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UNCONDITIONALLY OPTIMAL ERROR ESTIMATES OF LINEARIZED CRANK-NICOLSON VIRTUAL ELEMENT METHODS FOR QUASILINEAR PARABOLIC PROBLEMS ON GENERAL POLYGONAL MESHES

Yang Wang[1], Huaming Yi[2], Xiaohong Fan[2] and Guanrong Li[3,*]

Abstract. In this paper, we construct, analyze, and numerically validate a linearized Crank-Nicolson virtual element method (VEM) for solving quasilinear parabolic problems on general polygonal meshes. In particular, we consider the more general nonlinear term $a(\mathbf{x}, u)$, which does not require Lipschitz continuity or uniform ellipticity conditions. To ensure that the fully discrete solution remains bounded in L^{∞}-norm, we construct two novel elliptic projections and apply a new error splitting technique. With the help of the boundedness of numerical solution and delicate analysis of the nonlinear term, we derive the optimal error estimates for any k-order VEMs without any time-step restrictions. Numerical experiments on various polygonal meshes validate the accuracy of theoretical analysis and the unconditional convergence of the proposed scheme.

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1. Introduction

Nonlinear parabolic problems are mathematical models that describe the evolution of systems affected by both diffusion and convection effects. These problems are widely applied in various scientific and engineering fields, including heat conduction [15,21,25], fluid dynamics [9,19,33,38], reaction-diffusion systems [14,29] and so on, which providing useful mathematical tools to study and understand complex physical phenomena. The numerical solution of the nonlinear parabolic problem has garnered considerable interest from researchers over the past several decades. For example, finite difference methods, finite volume methods, finite element methods, among others, are proposed. This is just a small selection of available methods [9,15,19,30,31,33,38,39].

The aim of this paper is to investigate linearized Crank-Nicolson virtual element methods and present unconditionally error estimates for quasilinear parabolic problems. The virtual element method (VEM) [5,6] is a recently introduced technology that extends the capabilities of the finite element method (FEM). VEM is able to handle complex polygonal/polyhedral meshes, including elements with "hanging nodes" and nonconvex

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[1] School of Mathematics and Statistics, Huangshi Key Laboratory of Metaverse and Virtual Simulation, Hubei Normal University, Huangshi 435002, P.R. China.

[2] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, P.R. China.

[3] School of Mathematics and statistics, Lingnan Normal University, Zhanjiang, Guangdong 524048, P.R. China.

*Corresponding author: liguanrong88@126.com

shapes [1,3,9,27,36]. This flexibility makes VEM suitable for a wide range of applications. Another advantage of VEM is its ability to easily implement highly regular conforming discrete spaces. This property makes the method particularly suitable for solving nonlinear problems.

In [36], Vacca *et al.* proposed, for the first time, the VEMs for linear parabolic problems. They conducted an analysis on the convergence of both semi-discrete and fully-discrete solutions, and derived error estimates in L^2 and H^1 norms. In [1], conforming VEMs were proposed for semilinear parabolic problems by Adak *et al.* They present *a priori* error estimates for the proposed schemes in H^1 and L^2 norms under the assumption that the source term f is Lipschitz continuous. Anaya *et al.* [3] proposed a VEM for a nonlocal reactiondiffusion system of cardiac electric field. Optimal order space-time error estimates in L^2 norm were obtained. In [4], Arrutselvi *et al.* considered stabilization of VEM for a system of time-dependent nonlinear convection-diffusion-reaction equations and derived the optimal convergence estimates in the H^1 semi-norm. However, when dealing with strongly nonlinear problems, the crucial aspect in obtaining optimal error estimates for numerical schemes lies in demonstrating the boundedness of numerical solutions in L^∞ norm (or stronger norm) [21,23,24]. In classical FE analysis, the L^∞ boundedness of numerical solutions are usually estimated by

$$\begin{aligned}\|u_h^n\|_{L^\infty(\Omega)} &\le \|R_h u^n\|_{L^\infty(\Omega)} + \|R_h u^n - u_h^n\|_{L^\infty(\Omega)}\\ &\le \|R_h u^n\|_{L^\infty(\Omega)} + Ch^{-1}\|R_h u^n - u_h^n\|_{L^2(\Omega)}\\ &\le \|R_h u^n\|_{L^\infty(\Omega)} + Ch^{-1}(\tau^p + h^{r+1}),\end{aligned} \tag{1}$$

where u^n is the theoretical solution at time t^n, u_h^n is the fully-discrete solution, R_h denotes a projection operator, τ is the time-step size, h denotes spatial mesh size and p and $r+1$ stand for the convergence orders in temporal and spatial directions, respectively. Hence, in (1), we need a time-step restriction $\tau = \mathcal{O}(h^{1/p})$ (see Refs. [12,17,18,28,32]). These restrictions can result in the utilization of excessively small time steps, consequently making the computations significantly more time-consuming. We strongly believe that in most situations, these time-step restrictions may not be necessary.

In this paper, we will give the optimal convergence analysis with no time-step restriction. Drawing inspiration from the research conducted by Li and Sun [21,22], we decompose the numerical error into two components: one in the temporal direction and the other in the spatial direction. To accomplish this, we introduce a time-discrete system represented by equations (22)–(23). With the help of the regularity results of time-discrete solutions (see Thm. 4.5) and error bounds of the novel elliptic projections, we prove that the L^2 error between the virtual element discretization for the time-discrete system and solution on the time-discrete system is dependent on both h^2 and τh. This is a sharp contrast to previous results for the unconditional analysis of VEMs [26,37]. Moreover, if we can demonstrate that the solution of the time-discrete equations has suitable regularity, fully discrete solution in L^∞ norm can be unconditionally bounded by the induction assumption

$$\begin{aligned}\|u_h^n\|_{L^\infty(\Omega)} &\le \|R_h^n u_\tau^n\|_{L^\infty(\Omega)} + Ch^{-1}\|R_h^n u_\tau^n - u_h^n\|_{L^2(\Omega)}\\ &\le \|R_h^n u_\tau^n\|_{L^\infty(\Omega)} + Ch^{-1}(h^2 + \tau h)\\ &\le \|R_h^n u_\tau^n\|_{L^\infty(\Omega)} + C(h+\tau) \le C,\end{aligned}$$

where u_τ^n is the solution of time-discrete system (for specific details, please refer to the proof of Thm. 4.10) . With the L^∞ boundedness, we can establish optimal error estimates for the fully discrete formulation without any limitations on time-step size.

Furthermore, we do not assume that the nonlinear diffusion coefficient satisfies the usual Lipschitz continuity or ellipticity condition. Instead, we only require that the coefficient function $a(\mathbf{x}, u)$ is positive and smooth with respect to $u \in \mathbb{R}^2$ under Assumption 2.1. These conditions are more general than the usual Lipschitz continuity and ellipticity conditions. In addition, for the Crank-Nicolson scheme, the errors at certain time steps are one order lower than the average of errors at two consecutive time levels, as shown in equation (27a). The presence of nonlinear coefficients under such general conditions and the time-discrete scheme will inevitably make it difficult to analyze the boundedness of numerical solutions unconditionally. To address these challenges, we propose two new elliptic projections operator (*i.e.* $\mathcal{R}_h^n$ and $\mathcal{F}_h^n$, see Sect. 4.2) and analyze its boundedness and convergence properties. By employing the elliptic projection as well as conducting rigorous and meticulous analysis, we will overcome these fundamental problems. To the best of our knowledge, this is the first article that utilizes virtual elements in combination with the Crank-Nicolson scheme to solve a parabolic problem with such a general nonlinear diffusion coefficient. Finally, we provide numerical examples to validate the unconditional convergence results obtained in this paper.

The present paper is organized as follows. In Section 2, we present the model problem and the weak formulation. In Section 3, we give the fully discrete formulation of the quasilinear parabolic problem. The main contribution of this article is Section 4, where we analyze the L^∞ boundedness of fully discrete solution. Afterwards, in Section 5, we develop the aforementioned unconditionally optimal convergence analysis. Finally, numerical examples are given in Section 6 to validate the theoretical analysis.

2. The continuous problem and weak formulation

Let $\Omega \subset \mathbb{R}^2$ be a convex, polygonal domain with Lipschitz boundary $\partial\Omega$ and let T be the fixed final time. Consider the following quasilinear parabolic problem: find $u(\mathbf{x},t) \in \Omega \times (0,T]$ such that

$$\begin{cases} u_t(\mathbf{x},t) - \nabla \cdot (a(\mathbf{x},u)\nabla u(\mathbf{x},t)) = f(\mathbf{x},t), \text{ in } \Omega \times (0,T], \\ u(\mathbf{x},t) = 0, \text{ on } \partial\Omega \times (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \text{ in } \Omega. \end{cases} \tag{2}$$

For the nonlinearity $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, we essentially assume that $a(\mathbf{x},u)$ is a positive and continuous function with respect to $u \in \mathbb{R}$. But we do not require global Lipschitz continuity or uniform ellipticity conditions of the nonlinear term $a(\mathbf{x},u)$. The class of possible nonlinearities includes polynomial-type nonlinearities like $a(\mathbf{x},u) = \sum_{i=0}^n a_i u^i$ with $0 \le n < \infty$ and nonpolynomial-type nonlinearities like $a(\mathbf{x},u) = cos(u)$ or $a(\mathbf{x},u) = \frac{1}{1+u}$, where $a_i \in L^\infty(\Omega)$ for $i = 0, \cdots, n$. These conditions are more general than the usual global Lipschitz continuity and uniform ellipticity conditions [35] in the following sense:

- the nonlinearity $a(\mathbf{x},u)$ is allowed to approach zero as $u \to \infty$,
- the partial derivative $a_u(\mathbf{x},u)$ and $a_{uu}(\mathbf{x},u)$ may tend to ∞ as $u \to \infty$.

Specifically, the following set of assumptions holds throughout the article.

Assumption 2.1. *We assume that the nonlinear term* $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a positive and continuous function with respect to* $u \in \mathbb{R}$.

Furthermore, it follows that for any $M > 0$,

$$\mu_* \le a(\mathbf{x},u) \le \mu^*, \quad \text{and} \quad |a_u(\mathbf{x},u)| + |a_{uu}(\mathbf{x},u)| \le \mu^*, \quad \text{for } \|u\|_{L^\infty(\Omega)} \le M, \tag{3}$$

where constant M was defined in (19) and μ_*, μ^* are positive constants which depend on M. Note that the above assumptions of the nonlinearity $a(u)$ may be weakened in numerical simulations. For simplicity, the forcing term $f(\mathbf{x},t)$ is assumed to be a smooth function, but our analysis can be easily extended to problem (2) with $f = f(\mathbf{x},u)$ under the assumption that $f \in C^2(R)$ with bounded derivatives.

Throughout this paper, for any open subset $\omega \subseteq \Omega$, we use the standard Lebesgue space $L^p(\omega), 1 \le p \le \infty$ with norm $\|\cdot\|_{L^p(\omega)}$, and denote the $L^2(\omega)$ inner product by $(\cdot,\cdot)_\omega$. Let $W^{l,p}(\omega), l \ge 0$ be the l-th order of Sobolev space based on $L^p(\omega)$ and $H^l(\omega) := W^{l,2}(\omega)$, along with the corresponding norms $\|\cdot\|_{W^{l,p}(\omega)}$ and $\|\cdot\|_{H^l(\omega)}$, respectively. Set $H_0^1(\omega) := \left\{v \in H^1(\omega) : v|_{\partial\omega} = 0\right\}$. We denote, as usual, $H^{-1}(\omega)$ is the dual space of $H_0^1(\omega)$ equipped with the dual norm $\|\cdot\|_{-1}$, and duality product $\langle\cdot,\cdot\rangle_\omega$.

For functions $v(\mathbf{x},t)$ that have both a temporal variable $t \in [0,T]$ and a spatial variable $\mathbf{x} \in \Omega$, we also adopt the usual Bochner space $L^p\left(0,T;L^2(\Omega)\right)$ whose norms are defined as follows

$$\|v(\mathbf{x},t)\|_{L^p(0,T;L^2(\Omega))} = \begin{cases} \left(\int_0^T \|v(\mathbf{x},t)\|_{L^2(\Omega)}^p dt\right)^{1/p}, & 1 \le p < \infty, \\ \text{ess sup}_{t\in[0,T]} \|v(\mathbf{x},t)\|_{L^2(\Omega)}, & p = \infty. \end{cases}$$

We can further define the Sobolev space $L^p\left(0,T;H^l(\Omega)\right)$, for any integer $l > 0$. For the sake of readability, the space variable $\mathbf{x}$ and time variable t will be omitted as argument of all functions (for example, $a(u)$ means $a(\mathbf{x},u)$).

By virtue of Green's formula, we have the following variational formulation for quasilinear parabolic problem (2): find $u \in L^2(0,T;H_0^1(\Omega))$ with $u_t \in L^2(0,T;H^{-1}(\Omega))$, such that

$$\begin{cases} \mathcal{M}(u_t,v) + \mathcal{A}(u;u,v) = \langle f,v\rangle, \forall v \in H_0^1(\Omega), \\ u(0) = u_0, \end{cases} \tag{4}$$

where $\mathcal{M}(v,w) = \langle v,w\rangle$ and $\mathcal{A}(z;v,w) = (a(z)\nabla v,\nabla w), \forall v,w,z \in H_0^1(\Omega)$. And then we present some properties of bilinear form and the proof is omitted.

Lemma 2.2. *For $\mathcal{M}(\cdot,\cdot)$, it holds, for all $v,w \in H_0^1(\Omega)$,*

$$\mathcal{M}(v,w) \le \|v\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}, \quad \mathcal{M}(v,v) \ge \|v\|_{L^2(\Omega)}^2. \tag{5}$$

Concerning $\mathcal{A}(\cdot;\cdot,\cdot)$, for all $z \in L^\infty(\Omega)$ and $v,w \in H_0^1(\Omega)$, we have

$$\mathcal{A}(z;v,w) \le \mu^*|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}, \quad \mathcal{A}(z;v,v) \ge \mu_*|v|_{H^1(\Omega)}^2. \tag{6}$$

where μ^* and μ_* are positive constant defined in (3).

Moreover, well-posedness of the problem (4) can be shown by combining the Lemma 2.2 with the results in [35].

3. Virtual element discretization

In this section, we derive a virtual element formulation for the variational problem (4). For this purpose, we firstly describes the concept of polygonal decompositions of Ω in Section 3.1. And then, we introduce the virtual element space, discrete bilinear forms, projectors operators and their approximation properties in Section 3.2. We refer to the papers [2,5,11] for more details. With these preparations, we propose a linearized Crank-Nicolson virtual element methods for solving the quasilinear parabolic problem in Section 3.3.

