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THE NONCONFORMING VIRTUAL ELEMENT METHOD FOR OSEEN'S EQUATION USING A STREAM-FUNCTION FORMULATION

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Abstract. We approximate the solution of the stream function formulation of the Oseen equations on general domains by designing a nonconforming Morley-type virtual element method. Under a suitable assumption on the continuous problem's coefficients, the discrete scheme is well-posed. By introducing an *enriching operator*, we derive an *a priori* estimate of the error in a discrete H^2 norm. By post-processing the discrete stream function, we compute the discrete velocity and vorticity fields. Furthermore, we recover an approximate pressure field by solving a Stokes-like problem in a nonconforming *Crouzeix–Raviart*-type virtual element space that is in a Stokes-complex relation with the Morley-type space of the virtual element approximation. Finally, we confirm our theoretical estimates by solving benchmark problems that include a convex and a nonconvex domain.

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1. Introduction

The incompressible Oseen equations are a linearized version of the Navier–Stokes equations. For the velocity vector field $\boldsymbol{u}$ and the scalar pressure field p they read as

$$-\nu\Delta\boldsymbol{u} + (\nabla\boldsymbol{u})\boldsymbol{\beta} + \gamma\boldsymbol{u} + \nabla p = f \quad \text{in } \Omega, \tag{1.1a}$$

$$\operatorname{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma, \tag{1.1c}$$

$$(p,1) = 0, \tag{1.1d}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2$, is a simply connected domain with boundary Γ, $\nu > 0$ is the viscosity coefficient, $\boldsymbol{\beta}$ is the convection vector field, γ is a suitable positive coefficient, and $(p,1)$ in (1.1d) stands for the integral of p over Ω. Hereafter, we consider a polygonal domain Ω, and, for simplicity of exposition, we assume that ν, $\boldsymbol{\beta}$ and γ are constant fields.

Several types of numerical schemes were designed and proposed in the finite element literature to approximate the Oseen equations. See for instance, to the works of references [5–7, 12, 13, 26, 30, 31] for a presentation of

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such schemes and their convergence analysis. In most of these articles, the Authors propose mixed schemes to approximate the velocity–pressure pair $(\boldsymbol{u}, p)$. Instead, following [8], we consider a streamline formulation and we propose and analyze a nonconforming Virtual Element Method (VEM). This combination yields a very effective way of solving the Oseen equations. In fact, in the one hand, the stream function formulation reduces (1.1a)–(1.1d) to a scalar problem that does not contain the pressure variable. The incompressibility condition is automatically satisfied by setting the velocity equal to the curl of the stream function. On the other hand, the VEM [10, 14] is a very robust and accurate discretization technique for solving partial differential equations (PDEs) on unstructured polygonal and polyhedral meshes admitting very general shaped (even nonconvex and degenerate) elements [49–51], elements with small edges and faces [28], and elements with curved faces [19, 20]. An incomplete list of works pertinent with the topic of this paper includes conforming VEMs for the Stokes equations [8,16,21,24,40,45,46], the Navier–Stokes equations [1,17,18], the Oseen equations [6,22,48], the Darcy and Brinkman equations [47,52]. The VEM for the Navier–Stokes equations is also applied in the discretization of 2D and 3D magnetohydrodynamics models, *cf.* [23].

The nonconforming VEM was initially proposed in [11] for elliptic problems, later extended to Stokes equations [32, 55], eigenvalue problems [37, 38, 41], plate bending problems [53, 54], biharmonic equations [9] and, eventually, generalized to polyharmonic problems in any number of spatial dimensions [39].

We summarize our contributions to the technical literature of virtual elements and numerical approximation of Oseen equations as follows:

– we design and investigate a Morley-type nonconforming VEM to approximate the stream function formulation of the Oseen equations on generally shaped domain and unstructured polygonal meshes;
– we derive an *a priori* error estimate for the discretization of the Oseen equations with less regular exact solutions by introducing an enriching operator remapping the nonconforming virtual element space onto a conforming counterpart;
– we discuss a post-processing technique for velocity and vorticity based on computable projection operators;
– we recover the pressure variable by solving a Stokes-like problem using a Crouzeix–Raviart-like space in a Stokes complex relation with the Morley-type nonconforming space.

While a post-processing technique provides the velocity and vorticity fields, pressure recovery is not straightforward and deserves special attention. Notably, the pressure recovery technique proposed in [47] is inadequate for our approach since it relies on a C^1-regular virtual element formulation, and requires at least $H^3(\Omega)$ regularity of the exact solution. This regularity is usually not available if the model problem is formulated over a nonconvex domain. Therefore, inspired by related works on FEM, see [35, 36], we recover the pressure variable from the discrete stream function by solving a Stokes-like problem. To this end, we introduce a Crouzeix–Raviart-type space that forms a Stokes complex with the Morley-type space of the stream function approximation and post-process the pressure from the discrete stream function. Since we are interested in the approximation of solution with low regularity, we focus onto the low-order case. Assuming more regularity of the exact solution, the VEM can be extended to the arbitrary order almost straightforwardly and the pressure can be recovered in a simpler way.

The paper is organised as follows. In Section 2, we present the variational stream function formulation of model problem (1.1a)–(1.1d), and discuss its well-posedness. In Section 3, we introduce lowest-order Morley-type and Crouzeix–Raviart-type nonconforming virtual element spaces for the approximation of the stream function and the pressure variable, and present a virtual element method for the stream function formulation of the Oseen equations. In Section 4, we introduce the *enriching operator* and derive an *a priori* error estimate in a broken H^2-like mesh dependent norm. In Section 5, we post-process the discrete stream function to obtain a numerical approximation of the velocity and vorticity fields. In Section 6, we present the pressure recovery algorithm. In Section 7, we assess the behavior of the VEM through numerical experiments confirming our theoretical expectations. In Section 8, we offer our final remarks and discuss possible future developments that could stem out of this work.

2. Preliminaries and weak formulation of the problem

2.1. Notation

Throughout this paper, we adopt the notation of Sobolev spaces of reference [4]. Accordingly, we denote the space of square integrable functions defined on any open, bounded, connected domain $\mathcal{D} \subset \mathbb{R}^2$ with boundary $\partial\mathcal{D}$ by $L^2(\mathcal{D})$, and the Hilbert space of functions in $L^2(\mathcal{D})$ with all partial derivatives up to a positive integer m also in $L^2(\mathcal{D})$ by $H^m(\mathcal{D})$, *cf.* [4]. For $\mathcal{D} = \Omega$, we also consider the space $L^2_0(\Omega)$, which is the subspace of functions in $L^2(\Omega)$ having zero average on Ω. We endow $H^m(\mathcal{D})$ with a norm and a seminorm that we denote as $\|\cdot\|_{m,\mathcal{D}}$ and $|\cdot|_{m,\mathcal{D}}$, respectively. We denote the space of polynomials of degree up to a given integer $\ell \geq 0$ and defined on $\mathcal{D}$ by $\mathbb{P}_\ell(\mathcal{D})$, and, for $\ell = -1$, we conventionally assume that $\mathbb{P}_{-1} = \{0\}$. We denote the unit vector that is orthogonal to $\partial\mathcal{D}$ and pointing out of $\mathcal{D}$ by $\mathbf{n}_\mathcal{D} = (n_1, n_2)^T$, and the unit vector that is tangent to $\partial\mathcal{D}$ by $\mathbf{t}_\mathcal{D} = (t_1, t_2)^T$ and oriented such that $t_1 = -n_2$ and $t_2 = n_1$. For simplicity, we write $\partial_i\phi = \partial\phi/\partial x_i$, $i = 1, 2$, to denote the i-th directional derivative of the scalar field ϕ along the axis x and y, and $\partial_\mathbf{n}\phi = \mathbf{n}_e \cdot \nabla\phi$ and $\partial_\mathbf{t}\phi = \mathbf{t}_e \cdot \nabla\phi$ to denote the normal and tangential derivatives along edge e with unit normal and tangential vectors $\mathbf{n}_e$ and $\mathbf{t}_e$, respectively. For any scalar field ϕ and vector field $\boldsymbol{u} = (u_1, u_2)^T$, we introduce the following differential operators: the *curl of* ϕ, defined as $\mathbf{curl}\,\phi = (\partial_2\phi, -\partial_1\phi)^T$; the *Hessian matrix of* ϕ, defined as $\mathfrak{D}^2\phi := (\partial_{ij}\phi)_{1\leq i,j\leq 2}$; the *gradient of* $\boldsymbol{u}$, defined as the matrix $\nabla\boldsymbol{u} = (\partial_j u_i)_{i,j,=1,2}$; the *rotor of* $\boldsymbol{u}$, defined as $\mathbf{rot}\,\boldsymbol{u} = \partial_2 u_1 - \partial_1 u_2$. These definitions imply that $\mathbf{rot}\,\mathbf{curl}\,\phi = (\partial_{11} + \partial_{22})\phi = \Delta\phi$ and $\operatorname{div}\mathbf{curl}\,\phi = 0$. We denote the scalar product of any pair of tensors $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,2}$ and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2}$ by $\boldsymbol{\tau} : \boldsymbol{\sigma} = \sum_{i,j=1}^2 \tau_{ij}\sigma_{ij}$. Furthermore, for any pair of sufficiently regular vector-valued fields $\boldsymbol{v} = (v_1, v_2)^T$ and $\boldsymbol{w} = (w_1, w_2)^T$, the i-th component of vector $(\nabla\boldsymbol{v})\boldsymbol{w}$ for $i = 1, 2$ is equal to $\sum_{j=1,2} w_j\partial_j v_i$. For any pair of scalar fields $\phi, \psi \in L^2(\mathcal{D})$, vector fields $\boldsymbol{u}, \boldsymbol{v} \in \left[L^2(\mathcal{D})\right]^2$, and tensor fields $\boldsymbol{\tau}, \boldsymbol{\sigma} \in \left[L^2(\mathcal{D})\right]^{2\times 2}$, we consider the L^2-inner products

$$(\phi, \psi)_\mathcal{D} = \int_\mathcal{D} \phi\,\psi, \quad (\boldsymbol{u}, \boldsymbol{v})_\mathcal{D} = \int_\mathcal{D} \boldsymbol{u}\cdot\boldsymbol{v}, \quad (\boldsymbol{\tau}, \boldsymbol{\sigma})_\mathcal{D} = \int_\mathcal{D} \boldsymbol{\tau} : \boldsymbol{\sigma}. \tag{2.1}$$

In (2.1), we omit the subscript Ω on the left if $\mathcal{D} = \Omega$. Finally, we recall that the letter C, possibly with a subindex, a superindex, or a modifier on top such as "$\widetilde{C}$" or "$\widehat{C}$", denotes a costant whose value can be different at any instance and that is independent of h but may depend on the other parameters of the problem and the discretization.

2.2. Weak formulation

The *velocity–pressure* variational formulation of the Oseen problem reads as follows: *Find* $(\boldsymbol{u}, p) \in \left[H^1_0(\Omega)\right]^2 \times L^2_0(\Omega)$ *such that*

$$\nu\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v} + \int_\Omega (\nabla\boldsymbol{u})\boldsymbol{\beta}\cdot\boldsymbol{v} + \int_\Omega \gamma\boldsymbol{u}\cdot\boldsymbol{v} - \int_\Omega p\,\operatorname{div}\boldsymbol{v} = \int_\Omega \mathbf{f}\cdot\boldsymbol{v} \qquad \forall\boldsymbol{v} \in \left[H^1_0(\Omega)\right]^2, \tag{2.2a}$$

$$\int_\Omega q\,\operatorname{div}\boldsymbol{u} = 0 \qquad \forall q \in L^2_0(\Omega). \tag{2.2b}$$

Next, we introduce the Hilbert space of divergence-free functions defined as $\mathbf{V} := \{\boldsymbol{v} \in [H^1_0(\Omega)]^2 : \operatorname{div}\boldsymbol{v} = 0\}$. Since Ω is a simply-connected domain, we can represent every function $\boldsymbol{v} \in \mathbf{V}$ through a *stream function* $\varphi \in H^2(\Omega)$ such that $\boldsymbol{v} = \mathbf{curl}\,\varphi \in \mathbf{V}$. Consider the Hilbert space $\mathfrak{U} = \{\varphi \in H^2(\Omega) : \varphi = 0,\ \partial_\mathbf{n}\varphi = 0 \text{ on } \Gamma\}$, endowed with the norm $\|\varphi\|_\mathfrak{U} := |\varphi|_{2,\Omega} = \|\mathfrak{D}^2\varphi\|_{0,\Omega}$, which is equivalent to the H^2-norm $\|\psi\|_{2,\Omega}$ for $\psi \in \mathfrak{U}$ because of the Poincaré inequality. By introducing the stream function ψ, we write the variational formulation as: *Find* $\psi \in \mathfrak{U}$ *such that*

$$\int_\Omega \nu\mathfrak{D}^2\psi : \mathfrak{D}^2\phi + \int_\Omega (\nabla\,\mathbf{curl}\,\psi)\boldsymbol{\beta}\cdot\mathbf{curl}\,\phi + \int_\Omega \gamma\mathbf{curl}\,\psi\cdot\mathbf{curl}\,\phi = \int_\Omega \mathbf{f}\cdot\mathbf{curl}\,\phi \qquad \forall\phi \in \mathfrak{U}. \tag{2.3}$$

The divergence-free condition (2.2b) is automatically satisfied because setting $\boldsymbol{u} = \mathbf{curl}\,\psi$ implies that $\operatorname{div} \boldsymbol{u} = \operatorname{div} \mathbf{curl}\,\psi = 0$. Similarly, the divergence term in (2.2a) is absent in (2.3).

We rewrite the variational formulation (2.3) in the abstract form:

$$\text{Find } \psi \in \mathfrak{U} \text{ such that } \quad \mathfrak{A}(\psi, \phi) = \mathcal{F}(\phi) \qquad \forall \phi \in \mathfrak{U}, \tag{2.4}$$

where we define the bilinear form $\mathfrak{A}(\cdot,\cdot) : \mathfrak{U} \times \mathfrak{U} \to \mathbb{R}$ in the left-hand side of (2.4) as

$$\mathfrak{A}(\psi, \phi) := \mathcal{A}(\psi, \phi) + \mathcal{B}(\psi, \phi) + \mathcal{C}(\psi, \phi) \tag{2.5}$$

by using the three bilinear forms $\mathcal{A}(\cdot,\cdot), \mathcal{B}(\cdot,\cdot), \mathcal{C}(\cdot,\cdot) : \mathfrak{U} \times \mathfrak{U} \to \mathbb{R}$ such that

$$\mathcal{A}(\psi, \phi) := \int_\Omega \nu \mathfrak{D}^2 \psi : \mathfrak{D}^2 \phi, \quad \mathcal{B}(\psi, \phi) := \int_\Omega (\nabla\, \mathbf{curl}\,\psi) \boldsymbol{\beta} \cdot \mathbf{curl}\,\phi, \tag{2.6}$$

$$\mathcal{C}(\psi, \phi) := \int_\Omega \gamma \mathbf{curl}\,\psi \cdot \mathbf{curl}\,\phi, \tag{2.7}$$

and the linear functional $\mathcal{F}(\cdot) : \mathfrak{U} \to \mathbb{R}$ in the right-hand-side (2.4) as

$$\mathcal{F}(\phi) := \int_\Omega \mathbf{f} \cdot \mathbf{curl}\,\phi. \tag{2.8}$$

A straightforward calculation shows that $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{C}(\cdot,\cdot)$ are coercive and continuous with respect to the norms $\|\cdot\|_{\mathfrak{U}}$ and $\|\cdot\|_{1,\Omega}$, so that

$$\mathcal{A}(\phi, \phi) \geq C_1(\nu) \|\phi\|^2_{\mathfrak{U}} \quad \text{and} \quad |\mathcal{A}(\psi, \phi)| \leq C_2(\nu) \|\psi\|_{\mathfrak{U}} \, \|\phi\|_{\mathfrak{U}},$$
$$\mathcal{C}(\phi, \phi) \geq \ C_1(\gamma) \ \|\phi\|^2_{1,\Omega}, \quad \text{and} \quad |\mathcal{C}(\psi, \phi)| \leq C_2(\gamma) \ \|\psi\|_{\mathfrak{U}} \|\phi\|_{\mathfrak{U}},$$

for some strictly positive constants $C_1(\nu)$, $C_1(\gamma)$, $C_2(\nu)$, and $C_2(\gamma)$ (note that for all $\psi \in \mathfrak{U}$, it holds that $|\psi|_{1,\Omega} \leq \|\psi\|_{1,\Omega} \leq \|\psi\|_{2,\Omega}$ and this latter is equivalent to $|\psi|_{2,\Omega} = \|\psi\|_{\mathfrak{U}}$). Similarly, the bilinear form $\mathcal{B}(\cdot,\cdot)$ and the linear functional $\mathcal{F}(\cdot)$ are continuous with respect to the norm $\|\cdot\|_{\mathfrak{U}}$, so that

$$|\mathcal{B}(\psi, \phi)| \leq C_\beta \ \|\psi\|_{\mathfrak{U}} \|\phi\|_{\mathfrak{U}} \quad \text{and} \quad |\mathcal{F}(\phi)| \leq \|\mathcal{F}\|_{-2,\Omega} \|\phi\|_{\mathfrak{U}}, \tag{2.9}$$

for some strictly positive constants C_β, which may depend on $\beta = \|\boldsymbol{\beta}\|$, the Euclidean norm of the constant vector $\boldsymbol{\beta}$, and the usual definition of the dual norm $\|\mathcal{F}\|_{-2,\Omega} = \sup_{\phi \in H^2(\Omega)} |\mathcal{F}(\phi)|/|\phi|_{2,\Omega}$. An application of the Lax–Milgram theorem confirms the well-posedness of problem (2.3), and the regularity result that we state as follows.

Theorem 2.1. *The solution to problem* (2.3) *exists and is unique, and is such that*

$$\|\psi\|_{\mathfrak{U}} \leq C \|\mathcal{F}\|_{-2,\Omega} \tag{2.10}$$

for some strictly positive constant C, which depends on ν, $\boldsymbol{\beta}$, and γ.

Now, we move to assert an additional regularity result for the solution of problem (2.3).

Theorem 2.2. *Let $\psi \in \mathfrak{U}$ be the unique solution of problem* (2.3)*. If $\mathcal{F} \in H^{-1}(\Omega)$, then there exist $r \in (1/2, 1]$, and $C > 0$ such that $\psi \in H^{2+r}(\Omega)$ satisfying*

$$\|\psi\|_{2+r,\Omega} \leq C \|\mathcal{F}\|_{-1,\Omega}.$$

Interested readers are requested to refer to [25] for a detailed proof.

Remark 2.1. By using the Green theorem, we can split equation (2.3) into a symmetric and a skew-symmetric part [22]. If the convective field $\boldsymbol{\beta}$ is constant, so that $\nabla \cdot \boldsymbol{\beta} = 0$, and noting that the skew-symmetric part vanishes when we choose $\phi = \psi$, the coercivity of (2.3) directly follows from the coercivity of $\mathcal{A}(\cdot,\cdot)$, and $\mathcal{C}(\cdot,\cdot)$. If $\boldsymbol{\beta}$ is not constant and $\nabla \cdot \boldsymbol{\beta} \neq 0$, the term $(\nabla \cdot \boldsymbol{\beta})\,(\mathbf{curl}\,\psi \cdot \mathbf{curl}\,\phi)$ can be absorbed by the reaction term, and the coercivity follows from the coercivity of $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{C}(\cdot,\cdot)$.

