}}{V_{m-4}} + B_4 \frac{C_{m-4}}{V_{m-4}} + C_4 \frac{B_{m-4}}{V_{m-4}} + \sum_{p+q=m,\, p,q>4} \frac{B_p C_q}{V_{m-4}}\right) \\ &\leq 6\left(\frac{6k_b}{m} + \frac{1}{m}\left[m\left(B_2 r_{C_{m-2}} + C_2 r_{B_{m-2}} + B_4 r_{C_{m-4}} + C_4 r_{B_{m-4}}\right)\right]_{m=200} + S_{4,0}^{B,\frac{1}{4},C,8}(m)\right) \\ &= \frac{745.2\cdots}{m} \leq \frac{k_c}{m}. \qquad (18)\end{aligned}$$

Applying Lemma A.1 to inequalities (17) and (18) gives

$$\frac{B_m}{V_m} \leq \left[\left(\frac{1}{6} + \frac{k_b}{m}\right)\frac{6}{m-2}\right]_{m=200} \frac{6}{m} \leq \frac{0.05}{m} \leq \frac{1}{4}, \qquad \frac{C_m}{V_m} \leq \left[\left(6 + \frac{k_c}{m}\right)\frac{6}{m-2}\right]_{m=200} \frac{6}{m} \leq \frac{1.8}{m} \leq 8. \qquad (19)$$

Together inequalities (17), (18) and (19) imply that H_{n-2}^{bc} holds.

Step 3: Showing that $H_{\le n-4} \Rightarrow H_{n-2}$. Suppose $n-4 \ge 200$ and that $H_{\le n-4}$ holds. We now bound the sums $R^n_{\cdot}$ defined in (12). Lemma A.1 gives

$$
\begin{aligned}
|u_4 u_{n-4} + v_4 v_{n-6} + u_6 u_{n-6}| \frac{1}{V_{n-4}} &\le \left(|u_4 - v_4| + U_4 \frac{K}{(n-4)^2} + U_6 \left(1 + \frac{K}{(n-6)^2}\right) \frac{6}{n-6} \right) \frac{6}{n-4} =: T_1^n, \\
|u_2 u_{n-4} + v_2 v_{n-6} + u_4 u_{n-6}| \frac{1}{V_{n-4}} &\le \left(|u_2 - v_2| + U_2 \frac{K}{(n-4)^2} + U_4 \left(1 + \frac{K}{(n-6)^2}\right) \frac{6}{n-6} \right) \frac{6}{n-4} =: T_2^n, \\
\frac{U_{n-4}}{V_{n-4}} &\le \left(1 + \frac{K}{(n-4)^2}\right) \frac{6}{n-4} =: T_3^n,
\end{aligned}
$$

and the penultimate steps of (19) give

$$
\frac{B_{n-4}}{V_{n-4}} \le \frac{0.05}{n-4} =: T_4^n, \qquad \frac{C_{n-4}}{V_{n-4}} \le \frac{1.8}{n-4} =: T_5^n.
$$

Applying Lemma A.3 to the sums in the following expressions, we obtain

$$
\begin{aligned}
\left| \frac{R^n_{a_n}}{v_{n-4}} + \frac{49}{180} \right| &\le 2T_1^n + \sum_{p+q=n,\, p,q>6} \frac{U_p U_q}{V_{n-4}} + \left| 2v_2 + \frac{49}{180} \right| + \sum_{p+q=n-2,\, p,q>4} \frac{V_p V_q}{V_{n-4}} \le \frac{1.33\cdots}{n}, \\
\left| \frac{R^n_{a_{n-2}}}{v_{n-4}} - \frac{1}{3} \right| &\le 2T_2^n + \sum_{p+q=n-2,\, p,q>4} \frac{U_p U_q}{V_{n-4}} + \left| 2v_0 - \frac{1}{3} \right| + \sum_{p+q=n-4,\, p,q>2} \frac{V_p V_q}{V_{n-4}} \le \frac{0.519\cdots}{n}, \\
\left| \frac{R^n_{b_n}}{v_{n-4}} \right| &\le 2B_4 T_4^n + \sum_{p+q=n,\, p,q>4} \frac{B_p B_q}{V_{n-4}} \le \frac{0.035\cdots}{n} \\
\left| \frac{R^n_{b_{n-2}}}{v_{n-4}} \right| &\le 2B_2 T_4^n + \sum_{p+q=n-2,\, p,q>2} \frac{B_p B_q}{V_{n-4}} \le \frac{0.024}{n} \\
\left| \frac{R^n_{c_n}}{v_{n-4}} \right| &\le B_4 T_5^n + C_4 T_4^n + \sum_{p+q=n,\, p,q>4} \frac{B_p C_q}{V_{n-4}} \le \frac{1.23\cdots}{n} \\
\left| \frac{R^n_{c_{n-2}}}{v_{n-4}} \right| &\le B_2 T_5^n + C_2 T_4^n + \sum_{p+q=n-2,\, p,q>2} \frac{B_p C_q}{V_{n-4}} \le \frac{0.645\cdots}{n} \\
\left| \frac{R^n_{d_n}}{v_{n-4}} \right| &\le C_4 T_3^n + U_4 T_5^n + \sum_{p+q=n,\, p,q>4} \frac{U_p C_q}{V_{n-4}} \le \frac{12.765\cdots}{n} \\
\left| \frac{R^n_{f_{n-1}}}{v_{n-4}} - \frac{9}{10} \right| &\le \left| c_2 - \frac{9}{10} \right| + V_2 T_5^n + \sum_{p+q=n-2,\, p,q>2} \frac{V_p C_q}{V_{n-4}} \le \frac{9.295\cdots}{n}.
\end{aligned}
$$