3.1. Polygonal decompositions

Let $\mathcal{T}_h, 0 < h < 1$, be a sequence of decompositions of the domain Ω into polygons element K with

$$h_K := \text{diameter}(K), \quad h := \max_{K\in\mathcal{T}_h} h_K,$$

and h_e indicates the length of edge e. Denote the set of all edges in $\mathcal{T}_h$ by $\mathcal{E}_h$, and denote the set of edges belonging to K by $\mathcal{E}^K$.

We suppose that for all h, each element $K \in \mathcal{T}_h$, there exists positive constant ρ_0 and $\rho_1 > 0$, such that

(**A1**) K satisfies star-shaped condition with respect to a ball of radius $\rho \ge \rho_0 h_K$;
(**A2**) For each edge $e \in \mathcal{E}^K$, it holds that $h_e \ge \rho_0 h_K$;

(**A3**) $h_K \geq \rho_1 h$ for all $K \in \mathcal{T}_h$.

The mesh quasi-uniformity assumption (**A3**) ensures that the element sizes across the mesh do not vary significantly, thus guaranteeing the validity of inverse estimates (see Lem. 3.3) within the virtual element framework.

Given $\mathcal{T}_h$, we introduce the broken Sobolev space for any $s > 0$

$$H^s(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_K \in H^s(K),\ \forall K \in \mathcal{T}_h \right\},$$

together with the corresponding broken seminorms and norms

$$|v|^2_{H^s(\mathcal{T}_h)} := \sum_{K \in \mathcal{T}_h} |v|^2_{H^s(K)}, \quad \|v\|_{H^s(\mathcal{T}_h)} := \sum_{K \in \mathcal{T}_h} \|v\|^2_{H^s(K)}.$$

3.2. Virtual element space

In this subsection, we discuss the construction of two dimensional virtual element spaces which were originally introduced in [2,5]. Different from the finite element space, the virtual element space consists of both polynomial functions and implicitly defined non-polynomial functions, as well as the numerical schemes dispense with any explicit construction of the discrete shape functions. Non-polynomial parts of the discrete bilinear forms are approximated by suitable projection operators, which can be computed by only the degrees of freedoms (DoFs) associated with the virtual element space.

For given degree of accuracy $k \in \mathbb{N}$, we define the preliminary local conforming virtual element space on $K \in \mathcal{T}_h$ as follows:

$$\widetilde{V_h}(K) := \left\{ v_h \in H^1(K) \cap C^0(\partial K) : v_{h|e} \in \mathbb{P}_k(e), \forall e \subset \partial K, \Delta v_h \in \mathbb{P}_k(K) \right\}, \tag{7}$$

with the usual convention that $\mathbb{P}_{-1}(K) = \{0\}$. Obviously, we have $\mathbb{P}_k(K) \subseteq \widetilde{V_h}(K)$. Let $\Pi_k^{\nabla,K} : H^1(K) \to \mathbb{P}_k(K)$ be the projection operator, such that

$$\begin{cases} \left(\nabla \Pi_k^{\nabla,K} v, \nabla p_k\right)_{0,K} = (\nabla v, \nabla p_k)_{0,K}, & \forall p_k \in \mathbb{P}_k(K) \\ \dfrac{1}{|\partial K|} \displaystyle\int_{\partial K} \Pi_k^{\nabla,K} v \ \mathrm{d}s = \dfrac{1}{|\partial K|} \displaystyle\int_{\partial K} v \ \mathrm{d}s, & \end{cases}$$

where the second identity is needed to fix the constants.

The unisolvent set of local DoFs for the local space $\widetilde{V_h}(K)$ can be chosen as

(D_1) the values of v_h at each vertex of K,

and for $k \geq 2$:

(D_2) the edge moments $\int_e v p_{k-2} \ \mathrm{d}s,\ p_{k-2} \in \mathbb{P}_{k-2}(e)$, on each edge e of K,
(D_3) the internal moments $\int_K v p_{k-2} \ \mathrm{d}x,\ p_{k-2} \in \mathbb{P}_{k-2}(K)$.

The following extra DoFs which are taken as the moments of v of $(k-1)$- and k-order inside the element K:

$$\int_K v p_l \ \mathrm{d}x,\ p_l \in \mathbb{P}_k(K)/\mathbb{P}_{k-2}(K), \tag{8}$$

where $\mathbb{P}_k(K)/\mathbb{P}_{k-2}(K)$ is the space of polynomials in $\mathbb{P}_k(K)$ which are $L^2(K)$ orthogonal to $\mathbb{P}_{k-2}(K)$.

To avoid the requirements of the extra DoFs (8), we can use the operator $\Pi_k^{\nabla,K}$ to pinpoint the local enhanced space [2]

$$V_h(K) := \left\{ v_h \in \widetilde{V_h}(K) : (v_h, q)_K = \left(\Pi_k^{\nabla,K} v_h, q\right)_K, \forall q \in \mathbb{P}_k(K)/\mathbb{P}_{k-2}(K) \right\}. \tag{9}$$

Clearly, the DoFs uniquely identifying a function $v \in V_h(K)$ can be choosen as (D_1)-(D_3). For any integer $k \in \mathbb{N}_0$, the global discrete space is given by

$$V_h := \left\{ v_h \in L^2(\Omega) : v_h|_K \in V_h(K),\ \forall K \in \mathcal{T}_h \right\},$$

and the global DoFs for the space V_h are the direct extension of the local DoFs (D_1)-(D_3).

Then, we introduce the following L^2 projection operators which are key ingredient in virtual element construction. For any $v \in L^2(K)$, we define $\Pi_k^{0,K} : L^2(K) \to \mathbb{P}_k(K)$ is L^2-projection for scalar functions given by

$$\left(\Pi_k^{0,K} v, p_k\right)_K = (v, p_k)_K\,, \quad \forall p_k \in \mathbb{P}_k(K).$$

Similarly, the L^2-projector $\boldsymbol{\Pi}_{k-1}^{\boldsymbol{0},\boldsymbol{K}} : \left[L^2(K)\right]^2 \to [\mathbb{P}_{k-1}(K)]^2$ for vector valued is defined as, given $v \in \left[L^2(\Omega)\right]^2$,

$$\left(\boldsymbol{\Pi}_{k-1}^{\boldsymbol{0},\boldsymbol{K}} \boldsymbol{v}, \boldsymbol{p}_{k-1}\right)_K = \left(\boldsymbol{v}, \boldsymbol{p}_{k-1}\right)_K \quad \forall \boldsymbol{p}_{k-1} \in [\mathbb{P}_{k-1}(K)]^2 .$$

For the sake of simplicity, we denote by Π_k^∇, Π_k^0 and $\boldsymbol{\Pi}_{k-1}^{\boldsymbol{0}}$, the global projectors which are defined elementwise as the corresponding local projector operators. It is worth mentioning that the advantage of the space $V_h(K)$, when compared to $\widetilde{V}_h(K)$, can be using DoFs (D_1)-(D_3) to calculate the L^2 projection $\Pi_k^{0,K}$ and $\boldsymbol{\Pi}_{k-1}^{\boldsymbol{0},\boldsymbol{K}}$.

For all $K \in \mathcal{T}_h$, base on projection operators $\Pi_k^{\nabla,K}, \Pi_k^{0,K}$ and $\boldsymbol{\Pi}_{k-1}^{\boldsymbol{0},\boldsymbol{K}}$, we define the local form $\mathcal{M}_h^K(\cdot,\cdot)$ and $\mathcal{A}_h^K(\cdot;\cdot,\cdot)$ on the space $V_h(K)$ that approximate the exact forms $\mathcal{M}^K(\cdot,\cdot)$ and $\mathcal{A}^K(\cdot;\cdot,\cdot)$, respectively, in the following manner

$$\mathcal{M}_h^K(u,v) := \left(\Pi_k^{0,K} u, \Pi_k^{0,K} v\right)_K + S_{\mathcal{M}}^K\left((I-\Pi_k^{0,K})u, (I-\Pi_k^{0,K})v\right), \tag{10a}$$

$$\begin{aligned} \mathcal{A}_h^K(w;u,v) := &\left(a(\Pi_k^{0,K} w)\boldsymbol{\Pi}_{k-1}^{\boldsymbol{0},\boldsymbol{K}}(\nabla u), \boldsymbol{\Pi}_{k-1}^{\boldsymbol{0},\boldsymbol{K}}(\nabla v)\right)_K \\ &+ \nu_{\mathcal{A}}^K(w)\, S_{\mathcal{A}}^K\left((I-\Pi_k^{\nabla,K})u, (I-\Pi_k^{\nabla,K})v\right), \end{aligned} \tag{10b}$$

where stability constants $\nu_{\mathcal{A}}^K(w) = \Pi_0^{0,K}\left(a(\Pi_k^{0,K} w)\right)$ and $\Pi_0^{0,K} : L^2(K) \to \mathbb{P}_0(K)$ is the L^2 projector onto scalar valued constants.

Then, $S_{\mathcal{M}}^K, S_{\mathcal{A}}^K$ are the symmetric and positive definite bilinear form defined by

$$S_{\mathcal{M}}^K(u,v) := |K| \sum_{i=1}^{N_V^K} \chi_i(u)\chi_i(v), \quad S_{\mathcal{A}}^K(u,v) := \sum_{i=1}^{N_V^K} \chi_i(u)\chi_i(v),$$

where $N_V^K = dim\ V_h(K)$ and $\chi_i(v)$ denotes the i-th local DoFs on $V_h(K)$. According to literature [8,10,11,13], the local stability terms $S_{\mathcal{M}}^K$ and $S_{\mathcal{A}}^K$ satisfies the following properties: for all $v \in V_h \cap \ker(\Pi_k^{0,K})$

$$\begin{aligned} M_0^{\mathcal{M}} \|v\|_{L^2(K)}^2 \le S_{\mathcal{M}}^K(v,v) \le M_1^{\mathcal{M}} \|v\|_{L^2(K)}^2, \\ M_0^{\mathcal{A}} \|\nabla v\|_{L^2(K)}^2 \le S_{\mathcal{A}}^K(v,v) \le M_1^{\mathcal{A}} \|\nabla v\|_{L^2(K)}^2, \end{aligned} \tag{11}$$

where the positive constants $M_0^{\mathcal{M}}, M_1^{\mathcal{M}}, M_0^{\mathcal{A}}, M_1^{\mathcal{A}}$ are independent of h and K.

Next, we define the corresponding global bilinear forms by assembling the local counterpart as

$$\mathcal{M}_h(u,v) := \sum_{K\in\mathcal{T}_h} \mathcal{M}_h^K(u,v), \quad \mathcal{A}_h(w;u,v) := \sum_{K\in\mathcal{T}_h} \mathcal{A}_h^K(w;u,v).$$

Finally, we present several useful lemmas as follows, including local approximation properties by virtual element functions[7] and polynomial functions, as well as the inverse inequality for virtual element spaces [13,20,26].

Let $\mathcal{I}_h v \in V_h$ be the virtual element interpolation function of v, and on each element $K \in \mathcal{T}_h$, the interpolation function $\mathcal{I}_h v$ satisfies

$$\chi_i(\mathcal{I}_h v) = \chi_i(v), \quad \text{for } i = 1, \dots, N_V^K. \tag{12}$$

The interpolant function $\mathcal{I}_h v \in V_h$ satisfies the following estimate [5,8].

Lemma 3.1. *Let $v \in H_0^1(\Omega) \cap H^{k+1}(\Omega)$. Under the assumption **(A1)**–**(A2)** on the decomposition $\mathcal{T}_h$, there exists $\mathcal{I}_h v \in V_h$, such that*

$$\|v - \mathcal{I}_h v\|_{L^2(\Omega)} + h\|v - \mathcal{I}_h v\|_{H^1(\mathcal{T}_h)} \leq C_{\mathcal{I}} h^{k+1} \|v\|_{H^{k+1}(\mathcal{T}_h)}, \tag{13}$$

where the constant $C_{\mathcal{I}}$ is independent of the discretization parameters h.

Lemma 3.2. (Projection error) *Suppose that ψ and $\boldsymbol{\psi}$ are sufficiently smooth scalar and vector valued functions on $K \in \mathcal{T}_h$, respectively. Then, it holds, for all $k \in \mathbb{N}_0$,*

$$\begin{aligned}
&\left\|\psi - \Pi_k^{\nabla,K}\psi\right\|_{H^\ell(K)} \leq C_{\mathcal{P}} h_K^{s-\ell} |\psi|_{H^s(K)}, \quad 0 \leq \ell \leq s \leq k+1, s \geq 1, \\
&\left\|\psi - \Pi_k^{0,K}\psi\right\|_{H^\ell(K)} \leq C_{\mathcal{P}} h_K^{s-\ell} |\psi|_{H^s(K)}, \quad 0 \leq \ell \leq s \leq k+1 \\
&\left\|\boldsymbol{\psi} - \boldsymbol{\Pi}_k^{\mathbf{0},\boldsymbol{K}}\boldsymbol{\psi}\right\|_{H^\ell(K)} \leq C_{\mathcal{P}} h_K^{s-\ell} |\boldsymbol{\psi}|_{H^s(K)}, \quad 0 \leq \ell \leq s \leq k+1
\end{aligned} \tag{14}$$

where the constant $C_{\mathcal{P}}$ only depends on the polynomial of degree k and mesh regularity ρ_0.

Lemma 3.3. (Inverse inequality) *For $v \in V_h(K)$, it holds that*

$$\|\nabla v\|_{L^2(K)} \leq C_{\mathrm{Inv}} h^{-1} \|v\|_{L^2(K)}, \qquad \|v\|_{L^\infty(K)} \leq C_{\mathrm{Inv}} h^{-1} \|v\|_{L^2(K)}. \tag{15}$$

where positive constant C_{Inv} is independent of h.

3.3. Fully discrete formulation

With these preparations completed, we employ a Crank-Nicolson scheme and a linear extrapolation in time formula with the explicit treatment of the nonlinear terms for the quasilinear parabolic problem (2).

Assume that $\{t_n | t_n = n\tau; 0 \leq n \leq N\}$ is uniform partition of $[0,T]$ with time step-size $\tau = T/N$, and we use the notations $u^n = u(\mathbf{x}, t_n), u^{n-\frac{1}{2}} = u(\mathbf{x}, t_{n-\frac{1}{2}})$.