3. Nonconforming virtual element method

In this section, we introduce the Morley-type fully nonconforming virtual element space that we will use to approximate the variational stream function formulation (2.3) of the Oseen equations. Nonconforming virtual element methods for fourth-order problems were first proposed in the literature in references [9, 53–55]. Herein, we mainly consider the formulation of reference [54] for the approximation of biharmonic stream function formulation, and [55] for the recovery of pressure variable by using a post-processing technique. We also briefly introduce the vector-valued Crouzeix–Raviart-type nonconforming space that we will use in Section 6 to approximate the pressure variable. The virtual element approximation of the variational formulation (2.4) reads as

$$\text{Find } \psi_h \in \mathfrak{U}_h \text{ such that} \quad \mathfrak{A}_h(\psi_h, \phi_h) = \mathcal{F}_h(\phi_h) \qquad \forall \phi_h \in \mathfrak{U}_h. \tag{3.1}$$

In (3.1), ψ_h is the virtual element approximation of ψ and belongs to $\mathfrak{U}_h$, the Morley-type nonconforming virtual element space that approximates $\mathfrak{U}$; ϕ_h is the test function; $\mathfrak{A}_h(\cdot,\cdot)$ and $\mathcal{F}_h(\cdot)$ are the virtual element approximations of the bilinear form $\mathfrak{A}(\cdot,\cdot)$ and the functional $\mathcal{F}(\cdot)$, respectively. In the rest of this section, we introduce some basic notations and the mesh regularity assumptions, the functional space $\mathfrak{U}_h$, which defines the virtual element method, and the functional space $\mathfrak{V}_h$, which we will use to formulate the pressure recovery technique in Section 6. Then, we introduce the polynomial projection operators, detail the construction of the discrete bilinear form $\mathfrak{A}_h(\cdot,\cdot)$ and the linear functional $\mathcal{F}_h(\cdot)$, and discuss the well-posedness of the method.

3.1. Mesh notation and mesh regularity assumptions

Let $\{\Omega_h\}_{0<h\leq 1}$ be a sequence of mesh decompositions consisting of star-shaped, polygonal elements K with boundary ∂K and diameter $h_K := \max_{\mathbf{x}_1,\mathbf{x}_2\in K} |\mathbf{x}_1 - \mathbf{x}_2|$. We label every mesh with the subindex $h := \max_{K\in\Omega_h} h_K$. We denote a mesh edge by e, the set of mesh edges of Ω_h by $\mathcal{E}_h = \mathcal{E}_h^{\text{int}} \cup \mathcal{E}_h^{\text{bdy}}$, where $\mathcal{E}_h^{\text{int}}$ and $\mathcal{E}_h^{\text{bdy}}$ are the subsets of internal and boundary edges, and the set of polygonal edges of the elemental boundary ∂K by $\mathcal{E}_h^K$. We denote a mesh vertex by $\boldsymbol{\chi}$, the set of mesh vertices by $\boldsymbol{\mathcal{V}}_h = \boldsymbol{\mathcal{V}}_h^{\text{int}} \cup \boldsymbol{\mathcal{V}}_h^{\text{bdy}}$, where $\boldsymbol{\mathcal{V}}_h^{\text{int}}$ and $\boldsymbol{\mathcal{V}}_h^{\text{bdy}}$ are the subsets of internal and boundary vertices, and the set of polygonal vertices of the elemental boundary ∂K by $\boldsymbol{\mathcal{V}}_h^K$. For the convergence analysis, we assume that the mesh family $\{\Omega_h\}_{0<h\leq 1}$ satisfies the following regularity condition.

Assumption 1 (Mesh regularity)**.** *There exists a positive real number ρ independent of h such that for every $K \in \Omega_h$, it holds that*

(A1) ***star-shapedness****: K is star-shaped with respect to an internal ball with radius bigger than ρh_K;*
(A2) ***uniform scaling****: the edge length h_e for all $e \in \mathcal{E}_h^K$ is bounded from below by ρh_K, i.e., $h_e \geq \rho h_K$.*

Next, for any integer $m > 0$, we introduce the broken Sobolev space of degree m on Ω_h, which is given by

$$H^m(\Omega_h) := \left\{\phi \in L^2(\Omega) : \phi_{|K} \in H^m(K) \qquad \forall K \in \Omega_h\right\}. \tag{3.2}$$

We endow this functional space with the broken seminorm

$$|\phi|_{m,h} := \left(\sum_{K\in\Omega_h} |\phi|^2_{m,K}\right)^{1/2}. \tag{3.3}$$

The functions in $H^m(\Omega_h)$ can be discontinuous across the elemental boundaries ∂K, so we introduce the concept of *jump across an edge*. To this end, for any internal edge $e \in \mathcal{E}_h^{\text{int}}$, we denote the two elements sharing e by $K^\pm$, so that $e \subseteq \partial K^+ \cap \partial K^-$ and the unit normal vector $\mathbf{n}_e$ points from K^+ to K^-. We also denote the traces of a function ϕ on e taken inside $K^\pm$ by $\phi_{|e,K^\pm}$. Then, the jump of ϕ across e is $[\![\phi]\!] := \phi_{|e,K^+} - \phi_{|e,K^-}$. Conventionally, we identify the jump of ϕ at a boundary edge $e \in \mathcal{E}_h^{\text{bdy}}$ as the trace of ϕ along e, *i.e.*, $[\![\phi]\!] := \phi_{|e}$. Then, we introduce $H^{2,\mathrm{NC}}(\Omega_h)$, the nonconforming subspace of $H^2(\Omega_h)$ [9] by requiring some additional regularity such as

$$H^{2,\mathrm{NC}}(\Omega_h) := \Big\{ \phi \in H^2(\Omega_h) : \phi \text{ is continuous at the internal vertices } \boldsymbol{\chi} \in \boldsymbol{\mathcal{V}}_h^{\mathrm{int}}, \\ \phi(\boldsymbol{\chi}) = 0 \ \forall \boldsymbol{\chi} \in \boldsymbol{\mathcal{V}}_h^{\mathrm{bdy}}, \text{ and } \int_e [\![\partial_n \phi]\!] = 0 \ \forall e \in \mathcal{E}_h \Big\}.$$

Finally, we note that we can decompose the bilinear forms defined in (2.6) and (2.7) and the linear functional defined in (2.8) as the sum of local (elemental) bilinear forms $\mathcal{A}^K(\cdot,\cdot), \mathcal{B}^K(\cdot,\cdot), \mathcal{C}^K(\cdot,\cdot) : H^2(K) \times H^2(K) \to \mathbb{R}$ such that

$$\mathcal{A}(\psi,\phi) = \sum_{K\in\Omega_h} \mathcal{A}^K(\psi,\phi) = \sum_{K\in\Omega_h} \int_K \nu\, \mathfrak{D}^2\psi : \mathfrak{D}^2\phi, \tag{3.4a}$$

$$\mathcal{B}(\psi,\phi) = \sum_{K\in\Omega_h} \mathcal{B}^K(\psi,\phi) = \sum_{K\in\Omega_h} \int_K (\nabla\, \mathbf{curl}\, \psi)\boldsymbol{\beta} \cdot \mathbf{curl}\, \phi, \tag{3.4b}$$

$$\mathcal{C}(\psi,\phi) = \sum_{K\in\Omega_h} \mathcal{C}^K(\psi,\phi) = \sum_{K\in\Omega_h} \int_K \gamma \mathbf{curl}\, \psi \cdot \mathbf{curl}\, \phi, \tag{3.4c}$$

$$\mathcal{F}(\psi) = \sum_{K\in\Omega_h} \mathcal{F}^K(\psi) = \sum_{K\in\Omega_h} \int_K \mathbf{f} \cdot \mathbf{curl}\, \psi, \tag{3.4d}$$

for $\psi, \phi \in \mathfrak{U}$. By using these local decompositions, we reformulate the bilinear form $\mathfrak{A}(\cdot,\cdot)$ introduced in (2.4) as

$$\mathfrak{A}(\psi,\phi) = \sum_{K\in\Omega_h} \mathfrak{A}^K(\psi,\phi) = \sum_{K\in\Omega_h} \Big(\mathcal{A}^K(\psi,\phi) + \mathcal{B}^K(\psi,\phi) + \mathcal{C}^K(\psi,\phi) \Big) \qquad \forall \psi,\, \phi \in \mathfrak{U}.$$

3.2. Crouzeix–Raviart-like virtual element space

Let $K \in \Omega_h$ be a convex, polygonal element with interior angles different from π. According to [55], to define the nonconforming Crouzeix–Raviart-type virtual element space that we will use in Section 6 for the pressure recovery, we first need to introduce the following auxiliary finite-dimensional spaces

$$\widehat{\mathfrak{M}}(K) := \Big\{ \boldsymbol{v}_h \in \big[H^1(K)\big]^2 : \operatorname{div} \boldsymbol{v}_h \in \mathbb{P}_0(K),\, \mathbf{rot}\, \boldsymbol{v}_h \in \mathbb{P}_0(K),\, \boldsymbol{v}_h \cdot \mathbf{n}_{K|e} \in \mathbb{P}_1(e) \qquad \forall e \in \mathcal{E}_h^K \Big\}, \tag{3.5}$$

$$\mathfrak{M}_0(K) := \Big\{ \phi_h \in H^2(K) : \Delta^2 \phi_h = 0,\, \phi_{h|e} = 0,\, \Delta\phi_{h|e} \in \mathbb{P}_0(e) \qquad \forall e \in \mathcal{E}_h^K \Big\}. \tag{3.6}$$

By adding space (3.5) and the curl of space (3.6), we define

$$\widetilde{\mathfrak{V}}(K) := \widehat{\mathfrak{M}}(K) + \mathbf{curl}\; \mathfrak{M}_0(K). \tag{3.7}$$

Then, we introduce the set of bounded, vector-valued, linear functionals $\big(\mathbf{F}_e^{\mathrm{CR}}\big)_{e\in\mathcal{E}_h^K}$ on $\widetilde{\mathfrak{V}}(K)$ such that

$$\mathbf{F}_e^{\mathrm{CR}}(\boldsymbol{v}_h) = \frac{1}{h_e} \int_e \boldsymbol{v}_h \cdot \mathbf{q} \tag{3.8}$$

for $\mathbf{q} \in \big[\mathbb{P}_0(e)\big]^2$ and $\boldsymbol{v}_h \in \widetilde{\mathfrak{V}}(K)$. We observe that $\big[\mathbb{P}_1(K)\big]^2 \subset \widehat{\mathfrak{M}}(K) \subset \widetilde{\mathfrak{V}}(K)$, and we introduce the elliptic projection operator $\mathbf{\Pi}^{\nabla,K} : \widetilde{\mathfrak{V}}(K) \to \big[\mathbb{P}_1(K)\big]^2$ such that the vector polynomial $\mathbf{\Pi}^{\nabla,K}\boldsymbol{v}_h \in \big[\mathbb{P}_1(K)\big]^2$ for any $\boldsymbol{v}_h \in \widetilde{\mathfrak{V}}(K)$ is the solution to the variational problem

$$\int_K \nabla\Big(\mathbf{\Pi}^{\nabla,K}\boldsymbol{v}_h - \boldsymbol{v}_h \Big) : \nabla \mathbf{q} = 0 \qquad \forall \mathbf{q} \in [\mathbb{P}_1(K)]^2, \tag{3.9}$$

$$\int_{\partial K}\Big(\boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h - \boldsymbol{v}_h\Big) = 0. \tag{3.10}$$

Through an integration by parts on (3.9), we can prove that the polynomial projection $\boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h$ for all $\boldsymbol{v}_h \in \widetilde{\mathfrak{V}}(K)$ is computable from the values $(\mathbf{F}_e^{\mathrm{CR}}(\boldsymbol{v}_h))_{e\in\mathcal{E}_h^K}$, *cf.* [55]. By employing the projection operator $\boldsymbol{\Pi}^{\nabla,K}$, we define the local nonconforming virtual element space

$$\mathfrak{V}_h(K) := \left\{\boldsymbol{v}_h \in \widetilde{\mathfrak{V}}(K) : \int_e \boldsymbol{v}_h\cdot\mathbf{n}_e q = \int_e \boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h\cdot\mathbf{n}_e q \qquad \forall q\in\mathbb{P}_1(K)/\mathbb{P}_0(K), \qquad \forall e\in\mathcal{E}_h^K\right\}. \tag{3.11}$$

The virtual element functions $\boldsymbol{v}_h$ in $\mathfrak{V}_h(K)$ are uniquely characterized by the values $\big(\mathbf{F}_e^{\mathrm{CR}}(\boldsymbol{v}_h)\big)_{e\in\mathcal{E}_h^K}$, which we take as the degrees of freedom of $\boldsymbol{v}_h$.

Remark 3.1. The nonconforming VEM space defined in (3.11) coincides with the *Crouzeix–Raviart* finite element space whenever K is a triangle. Therefore, we can see this space as an extension of the Crouzeix–Raviart space from triangular to polygonal meshes in the virtual element context.

The global Crouzeix–Raviart-like space is naturally defined as

$$\mathfrak{V}_h := \left\{\boldsymbol{v}_h \in \big[L^2(\Omega)\big]^2 : \boldsymbol{v}_h|_K \in \mathfrak{V}(K) \qquad \forall K\in\Omega_h,\ \int_e [\![\boldsymbol{v}_h]\!]\cdot\mathbf{q} = 0 \quad \forall \mathbf{q}\in[\mathbb{P}_0(e)]^2 \qquad \forall e\in\mathcal{E}_h\right\}. \tag{3.12}$$

The dimension of $\mathfrak{V}_h$ is equal to $2N^{\mathcal{E}_h}$, where $N^{\mathcal{E}_h}$ is the total number of mesh edges, and the degrees of freedom characterizing the functions in $\mathfrak{V}_h$ are the edge values $(\mathbf{F}_e^{\mathrm{CR}}(\boldsymbol{v}_h))_{e\in\mathcal{E}_h}$.

3.3. Local and global nonconforming Morley-type virtual element spaces

For every polygonal element $K\in\Omega_h$, we first consider the local, auxiliary space [53, 54] defined as

$$\widehat{\mathfrak{U}}_h(K) := \Big\{\phi_h \in H^2(K) : \Delta^2\phi_h \in \mathbb{P}_2(K),\ \phi_{h|e}\in\mathbb{P}_2(e),\ \Delta\phi_{h|e}\in\mathbb{P}_0(e) \qquad \forall e\in\mathcal{E}_h^K\Big\}, \tag{3.13}$$

where Δ^2 is the biharmonic operator. Then, we introduce the two sets of bounded, linear functionals on $\widehat{\mathfrak{U}}_h(K)$, denoted as $\boldsymbol{\Xi}_1 = \big(\boldsymbol{\Xi}_{1,\boldsymbol{\chi}}\big)_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h^K}$ and $\boldsymbol{\Xi}_2 = \big(\boldsymbol{\Xi}_{2,e}\big)_{e\in\mathcal{E}_h^K}$, such that

- $\boldsymbol{\Xi}_{1,\boldsymbol{\chi}}(\phi_h)$ is the vertex value $\phi_h(\boldsymbol{x}_{\boldsymbol{\chi}})$ for $\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h^K$ with position vector $\boldsymbol{x}_{\boldsymbol{\chi}}$;
- $\boldsymbol{\Xi}_{2,\mathsf{e}}(\phi_h)$ is the zero-th order moment of the normal derivative of ϕ_h on edge $e\in\mathcal{E}_h^K$, *i.e.*, $\boldsymbol{\Xi}_{2,\mathsf{e}}(\phi_h) = \int_e \partial_{\mathbf{n}}\phi_h$.

For each polygonal element K, we define the elliptic projection operator $\Pi^{\Delta,K} : \widehat{\mathfrak{U}}_h(K)\to\mathbb{P}_2(K)$, such that for every $\phi_h\in\widehat{\mathfrak{U}}_h(K)$, the quadratic polynomial $\Pi^{\Delta,K}(\phi_h)$ is the solution to the local variational problem

$$\mathcal{A}^K\big(\Pi^{\Delta,K}(\phi_h) - \phi_h, q\big) = 0 \qquad \forall q\in\mathbb{P}_2(K), \tag{3.14a}$$

$$\widehat{\Pi^{\Delta,K}(\phi_h)} = \widehat{\phi_h} \quad\text{and}\quad \int_e \nabla\Pi^{\Delta,K}(\phi_h) = \int_e \nabla\phi_h, \tag{3.14b}$$

where $\widehat{\phi_h}$ is the vertex average of ϕ_h defined by

$$\widehat{\phi_h} := \frac{1}{N^{\mathcal{V}_h^K}}\sum_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h^K}\phi_h(\boldsymbol{x}_{\boldsymbol{\chi}}), \tag{3.15}$$

and $N^{\mathcal{V}_h^K}$ is the number of vertices of element K. As stated in the following lemma, the projection $\Pi^{\Delta,K}\phi_h$ is computable using only the values of the functionals $\boldsymbol{\Xi}_1$–$\boldsymbol{\Xi}_2$ of ϕ_h associated with the local space $\widehat{\mathfrak{U}}_h(K)$. The proof is based on an integration by parts of (3.14a) and noting that all terms including the right-hand sides of (3.14b) are computable using the values provided by $\boldsymbol{\Xi}_1$–$\boldsymbol{\Xi}_2$, and can be found in [54]. Furthermore, we stress that $\mathbf{curl}\,\phi_h\in\widetilde{\mathfrak{V}}(K)$ and $\boldsymbol{\Pi}^{\nabla,K}(\mathbf{curl}\,\phi_h)$ for all $\phi_h\in\widehat{\mathfrak{U}}_h(K)$, are computable from the values of the functionals $\boldsymbol{\Xi}_1$–$\boldsymbol{\Xi}_2$, see Lemmas 15 and 18 from reference [55]. We summarize these results in the following lemma.

Lemma 3.1.

- *For every polygon $K \in \Omega_h$, the projection operator $\Pi^{\Delta,K} : \widehat{\mathfrak{U}}_h(K) \to \mathbb{P}_2(K)$ is computable on using the values of the functionals $\boldsymbol{\Xi}_1$–$\boldsymbol{\Xi}_2$;*
- *For each $K \in \Omega_h$, $\mathbf{curl}\,\widehat{\mathfrak{U}}_h(K) \subset \widetilde{\mathfrak{V}}(K)$; for all $\phi_h \in \widehat{\mathfrak{U}}_h(K)$, $\boldsymbol{\Pi}^{\nabla,K}\mathbf{curl}\,\phi_h$ is computable on using the values of the functionals $\boldsymbol{\Xi}_1$–$\boldsymbol{\Xi}_2$;*

By employing the projection operators and $\boldsymbol{\Pi}^{\nabla,K}$ and $\Pi^{\Delta,K}$, we define *the nonconforming virtual element space* as

$$\mathfrak{U}_h(K) := \Big\{ \phi_h \in \widehat{\mathfrak{U}}_h(K) : \int_e \mathbf{curl}\,\phi_h \cdot \mathbf{n}_e q = \int_e \boldsymbol{\Pi}^{\nabla,K}(\mathbf{curl}\,\phi_h) \cdot \mathbf{n}_e q \qquad \forall q \in \mathbb{P}_1(e)/\mathbb{P}_0(e)\ \forall e \in \mathcal{E}_h^K, \\ \int_K \phi_h q = \int_K \Pi^{\Delta,K}\phi_h q \qquad \forall q \in \mathbb{P}_2(K) \Big\}. \tag{3.16}$$

The virtual element space $\mathfrak{U}_h(K)$ has some major properties that are fundamental in our virtual element formulation. We summarize these properties in the following lemma. Their proof can be obtained using the arguments in [53, 55].

Lemma 3.2.