As the solution to linear equation (10) for u_{n-2} is

$$
\begin{aligned}
u_{n-2} &= \frac{m_n^u + m_n^v}{n+1} \\
&= \frac{1}{n+1} \left(-n v_{n-4} - R^n_{f_{n-1}} - R^n_{d_n} + \frac{1}{2} R^n_{c_{n-2}} + R^n_{c_n} - \frac{72}{5} R^n_{b_{n-2}} - 18 R^n_{b_n} + \frac{72}{5} R^n_{a_{n-2}} + 18 R^n_{a_n} \right)
\end{aligned}
$$

and we have $\frac{9}{10} - \frac{72}{5} \times \frac{1}{3} + \frac{49}{10} = 1$, substituting the above bounds on $R^n_\cdot$ gives

$$\begin{aligned}\left|\frac{u_{n-2}}{v_{n-4}} + 1\right| &\le \frac{1}{n+1}\Bigg(\left|\frac{R^n_{f_{n-1}}}{v_{n-4}} - \frac{9}{10}\right| + \left|\frac{R^n_{d_n}}{v_{n-4}}\right| + \frac{1}{2}\left|\frac{R^n_{c_{n-2}}}{v_{n-4}}\right| + \left|\frac{R^n_{c_n}}{v_{n-4}}\right| + \frac{72}{5}\left|\frac{R^n_{b_{n-2}}}{v_{n-4}}\right| + 18\left|\frac{R^n_{b_n}}{v_{n-4}}\right| \\ &\qquad + \frac{72}{5}\left|\frac{R^n_{a_{n-2}}}{v_{n-4}} - \frac{1}{3}\right| + 18\left|\frac{R^n_{a_n}}{v_{n-4}} + \frac{49}{180}\right|\Bigg) \\ &= \frac{56.06\cdots}{n(n+1)} \le \frac{K}{(n-2)^2}. \end{aligned} \tag{20}$$

Similarly, the solution for v_{n-2} is

$$v_{n-2} = u_{n-2} + \frac{m^v_n}{6}, \qquad \text{where} \qquad m^v_n = -nv_{n-4} - R^n_{f_{n-1}} + R^n_{c_{n-2}} + 18\left(R^n_{a_{n-2}} - R^n_{b_{n-2}}\right),$$

and substituting the above bounds on $R^n_\cdot$ and u_{n-2} into this expression gives

$$\begin{aligned}\left|\frac{v_{n-2}}{v_{n-4}} + \frac{n-2+2.9}{6}\right| &\le \left|\frac{u_{n-2}}{v_{n-4}} + 1\right| + \left|-1 + \frac{1}{6}\left(\frac{m^v_n}{v_{n-4}} + n + 0.9\right)\right| \\ &\le \frac{K}{(n-2)^2} + \frac{1}{6}\Bigg(\left|\frac{R^n_{f_{n-1}}}{v_{n-4}} - \frac{9}{10}\right| + \left|\frac{R^n_{c_{n-2}}}{v_{n-4}}\right| + 18\left|\frac{R^n_{a_{n-2}}}{v_{n-4}} - \frac{1}{3}\right| + 18\left|\frac{R^n_{b_{n-2}}}{v_{n-4}}\right|\Bigg) \\ &= \frac{3.59\cdots}{n} \le \frac{K_v}{n-2}. \end{aligned} \tag{21}$$