Let $\{\omega^n\}_{n=0}^N$ is a sequence of functions, we denote, for $1 \leq n \leq N$,

$$D_\tau \omega^n = \frac{1}{\tau}\left(\omega^n - \omega^{n-1}\right), \quad \overline{\omega}^n = \frac{1}{2}\left(\omega^n + \omega^{n-1}\right),$$

and the second-order linear extrapolation operator is defined as

$$\widehat{\omega}^n = \frac{3}{2}\omega^{n-1} - \frac{1}{2}\omega^{n-2}, \quad \text{for } 2 \leq n \leq N. \tag{16}$$

With above notations and the discrete bilinear forms introduced earlier, a linearized Crank-Nicolson virtual element method is : to find $u_h^n \in V_h$ such that

$$\begin{cases}
\mathcal{M}_h\left(D_\tau u_h^n, v_h\right) + \mathcal{A}_h\left(\widehat{u}_h^n; \overline{u}_h^n, v_h\right) = (\Pi_k^0 \overline{f}^n, v_h), \\
u_h^0 := \mathcal{I}_h u_0,
\end{cases} \tag{17}$$

for all $v_h \in V_h$ and $n = 1, \dots, N$, where $I_h u_0$ is virtual element interpolant of u_0 in space V_h.

The numerical scheme (17) is semi-implicit and we only need to solve a linear (elliptic) system at each time level. To maintains the second-order accuracy of temporal discretization, some starting approximations should be given. So we choose the most commonly used start-up procedure: let $\widehat{u}_h^1 = \frac{1}{2}(u_h^{1**} + u_h^0)$ and u_h^{1*} is the solution of the following system

$$\mathcal{M}_h\left(D_\tau u_h^{1*}, v_h\right) + \mathcal{A}_h\left(u_h^0; u_h^{1*}, v_h\right) = (\Pi_k^0 f^1, v_h), \tag{18a}$$
$$\mathcal{M}_h\left(D_\tau u_h^{1**}, v_h\right) + \mathcal{A}_h\left(u_h^{1*}; u_h^{1**}, v_h\right) = (\Pi_k^0 f^1, v_h), \tag{18b}$$

for all $v_h \in V_h$, where $D_\tau u_h^{1*} := \frac{1}{\tau}\left(u_h^{1*} - u_h^0\right)$ and $D_\tau u_h^{1**} := \frac{1}{\tau}\left(u_h^{1**} - u_h^0\right)$.

We will obtain the unconditionally optimal convergence result for the quasilinear parabolic problem (2) based on the following regularity assumption of the exact solution,

$$\begin{aligned}\|u_0\|_{H^{k+1}(\Omega)} &+ \|u\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + \|u\|_{L^\infty(0,T;W^{2,p}(\Omega))} + \|u\|_{L^\infty(0,T;H^{k+1}(\Omega))} \\ &+ \|u_t\|_{L^2(0,T;H^{k+1}(\Omega))} + \|u_{tt}\|_{L^2(0,T;H^2(\Omega))} + \|u_{ttt}\|_{L^2(0,T;L^p(\Omega))} \leq M,\end{aligned} \tag{19}$$

here, $2 \leq p \leq 4$ and the positive constant M independent of the discretization parameters h and τ. Although the exact solution of the problem (2) is satisfying the regularity assumption (19) in most cases, these local conditions of $a(u)$ (that is, Assumption 2.1) will impose difficulties on the analysis of the numerical solution, which is not known to be bounded *a priori*. A challenge is to prove that the uniform (pointwise) boundedness of the discrete approximation u_h^n to u^n. For this purpose, we adopt the error splitting technique in Li and Sun [21, 23] to prove the pointwise boundedness of the numerical solution in the next section. Based on this result, we obtain the unconditionally optimal error estimates of the linearized Crank-Nicolson virtual element method for the quasilinear parabolic problem (2) in Section 5.

With the above discussion, the following continuity and coercivity properties for $\mathcal{M}_h(\cdot,\cdot)$ and $\mathcal{A}_h(\cdot,\cdot)$, defined (10a) and (10b), respectively, hold true.

Lemma 3.4. *For $\mathcal{M}_h(\cdot,\cdot)$, it holds true that , for all $v_h, w_h \in V_h$,*

$$\mathcal{M}_h\left(v_h, w_h\right) \leq \alpha^* \left\|v_h\right\|_{L^2(\Omega)} \left\|w_h\right\|_{L^2(\Omega)} \quad \text{and} \quad \mathcal{M}_h\left(v_h, v_h\right) \geq \alpha_* \left\|v_h\right\|_{L^2(\Omega)}^2. \tag{20}$$

Concerning $\mathcal{A}_h(\cdot,\cdot)$, if $z_h \in L^\infty(\Omega)$, and for all $v_h, w_h \in V_h$, it holds true that

$$\mathcal{A}_h\left(z_h; v_h, w_h\right) \leq \beta^* |v_h|_{H^1(\mathcal{T}_h)} |w_h|_{H^1(\mathcal{T}_h)} \text{ and } \mathcal{A}_h\left(z_h; v_h, v_h\right) \geq \beta_* |v_h|_{H^1(\mathcal{T}_h)}^2. \tag{21}$$

Proof. To show (20), from the definition (10a) of $\mathcal{M}_h^K$ and properties (11), we have

$$\mathcal{M}_h^K\left(v_h, v_h\right) \leq \left\|\Pi_k^{0,K} v_h\right\|_{L^2(K)}^2 + M_1^{\mathcal{M}} \left\|\left(I - \Pi_k^{0,K}\right) v_h\right\|_{L^2(K)}^2 \leq \alpha^* \|v_h\|_{L^2(K)}^2,$$

where the positive constant $\alpha^* = \max\{1, M_1^{\mathcal{M}}\}$. Then, by using Cauchy-Schwarz inequality, we can directly get the following result

$$\mathcal{M}_h\left(v_h, w_h\right) \leq \sum_{K \in \mathcal{T}_h} \left(\mathcal{M}_h^K\left(v_h, v_h\right)\right)^{1/2} \left(\mathcal{M}_h^K\left(w_h, w_h\right)\right)^{1/2} \leq \alpha^* \left\|v_h\right\|_{L^2(\Omega)} \left\|w_h\right\|_{L^2(\Omega)}.$$

And by utilizing the properties (11) and the Pythagorean theorem, we can obtain the following coercivity bound

$$\mathcal{M}_h\left(v_h, v_h\right) \geq \|\Pi_k^{0,K} v_h\|_{L^2(\Omega)}^2 + M_0^{\mathcal{M}} \|(I - \Pi_{k+1}^{0,K}) v_h\|_{L^2(\Omega)}^2 \geq \alpha_* \left\|v_h\right\|_{L^2(\Omega)}^2,$$

where $\alpha_* = \min\{1, M_0^{\mathcal{M}}\}$ is a positive constant.

Applying the inverse estimate (15), the continuity of the L^2 projector, and the Hölder's inequality, it follows that for all $z_h \in L^\infty(K)$

$$\left\|\Pi_k^{0,K} z_h\right\|_{L^\infty(K)} \leq C_{\mathrm{Inv}} h_K^{-1} \left\|\Pi_k^{0,K} z_h\right\|_{L^2(K)} \leq C_{\mathrm{Inv}} h_K^{-1} \|z_h\|_{L^2(K)} \leq C_{\mathrm{Inv}} \|z_h\|_{L^\infty(K)} \leq M,$$

and this implies that assumption (3) is satisfied. Then, recalling the definition (10b) of $\mathcal{A}_h^K$ and the stability properties (11), we have

$$\begin{aligned}\mathcal{A}_h^K(z_h; v_h, v_h) &\leq \mu^* \left\|\mathbf{\Pi}_{k-1}^{\mathbf{0},\boldsymbol{K}}(\nabla v_h)\right\|_{L^2(K)}^2 + \mu^* M_1^{\mathcal{A}} \left\|\nabla(I - \Pi_k^{\nabla,K}) v_h\right\|_{L^2(K)}^2 \\ &\leq \max\{1, M_1^{\mathcal{A}}\} \mu^* \|\nabla v_h\|_{L^2(K)}^2 .\end{aligned}$$

Setting $\beta^* := \max\{\mu^*, \mu^* M_1^{\mathcal{A}}\}$, using the Hölder's inequality, and summing over all elements, the continuity bound in (21) holds. And then, from the definition of $\mathcal{A}_h^K$ given in (10b), we can deduce that

$$\mathcal{A}_h^K(z_h; v_h, v_h) \geq \mu_* \left\|\mathbf{\Pi}_{k-1}^{\mathbf{0},\boldsymbol{K}}(\nabla v_h)\right\|_{L^2(K)}^2 + \mu_* M_0^{\mathcal{A}} \left\|\nabla(I - \Pi_{k-1}^{\nabla,K}) v_h\right\|_{L^2(K)}^2 .$$

Noting that $\|\nabla(I - \Pi_k^{\nabla,K}) v_h\|_{L^2(K)} \geq \|(\boldsymbol{I} - \mathbf{\Pi}_{k-1}^{\mathbf{0},\boldsymbol{K}}) \nabla v_h\|_{L^2(K)}$ and using the Pythagorean theorem, we have

$$\begin{aligned}\mathcal{A}_h(z_h; v_h, v_h) &\geq \sum_{K \in \mathcal{T}_h} \mu_* \left\|\mathbf{\Pi}_{k-1}^{\mathbf{0},\boldsymbol{K}}(\nabla v_h)\right\|_{L^2(K)}^2 + \mu_* M_0^{\mathcal{A}} \left\|(\boldsymbol{I} - \mathbf{\Pi}_{k-1}^{\mathbf{0},\boldsymbol{K}}) \nabla v_h\right\|_{L^2(K)}^2 \\ &\geq \sum_{K \in \mathcal{T}_h} \beta_* \|\nabla v_h\|_{L^2(K)}^2 = \beta_* |v_h|_{H^1(\mathcal{T}_h)}^2 .\end{aligned}$$

where $\beta_* = \min\{\mu_*, \mu_* M_0^{\mathcal{A}}\}$. This completes the proof. □

4. The L^∞ boundedness of the fully discrete solution

The key point in this paper, to deal with nonlinearity $a(u)$ in (2), is the L^∞ boundedness of the numerical solution u_h^n. To this end, we introduce the corresponding time-discrete system to split the error into two parts: the temporal discretization error and the spatial discretization error. Since the spatial discretization error is only slightly dependent on time step size τ, we can prove that the numerical solution is bounded in the L^∞ norms by the inverse inequalities.

4.1. Time-discrete system and temporal error estimates

In this subsection, we adopt the error splitting argument in [21,23] by introducing the semi-discrete linearized Crank-Nicolson system of the quasilinear parabolic problem (2). Then we establish uniform L^∞ boundedness and optimal error estimations for the time-discrete numerical solution in the Hilbert space setting.

The semi-discrete Crank-Nicolson approximation u_τ^n to $u^n, 1 \leq n \leq N$, are given recursively as the solution to the following system

$$\begin{cases} D_\tau u_\tau^n - \nabla \cdot (a(\widehat{u}_\tau^n) \nabla \overline{u}_\tau^n) = \overline{f}^n, \text{ in } \Omega \\ u_\tau^n(\mathbf{x}) = 0, \text{ on } \partial\Omega \times (0, T], \\ u_\tau^0(\mathbf{x}) = u_0(\mathbf{x}), \text{ in } \Omega, \end{cases} \tag{22}$$

where $\widehat{u}_\tau^1 = \frac{1}{2}(u_\tau^{1**} + u_\tau^0)$ and u_τ^{1**} are defined by

$$D_\tau u_\tau^{1*} - \nabla \cdot \left(a(u_\tau^0) \nabla u_\tau^{1*}\right) = f^1, \tag{23a}$$

$$D_\tau u_\tau^{1**} - \nabla \cdot \left(a(u_\tau^{1*}) \nabla u_\tau^{1**}\right) = f^1, \tag{23b}$$

with the homogenous Dirichlet boundary condition. Noting that the fully discrete solution u_h^n is identical to the virtual element solution of (22)–(23), and the error estimates in arbitrary norms $\|\cdot\|_*$ can be decomposed as

$$\|u^n - u_h^n\|_* \leq \|u^n - u_\tau^n\|_* + \|u_\tau^n - u_h^n\|_*.$$

Under the regularity assumption of exact solution (19), we define

$$M_1 := \max_{0\leq n\leq N} \|u^n\|_{H^2(\Omega)} + 1. \tag{24}$$

Now, we introduce the following lemmas, which is critical in the error estimates.

Lemma 4.1. (Gagliardo-Nirenberg inequality [19]) *Let v be a function defined on Ω and set $2 \leq p < \infty$. Then there exists a constant C_Ω depending on Ω and p such that*

$$\|v\|_{L^p(\Omega)} \leq C_\Omega \|v\|_{L^2(\Omega)}^{\frac{2}{p}} \|\nabla v\|_{L^2(\Omega)}^{1-\frac{2}{p}}, \quad \forall v \in H^1(\Omega). \tag{25}$$

Moreover, for $p = \infty$, we have the following Agmon's inequality

$$\|v\|_{L^\infty(\Omega)} \leq C_\Omega \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \|v\|_{H^2(\Omega)}^{\frac{1}{2}}, \quad \forall v \in H^1(\Omega) \cap H^2(\Omega). \tag{26}$$

Lemma 4.2. *Let $\{\omega^i\}_{i=0}^N$ and $\{v^i\}_{i=0}^N$ be two sequences of functions in domain Ω. Then, it holds that, for any norm $\|\cdot\|_*$*

$$\tau\|\omega^n\|_* \leq 2\tau \sum_{i=1}^n \|\overline{\omega}^n\|_* + \tau\|\omega^0\|_* \leq 2\sqrt{T}\left(\tau \sum_{i=1}^n \|\overline{\omega}^n\|_*^2 + \tau\|\omega^0\|_*^2\right)^{\frac{1}{2}}, \tag{27a}$$

$$\tau \sum_{i=1}^n \left(D_\tau \omega^i, \overline{v}^i\right) = \frac{\tau}{2}\left(D_\tau \omega^1, v^0\right) + \tau \sum_{i=2}^n \left(D_\tau \overline{\omega}^i, v^{i-1}\right) + \frac{\tau}{2}\left(D_\tau \omega^n, v^n\right). \tag{27b}$$

Lemma 4.3. *Suppose that ϕ is a solution of the following the linear elliptic problem:*

$$\begin{cases} -\nabla\cdot(b(\mathbf{x})\nabla\phi) = f, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $b(\mathbf{x}) \in C^1(\mathbb{R})$ satisfies

$$C_\iota^{-1} \leq b(\mathbf{x}) \leq C_\iota, \quad \forall x \in \Omega,$$

for some positive constant C_ι. Then, it holds that (see [16])

$$\|\phi\|_{W^{2,p}(\Omega)} \leq C_\Omega \|f\|_{L^p(\Omega)}$$

for $2 \leq p \leq 4$, where C_Ω is a positive constant merely related to the domain Ω.