- *$\mathbb{P}_2(\mathbb{K}) \subset \mathfrak{U}_h(K)$;*
- *the set of linear operators $\boldsymbol{\Xi}_1$–$\boldsymbol{\Xi}_2$ uniquely characterize any virtual element function of $\mathfrak{U}_h(K)$ and can be taken as the degrees of freedom for this space;*
- *the polynomial projection operator $\Pi^{\Delta,K} : \widehat{\mathfrak{U}}_h(K) \to \mathbb{P}_2(K)$ is computable for every $\phi_h \in \mathfrak{U}_h(K)$ from the degrees of freedom of ϕ_h. Moreover, $\Pi^{\Delta,K} q = q$ for every $q \in \mathbb{P}_2(K)$;*
- *$\dim \mathfrak{U}_h(K) = N^{\mathcal{V}_h^K} + N^{\mathcal{E}_h^K}$, where $N^{\mathcal{V}_h^K}$ and $N^{\mathcal{E}_h^K}$ are the number of vertices and edges of K.*

Remark 3.2. In the construction of the modified virtual element space $\mathfrak{U}_h(K)$, we impose that $\int_e \mathbf{curl}\,\phi_h \cdot \mathbf{n}_e q = \int_e \boldsymbol{\Pi}^{\nabla,K}(\mathbf{curl}\,\phi_h) \cdot \mathbf{n}_e q$ where $q \in \mathbb{P}_1(e)/\mathbb{P}_0(e)$, and $e \subset \partial K$. By following Lemma 18 of [55], we deduce that the term $\int_e \mathbf{curl}\,\phi_h \cdot \mathbf{n}_e q$ can be computed by using $\boldsymbol{\Xi}_1 - \boldsymbol{\Xi}_2$, and the functional $\int_e \phi_h$. However, considering $\int_e \mathbf{curl}\,\phi_h \cdot \mathbf{n}_e q$ as degrees of freedom may increase the complexity of the construction of the discrete space $\widehat{\mathfrak{U}}_h(K)$ and consequently, we define the degrees of freedom as in Section 3.4 of reference [44].

Finally, we define the *global nonconforming virtual element space* built on the mesh decomposition Ω_h:

$$\mathfrak{U}_h := \Big\{ \phi_h \in H^{2,\mathrm{NC}}(\Omega_h) : \phi_{h|K} \in \mathfrak{U}_h(K) \qquad \forall K \in \Omega_h \Big\}. \tag{3.17}$$

The nonconforming VEM space $\mathfrak{U}_h$ is not a subset of the space $H_0^2(\Omega)$ but of the nonconforming space $H^{2,\mathrm{NC}}(\Omega_h)$, and, in particular does not require C^0- or C^1-regularity. Moreover, the following lemma also holds. The proof is established in Lemma 5.1 of [54].

Lemma 3.3. *For all $\varphi_h \in \mathfrak{U}_h$, the following inequality holds*

$$\|\phi_h\|_{0,\Omega} + |\phi_h|_{1,h} \leq C\ |\phi_h|_{2,h}, \tag{3.18}$$

for some positive constant C, which is independent of h but may depend on the mesh regularity parameter ρ of Assumption 1. Moreover, the broken semi-norm $|\varphi_h|_{2,h}$ is a norm on $\mathfrak{U}_h$.

Remark 3.3. In this article, we consider two lowest order nonconforming VEM spaces such as the fully-nonconforming Morley-type VEM space ($k = 2$), and the Crouzeix–Raviart-like VEM space ($k = 1$). We use the Morley-type space to approximate the biharmonic discrete equation in the stream function formulation (3.1), and the Crouzeix–Raviart-like space for the pressure recovery. We highlight that the degrees of freedom defined in (3.7) are unisolvent for the Crouzeix–Raviart-like VEM space $\mathfrak{V}_h$ [55]. In addition, the degrees of freedom defined in $\mathbf{\Xi}_1$–$\mathbf{\Xi}_2$ are unisolvent for the discrete space $\mathfrak{U}_h$ [54, 55]. Precisely, for each $K \in \Omega_h$ and $e \in \partial K$, we infer that $v \in \widehat{\mathfrak{U}}_h(K)$ implies that $v|_e \in \mathbb{P}_2(e)$, $\Delta v|_e \in \mathbb{P}_0(e)$, and $\Delta^2 v \in \mathbb{P}_2(K)$, and consequently $\dim(\widehat{\mathfrak{U}}_h(K)) = N^{\mathcal{V}_h^K} + 2N^{\mathcal{E}_h^K} + \dim(\mathbb{P}_2(K))$. Furthermore, due to the restriction, it holds that $\dim(\mathfrak{U}_h(K)) \geq \dim(\widehat{\mathfrak{U}}_h(K)) - N^{\mathcal{E}_h^K} - \dim(\mathbb{P}_2(K)) = N^{\mathcal{V}_h^K} + N^{\mathcal{E}_h^K}$ ([3], Sect. 3.4, [44]).

Remark 3.4. We assume that K is a convex polygon with all interior angles different from π. However, if the polygon possesses a hanging node and we consider the side containing the hanging node as a single edge, the additional condition could be removed since the evaluation of the function at the edge midpoint meets the requirement for the unisolvency of the discrete space. Interested readers may review [33] for a better explanation of the Morley-type discrete space construction on a polygonal element with hanging nodes. However, the beauty of the VEM is that a hanging node does not require any specific treatment or modification of the scheme. Citing reference [34], pages 597, 598, we can claim that "*The adaptive refinement generates hanging nodes because of the way refinement strategy is defined, but this is not troublesome in VEM setting as a hanging node can be treated as just another vertex in the decomposition of the domain.*"

3.4. Other polynomial projections

The formulation of the discrete bilinear forms that approximate the local continuous bilinear forms in (3.4) requires a few other L^2 orthogonal polynomial projections, which are computable from the degrees of freedom $\mathbf{\Xi}_1$–$\mathbf{\Xi}_2$ of the virtual element functions of $\mathfrak{U}_h(K)$. These projection operators are the two scalar projectors $\Pi_K^{2,\nabla^\perp}$ and $\Pi^{2,K}$, and the vector-valued projector $\mathbf{\Pi}^{1,K}$ applied to the **curl** operator.

The projection operator $\Pi_K^{2,\nabla^\perp} : \mathfrak{U}_h(K) \to \mathbb{P}_2(K)$ is such that for all $\phi_h \in \mathfrak{U}_h(K)$, the quadratic polynomial $\Pi_K^{2,\nabla^\perp}\phi_h$ is the solution to the variational problem

$$\int_K \mathbf{curl}\left(\phi_h - \Pi_K^{2,\nabla^\perp}\phi_h\right) \cdot \mathbf{curl}\, q = 0 \qquad \forall q \in \mathbb{P}_2(K), \tag{3.19a}$$

$$\widehat{\Pi_K^{2,\nabla^\perp}\phi_h} = \widehat{\phi_h}, \tag{3.19b}$$

where $\widehat{\phi_h}$ is the vertex average (3.15). The projection operator $\Pi^{2,K} : L^2(K) \to \mathbb{P}_2(K)$ is such that the quadratic polynomial $\Pi^{2,K}\phi$ for all $\phi \in L^2(K)$ is the solution to the variational problem

$$\int_K (\phi_h - \Pi^{2,K}\phi_h) q = 0 \qquad \forall q \in \mathbb{P}_2(K). \tag{3.20}$$

Finally, the projection operator $\mathbf{\Pi}^{1,K} : \left[L^2(K)\right]^2 \to \left[\mathbb{P}_1(K)\right]^2$, is such that the linear vector polynomial $\mathbf{\Pi}^{1,K}\boldsymbol{v}$ for all $\boldsymbol{v} \in \left[L^2(K)\right]^2$ is the solution to the variational problem

$$\int_K \left(\boldsymbol{v} - \mathbf{\Pi}^{1,K}\boldsymbol{v}\right) \cdot \mathbf{q} = 0 \qquad \forall \mathbf{q} \in \left[\mathbb{P}_1(K)\right]^2. \tag{3.21}$$

For future reference, we formally state the computability of the polynomial projections defined above in the following lemma. We omit the proof that is obtained on using the arguments of [47], and Lemma 4.1 of reference [48].

Lemma 3.4. *Let $K \in \Omega_h$ be a polygonal element and $\mathfrak{U}_h(K)$ the virtual element space defined in* (3.16). *Then, the polynomial projections $\Pi_K^{2,\nabla^\perp}\phi_h$, $\Pi^{2,K}\phi_h$ and $\mathbf{\Pi}^{1,K}\mathbf{curl}\,\phi_h$ are computable for any scalar function ϕ_h in $\mathfrak{U}_h(K)$ using only the values of the functionals $(\mathbf{\Xi}_1(\phi_h))$ and $(\mathbf{\Xi}_2(\phi_h))$. Moreover, there exists a real constant $C_N > 0$, independent of h such that*

$$\left|\mathbf{\Pi}^{1,K}\mathbf{curl}\,\phi_h\right|_{1,K} \leq C_N|\phi|_{2,K} \qquad \forall \phi_h \in \mathfrak{U}_h(K);$$

$$\left\| \boldsymbol{v} - \boldsymbol{\Pi}^{1,K}\boldsymbol{v} \right\|_{l,K} \leq C_N h_K^{1+r-l} |\boldsymbol{v}|_{1+r,K}, \quad l = 0,1 \quad r \in [0,1], \quad \text{and} \qquad \forall \boldsymbol{v} \in \left[H^{1+r}(K)\right]^2.$$

Remark 3.5. To compute the projection operators $\Pi_K^{2,\nabla^\perp}$, and $\boldsymbol{\Pi}^{1,K}$, we need the explicit polynomial representation of the trace of the basis functions of $\mathfrak{U}_h(K)$ that we obtain on interpolating the degrees of freedom $\boldsymbol{\Xi}_1$, and the restriction of the basis functions on the elemental edges as defined in (3.16), see [55].

3.5. The virtual element bilinear forms and right-hand side functional

Let ϕ_h, $\psi_h \in \mathfrak{U}_h(K)$. We define the local, discrete bilinear forms $\mathcal{A}_h^K(\cdot,\cdot)$, $\mathcal{B}_h^K(\cdot,\cdot)$, $\mathcal{C}_h^K(\cdot,\cdot) : \mathfrak{U}_h(K) \times \mathfrak{U}_h(K) \to \mathbb{R}$ as

$$\begin{aligned}
\mathcal{A}_h^K(\psi_h,\phi_h) &:= \int_K \mathfrak{D}^2\Pi^{\Delta,K}\psi_h : \mathfrak{D}^2\Pi^{\Delta,K}\phi_h + \mathfrak{S}_{\mathcal{A}}^K\big(\psi_h - \Pi^{\Delta,K}\psi_h, \phi_h - \Pi^{\Delta,K}\phi_h\big),\\
\mathcal{B}_h^K(\psi_h,\phi_h) &:= \int_K \Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h\Big)\boldsymbol{\beta}\cdot\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\phi_h,\\
\mathcal{C}_h^K(\psi_h,\phi_h) &:= \int_K \boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h\cdot\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\phi_h + \mathfrak{S}_{\mathcal{C}}^K\Big(\psi_h - \Pi_K^{2,\nabla^\perp}\psi_h, \phi_h - \Pi_K^{2,\nabla^\perp}\phi_h\Big),
\end{aligned}$$

where $\mathfrak{S}_{\mathcal{A}}^K(\cdot,\cdot), \mathfrak{S}_{\mathcal{C}}^K(\cdot,\cdot) : \mathfrak{U}_h(K)\times\mathfrak{U}_h(K) \to \mathbb{R}$, are symmetric, positive-definite bilinear forms satisfying the following conditions

$$\begin{aligned}
C_{0,\mathcal{A}}\mathcal{A}(\phi_h,\phi_h) \leq \mathfrak{S}_{\mathcal{A}}^K(\phi_h,\phi_h) \leq C^{0,\mathcal{A}}\mathcal{A}(\phi_h,\phi_h) \qquad &\forall\phi_h \in \mathfrak{U}_h(K)\cap\ker\big(\Pi^{\Delta,K}\big),\\
C_{0,\mathcal{C}}\mathcal{C}(\phi_h,\phi_h) \leq \mathfrak{S}_{\mathcal{C}}^K(\phi_h,\phi_h) \leq C^{0,\mathcal{C}}\mathcal{C}(\phi_h,\phi_h) \qquad &\forall\phi_h \in \mathfrak{U}_h(K)\cap\ker\Big(\Pi_K^{2,\nabla^\perp}\Big),
\end{aligned}$$

for some choice of strictly positive constants $C_{0,\mathcal{A}}$, $C^{0,\mathcal{A}}$, $C_{0,\mathcal{C}}$, and $C^{0,\mathcal{C}}$, which are independent of K and h. We define the global bilinear forms $\mathcal{A}_h(\cdot,\cdot), \mathcal{B}_h(\cdot,\cdot), \mathcal{C}_h(\cdot,\cdot) : \mathfrak{U}_h(K)\times\mathfrak{U}_h(K) \to \mathbb{R}$ as the sum of the local ones

$$\begin{aligned}
\mathcal{A}_h(\psi_h,\phi_h) &:= \sum_{K\in\Omega_h}\mathcal{A}_h^K(\psi_h,\phi_h), \quad \mathcal{B}_h(\psi_h,\phi_h) := \sum_{K\in\Omega_h}\mathcal{B}_h^K(\psi_h,\phi_h),\\
\mathcal{C}_h(\psi_h,\phi_h) &:= \sum_{K\in\Omega_h}\mathcal{C}_h^K(\psi_h,\phi_h),
\end{aligned}$$

for all $\psi_h, \phi_h \in \mathfrak{U}_h(K)$. By using the vector-valued L^2-orthogonal projector $\boldsymbol{\Pi}^{1,K}$, we discretize the load term as

$$\mathcal{F}_h(\phi_h) := \sum_{K\in\Omega_h}\int_K \boldsymbol{\Pi}^{1,K}\mathbf{f}\cdot\mathbf{curl}\,\phi_h = \sum_{K\in\Omega_h}\int_K \mathbf{f}\cdot\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\phi_h. \tag{3.22}$$

Remark 3.6. We would like to highlight that in the discretization of the bilinear forms $\mathcal{A}_h^K(\cdot,\cdot)$, $\mathcal{B}_h^K(\cdot,\cdot)$, and $\mathcal{C}_h^K(\cdot,\cdot)$, we have used different projection operators such as $\Pi^{\Delta,K}$, $\boldsymbol{\Pi}^{1,K}$, and $\Pi_K^{2,\nabla^\perp}$. By employing the projection operator $\Pi^{\Delta,K}$, we can discretize the bilinear form $\mathcal{B}_h^K$ which is computable from the degrees of freedom associated with the discrete space. However, it is not straightforward to derive the convergence analysis. In fact, to derive the boundedness of $\Pi^{\Delta,K}\phi_h$ in H^1 norm is not known. On the other hand, by employing the vector-valued L^2 projection operator $\boldsymbol{\Pi}^{1,K}$, we can bound the discrete bilinear from $\mathcal{B}_h^K(\cdot,\cdot)$ (Lem. 3.4), and derive the convergence analysis. Moreover, the choice $\mathbf{curl}\,\Pi^{\Delta,K}\psi_h$ can lead to heavy losses in the order of convergence, as mentioned in [15]. To avoid such difficulties, we have chosen suitable projection operators for the discretization of the bilinear forms $\mathcal{A}_h^K(\cdot,\cdot)$, $\mathcal{B}_h^K(\cdot,\cdot)$, and $\mathcal{C}_h^K(\cdot,\cdot)$. Further, we remark that the discretization $\nabla\mathrm{curl}\;\Pi^{\Delta,K}\psi_h$ affects the convergence order mildly for lowest order VEM space. However, we avoid such type of discretization to propose a discrete scheme that will work for general order k.

Moreover, the discrete bilinear forms satisfy the consistency and stability properties stated in the following lemma.

Lemma 3.5. *For $0 < h \leq 1$, let Ω_h be a domain discretization satisfying Assumption 1 and $K \in \Omega_h$ be a polygonal element. Then, we have*

- ***Polynomial consistency**: for all $\psi_h \in \mathfrak{U}_h(K)$, it holds that*

$$\chi_h^K(\psi_h, q) = \chi^K(\psi_h, q) \qquad \forall q \in \mathbb{P}_2(K), \quad \chi \in \{\mathcal{A}, \mathcal{C}\}.$$

- ***Stability**:*

$$C_\chi \chi^K(\varphi_h, \varphi_h) \leq \chi_h^K(\varphi_h, \varphi_h) \leq C^\chi \chi^K(\varphi_h, \varphi_h) \qquad \forall \varphi_h \in \mathfrak{U}_h(K), \quad \chi \in \{\mathcal{A}, \mathcal{C}\}$$

for some pairs of positive constants (C_χ, C^χ) that are independent of h.

The well-posedness of the VEM follows from the Lax–Milgram theorem, *cf.* [27]. To apply it, we need the coercivity of the discrete bilinear form $\mathfrak{A}_h(\cdot,\cdot)$, which holds under a condition on the viscosity coefficient ν, *cf.* Lemma 3.5 of reference [48].

Lemma 3.6. *For all $0 < h \leq 1$, let Ω_h be a mesh partitioning of domain Ω satisfying the mesh regularity assumptions* (**A1**) *and* (**A2**), *cf. Assumption 1. Let $\widehat{C} := C_{\mathcal{A}} C_1(\nu) - C_N^2 C_\beta^2 / 2C_1(\gamma) C_{\mathcal{C}}$ be a constant independent of h, but possibly dependent on the problem coefficients $\boldsymbol{\beta}$, γ, ν, and the mesh regularity coefficient ρ through the various stability constants in its definition. If $\widehat{C} > 0$, it holds that*

$$\mathfrak{A}_h(\phi_h, \phi_h) \geq \widehat{C} |\phi_h|_{2,h}^2.$$

The proof follows the same argument of Lemma 3.5 from [48]; hence, we omit it. Further, we choose the coefficients ν, $\boldsymbol{\beta}$, and γ so that $C_{\mathcal{A}} C_1(\nu) > C_N^2 C_\beta^2 / 2C_1(\gamma) C_{\mathcal{C}}$ to ensure that $\widehat{C} > 0$.

Remark 3.7. In Lemma 3.6, we stated the coercivity of the discrete bilinear form $\mathfrak{A}_h(\cdot,\cdot)$. To establish the constant $\widehat{C} > 0$, we need some assumptions on the coefficients ν, $\boldsymbol{\beta}$, and γ. Accordingly, we assume that ν and γ are sufficiently big and $\boldsymbol{\beta}$ is sufficiently small. We emphasize that such choice of the coefficients produces a stable numerical solution for the discrete scheme. For small values of diffusion coefficient the numerical solution produces nonphysical oscillations, so that we need an additional stabilizer to have stable numerical solutions. [22].

Remark 3.8. We derived the convergence analysis for the nonconforming VEM discretization for the Oseen equations assuming a minimum regularity of the exact solution, *i.e.*, $u \in H^{2+r}(\Omega)$, where $r \in [1/2, 1]$. To derive the optimal order of convergence in the nonconforming VEM setting, we constructed an enriching operator. We can extend our analysis straightforwardly to higher-order VEM spaces if we assume that the exact solution has higher regularity and the computational domain has a convex shape. The discrete scheme and the apriori error analysis for general order k can be achieved by employing analogous arguments as provided in [53, 54].

4. Error analysis

This section first introduces specific theoretical tools and concepts that will be useful for the convergence analysis. Since we are interested in proving the convergence of the method for low regular solutions on nonconvex domains, we need an enriching operator [43] to map the nonconforming virtual element space $\mathfrak{U}_h$ to its C^1-conforming counterpart. Then, we introduce and estimate the nonconforming error, *cf.* Theorem 4.1, and, eventually, prove the convergence of the method and estimate the approximation error, *cf.* Theorem 4.2, which is the most important result of this section.

4.1. Preliminary results

In the next two lemmas, we summarize some approximation properties from references [54, 55].

Lemma 4.1. *Under mesh-regularity assumptions* (**A1**) *and* (**A2**)*, cf. Assumption 1, there exists a constant* $C > 0$ *such that for all* $\phi \in H^{2+r}(K)$, $r \in [0, 1]$,

– *there exists a polynomial* $\phi_\pi \in \mathbb{P}_2(K)$ *such that*

$$\|\phi - \phi_\pi\|_{l,K} \leq C\ h_K^{2+r-l}\ |\phi|_{2+r,K}, \qquad l = 0, 1, 2; \tag{4.1}$$

– *there exists a virtual element function* $\phi_I \in \mathfrak{U}_h(K)$ *such that*

$$\|\phi - \phi_I\|_{l,K} \leq C\ h_K^{2+r-l} |\phi|_{2+r,K}, \qquad l = 0, 1, 2. \tag{4.2}$$

Lemma 4.2. *The projection operator* $\mathfrak{T}_e^0 : L^2(K) \to \mathbb{P}_0(e)$ *defined by the average on* e*, i.e.,* $\mathfrak{T}_e^0 \varphi := \frac{1}{|e|}\int_e \varphi$*, satisfies the approximation property*

$$\left\|\varphi - \mathfrak{T}_e^0\varphi\right\|_{0,e} \leq C\ h_K^{1/2}|\varphi|_{1,K} \qquad \forall \varphi \in H^1(K), \quad \text{and } \forall e \in \partial K \tag{4.3}$$

where C *is a real, positive constant independent of* h.

