Mathematical Modelling and Numerical Analysis Will be set by the publisher
Modélisation Mathématique et Analyse Numérique

EXPONENTIAL STABILIZATION FOR CARBON NANOTUBES MODELED AS TIMOSHENKO BEAMS WITH THERMOELASTIC EFFECTS

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Abstract. In this article we consider the problem of heat conduction in carbon nanotubes modeled like Timoshenko beams, inspired by the work of J. Yoon *et al.* (Composites Part B: Engineering, **35**(2), 87–93. 2004). Using the theory of semigroups of linear operators, we prove the well-posedness of the problem and the exponential stabilization of the total energy of the system of differential equations, partially damped, without assuming the known relationship of equality of wave velocities. Furthermore, we analyze the fully discrete problem using a finite difference scheme, introduced by a spatiotemporal discretization that combines explicit and implicit integration methods. We show the construction of numerical energy and simulations that prove our theoretical exponential decay results and display the convergence rates.

1991 Mathematics Subject Classification. 93D22, 65M06, 65M15.

Received xx. Revised xx. Accepted xx.

1. INTRODUCTION

In 1991, Iijima [1] reported the discovery of a new type of finite carbon structure consisting of needle-like tubes. The structure is produced using an arc-shaped discharge evaporation method, similar to that used in the synthesis of fullerene – a type of closed carbon structure shaped like a football (geodetic dome). With the use of electron microscopes, it was possible to observe that each needle-type tube comprises from 2 to 50 coaxial tubes of graphite sheets. An interesting point in the formation of these needles lies in the length of their diameter, which can vary from a few units to a few tens of nanometers, a number much higher than that found in the structure of fullerenes. According to Lin *et al.* [2], the carbon nanotubes (CNTs) they can be metallic or semiconductor, depending on their diameters and the arrangement of the hexagonal rings along the length of the tube. In addition to the interesting electronic characteristics, CNTs have excellent mechanical and thermal properties [3,4] that make it a very promising material. In Dai [5], we find a complete description of how graphene sheets are rolled to provide tubes of the type chair, zigzag and chiral. Another pertinent question concerns the amount of walls present in each tube. According to Shen and Brozena [6], CNTs are classified in three ways: single wall carbon nanotubes (SWCNTs), double wall carbon nanotubes (DWCNTs) and multi wall carbon nanotubes (MWCNTs), where the concentric cylinders interact with each other through the Van der Walls force. The

Keywords and phrases: Carbon nanotubes, Exponential stability, Numerical analysis, Numerical simulation, Convergence order.

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This provisional PDF is the accepted version. The article should be cited as: ESAIM: M2AN, doi: 10.1051/m2an/2023002

authors also point out that DWCNTs (see Figure 1) are an emerging class of carbon nanostructures and represent the simplest way to study the physical effects of coupling between the walls of carbon nanotubes.

The Timoshenko beam theory is the most used to dealing with the modeling of DWCNTs, by partial differential equations, [7–13]. In [11] Yoon *et al.* the effects of rotational inertia and shear deformation on the propagation of transverse waves in nanotubes in the terahertz range was studied. The authors analyzed the transverse wave velocities of DWCNTs, based on the Euler-Bernoulli and Timoshenko beam models. They observed that the effects of rotational inertia and shear deformation are negligible and the transverse wave propagation can be described satisfactorily by the Euler-Bernoulli beam model only when the frequency is far below the lowest critical frequency. However, when the frequency is below but close to the lowest critical frequency, rotational inertia and shear deformation significantly affect the wave velocity. Furthermore, when the frequency is greater than the lowest critical frequency, there is more than one wave velocity, and transverse waves of a given frequency can propagate at various velocities that are considerably different from the velocity predicted by the Euler-Bernoulli beam model. With this, they concluded that rotational inertia and shear deformation have a significant effect on both wave speeds and critical frequencies, especially for nanotubes with larger radius. Therefore, the propagation of transverse waves in the terahertz range in nanotubes is best modeled by the Timoshenko beam model.

Timoshenko's beam model [14] is given by the following system of partial differential equations

$$\rho A\frac{\partial^2 Y}{\partial t^2}-kAG\left(\frac{\partial^2 Y}{\partial x^2}-\frac{\partial\varphi}{\partial x}\right) = 0, \tag{1}$$
$$\rho I\frac{\partial^2\varphi}{\partial t^2}-EI\frac{\partial^2\varphi}{\partial x^2}-kAG\left(\frac{\partial Y}{\partial x}-\varphi\right) = 0, \tag{2}$$

where the functions Y and φ denote the vertical displacement of the beam centerline and the rotation of the vertical filament in the beam. The positive constants ρ, A, I, E, G, k denote, respectively, the mass density of the material, the cross-sectional area, the moment of inertia of the cross-section, the Young's modulus, the stiffness modulus and the shear factor. In [15] Soufyane considered the Timoshenko dissipative beam system given by

$$\rho A\frac{\partial^2 Y}{\partial t^2}-kAG\left(\frac{\partial^2 Y}{\partial x^2}-\frac{\partial\varphi}{\partial x}\right) = 0, \tag{3}$$
$$\rho I\frac{\partial^2\varphi}{\partial t^2}-EI\frac{\partial^2\varphi}{\partial x^2}-kAG\left(\frac{\partial Y}{\partial x}-\varphi\right)+b(x)\frac{\partial\varphi}{\partial t} = 0, \tag{4}$$

with variable coefficient $b(x)$ satisfying the relationship $0<b_0\leq b(x)\leq b_1$ and homogeneous Dirichlet boundary conditions. He proved the exponential decay of the system if and only if the relation

$$\frac{kAG}{\rho A}=\frac{EI}{\rho I} \quad\Leftrightarrow\quad G=\frac{E}{k}, \tag{5}$$

is satisfied. In [16] Rivera and Racke (see Sect. 2) considered the thermoelastic system of Timoshenko beams given by

$$\rho A\frac{\partial^2 Y}{\partial t^2}-kAG\left(\frac{\partial^2 Y}{\partial x^2}+\frac{\partial\varphi}{\partial x}\right) = 0, \tag{6}$$
$$\rho I\frac{\partial^2\varphi}{\partial t^2}-EI\frac{\partial^2\varphi}{\partial x^2}+kAG\left(\frac{\partial Y}{\partial x}+\varphi\right)+\gamma\frac{\partial\theta}{\partial x} = 0, \tag{7}$$
$$\rho_0\frac{\partial\theta}{\partial t}-K\frac{\partial^2\theta}{\partial x^2}+\gamma\frac{\partial^2\varphi}{\partial x\partial t} = 0. \tag{8}$$

Using the energy method, they proved exponential decay depending on the relationship (5). Since then, several studies have appeared considering the damping in a single equation of the system, among them we can mention [17–19], and in them, the relationship was used (5). However, we emphasize that this relationship is incompatible with the physical nature of the problem, so $G := E/2(1+\mu)$ implies that

$$k = 2(1+\mu) > 2, \tag{9}$$

which is not true. The Poisson ratio μ has values in the range $(0, 1/2)$ and the constant k, which depends on the geometry of the cross section, has values smaller than 1 (one).

Now turning our attention to carbon nanotubes, we highlight once again the pioneering work of Yoon *et al.* [11], where the authors proposed a coupled system of partial differential equations inspired by the Timoshenko beam model to model DWCNTs. The model consists of the following equations

$$\begin{aligned}
\rho A_1 \frac{\partial^2 Y_1}{\partial t^2} - kGA_1\left(\frac{\partial^2 Y_1}{\partial x^2} - \frac{\partial \varphi_1}{\partial x}\right) - P &= 0, &(10)\\
\rho I_1 \frac{\partial^2 \varphi_1}{\partial t^2} - EI_1\frac{\partial^2 \varphi_1}{\partial x^2} - kGA_1\left(\frac{\partial Y_1}{\partial x} - \varphi_1\right) &= 0, &(11)\\
\rho A_2 \frac{\partial^2 Y_2}{\partial t^2} - kGA_2\left(\frac{\partial^2 Y_2}{\partial x^2} - \frac{\partial \varphi_2}{\partial x}\right) + P &= 0, &(12)\\
\rho I_2 \frac{\partial^2 \varphi_2}{\partial t^2} - EI_2\frac{\partial^2 \varphi_2}{\partial x^2} - kGA_2\left(\frac{\partial Y_2}{\partial x} - \varphi_2\right) &= 0, &(13)
\end{aligned}$$

where Y_i and φ_i $(i = 1,2)$ represent respectively the total deflection and the inclination due to the bending of the nanotube i and the constants I_i, A_i denote the moment of inertia and the cross-sectional area of the tube i respectively and P is the Van der Waals force acting on the interaction between the two tubes per unit of axial length. Also according to [11], it can be seen that the deflections of the two tubes are coupled through the Van der Waals interaction P (see (14)) between the two tubes, and as the tubes inside and outside of a DWCNT are originally concentric, the Van der Waals interaction is determined by the spacing between the layers. Therefore, for a small-amplitude linear vibration, the interaction pressure at any point between the two tubes linearly depends on the difference in their deflection curves at that point, that is, it depends on the term

$$P = \mathscr{C}\big(Y_2 - Y_1\big). \tag{14}$$

In particular, the Van der Waals interaction coefficient $\mathscr{C}$ for the interaction pressure per unit axial length can be estimated based on an effective interaction width of the tubes as found in [9, 20]. Thus, this model treats each of the nested and concentric nanotubes as individual Timoshenko beams interacting in the presence of Van der Waals forces (see Figure 1).

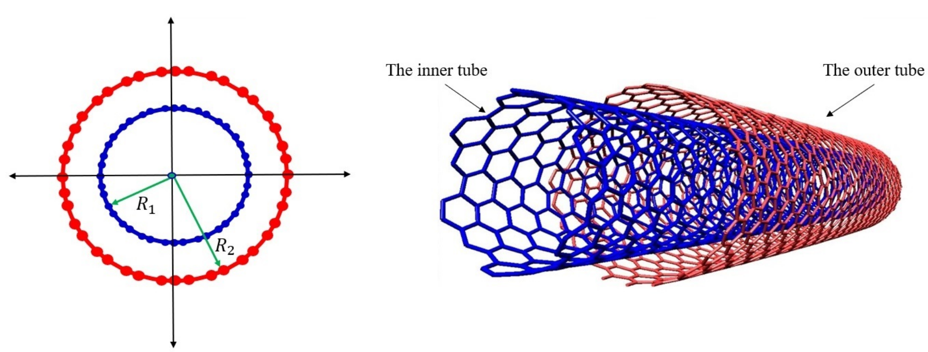


FIGURE 1. Double wall carbon nanotubes - DWCNTs

As we can see in the references cited above, there are many works that deal with the physical and mathematical properties of Timoshenko beam systems, however, it is worth mentioning that we are not aware of any work in the literature that addresses issues related to the asymptotic stabilization of DWCNTs modeled as beams of Timoshenko. Therefore, this work represents a new contribution, once the thermal effect is considered, and providing results regarding the well-posedness of the problem and the exponential stabilization of the solutions of the system of partial differential equations that model carbon nanotubes as Timoshenko beams.

The relationship (5) has no physical meaning and taking into account the high thermal conductivity of CNTs (see [21]), we incorporate the thermal effect acting on the inner tube into the formulation of Timoshenko's beam theory , with the intention of eliminating the need to use (5). In this way, we started to consider a more realistic system from the physical point of view.