Next, we have the following uniform L^∞ boundedness and optimal convergence result for the time-discrete solution.

Lemma 4.4. *Suppose that the problem (2) has a unique solution u satisfying (19). Then, there exists a constant $\tau_0' > 0$ such that when $\tau \leq \tau_0'$, the time-discrete system (23) admits a unique solution u_τ^{1*} and u_τ^{1**}, which satisfies*

$$\|e_\tau^{1**}\|_{L^2(\Omega)} + \|e_\tau^{1**}\|_{H^1(\Omega)} + \tau\|e_\tau^{1**}\|_{H^2(\Omega)} + \tau^{\frac{1}{2}}\|e_\tau^{1**}\|_{L^\infty(\Omega)} \leq C_0'\tau^2, \tag{28a}$$

$$\|u_\tau^{1*}\|_{H^2(\Omega)} + \|u_\tau^{1*}\|_{L^\infty(\Omega)} + \|\nabla u_\tau^{1*}\|_{L^\infty(\Omega)} + \tau\|D_\tau u_\tau^{*}\|_{H^2(\Omega)} \leq C_0', \tag{28b}$$

$$\|u_\tau^{1**}\|_{H^2(\Omega)} + \|u_\tau^{1**}\|_{L^\infty(\Omega)} + \|\nabla u_\tau^{1**}\|_{L^\infty(\Omega)} + \tau\|D_\tau u_\tau^{**}\|_{H^2(\Omega)} \leq C_0', \tag{28c}$$

where C_0' is a positive constant independent of τ.

Proof. For the purpose of readability, we provide the proof in Appendix A.1. □

Theorem 4.5. *Suppose that the quasilinear parabolic problem* (2) *has a unique solution u satisfying* (19). *Then, there exists a constant $\tau_0' > 0$ such that when $\tau \le \tau_1'$, the time-discrete system* (22)–(23) *admits a unique solution $u_\tau^n, n = 1, \cdots, N$, which satisfies*

$$\|u_\tau^n\|_{H^2(\Omega)} + \|u_\tau^n\|_{L^\infty(\Omega)} + \|\nabla u_\tau^n\|_{L^\infty(\Omega)} + \tau\|D_\tau u_\tau^1\|_{H^2(\Omega)} + \tau\sum_{i=2}^{N}\|D_\tau u_\tau^i\|_{H^2(\Omega)}^2 \le C_1', \tag{29a}$$

where C_1' is a positive constant independent of τ.

Proof. To start with, we denote $e_\tau^n := u_\tau^n - u^n$ and $e_\tau^0 = 0$. By taking $t = t_{n-\frac{1}{2}}$ in (2) and using time-discrete system (22), we obtain the following error equations, for $n = 1, 2, \ldots, N$,

$$D_\tau e_\tau^n - \nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n) = E^n + \nabla\cdot H^n \tag{30}$$

with the boundary condition $e_\tau^n = 0$ on $\partial\Omega$, where the truncation errors are given by

$$\begin{aligned} &E^n = u_t^{n-\frac{1}{2}} - D_\tau u^n + \overline{f}^n - f^{n-\frac{1}{2}},\\ &H^n := \left(a(\widehat{u}_\tau^n) - a(u^{n-\frac{1}{2}})\right)\nabla\overline{u}^n + a(u^{n-\frac{1}{2}})\nabla\left(\overline{u}^n - u^{n-\frac{1}{2}}\right) := H_A^n + H_B^n. \end{aligned}$$

In terms of the Taylor expansion, the truncation errors can be bounded by

$$\|E^1\|_{L^p(\Omega)}^p \le (C_\kappa)^p\tau^{2p}\left(\|u_{ttt}(t_*)\|_{L^p(\Omega)}^p + \|f_{tt}(t_*)\|_{L^p(\Omega)}^p\right), \tag{31a}$$

$$\sum_{i=2}^{n}\|E^i\|_{L^p(\Omega)}^p \le (C_\kappa)^p\tau^{2p-1}\int_{t_2}^{t_n}\left(\|u_{ttt}(s)\|_{L^p(\Omega)}^p + \|f_{tt}(s)\|_{L^2(\Omega)}^p\right)ds, \tag{31b}$$

for $2 \le p \le 4$, where $t_0 \le t_* \le t_1$ and C_κ is a positive constant independent of τ.

As for the subsequent proof, taking into account the complications arising from the time-discrete system, for convenience, we divide it into two steps.

Step 1: Error bounds for e_τ^1. With the boundedness of $\|u_\tau^{1**}\|_{L^\infty(\Omega)}$, the existence and uniqueness of solution to (22) at t_1 follow immediately since it is a linear elliptic equation.

Using the definition of H_A^1 and H_B^1 together with the Taylor expansion and (28a) yields

$$\begin{aligned} \|\nabla\cdot H_A^1\|_{L^2(\Omega)} &\le \left\|\nabla\left(a(\widehat{u}_\tau^1) - a(u^{\frac{1}{2}})\right)\right\|_{L^2(\Omega)}\|\nabla\overline{u}^1\|_{L^\infty(\Omega)}\\ &\quad + \|a(\widehat{u}_\tau^1) - a(u^{\frac{1}{2}})\|_{L^\infty(\Omega)}\left\|\Delta\overline{u}^1\right\|_{L^2(\Omega)}\\ &\le \mu^* M\left(\|\nabla e_\tau^{1**}\|_{L^2(\Omega)} + \|\nabla(\overline{u}^1 - u^{\frac{1}{2}})\|_{L^2(\Omega)}\right)\\ &\quad + \mu^* M\left(\|e_\tau^{1**}\|_{L^\infty(\Omega)} + \|\overline{u}^1 - u^{\frac{1}{2}}\|_{L^\infty(\Omega)}\right)\\ &\le \mu^* M\left(2C_0' + C_\kappa\|u_{tt}(t_*)\|_{H^1(\Omega)} + C_\kappa\|u_{tt}(t_*)\|_{L^\infty(\Omega)}\right)\tau^{\frac{3}{2}}\\ \|\nabla\cdot H_B^1\|_{L^2(\Omega)} &\le \left\|\nabla a(u_\tau^{\frac{1}{2}})\cdot\nabla(\overline{u}^1 - u^{\frac{1}{2}})\right\|_{L^2(\Omega)} + \left\|a(u^{\frac{1}{2}})\Delta(\overline{u}^1 - u^{\frac{1}{2}})\right\|_{L^2(\Omega)}\\ &\le 2\mu^* M C_\kappa\tau^2\|u_{tt}(t_*)\|_{H^2(\Omega)}. \end{aligned} \tag{32}$$

Multiplying the equation (30) for $n = 1$ by $D_\tau e_\tau^1$, and integrating the resulting equation in the domain Ω, we get

$$\frac{1}{\tau^2}\left\|e_\tau^1\right\|_{L^2(\Omega)}^2 + \frac{\mu_*}{\tau}\left\|\nabla e_\tau^1\right\|_{L^2(\Omega)}^2 \le \frac{1}{2\tau^2}\left\|e_\tau^1\right\|_{L^2(\Omega)}^2 + \|E^1\|_{L^2(\Omega)}^2 + \|\nabla\cdot H_A^1\|_{L^2(\Omega)}^2,$$

from which, in view of (31a) and (32), we conclude that

$$\|e_\tau^1\|_{L^2(\Omega)} \leq \sqrt{2}C_1\tau^2, \tag{33}$$

where $C_1 = 2\mu^* M(C_0' + (1+C_\Omega)C_\kappa\|u_{tt}(t_*)\|_{H^2(\Omega)}) + C_\kappa(\|u_{ttt}(t_*)\|_{L^2(\Omega)} + \|f_{tt}(t_*)\|_{L^2(\Omega)})$. Then, integrating (30) for $n=1$ against $-\nabla\cdot(a(\widehat{u}_\tau^1)\nabla e_\tau^1)$ in the over domain Ω, and applying the Cauchy-Schwarz inequality and (32), we have

$$\mu_*\|\nabla e_\tau^1\|_{L^2(\Omega)}^2 + \frac{\tau}{2}\|\nabla\cdot(a(\widehat{u}_\tau^1)\nabla e_\tau^1)\|_{L^2(\Omega)}^2 \leq (C_1)^2\tau^4,$$

which implies that

$$\|\nabla e_\tau^1\|_{L^2(\Omega)} \leq \mu_*^{-\frac{1}{2}}C_1\tau^2, \quad \|\nabla\cdot(a(\widehat{u}_\tau^1)\nabla e_\tau^1)\|_{L^2(\Omega)} \leq \sqrt{2}C_1\tau^{\frac{3}{2}}, \tag{34}$$

and together with Lemma 4.3 gives

$$\begin{aligned}\|e_\tau^1\|_{H^2(\Omega)} &\leq C_\Omega\|E^1 + \nabla\cdot H^1 - D_\tau e_\tau^1\|_{L^2(\Omega)}\\ &= C_\Omega\|\nabla\cdot(a(\widehat{u}_\tau^1)\nabla e_\tau^1)\|_{L^2(\Omega)} \leq \sqrt{2}C_\Omega C_1\tau^{\frac{3}{2}}.\end{aligned} \tag{35}$$

And owing to the Agmon's inequality (26), we obtain that

$$\|e_\tau^1\|_{L^\infty(\Omega)} \leq C_\Omega\|e_\tau^1\|_{L^2(\Omega)}^{\frac{1}{2}}\|e_\tau^1\|_{H^2(\Omega)}^{\frac{1}{2}} \leq \sqrt{2C_4}(C_\Omega)^{\frac{3}{2}}\tau^{\frac{3}{2}}. \tag{36}$$

We apply Lemma 4.3 to the error equation (30) for $n=1$,

$$\left\|e_\tau^1\right\|_{W^{2,4}(\Omega)} \leq 2C_\Omega\left(\left\|E^1\right\|_{L^4(\Omega)} + \left\|\nabla\cdot H^1\right\|_{L^4(\Omega)} + \left\|D_\tau e_\tau^1\right\|_{L^4(\Omega)}\right). \tag{37}$$

Then, following the steps used in (32) and using the Gagliardo-Nirenberg inequality (25), the Sobolev embedding theorem, and estimate (28), we obtain that

$$\begin{aligned}&\left\|\nabla\cdot H^1\right\|_{L^4(\Omega)} = \left\|\nabla\cdot H_A^1\right\|_{L^4(\Omega)} + \left\|\nabla\cdot H_B^1\right\|_{L^4(\Omega)}\\ &\leq \mu^* M\left(\|\nabla e_\tau^{1**}\|_{L^4(\Omega)} + \|\nabla(\overline{u}^1 - u^{\frac{1}{2}})\|_{L^4(\Omega)} + \|e_\tau^{1**}\|_{L^\infty(\Omega)} + \|\overline{u}^1 - u^{\frac{1}{2}}\|_{L^\infty(\Omega)}\right)\\ &+ \mu^* M\left(\|\nabla(\overline{u}^1 - u^{\frac{1}{2}})\|_{L^4(\Omega)} + \|\Delta(\overline{u}^1 - u^{\frac{1}{2}})\|_{L^4(\Omega)}\right)\\ &\leq \mu^* M\bigg(\|e_\tau^{1**}\|_{H^2(\Omega)} + \|e_\tau^{1**}\|_{L^\infty(\Omega)} + C_\kappa\tau^2\left(2\|u_{tt}(t_*)\|_{H^2(\Omega)} + \|u_{tt}(t_*)\|_{L^\infty(\Omega)}\right)\bigg)\\ &\leq 2\mu^* M\left(C_0' + (1+C_\Omega)C_\kappa\|u_{tt}(t_*)\|_{H^2(\Omega)}\right)\tau.\end{aligned}$$

Together with the estimate

$$\left\|D_\tau e_\tau^1\right\|_{L^4(\Omega)} \leq C_\Omega\left\|D_\tau e_\tau^1\right\|_{H^1(\Omega)} \leq (\mu_*)^{-\frac{1}{2}}C_\Omega C_1\tau$$

and using the Taylor expansion and the Sobolev embedding theorem, we further have

$$\left\|\nabla e_\tau^1\right\|_{L^\infty(\Omega)} \leq C_\Omega\left\|e_\tau^1\right\|_{W^{2,4}(\Omega)} \leq C_\Omega C_2\tau, \tag{38}$$

where $C_2 = 4\mu^* M(C_0' + (1+C_\Omega)C_\kappa\|u_{tt}(t_*)\|_{H^2(\Omega)}) + (\mu_*)^{-\frac{1}{2}}C_\Omega C_1 + C_\kappa(\|u_{ttt}(t_*)\|_{L^4(\Omega)} + \|f_{tt}(t_*)\|_{L^4(\Omega)})$.

Hence, there exists a positive constant τ_1 such that when $\tau < \tau_1$

$$\begin{aligned}\|u_\tau^1\|_{H^2(\Omega)} &\leq \|u^1\|_{H^2(\Omega)} + \|e_\tau^1\|_{H^2(\Omega)} \leq M_1,\\ \tau\|D_\tau u_\tau^1\|_{H^2(\Omega)} &\leq \tau\|D_\tau u^1\|_{H^2(\Omega)} + \tau\|D_\tau e_\tau^1\|_{H^2(\Omega)} \leq M_1,\\ \|u_\tau^1\|_{L^\infty(\Omega)} &\leq \|u^1\|_{L^\infty(\Omega)} + \|e_\tau^1\|_{L^\infty(\Omega)} \leq M_1,\\ \|\nabla u_\tau^1\|_{L^\infty(\Omega)} &\leq \|\nabla u^1\|_{L^\infty(\Omega)} + \|\nabla e_\tau^1\|_{L^\infty(\Omega)} \leq M_1,\end{aligned} \tag{39}$$

where constant $\tau_1 = \min\{(\sqrt{2}C_\Omega C_1)^{-\frac{2}{3}}, (C_\Omega)^{-1}(2C_1)^{-\frac{1}{3}}, (C_\Omega C_2)^{-1}\}$ independent of τ.