4.2. Enriching operator

In this section, we present the construction of the *enriching operator* $\mathfrak{T}_h$ and discuss its properties. A similar construction was proposed in [3] in the virtual element literature and in [42] for finite elements. Further, we highlight that *enriching operator* was constructed on fully nonconforming VEM space [9] in [43] to derive medius error estimates. We follow the construction of the enrichment operator derived in [3]. First, for each mesh element $K \in \Omega_h$, we introduce the auxiliary space

$$\begin{aligned}\widehat{\mathfrak{U}_h^C}(K) := \Big\{ &\phi_h \in H^2(K) : \Delta^2\phi_h \in \mathbb{P}_2(K),\ \phi_{h|\partial K} \in C^0(\partial K), \\ &\phi_{h|\mathsf{e}} \in \mathbb{P}_3(\mathsf{e})\ \forall \mathsf{e} \subset \partial K, (\nabla\phi_h)_{|\partial K} \in \left[C^0(\partial K)\right]^2,\ \partial_{\mathbf{n}_\mathsf{e}}\phi_h \in \mathbb{P}_1(\mathsf{e})\ \forall \mathsf{e} \subset \partial K \Big\}.\end{aligned} \tag{4.4}$$

A crucial property, which is a consequence of definition (4.4), is that $\widehat{\mathfrak{U}_h^C}(K)$ contains $\mathbb{P}_2(K)$, the space of polynomials of degree at most 2 defined on K.

Let $\psi_h \in \widehat{\mathfrak{U}_h^C}(K)$. We consider the two sets of linear, bounded functionals $\boldsymbol{\Xi}_1^c = (\boldsymbol{\Xi}_{1,\boldsymbol{\chi}}^c)_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h^K}$ and $\boldsymbol{\Xi}_2^c = (\boldsymbol{\Xi}_{2,\boldsymbol{\chi}}^c)_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h^K}$ acting on $\widehat{\mathfrak{U}}_h(K)$, that for all $\phi_h \in \widehat{\mathfrak{U}}_h(K)$, are such that

– $\boldsymbol{\Xi}_{1,\boldsymbol{\chi}}^c(\phi_h)$ is the vertex value $\phi_h(\boldsymbol{x}_{\boldsymbol{\chi}})$ for all $\boldsymbol{\chi} \in \boldsymbol{\mathcal{V}}_h^K$ with position vector $\boldsymbol{x}_{\boldsymbol{\chi}}$;
– $\boldsymbol{\Xi}_{2,\boldsymbol{\chi}}^c(\phi_h)$ consists of the values of $h_{\boldsymbol{\chi}}\partial_1\phi_h(\boldsymbol{x}_{\boldsymbol{\chi}})$, and $h_{\boldsymbol{\chi}}\partial_2\phi_h(\boldsymbol{x}_{\boldsymbol{\chi}})$ for all $\boldsymbol{\chi} \in \boldsymbol{\mathcal{V}}_h^K$ with position vector $\boldsymbol{x}_{\boldsymbol{\chi}}$ and associated *characteristic length* $h_{\boldsymbol{\chi}}$.

In the definition of $\boldsymbol{\Xi}_2^c$, we introduced a characteristic vertex length to remove the scaling of $\nabla\phi_h(\boldsymbol{x}_{\boldsymbol{\chi}})$ with respect to h. The characteristic length associated with vertex $\boldsymbol{\chi}$ can be defined in different ways, for example, by taking the square root of the area of all the elements to which vertex $\boldsymbol{\chi}$ belongs or the average of the distance from $\boldsymbol{\chi}$ to the connected vertices. However, the theoretical properties of the VEM are almost insensitive to this definition if the quantity $h_{\boldsymbol{\chi}}$, which is geometrically associated with vertex $\boldsymbol{\chi}$, scales like h.

Let $\Pi_C^{0,K} : \widehat{\mathfrak{U}_h^C}(K) \to \mathbb{P}_2(K) \subset \widehat{\mathfrak{U}_h^C}(K)$ be the L^2-orthogonal projection operator. For all $\phi_h \in \widehat{\mathfrak{U}_h^C}(K)$, the projection $\Pi_C^{0,K}\phi_h$ is computable from the values of the functionals $\boldsymbol{\Xi}_1^c$–$\boldsymbol{\Xi}_2^c$ applied to ϕ_h, *cf.* [2]. Now, for all $K \in \Omega_h$, we define the C^1-regular, conforming virtual element space

$$\mathfrak{U}_h^C(K) := \left\{ \phi_h \in \widehat{\mathfrak{U}_h^C}(K) : \int_K \left(\phi_h - \Pi_C^{0,K}\phi_h\right) q = 0 \qquad \forall q \in \mathbb{P}_2(K) \right\}. \tag{4.5}$$

The functionals Ξ_1^c–Ξ_2^c are unisolvent in the finite-dimensional space $\mathfrak{U}_h^C(K)$, and we can take them as the degrees of freedom of the functions of this space. We define the *global*, C^1-regular, conforming virtual element space as

$$\mathfrak{U}_h^C := \Big\{\phi_h \in H^2(\Omega) : \phi_{h|K} \in \mathfrak{U}_h^C(K)\Big\}. \tag{4.6}$$

The degrees of freedom of $\mathfrak{U}_h^C$, which we still denote as Ξ_1^c–Ξ_2^c with some abuse of notation, are given by collecting the elemental degrees of freedom.

Let $(\psi_{1,\boldsymbol{\chi}})_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h}$ and $(\psi_{2,\boldsymbol{\chi}})_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h}$ be the canonical basis functions associated with the degrees of freedom $\Xi_{1,\boldsymbol{\chi}}^c$ and $\Xi_{2,\boldsymbol{\chi}}^c$, respectively. We define the enriching operator $\mathfrak{T}_h : \mathfrak{U}_h \to \mathfrak{U}_h^C$ as

$$\mathfrak{T}_h(\phi_h)(x) := \sum_{\boldsymbol{\chi}\in\boldsymbol{\mathcal{V}}_h} \big(\Xi_{1,\boldsymbol{\chi}}^c(\mathfrak{T}_h(\phi_h))\psi_{1,\boldsymbol{\chi}}(x) + \Xi_{2,\boldsymbol{\chi}}^c(\mathfrak{T}_h(\phi_h))\psi_{2,\boldsymbol{\chi}}(x)\big) \qquad \forall \phi_h \in \mathfrak{U}_h, \tag{4.7}$$

so that for all $\phi_h \in \mathfrak{U}_h$, $\mathfrak{T}_h(\phi_h) \in \mathfrak{U}_h^C$ is uniquely characterized through its vertex values and vertex gradient values. To define such values, we consider the *patch* of a vertex $\boldsymbol{\chi}$, denoted by $\mathfrak{W}(\boldsymbol{\chi})$, which is the union of the elements of Ω_h sharing vertex $\boldsymbol{\chi}$. Let $\mathfrak{N}(\boldsymbol{\chi})$ denote the total number of elements contributing to patch $\mathfrak{W}(\boldsymbol{\chi})$. We define the degrees of freedom of $\mathfrak{T}_h(\phi_h)$ by using the projection operator $\Pi^{2,K}$ on the nonconforming space defined in (3.20), so that

- $\Xi_{1,\boldsymbol{\chi}}^c(\mathfrak{T}_h(\phi_h)) := \mathfrak{T}_h(\phi_h)(\boldsymbol{x}_{\boldsymbol{\chi}}) = \frac{1}{\mathfrak{N}(\boldsymbol{\chi})}\sum_{\widehat{K}\in\mathfrak{W}(\boldsymbol{\chi})} \Pi^{2,\widehat{K}}\phi_{h|\widehat{K}}(\boldsymbol{x}_{\boldsymbol{\chi}})$ for all internal vertices $\boldsymbol{\chi}$;
- $\Xi_{2,\boldsymbol{\chi}}^c(\mathfrak{T}_h(\phi_h)) := h_{\boldsymbol{\chi}}\nabla\mathfrak{T}_h(\phi_h)(\boldsymbol{x}_{\boldsymbol{\chi}}) = \frac{1}{\mathfrak{N}(\boldsymbol{\chi})}\sum_{\widehat{K}\in\mathfrak{W}(\boldsymbol{\chi})} h_{\boldsymbol{\chi}}\nabla(\Pi^{2,\widehat{K}}\phi_{h|\widehat{K}})(\boldsymbol{x}_{\boldsymbol{\chi}})$ for all internal vertices $\boldsymbol{\chi}$.

We state the approximation property of $\mathfrak{T}_h$, see [3], in the following lemma.

Lemma 4.3. *For all $\varphi_h \in \mathfrak{U}_h$, there exists a real, positive constant C, which is independent of h, such that*

$$\|\varphi_h - \mathfrak{T}_h\varphi_h\|_{0,\Omega} + h|\varphi_h - \mathfrak{T}_h\varphi_h|_{1,h} + h^2|\mathfrak{T}_h\varphi_h|_{2,h} \leq C\ h^2\ |\varphi_h|_{2,h}. \tag{4.8}$$

Remark 4.1. In [43], the enrichment operator was first developed to derive medius error analysis of fourth-order problems for the fully nonconforming VEM space of reference [9]. In [3], the construction and approximation properties of the enrichment operator are derived on the fully nonconforming Morley-type VEM space as [54]. Furthermore, we emphasize that the construction of the enrichment operator in [3] is different from that in [43].

4.3. *A priori* error estimates

In this section, we prove the convergence of the method and derive an error estimate in the *broken* H^2-seminorm (3.3). To this end, we need a few technical results. First, by employing the degrees of freedom associated with the discrete space, and proceeding as in Lemmas 4.10 and 4.11 of reference [3], we deduce the results that we formally state in this lemma.

Lemma 4.4.

- *Let $\psi \in H^{2+r}(\Omega)$, $r \in [0,1]$. Then, it holds that*

$$\mathcal{A}(\psi, \phi_h - \mathfrak{T}_h\phi_h) \leq C\ h^r\ \|\psi\|_{2+r,\Omega}\ |\phi_h|_{2,h} \qquad \forall \phi_h \in \mathfrak{U}_h,$$

  for some positive constant C independent of h.
- *Let $\varphi, \eta \in H^{2+r}(\Omega)$, $r \in [0,1]$. Then, it holds that*

$$\mathcal{A}(\varphi, \eta - \eta_I) \leq C\ h^{2r}\ \|\varphi\|_{2+r,\Omega}\ \|\eta\|_{2+r,\Omega},$$

  for some positive constant C independent of h, and where η_I is the interpolant of η in the virtual element space $\mathfrak{U}_h$ with the approximation property stated in Lemma 4.1.

Then, we introduce the consistency error that for the scalar function $u \in H^2(\Omega)$ is given by

$$\mathcal{N}_h(u, \phi_h) = \mathfrak{A}(u, \phi_h) - \mathcal{F}(\phi_h) \qquad \forall \phi_h \in \mathfrak{U}_h. \tag{4.9}$$

By employing the enriching operator $\mathfrak{T}_h$, we bound the consistency error $\mathcal{N}_h(\cdot, \cdot)$ as in the following theorem.

Theorem 4.1. *Let $\psi \in H^{2+r}(\Omega)$, $r \in (1/2, 1]$, be the weak solution of (2.3). Then, there exists a constant $C > 0$ independent of h such that*

$$\mathcal{N}_h(\psi, \phi_h) \leq Ch^r (\|\psi\|_{2+r,\Omega} + \|\mathbf{f}\|_{0,\Omega})\ |\phi_h|_{2,h},$$

for each virtual element functions $\phi_h \in \mathfrak{U}_h$. The constant C is independent of h, but depends on the stability constants of $\mathcal{A}(\cdot, \cdot)$ and $\mathcal{C}(\cdot, \cdot)$, and the mesh regularity parameter γ.

Proof. By using the definition of $\mathcal{N}_h(\cdot, \cdot)$ and $\mathfrak{A}(\cdot, \cdot)$, adding and subtracting $\mathcal{F}(\mathfrak{T}_h \phi_h)$ and using equation (3.1), we have

$$\begin{aligned} \mathcal{N}_h(\psi, \phi_h) &:= \mathfrak{A}(\psi, \phi_h) - \mathcal{F}(\phi_h) = \mathfrak{A}(\psi, \phi_h) - \mathcal{F}(\mathfrak{T}_h \phi_h) + \mathcal{F}(\mathfrak{T}_h \phi_h) - \mathcal{F}(\phi_h) \\ &= \mathcal{A}(\psi, \phi_h - \mathfrak{T}_h \phi_h) + \mathcal{B}(\psi, \phi_h - \mathfrak{T}_h \phi_h) + \mathcal{C}(\psi, \phi_h - \mathfrak{T}_h \phi_h) + \mathcal{F}(\mathfrak{T}_h \phi_h) - \mathcal{F}(\phi_h). \end{aligned} \tag{4.10}$$

By using Lemma 4.4, we bound the term $\mathcal{A}(\psi, \phi_h - \mathfrak{T}_h \phi_h)$ as follow

$$\mathcal{A}(\psi, \phi_h - \mathfrak{T}_h \phi_h) \leq C\ h^r\ |\psi|_{2+r,\Omega}\ |\phi_h|_{2,h}. \tag{4.11}$$

An application of the approximation properties of $\mathfrak{T}_h$ yields the the following estimate

$$\mathcal{B}(\psi, \phi_h - \mathfrak{T}_h \phi_h) \leq C_\beta\ \|\psi\|_{2,\Omega}\ |\phi_h - \mathfrak{T}_h \phi_h|_{1,h} \leq \widetilde{C_\beta}\ \|\psi\|_{2,\Omega}\ h\ |\phi_h|_{2,h}, \tag{4.12}$$

where $\widetilde{C_\beta}$ contains C_β, and the constant C appeared in Lemma 4.3. Upon using the approximation property of $\mathfrak{T}_h$ stated in Lemma 4.3, we bound the term $\mathcal{C}(\cdot, \cdot)$ as follows

$$\mathcal{C}(\psi, \phi_h - \mathfrak{T}_h \phi_h) \leq C|\psi|_{1,\Omega}\ |\phi_h - \mathfrak{T}_h \phi_h|_{1,h} \leq Ch\|\psi\|_{1,\Omega}\ |\phi_h|_{2,h}. \tag{4.13}$$

Using the approximation property of $\mathfrak{T}_h$, we bound the load term as follows

$$\begin{aligned} |\mathcal{F}(\mathfrak{T}_h \phi_h) - \mathcal{F}(\phi_h)| &\leq \left| \sum_{K \in \Omega_h} \int_K \mathbf{f} \cdot \mathbf{curl}\,(\mathfrak{T}_h \phi_h - \phi_h) \right| \leq \|\mathbf{f}\|_{0,\Omega}\ |\mathfrak{T}_h \phi_h - \phi_h|_{1,h} \\ &\leq C\ \|\mathbf{f}\|_{0,\Omega}\ h\ |\phi_h|_{2,h}. \end{aligned} \tag{4.14}$$

Inserting the estimates (4.11)–(4.14) in (4.10), we obtain the assertion of the lemma. $\square$

Then, we state two different bounds for the convective bilinear form $\mathcal{B}(\cdot, \cdot)$.

Lemma 4.5. *For $r \in [0, 1]$, we let $\psi \in H^{2+r}(\Omega)$ be the weak solution of (2.3) and $\mathfrak{T}_h$ be the enriching operator defined in (4.7). Then, for each discrete function $\phi_h \in \mathfrak{U}_h$, the following bound holds*

$$\mathcal{B}(\phi_h - \mathfrak{T}_h \phi_h, \psi) \leq C\ h^r \|\psi\|_{2+r,\Omega}\ |\phi_h|_{2,h} \qquad \forall r \in [0, 1], \tag{4.15}$$

where C is a real, positive constant independent of h.

Proof. According to Section 4.1 of [29], the theorem is true for all $r \in [0,1]$ if it is true for $r=0$ and $r=1$. By choosing $r=0$, we bound the left-hand side of (4.15) as follows

$$\begin{aligned}\mathcal{B}(\phi_h - \mathfrak{T}_h\phi_h, \psi) &= \sum_{K\in\Omega_h} \int_K (\nabla \mathbf{curl}\,(\phi_h - \mathfrak{T}_h\phi_h))\boldsymbol{\beta}\cdot \mathbf{curl}\,\psi \\ &\leq C_\beta \|\phi_h - \mathfrak{T}_h\phi_h\|_{2,h} |\psi|_{1,h} \leq C_\beta \|\psi\|_{2,\Omega}\ \|\phi_h\|_{2,h}.\end{aligned}$$

Then, we assume $r=1$, so that $\psi \in H^3(\Omega)$. For $\phi_h \in \mathfrak{U}_h$, from an integration by parts we obtain

$$\begin{aligned}\mathcal{B}(\phi_h - \mathfrak{T}_h\phi_h, \psi) &= \sum_{K\in\Omega_h} \int_K (\nabla \mathbf{curl}\,(\phi_h - \mathfrak{T}_h\phi_h))\boldsymbol{\beta}\cdot \mathbf{curl}\,\psi \\ &= \sum_{K\in\Omega_h} \Big(-\int_K \mathbf{curl}\,(\phi_h - \mathfrak{T}_h\phi_h)\cdot(\nabla\mathbf{curl}\,\psi)\boldsymbol{\beta} \\ &\quad + \int_{\partial K} (\mathbf{curl}\,(\phi_h - \mathfrak{T}_h\phi_h)\cdot \mathbf{curl}\,\psi)\,(\boldsymbol{\beta}\cdot\mathbf{n}_K)\Big) =: T_1 + T_2. \end{aligned} \tag{4.16}$$

The Cauchy–Schwartz inequality and Lemma 4.3 provide an upper bound on the first term

$$\begin{aligned} T_1 &\leq \Big(\sum_{K\in\Omega_h} |(\phi_h - \mathfrak{T}_h\phi_h)|^2_{1,K}\Big)^{1/2} \Big(\sum_{K\in\Omega_h} \|\nabla\mathbf{curl}\,\psi\boldsymbol{\beta}\|^2_{0,K}\Big)^{1/2} \\ &\leq C_\beta |\phi_h - \mathfrak{T}_h\phi_h|_{1,h}\ \|\psi\|_{2,\Omega} \leq \widetilde{C_\beta} h |\phi_h|_{2,h}\ \|\psi\|_{2,\Omega}. \end{aligned} \tag{4.17}$$

We rewrite the elemental summation in T_2 as a summation on the internal edges; then, we split all edge integrals into a tangential and a normal component as

$$\begin{aligned} T_2 &= \sum_{K\in\Omega_h} \int_{\partial K} (\nabla(\phi_h - \mathfrak{T}_h\phi_h)\cdot\nabla\psi)(\boldsymbol{\beta}\cdot\mathbf{n}_K) \\ &= \sum_{e\in\mathcal{E}_h^{\mathrm{int}}} (\boldsymbol{\beta}\cdot\mathbf{n}_K)\Big(\int_e [\![\nabla(\phi_h - \mathfrak{T}_h\phi_h)\cdot\mathbf{n}_e]\!]\nabla\psi\cdot\mathbf{n}_e + \int_e \Big[\!\!\Big[\frac{\partial(\phi_h - \mathfrak{T}_h\phi_h)}{\partial\mathbf{t}}\Big]\!\!\Big]\ \frac{\partial\psi}{\partial\mathbf{t}}\Big) \\ &= T_{2,1} + T_{2,2}. \end{aligned} \tag{4.18}$$