In this context, the system is described by the following evolution equations

$$\begin{aligned}
\rho_1\varphi_{tt}(x,t) - S^1_x(x,t) - P(x,t) &= 0 \quad \text{for} \quad (x,t)\in(0,l)\times(0,\infty),\\
\rho_2\psi_{tt}(x,t) - M^1_x(x,t) - S^1(x,t) &= 0 \quad \text{for} \quad (x,t)\in(0,l)\times(0,\infty), && (15)\\
\rho_3 y_{tt}(x,t) - S^2_x(x,t) + P(x,t) &= 0 \quad \text{for} \quad (x,t)\in(0,l)\times(0,\infty),\\
\rho_4 z_{tt}(x,t) - M^2_x(x,t) + S^2(x,t) &= 0 \quad \text{for} \quad (x,t)\in(0,l)\times(0,\infty), && (16)\\
\Psi_t(x,t) + Q_x(x,t) &= 0 \quad \text{for} \quad (x,t)\in(0,l)\times(0,\infty),
\end{aligned}$$

with the constitutive laws with temperature following the Fourier's law are given by

$$M^1(x,t) := b_1\psi_x(x,t) - \delta\theta_x(x,t), \quad M^2(x,t) := b_2\psi_x(x,t), \tag{17}$$
$$S^1(x,t) := \kappa_1(\varphi_x-\psi)(x,t), \quad S^2(x,t) := \kappa_2(\varphi_x-\psi)(x,t), \tag{18}$$
$$P(x,t) := \mathscr{C}\big(y-\varphi\big)(x,t), \quad \Psi(x,t) := \rho_5\theta(x,t) + \beta\psi(x,t), \quad Q(x,t) := -K\theta_x(x,t), \tag{19}$$

where S^i denotes the transverse shear force, M^i is the bending moment and Ψ is the entropy and Q is the heat flux with θ denoting the temperature. More precisely, we consider the partially dissipative system given by

$$\rho_1\varphi_{tt} - \kappa_1\big(\varphi_x-\psi\big)_x - \mathscr{C}\big(y-\varphi\big) + \gamma_1\varphi_t = 0 \quad \text{in} \quad (0,l)\times(0,\infty), \tag{20}$$
$$\rho_2\psi_{tt} - b_1\psi_{xx} - \kappa_1\big(\varphi_x-\psi\big) + \delta\theta_{xx} = 0 \quad \text{in} \quad (0,l)\times(0,\infty), \tag{21}$$
$$\rho_3 y_{tt} - \kappa_2\big(y_x-z\big)_x + \mathscr{C}\big(y-\varphi\big) + \gamma_2 y_t = 0 \quad \text{in} \quad (0,l)\times(0,\infty), \tag{22}$$
$$\rho_4 z_{tt} - b_2 z_{xx} - \kappa_2\big(y_x-z\big) + \gamma_3 z_t = 0 \quad \text{in} \quad (0,l)\times(0,\infty), \tag{23}$$
$$\rho_5\theta_t - K\theta_{xx} + \beta\psi_t = 0 \quad \text{in} \quad (0,l)\times(0,\infty). \tag{24}$$

In this work, we analyze the system (20)–(24) subject to the Dirichlet-Neumann boundary conditions

$$\varphi(0,t) = \varphi(l,t) = \psi(0,t) = \psi(l,t) = 0 \quad \text{for all} \quad t>0, \tag{25}$$
$$y(0,t) = y(l,t) = z(0,t) = z(l,t) = 0 \quad \text{for all} \quad t>0, \tag{26}$$
$$\theta(0,t) = \theta(l,t) = 0 \quad \text{for all} \quad t>0, \tag{27}$$

and the initial conditions

$$\varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x), \;\; \psi(x,0) = \psi_0(x), \;\; \psi_t(x,0) = \psi_1(x) \quad \text{in} \quad x\in(0,l), \tag{28}$$
$$y(x,0) = y_0(x), \;\; y_t(x,0) = y_1(x), \;\; z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x) \quad \text{in} \quad x\in(0,l), \tag{29}$$
$$\theta(x,0) = \theta_0(x) \quad \text{in} \quad x\in(0,l). \tag{30}$$

We introduced the total energy of the system (20)–(30) given by

$$
\begin{aligned}
\mathbb{E}(t) \quad := \quad & \frac{\rho_1}{2}\|\varphi_t\|^2+\frac{\rho_2}{2}\|\psi_t\|^2+\frac{\rho_3}{2}\|y_t\|^2+\frac{\rho_4}{2}\|z_t\|^2+\frac{b_1}{2}\|\psi_x\|^2+\frac{b_2}{2}\|z_x\|^2+\frac{\kappa_1}{2}\|\varphi_x-\psi\|^2 \\
& +\frac{\kappa_2}{2}\|y_x-z\|^2+\frac{\mathscr{C}}{2}\|y-\varphi\|^2+\frac{\rho_5\delta}{2\beta}\|\theta_x\|^2, \quad t\geq 0. \qquad (31)
\end{aligned}
$$

The results we get for this system are surprising, as we show that the partially dissipative thermoelastic system is exponentially stable. It is important to mention that the stability result obtained in this paper do not depend on the equal speed relation.

This paper is organized as follows. In Section 2 we prove the well-posedness of the problem. In Section 3, we proved the exponential decay result of the solutions without using the assumption of equal wave speeds. Finally in Section 4, we analyze the fully discrete model using the finite difference method and reproduce the exponential decay result proved in the previous section and prove the convergence rate.

2. Well-Posedness

We consider the following Hilbert space

$$
\mathscr{H} := \left[H_0^1(0,l)\times L^2(0,l)\right]^4\times H_0^1(0,l), \qquad (32)
$$

provided with the following inner product

$$
\begin{aligned}
\left\langle U_1,U_2\right\rangle_{\mathscr{H}} \quad := \quad & \rho_1\int_0^l u_1\overline{u}_2\,dx+\rho_2\int_0^l v_1\overline{v}_2\,dx+\rho_3\int_0^l s_1\overline{s}_2\,dx+\rho_4\int_0^l w_1\overline{w}_2\,dx+b_1\int_0^l \psi_{1,x}\overline{\psi}_{2,x}\,dx \\
& +b_2\int_0^l z_{1,x}\overline{z}_{2,x}\,dx+\kappa_1\int_0^l(\varphi_{1,x}-\psi_1)\overline{(\varphi_{2,x}-\psi_2)}\,dx+\kappa_2\int_0^l(y_{1,x}-z_1)\overline{(y_{2,x}-z_2)}\,dx \\
& +\mathscr{C}\int_0^l(y_1-\varphi_1)\overline{(y_2-\varphi_2)}dx+\frac{\rho_5\delta}{\beta}\int_0^l\theta_{1,x}\overline{\theta}_{2,x}dx, \qquad (33)
\end{aligned}
$$

for all $U_1=(\varphi_1,u_1,\psi_1,v_1,y_1,s_1,z_1,w_1,\theta_1)^\top$ and $U_2=(\varphi_2,u_2,\psi_2,v_2,y_2,s_2,z_2,w_2,\theta_2)^\top$ in $\mathscr{H}$ and norm given by

$$
||U_1||^2_{\mathscr{H}} := \left\langle U_1,U_1\right\rangle_{\mathscr{H}}. \qquad (34)
$$

In order to prove the existence and uniqueness of solutions, we will use the semigroup theory [22]. Then the system (20)–(30) can be rewritten as follows

$$
\Phi_t(t)=\mathscr{A}\Phi(t), \ \ t>0, \qquad (35)
$$

$$
\Phi(0)=\Phi_0, \qquad (36)
$$

where $\Phi(t)=\big(\varphi(t),\varphi_t(t),\psi(t),\psi_t(t),y(t),y_t(t),z(t),z_t(t),\theta(t)\big)^\top$ is the system solution, $\Phi_0=\big(\varphi_0,\varphi_1,\psi_0,\ \psi_1,y_0,y_1,z_0,z_1,\theta_0\big)^\top$ is the initial condition and $\mathscr{A}:D(\mathscr{A})\subset\mathscr{H}\to\mathscr{H}$ is the operator defined by

$$\mathscr{A}U:=\begin{pmatrix} u\\ \frac{\kappa_1}{\rho_1}\big(\varphi_x-\psi\big)_x+\frac{\mathscr{C}}{\rho_1}\big(y-\varphi\big)-\frac{\gamma_1}{\rho_1}u\\ v\\ \frac{b_1}{\rho_2}\psi_{xx}+\frac{\kappa_1}{\rho_2}\big(\varphi_x-\psi\big)-\frac{\delta}{\rho_2}\theta_{xx}\\ s\\ \frac{\kappa_2}{\rho_3}\big(y_x-z\big)_x-\frac{\mathscr{C}}{\rho_3}\big(y-\varphi\big)-\frac{\gamma_2}{\rho_3}s\\ w\\ \frac{b_2}{\rho_4}z_{xx}+\frac{\kappa_2}{\rho_4}\big(y_x-z\big)-\frac{\gamma_3}{\rho_4}w\\ \frac{K}{\rho_5}\theta_{xx}-\frac{\beta}{\rho_5}v \end{pmatrix},$$

where $U=(\varphi,u,\psi,v,y,s,z,w,\theta)^\top$. The domain of $\mathscr{A}$ is given by

$$D(\mathscr{A}):=\Big\{U=(\varphi,u,\psi,v,y,s,z,w,\theta)^\top\in\mathscr{H}\,:\,u,\,s,\,v,\,w\in H_0^1(0,l),\varphi,\ \psi,\,y,\,z\in H^2(0,l),\ \theta\in H^3(0,l)\Big\}. \tag{37}$$

In order to study the well-posedness of the problem given above, we start by showing that the operator $\mathscr{A}$ generates a C_0-semigroup of contractions over $\mathscr{H}$.

Lemma 2.1. *The operator $\mathscr{A}$ generates a C_0-semigroup $S(t)=e^{\mathscr{A}t}$ of contractions on the space $\mathscr{H}$.*

Proof. It is easy to see that $\mathscr{A}$ is dissipative. In fact, for each $U=(\varphi,u,\psi,v,y,s,z,w,\theta)^\top\in D(\mathscr{A})$ we have

$$\mathrm{Re}\Big\langle\mathscr{A}U,U\Big\rangle_{\mathscr{H}}=-\gamma_1\|u\|^2-\gamma_2\|s\|^2-\gamma_3\|w\|^2-\frac{\delta K}{\beta}\|\theta_{xx}\|^2\le 0. \tag{38}$$

Therefore, it is enough to show that $0\in\rho(\mathscr{A})$ (resolvent set of $\mathscr{A}$). To do that, let us take $F=\big(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9\big)^\top\in\mathscr{H}$, and look for a unique $U=\big(\varphi,u,\psi,v,y,s,z,w,\theta\big)^\top\in D(\mathscr{A})$ such that

$$-\mathscr{A}U=F. \tag{39}$$

Equivalently, we get $u=-f_1$, $v=-f_3$, $s=-f_5$, $w=-f_7$, $\theta_{xx}=-(\beta/K)f_3-(\rho_5/K)f_9$ and the following system

$$\begin{aligned} -\kappa_1\left(\varphi_x-\psi\right)_x-\mathscr{C}(y-\varphi) &= \gamma_1 f_1+\rho_1 f_2, &(40)\\ -b_1\psi_{xx}-\kappa_1\left(\varphi_x-\psi\right) &= \frac{\delta\beta}{K}f_3+\frac{\delta\rho_5}{K}f_9+\rho_2 f_4, &(41)\\ -\kappa_2\left(y_x-z\right)_x+\mathscr{C}(y-\varphi) &= \gamma_2 f_5+\rho_3 f_6, &(42)\\ -b_2 z_{xx}-\kappa_2\left(y_x-z\right) &= \gamma_3 f_7+\rho_4 f_8. &(43) \end{aligned}$$