Step 2: Error bounds for $e_\tau^n, 2 \le n \le N$. Noting that the linear elliptic equation (22) admits a unique bounded solution $u_\tau^n \in H_0^1, 2 \le n \le N$ at each time step if u_τ^{n-1} and u_τ^{n-2} is L^∞ bounded. For this purpose, we invoke a mathematical induction on

$$\|u^n - u_\tau^n\|_{L^2(\Omega)} + \tau\|u^n - u_\tau^n\|_{H^2(\Omega)} \le C_3\tau^2. \tag{40}$$

Assume that (40) holds for $1 \le n \le m$ (which is already proved for $m = 1$ when $\tau < \tau_1$) and try to prove that it also holds for $n = m+1$. Based on the above assumptions, and using the Agmon's inequality (26), we have

$$\|e_\tau^{n-1}\|_{L^\infty(\Omega)} \le C_\Omega\|e_\tau^{n-1}\|_{L^2(\Omega)}^{\frac{1}{2}}\|e_\tau^{n-1}\|_{H^2(\Omega)}^{\frac{1}{2}} \le C_\Omega C_3\tau^{\frac{3}{2}}. \tag{41}$$

Therefore, it holds that when $\tau < \tau_2$

$$\|u_\tau^{n-1}\|_{L^\infty(\Omega)} + \|\overline{u}_\tau^{n-1}\|_{L^\infty(\Omega)} + \|\widehat{u}_\tau^n\|_{L^\infty(\Omega)} \le M_1, \tag{42}$$

where the constant $\tau_2 = (C_\Omega C_3)^{-\frac{2}{3}}$ independent of τ. Next, we turn our attention to the estimate of e_τ^n. Integrating (30) against $\overline{e}_\tau^n$ in the over domain Ω, we have

$$(D_\tau e_\tau^n, \overline{e}_\tau^n)_\Omega + (a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n, \nabla\overline{e}_\tau^n)_\Omega = (E^n, \overline{e}_\tau^n)_\Omega + (H_A^n, \nabla\overline{e}_\tau^n)_\Omega + (H_B^n, \nabla\overline{e}_\tau^n)_\Omega\,. \tag{43}$$

Using the (2.1), Young's inequality, and the Taylor expansion, the last two terms in the above equation can be bounded by

$$\begin{aligned}
(H_A^n, \nabla\overline{e}_\tau^n)_\Omega &\le \|\nabla\overline{u}^n\|_{L^\infty(\Omega)}\|a(\widehat{u}_\tau^n) - a(u^{n-\frac{1}{2}})\|_{L^2(\Omega)}\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}\\
&\le \mu^* M\left(\|\widehat{e}_\tau^n\|_{L^2(\Omega)} + \|\widehat{u}^n - u^{n-\frac{1}{2}}\|_{L^2(\Omega)}\right)\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}\\
&\le \frac{(2\mu^* M)^2}{\mu_*}\left(\|\widehat{e}_\tau^n\|_{L^2(\Omega)}^2 + C_\kappa^2\tau^3\int_{t_{n-2}}^{t_{n-\frac{1}{2}}}\|u_{tt}(s)\|_{L^2(\Omega)}^2 ds\right) + \frac{\mu_*}{4}\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}^2,\\
(H_B^n, \nabla\overline{e}_\tau^n)_\Omega &\le \|a(u^{n-\frac{1}{2}})\|_{L^\infty(\Omega)}\|\nabla(\overline{u}^n - u^{n-\frac{1}{2}})\|_{L^2(\Omega)}\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}\\
&\le (\mu^* C_\kappa)^2\tau^3\int_{t_{n-1}}^{t_n}\|u_{tt}(s)\|_{H^1(\Omega)}^2 ds + \frac{\mu_*}{4}\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}^2,
\end{aligned}$$

Based on the above estimates, and applying the Cauchy-Schwarz inequality, (43) can be rewritten in the following form

$$\begin{aligned}
&\frac{1}{2\tau}\left(\|e_\tau^n\|_{L^2(\Omega)}^2 - \|e_\tau^{n-1}\|_{L^2(\Omega)}^2\right) + \mu_*\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}^2\\
&\le \frac{1}{2}\|E^n\|_{L^2(\Omega)}^2 + \frac{\mu_* + 3(2\mu^* M)^2}{2\mu_*}\sum_{i=0}^{2}\|e_\tau^{n-i}\|_{L^2(\Omega)}^2 + \frac{\mu_*}{2}\|\nabla\overline{e}_\tau^n\|_{L^2(\Omega)}^2\\
&+ \frac{1}{\mu_*}(4M^2 + \mu_*)(C_\kappa)^2\tau^3\int_{t_{n-2}}^{t_n}\|u_{tt}(s)\|_{H^1(\Omega)}^2 ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\frac{1}{2\tau}\left(\|e_\tau^n\|_{L^2(\Omega)}^2 - \|e_\tau^{n-1}\|_{L^2(\Omega)}^2\right) \le \frac{1}{2}\|E^n\|_{L^2(\Omega)}^2 + \frac{\mu_* + 3(2\mu^* M)^2}{2\mu_*}\sum_{i=0}^{2}\|e_\tau^{n-i}\|_{L^2(\Omega)}^2\\
+ \frac{1}{\mu_*}(4M^2 + \mu_*)(C_\kappa)^2\tau^3\int_{t_{n-2}}^{t_n}\|u_{tt}(s)\|_{H^1(\Omega)}^2 ds.
\end{aligned}$$

Multiplying both sides of the inequality above by 2τ and summing up from 2 to n, we obtain that

$$\|e_\tau^n\|^2_{L^2(\Omega)} \le \|e_\tau^1\|^2_{L^2(\Omega)} + 2\sum_{i=2}^{n} \tau\|E^n\|^2_{L^2(\Omega)} + \frac{\mu_* + 3(2\mu^* M)^2}{\mu_*} \sum_{i=0}^{n} \tau \|e_\tau^i\|^2_{L^2(\Omega)} + \frac{2}{\mu_*}(4M^2+\mu_*)(C_\kappa)^2\tau^4 \int_{t_0}^{t_n} \|u_{tt}(s)\|^2_{H^1(\Omega)} ds,$$

With the Gronwall's inequality, we obtain that when $\tau < \tau_3 = \frac{\mu_*}{2(4\mu^* M)^2}$,

$$\|e_\tau^n\|_{L^2(\Omega)} \le C_4\tau^2, \tag{44}$$

where the constant C_4 only is dependent on C_κ, μ_*, μ^* and regularity constant M but independent of time-discrete parameter τ and n.

By employing the following inequality

$$\begin{aligned}(D_\tau e_\tau^n, -\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n))_\Omega &= \frac{1}{2\tau}\left(a(\widehat{u}_\tau^n)\nabla(e_\tau^n - e_\tau^{n-1}), \nabla(e_\tau^n + e_\tau^{n-1})\right)_\Omega \\ &\ge \frac{\mu_*}{2\tau}\left(\|\nabla e_\tau^n\|^2_{L^2(\Omega)} - \|\nabla e_\tau^{n-1}\|^2_{L^2(\Omega)}\right),\end{aligned}$$

and integrating (30) against $-\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)$ in the over domain Ω, and applying the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}&\frac{\mu_*}{2\tau}\left(\|\nabla e_\tau^n\|^2_{L^2(\Omega)} - \|\nabla e_\tau^{n-1}\|^2_{L^2(\Omega)}\right) + \|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\|^2_{L^2(\Omega)} \\ &\le \frac{1}{2}\|E^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\overline{e}^n\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\|^2_{L^2(\Omega)} \\ &+ \|\nabla\cdot H_A^n\|^2_{L^2(\Omega)} + \|\nabla\cdot H_B^n\|^2_{L^2(\Omega)}.\end{aligned} \tag{45}$$

We now use the Hölder's inequality, regularity condition (19), the embedding theorem $H^1(\Omega)\hookrightarrow L^4(\Omega)$, and the truncation errors (31), to deduce

$$\begin{aligned}\|\nabla\cdot H_A^n\|_{L^2(\Omega)} &\le \left\|\nabla\left(a(\widehat{u}_\tau^n) - a(u^{n-\frac{1}{2}})\right)\right\|_{L^2(\Omega)} \|\nabla\overline{u}^n\|_{L^\infty(\Omega)} \\ &+ \|a(\widehat{u}_\tau^n) - a(u^{n-\frac{1}{2}})\|_{L^4(\Omega)} \|\Delta\overline{u}^n\|_{L^4(\Omega)} \\ &\le \mu^* M\left(\|\nabla\widehat{e}_\tau^n\|_{L^2(\Omega)} + \|\widehat{e}_\tau^n\|_{L^4(\Omega)}\right) \\ &+ \mu^* M C_\kappa \tau^{\frac{3}{2}} \int_{t_{n-2}}^{t_{n-\frac{1}{2}}} \|u_{tt}(s)\|_{H^1(\Omega)} + \|u_{tt}(s)\|_{L^\infty(\Omega)} ds \\ &\le \mu^* M\left(3\|\nabla e_\tau^{n-1}\|_{L^2(\Omega)} + \|\nabla e_\tau^{n-2}\|_{L^2(\Omega)}\right) \\ &+ \mu^* M C_\kappa \tau^{\frac{3}{2}} \int_{t_{n-2}}^{t_{n-\frac{1}{2}}} \|u_{tt}(s)\|_{H^1(\Omega)} + \|u_{tt}(s)\|_{L^\infty(\Omega)} ds,\end{aligned}$$

and

$$\|\nabla\cdot H_B^n\|_{L^2(\Omega)} \le \|a(u^{n-\frac{1}{2}})\|_{L^\infty(\Omega)} \|\nabla(\overline{u}^n - u^{n-\frac{1}{2}})\|_{L^2(\Omega)} \le \mu^* M C_\kappa \tau^{\frac{3}{2}} \int_{t_{n-2}}^{t_n} \|u_{tt}(s)\|_{H^1(\Omega)} ds.$$

With the above results, the inequality (45) can be reduced to

$$\begin{aligned}&\frac{\mu_*}{2\tau}\left(\|\nabla e_\tau^n\|^2_{L^2(\Omega)} - \|\nabla e_\tau^{n-1}\|^2_{L^2(\Omega)}\right) + \frac{1}{2}\|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\|^2_{L^2(\Omega)} \\ &\le \frac{1}{2}\|E^n\|^2_{L^2(\Omega)} + \frac{1+(6\mu^* M)^2}{4} \sum_{i=0}^{2} \|\nabla e_\tau^{n-i}\|^2_{L^2(\Omega)} \\ &+ (\mu^* M C_\kappa)^2 \tau^3 \int_{t_{n-2}}^{t_n} \|u_{tt}(s)\|^2_{H^1(\Omega)} + \|u_{tt}(s)\|^2_{L^\infty(\Omega)} ds\end{aligned} \tag{46}$$

Multiplying both sides of the above inequality by $\frac{2\tau}{\mu_*}$ and summing up from 2 to n, we obtain that

$$\begin{aligned}
&\|\nabla e_\tau^n\|_{L^2(\Omega)}^2 + \frac{\tau}{\mu_*}\sum_{i=2}^{n}\left\|\nabla\cdot\left(a(\widehat{u}_\tau^i)\nabla\overline{e}_\tau^i\right)\right\|_{L^2(\Omega)}^2 \\
&\le \|\nabla e_\tau^1\|_{L^2(\Omega)}^2 + \frac{\tau}{\mu_*}\sum_{i=2}^{n}\|E^i\|_{L^2(\Omega)}^2 + \frac{1+(6\mu^*M)^2}{\mu_*}\sum_{i=0}^{n}\tau\|\nabla e_\tau^i\|_{L^2(\Omega)}^2 \\
&+ \frac{2}{\mu_*}(\mu^*MC_\kappa)^2\tau^4\int_{t_0}^{t_n}\left(\|u_{tt}(s)\|_{H^1(\Omega)}^2 + \|u_{tt}(s)\|_{L^\infty(\Omega)}^2\right)ds.
\end{aligned}$$

By using (31) and (34), we further obtain

$$\|\nabla e_\tau^n\|_{L^2(\Omega)}^2 + \frac{\tau}{\mu_*}\sum_{i=2}^{n}\|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\|_{L^2(\Omega)}^2 \le \frac{1+(6\mu^*M)^2}{\mu_*}\sum_{i=0}^{n}\tau\|\nabla e_\tau^i\|_{L^2(\Omega)}^2 + (C_5)^2\tau^4,$$

the constant C_5 only is dependent on C_Ω, C_κ, μ_*, μ^* and regularity constant M but independent of time-discrete parameter τ and n. Applying Gronwall's inequality , we conclude that there exists $\tau < \tau_4 = \frac{\mu_*}{1+(6\mu^*M)^2}$ such that

$$\|\nabla e_\tau^n\|_{L^2(\Omega)}^2 + \frac{\tau}{\mu_*}\sum_{i=2}^{n}\|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\|_{L^2(\Omega)}^2 \le C_6^2\tau^4,$$

which implies that

$$\|\nabla e_\tau^n\|_{L^2(\Omega)} \le C_6\tau^2, \quad \sum_{i=2}^{n}\tau\left\|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\right\|_{L^2(\Omega)}^2 \le \sqrt{\mu_*}C_6\tau^4, \tag{47}$$

where C_6 is a τ-independent positive constant but dependent on constants μ^*, C_Ω, C_κ and regularity assumption (19). And applying the Lemma 4.3 and (30), we have

$$\tau\sum_{i=2}^{n}\|\overline{e}_\tau^n\|_{H^2(\Omega)}^2 \le \sum_{i=2}^{n}\tau\left\|\nabla\cdot(a(\widehat{u}_\tau^n)\nabla\overline{e}_\tau^n)\right\|_{L^2(\Omega)}^2 \le \mu_*C_6\tau^4, \tag{48}$$

and exploit the Lemma 4.2,

$$\|e_\tau^n\|_{H^2(\Omega)} \le \frac{2\sqrt{T}}{\tau}\left(\tau\sum_{i=2}^{n}\|\overline{e}_\tau^n\|_{H^2(\Omega)}^2 + \tau\|e_\tau^0\|_{H^2(\Omega)}^2\right)^{\frac{1}{2}} \le 2\sqrt{\mu_*T}C_\Omega C_6\tau. \tag{49}$$