Since $\mathfrak{T}_h\phi_h \in \mathfrak{U}_h^C \subset C^1(\Omega)$, and the normal derivatives of ϕ_h along $e \in \mathcal{E}^{\mathrm{int}}$, *i.e.*, $\partial_{\mathbf{n}}\phi_h = \mathbf{n}\cdot\nabla\phi_h$, are the degrees of freedom of the nonconforming space $\mathfrak{U}_h$, we find that

$$\int_e q_0 [\![\nabla(\phi_h - \mathfrak{T}_h\phi_h)\cdot\mathbf{n}_e]\!] = 0 \qquad \forall q_0 \in \mathbb{P}_0(e).$$

We choose $q_0 = \mathfrak{T}_e^0(\nabla\psi\cdot\mathbf{n}_e)$. Then, by employing the trace inequality from Lemma 4.2 and the estimate for enriching operator from Lemma 4.3, we rewrite $T_{2,1}$ as follows

$$\begin{aligned} T_{2,1} &= \sum_{e\in\mathcal{E}_h^{\mathrm{int}}} \int_e [\![\nabla(\phi_h - \mathfrak{T}_h\phi_h)\cdot\mathbf{n}_e]\!]\nabla\psi\cdot\mathbf{n}_e \\ &= \sum_{e\in\mathcal{E}_h^{\mathrm{int}}} \int_e [\![\nabla(\phi_h - \mathfrak{T}_h\phi_h)\cdot\mathbf{n}_e]\!]\big(\nabla\psi\cdot\mathbf{n}_e - \mathfrak{T}_e^0(\nabla\psi\cdot\mathbf{n}_e)\big) \\ &\leq C\ h\ \|\psi\|_{2,\Omega}\ |\phi_h|_{2,h}. \end{aligned} \tag{4.19}$$

Since $\mathfrak{T}_h\phi_h \in C^1(\Omega)$, and ϕ_h is continuous at the internal vertices, jump $[\![\phi_h - \mathfrak{T}_h\phi_h]\!]$ must be zero at every edge endpoint. Therefore, for all constant polynomial q_0, we find that

$$\int_e q_0 \left[\!\!\left[\frac{\partial(\phi_h - \mathfrak{T}_h\phi_h)}{\partial \mathbf{t}}\right]\!\!\right] = -\int_e \frac{\partial q_0}{\partial \mathbf{t}_e}[\![\phi_h - \mathfrak{T}_h\phi_h]\!] + ([\![\phi_h - \mathfrak{T}_h\phi_h]\!]q_0)(\boldsymbol{x}_{\boldsymbol{\chi}_2}) - ([\![\phi_h - \mathfrak{T}_h\phi_h]\!]q_0)(\boldsymbol{x}_{\boldsymbol{\chi}_1}) = 0,$$

where $\boldsymbol{x}_{\boldsymbol{\chi}_1}$ and $\boldsymbol{x}_{\boldsymbol{\chi}_2}$ are the position vectors of the vertices at the endpoints of edge e. We take $q_0 = -\mathfrak{T}_e^0(\partial\psi/\partial\mathbf{t})$ and use the trace inequality of Lemma 4.2, and we bound term $T_{2,2}$ as follows

$$\begin{aligned} T_{2,2} &= \sum_{e\in\mathcal{E}_h^{\text{int}}} \int_e \left[\!\!\left[\frac{\partial(\phi_h - \mathfrak{T}_h\phi_h)}{\partial \mathbf{t}}\right]\!\!\right] \frac{\partial\psi}{\partial\mathbf{t}} \\ &= \sum_{e\in\mathcal{E}_h^{\text{int}}} \int_e \left[\!\!\left[\frac{\partial(\phi_h - \mathfrak{T}_h\phi_h)}{\partial \mathbf{t}}\right]\!\!\right] \left(\frac{\partial\psi}{\partial\mathbf{t}} - \mathfrak{T}_e^0\left(\frac{\partial\psi}{\partial\mathbf{t}}\right)\right) \leq C\ h\ \|\psi\|_{2,\Omega}\ |\phi_h|_{2,h}. \end{aligned} \tag{4.20}$$

Finally, the theorem's assertion follows on using estimates (4.17), (4.19), and (4.20) in (4.16). ☐

Lemma 4.6. *For $r \in [0,1]$, we let $\xi \in H^{2+r}(\Omega)$ and $\eta \in H^{2+r}(\Omega) \cap \mathfrak{U}$. Then, the following estimate holds*

$$\mathcal{B}(\eta - \eta_I, \xi) \leq C\ h^{2r}\ \|\eta\|_{2+r,\Omega}\|\xi\|_{2+r,\Omega},$$

for some positive constant C independent of h.

Proof. As in the proof of Lemma 4.5, we only need to consider the lemma's statement for $r = 0$ and $r = 1$. For $r = 0$, the result follows immediately from the Cauchy–Schwarz inequality. For $r = 1$, we first integrate by parts and deduce that

$$\begin{aligned} \mathcal{B}(\eta - \eta_I, \xi) &= \sum_{K\in\Omega_h} \int_K (\nabla \mathbf{curl}\,(\eta - \eta_I))\boldsymbol{\beta} \cdot \mathbf{curl}\,\xi \\ &= -\sum_{K\in\Omega_h} \int_K \mathbf{curl}\,(\eta - \eta_I) \cdot (\nabla\mathbf{curl}\,\xi)\boldsymbol{\beta} \\ &\quad + \sum_{e\in\mathcal{E}_h^{\text{int}}} (\boldsymbol{\beta}\cdot\mathbf{n}_K)\left(\int_e [\![\nabla(\eta - \eta_I)\cdot\mathbf{n}_e]\!]\nabla\xi\cdot\mathbf{n}_e + \int_e \left[\!\!\left[\frac{\partial(\eta-\eta_I)}{\partial\mathbf{t}}\right]\!\!\right] \frac{\partial\xi}{\partial\mathbf{t}}\right) \\ &= T_1 + T_2. \end{aligned} \tag{4.21}$$

We first use the Cauchy–Schwarz inequality; then, we note that $\eta \in H^3(\Omega)$ and we apply the error estimate for the interpolation operator from Lemma 4.1, *cf.* equation 4.2, and we find that

$$\begin{aligned} |T_1| &\leq \sum_{K\in\Omega_h} \left|\int_K \mathbf{curl}\,(\eta - \eta_I)\cdot(\nabla\mathbf{curl}\,\xi)\boldsymbol{\beta}\right| \leq C_\beta|\eta - \eta_I|_{1,\Omega}\,|\xi|_{2,\Omega} \\ &\leq \widehat{C_\beta}\ h^2\ \|\eta\|_{3,\Omega}\ \|\xi\|_{2,\Omega}, \end{aligned}$$

where $\widehat{C_\beta}$ is the constant depends on C_β, and the constant appeared in the interpolation estimates in Lemma 4.1. By using the same argument to estimate term T_2 in the proof of Lemma 4.5, we find the following upper bound for the edge integrals

$$\begin{aligned} |T_2| &\leq \sum_{e\in\mathcal{E}_h^{\text{int}}} (\boldsymbol{\beta}\cdot\mathbf{n}_K)\left|\int_e [\![\nabla(\eta - \eta_I)\cdot\mathbf{n}_e]\!]\nabla\xi\cdot\mathbf{n}_e + \int_e \left[\!\!\left[\frac{\partial(\eta-\eta_I)}{\partial\mathbf{t}}\right]\!\!\right] \frac{\partial\xi}{\partial\mathbf{t}}\right| \\ &\leq \widehat{C_\beta}\ h^2\ \|\eta\|_{3,\Omega}\ \|\xi\|_{2,\Omega}, \end{aligned}$$

where again we used the fact that η is supposed to belong to $H^3(\Omega)$. ☐

Finally, we proceed to estimate the term $\|\psi-\psi_h\|_{2,h}$. In the next theorem, we first provide an abstract convergence result. Then, using the approximation estimates derived above, we show a bound for this error.

Theorem 4.2. *Let the mesh regularity Assumption 1 holds and $\psi \in \mathfrak{U}$ and $\psi_h \in \mathfrak{U}_h$ be the continuous and discrete solution of (2.3) and (3.1), respectively. Then, there exists a positive constant C such that*

$$\|\psi-\psi_h\|_{2,h} \leq C\biggl(|\psi-\psi_I|_{2,h}+|\psi-\psi_\pi|_{2,h}+\|\mathcal{F}-\mathcal{F}_h\|_{H^{-2}(\Omega)}+\sup_{\phi_h\neq 0\in\mathfrak{U}_h}\frac{\mathcal{N}(\psi,\phi_h)}{|\phi_h|_{2,h}}\biggr), \tag{4.22}$$

where $\mathcal{N}(\psi,\phi_h)$ is the consistency error defined in (4.9) and ψ_I and ψ_π can respectively be any *suitable interpolant of ψ in $\mathfrak{U}_h$ and* any *suitable piecewise-quadratic polynomial approximation of ψ defined on Ω_h.*

Furthermore, if $\psi \in H^{2+r}(\Omega)$, $r\in(1/2,1]$, we have the estimate

$$\|\psi-\psi_h\|_{2,h} \leq Ch^r\Bigl(\|\psi\|_{2+r,\Omega}+\|\mathbf{f}\|_{0,\Omega}\Bigr). \tag{4.23}$$

The constant C in (4.22) and (4.23) is independent of h, but may depend on the model coefficients ν and γ, constant C_β, the mesh regularity constant ρ, and the stability and coercivity constants of the bilinear forms used in the method's formulation.

Proof. By using the interpolant ψ_I, we split the term $\psi-\psi_h$ as $\psi-\psi_h=\psi-\psi_I+\psi_I-\psi_h$. Then, we denote $\delta_h=\psi_h-\psi_I$. By using the coercivity property of $\mathfrak{A}_h(\cdot,\cdot)$, *cf.* Lemma 3.6, adding and subtracting the interpolant $\psi_I\in\mathfrak{U}_h$ and the piecewise-quadratic polynomial approximation ψ_π built on mesh Ω_h, using the definition of the non-conforming error given in (4.9), and equation (3.1), and suitably splitting the resulting terms, we obtain

$$\begin{aligned}
\widehat{C}|\delta_h|^2_{2,h} &\leq \mathfrak{A}_h(\delta_h,\delta_h)=\mathfrak{A}_h(\psi_h,\delta_h)-\mathfrak{A}_h(\psi_I,\delta_h)\\
&=\mathcal{F}_h(\delta_h)-\sum_{K\in\Omega_h}\mathfrak{A}_h^K(\psi_I-\psi_\pi,\delta_h)-\sum_{K\in\Omega_h}\mathfrak{A}_h^K(\psi_\pi,\delta_h)\\
&=\mathcal{F}_h(\delta_h)-\sum_{K\in\Omega_h}\mathfrak{A}_h^K(\psi_I-\psi_\pi,\delta_h)-\sum_{K\in\Omega_h}\bigl(\mathfrak{A}_h^K(\psi_\pi,\delta_h)-\mathfrak{A}^K(\psi,\delta_h)\bigr)-\sum_{K\in\Omega_h}\mathfrak{A}^K(\psi,\delta_h)\\
&=[\mathcal{F}_h(\delta_h)-\mathcal{F}(\delta_h)]+[-\mathcal{N}_h(\psi,\delta_h)]+\Biggl[-\sum_{K\in\Omega_h}\mathfrak{A}_h^K(\psi_I-\psi_\pi,\delta_h)\Biggr]\\
&\quad+\Biggl[-\sum_{K\in\Omega_h}\bigl(\mathfrak{A}_h^K(\psi_\pi,\delta_h)-\mathfrak{A}^K(\psi,\delta_h)\bigr)\Biggr]\\
&=T_1+T_2+T_3+T_4.
\end{aligned}\tag{4.24}$$

The abstract convergence result (4.22) is an immediate consequence of (4.24). Further, by employing polynomial consistency (Lem. 3.5) of $\mathfrak{A}^K(\cdot,\cdot)$, and Cauchy–Schwarz inequality, we bound T_4 as follows.

$$|T_4|=\Biggl|\sum_{K\in\Omega_h}\bigl(\mathfrak{A}_h^K(\psi_\pi,\delta_h)-\mathfrak{A}^K(\psi,\delta_h)\bigr)\Biggr|=\Biggl|\sum_{K\in\Omega_h}\mathfrak{A}^K(\psi_\pi-\psi,\delta_h)\Biggr|\leq C|\psi-\psi_\pi|_{2,h}\;|\delta_h|_{2,h},$$

where C is a positive generic constant. To prove (4.23), we will separately bound all terms T_i, for $i=1,2,3,4$, and assume that the polynomial approximation ψ_π and the interpolant ψ_I have the approximation properties stated in Lemma 4.1. By employing the approximation properties of $\mathbf{\Pi}^{1,K}$, we bound the term

$$|T_1|=|\mathcal{F}_h(\delta_h)-\mathcal{F}(\delta_h)|\leq Ch\|\mathbf{f}\|_{0,\Omega}\;|\delta_h|_{2,h}. \tag{4.25}$$

Upon employing the result of Theorem 4.1, we can bound the term $\mathcal{N}_h(\psi,\delta_h)$ as follows

$$|T_2|=|\mathcal{N}_h(\psi,\delta_h)|\leq Ch^r(\|\psi\|_{2+r,\Omega}+\|\mathbf{f}\|_{0,\Omega})\;|\delta_h|_{2,h}. \tag{4.26}$$

By using the continuity of discrete bilinear form $\mathfrak{A}_h^K(\cdot,\cdot)$, adding and subtracting ψ, and using Lemma 4.1, we find that

$$\begin{aligned} |T_3| &\leq \sum_{K\in\Omega_h} \left|\mathfrak{A}_h^K(\psi_I-\psi_\pi,\delta_h)\right| \leq C \sum_{K\in\Omega_h} |\psi_I-\psi_\pi|_{2,K}\ |\delta_h|_{2,K} \\ &\leq C \sum_{K\in\Omega_h} \Big(|\psi_I-\psi|_{2,K} + |\psi-\psi_\pi|_{2,K} \Big) |\delta_h|_{2,K} \\ &\leq C\Big(|\psi_I-\psi|_{2,h} + |\psi-\psi_\pi|_{2,h}\Big) |\delta_h|_{2,h} \leq Ch^r \|\psi\|_{2+r,\Omega} |\delta_h|_{2,h}. \end{aligned} \tag{4.27}$$

The difference between continuous and discrete bilinear forms in (4.28) can be estimated as follows

$$\begin{aligned} |T_4| &\leq \sum_{K\in\Omega_h} \left|\mathfrak{A}_h^K(\psi_\pi,\delta_h) - \mathfrak{A}^K(\psi,\delta_h)\right| \\ &\leq \sum_{K\in\Omega_h} \big(\left|\mathcal{A}_h^K(\psi_\pi,\delta_h) - \mathcal{A}^K(\psi,\delta_h)\right| + \left|\mathcal{B}_h^K(\psi_\pi,\delta_h) - \mathcal{B}^K(\psi,\delta_h)\right| \\ &\quad + \left|\mathcal{C}_h^K(\psi_\pi,\delta_h) - \mathcal{C}^K(\psi,\delta_h)\right|\big). \end{aligned} \tag{4.28}$$

We can reduce the discrete bilinear forms $\mathcal{A}_h(\cdot,\cdot)$ and $\mathcal{C}_h(\cdot,\cdot)$ to their continuous counterpart, *i.e.*, $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{C}(\cdot,\cdot)$, respectively, for polynomial functions. On using the continuity and the approximation property of $\mathbf{\Pi}^{1,K}$ yields

$$\sum_{K\in\Omega_h} \left|\mathcal{A}_h^K(\psi_\pi,\delta_h) - \mathcal{A}^K(\psi,\delta_h)\right| \leq Ch^r \|\psi\|_{2+r,\Omega}\ |\delta_h|_{2,h}, \tag{4.29}$$

and, similarly,

$$\sum_{K\in\Omega_h} \left|\mathcal{C}_h^K(\psi_\pi,\delta_h) - \mathcal{C}^K(\psi,\delta_h)\right| \leq C\ h^r\ \|\psi\|_{2+r,\Omega}\ |\delta_h|_{2,h}. \tag{4.30}$$

Then, we estimate $|\mathcal{B}_h^K(\psi_\pi,\delta_h) - \mathcal{B}^K(\psi,\delta_h)|$, the local error on the convective term, as follows

$$\begin{aligned} \sum_{K\in\Omega_h} \left|\mathcal{B}_h^K(\psi_\pi,\delta_h) - \mathcal{B}^K(\psi,\delta_h)\right| &= \sum_{K\in\Omega_h} \left| \int_K \nabla \mathbf{\Pi}^{1,K}\mathbf{curl}\,\psi_\pi\boldsymbol{\beta}\cdot\mathbf{\Pi}^{1,K}\mathbf{curl}\,\delta_h - \int_K \nabla\mathbf{curl}\,\psi\boldsymbol{\beta}\cdot\mathbf{curl}\,\delta_h \right| \\ &= \sum_{K\in\Omega_h} \left| \int_K \Big(\mathbf{\Pi}^{1,K}(\nabla\mathbf{curl}\,\psi_\pi\boldsymbol{\beta}) - \mathbf{\Pi}^{1,K}(\nabla\mathbf{curl}\,\psi\boldsymbol{\beta}) + \mathbf{\Pi}^{1,K}(\nabla\mathbf{curl}\,\psi\boldsymbol{\beta}) \right. \\ &\qquad \left. - \nabla\mathbf{curl}\,\psi\boldsymbol{\beta}\Big)\cdot\mathbf{curl}\,\delta_h \right|. \end{aligned}$$

On employing the approximation properties of $\mathbf{\Pi}^{1,K}$, and boundedness of $\mathbf{\Pi}^{1,K}$ in L^2 norm, we obtain

$$\sum_{K\in\Omega_h} \left|\mathcal{B}_h^K(\psi_\pi,\delta_h) - \mathcal{B}^K(\psi,\delta_h)\right| \leq C(\boldsymbol{\beta})h^r\|\psi\|_{2+r,\Omega}\,\|\delta_h\|_{1,h}. \tag{4.31}$$

Using estimates (4.29)–(4.31) in (4.28) provides the upper bound for $|T_4|$. By using the bound for T_4 and estimates (4.25)–(4.27) in (4.24), we obtain (4.22), which is the assertion of the theorem. □

5. Post-processing of velocity and vorticity

Once the nonconforming virtual element approximation ψ_h of the stream function ψ is available, we can recover velocity $\boldsymbol{u}$, vorticity ω, and pressure p through suitable projection operators. In this section, we discuss the post-processing of velocity and vorticity. In the next section, we discuss the pressure recovery.

5.1. Velocity recovery

We recall that if $\psi \in \mathfrak{U}$ is the unique solution of the the stream function variational formulation (2.3), then, the velocity field is given by $\boldsymbol{u} = \mathbf{curl}\,\psi$. We define a piecewise-linear approximation of the velocity field $\boldsymbol{u}$ as $\boldsymbol{u}_h = (\boldsymbol{u}_h|K)$ with

$$\boldsymbol{u}_h|_K := \boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h,$$

for all $K \in \Omega_h$, where $\boldsymbol{\Pi}^{1,K}$ is the projection operator (3.21). In fact, the piecewise-linear vector polynomial $\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h$ is computable for all $\psi_h \in \mathfrak{U}_h(K)$ from the degrees of freedom of $\mathbf{curl}\,\psi_h \in \big[\mathfrak{U}_h(K)\big]^2$, *cf.* Lemma 3.4. We state the convergence of the velocity variable approximation in the next theorem.

Theorem 5.1. *Let $\psi \in H^{2+r}(\Omega)$, $r \in (1/2, 1]$, be the unique solution of the stream function variational formulation (2.3). Let $\boldsymbol{u}_h = \boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h$, where $\psi_h \in \mathfrak{U}_h$ is the virtual element approximation of ψ solving (3.1) under the mesh-regularity assumptions* $(\mathbf{A1})$–$(\mathbf{A2})$, *cf. Assumption 1. Then, it holds that*

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} \leq Ch^r(\|\psi\|_{2+r,\Omega} + \|\mathbf{f}\|_{0,\Omega}),$$

for some real, positive constant C, independent of h.