Multiplying (40) [resp. (41)] by $\overline{\varphi^*}\in H_0^1(0,l)$ [resp. by $\overline{\psi^*}\in H_0^1(0,l)$] and (42) [resp. (43)] by $\overline{y^*}\in H_0^1(0,l)$ [resp. by $\overline{z^*}\in H_0^1(0,l)$], integrating in $[0,l]$ and taking the sum, we obtain the equivalent variational problem:

$$\mathscr{B}\Big(\big(\varphi,\psi,y,z\big),\,\big(\varphi^*,\psi^*,y^*,z^*\big)\Big) \;=\; \mathscr{L}\big(\varphi^*,\psi^*,y^*,z^*\big), \tag{44}$$

where $\mathscr{B}(\cdot,\cdot)$ is the sesquilinear form in $\left[H_0^1(0,l)\right]^4$ given by

$$\begin{aligned}\mathscr{B}\Big((\varphi,\psi,y,z),\,(\varphi^*,\psi^*,y^*,z^*)\Big) \quad := \quad & \kappa_1\int_0^l(\varphi_x-\psi)\,\overline{(\varphi_x^*-\psi^*)}dx+\kappa_2\int_0^l(y_x-z)\,\overline{(y_x^*-z^*)}dx \\ & +b_1\int_0^l\psi_x\overline{\psi_x^*}dx+b_2\int_0^l z_x\overline{z_x^*}dx \\ & +\mathscr{C}\int_0^l(y-\varphi)\overline{(y^*-\varphi^*)}dx \end{aligned} \tag{45}$$

and $\mathscr{L}(\cdot,\cdot,\cdot,\cdot)$ is a continuous linear form in $\left[H_0^1(0,l)\right]^4$ given by

$$\begin{aligned}\mathscr{L}(\varphi^*,\psi^*,y^*,z^*) \quad := \quad & \gamma_1\int_0^l f_1\overline{\varphi^*}dx+\rho_1\int_0^l f_2\overline{\varphi^*}dx+\frac{\delta\beta}{K}\int_0^l f_3\overline{\psi^*}dx+\frac{\delta\rho_5}{K}\int_0^l f_9\overline{\psi^*}dx \\ & +\rho_2\int_0^l f_4\overline{\psi^*}dx+\gamma_2\int_0^l f_5\overline{y^*}dx+\rho_3\int_0^l f_6\overline{y^*}dx+\gamma_3\int_0^l f_7\overline{z^*}dx \\ & +\rho_4\int_0^l f_8\overline{z^*}dx. \end{aligned} \tag{46}$$

Since

$$\mathscr{B}\Big((\varphi,\psi,y,z),(\varphi,\psi,y,z)\Big) \quad := \quad \kappa_1\|\varphi_x-\psi\|^2+\kappa_2\|y_x-z\|^2+b_1\|\psi_x\|^2+b_2\|z_x\|^2+\mathscr{C}\|y-\varphi\|^2,$$

the sesquilinear form $\mathscr{B}(\cdot,\cdot)$ is strongly coercive on $\left[H_0^1(0,l)\right]^4$, and since (46) defines a continuous linear functional of $(\varphi^*,\psi^*,y^*,z^*)$, by Lax-Milgram's Theorem, problem (44) admits a unique solution $(\varphi,\psi,y,z)\in\left[H_0^1(0,l)\right]^4$. By taking test functions in the form $(\widetilde{\varphi},0,0,0)$, $(0,\widetilde{\psi},0,0)$, $(0,0,\widetilde{y},0)$ and $(0,0,0,\widetilde{z})$ with $\widetilde{\varphi},\widetilde{\psi},\widetilde{y},\widetilde{z}\in\mathscr{D}(0,l)$ (space of test functions), it is easy to see that (φ,ψ,y,z) satisfies Eqs. (40)–(43) in the distributional sense. This also shows that $\varphi,\psi,y,z\in H^2(0,l)$ because

$$\kappa_1\varphi_{xx} \;=\; \kappa_1\psi_x-\mathscr{C}(y-\varphi)-\gamma_1 f_1-\rho_1 f_2 \;\in\; L^2(0,l), \tag{47}$$

$$b_1\psi_{xx} \;=\; -\kappa_1(\varphi_x-\psi)-\frac{\delta\beta}{K}f_3-\frac{\delta\rho_5}{K}f_9-\rho_2 f_4 \;\in\; L^2(0,l), \tag{48}$$

$$\kappa_2 y_{xx} \;=\; \kappa_2 z_x+\mathscr{C}(y-\varphi)-\gamma_2 f_5-\rho_3 f_6 \;\in\; L^2(0,l), \tag{49}$$

$$b_2 z_{xx} \;=\; -\kappa_2(y_x-z)-\gamma_3 f_7-\rho_4 f_8 \;\in\; L^2(0,l). \tag{50}$$

Since $u=-f_1\in H_0^1(0,l)$, $v=-f_3\in H_0^1(0,l)$, $s=-f_5\in H_0^1(0,l)$, $w=-f_7\in H_0^1(0,l)$, $\theta_{xx}=-(\beta/K)f_3-(\rho_5/K)f_9\in H_0^1(0,l)$ we have shown that $\left(\varphi,u,\psi,v,y,s,z,w,\theta\right)^\top$ belongs to $D(\mathscr{A})$ and is a solution of $-\mathscr{A}U=F$. Therefore, we deduce that $0\in\rho(\mathscr{A})$. Then by the resolvent identity, for $\lambda>0$ small enough we have $R(\lambda I-\mathscr{A})=\mathscr{H}$ (see Theorem 1.2.4 in [23]). □

Theorem 2.2. *The operator $\mathscr{A}$ generates a C_0–semigroup $S(t)$ of contraction on $\mathscr{H}$. Thus, for any initial data $U_0\in\mathscr{H}$ the system* (20)–(30) *has a unique weak solution $U(t)\in C\Big([0,\infty),\mathscr{H}\Big)$. Moreover, if $U_0\in D(\mathscr{A})$, then $U(t)$ is strong solution of* (20)–(30)*, that is*

$$U(t)\in C\Big([0,\infty),D(\mathscr{A})\Big)\cap C^1\Big([0,\infty),\mathscr{H}\Big). \tag{51}$$

3. ASYMPTOTIC BEHAVIOR

In this section, we shall prove the main results. Along the the paper, C will always stand for a generic positive constant and we will routinely use the Young, Hölder and Poincaré inequalities without explicit mention. First, we recall the following standard result which are stated in a comparable way [24, 25].

Let $\mathscr{A}: D(\mathscr{A}) \subset \mathscr{H} \to \mathscr{H}$ generates a bounded C_0-semigroup $S(t) = e^{\mathscr{A}t}$ on Hilbert space $\mathscr{H}$. Assume that

$$i\mathbb{R} := \{i\lambda;\ \lambda \in \mathbb{R}\} \subset \rho(\mathscr{A}). \tag{52}$$

Semigroup $S(t) = e^{\mathscr{A}t}$ is exponentially stable if and only if

$$\limsup_{|\lambda|\to\infty} ||(i\lambda I - \mathscr{A})^{-1}||_{\mathscr{L}(\mathscr{H})} < \infty, \quad \lambda \in \mathbb{R}, \tag{53}$$

i.e., there are constants $M > 0$ and $\omega > 0$ such that,

$$||S(t)|| \leq Me^{-\omega t}, \quad \text{for all} \quad t > 0. \tag{54}$$

As we can see, the result require that the resolvent of the operator associated with our system contains the imaginary axis.

Remark 3.1. Since the embedding $D(\mathscr{A}) \hookrightarrow \mathscr{H}$ is compact, $\mathscr{A}$ has compact resolvent (see [26], Proposition 4.25).

Lemma 3.2. *Under the above notations we have*

$$i\mathbb{R} \subset \rho(\mathscr{A}). \tag{55}$$

Proof. Since $\mathscr{A}$ has a compact resolvent, its spectrum is discrete. To show that $i\mathbb{R} \subset \rho(\mathscr{A})$ we argue by contradiction. Assuming $\mathscr{A}$ has pure imaginary eigenvalues, we consider $U = (\varphi, u, \psi, v, y, s, z, w, \theta)^\top \in D(\mathscr{A}) \setminus \{0\}$ and the resolving equation

$$(i\lambda I - \mathscr{A})U = 0, \quad \lambda \in \mathbb{R}. \tag{56}$$

That is,

$$\begin{aligned}
i\lambda\varphi - u &= 0, && (57)\\
i\lambda u - \frac{\kappa_1}{\rho_1}(\varphi_x - \psi)_x - \frac{\mathscr{C}}{\rho_1}(y - \varphi) + \frac{\gamma_1}{\rho_1}u &= 0, && (58)\\
i\lambda\psi - v &= 0, && (59)\\
i\lambda v - \frac{b_1}{\rho_2}\psi_{xx} - \frac{\kappa_1}{\rho_2}(\varphi_x - \psi) + \frac{\delta}{\rho_2}\theta_{xx} &= 0, && (60)\\
i\lambda y - s &= 0, && (61)\\
i\lambda s - \frac{\kappa_2}{\rho_3}(y_x - z)_x + \frac{\mathscr{C}}{\rho_3}(y - \varphi) + \frac{\gamma_2}{\rho_3}s &= 0, && (62)\\
i\lambda z - w &= 0, && (63)\\
i\lambda w - \frac{b_2}{\rho_4}z_{xx} - \frac{\kappa_2}{\rho_4}(y_x - z) + \frac{\gamma_3}{\rho_4}w &= 0, && (64)\\
i\lambda\theta - \frac{K}{\rho_5}\theta_{xx} + \frac{\beta}{\rho_5}v &= 0. && (65)
\end{aligned}$$

To prove (55) it is enough to show that $i\lambda U = \mathscr{A}U$ implies $U = 0$. Therefore, from (38), $u = s = w = \theta_{xx} = 0$. Consequently, using Poincaré's inequality, we obtain $\theta = \theta_x = 0$. Using this, from (57), (61), (63) we get $\varphi = y = z = 0$ and from

$\theta=\theta_{xx}=0$ in (65), implies in $v=0$. Finally, from $v=0$ and (59) implies in $\psi=0$. Thus, $U=(\varphi,u,\psi,v,y,s,z,w,\theta)^\top=0$ and our conclusion follows. □

Once we have proved (55), we are interested in the exponential stabilization of the system (20)–(30). For this we use the semigroup theory of linear operators [22, 23].