From (44) and (49), and using the Agmon's inequality (26), we have

$$\|e_\tau^n\|_{L^\infty(\Omega)} \le C_\Omega\|e_\tau^n\|_{L^2(\Omega)}^{\frac{1}{2}}\|e_\tau^n\|_{H^2(\Omega)}^{\frac{1}{2}} \le C_7\tau^{\frac{3}{2}}, \tag{50}$$

where the constant $C_6 = \left(2\sqrt{\mu_*T}C_4C_6\right)^{\frac{1}{2}}(C_\Omega)^{\frac{3}{2}}$. Likewise, we apply Lemma 4.3 to the error equation (30),

$$\|\overline{e}_\tau^n\|_{W^{2,4}(\Omega)} \le C_\Omega\left(\|E^n\|_{L^4(\Omega)} + \|\nabla\cdot H^n\|_{L^4(\Omega)} + \|D_\tau e_\tau^n\|_{L^4(\Omega)}\right). \tag{51}$$

As for the term $\|\nabla\cdot H^n\|_{L^4(\Omega)}$, we use the same idea as for $\|\nabla\cdot H^1\|_{L^4(\Omega)}$ to get

$$\begin{aligned}\|\nabla\cdot H^n\|_{L^4(\Omega)} &\le \mu^* M\left(\|\nabla\widehat{e}_\tau^n\|_{L^4(\Omega)} + \|\nabla(\overline{u}^n - u^{n-\frac{1}{2}})\|_{L^4(\Omega)}\right)\\ &+ \mu^* M\left(\|e_\tau^n\|_{L^\infty(\Omega)} + \|\overline{u}^n - u^{n-\frac{1}{2}}\|_{L^\infty(\Omega)}\right)\\ &+ \mu^* M\left(\|\nabla(\overline{u}^n - u^{n-\frac{1}{2}})\|_{L^4(\Omega)} + \|\Delta(\overline{u}^n - u^{n-\frac{1}{2}})\|_{L^4(\Omega)}\right)\\ &\le 2\mu^* M\sqrt{\mu^* T}C_\Omega C_6\tau + \mu^* M C_7\tau^{\frac{3}{2}}\\ &+ \mu^* M C_\kappa\tau^2\int_{t_{n-2}}^{t_n}\|u_{tt}(s)\|_{H^2(\Omega)} + \|u_{tt}(s)\|_{W^{2,4}(\Omega)} + \|u_{tt}(s)\|_{L^\infty(\Omega)}ds.\end{aligned}$$

Again employing the Gagliardo-Nirenberg inequality (25) for $\|D_\tau e_\tau^n\|_{L^4(\Omega)}$:

$$\|D_\tau e_\tau^n\|_{L^4(\Omega)} \le C_\Omega\|D_\tau e_\tau^n\|_{H^1(\Omega)} \le C_\Omega C_6\tau$$

and using the Taylor expansion, we further deduce that

$$\|\overline{e}_\tau^n\|_{W^{2,4}(\Omega)} \le C_8\tau, \tag{52}$$

which implies that

$$\|\nabla e_\tau^n\|_{L^\infty(\Omega)} \le C_\Omega\|e_\tau^n\|_{W^{2,4}(\Omega)} \le C_\Omega\sum_{i=2}^n\|\overline{e}_\tau^i\|_{W^{2,4}(\Omega)} + C_\Omega\|e_\tau^1\|_{W^{2,4}(\Omega)} \le C_\Omega\left(TC_8 + C_2\right), \tag{53}$$

where C_8 is positive constant only dependent on constants μ^*, C_Ω, C_κ and regularity assumption (19).

Therefore, for $\tau < \tau_5$, it holds that

$$\begin{aligned}\|u_\tau^n\|_{H^2(\Omega)} &\le \|u^n\|_{H^2(\Omega)} + \|e_\tau^n\|_{H^2(\Omega)} \le M_1,\\ \|u_\tau^n\|_{L^\infty(\Omega)} &\le \|u^n\|_{L^\infty(\Omega)} + \|e_\tau^n\|_{L^\infty(\Omega)} \le M_1,\\ \|\nabla u_\tau^n\|_{L^\infty(\Omega)} &\le \|\nabla u^n\|_{L^\infty(\Omega)} + \|\nabla e_\tau^n\|_{L^\infty(\Omega)} \le M + C_\Omega\left(TC_8 + C_2\right),\\ \tau\sum_{i=2}^n\|D_\tau u_\tau^i\|_{H^2(\Omega)}^2 &\le \tau\sum_{i=2}^n\|D_\tau u^n\|_{H^2(\Omega)}^2 + \tau\sum_{i=2}^n\|D_\tau e_\tau^n\|_{H^2(\Omega)}^2 \le M + 4\mu_*\left(TC_\Omega C_6\right)^2,\end{aligned}$$

where the constant $\tau_5 = \min\{(2\sqrt{\mu_* T}C_\Omega C_6)^{-1}, (C_7)^{-1}\}$ independent of τ. Thus, the mathematical induction is closed for $\tau < \tau_0' = \min_{1\le i\le 5}\{\tau_i\}$. Moreover, taking $C_0' = \max\{M_1, M + 4\mu_*(TC_\Omega C_6)^2, M + C_\Omega(TC_8 + C_2)\}$, the proof of Theorem 4.5 is completed. □

4.2. The elliptic projection and error estimates

To prove the uniform L^∞ boundedness for the solution of the proposed scheme (17) - (18), we first introduce a novel virtual element approximation projection for a steady-state problem.

Associated with the conforming virtual element space V_h, we define the elliptic projection $\mathcal{R}_h u \in V_h$ of the exact solution u, such that, for fixed $u(t) \in H_0^1$, $t \in (0, T]$,

$$\mathcal{A}_h(u; \mathcal{R}_h u, v_h) = \mathcal{A}(u; u, v_h), \quad \forall v_h \in V_h, \tag{54}$$

and $\mathcal{R}_h u(0) = \mathcal{I}_h u(0)$.

Before proving the error estimates of $\mathcal{R}_h$, we mention a preliminary result that are useful for our purposes. Readers can see proof of Lemma 4.6 in Appendix A.2.

Lemma 4.6. *Assume that $w, v \in H^k(\Omega) \cap L^\infty(\Omega), k \geq 1$, and $\phi \in H^2(\Omega)$. Then there exists a positive constant C_ℓ independent of h such that*

$$\begin{aligned}&\mathcal{A}_h(w; \mathcal{R}_h v, \mathcal{I}_h \phi) - \mathcal{A}(w; \mathcal{R}_h v, \mathcal{I}_h \phi)\\ &\leq C_\ell \left(h^{k+1} \left(\|v\|_{H^{k+1}(\mathcal{T}_h)} + \|w\|_{H^{k+1}(\mathcal{T}_h)} \right) + h \left\| \nabla \left(\mathcal{R}_h v - v \right) \right\|_{L^2(\Omega)} \right) |\phi|_{H^2(\mathcal{T}_h)}\end{aligned}$$

where $\mathcal{I}_h \phi$ is the virtual element approximation of ϕ given by (12).

Then, we have the following main result for the error estimates of the elliptic projection $\mathcal{R}_h u$.

Lemma 4.7. *For $k \geq 1$, let $u(t) \in H^k(\Omega) \cap L^\infty(\Omega)$, then there exists $C_\mathcal{R} > 0$ independent of discretization parameters h such that*

$$\|u - \mathcal{R}_h u\|_{L^2(\Omega)} + h \|u - \mathcal{R}_h u\|_{H^1(\mathcal{T}_h)} \leq C_\mathcal{R} h^{k+1} \|u\|_{H^{k+1}(\mathcal{T}_h)}. \tag{55}$$

Proof. With the L^∞-boundedness of u, the existence and uniqueness of the solution to (54) follow immediately since the bilinear form $\mathcal{A}_h(\cdot; \cdot, \cdot)$ is continuous and coercive, as well as the bilinear form $\mathcal{A}(\cdot; \cdot, \cdot)$ is continuous on V_h.

Setting $\delta_h = \mathcal{R}_h u - \mathcal{I}_h u$, one obtains with the coercivity (21) of $\mathcal{A}_h(\cdot; \cdot, \cdot)$ and the definition of $\mathcal{R}_h u$ in (54),

$$\begin{aligned}\beta_* |\delta_h|^2_{H^1(\mathcal{T}_h)} &\leq \mathcal{A}_h(u; \delta_h, \delta_h) = \mathcal{A}_h(u; \mathcal{R}_h u, \delta_h) - \mathcal{A}_h(u; \mathcal{I}_h u, \delta_h)\\ &= [\mathcal{A}(u; u, \delta_h) - \mathcal{A}_h(u; u, \delta_h)] + \mathcal{A}_h(u; u - \mathcal{I}_h u, \delta_h) := S_1 + S_2.\end{aligned} \tag{56}$$

From the definition of $\mathcal{A}_h^K$ given in (10b), the term S_1 is split as follows:

$$\begin{aligned}S_1 = &\left((a(u) \nabla u, \nabla \delta_h)_\Omega - \left(a(\Pi_k^0 u) \mathbf{\Pi}_{k-1}^{\mathbf{0}} (\nabla u), \mathbf{\Pi}_{k-1}^{\mathbf{0}} (\nabla \delta_h) \right)_\Omega \right)\\ &- \sum_{K \in \mathcal{T}_h} \nu_\mathcal{A}^K(u) S_\mathcal{A}^K \left(\left(I - \Pi_k^{\nabla, K} \right) u, \left(I - \Pi_k^{\nabla, K} \right) \delta_h \right) := S_1^A + S_1^B.\end{aligned}$$

Noticing that

$$\|\Pi_k^0 u\|_{L^\infty(K)} \leq C_{\mathrm{Inv}} h^{-1} \|\Pi_k^0 u\|_{L^2(K)} \leq C_{\mathrm{Inv}} h^{-1} \|u\|_{L^2(K)} \leq C_{\mathrm{Inv}} \|u\|_{L^\infty(K)}, \tag{57}$$

one applies the bounded condition of $a(u)$ in (3), the Sobolev embedding theorem $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and Lemma 3.2 to derive that

$$\begin{aligned}S_1^A = &\left(\left(a(u) - a(\Pi_k^0 u) \right) \nabla u, \nabla \delta_h \right)_\Omega + \Big[\left(a(\Pi_k^0 u) \nabla u, \nabla \delta_h \right)_\Omega\\ &- \left(a(\Pi_k^0 u) \mathbf{\Pi}_{k-1}^{\mathbf{0}} (\nabla u), \mathbf{\Pi}_{k-1}^{\mathbf{0}} (\nabla \delta_h) \right)_\Omega \Big]\\ &\leq \left\| a(u) - a(\Pi_k^0 u) \right\|_{L^4(\Omega)} \|\nabla u\|_{L^4(\Omega)} \|\nabla \delta_h\|_{L^2(\Omega)}\\ &+ \left\| a(\Pi_k^0 u) \right\|_{L^\infty(\Omega)} \left\| \left(\boldsymbol{I} - \mathbf{\Pi}_{k-1}^{\mathbf{0}} \right) \nabla u \right\|_{L^2(\Omega)} \|\nabla \delta_h\|_{L^2(\Omega)}\\ &\leq \mu^* C_\mathcal{P} (M C_\Omega + 1) h^k \|u\|_{H^{k+1}(\mathcal{T}_h)} \|\nabla \delta_h\|_{L^2(\Omega)}.\end{aligned}$$

For term S_1^B, it holds with the stability property (11), (57) and Lemma 3.2,

$$\begin{aligned}S_1^B &\leq \left\| \Pi_0^0 \left(a(\Pi_k^0 u) \right) \right\|_{L^\infty(\Omega)} \left\| (I - \Pi_k^\nabla) \nabla u \right\|_{L^2(\Omega)} \left\| (I - \Pi_k^\nabla) \nabla \delta_h \right\|_{L^2(\Omega)}\\ &\leq \mu^* C_\mathcal{P}^2 h^k \|u\|_{H^{k+1}(\mathcal{T}_h)} \|\nabla \delta_h\|_{L^2(\Omega)}.\end{aligned}$$

With the above estimates, we obtain that

$$S_1 \leq \mu^* C_{\mathcal{P}} \left(M C_\Omega + C_{\mathcal{P}} + 1\right) h^k \|u\|_{H^{k+1}(\mathcal{T}_h)} \|\nabla \delta_h\|_{L^2(\Omega)} . \tag{58}$$

As for the term S_2, we use the continuity (21) of $\mathcal{A}_h(\cdot;\cdot,\cdot)$ together with the interpolation results (13) to derive

$$S_2 \leq \beta^* \left\|\nabla\left(u - \mathcal{I}_h u\right)\right\|_{L^2(\Omega)} \|\nabla \delta_h\|_{L^2(\Omega)} \leq \beta^* C_{\mathcal{I}} h^k \|u\|_{H^{k+1}(\mathcal{T}_h)} \|\nabla \delta_h\|_{L^2(\Omega)} . \tag{59}$$

Then, inserting (58)–(59) into (56), we have

$$\|\nabla \delta_h\|_{L^2(\Omega)} \leq \frac{1}{\beta_*} \left(\mu^* C_{\mathcal{P}} \left(M C_\Omega + C_{\mathcal{P}} + 1\right) + \beta^* C_{\mathcal{I}}\right) h^k \|u\|_{H^{k+1}(\mathcal{T}_h)}.$$

Therefore, applying the triangle inequality and (13), we obtain that

$$\begin{aligned}
&\|u - \mathcal{R}_h u\|_{H^1(\mathcal{T}_h)} \leq \|u - \mathcal{I}_h u\|_{H^1(\mathcal{T}_h)} + \|\nabla \delta_h\|_{L^2(\Omega)} \\
&\leq \left(C_{\mathcal{I}} + \frac{1}{\beta_*} \left(\mu^* C_{\mathcal{P}} \left(M C_\Omega + C_{\mathcal{P}} + 1\right) + \beta^* C_{\mathcal{I}}\right)\right) h^k \|u\|_{H^{k+1}(\mathcal{T}_h)} := C_9 h^k \|u\|_{H^{k+1}(\mathcal{T}_h)}.
\end{aligned}$$

Next, we derive an error bounded for $\|u - \mathcal{R}_h u\|_{L^2(\Omega)}$ by applying the Aubin-Nitsche duality argument [36]. For fixed $u \in H_0^1(\Omega)$, we consider the following (linear) auxiliary problem: find $\phi \in H^2 \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (a(u) \nabla \phi) = u - \mathcal{R}_h u, \text{ in } \Omega, \\ \phi(\mathbf{x}) = 0, \text{ on } \partial\Omega. \end{cases}$$