Coupling inequalities (20) and (21) gives

$$\frac{U_{n-2}}{V_{n-4}} \le 1 + \frac{K}{(n-2)^2} \le \frac{n-2+2.9}{6} - \frac{K_v}{n-2} \le \frac{V_{n-2}}{V_{n-4}}. \tag{22}$$

As inequality (21) holds, we may argue as in Lemmas A.2 and A.1 to get

$$\frac{V_{n-2}}{V_{n-4}} \ge \frac{n-2}{5.9} \ge \frac{n-4}{6} \ge \frac{V_{n-4}}{V_{n-6}}. \tag{23}$$

As H^{bc}_{n-2} holds (Step 2), inequalities (20), (21), (22) and (23) imply H_{n-2} holds, completing the induction.

Step 4: Conclusion. As H_p holds for all even p, we have

$$\lim_{p\to\infty,\, p \text{ even}} \left|\frac{v_p t^p}{v_{p-2}t^{p-2}}\right| = \lim_{p\to\infty} \frac{p+2.9}{6} t^2 = \infty$$

for any $t \neq 0$. By d'Alembert's ratio test, it follows that the series $\sum_{p=0,\, p \text{ even}}^{\infty} v_p t^p$ diverges for all non-zero t. Therefore this series is not the Taylor series at $t = 0$ of an analytic function. This completes the proof. □

Acknowledgements. The author thanks Tomi Silander and Patrick Ballard for their comments on an early version of this manuscript; and Michel Gastaldo, Florent Perronnin and Sangok Seok for their encouragement and support.

References

[1] P. Ballard. The dynamics of discrete mechanical systems with perfect unilateral constraints. *Archive for Rational Mechanics and Analysis*, 154(3):199–274, 2000.

[2] P. Ballard and S. Basseville. Existence and uniqueness for dynamical unilateral contact with Coulomb friction: a model problem. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(1):59–77, 2005.

[3] P. Ballard and A. Charles. An overview of the formulation, existence and uniqueness issues for the initial value problem raised by the dynamics of discrete systems with unilateral contact and dry friction. *Comptes Rendus Mécanique*, 346(3):222–236, 2018.

[4] A. Bressan. Incompatibilità dei teoremi di esistenza e di unicità del moto per un tipo molto comune e regolare di sistemi meccanici. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, 14(4):333–348, 1960.

[5] A. Charles and P. Ballard. Existence and uniqueness of solutions to dynamical unilateral contact problems with Coulomb friction: the case of a collection of points. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(1):1–25, 2014.

[6] F. Dubois, V. Acary, and M. Jean. The contact dynamics method: a nonsmooth story. *Comptes Rendus Mécanique*, 346(3):247–262, 2018.

[7] J. Hwangbo, J. Lee, and M. Hutter. Per-contact iteration method for solving contact dynamics. *IEEE Robotics and Automation Letters*, 3(2):895–902, 2018.

[8] S. Kepley and T. Zhang. A constructive proof of the Cauchy-Kovalevskaya theorem for ordinary differential equations. *Journal of Fixed Point Theory and Applications*, 23(1):1–23, 2021.

[9] M. Loday-Richaud. Divergent series, summability and resurgence II. *Lecture Notes in Mathematics*, 2154, 2016.

[10] Z. Manchester and S. Kuindersma. Variational contact-implicit trajectory optimization. In *Robotics Research*, pages 985–1000. Springer, 2020.

[11] M. D. P. Monteiro Marques. Inelastic shocks with or without friction: Existence results. In *Differential Inclusions in Nonsmooth Mechanical Problems*, pages 72–111. Springer, 1993.

[12] J.-J. Moreau. Application of convex analysis to some problems of dry friction. In *Trends in Applications of Pure Mathematics to Mechanics*, pages 263–280. Pitman, 1977.

[13] D. Percivale. Uniqueness in the elastic bounce problem, II. *Journal of Differential Equations*, 90(2):304–315, 1991.

[14] M. Schatzman. A class of nonlinear differential equations of second order in time. *Nonlinear Analysis: Theory, Methods and Applications*, 2(3):355–373, 1978.