3.1. Exponential stability

This subsection is devoted to the study of the exponential stability of system (20)–(30). To do this, let us consider the resolvent equation

$$(i\lambda I-\mathscr{A})U=F, \tag{66}$$

with $U=(\varphi,u,\psi,v,y,s,z,w,\theta)^\top\in D(\mathscr{A})$, $F=(f_1,f_2,f_3,f_4,f_5,f_6,\ f_7,f_8,f_9)^\top\in\mathscr{H}$ and $\lambda\in\mathbb{R}\setminus\{0\}$. Rewriting the resolvent equation (66) in term of its components, we have

$$\begin{aligned}
i\lambda\varphi-u &= f_1 \in H_0^1(0,l), && (67)\\
i\lambda u-\frac{\kappa_1}{\rho_1}(\varphi_x-\psi)_x-\frac{\mathscr{C}}{\rho_1}(y-\varphi)+\frac{\gamma_1}{\rho_1}u &= f_2 \in L^2(0,l), && (68)\\
i\lambda\psi-v &= f_3 \in H_0^1(0,l), && (69)\\
i\lambda v-\frac{b_1}{\rho_2}\psi_{xx}-\frac{\kappa_1}{\rho_2}(\varphi_x-\psi)+\frac{\delta}{\rho_2}\theta_{xx} &= f_4 \in L^2(0,l), && (70)\\
i\lambda y-s &= f_5 \in H_0^1(0,l), && (71)\\
i\lambda s-\frac{\kappa_2}{\rho_3}(y_x-z)_x+\frac{\mathscr{C}}{\rho_3}(y-\varphi)+\frac{\gamma_2}{\rho_3}s &= f_6 \in L^2(0,l), && (72)\\
i\lambda z-w &= f_7 \in H_0^1(0,l), && (73)\\
i\lambda w-\frac{b_2}{\rho_4}z_{xx}-\frac{\kappa_2}{\rho_4}(y_x-z)+\frac{\gamma_3}{\rho_4}w &= f_8 \in L^2(0,l), && (74)\\
i\lambda\theta-\frac{K}{\rho_5}\theta_{xx}+\frac{\beta}{\rho_5}v &= f_9 \in H_0^1(0,l). && (75)
\end{aligned}$$

Before proving the technical lemmas that will support our propositions, consider the following definition.

Definition 3.3. Let $U=(u_1,u_2,u_3,u_4,u_5,u_6,,u_7,u_8,u_9)^\top\in D(\mathscr{A})$ and $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,\ f_8,f_9)^\top\in\mathscr{H}$. Then we define the functional class $\mathfrak{R}$ given by

$$\mathfrak{R}:=\left\{\mathscr{R}=\int_0^l f_iu_j\,dx;\ \left|\mathscr{R}\right|\le C||U||_{\mathscr{H}}||F||_{\mathscr{H}},\ i,j\in\{1,2,\ldots,9\}\right\}. \tag{76}$$

Lemma 3.4. *Under the above notations there exists a positive constant C, such that*

$$\|u\|^2,\ \|s\|^2,\ \|w\|^2,\ \|\theta\|^2,\ \|\theta_x\|^2,\ \|\theta_{xx}\|^2\le C||U||_{\mathscr{H}}||F||_{\mathscr{H}}. \tag{77}$$

Proof. The proof follows immediately from (38). □

Lemma 3.5. *For every $\lambda\in\mathbb{R}\setminus\{0\}$ the inequality*

$$\|y-\varphi\|^2\le\frac{C}{\lambda^2}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+\frac{C}{|\lambda|}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \tag{78}$$

holds for some structural constant $C>0$ independent of λ.

Proof. Combining Eqs. (67) and (71) we have

$$i\lambda(y-\varphi)-(s-u)=f_5-f_1. \tag{79}$$

Multiplying the above equation by $\overline{(y-\varphi)}$, integrating by parts on $[0,l]$ and using Lemma 3.4 we get

$$\|y-\varphi\|^2 \leq \frac{C}{\lambda^2}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+\frac{C}{|\lambda|}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \tag{80}$$

leading to the desired conclusion. □

Lemma 3.6. *For every $\lambda \in \mathbb{R}\setminus\{0\}$ the inequality*

$$\|\varphi_x-\psi\|^2+\|y_x-z\|^2 \leq \frac{C}{\lambda^2}\|v\|^2+\frac{C}{\lambda^4}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+\frac{C}{|\lambda|^3}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \tag{81}$$

holds for some structural constant $C>0$ independent of λ.

Proof. Combining Eqs. (67) and (69) we have

$$i\lambda(\varphi_x-\psi)-(u_x-v)=f_{1,x}-f_3. \tag{82}$$

Multiplying the above equation by $\overline{(\varphi_x-\psi)}$ and integrating by parts on $[0,l]$ we get yields

$$i\lambda\|\varphi_x-\psi\|^2+\underbrace{\int_0^l u\overline{(\varphi_x-\psi)_x}dx}_{\mathscr{I}_2:=}+\int_0^l v\overline{(\varphi_x-\psi)}dx=\mathscr{R}. \tag{83}$$

Substituting $\overline{(\varphi_x-\psi)_x}$ given by (68) into $\mathscr{I}_2$, we obtain

$$i\lambda\|\varphi_x-\psi\|^2=i\lambda\frac{\rho_1}{\kappa_1}\|u\|^2-\frac{\mathscr{C}}{\kappa_1}\int_0^l u\overline{(y-\varphi)}dx+\frac{\gamma_1}{\kappa_1}\|u\|^2+\int_0^l v\overline{(\varphi_x-\psi)}dx+\mathscr{R}. \tag{84}$$

Taking the imaginary part we have

$$\|\varphi_x-\psi\|^2 \leq C\|u\|^2+\frac{C}{\lambda^2}\left(\|y-\varphi\|^2+\|v\|^2\right)+\mathscr{R}. \tag{85}$$

Similarly for Eqs. (71) and (73) we have

$$\|y_x-z\|^2 \leq C\|s\|^2+\frac{C}{\lambda^2}\left(\|y-\varphi\|^2+\|w\|^2\right)+\mathscr{R}. \tag{86}$$

Adding Eqs. (85), (86), using Lemmas 3.4 and 3.5 we get the required result. □

Lemma 3.7. *For every $\lambda \in \mathbb{R}\setminus\{0\}$ the inequality*

$$b_1\|\psi_x\|^2+b_2\|z_x\|^2 \leq C\left(1+\frac{1}{\lambda^2}\right)\|v\|^2+\frac{C}{\lambda^4}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+\frac{C}{|\lambda|^3}\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}+C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}, \tag{87}$$

holds for some structural constant $C>0$ independent of λ.

Proof. Multiplying (70) by $\overline{\psi}$ we obtain

$$i\lambda\rho_2\int_0^l v\overline{\psi}dx+b_1\|\psi_x\|^2-\kappa_1\int_0^l(\varphi_x-\psi)\overline{\psi}dx+\delta\int_0^l\theta_{xx}\overline{\psi}dx=\mathscr{R}.$$

From (69) we get

$$b_1\|\psi_x\|^2=\rho_2\|v\|^2+\kappa_1\int_0^l(\varphi_x-\psi)\overline{\psi}dx-\delta\int_0^l\theta_{xx}\overline{\psi}dx+\mathscr{R}. \tag{88}$$

Similarly multiplying Eq. (74) by $\bar{z}$ we get

$$b_2\|z_x\|^2=\rho_4\|w\|^2+\kappa_2\int_0^l(y_x-z)\bar{z}dx-\gamma_3\int_0^l w\bar{z}dx+\mathscr{R}. \tag{89}$$

Adding Eqs. (88), (89) it follows that

$$b_1\|\psi_x\|^2+b_2\|z_x\|^2\le C\|v\|^2+C\|w\|^2+C\|\varphi_x-\psi\|^2+C\|y_x-z\|^2+C\|\theta_{xx}\|^2+C\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}}.$$

Then, using Lemmas 3.4 and 3.6 our conclusion follows. □

Lemma 3.8. *For every $\lambda\in\mathbb{R}\setminus\{0\}$ the inequality*

$$\|v\|^2\le\varepsilon\|U\|^2_{\mathscr{H}}+C_\varepsilon\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}},\quad \textit{for all}\quad \varepsilon>0, \tag{90}$$

holds for some structural constant $C>0$ independent of λ.

Proof. Multiplying Eq. (75) by $\bar{v}$ and integrating over $[0,L]$ we have

$$i\lambda\rho_5\int_0^l\theta\bar{v}dx-K\int_0^l\theta_{xx}\overline{\psi}dx+\beta\|v\|^2=\mathscr{R}. \tag{91}$$

By using Eq. (70) that

$$\beta\|v\|^2=-\frac{\rho_5 b_1}{\rho_2}\int_0^l\theta_x\overline{\psi}_xdx+\frac{\rho_5\kappa_1}{\rho_2}\int_0^l\theta\overline{(\varphi_x-\psi)}dx+\frac{\rho_5\delta}{\rho_2}\|\theta_x\|^2-K\int_0^l\theta_x\overline{\psi}_xdx+\mathscr{R}. \tag{92}$$

By using Lemma 3.4 we have

$$\|v\|^2\le\varepsilon\|U\|^2_{\mathscr{H}}+C_\varepsilon\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}},\quad \text{for all}\quad \varepsilon>0. \tag{93}$$

□

Theorem 3.9. *The semigroup associated with the system (20)–(30) is exponentially stable regardless of any relationship between system parameters.*

Proof. In fact, using the inequalities of Lemmas 3.4–3.8, we obtain

$$\|U\|^2_{\mathscr{H}}\le\varepsilon C\|U\|^2_{\mathscr{H}}+C_\varepsilon\|U\|_{\mathscr{H}}\|F\|_{\mathscr{H}},\quad \text{for all}\quad \varepsilon>0, \tag{94}$$

since $|\lambda|>1$. We can conclude that there exist a positive constant C such that

$$\|U\|_{\mathscr{H}}\le C\|F\|_{\mathscr{H}}. \tag{95}$$

Then using the Prüss's result [24] one has the conclusion of theorem. □

4. NUMERICAL APPROACH

In the numerical approach of this work, we first want to validate the results of the numerical scheme given through the inhomogeneous problem, where the functions φ, ψ, y, z, θ are chosen to perform numerical simulations. After validating the results, simulations will be performed for the original homogeneous problem.

4.1. **Non-homogeneous problem**

Consider the following inhomogeneous system, where the external forces $f_i,\ i=1,\cdots,5$ are conveniently chosen in such a way that the exact solution of the system is known.

$$\rho_1\varphi_{tt}-\kappa_1\big(\varphi_x-\psi\big)_x-\mathscr{C}\big(y-\varphi\big)+\gamma_1\varphi_t = f_1(x,t) \quad \text{in} \quad (0,l)\times(0,T), \tag{96}$$

$$\rho_2\psi_{tt}-b_1\psi_{xx}-\kappa_1\big(\varphi_x-\psi\big)+\delta\theta_{xx} = f_2(x,t) \quad \text{in} \quad (0,l)\times(0,T), \tag{97}$$

$$\rho_3 y_{tt}-\kappa_2\big(y_x-z\big)_x+\mathscr{C}\big(y-\varphi\big)+\gamma_2 y_t = f_3(x,t) \quad \text{in} \quad (0,l)\times(0,T), \tag{98}$$

$$\rho_4 z_{tt}-b_2 z_{xx}-\kappa_2\big(y_x-z\big)+\gamma_3 z_t = f_4(x,t) \quad \text{in} \quad (0,l)\times(0,T), \tag{99}$$

$$\rho_5\theta_t-K\theta_{xx}+\beta\psi_t = f_5(x,t) \quad \text{in} \quad (0,l)\times(0,T), \tag{100}$$

with boundary conditions,

$$\varphi(0,t)=\varphi(l,t)=\psi(0,t)=\psi(l,t)=0, \quad \text{for all} \quad t\in[0,T], \tag{101}$$

$$y(0,t)=y(l,t)=z(0,t)=z(l,t)=\theta(0,t)=\theta(l,t)=0, \quad \text{for all} \quad t\in[0,T], \tag{102}$$

and the following initial data,

$$\varphi(x,0)=\varphi_0(x),\ \ \varphi_t(x,0)=\varphi_1(x), \psi(x,0)=\psi_0(x),\ \ \psi_t(x,0)=\psi_1(x), \quad x\in(0,l), \tag{103}$$

$$y(x,0)=y_0(x),\ \ y_t(x,0)=y_1(x),\ \ z(x,0)=z_0(x),\ \ z_t(x,0)=z_1(x), \quad x\in(0,l), \tag{104}$$

$$\theta(x,0)=\theta_0(x), \quad x\in(0,l). \tag{105}$$

4.1.1. *Non-homogeneous numerical scheme*

Let $\Delta x = l/(J+1)$ and $\Delta t = T/(N+1)$, for $J,N\in\mathbb{N}$ and we introduce the mesh:

$$0=x_0<x_1=\Delta x<\ldots<x_j=j\Delta x<\ldots<x_J<x_{J+1}=(J+1)\Delta x=l,$$
$$0=t_0<t_1=\Delta t<\ldots<t_n=n\Delta t<\ldots<t_N<t_{N+1}=(N+1)\Delta t=T,$$

for all $j=0,1,2,\ldots,J+1$ and $n=0,1,2,\ldots,N+1$.