From Lemma 4.3, we have the following regularity estimate

$$\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u - \mathcal{R}_h u\|_{L^2(\Omega)}, \quad \forall \phi \in H^2(\Omega). \tag{60}$$

Let $\mathcal{I}_h \phi$ be the interpolation function of ϕ. Then, using the previous estimates, the definition of $\mathcal{R}_h u$ in (54), and Lemma 4.6, we have

$$\begin{aligned}
\|u - \mathcal{R}_h u\|_{L^2(\Omega)}^2 &= \mathcal{A}\left(u; u - \mathcal{R}_h u, \phi\right) = \mathcal{A}\left(u; u - \mathcal{R}_h u, \phi - \mathcal{I}_h \phi\right) + \mathcal{A}\left(u; u - \mathcal{R}_h u, \mathcal{I}_h \phi\right) \\
&= \mathcal{A}\left(u; u - \mathcal{R}_h u, \phi - \mathcal{I}_h \phi\right) + \left[\mathcal{A}\left(u; u, \mathcal{I}_h \phi\right) - \mathcal{A}\left(u; \mathcal{R}_h u, \mathcal{I}_h \phi\right)\right] \\
&= \mathcal{A}\left(u; u - \mathcal{R}_h u, \phi - \mathcal{I}_h \phi\right) + \left[\mathcal{A}_h\left(u; \mathcal{R}_h u, \mathcal{I}_h \phi\right) - \mathcal{A}\left(u; \mathcal{R}_h u, \mathcal{I}_h \phi\right)\right] \\
&\leq \left(\left(C_\ell + \beta^* C_{\mathcal{P}}\right) h \left\|\nabla\left(u - \mathcal{R}_h u\right)\right\|_{L^2(\Omega)} + 2 C_\ell h^{k+1} \|u\|_{H^{k+1}(\mathcal{T}_h)}\right) |\phi|_{H^2(\mathcal{T}_h)} \\
&\leq \left(C_\ell + C_9 \left(C_\ell + \beta^* C_{\mathcal{P}}\right)\right) h^{k+1} \|u\|_{H^{k+1}(\mathcal{T}_h)} |\phi|_{H^2(\mathcal{T}_h)} ,
\end{aligned}$$

which implies that

$$\|u - \mathcal{R}_h u\|_{L^2(\Omega)} \leq C_\Omega \left(C_\ell + C_9 \left(C_\ell + \beta^* C_{\mathcal{P}}\right)\right) h^{k+1} \|u\|_{H^{k+1}(\mathcal{T}_h)}.$$

Taking $C_{\mathcal{R}} = \max\{C_9, C_\Omega(C_\ell + C_9(C_\ell + \beta^* C_{\mathcal{P}}))\}$, the proof of Lemma 4.7 is completed. □

We shall also need the boundedness of $\mathcal{R}_h u$.

Lemma 4.8. *Let $R_h u^n, n = 0, \ldots, N$, $R_h u^{1*}$, and $R_h u^{1**}$ be the elliptic projections defined in (54). Then, it holds that*

$$\|\nabla \mathcal{R}_h u\|_{L^\infty(\Omega)} + \|\mathbf{\Pi}_{k-1}^{\mathbf{0}} \nabla \mathcal{R}_h u\|_{L^\infty(\Omega)} \leq C_M, \tag{61a}$$

$$\|\mathcal{R}_h u\|_{L^\infty(\Omega)} + \|\Pi_k^0 \mathcal{R}_h u\|_{L^\infty(\Omega)} \leq C_M, \tag{61b}$$

where the constant C_M independent of discretization parameters h.

Proof. Using the inverse estimate (15) together with Lemma 4.7 and the known error estimate for $\mathcal{I}_h u$, we obtain that

$$\begin{aligned}&\|\nabla(\mathcal{R}_h u-\mathcal{I}_h u)\|_{L^\infty(K)}\le C_{\mathrm{Inv}}h^{-1}\|\nabla(\mathcal{R}_h u-\mathcal{I}_h u)\|_{L^2(K)}\\&\le C_{\mathrm{Inv}}h^{-1}\left(\|\nabla(\mathcal{R}_h u-u)\|_{L^2(K)}+\|\nabla(\mathcal{I}_h u-u)\|_{L^2(K)}\right)\le C_{\mathrm{Inv}}\left(C_{\mathcal{R}}+C_{\mathcal{I}}\right)\|u\|_{H^2(K)}.\end{aligned}$$

Since it is easy to see that $\|\nabla\mathcal{I}_h u\|_{L^\infty(\Omega)}\le C_\Omega\|\nabla u\|_{L^\infty(\Omega)}$, the result follows:

$$\|\nabla\mathcal{R}_h u\|_{L^\infty(\Omega)}\le\|\nabla\mathcal{I}_h u\|_{L^\infty(\Omega)}+\|\nabla(\mathcal{R}_h u-\mathcal{I}_h u)\|_{L^\infty(K)}\le C_M,$$

Further, using the stability property of $\mathbf{\Pi}^{\mathbf{0}}_{k-1}$ and the inverse estimate (15), we obtain

$$\begin{aligned}\|\mathbf{\Pi}^{\mathbf{0}}_{k-1}\nabla\mathcal{R}_h u\|_{L^\infty(\Omega)}&\le C_{\mathrm{Inv}}h^{-1}\sum_{K\in\mathcal{T}_h}\|\mathbf{\Pi}^{\mathbf{0}}_{k-1}\nabla\mathcal{R}_h u\|_{L^2(K)}\le C_\Omega C_{\mathrm{Inv}}h^{-1}\sum_{K\in\mathcal{T}_h}\|\nabla\mathcal{R}_h u\|_{L^2(K)}\\&\le C_\Omega C_{\mathrm{Inv}}\|\nabla\mathcal{R}_h u\|_{L^\infty(K)}.\end{aligned}$$

Thus, (61a) holds. The proof of (61b) can be provided in a similar fashion as above. ◻

It is worth mentioning that the semi-discrete solution $u_\tau^n, n=0,1,\dots,N$, or $1*$, or $1**$ also holds for elliptic operators $\mathcal{R}_h^n$ (the discrete analogue of $\mathcal{R}_h$) defined by (54) and satisfies the Lemma 4.7 with $k=1$, namely

$$\|u_\tau^n-\mathcal{R}_h^n u_\tau^n\|_{L^2(\Omega)}+h\,\|u_\tau^n-\mathcal{R}_h^n u_\tau^n\|_{H^1(\mathcal{T}_h)}\le C_{\mathcal{R}}h^2\|u_\tau^n\|_{H^2(\mathcal{T}_h)},\tag{62}$$

and L^∞-boundedness conditions

$$\|\nabla\mathcal{R}_h^n u_\tau^n\|_{L^\infty(\Omega)}+\|\mathbf{\Pi}^{\mathbf{0}}_{k-1}\nabla\mathcal{R}_h^n u_\tau^n\|_{L^\infty(\Omega)}\le C_M,\tag{63a}$$

$$\|\mathcal{R}_h^n u_\tau^n\|_{L^\infty(\Omega)}+\|\Pi_k^0\mathcal{R}_h^n u_\tau^n\|_{L^\infty(\Omega)}\le C_M.\tag{63b}$$

Finally, we also need the following modified projection operator during the spatial error analysis. Let $\mathcal{F}_h^n: H_0^1(\Omega)\to V_h, n=0,1,\dots,N$, be projection operator defined by the following linear elliptic problems:

$$\begin{cases}\mathcal{A}_h(u_\tau^0;\mathcal{F}_h^0\omega,v_h)=\mathcal{A}(u_\tau^0;\omega,v_h), & \forall v_h\in V_h,\\ \mathcal{A}_h(u_\tau^{1*};\mathcal{F}_h^{1*}\omega,v_h)=\mathcal{A}(u_\tau^{1*};\omega,v_h), & \forall v_h\in V_h,\\ \mathcal{A}_h(\widehat{u}_\tau^n;\mathcal{F}_h^n\omega,v_h)=\mathcal{A}(\widehat{u}_\tau^n;\omega,v_h), & \forall v_h\in V_h, n=1,\dots,N.\end{cases}\tag{64}$$

From the coercivity conditions (21) and using the Lax-Milgram lemma, there exists a unique solution $\mathcal{F}_h^n\omega\in V_h$ for $n=0,1,\dots,N$.

With the help of Lemma 4.7, we can derive the following error estimates for $\mathcal{F}_h^n$ and $\mathcal{R}_h^n$.

Lemma 4.9. *With the regularity of u_τ^n given in Theorem 4.5 and the boundedness given in (63a)–(63b), it holds that*

$$\left\|\nabla\left(\mathcal{F}_h^0 u_\tau^{1*}-\mathcal{R}_h^{1*}u_\tau^{1*}\right)\right\|_{L^2(\Omega)}+\left\|\nabla\left(\mathcal{F}_h^{1*}u_\tau^{1**}-\mathcal{R}_h^{1**}u_\tau^{1**}\right)\right\|_{L^2(\Omega)}\le C_{\mathcal{F}}(h^2+\tau h),\tag{65a}$$

$$\left\|\nabla\left(\mathcal{F}_h^n\overline{u}_\tau^n-\overline{\mathcal{R}_h u_\tau}^n\right)\right\|_{L^2(\Omega)}\le C_{\mathcal{F}}(h^2+\tau h),\tag{65b}$$

for $n=1,\dots,N$, where $C_{\mathcal{F}}$ is a positive constant independent of τ and h.

Proof. From the definition (54) and (64), it is easy to verify that $\overline{\mathcal{R}_h u_\tau}^n=\frac{1}{2}(\mathcal{R}_h^n u_\tau^n+\mathcal{R}_h^{n-1}u_\tau^{n-1})$ satisfies

$$\begin{aligned}\mathcal{A}_h\left(\widehat{u}_\tau^n;\mathcal{F}_h^n\overline{u}_\tau^n-\overline{\mathcal{R}_h u_\tau}^n,v_h\right)&=\mathcal{A}_h\left(\overline{u}_\tau^n;\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n,v_h\right)-\mathcal{A}_h\left(\widehat{u}_\tau^n;\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n,v_h\right)\\&+\mathcal{A}_h\left(\overline{u}_\tau^n;\overline{u}_\tau^n,v_h\right)-\mathcal{A}_h\left(\widehat{u}_\tau^n;\overline{u}_\tau^n,v_h\right)-\mathcal{A}\left(\overline{u}_\tau^n;\overline{u}_\tau^n,v_h\right)+\mathcal{A}\left(\widehat{u}_\tau^n;\overline{u}_\tau^n,v_h\right).\end{aligned}\tag{66}$$

For the first and second terms in equation (66), we can use the definition (10b) and continuity (21) of $\mathcal{A}_h(\cdot;\cdot,\cdot)$ along with the estimate (62), (67), to obtain

$$\begin{aligned}
&\mathcal{A}_h\left(\overline{u}_\tau^n;\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n,v_h\right)-\mathcal{A}_h\left(\widehat{u}_\tau^n;\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n,v_h\right)\\
&\le\left(\left(a\left(\Pi_k^0\overline{u}_\tau^n\right)-a\left(\Pi_k^0\widehat{u}_\tau^n\right)\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\nabla\left(\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n\right),\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega\\
&+\sum_{K\in\mathcal{T}_h}\left|\nu_{\mathcal{A}}^K\left(\overline{u}_\tau^n\right)-\nu_{\mathcal{A}}^K\left(\widehat{u}_\tau^n\right)\right|S_{\mathcal{A}}^K\left((I-\Pi_k^{\nabla,K})\left(\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n\right),(I-\Pi_k^{\nabla,K})v_h\right)\\
&\le 2\mu^*C_{\mathrm{Inv}}\left(C_{\mathcal{P}}\right)^3\left\|\widehat{u}_\tau^n-\overline{u}_\tau^n\right\|_{L^\infty(\Omega)}\left\|\nabla\left(\overline{\mathcal{R}_h u_\tau}^n-\overline{u}_\tau^n\right)\right\|_{L^2(\Omega)}\left\|\nabla v_h\right\|_{L^2(\Omega)}\\
&\le\left(2\mu^*C_{\mathrm{Inv}}\left(C_{\mathcal{P}}\right)^3C_{\mathcal{R}}C_{11}C_1'\right)\tau^{\frac32}h\|\nabla v_h\|_{L^2(\Omega)}:=C_{13}\tau^{\frac32}h\|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned}$$

By employing the Taylor expansion and utilizing estimate (49), (50), we have

$$\begin{aligned}
&\left\|\left(\mathbf{\Pi}_{k-1}^{\mathbf{0}}-\boldsymbol{I}\right)\left(a\left(\overline{u}_\tau^n\right)-a\left(\widehat{u}_\tau^n\right)\right)\right\|_{L^2(\Omega)}\le\mu^*C_{\mathcal{P}}h\left\|\overline{u}_\tau^n-\widehat{u}_\tau^n\right\|_{H^2(\mathcal{T}_h)}\\
&\le\mu^*C_{\mathcal{P}}h\left(\left\|\overline{u}_\tau^n-\overline{u}^n\right\|_{H^2(\mathcal{T}_h)}+\left\|\widehat{u}_\tau^n-\widehat{u}^n\right\|_{H^2(\mathcal{T}_h)}+\left\|\overline{u}^n-\widehat{u}^n\right\|_{H^2(\mathcal{T}_h)}\right)\\
&\le\mu^*C_{\mathcal{P}}\left(4\sqrt{\mu_*T}C_\Omega C_6+C_\kappa\left\|u_{tt}\right\|_{L^1(t_n,t_{n-2};H^2(\mathcal{T}_h))}\right)\tau h:=C_{10}\tau h,
\end{aligned}$$

and

$$\begin{aligned}
\left\|\widehat{u}_\tau^n-\overline{u}_\tau^n\right\|_{L^\infty(\Omega)}&\le\left\|\overline{u}_\tau^n-\overline{u}^n\right\|_{L^\infty(\Omega)}+\left\|\widehat{u}_\tau^n-\widehat{u}^n\right\|_{L^\infty(\Omega)}+\left\|\overline{u}^n-\widehat{u}^n\right\|_{L^\infty(\Omega)}\\
&\le 3C_7\tau^{\frac32}+C_\kappa\tau^2\left\|u_{tt}\right\|_{L^1(t_n,t_{n-2};L^\infty(\Omega))}:=C_{11}\tau^{\frac32}.
\end{aligned}\tag{67}$$