Proof. We add and subtract $\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi$, use the approximation properties of $\boldsymbol{\Pi}^{1,K}$ from Lemma 3.4, and find that

$$\begin{aligned}
|\boldsymbol{u} - \boldsymbol{u}_h|^2_{1,h} &= \Big|\mathbf{curl}\,\psi - \boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h\Big|^2_{1,h} \\
&\leq \sum_{K\in\Omega_h} \Big|\mathbf{curl}\,\psi - \boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi\Big|^2_{1,K} + \Big|\boldsymbol{\Pi}^{1,K}(\mathbf{curl}\,\psi - \mathbf{curl}\,\psi_h)\Big|^2_{1,K} \\
&\leq Ch^{2r}\big(|\psi|^2_{2+r,\Omega} + \|\,\mathbf{f}\,\|^2_{0,\Omega}\big).
\end{aligned}$$

$\square$

5.2. Vorticity recovery

By applying the projection operator Π^0_K and a post-processing technique, we compute an approximation of the fluid vorticity $\omega = \mathbf{rot}\,\boldsymbol{u}$. Using the identity $\boldsymbol{u} = \mathbf{curl}\,\psi$ and the definitions of the differential operators $\mathbf{curl}$ and $\mathbf{rot}$, see Section 2.1, yields

$$\omega = \mathbf{rot}\,\boldsymbol{u} = \mathbf{rot}\,(\mathbf{curl}\,\psi) = \Delta\psi.$$

Let $\psi_h \in \mathfrak{U}_h$ be the unique solution to (3.1). We define the local post-processed vorticity as

$$\omega^K_h := \Pi^0_K(\Delta\psi_h) = \frac{1}{|K|}\int_K \Delta\psi_h = \frac{1}{|K|}\sum_{e\subset\partial K}\int_e \partial_{\mathbf{n}_e}\psi_h, \tag{5.1}$$

by noting that $\Delta\psi_h = \nabla\cdot\nabla\psi_h$ and applying the Divergence Theorem, where we recall that $|K|$ is the area of the polygon K. The local vorticity ω^K_h is computable from the degrees of freedom $\boldsymbol{\Xi}_2$. In the next theorem, we will estimate the error of the approximation ω_h in L^2 norm.

Theorem 5.2. *Let $\psi \in H^{2+r}(\Omega)$, $r \in (1/2, 1]$, be the unique solution of the stream function variational formulation (2.3). Let $\psi_h \in \mathfrak{U}_h$ be the virtual element approximation of ψ solving (3.1) under the mesh-regularity assumptions* $(\mathbf{A1})$–$(\mathbf{A2})$, *cf. Assumption 1. Then, there exists a real, positive constant C, independent of h, such that*

$$\|\omega - \omega_h\|_{0,\Omega} \leq Ch^r(\|\psi\|_{2+r,\Omega} + \|\mathbf{f}\|_{0,\Omega}).$$

Proof. By using the definition of the continuous and discrete vorticity, the boundedness of the projection operator Π^0_K, and the result of Theorem 4.2, we obtain

$$\begin{aligned}\|\omega-\omega_h\|^2_{0,\Omega} &= \sum_{K\in\Omega_h}\left\|\omega-\omega_h^K\right\|^2_{0,K} = \sum_{K\in\Omega_h}\left\|\Delta\psi-\Pi^0_K\Delta\psi_h\right\|^2_{0,K}\\ &\leq \sum_{K\in\Omega_h} 2\Big(\left\|\Delta\psi-\Pi^0_K\Delta\psi\right\|^2_{0,K}+\left\|\Pi^0_K(\Delta\psi-\Delta\psi_h)\right\|^2_{0,K}\Big)\\ &\leq C\ h^{2r}\big(|\psi|^2_{2+r,\Omega}+\|\mathbf{f}\|^2_{0,\Omega}\big).\end{aligned}$$

□

6. Post-processing of pressure

In this section we present a post-processing technique that allows us to approximate the pressure variable from the virtual element stream function ψ_h solving (3.1), *cf.* Section 3. Such an approximation is found in the discrete space

$$\mathcal{Q}_h := \Big\{q_h\in L^2_0(\Omega) : q_{h|K}\in\mathbb{P}_0(K)\qquad\forall K\in\Omega_h\Big\}. \tag{6.1}$$

To this end, we need to introduce the auxiliary space

$$\widehat{\mathfrak{V}_h} := \left\{\boldsymbol{v}_h\in\mathfrak{V}_h : \sum_{K\in\Omega_h}\int_K q^h\operatorname{div}\boldsymbol{v}_h = 0\qquad\forall q^h\in\mathcal{Q}_h\right\}, \tag{6.2}$$

which is obviously a subspace of the virtual element space $\mathfrak{V}_h$ defined in (3.12).

First, we prove that the virtual element space $\mathfrak{U}_h$ defined in (3.17) for the stream function approximation and the velocity space $\mathfrak{V}_h$ defined in (3.12) are in a Stokes complex relation.

Lemma 6.1. *Let $\mathfrak{U}_h$ be the virtual element space defined in (3.17) and $\mathfrak{V}_h$ the nonconforming virtual element space space defined in (3.12). Then, it holds that*

$$\mathbf{curl}\ \mathfrak{U}_h\subset\mathfrak{V}_h. \tag{6.3}$$

Moreover, it also holds that

$$\mathbf{curl}\ \mathfrak{U}_h=\widehat{\mathfrak{V}_h}, \tag{6.4}$$

where $\widehat{\mathfrak{V}_h}$ is the subspace of $\mathfrak{V}_h$ defined in (6.2).

Proof. Relation (6.3) follows from Lemma 21 of [55]. To prove (6.4), we first note that any vector-valued field $\boldsymbol{v}_h\in\mathbf{curl}\ \mathfrak{U}_h$ is (trivially) divergence-free, *i.e.*, $\operatorname{div}\boldsymbol{v}_h=0$, and necessarily belongs to $\widehat{\mathfrak{V}_h}$. Consequently, it holds that $\mathbf{curl}\ \mathfrak{U}_h\subseteq\widehat{\mathfrak{V}_h}$, and we only need to prove that the two spaces have the same dimension. Let $N^{\mathcal{V}_h}$, $N^{\mathcal{E}_h}$, and $N^{\mathcal{P}_h}$ denote the total number of vertices, edges, and polygonal cells of mesh Ω_h, respectively. The space definition (3.12) implies that

$$\dim\Big(\widehat{\mathfrak{V}_h}\Big)=2N^{\mathcal{E}_h}-N^{\mathcal{P}_h}. \tag{6.5}$$

Since $\operatorname{div}\mathbf{curl}\ \mathfrak{U}_h=0$, we infer that

$$\dim(\mathbf{curl}\ \mathfrak{U}_h)=\dim(\mathfrak{U}_h)-1=N^{\mathcal{V}_h}+N^{\mathcal{E}_h}-1. \tag{6.6}$$

Using (6.6) and (6.5), and the Euler formula for a polygonal element, *i.e.*, $N^{\mathcal{V}_h} + N^{\mathcal{P}_h} = N^{\mathcal{E}_h} + 1$, we deduce that

$$\begin{aligned} \dim(\mathbf{curl}\, \mathfrak{U}_h) - \dim\left(\widehat{\mathfrak{V}_h}\right) &= \left(N^{\mathcal{V}_h} + N^{\mathcal{E}_h} - 1\right) - \left(2N^{\mathcal{E}_h} - N^{\mathcal{P}_h}\right) \\ &= N^{\mathcal{V}_h} + N^{\mathcal{P}_h} - N^{\mathcal{E}_h} - 1 = 0, \end{aligned}$$

which implies the assertion of the lemma. □

Then, we rewrite (2.2a) as follows

$$b(\boldsymbol{v}, p) := \int_\Omega p \ \operatorname{div} \boldsymbol{v} = \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + \int_\Omega (\nabla \boldsymbol{u}) \boldsymbol{\beta} \cdot \boldsymbol{v} + \gamma \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} - \int_\Omega \mathbf{f} \cdot \boldsymbol{v}. \tag{6.7}$$

Consider the functional $\mathfrak{F}(\psi, \mathbf{f})(\cdot) : \left[H^1(\Omega)\right]^2 \to \mathbb{R}$ such that

$$\mathfrak{F}(\psi, \mathbf{f})(\boldsymbol{v}) := \nu \int_\Omega \nabla \mathbf{curl}\, \psi : \nabla \boldsymbol{v} + \int_\Omega (\nabla \mathbf{curl}\, \psi) \boldsymbol{\beta} \cdot \boldsymbol{v} + \gamma \int_\Omega \mathbf{curl}\, \psi \cdot \boldsymbol{v} - \int_\Omega \mathbf{f} \cdot \boldsymbol{v}. \tag{6.8}$$

Using definition (6.8) in (6.7), we reformulate (2.2a) as a variational problem for the pressure variable: *For any given* $\psi \in \mathfrak{U}$ *solving* (2.4) *with* $\mathbf{f} \in L^2(\Omega)$, *find* $p \in L_0^2(\Omega)$ *such that*

$$b(\boldsymbol{v}, p) = \mathfrak{F}(\psi, \mathbf{f})(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \left[H_0^1(\Omega)\right]^2. \tag{6.9}$$

According to the result of Theorem 5.1 of reference [36], the solution p exists and is unique. Then, Lemma 6.1 implies that the discrete analog of (6.9) is also uniquely solvable. Now, we focus on redefining (6.9) into an equivalent formulation that we know it is numerically solvable from the theory of the Stokes equations. To this end, we consider the saddle-point problem: *Find* $(\boldsymbol{w}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ *such that*

$$a(\boldsymbol{w}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \mathfrak{F}(\psi, \mathbf{f})(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{H}_0^1(\Omega), \tag{6.10a}$$

$$b(\boldsymbol{w}, q) = 0 \qquad \forall q \in L_0^2(\Omega), \tag{6.10b}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}$ is an inner product on $[H_0^1(\Omega)]^2$.

6.1. Discretization of the Stokes problem (6.10)

By employing the projection operator $\boldsymbol{\Pi}^{\nabla,K}$, we discretize the bilinear form $a(\cdot,\cdot)$ through the bilinear form $a_h : \mathfrak{V}_h \times \mathfrak{V}_h \to \mathbb{R}$, which is such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) := \sum_{K \in \Omega_h} a_h^K(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{K \in \Omega_h} \left(a^K\left(\boldsymbol{\Pi}^{\nabla,K} \boldsymbol{u}_h, \boldsymbol{\Pi}^{\nabla,K} \boldsymbol{v}_h\right) + S^K\left(\boldsymbol{u}_h - \boldsymbol{\Pi}^{\nabla,K} \boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{\Pi}^{\nabla,K} \boldsymbol{v}_h\right)\right),$$

where $S^K(\cdot,\cdot)$ is a symmetric, positive-definite bilinear form satisfying the stability condition

$$C_\# a^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le S^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \le C^\# a^K(\boldsymbol{v}_h, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \mathfrak{V}_h(K) \cap \ker\left(\boldsymbol{\Pi}^{\nabla,K}\right),$$

for some pair of strictly positive, real constants $C_\#$ and $C^\#$. Then, we define the bilinear form $b_h : \mathfrak{V}_h \times Q_h \to \mathbb{R}$ as

$$b_h(\boldsymbol{v}_h, p) := \sum_{K \in \Omega_h} \int_K p \operatorname{div} \boldsymbol{v}_h. \tag{6.11}$$

Following [55], we define the discrete functional $\mathfrak{F}_h(\psi_h,\mathbf{f})(\cdot):\mathfrak{V}_h\rightarrow\mathbb{R}$ such that

$$\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h):=\sum_{K\in\Omega_h}\int_K\Big(\nu\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h:\nabla\boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h$$
$$+\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h\Big)\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h+\gamma\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi_h\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h-\mathbf{f}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\Big),\tag{6.12}$$

where we introduced the orthogonal projection $\boldsymbol{\mathcal{P}}_0^K:\mathfrak{V}_h(K)\rightarrow[\mathbb{P}_0(K)]^2$ such that the constant vector field $\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h$ for all $\boldsymbol{v}_h\in\mathfrak{V}_h(K)$ is computable from the degrees of freedom $\mathbf{F}_e^{\mathrm{CR}}(\boldsymbol{v}_h)$ of $\boldsymbol{v}_h$, see (3.8). Note that $|\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h)|\leq C\,|\boldsymbol{v}_h|_{1,h}$ for some real, positive constant C independent of h.

Finally, we consider the virtual element discretization of the Stokes problem (6.10) that reads as: *Find* $(\boldsymbol{w}_h,p_h)\in\mathfrak{V}_h\times\mathcal{Q}_h$ *such that*

$$a_h(\boldsymbol{w}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,p_h)=\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h)\qquad\forall\boldsymbol{v}_h\in\mathfrak{V}_h,\tag{6.13a}$$
$$b_h(\boldsymbol{w}_h,q_h)=0\qquad\forall q_h\in\mathcal{Q}_h,\tag{6.13b}$$

where $\mathcal{Q}_h$ is the space defined in (6.1). The scheme (6.13a) is well-posed since $a_h(\cdot,\cdot)$ is coercive and continuous, and $b_h(\cdot,\cdot)$ is continuous and satisfies a discrete inf-sup condition on the pair of functional spaces $\mathfrak{V}_h$–$\mathcal{Q}_h$ according to the following lemma, *cf.* equation (5.7) of reference [55].

Lemma 6.2. *Let $b_h(\cdot,\cdot)$ be the discrete bilinear form defined in (6.11). Then, there exists a strictly positive, real constant $C_b>0$ such that*

$$\sup_{\boldsymbol{v}_h\in\mathfrak{V}_h\setminus\{\mathbf{0}\}}\frac{b_h(\boldsymbol{v}_h,q_h)}{|\boldsymbol{v}_h|_{1,h}}\geq C_b\|q_h\|\qquad\forall q_h\in\mathcal{Q}_h.\tag{6.14}$$

Now, we focus on deriving a bound on the difference between the functional (6.12) applied to the stream function ψ solving the continuous variational formulation (2.4) and its virtual element approximation solving (3.1).

Lemma 6.3. *Let $\mathfrak{F}_h(\psi;\mathbf{f})$ and $\mathfrak{F}_h(\psi_h;\mathbf{f})$ be the functional defined in (6.12) respectively applied to $\psi\in\mathfrak{U}$, the solution to (2.4), and $\psi_h\in\mathfrak{U}_h$, the solution to (3.1). Then, there exists a real, positive constant $C_\lambda>0$, independent of h, such that*

$$|\mathfrak{F}_h(\psi;\mathbf{f})(\boldsymbol{v}_h)-\mathfrak{F}_h(\psi_h;\mathbf{f})(\boldsymbol{v}_h)|\leq C_\lambda\;|\psi-\psi_h|_{2,h}\;|\boldsymbol{v}_h|_{1,h}.\tag{6.15}$$

Proof. Upon employing the definition (6.12), we obtain

$$|\mathfrak{F}_h(\psi;\mathbf{f})(\boldsymbol{v}_h)-\mathfrak{F}_h(\psi_h;\mathbf{f})(\boldsymbol{v}_h)|\leq\nu\sum_{K\in\Omega_h}\left|\int_K\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi-\psi_h):\nabla\boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h\right|$$
$$+\sum_{K\in\Omega_h}\left|\int_K\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi-\psi_h)\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\right|$$
$$+\gamma\sum_{K\in\Omega_h}\left|\int_K\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi-\psi_h)\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\right|.\tag{6.16}$$

Since $\boldsymbol{\Pi}^{1,K}$ is a continuous operator with respect to the H^1-semi-inner product, we bound the first term as follows

$$\nu\sum_{K\in\Omega_h}\left|\int_K\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi_h-\psi):\nabla\boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h\right|\leq C(\nu)|\psi_h-\psi|_{2,h}\;|\boldsymbol{v}_h|_{1,h}.\tag{6.17}$$

Likewise, using the continuity of the projection operators $\boldsymbol{\mathcal{P}}_0^K$ and $\boldsymbol{\Pi}^{1,K}$, and applying the Poincaré inequality yields the following bound on the second integral term

$$\sum_{K\in\Omega_h}\left|\int_K\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi_h-\psi)\Big)\beta\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\right|\leq C(C_\beta,C_p)|\psi_h-\psi|_{2,h}\,|\boldsymbol{v}_h|_{1,h},\tag{6.18}$$

where we recall that C_β is the constant introduced in (2.9), and C_p is the constant appeared in the Poincaré inequality. Finally, applying the Poincaré inequality and the continuity of $\boldsymbol{\Pi}^{1,K}$ yields the following bound the third integral term

$$\begin{aligned}\gamma\sum_{K\in\Omega_h}\left|\int_K\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi_h-\psi)\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\right|&\leq C(\gamma)\|\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi_h-\psi)\|_{0,\Omega}\,\|\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\|_{0,\Omega}\\&\leq C(\gamma)\|\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,(\psi_h-\psi)\|_{1,h}\,\|\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\|_{0,\Omega}\\&\leq C(C(\gamma),C_p)|\psi_h-\psi|_{2,h}\,|\boldsymbol{v}_h|_{1,h}.\end{aligned}\tag{6.19}$$

The assertion of the lemma follows on inserting estimates (6.17)–(6.19) into (6.16). ☐

Before deriving an estimate for the pressure approximation error, we need to introduce the consistency error. Let $\mathbf{f}$ and p be the right-hand side forcing term and the pressure solution to the variational formulation (2.2). Then, the consistency error $\mathfrak{Y}_h(\cdot,\cdot)$ is given by

$$\mathfrak{Y}_h(\psi,\boldsymbol{v}_h):=\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{v}_h)-b_h(\boldsymbol{v}_h,p)\qquad\forall\psi\in H^2(\Omega),\ \forall\boldsymbol{v}_h\in\mathfrak{V}_h.$$

The consistency error can be controlled as stated by the following lemma.