Considering the system (96)–(105), our discrete problem consists of obtaining $(\varphi_j^n,\ \psi_j^n,\ y_j^n,\ z_j^n,\ \theta_j^n)$ such that

$$\rho_1\nabla_t\overline{\nabla}_t\varphi_j^n-\kappa_1\nabla_x\overline{\nabla}_x\varphi_j^n+\kappa_1\frac{\nabla_x+\overline{\nabla}_x}{2}\psi_j^n-\mathscr{C}(y_j^n-\varphi_j^n)+\gamma_1\frac{\nabla_t+\overline{\nabla}_t}{2}\varphi_j^n = \Upsilon_4(f_1)_j^n, \tag{106}$$

$$\rho_2\nabla_t\overline{\nabla}_t\psi_j^n-b_1\nabla_x\overline{\nabla}_x\psi_j^n-\kappa_1\frac{\nabla_x+\overline{\nabla}_x}{2}\varphi_j^n+\kappa_1\Upsilon_4\psi_j^n+\delta\nabla_x\overline{\nabla}_x\theta_j^n = \Upsilon_4(f_2)_j^n, \tag{107}$$

$$\rho_3\nabla_t\overline{\nabla}_t y_j^n-\kappa_2\nabla_x\overline{\nabla}_x y_j^n+\kappa_2\frac{\nabla_x+\overline{\nabla}_x}{2}z_j^n+\mathscr{C}(y_j^n-\varphi_j^n)+\gamma_2\frac{\nabla_t+\overline{\nabla}_t}{2}y_j^n = \Upsilon_4(f_3)_j^n, \tag{108}$$

$$\rho_4\nabla_t\overline{\nabla}_t z_j^n-b_2\nabla_x\overline{\nabla}_x z_j^n-\kappa_2\frac{\nabla_x+\overline{\nabla}_x}{2}y_j^n+\kappa_2\Upsilon_4 z_j^n+\gamma_3\frac{\nabla_t+\overline{\nabla}_t}{2}z_j^n = \Upsilon_4(f_4)_j^n, \tag{109}$$

$$\rho_5\overline{\nabla}_t\theta_j^n-k\nabla_x\overline{\nabla}_x\theta_j^n+\beta\frac{\nabla_t+\overline{\nabla}_t}{2}\psi_j^n = \Upsilon_4(f_5)_j^n, \tag{110}$$

for all $j=1,2,...,J$ and $n=1,2,...,N$, where we assume the following numerical operators of finite differences

$$\nabla_t u_j^n := \frac{u_j^{n+1}-u_j^n}{\Delta t}, \qquad \overline{\nabla}_t u_j^n := \frac{u_j^n-u_j^{n-1}}{\Delta t}, \qquad \nabla_t\overline{\nabla}_t u_j^n := \frac{u_j^{n+1}-2u_j^n+u_j^{n-1}}{\Delta t^2},$$

$$\nabla_x u_j^n := \frac{u_{j+1}^n-u_j^n}{\Delta x}, \qquad \overline{\nabla}_x u_j^n := \frac{u_j^n-u_{j-1}^n}{\Delta x}, \qquad \nabla_x\overline{\nabla}_x u_j^n := \frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2},$$

$$\frac{\nabla_t+\overline{\nabla}_t}{2}u_j^n := \frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}, \qquad \frac{\nabla_x+\overline{\nabla}_x}{2}u_j^n := \frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}, \qquad \Upsilon_4 u_j^n := \frac{u_{j+1}^n+2u_j^n+u_{j-1}^n}{4}.$$

For the initial conditions, we use

$$\varphi_j^0=\varphi_{0j}, \quad \frac{\varphi_j^1-\varphi_j^{-1}}{2\Delta t}=\varphi_{1j}, \quad \psi_j^0=\psi_{0j}, \quad \frac{\psi_j^1-\psi_j^{-1}}{2\Delta t}=\psi_{1j}, \quad j=1,2,...,J, \tag{111}$$

$$y_j^0=y_{0j}, \quad \frac{y_j^1-y_j^{-1}}{2\Delta t}=y_{1j}, \quad z_j^0=z_{0j}, \quad \frac{z_j^1-z_j^{-1}}{2\Delta t}=z_{1j}, \quad j=1,2,...,J, \tag{112}$$

$$\theta_j^n=\theta_{0j}, \quad j=1,2,...,J, \tag{113}$$

and for the boundary conditions we adopted

$$\varphi_0^n=\varphi_{J+1}^n=\psi_0^n=\psi_{J+1}^n=y_0^n=y_{J+1}^n=z_0^n=z_{J+1}^n=\theta_0^n=\theta_{J+1}^n=0, \tag{114}$$

for all $n=0,1,...,N+1$.

4.1.2. *Numerical simulation for non-homogeneous problem*

In this section, our objective is to study the numerical scheme (106)–(114) by displaying the results of numerical simulations that we carried out using the explicit method derived from (106)–(110) in finite differences in order to validate through simulations the analytical results established in the previous sections. For the non-homogeneous problem, any sufficiently regular function satisfying the conditions on the initial data may be the exact solution of the model. To do this, simply define the initial data and the external forces $f_i,\ i=1,\cdots,5$ the external function from the chosen function.

For this we assume that

$$\begin{aligned}
&\varphi(x,t)=\frac{3}{256}\left[\frac{l}{2}\cos\left(\frac{\nu\pi x}{l}\right)+x-\frac{l}{2}\right]e^{-t}, \quad \psi(x,t)=\frac{3}{400}\left[\frac{l^2}{2\pi}\sin\left(\frac{\nu\pi x}{l}\right)+\frac{x}{2}(x-l)\nu\right]e^{-t},\\
&y(x,t)=\frac{1}{64}\left[\frac{l}{2}\cos\left(\frac{\nu\pi x}{l}\right)+x-\frac{l}{2}\right]e^{-t}, \quad z(x,t)=\frac{1}{800}\left[\frac{l^2}{2\pi}\sin\left(\frac{\nu\pi x}{l}\right)+\frac{x}{2}(x-l)\nu\right]e^{-t},\\
&\theta(x,t)=\frac{1}{128}\sin\left(\frac{\nu\pi x}{l}\right)e^{-t}.
\end{aligned} \tag{115}$$

Let the following values for the coefficients

$$\begin{aligned}
&\rho_1=2.8500, \quad \rho_2=3.0115\cdot10^{-4}, \quad \rho_3=7.3550, \quad \rho_4=6.4956\cdot10^{-4},\\
&\kappa_1=2.0829\cdot10^1, \quad \kappa_2=5.5348\cdot10^1, \quad \mathscr{C}=3.7195\cdot10^{-8}, \quad b_1=2.2312\cdot10^{-3}, \quad b_2=4.9637\cdot10^{-3},\\
&\rho_5=1, \quad k=1.6, \quad \beta=4.6, \quad \delta=1.2, \quad \gamma_1=2\cdot10^1, \quad \gamma_2=1.2\cdot10^1, \quad \gamma_3=3.2\cdot10^1.
\end{aligned} \tag{116}$$

The numerical solutions are computed for different discretizations and the corresponding numerical errors are calculated in the $L^\infty\big(0,T;L^2(0,1)\big)$ norm. Thus, one can compare the approximation error between the approximate and exact solutions and calculate the convergence order. The discrete norm is defined:

$$E = \|E\|_{L^\infty(0,T,L^2(0,1))} = \max_{t_n \in [0,T]} \left(\sum_{j=2}^{m} h_j |E_j^n|^2 \right)^{1/2}, \tag{117}$$

where E_j is the error associated with mesh $h_j = x_{j+1} - x_j$.

Moreover, we consider two consecutive computations for the same problem with discrete error norms E_j and E_{j+1} calculated according to (117), for $j = 1, 2, \ldots$, where j is related to the space mesh size. Thus, the convergence order p is defined as $p = \ln(\alpha_j)/\ln(h_j/h_{j+1})$, where $\alpha_j = E_j/E_{j+1}$. Note that when the space mesh has the value 2^{-j}, we have $p = \ln(\alpha_j)/\ln(2)$.

The Tables 1 and 2 show the errors for $l = 1$, $T = 2$ and $\nu = 1$ in the $L^\infty(0,T;L^2(0,1))$ norm and the convergence orders:

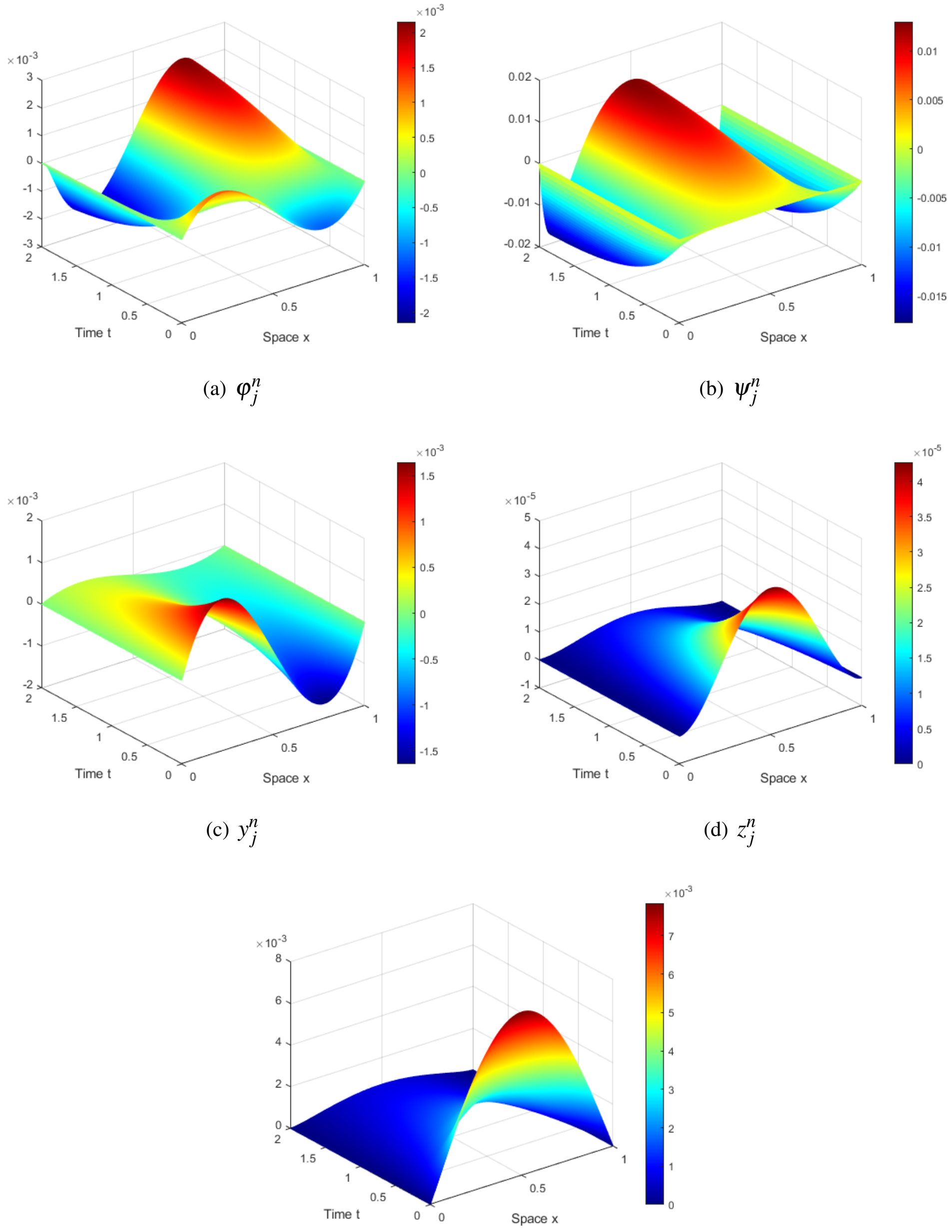
Δx	Δt	E_φ	E_ψ	E_y	E_z	E_θ
2^{-6}	2^{-10}	0.017306	0.120886	7.85e-07	2.10e-06	0.0136003
2^{-7}	2^{-11}	0.007397	0.047986	1.96e-07	5.21e-07	0.0066325
2^{-8}	2^{-12}	0.003464	0.022501	4.88e-08	1.28e-07	0.0032205
2^{-9}	2^{-13}	0.001614	0.010514	1.24e-08	3.11e-08	0.0015460

TABLE 1. Non-homogeneous problem - Errors

Δx	Δt	p_φ	p_ψ	p_y	p_z	p_θ
2^{-6}	2^{-10}	——	——	——	——	——
2^{-7}	2^{-11}	1.226359	1.332948	2.003855	2.009805	1.036021
2^{-8}	2^{-12}	1.094483	1.092628	2.002928	2.023216	1.042244
2^{-9}	2^{-13}	1.101692	1.097621	1.975316	2.044120	1.058762

TABLE 2. Non-homogeneous problem - Convergence orders

The graphs for numerical solutions at $\Delta x = 2^{-9}$ and $\Delta t = 2^{-13}$ are shown in the Figure 2.



(a) φ_j^n (b) ψ_j^n

(c) y_j^n (d) z_j^n

(e) θ_j^n

FIGURE 2. Non-homogeneous problem - Solution

4.2. **Homogeneous numerical scheme**

For the original homogeneous problem, that is, when $f_i = 0,\ i = 1, \cdots, 5$, we do not know the exact solution. In this case, a standard procedure is to consider as an "exact solution" a numerical solution generated with a sufficiently refined discretization of space and time.

In fact, consider the system

$$\rho_1\nabla_t\overline{\nabla}_t\varphi_j^n-\kappa_1\nabla_x\overline{\nabla}_x\varphi_j^n+\kappa_1\frac{\nabla_x+\overline{\nabla}_x}{2}\psi_j^n-\mathscr{C}(y_j^n-\varphi_j^n)+\gamma_1\frac{\nabla_t+\overline{\nabla}_t}{2}\varphi_j^n = 0, \quad (118)$$

$$\rho_2\nabla_t\overline{\nabla}_t\psi_j^n-b_1\nabla_x\overline{\nabla}_x\psi_j^n-\kappa_1\frac{\nabla_x+\overline{\nabla}_x}{2}\varphi_j^n+\kappa_1\Upsilon_4\psi_j^n+\delta\nabla_x\overline{\nabla}_x\theta_j^n = 0, \quad (119)$$

$$\rho_3\nabla_t\overline{\nabla}_t y_j^n-\kappa_2\nabla_x\overline{\nabla}_x y_j^n+\kappa_2\frac{\nabla_x+\overline{\nabla}_x}{2}z_j^n+\mathscr{C}(y_j^n-\varphi_j^n)+\gamma_2\frac{\nabla_t+\overline{\nabla}_t}{2}y_j^n = 0, \quad (120)$$

$$\rho_4\nabla_t\overline{\nabla}_t z_j^n-b_2\nabla_x\overline{\nabla}_x z_j^n-\kappa_2\frac{\nabla_x+\overline{\nabla}_x}{2}y_j^n+\kappa_2\Upsilon_4 z_j^n+\gamma_3\frac{\nabla_t+\overline{\nabla}_t}{2}z_j^n = 0, \quad (121)$$

$$\rho_5\overline{\nabla}_t\theta_j^n-k\nabla_x\overline{\nabla}_x\theta_j^n+\beta\frac{\nabla_t+\overline{\nabla}_t}{2}\psi_j^n = 0, \quad (122)$$

for all $j=1,2,...,J$ and $n=1,2,...,N$. For the initial conditions, we use

$$\varphi_j^0=\varphi_{0j},\quad \frac{\varphi_j^1-\varphi_j^{-1}}{2\Delta t}=\varphi_{1j},\quad \psi_j^0=\psi_{0j},\quad \frac{\psi_j^1-\psi_j^{-1}}{2\Delta t}=\psi_{1j},\quad j=1,2,...,J, \quad (123)$$

$$y_j^0=y_{0j},\quad \frac{y_j^1-y_j^{-1}}{2\Delta t}=y_{1j},\quad z_j^0=z_{0j},\quad \frac{z_j^1-z_j^{-1}}{2\Delta t}=z_{1j},\quad j=1,2,...,J, \quad (124)$$

$$\theta_j^0=\theta_{0j},\quad j=1,2,...,J, \quad (125)$$

and for the boundary conditions we adopted

$$\varphi_0^n=\varphi_{J+1}^n=\psi_0^n=\psi_{J+1}^n=y_0^n=y_{J+1}^n=z_0^n=z_{J+1}^n=\theta_0^n=\theta_{J+1}^n=0, \quad (126)$$

for all $n=1,2,...,N$.

The discrete energy associated with the numerical scheme (118)–(126) is given by

$$\begin{aligned}\mathbb{E}^n := \frac{\Delta x}{2}\sum_{j=0}^{J}\Bigg[&\rho_1\left(\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t}\right)^2+\rho_2\left(\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t}\right)^2+\rho_3\left(\frac{y_j^{n+1}-y_j^n}{\Delta t}\right)^2+\rho_4\left(\frac{z_j^{n+1}-z_j^n}{\Delta t}\right)^2\\ &+b_1\left(\frac{\psi_{j+1}^{n+1}-\psi_j^{n+1}}{\Delta x}\right)\left(\frac{\psi_{j+1}^n-\psi_j^n}{\Delta x}\right)+b_2\left(\frac{z_{j+1}^{n+1}-z_j^{n+1}}{\Delta x}\right)\left(\frac{z_{j+1}^n-z_j^n}{\Delta x}\right)\\ &+\kappa_1\left(\frac{\varphi_{j+1}^{n+1}-\varphi_j^{n+1}}{\Delta x}-\frac{\psi_{j+1}^{n+1}+\psi_j^{n+1}}{2}\right)\left(\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}-\frac{\psi_{j+1}^n+\psi_j^n}{2}\right)\\ &+\kappa_2\left(\frac{y_{j+1}^{n+1}-y_j^{n+1}}{\Delta x}-\frac{z_{j+1}^{n+1}+z_j^{n+1}}{2}\right)\left(\frac{y_{j+1}^n-y_j^n}{\Delta x}-\frac{z_{j+1}^n+z_j^n}{2}\right)\\ &+\mathscr{C}\left(y_j^{n+1}-\varphi_j^{n+1}\right)\left(y_j^n-\varphi_j^n\right)+\frac{\rho_5\delta}{\beta}\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\right)^2\Bigg]. \end{aligned} \quad (127)$$

The following theorem deals with this result.

Theorem 4.1. *Let* $(\varphi_j^n,\psi_j^n,y_j^n,z_j^n,\theta_j^n)$ *the numerical solution of the problem (118)–(126). Thus, for all* $\Delta x,\Delta t\in(0,1)$ *the rate of change of the numerical energy* $\mathbb{E}^n$ *in* (127) *at the instant* t_n *is given by*

$$\begin{aligned}\frac{\mathbb{E}^n-\mathbb{E}^{n-1}}{\Delta t} \quad \leq \quad & -\gamma_1\Delta x\sum_{j=1}^{J}\left|\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right|^2-\gamma_2\Delta x\sum_{j=1}^{J}\left|\frac{y_j^{n+1}-y_j^{n-1}}{2\Delta t}\right|^2-\gamma_3\Delta x\sum_{j=1}^{J}\left|\frac{z_j^{n+1}-z_j^{n-1}}{2\Delta t}\right|^2\\ & -\frac{\delta k}{\beta}\Delta x\sum_{j=1}^{J}\left|\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right|^2\leq 0 \qquad \textit{for all} \qquad n=1,2,\ldots,N. \end{aligned} \tag{128}$$

Proof. Multiplying the Eq. (118) by $\Delta x\Big(\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\Big)$ and adding to $j=1,2,\ldots,J$ we have

$$\begin{aligned}&\rho_1\Delta x\underbrace{\sum_{j=1}^{J}\left(\frac{\varphi_j^{n+1}-2\varphi_j^n+\varphi_j^{n-1}}{\Delta t^2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right)}_{\mathscr{R}_1^n:=}-\kappa_1\Delta x\underbrace{\sum_{j=1}^{J}\left(\frac{\varphi_{j+1}^n-2\varphi_j^n+\varphi_{j-1}^n}{\Delta x^2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right)}_{\mathscr{R}_2^n:=}\\ &+\kappa_1\Delta x\underbrace{\sum_{j=1}^{J}\left(\frac{\psi_{j+1}^n-\psi_{j-1}^n}{2\Delta x}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right)}_{\mathscr{R}_3^n:=}-\mathscr{C}\frac{\Delta x}{2\Delta t}\underbrace{\sum_{j=1}^{J}(y_j^n-\varphi_j^n)(\varphi_j^{n+1}-\varphi_j^{n-1})}_{\mathscr{R}_4^n:=}\\ &+\gamma_1\Delta x\sum_{j=1}^{J}\left|\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right|^2=0.\end{aligned} \tag{129}$$

Making some simplifications in the terms of (129) we have

$$\begin{aligned}\mathscr{R}_1^n \quad :=\quad & \sum_{j=1}^{J}\left(\frac{\varphi_j^{n+1}-2\varphi_j^n+\varphi_j^{n-1}}{\Delta t^2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right)\\ =\quad & \frac{1}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta t}\right)-\frac{1}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\varphi_j^n-\varphi_j^{n-1}}{\Delta t}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta t}\right)\\ =\quad & \frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t}\right)^2-\frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_j^n-\varphi_j^{n-1}}{\Delta t}\right)^2.\end{aligned} \tag{130}$$