Lemma 6.4. *Let $\psi\in H^{2+r}(\Omega)$, $r\in[0,1]$, be the solution to the stream function formulation (2.3). Then, there exists a strictly positive, real constant $C>0$, independent of h, such that*

$$|\mathfrak{Y}_h(\psi,\boldsymbol{v}_h)|\leq C\,h^r\,|\psi|_{2+r}\,|\boldsymbol{v}_h|_{1,h}\qquad\forall\boldsymbol{v}_h\in\mathfrak{V}_h.\tag{6.20}$$

Proof. Upon employing (6.12), we obtain

$$\begin{aligned}\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{v}_h)-b_h(\boldsymbol{v}_h,p)=\nu\sum_{K\in\Omega_h}\int_K\Big(&\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi:\nabla\boldsymbol{\Pi}^{\nabla,K}\boldsymbol{v}_h+\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi\Big)\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\\&+\gamma\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h-\mathbf{f}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\Big)-\sum_{K\in\Omega_h}\int_K p\,\mathrm{div}\,\boldsymbol{v}_h.\end{aligned}\tag{6.21}$$

Reasoning as in the proof of Lemma 4.5, we need to prove that (6.20) is true for $r=0$ and $r=1$. The case for $r=0$ follows directly from the continuity of the projection operators. Then, we consider the case for $r=1$. First, an integration by parts yields

$$\int_K p\,\mathrm{div}\,\boldsymbol{v}_h=-\int_K\boldsymbol{v}_h\cdot\nabla p+\sum_{e\subset\partial K}\int_e\boldsymbol{v}_h\cdot\mathbf{n}_e p.$$

Upon using (1.1a), we rewrite the integration over K as

$$\begin{aligned}\int_K\boldsymbol{v}_h\cdot\nabla p&=\int_K\boldsymbol{v}_h\cdot\big(\mathbf{f}+\nu\Delta\boldsymbol{u}-(\nabla\boldsymbol{u})\boldsymbol{\beta}-\gamma\boldsymbol{u}\big)\\&=\int_K\boldsymbol{v}_h\cdot\mathbf{f}+\nu\int_K\boldsymbol{v}_h\cdot\Delta\boldsymbol{u}-\int_K\boldsymbol{v}_h\cdot(\nabla\boldsymbol{u})\boldsymbol{\beta}-\gamma\int_K\boldsymbol{v}_h\cdot\boldsymbol{u}.\end{aligned}$$

We integrate by parts the diffusion term, and we find that

$$\int_K \Delta \boldsymbol{u}\cdot\boldsymbol{v}_h = -\int_K \nabla\boldsymbol{u}:\nabla\boldsymbol{v}_h + \sum_{e\in\partial K}\int_e (\nabla\boldsymbol{u})\mathbf{n}_e\cdot\boldsymbol{v}_h.$$

We sum over all the mesh elements K and use the fact that $\boldsymbol{u}=\mathbf{curl}\,\psi$ to obtain

$$\begin{aligned}\sum_{K\in\Omega_h}\int_K p\,\mathrm{div}\,\boldsymbol{v}_h = &-\sum_{K\in\Omega_h}\Bigg(\int_K \boldsymbol{v}_h\cdot\mathbf{f} + \nu\int_K \nabla\mathbf{curl}\,\psi:\nabla\boldsymbol{v}_h + \int_K(\nabla\mathbf{curl}\,\psi)\boldsymbol{\beta}\cdot\boldsymbol{v}_h + \gamma\int_K \mathbf{curl}\,\psi\cdot\boldsymbol{v}_h\\ &+\sum_{e\in\partial K}\int_e(p\boldsymbol{I}-\nu\nabla\mathbf{curl}\,\psi)\mathbf{n}_e\cdot\boldsymbol{v}_h\Bigg),\end{aligned}\tag{6.22}$$

$\boldsymbol{I}$ being the identity matrix. We substitute (6.22) in (6.21), rewrote the double sum "$\sum_{K\in\Omega_h}\sum_{e\in\partial K}$" as a sum over all the mesh edges, and obtain

$$\begin{aligned}\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h)-b_h(\boldsymbol{v}_h,p) = \sum_{K\in\Omega_h}\nu\int_K&\Big(\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi-\nabla\mathbf{curl}\,\psi\Big):\nabla\boldsymbol{v}_h+\Big(\mathbf{f}-\boldsymbol{\mathcal{P}}_0^K\mathbf{f}\Big)\boldsymbol{v}_h\\ &+\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi\Big)\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h-(\nabla\mathbf{curl}\,\psi)\boldsymbol{\beta}\cdot\boldsymbol{v}_h\Big)\\ &-\sum_{e\in\mathcal{E}_h}\int_e(p\boldsymbol{I}-\nu\nabla\mathbf{curl}\,\psi)\mathbf{n}_e\cdot[\![\boldsymbol{v}_h]\!].\end{aligned}\tag{6.23}$$

In (6.23), we introduced the jump of a vector, which is defined as $[\![\boldsymbol{v}_h]\!]=\boldsymbol{v}_h^+-\boldsymbol{v}_h^-$, where $\boldsymbol{v}_h^\pm$ is the trace of $\boldsymbol{v}_h$ along edge e from the two opposite elements K^+, and K^- sharing e. The elements are labeled so that $\mathbf{n}_e$ points from K^+ to K^-. We denote the edge integrals as

$$T_B=\sum_{e\in\mathcal{E}_h}\int_e\big(p\boldsymbol{I}-\nu\nabla\mathbf{curl}\,\psi\big)\mathbf{n}_e\cdot[\![\boldsymbol{v}_h]\!],$$

and, to ease the notation, we introduce the symbol $\boldsymbol{\mu}=\big(p\boldsymbol{I}-\nu\nabla\mathbf{curl}\,\psi\big)\cdot\mathbf{n}_e$. The definition of the nonconforming space $H^{2,\mathrm{NC}}(\Omega_h)$, *cf.* (3.4a), and the fact that $\mathfrak{V}_h(K)\subset H^{2,\mathrm{NC}}(\Omega_h)$ implies that

$$\int_e \mathbf{q}_{k-1}\cdot[\![\boldsymbol{v}_h]\!]=0\quad\forall\boldsymbol{v}_h\in\mathfrak{V}_h,\ \forall\mathbf{q}_{k-1}\in\big[\mathbb{P}_{k-1}(K)\big]^2,$$

which holds for every mesh edge $e\in\mathcal{E}_h$. Consequently, we have

$$|T_B|=\left|\sum_{e\in\mathcal{E}_h}\int_e(\boldsymbol{\mu}-\boldsymbol{\mathcal{P}}_0^e\boldsymbol{\mu})\cdot[\![\boldsymbol{v}_h-\boldsymbol{\mathcal{P}}_0^e\boldsymbol{v}_h]\!]\right|\le\sum_{e\in\mathcal{E}_h}\|\boldsymbol{\mu}-\boldsymbol{\mathcal{P}}_0^e\boldsymbol{\mu}\|_{0,e}\ \|[\![\boldsymbol{v}_h-\boldsymbol{\mathcal{P}}_0^e\boldsymbol{v}_h]\!]\|_{0,e},\tag{6.24}$$

where $\boldsymbol{\mathcal{P}}_0^e$ is the vector-valued L^2-orthogonal projection operator on edge e. By using the approximation property of $\boldsymbol{\mathcal{P}}_0^e$, we estimate

$$\|\boldsymbol{\mu}-\boldsymbol{\mathcal{P}}_{\mathbf{0}}^{\boldsymbol{e}}\boldsymbol{\mu}\|_{0,e}\le C\ h^{1/2}|\boldsymbol{\mu}|_{1,K},\tag{6.25a}$$

$$\|[\![\boldsymbol{v}_h-\boldsymbol{\mathcal{P}}_0^e\boldsymbol{v}_h]\!]\|_{0,e}\le C\ h^{1/2}\Big(|\boldsymbol{v}_h|_{1,K^+}+|\boldsymbol{v}_h|_{1,K^-}\Big),\tag{6.25b}$$

where e is shared by two elements K^+, and K^-. Inserting (6.25) into (6.24), we derive

$$|T_B|\le C\ h\ (|p|_{1,\Omega}+|\psi|_{3,\Omega})|\boldsymbol{v}_h|_{1,h}.\tag{6.26}$$

By using the orthogonality and approximation properties of the projection operator $\boldsymbol{\mathcal{P}}_0^K$, we find that

$$\sum_{K\in\Omega_h}\left|\int_K\Big(\mathbf{f}-\boldsymbol{\mathcal{P}}_0^K\mathbf{f}\Big)\boldsymbol{v}_h\right|\le C\ h\ |\mathbf{f}|_{0,\Omega}\,|\boldsymbol{v}_h|_{1,h}. \tag{6.27}$$

Furthermore, by using the approximation properties of $\boldsymbol{\Pi}^{1,K}$, we derive the bound

$$\sum_{K\in\Omega_h}\left|\nu\int_K\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi-\nabla\mathbf{curl}\,\psi\Big):\nabla\boldsymbol{v}_h\right|\le C(\nu)h|\psi|_{3,\Omega}\ |\boldsymbol{v}_h|_{1,h}. \tag{6.28}$$

By splitting and employing the approximation property of $\boldsymbol{\Pi}^{1,K}$ defined in Lemma 3.4, and $\boldsymbol{\mathcal{P}}_0^K$, we bound the term as

$$\begin{aligned}\sum_{K\in\Omega_h}\int_K\Big(\nabla\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi\Big)\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h-(\nabla\mathbf{curl}\,\psi)\boldsymbol{\beta}\cdot\boldsymbol{v}_h&=\int_K\Big(\nabla\Big(\boldsymbol{\Pi}^{1,K}\mathbf{curl}\,\psi-\mathbf{curl}\,\psi\Big)\Big)\boldsymbol{\beta}\cdot\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h\\&+\int_K(\nabla\mathbf{curl}\,\psi)\boldsymbol{\beta}\Big(\boldsymbol{\mathcal{P}}_0^K\boldsymbol{v}_h-\boldsymbol{v}_h\Big)\le C(C_\beta,C_p)\ h|\psi|_{3,\Omega}\ |\boldsymbol{v}_h|_{1,h}.\end{aligned} \tag{6.29}$$

Inequality (6.20) follows on using estimates (6.26)–(6.29) into (6.23). □

Finally, we estimate the approximation error for the pressure variable. The proof of this theorem is based on the previous lemmas.

Theorem 6.1. *Let $(\boldsymbol{w}_h,p_h)\in\mathfrak{V}_h\times\mathcal{Q}_h$ be the solution of the variational problem (6.13a). Then, for $p\in H^r(\Omega)$ and $\psi\in H^{2+r}(\Omega)$, $r\in(1/2,1]$, it holds that*

$$|p-p_h|_{0,\Omega}\le C\ h^r\ \Big(|p|_{r,\Omega}+|\psi|_{2+r,\Omega}+|\mathbf{f}|_{0,\Omega}\Big). \tag{6.30}$$

Proof. We add and subtract $\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{v}_h)$ and $b_h(\boldsymbol{v}_h,p)$ in (6.13a), and we have

$$\begin{aligned}a_h(\boldsymbol{w}_h,\boldsymbol{v}_h)&=\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h)-b_h(\boldsymbol{v}_h,p_h)\\&=\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h)-\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{v}_h)+\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{v}_h)\\&\quad-b_h(\boldsymbol{v}_h,p)+b_h(\boldsymbol{v}_h,p-p_h).\end{aligned} \tag{6.31}$$

By inserting $\boldsymbol{v}_h=\boldsymbol{w}_h$ in (6.31), and noting that $b_h(\boldsymbol{w}_h,p_h)=b_h(\boldsymbol{v}_h,p_h)=0$, we find that

$$C|\boldsymbol{w}_h|_{1,h}^2\le a_h(\boldsymbol{w}_h,\boldsymbol{w}_h)=\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{w}_h)-\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{w}_h)+\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{w}_h)-b_h(\boldsymbol{w}_h,p)+b_h(\boldsymbol{w}_h,p-q_h)$$

for all $q_h\in\mathcal{Q}_h$. By using Lemmas 6.3 and 6.4, we find the estimate

$$|\boldsymbol{w}_h|_{1,h}\le C\ \left(|\psi-\psi_h|_{2,h}+|p-q_h|_{0,\Omega}+\sup_{\boldsymbol{v}_h\in\mathfrak{V}_h\setminus\{\mathbf{0}\}}\frac{|\mathfrak{Y}_h(\psi,\boldsymbol{v}_h)|}{|\boldsymbol{v}_h|_{1,h}}\right). \tag{6.32}$$

Moreover, for any $q_h\in\mathcal{Q}_h$, we have

$$\begin{aligned}b_h(\boldsymbol{v}_h,q_h-p_h)=b_h(\boldsymbol{v}_h,q_h-p)+b_h(\boldsymbol{v}_h,p-p_h)&=b_h(\boldsymbol{v}_h,q_h-p)+a_h(\boldsymbol{w}_h,\boldsymbol{v}_h)\\&-\mathfrak{Y}_h(\psi,\boldsymbol{v}_h)-\mathfrak{F}_h(\psi_h,\mathbf{f})(\boldsymbol{v}_h)+\mathfrak{F}_h(\psi,\mathbf{f})(\boldsymbol{v}_h).\end{aligned}$$

By using the inf-sup condition from Lemma 6.2, we obtain

$$|q_h-p_h|_{0,\Omega}\le|p-q_h|_{0,\Omega}+|\boldsymbol{w}_h|_{1,h}+|\psi-\psi_h|_{2,h}+\sup_{\boldsymbol{v}_h\in\mathfrak{V}_h\setminus\{\mathbf{0}\}}\frac{|\mathfrak{Y}_h(\psi,\boldsymbol{v}_h)|}{|\boldsymbol{v}_h|_{1,h}}.$$

Upon bounding the term $|\boldsymbol{w}_h|_{1,h}$ with (6.32), we have

$$\begin{aligned}|p-p_h|_{0,\Omega} &\leq |p-q_h|_{0,\Omega} + |p_h-q_h|_{0,\Omega}\\ &\leq |p-q_h|_{0,\Omega} + |\psi-\psi_h|_{2,h} + \sup_{\boldsymbol{v}_h\in\mathfrak{V}_h\setminus\{\mathbf{0}\}} \frac{|\mathfrak{Y}_h(\psi,\boldsymbol{v}_h)|}{|\boldsymbol{v}_h|_{1,h}}.\end{aligned}$$

By using the estimate for the consistency error from Lemma 6.4, and the H^2-error estimates of ψ from Theorem 4.2 and choosing $q_h = \mathcal{P}_0^K p$ we obtain

$$|p-p_h|_{0,\Omega} \leq Ch^r\Big(|p|_{r,\Omega} + |\psi|_{2+r,\Omega} + \|\mathbf{f}\|_{0,\Omega}\Big),$$

which is the theorem's assertion. □

Remark 6.1. In this remark, we highlight the regularity of the stream function ψ satisfying 2.3, in particular that that $\psi \in H^{2+r}(\Omega)$, where $r \in (1/2, 1]$. To derive the estimates of Theorems 4.1, 4.2, 5.1, 5.2, and 6.1, we assumed that $\psi \in H^{2+r}(\Omega)$, where *cf.*, Theorem 2.2. However, in Lemmas 4.1, 4.4, 4.5, 4.6, and 6.4, we derived an estimates for general regularity of ψ, *i.e.*, $\psi \in H^{2+r}(\Omega)$, $r \in [0, 1]$ by employing real methods of interpolation ([29], Sect. 4.1).

7. Numerical experiments

This section presents and discusses the results of two numerical benchmarks to confirm our theoretical estimates and assess the method's behavior. We first approximate the stream function in both tests and recover the velocity field and vorticity through the post-processing technique of Section 5. Then, we recover the approximation of the pressure variable from the solution of the stream function formulation using the method discussed in Section 6.

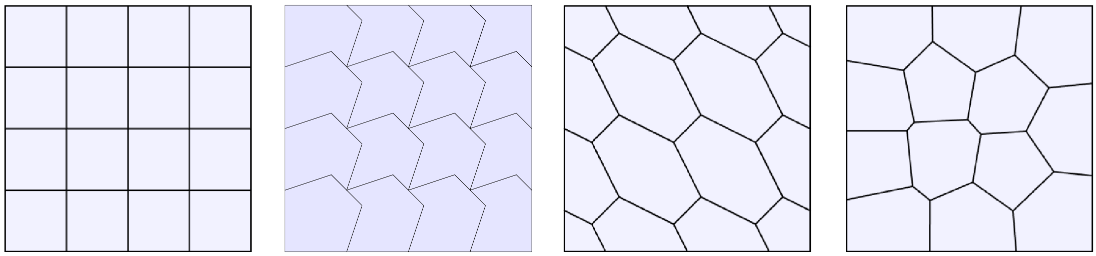
We quantify the approximation errors for stream function, velocity, and vorticity fields, and pressure by using the projection operators, $\Pi^{K,\Delta}$, $\Pi^{1,K}$, and Π^0_K, according to the following definitions:

$$\begin{aligned}&\mathcal{E}_2(\psi) := \sum_{K\in\Omega_h}\big(|\psi-\Pi^{K,\Delta}\psi_h|^2_{2,K}\big)^{1/2}; \quad \mathcal{E}_1(\mathbf{u}) := \sum_{K\in\Omega_h}\Big(\big|\mathbf{u}-\Pi^{1,K}\mathbf{u}_h\big|^2_{1,K}\Big)^{1/2};\\ &\mathcal{E}_0(\omega) := \sum_{K\in\Omega_h}\Big(\big|\omega-\Pi^0_K\omega_h\big|^2_{0,K}\Big)^{1/2}; \quad \mathcal{E}_0(p) := \sum_{K\in\Omega_h}\Big(|p-p_h|^2_{0,K}\Big)^{1/2}.\end{aligned} \tag{7.1}$$

Furthermore, we let $\mathcal{R}_i(\eta)$, $i \in \{0, 1, 2\}$, $\eta \in \{\psi, \mathbf{u}, \omega, p\}$, denote the rate of convergence for the approximation errors measured using the L^2 norm and the broken H^1, and H^2 semi-norms. Further, we emphasize that for the numerical experiments, we have considered the following closed forms of the stabilizers such as follows:

$$\begin{aligned}\mathfrak{S}^K_{\mathcal{A}}(\phi_h,\xi_h) &:= h_K^{-2}\sum_{i=1}^{N^K_{\text{dof}}}\text{dof}_i(\phi_h)\text{dof}_i(\xi_h) \qquad \forall \phi_h,\xi_h \in \mathfrak{U}_h(K),\\ \mathfrak{S}^K_{\mathcal{C}}(\phi_h,\xi_h) &:= h_K^{2}\sum_{i=1}^{N^K_{\text{dof}}}\text{dof}_i(\phi_h)\text{dof}_i(\xi_h) \qquad \forall \phi_h,\xi_h \in \mathfrak{U}_h(K),\end{aligned}$$

where N^K_{dof} denotes total number of degrees of freedom of a polygon K, and the stabilizers are completely computable from DoFs associated with the discrete space.



(a) (b) (c) (d)

FIGURE 1. We discretize the computational domain using four different mesh families: (a) square mesh, (b) non-convex mesh, (c) smoothly remapped polygons, and (d) Voronoi mesh.

7.1. Test 1. Smooth solution in a convex domain

In this example, we solve the Oseen equations (1.1a)–(1.1d) on the unit square domain $\Omega = (0,1) \times (0,1)$ with the following coefficients: $\nu = 1$, $\gamma = 100$. For convective field and the exact divergence-free velocity field, we choose

$$\boldsymbol{\beta} := \begin{bmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{bmatrix}, \quad \boldsymbol{u} := \begin{bmatrix} 2x^2(1-x)^2 y(y-1)(2y-1) \\ -2y^2(1-y)^2 x(x-1)(2x-1) \end{bmatrix}. \tag{7.2}$$

The pressure variable is $p = x^3 + y^3 - \frac{1}{2}$, and $\mathbf{f}$ is computed by inserting $\mathbf{u}$, and p into (1.1a). Further, the associated stream function is given by $\psi = x^2(1-x)^2 y^2 (1-y)^2$.

The errors are evaluated by solving the discrete scheme (3.1) on the meshes of Figure 1 and considering four further refinements.

Figure 2 shows the log–log plots *versus* the mesh size parameter h of the error curves for the virtual element approximations of the stream function, and the post-processed velocity, vorticity and pressure field. The convergence rates are reflected by the slopes of the error curves. We explicitly report the rate of the last refinement on each figure and we show the corresponding slope with a triangle. Figure 3 shows a snapshot of these fields.

All convergence rates are in agreement with our theoretical expectations.

7.2. Test 2. Nonsmooth solution on L shaped domain

In this section, we focus on examining the convergence nature of the discrete stream function, discrete velocity field, and vorticity on a non-convex L shaped domain where the exact solution ψ is not a smooth function. For the computational domain, we consider $\Omega = (-1,1) \times (-1,1) \setminus (0,1) \times (-1,0)$. The ground-truth solution for the stream function is given by $\psi(x,y) = \bar{r}^{5/3} \sin(\frac{5\theta}{3})$, where $\bar{r}$ is the radial distance from the origin and θ is the angle with the vertical axis. The radial derivative of ψ is unbounded near the origin; consequently, the solution has a weak singularity at that location. Furthermore, $\psi(x,y) \in H^{2+r}(\Omega)$, where $r \leq \frac{2}{3}$, and the velocity vector is given by $\boldsymbol{u} = \mathbf{curl}\,\psi$. We choose the pressure variable as $p(x,y) = x^2 + y^2 - 2$, so that p satisfies the zero mean condition on the computational domain Ω, *i.e.*, $\int_\Omega p = 0$. We also set the coefficients ν, γ, and the convective field $\boldsymbol{\beta}$ as in Test 1.

Table 1 illustrates the convergence behavior of the numerical approximation of the stream function, the velocity field, and the vorticity field. Furthermore, the less regularity of the stream function affects the convergence rate of the pressure variable approximation as shown in Table 1.