Similarly, after some simplifications, we conclude that

$$\begin{aligned}\mathscr{R}_2^n \quad :=\quad & \sum_{j=1}^{J}\left(\frac{\varphi_{j+1}^n-2\varphi_j^n+\varphi_{j-1}^n}{\Delta x^2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right)\\ =\quad & \frac{1}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right)-\frac{1}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\varphi_j^n-\varphi_{j-1}^n}{\Delta x}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right)\\ =\quad & \frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right)-\frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}\,\frac{\varphi_{j+1}^{n+1}-\varphi_{j+1}^{n-1}}{\Delta x}\right)\\ =\quad & -\frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^{n+1}-\varphi_j^{n+1}}{\Delta x}\,\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}-\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}\,\frac{\varphi_j^{n-1}-\varphi_j^{n-1}}{\Delta x}\right).\end{aligned} \tag{131}$$

We also have

$$
\begin{aligned}
\mathscr{R}_3^n &:= \sum_{j=1}^{J}\left(\frac{\psi_{j+1}^n-\psi_{j-1}^n}{2\Delta x}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right)\\
&= \frac{1}{2\Delta t}\sum_{j=1}^{J}\left[\frac{(\psi_{j+1}^n+\psi_j^n)-(\psi_j^n+\psi_{j-1}^n)}{2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right]\\
&= \frac{1}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\psi_{j+1}^n+\psi_j^n}{2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right)-\frac{1}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\psi_j^n+\psi_{j-1}^n}{2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right)\\
&= \frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^n+\psi_j^n}{2}\,\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{\Delta x}\right)-\frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^n+\psi_j^n}{2}\,\frac{\varphi_{j+1}^{n+1}-\varphi_{j+1}^{n-1}}{\Delta x}\right)\\
&= -\frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^n+\psi_j^n}{2}\,\frac{\varphi_{j+1}^{n+1}-\varphi_j^{n+1}}{\Delta x}\right)+\frac{1}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^n+\psi_j^n}{2}\,\frac{\varphi_{j+1}^{n-1}-\varphi_{j+1}^{n-1}}{\Delta x}\right).
\end{aligned}
$$

From $\mathscr{R}_4^n$ we have

$$
\mathscr{R}_4^n := \sum_{j=1}^{J}(y_j^n-\varphi_j^n)(\varphi_j^{n+1}-\varphi_j^{n-1}) = \sum_{j=0}^{J}(y_j^n-\varphi_j^n)\varphi_j^{n+1}-\sum_{j=0}^{J}(y_j^n-\varphi_j^n)\varphi_j^{n-1}.
$$

Replacing (130), (131), (132) and (132) in the Eq. (129) we arrive at

$$
\begin{aligned}
&\rho_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t}\right)^2-\rho_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_j^{n}-\varphi_j^{n-1}}{\Delta t}\right)^2+\mathscr{C}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}(y_j^n-\varphi_j^n)\varphi_j^{n+1}\\
&+\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}-\frac{\psi_{j+1}^n+\psi_j^n}{2}\right)\frac{\varphi_{j+1}^{n+1}-\varphi_j^{n+1}}{\Delta x}-\mathscr{C}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}(y_j^n-\varphi_j^n)\varphi_j^{n-1}\\
&-\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_j^n}{\Delta x}-\frac{\psi_j^n+\psi_{j+1}^n}{2}\right)\frac{\varphi_{j+1}^{n-1}-\varphi_j^{n-1}}{\Delta x}\\
&+\gamma_1\Delta x\sum_{j=1}^{J}\left|\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right|^2=0. \qquad (132)
\end{aligned}
$$

Analogously, multiplying the Eq. (119) by $\Delta x\left(\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)$ we obtain

$$
\begin{aligned}
&\rho_2\Delta x\sum_{j=1}^{J}\left(\frac{\psi_j^{n+1}-2\psi_j^n+\psi_j^{n-1}}{\Delta t^2}\,\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)-b_1\Delta x\sum_{j=1}^{J}\left(\frac{\psi_{j+1}^n-2\psi_j^n+\psi_{j-1}^n}{\Delta x^2}\,\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)\\
&-\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_{j-1}^n}{\Delta x}\,\frac{\psi_j^{n+1}+\psi_j^n}{2}\right)+\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=1}^{J}\left(\frac{\varphi_{j+1}^n-\varphi_{j-1}^n}{\Delta x}\,\frac{\psi_j^{n}+\psi_j^{n-1}}{2}\right)\\
&+\kappa_1\Delta x\sum_{j=1}^{J}\left(\frac{\psi_{j+1}^n+\psi_j^n}{4}\,\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)+\kappa_1\Delta x\sum_{j=1}^{J}\left(\frac{\psi_j^n+\psi_{j-1}^n}{4}\,\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)\\
&+\delta\Delta x\sum_{j=1}^{J}\left(\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\,\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)=0.
\end{aligned}
$$

Consequently,

$$\begin{aligned}
&\rho_2 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t}\right)^2-\rho_2 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_j^{n}-\psi_j^{n-1}}{\Delta t}\right)^2+b_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^{n+1}-\psi_j^{n+1}}{\Delta x}\frac{\psi_{j+1}^{n}-\psi_j^{n}}{\Delta x}\right)\\
&-b_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^{n}-\psi_j^{n}}{\Delta x}\frac{\psi_{j+1}^{n-1}-\psi_j^{n-1}}{\Delta x}\right)-\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^{n}-\varphi_j^{n}}{\Delta x}-\frac{\psi_{j+1}^{n}+\psi_j^{n}}{2}\right)\frac{\psi_{j+1}^{n+1}-\psi_j^{n+1}}{2}\\
&+\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^{n}-\varphi_j^{n}}{\Delta x}-\frac{\psi_{j+1}^{n}+\psi_j^{n}}{2}\right)\frac{\psi_{j+1}^{n-1}-\psi_j^{n-1}}{2}+\delta\Delta x\sum_{j=1}^{J}\left(\frac{\theta_{j+1}^{n}-2\theta_j^{n}+\theta_{j-1}^{n}}{\Delta x^2}\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)=0. \qquad (133)
\end{aligned}$$

Adding Eqs. (132) and (133) we have

$$\begin{aligned}
&\rho_1 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_j^{n+1}-\varphi_j^n}{\Delta t}\right)^2-\rho_1 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_j^{n}-\varphi_j^{n-1}}{\Delta t}\right)^2+\rho_2 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_j^{n+1}-\psi_j^n}{\Delta t}\right)^2\\
&-\rho_2 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_j^{n}-\psi_j^{n-1}}{\Delta t}\right)^2-\mathscr{C}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}(y_j^n-\varphi_j^n)\varphi_j^{n+1}+\mathscr{C}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}(y_j^n-\varphi_j^n)\varphi_j^{n-1}\\
&+b_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^{n+1}-\psi_j^{n+1}}{\Delta x}\frac{\psi_{j+1}^{n}-\psi_j^{n}}{\Delta x}\right)-b_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\psi_{j+1}^{n}-\psi_j^{n}}{\Delta x}\frac{\psi_{j}^{n-1}-\psi_j^{n-1}}{\Delta x}\right)\\
&+\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^{n}-\varphi_j^{n}}{\Delta x}-\frac{\psi_{j+1}^{n}+\psi_j^{n}}{2}\right)\left(\frac{\varphi_{j+1}^{n+1}-\varphi_j^{n+1}}{\Delta x}-\frac{\psi_{j+1}^{n+1}-\psi_j^{n+1}}{2}\right)\\
&-\kappa_1\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{\varphi_{j+1}^{n}-\varphi_j^{n}}{\Delta x}-\frac{\psi_{j}^{n}+\psi_{j+1}^{n}}{2}\right)\left(\frac{\varphi_{j+1}^{n-1}-\varphi_j^{n-1}}{\Delta x}-\frac{\psi_{j+1}^{n-1}-\psi_j^{n-1}}{2}\right)\\
&+\delta\Delta x\sum_{j=1}^{J}\left(\frac{\theta_{j+1}^{n}-2\theta_j^{n}+\theta_{j-1}^{n}}{\Delta x^2}\frac{\psi_j^{n+1}-\psi_j^{n-1}}{2\Delta t}\right)+\gamma_1\Delta x\sum_{j=1}^{J}\left|\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right|^2=0. \qquad (134)
\end{aligned}$$

Proceeding in a similar way for Eqs. (120) and (121), we have

$$\begin{aligned}
&\rho_3 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{y_j^{n+1}-y_j^n}{\Delta t}\right)^2-\rho_3 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{y_j^{n}-y_j^{n-1}}{\Delta t}\right)^2+\rho_4 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{z_j^{n+1}-z_j^n}{\Delta t}\right)^2\\
&-\rho_4 \frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{z_j^{n}-z_j^{n-1}}{\Delta t}\right)^2+\mathscr{C}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}(y_j^n-\varphi_j^n)y_j^{n+1}-\mathscr{C}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}(y_j^n-\varphi_j^n)y_j^{n-1}\\
&+b_2\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{z_{j+1}^{n+1}-z_j^{n+1}}{\Delta x}\frac{z_{j+1}^{n}-z_j^{n}}{\Delta x}\right)-b_2\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{z_{j+1}^{n}-z_j^{n}}{\Delta x}\frac{z_{j}^{n-1}-z_j^{n-1}}{\Delta x}\right)\\
&+\kappa_2\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{y_{j+1}^{n}-y_j^{n}}{\Delta x}-\frac{z_{j+1}^{n}+z_j^{n}}{2}\right)\left(\frac{y_{j+1}^{n+1}-y_j^{n+1}}{\Delta x}-\frac{z_{j+1}^{n+1}-z_j^{n+1}}{2}\right)\\
&-\kappa_2\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{y_{j+1}^{n}-y_j^{n}}{\Delta x}-\frac{z_{j}^{n}+z_{j+1}^{n}}{2}\right)\left(\frac{y_{j+1}^{n-1}-y_j^{n-1}}{\Delta x}-\frac{z_{j+1}^{n-1}-z_j^{n-1}}{2}\right)\\
&+\gamma_2\Delta x\sum_{j=1}^{J}\left|\frac{y_j^{n+1}-y_j^{n-1}}{2\Delta t}\right|^2+\gamma_3\Delta x\sum_{j=1}^{J}\left|\frac{z_j^{n+1}-z_j^{n-1}}{2\Delta t}\right|^2=0. \qquad (135)
\end{aligned}$$

Now, multiplying the Eq. (122) by $-\frac{\delta\Delta x}{\beta}\left(\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right)$ and adding to $j=1,...,J$, we have

$$-\frac{\rho_5\delta}{\beta}\Delta x\underbrace{\sum_{j=1}^{J}\left(\frac{\theta_j^n-\theta_j^{n-1}}{\Delta t}\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right)}_{\mathscr{R}_5^n:=}+\frac{\delta k}{\beta}\Delta x\sum_{j=1}^{J}\left|\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right|^2$$
$$-\delta\Delta x\sum_{j=1}^{J}\left(\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\frac{v_j^{n+1}-v_j^{n-1}}{2\Delta t}\right)=0.$$

From $\mathscr{R}_5^n$ we have

$$\begin{aligned}
\mathscr{R}_5^n &:= \sum_{j=1}^{J}\left(\frac{\theta_j^n-\theta_j^{n-1}}{\Delta t}\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right)\\
&= \frac{1}{\Delta t}\sum_{j=1}^{J}\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\frac{\theta_j^n-\theta_j^{n-1}}{\Delta x}\right)-\frac{1}{\Delta t}\sum_{j=1}^{J}\left(\frac{\theta_j^n-\theta_{j-1}^n}{\Delta x}\frac{\theta_j^n-\theta_j^{n-1}}{\Delta x}\right)\\
&= \frac{1}{\Delta t}\sum_{j=0}^{J}\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\frac{\theta_j^n-\theta_j^{n-1}}{\Delta x}\right)-\frac{1}{\Delta t}\sum_{j=0}^{J}\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\frac{\theta_{j+1}^n-\theta_{j+1}^{n-1}}{\Delta x}\right)\\
&= -\frac{1}{\Delta t}\sum_{j=0}^{J}\left[\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\right)^2-\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\frac{\theta_{j+1}^{n-1}-\theta_j^{n-1}}{\Delta x}\right)\right]\\
&\leq -\frac{1}{2\Delta t}\sum_{j=0}^{J}\left[\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\right)^2-\left(\frac{\theta_{j+1}^{n-1}-\theta_j^{n-1}}{\Delta x}\right)^2\right].
\end{aligned} \tag{136}$$

Replacing (136) in the Eq. (134) we arrive at

$$\frac{\rho_5\delta}{\beta}\frac{\Delta x}{2\Delta t}\sum_{j=0}^{J}\left[\left(\frac{\theta_{j+1}^n-\theta_j^n}{\Delta x}\right)^2-\left(\frac{\theta_{j+1}^{n-1}-\theta_j^{n-1}}{\Delta x}\right)^2\right]+\frac{\delta k}{\beta}\Delta x\sum_{j=1}^{J}\left|\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right|^2$$
$$-\delta\Delta x\sum_{j=1}^{J}\left(\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\frac{v_j^{n+1}-v_j^{n-1}}{2\Delta t}\right)\leq 0. \tag{137}$$

Now, adding the Eqs. (132), (133), (134), (135) and (137) we get

$$\begin{aligned}
\frac{\mathbb{E}^n-\mathbb{E}^{n-1}}{\Delta t} \leq\ & -\gamma_1\Delta x\sum_{j=1}^{J}\left|\frac{\varphi_j^{n+1}-\varphi_j^{n-1}}{2\Delta t}\right|^2-\gamma_2\Delta x\sum_{j=1}^{J}\left|\frac{y_j^{n+1}-y_j^{n-1}}{2\Delta t}\right|^2-\gamma_3\Delta x\sum_{j=1}^{J}\left|\frac{z_j^{n+1}-z_j^{n-1}}{2\Delta t}\right|^2\\
&-\frac{\delta k}{\beta}\Delta x\sum_{j=1}^{J}\left|\frac{\theta_{j+1}^n-2\theta_j^n+\theta_{j-1}^n}{\Delta x^2}\right|^2 \quad \text{for all} \quad n=1,2,...,N.
\end{aligned} \tag{138}$$

□

4.2.1. **Numerical simulation for homogeneous problem**

For initial conditions, we assume that

$$
\begin{aligned}
&\varphi_{0j} = \frac{3}{256}\left[\frac{l}{2}\cos\left(\frac{\nu\pi x_j}{l}\right) + x_j - \frac{l}{2}\right], \quad \varphi_{1j} = -\frac{3}{256}\left[\frac{l}{2}\cos\left(\frac{\nu\pi x_j}{l}\right) + x_j - \frac{l}{2}\right], \quad \text{for all} \quad \nu \in \mathbb{N},\\
&\psi_{0j} = \frac{3}{400}\left[\frac{l^2}{2\pi}\sin\left(\frac{\nu\pi x_j}{l}\right) + \frac{x_j}{2}(x_j - l)\nu\right], \quad \psi_{1j} = -\frac{3}{400}\left[\frac{l^2}{2\pi}\sin\left(\frac{\nu\pi x_j}{l}\right) + \frac{x_j}{2}(x_j - l)\nu\right], \quad \text{for all} \quad \nu \in \mathbb{N},\\
&y_{0j} = \frac{1}{64}\left[\frac{l}{2}\cos\left(\frac{\nu\pi x_j}{l}\right) + x_j - \frac{l}{2}\right], \quad y_{1j} = -\frac{1}{64}\left[\frac{l}{2}\cos\left(\frac{\nu\pi x_j}{l}\right) + x_j - \frac{l}{2}\right], \quad \text{for all} \quad \nu \in \mathbb{N},\\
&z_{0j} = \frac{1}{800}\left[\frac{l^2}{2\pi}\sin\left(\frac{\nu\pi x_j}{l}\right) + \frac{x_j}{2}(x_j - l)\nu\right], \quad z_{1j} = -\frac{1}{800}\left[\frac{l^2}{2\pi}\sin\left(\frac{\nu\pi x_j}{l}\right) + \frac{x_j}{2}(x_j - l)\nu\right], \quad \text{for all} \quad \nu \in \mathbb{N},\\
&\theta_{0j} = \frac{1}{128}\sin\left(\frac{\nu\pi x_j}{l}\right), \quad \text{for all} \quad \nu \in \mathbb{N}.
\end{aligned} \tag{139}
$$

In the Tables 3 and 4, we are showing the behavior of the approximate numerical solution error and the convergence order, fixing a sufficiently refined mesh $\Delta x = 2^{-10}$ and $\Delta t = 2^{-14}$ and comparing the error behavior and the order of convergence between the approximate solutions in a sequence of refined meshes in space and time, respectively, for $l = 1$, $T = 2$ and $\nu = 1$ in the $L^\infty(0,T;L^2(0,1))$ norm.

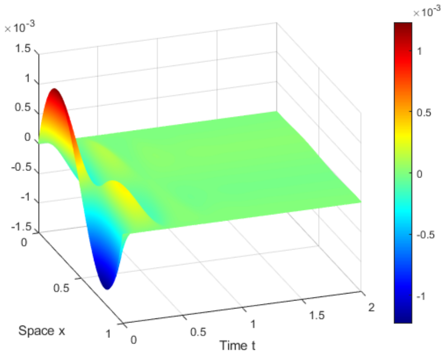
Δx	Δt	E_φ	E_ψ	E_y	E_z	E_θ
2^{-6}	2^{-10}	8.85e-06	0.000138	9.20e-06	1.12e-05	0.000365
2^{-7}	2^{-11}	3.67e-06	7.24e-05	4.19e-06	7.47e-06	0.000266
2^{-8}	2^{-12}	1.12e-06	3.70e-05	1.73e-06	2.65e-06	0.000160
2^{-9}	2^{-13}	2.53e-07	1.52e-05	5.61e-07	6.69e-07	6.77e-05

TABLE 3. Homogeneous problem - Errors

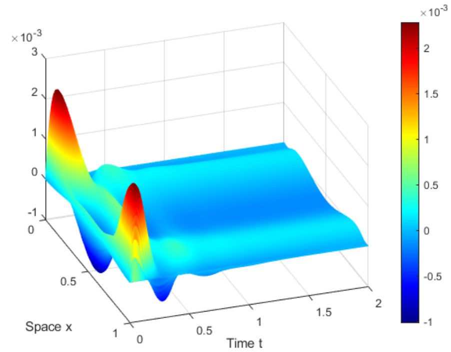
Δx	Δt	p_φ	p_ψ	p_y	p_z	p_θ
2^{-6}	2^{-10}	——	——	——	——	——
2^{-7}	2^{-11}	1.271102	0.9327397	1.134103	0.590049	0.4584155
2^{-8}	2^{-12}	1.712556	0.9667707	1.274473	1.492280	0.7325816
2^{-9}	2^{-13}	2.145142	1.282408	1.627195	1.988789	1.242428

TABLE 4. Homogeneous problem - Convergence orders

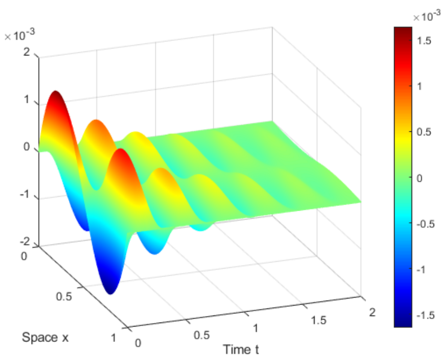
The graphs for numerical solutions are shown in the Figure 3.



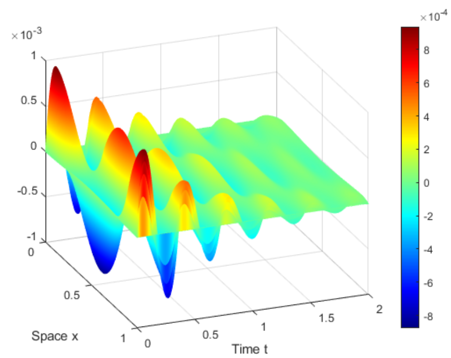
(a) φ_j^n



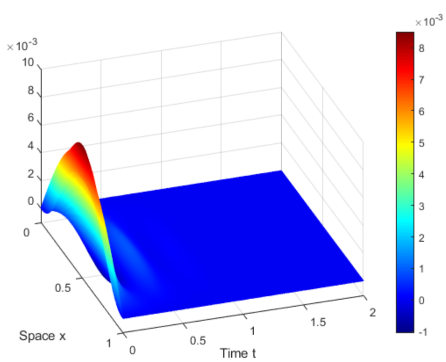
(b) ψ_j^n



(c) y_j^n



(d) z_j^n



(e) θ_j^n

FIGURE 3. Homogeneous problem - Solution

The Figure 4 shows the energy decay and its logarithm for $T = 2$ and $\Delta x = 2^{-9}$, $\Delta t = 2^{-13}$.

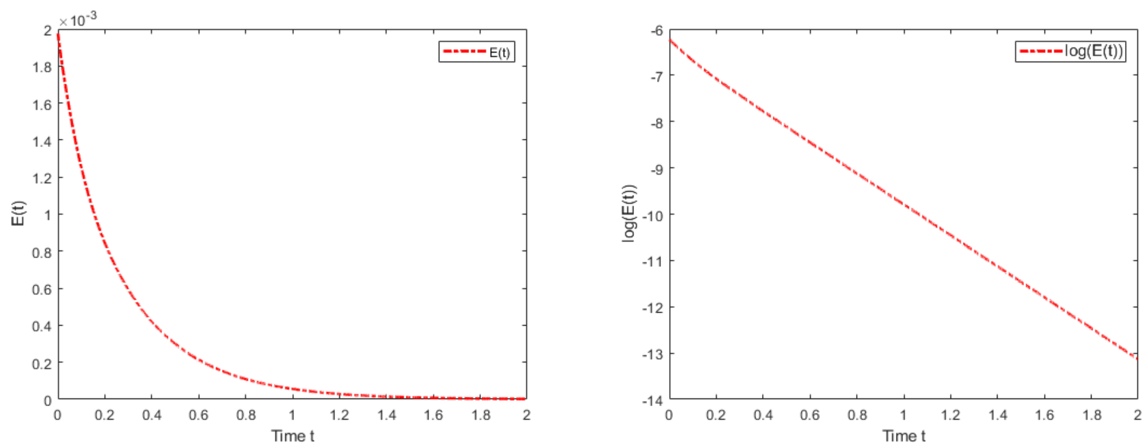


FIGURE 4. Homogeneous problem - Exponential decay

Acknowledgment:

The authors A. J. A. Ramos, M. A. Rincon and M. M. Freitas are partially supported by CNPQ- Brazil and M. A. Rincon is also partially supported by FAPERJ- Brazil.

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