Remark 7.1. We stress that for the development of the theoretical analysis, we have assumed that the coefficients ν, $\boldsymbol{\beta}$, and γ are constant. In fact, the proposed analysis can be extended for the variable coefficients by imposing the admissible conditions on the norms of the coefficients. For numerical evidence, we have verified the

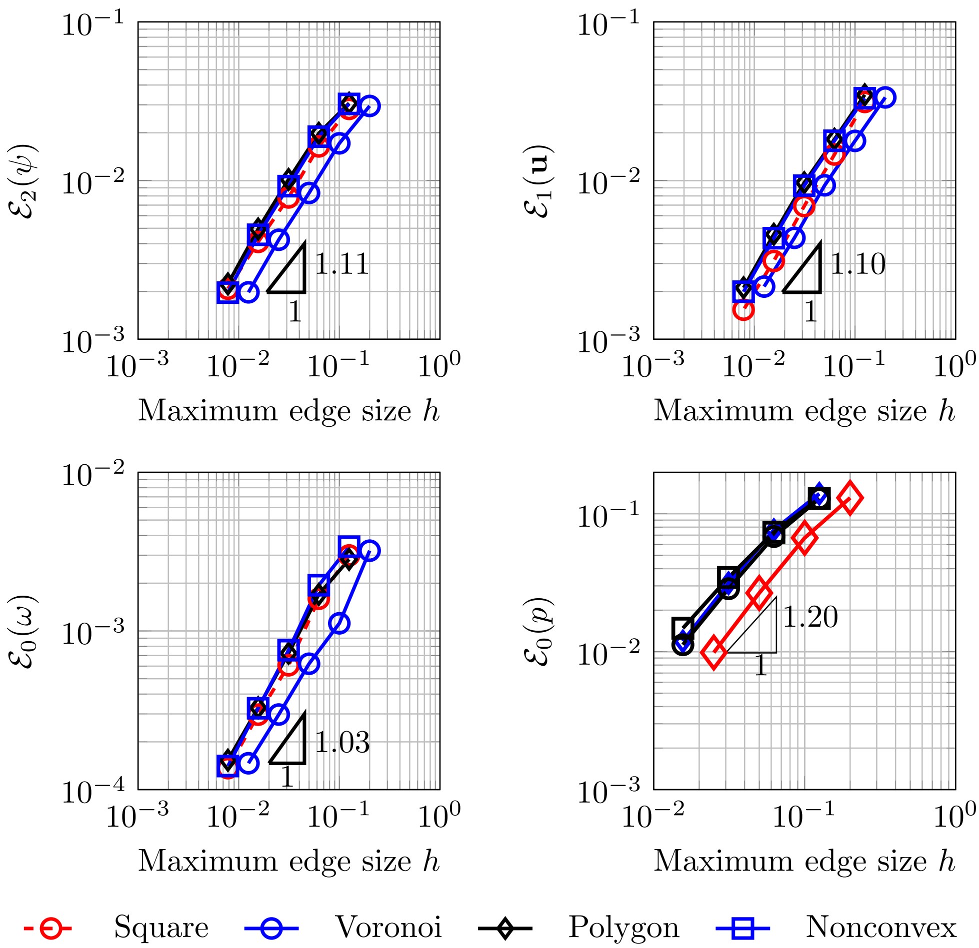


FIGURE 2. Test 1. Error curves with respect to the mesh size parameter h and convergence rates for the approximation of ψ, $\mathbf{u}$, ω and p using the broken H^2- and H^1-seminorm and the L^2-norm.

theory by considering examples that have variable coefficients. Indeed, the numerical experiments performed for a more general model problem with variable coefficients shows the wide applicability of the proposed numerical scheme.

8. Conclusions

We considered the virtual element approximation of the stream function formulation of the Oseen equations on convex and non-convex domains. In particular, we approximated the reduced biharmonic model problem by using a Morley-type nonconforming VEM. We discussed the post-proessing of the velocity and vorticity fields based on suitable polynomial projection operators on Morley space. Furthermore, to post-process the pressure variable, we introduced an auxiliary Stokes-like problem that we discretized on a Crouzeix–Raviart-type discrete velocity space. This latter space is in a Stokes-complex relationship with the Morley-type nonconforming virtual element space. Two representative benchmark problems numerically show the performance of the method. A numerical approximation of the eigenvalue problem for the Oseen equation would be a further field of interest.

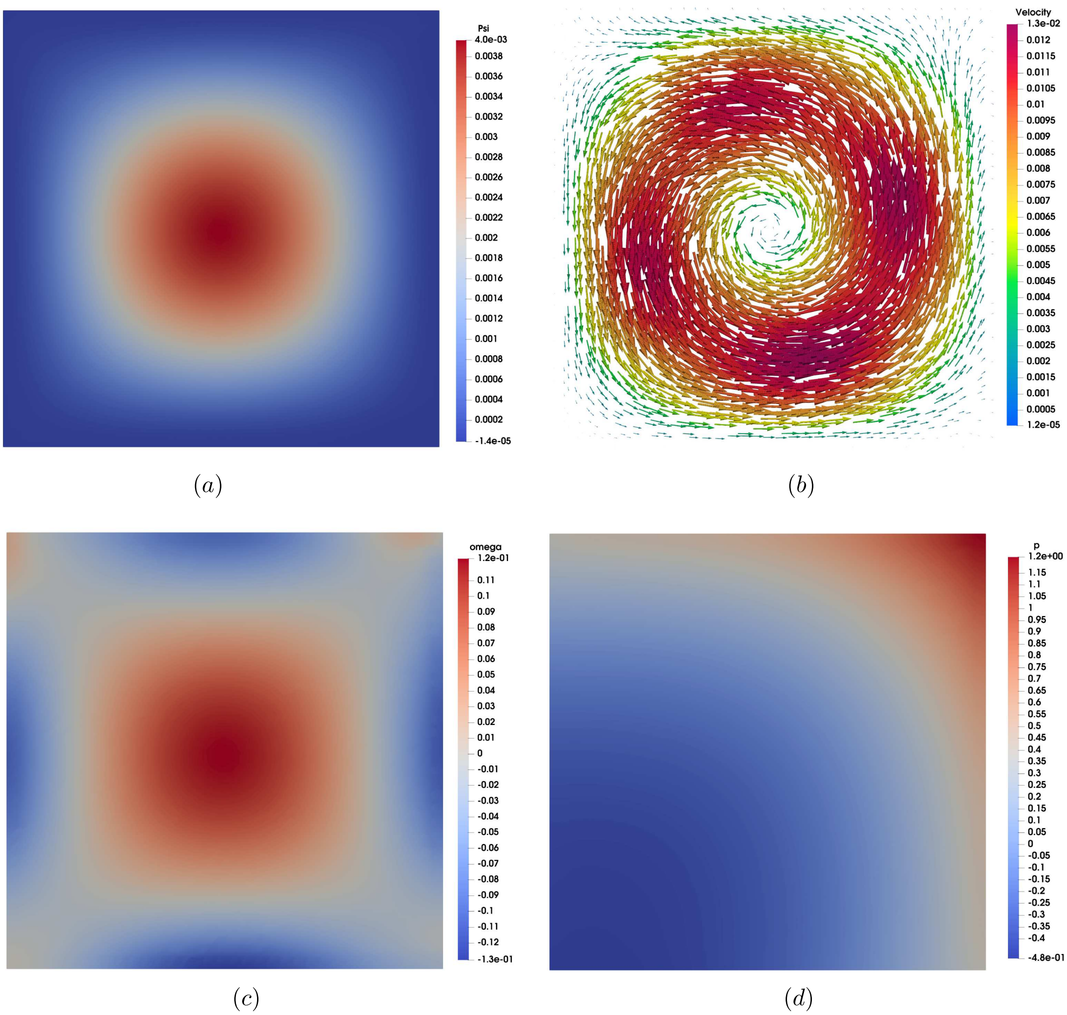


FIGURE 3. Test 1: "Snapshots" of the approximate stream function, and the post-processed velocity, vorticity, and pressure. (a) Discrete stream function. (b) Discrete velocity. (c) Discrete vorticity. (d) Discrete pressure.

TABLE 1. Test 2. Approximation errors $\mathcal{E}_i(\cdot)$ and corresponding comvergence rates $\mathcal{R}_i(\cdot)$ for the stream-function, velocity, vorticity, and pressure fields using the broken H^2- and H^1-seminorms and the L^2-norm.

h	$\mathcal{E}_2(\psi)$	$\mathcal{R}_2(\psi)$	$\mathcal{E}_1(\boldsymbol{u})$	$\mathcal{R}_1(\boldsymbol{u})$	$\mathcal{E}_0(\omega)$	$\mathcal{R}_0(\omega)$	$\mathcal{E}_0(p)$	$\mathcal{R}_0(p)$
1/8	2.37144e–1	–	2.2810e–1	–	7.3410e–2	–	1.6603e–1	–
1/16	1.54398e–1	0.61	1.5896e–1	0.52	5.8825e–2	0.31	8.9592e–2	0.89
1/32	9.7490e–2	0.66	1.0256e–1	0.63	3.8541e–2	0.61	5.2175e–2	0.78
1/64	6.2756e–2	0.64	6.5270e–2	0.65	2.4392e–2	0.66	3.1239e–2	0.74
1/128	3.9843e–2	0.65	4.1022e–2	0.67	1.5437e–2	0.66	1.8834e–2	0.73

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References

[1] D. Adak, D. Mora, S. Natarajan and A. Silgado, A virtual element discretization for the time dependent Navier–Stokes equations in stream-function formulation. *ESAIM: Math. Model. Numer. Anal.* **55** (2021) 2535–2566.

[2] D. Adak, D. Mora and S. Natarajan, Convergence analysis of virtual element method for nonlinear nonlocal dynamic plate equation. *J. Sci. Comput.* **91** (2022) 1–37.

[3] D. Adak, D. Mora and A Silgado, A Morley-type virtual element approximation for a wind-driven ocean circulation model on polygonal meshes. *J. Comput. Appl. Math.* **425** (2023) 115026.

[4] R.A. Adams and J.J.F. Fournier, Sobolev Spaces. *Pure and Applied Mathematics*, 2 edition. Academic Press (2003).

[5] J. Aghili and D.A. Di Pietro, An advection-robust hybrid high-order method for the Oseen problem. *J. Sci. Comput.* **77** (2018) 1310–1338.

[6] N. Ahmed, G.R. Barrenechea, E. Burman, J. Guzmán, A. Linke and C. Merdon, A pressure-robust discretization of Oseen's equation using stabilization in the vorticity equation. *SIAM J. Numer. Anal.* **59** (2021) 2746–2774.

[7] V. Anaya, A. Bouharguane, D. Mora, C. Reales, R. Ruiz-Baier, N. Seloula and H. Torres, Analysis and approximation of a vorticity–velocity–pressure formulation for the Oseen equations. *J. Sci. Comput.* **80** (2019) 1577–1606.

[8] P.F. Antonietti, L. Beirão da Veiga, D. Mora and M. Verani, A stream virtual element formulation of the Stokes problem on polygonal meshes. *SIAM J. Numer. Anal.* **52** (2014) 386–404.

[9] P.F. Antonietti, G. Manzini and M. Verani, The fully nonconforming virtual element method for biharmonic problems. *Math. Models Methods Appl. Sci.* **28** (2018) 387–407.

[10] P.F. Antonietti, L. Beirão da Veiga and G. Manzini, editors. The Virtual Element Method and its Applications. Vol. 31 of *SEMA SIMAI Springer Series*. Springer Nature, Switzerland AG (2022).

[11] B. Ayuso de Dios, K. Lipnikov and G. Manzini, The non-conforming virtual element method. *ESAIM Math. Model. Numer.* **50** (2016) 879–904.

[12] G.R. Barrenechea and A. Wachtel, Stabilised finite element methods for the Oseen problem on anisotropic quadrilateral meshes. *ESAIM Math. Model. Numer.* **52** (2018) 99–122.

[13] T.P. Barrios, J.M. Cascón and M. González, Augmented mixed finite element method for the Oseen problem: a priori and a posteriori error analyses. *Comput. Methods Appl. Mech. Eng.* **313** (2017) 216–238.

[14] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini and A. Russo, Basic principles of virtual element methods. *Math. Models Methods Appl. Sci.* **23** (2013) 199–214.

[15] L. Beirão da Veiga, F. Brezzi, L.D. Marini and A. Russo, Virtual element method for general second-order elliptic problems on polygonal meshes. *Math. Models Methods Appl. Sci.* **26** (2016) 729–750.

[16] L. Beirão da Veiga, C. Lovadina and G. Vacca, Divergence free virtual elements for the Stokes problem on polygonal meshes. *ESAIM: Math. Model. Numer. Anal.* **51** (2017) 509–535.

[17] L. Beirão da Veiga, C. Lovadina and G. Vacca, Virtual elements for the Navier–Stokes problem on polygonal meshes. *SIAM J. Numer. Anal.* **56** (2018) 1210–1242.

[18] L. Beirão da Veiga, D. Mora and G. Vacca, The Stokes complex for virtual elements with application to Navier–Stokes flows. *J. Sci. Comput.* **81** (2019) 990–1018.

[19] L. Beirão da Veiga, A. Russo and G. Vacca, The virtual element method with curved edges. *ESAIM: Math. Model. Numer. Anal.* **53** (2019) 375–404.

[20] L. Beirão da Veiga, F. Brezzi, L.D. Marini and A. Russo, Polynomial preserving virtual elements with curved edges. *Math. Models Methods Appl. Sci.* **30** (2020) 1555–1590.

[21] L. Beirão da Veiga, F. Dassi and G. Vacca, The Stokes complex for virtual elements in three dimensions. *Math. Models Methods Appl. Sci.* **30** (2020) 477–512.

[22] L. Beirão da Veiga, F. Dassi and G. Vacca, Vorticity-stabilized virtual elements for the Oseen equation. *Math. Models Methods Appl. Sci.* **31** (2021) 3009–3052.

[23] L. Beirão da Veiga, F. Dassi, G. Manzini and L. Mascotto, The virtual element method for the 3D resistive magnetohydrodynamic model. *Math. Models Methods Appl. Sci.* **33** (2023) 643–686.

[24] L. Beirão da Veiga, L. Mascotto and J. Meng, Stability and interpolation properties for Stokes-like virtual element spaces. *J. Sci. Comput.* **94** (2023) 56.

[25] H. Blum, R. Rannacher and R. Leis, On the boundary value problem of the biharmonic operator on domains with angular corners. *Math. Method Appl. Sci.* **2** (1980) 556–581.

[26] M. Braack, E. Burman, V. John and G. Lube, Stabilized finite element methods for the generalized Oseen problem. *Comput. Methods Appl. Mech. Eng.* **196** (2007) 853–866.

[27] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods. Vol. 15 of *Texts in Applied Mathematics*, 3rd edition. Springer, New York (2008).

[28] S.C. Brenner and L.-Y. Sung, Virtual element methods on meshes with small edges or faces. *Math. Models Methods Appl. Sci.* **28** (2018) 1291–1336.

[29] S.C. Brenner, L. Sung, H. Zhang and Y. Zhang, A Morley finite element method for the displacement obstacle problem of clamped Kirchhoff plates. *J. Comput. Appl. Math.* **254** (2013) 31–42.

[30] A.N. Brooks and T.J.R. Hughes, Streamline upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations. *Comput. Methods Appl. Mech. Eng.* **32** (1982) 199–259.

[31] E. Burman, A. Ern and M.A. Fernández, Fractional-step methods and finite elements with symmetric stabilization for the transient Oseen problem. *ESAIM: Math. Model. Numer. Anal.* **51** (2017) 487–507.

[32] A. Cangiani, V. Gyrya and G. Manzini, The non-conforming virtual element method for the Stokes equations. *SIAM J. Numer. Anal.* **54** (2016) 3411–3435.

[33] C. Carstensen, R. Khot and A.K. Pani, Nonconforming virtual elements for the biharmonic equation with Morley degrees of freedom on polygonal meshes. Preprint: arxiv:2205.08764 [math.NA] (2022).

[34] C. Carstensen, R. Khot and A.K. Pani, A priori and a posteriori error analysis of the lowest-order NCVEM for second-order linear indefinite elliptic problems. *Numer. Math.* **151** (2022) 551–600.

[35] M.E. Cayco and R.A. Nicolaides, Finite element technique for optimal pressure recovery from stream function formulation of viscous flows. *Math. Comput.* **46** (1986) 371–377.

[36] M.E. Cayco and R.A. Nicolaides, Analysis of nonconforming stream function and pressure finite element spaces for the Navier–Stokes equations. *Comput. Math. Appl.* **18** (1989) 745–760.

[37] O. Certik, F. Gardini, G. Manzini and G. Vacca, The virtual element method for eigenvalue problems with potential terms on polytopic meshes. *Appl. Math.* **63** (2018) 333–365.

[38] O. Certik, F. Gardini, G. Manzini, L. Mascotto and G. Vacca, The p- and hp-versions of the virtual element method for elliptic eigenvalue problems. *Comput. Math. Appl.* **79** (2020) 2035–2066.

[39] L. Chen and X. Huang, Nonconforming virtual element method for $2m$th order partial differential equations in $\mathbb{R}^n$. *Math. Comput.* **89** (2020) 1711–1744.

[40] A. Chernov, C. Marcati and L. Mascotto, p- and hp-virtual elements for the stokes problem. *Adv. Comput. Math.* **47** (2021) 1–31.

[41] F. Gardini, G. Manzini and G. Vacca, The nonconforming virtual element method for eigenvalue problems. *ESAIM: Math. Model. Numer. Anal.* **53** (2019) 749–774.

[42] T. Gudi, A new error analysis for discontinuous finite element methods for linear elliptic problems. *Math. Comput.* **79** (2010) 2169–2189.

[43] J. Huang and Y. Yu, A medius error analysis for nonconforming virtual element methods for Poisson and biharmonic equations. *J. Comput. Appl. Math.* **386** (2021) 113229.

[44] M. Li, J. Zhao, C. Huang and S. Chen, Conforming and nonconforming vems for the fourth-order reaction–subdiffusion equation: a unified framework. *IMA J. Numer. Anal.* **42** (2022) 2238–2300.

[45] G. Manzini and A. Mazzia, Conforming virtual element approximations of the two-dimensional Stokes problem. *Appl. Numer. Math.* **181** (2022) 176–203.

[46] G. Manzini and A. Mazzia, A virtual element generalization on polygonal meshes of the Scott–Vogelius finite element method for the 2-D Stokes problem. *J. Comput. Dyn.* **9** (2022) 207–238.

[47] D. Mora, C. Reales and A. Silgado, AC 1-virtual element method of high order for the Brinkman equations in stream function formulation with pressure recovery. *IMA J. Numer. Anal.* **42** (2022) 3632–3674.

[48] D. Mora and A. Silgado, Virtual element methods for a stream-function formulation of the Oseen equations, in The Virtual Element Method and its Applications. Springer International Publishing, Cham (2022) 321–361.

[49] T. Sorgente, S. Biasotti, G. Manzini and M. Spagnuolo, The role of mesh quality and mesh quality indicators in the virtual element method. *Adv. Comput. Math.* **48** (2021) 3.

[50] T. Sorgente, D. Prada, D. Cabiddu, S. Biasotti, G. Patane, M. Pennacchio, S. Bertoluzza, G. Manzini and M. Spagnuolo, VEM and the Mesh. Vol. 31 of *SEMA SIMAI Springer Series*, Chapter 1. Springer Nature, Switzerland AG (2021) 1–54.

[51] T. Sorgente, S. Biasotti, G. Manzini and M. Spagnuolo, Polyhedral mesh quality indicator for the virtual element method. *Comput. Math. Appl.* **114** (2022) 151–160.

[52] G. Vacca, An H^1-conforming virtual element for Darcy and Brinkman equations. *Math. Models Methods Appl. Sci.* **28** (2018) 159–194.

[53] J. Zhao, S. Chen and B. Zhang, The nonconforming virtual element method for plate bending problems. *Math. Models Methods Appl. Sci.* **26** (2016) 1671–1687.

[54] J. Zhao, B. Zhang, S. Chen and S. Mao, The Morley-type virtual element for plate bending problems. *J. Sci. Comput.* **76** (2018) 610–629.

[55] J. Zhao, B. Zhang, S. Mao and S. Chen, The divergence-free nonconforming virtual element for the Stokes problem. *SIAM J. Numer. Anal.* **57** (2019) 2730–2759.

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