Next, in view of the definitions of $\mathcal{A}(\cdot;\cdot,\cdot)$, $\mathcal{A}_h(\cdot;\cdot,\cdot)$, and Π_k^0, as well as using the Hölder's inequality and L^∞-boundedness conditions (63a)–(63b), we can prove that

$$\begin{aligned}
&\mathcal{A}_h\left(\overline{u}^n;\overline{u}_\tau^n,v_h\right)-\mathcal{A}_h\left(\widehat{u}^n;\overline{u}_\tau^n,v_h\right)-\mathcal{A}\left(\overline{u}^n;\overline{u}_\tau^n,v_h\right)+\mathcal{A}\left(\widehat{u}^n;\overline{u}_\tau^n,v_h\right)\\
&=\left[\left(a\left(\Pi_k^0\overline{u}_\tau^n\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla\overline{u}_\tau^n\right),\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega-\left(a(\overline{u}_\tau^n)\nabla\overline{u}_\tau^n,\nabla v_h\right)_\Omega\right]\\
&-\left[\left(a\left(\Pi_k^0\widehat{u}_\tau^n\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla\overline{u}_\tau^n\right),\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega-\left(a(\widehat{u}_\tau^n)\nabla\overline{u}_\tau^n,\nabla v_h\right)_\Omega\right]\\
&+\sum_{K\in\mathcal{T}_h}\left|\nu_{\mathcal{A}}^K\left(\overline{u}_\tau^n\right)-\nu_{\mathcal{A}}^K\left(\widehat{u}_\tau^n\right)\right|S_{\mathcal{A}}^K\left((I-\Pi_k^{\nabla,K})\overline{u}_\tau^n,(I-\Pi_k^{\nabla,K})v_h\right)\\
&=\left(\left(a\left(\Pi_k^0\overline{u}_\tau^n\right)-a\left(\overline{u}_\tau^n\right)\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla\overline{u}_\tau^n\right),\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega\\
&+\left(\left(a\left(\widehat{u}_\tau^n\right)-a\left(\Pi_k^0\widehat{u}_\tau^n\right)\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla\overline{u}_\tau^n\right),\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega\\
&+\left(\left(a\left(\overline{u}_\tau^n\right)-a\left(\widehat{u}_\tau^n\right)\right)\left(\mathbf{\Pi}_{k-1}^{\mathbf{0}}-\boldsymbol{I}\right)\left(\nabla\overline{u}_\tau^n\right),\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega\\
&+\left(\left(\mathbf{\Pi}_{k-1}^{\mathbf{0}}-\boldsymbol{I}\right)\left(a\left(\overline{u}_\tau^n\right)-a\left(\widehat{u}_\tau^n\right)\right)\nabla\overline{u}_\tau^n,\nabla v_h\right)_\Omega\\
&+\sum_{K\in\mathcal{T}_h}\left|\nu_{\mathcal{A}}^K\left(\overline{u}_\tau^n\right)-\nu_{\mathcal{A}}^K\left(\widehat{u}_\tau^n\right)\right|S_{\mathcal{A}}^K\left((I-\Pi_k^{\nabla,K})\left(I-\Pi_k^{0,K}\right)\overline{u}_\tau^n,(I-\Pi_k^{\nabla,K})v_h\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|a\left(\Pi_k^0 \overline{u}_\tau^n\right)-a\left(\overline{u}_\tau^n\right)\right\|_{L^2(\Omega)}\left\|\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla \overline{u}_\tau^n\right)\right\|_{L^\infty(\Omega)}\left\|\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right\|_{L^2(\Omega)} \\
&+\left\|a\left(\Pi_k^0 \widehat{u}_\tau^n\right)-a\left(\widehat{u}_\tau^n\right)\right\|_{L^2(\Omega)}\left\|\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla \overline{u}_\tau^n\right)\right\|_{L^\infty(\Omega)}\left\|\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right\|_{L^2(\Omega)} \\
&+\left\|a\left(\overline{u}_\tau^n\right)-a\left(\widehat{u}_\tau^n\right)\right\|_{L^\infty(\Omega)}\left\|\left(\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}-\boldsymbol{I}\right)\left(\nabla \overline{u}_\tau^n\right)\right\|_{L^2(\Omega)}\left\|\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right\|_{L^2(\Omega)} \\
&+\left\|\left(\boldsymbol{\Pi}_{k-1}^{\mathbf{0}}-\boldsymbol{I}\right)\left(a\left(\overline{u}_\tau^n\right)-a\left(\widehat{u}_\tau^n\right)\right)\right\|_{L^2(\Omega)}\left\|\nabla \overline{u}_\tau^n\right\|_{L^\infty(\Omega)}\left\|\nabla v_h\right\|_{L^2(\Omega)} \\
&+M_1^{\mathcal{A}} C_{\mathrm{Inv}}\left\|a\left(\Pi_k^0 \overline{u}_\tau^n\right)-a\left(\Pi_k^0 \widehat{u}_\tau^n\right)\right\|_{L^\infty(\Omega)}\left\|\nabla\left(I-\Pi_k^{0, K}\right) \overline{u}_\tau^n\right\|_{L^2(\Omega)}\left\|\nabla v_h\right\|_{L^2(\Omega)} \\
&\leq C_0^{\prime}\left(C_{10}+3 C_M\left(C_{\mathcal{P}}\right)^2+\left(1+\mu^* M_1^{\mathcal{A}} C_{\mathrm{Inv}}\right) C_{\mathcal{P}} C_{11}\right)\left(h^2+\tau h\right)\left\|\nabla v_h\right\|_{L^2(\Omega)} \\
&:=C_{12}\left(h^2+\tau h\right)\left\|\nabla v_h\right\|_{L^2(\Omega)} .
\end{aligned}
$$

With the above results, and taking $v_h=\mathcal{F}_h^n \overline{u}_\tau^n-\overline{\mathcal{R}_h u_\tau}^n$, we conclude that

$$
\left\|\nabla\left(\mathcal{F}_h^n \overline{u}_\tau^n-\overline{\mathcal{R}_h u_\tau}^n\right)\right\|_{L^2(\Omega)} \leq \frac{1}{\beta_*}\left(C_{12}+C_{13}\right)\left(h^2+\tau h\right):=C_{\mathcal{F}}\left(h^2+\tau h\right).
$$

Also, using the definition (54) and (64), we obtain

$$
\begin{aligned}
\mathcal{A}_h\left(u_\tau^0 ; \mathcal{F}_h^0 u_\tau^{1 *}-\mathcal{R}_h^{1 *} u_\tau^{1 *}, v_h\right)= & \mathcal{A}_h\left(u_\tau^{1 *} ; \mathcal{R}_h^{1 *} u_\tau^{1 *}-u_\tau^{1 *}, v_h\right)-\mathcal{A}_h\left(u_\tau^0 ; \mathcal{R}_h^{1 *} u_\tau^{1 *}-u_\tau^{1 *}, v_h\right) \\
& +\mathcal{A}_h\left(u_\tau^{1 *} ; u_\tau^{1 *}, v_h\right)-\mathcal{A}_h\left(u_\tau^0 ; u_\tau^{1 *}, v_h\right)-\mathcal{A}\left(u_\tau^{1 *} ; u_\tau^{1 *}, v_h\right)+\mathcal{A}\left(u_\tau^0 ; u_\tau^{1 *}, v_h\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_h\left(u_\tau^{1 *} ; \mathcal{F}_h^{1 *} u_\tau^{1 * *}-\mathcal{R}_h^{1 * *} u_\tau^{1 * *}, v_h\right)= & \mathcal{A}_h\left(u_\tau^{1 * *} ;\left(\mathcal{R}_h^{1 * *}-I\right) u_\tau^{1 * *}, v_h\right) \\
& -\mathcal{A}_h\left(u_\tau^{1 *} ;\left(\mathcal{R}_h^{1 * *}-I\right) u_\tau^{1 * *}, v_h\right)+\mathcal{A}_h\left(u_\tau^{1 * *} ; u_\tau^{1 * *}, v_h\right) \\
& -\mathcal{A}_h\left(u_\tau^{1 *} ; u_\tau^{1 * *}, v_h\right)-\mathcal{A}\left(u_\tau^{1 * *} ; u_\tau^{1 * *}, v_h\right)+\mathcal{A}\left(u_\tau^{1 *} ; u_\tau^{1 * *}, v_h\right) .
\end{aligned}
$$

By applying the Theorem 4.5 and (62)–(63b), the above equation can be proved similarly. The proof of Lemma 4.9 is complete. □

4.3. Spatial error analysis

In this subsection, we prove the L^∞-boundedness of the fully discrete numerical solutions. To this purpose, we first consider the weak form of the time-discrete system (22)–(23) given as, for $n=1, \ldots, N$

$$
\mathcal{M}\left(D_\tau u_\tau^{1 *}, v\right)+\mathcal{A}\left(u_\tau^0 ; u_\tau^{1 *}, v\right)=\left(f^1, v\right)_{\Omega}, \tag{68a}
$$

$$
\mathcal{M}\left(D_\tau u_\tau^{1 * *}, v\right)+\mathcal{A}\left(u_\tau^{1 *} ; u_\tau^{1 * *}, v\right)=\left(f^1, v\right)_{\Omega}, \tag{68b}
$$

$$
\mathcal{M}\left(D_\tau u_\tau^n, v\right)+\mathcal{A}\left(\widehat{u}_\tau^n ; \overline{u}_\tau^n, v\right)=\left(\overline{f}^n, v\right)_{\Omega}, \tag{68c}
$$

for all $v \in H_0^1(\Omega)$, and $u_\tau^0:=u_0$. Noting that the virtual element approximation solutions of above system are consistent with fully discrete solutions u_h^n, and then the boundedness of u_h^n can be further decomposed into

$$
\begin{aligned}
\left\|u_h^n\right\|_{L^\infty(K)} & \leq\left\|u^n\right\|_{L^\infty(K)}+\left\|u^n-u_\tau^n\right\|_{L^\infty(K)}+\left\|u_\tau^n-R_h^n u_\tau^n\right\|_{L^\infty(K)}+\left\|R_h^n u_\tau^n-u_h^n\right\|_{L^\infty(K)} \\
& \leq C_{\Omega}\left\|u^n\right\|_{H^2(K)}+C_{\Omega}\left\|u^n-u_\tau^n\right\|_{H^2(K)}+\left\|u_\tau^n\right\|_{L^\infty(K)}+\left\|R_h^n u_\tau^n\right\|_{L^\infty(K)}+\left\|R_h^n u_\tau^n-u_h^n\right\|_{L^\infty(K)} \\
& \leq 2 C_{\Omega}\left(\left\|u^n\right\|_{H^2(K)}+\left\|u_\tau^n\right\|_{H^2(K)}\right)+\left\|R_h^n u_\tau^n\right\|_{L^\infty(K)}+C_{\mathrm{Inv}} h^{-1}\left\|R_h^n u_\tau^n-u_h^n\right\|_{L^2(K)},
\end{aligned}
$$

where the time-discrete solution $\|u_\tau^n\|_{H^2(\Omega)}$ has been estimated in the Theorem 4.5, and $\|R_h^n u_\tau^n\|_{L^\infty(K)}$ can be controlled by Lemma 4.8. Hence, we only focus on analyzing the convergence of $\|\mathcal{R}_h^n u_\tau^n-u_h^n\|_{L^2(\Omega)}$. Once $\|\mathcal{R}_h^n u_\tau^n-u_h^n\|_{L^2(\Omega)} \leq C(h^2+\tau h)$ is proved, we can obtain the L^∞-boundedness of the fully discrete numerical solutions by the inverse inequality.

Theorem 4.10. *Assume that the time-discrete system* (22)–(23) *has a unique solution* u_τ^{1*} *and* $u_\tau^n, n = 0, 1, \ldots, N$. *Then there exist positive constants* h_2' *and* τ_2' *such that when* $h < h_2'$ *and* $\tau \le \tau_2'$, *the fully discrete system defined in* (17)–(18) *admits a unique solution* u_h^{1*} *and* $u_h^n, n = 0, 1, \ldots, N$, *which satisfies*

$$\|u_h^{1*}\|_{L^\infty(\Omega)} + \|u_h^{1**}\|_{L^\infty(\Omega)} + \|u_h^n\|_{L^\infty(\Omega)} \le C_2', \tag{69}$$

where the constant C_2' *defined as, for any* $n = 0, 1, \ldots, N$,

$$C_2' := \max\left\{ \left\|\mathcal{R}_h^{1*} u_\tau^{1*}\right\|_{L^\infty(\Omega)}, \left\|\mathcal{R}_h^{1*} u_\tau^{1**}\right\|_{L^\infty(\Omega)}, \|\mathcal{R}_h^n u_\tau^n\|_{L^\infty(\Omega)} \right\} + 1,$$

Proof. The existence and uniqueness of fully discrete system (17)–(18) follow immediately since the corresponding homogenous problem only admits zero solutions [16].

From the proof in Appendix A.3, it immediately follows that

$$\|u_\tau^{1**} - u_h^{1**}\|_{L^2} \le \widehat{C}_7\left(h^2 + \tau h\right), \quad \text{and} \quad \|u_h^{1*}\|_{L^\infty(\Omega)} + \|u_h^{1**}\|_{L^\infty(\Omega)} \le C_2'. \tag{70}$$

Next, we turn our attention to the estimate of $e_h^n := u_h^n - \mathcal{R}_h^n u_\tau^n, n = 1, \ldots, N$. From the fully discrete system (17) and the time-discrete system (68c), we have the following error equations, for $v_h \in V_h$, and $n = 1, \ldots, N$

$$\begin{aligned}
&\mathcal{M}_h\left(D_\tau e_h^n, v_h\right) + \mathcal{A}_h\left(\widehat{u}_h^n; \overline{e}_h^n, v_h\right) \\
&= \left(\Pi_k^0 \overline{f}^n - \overline{f}^n, v_h\right) + \left[\mathcal{M}\left(D_\tau u_\tau^n, v_h\right) - \mathcal{M}_h\left(D_\tau \mathcal{R}_h^n u_\tau^n, v_h\right)\right] \\
&+ \left[\mathcal{A}_h\left(\widehat{u}_\tau^n; \mathcal{F}_h^n \overline{u}_\tau^n, v_h\right) - \mathcal{A}_h\left(\widehat{u}_\tau^n; \overline{\mathcal{R}_h u_\tau}^n, v_h\right)\right] \\
&+ \left[\mathcal{A}_h\left(\widehat{u}_\tau^n; \overline{\mathcal{R}_h u_\tau}^n, v_h\right) - \mathcal{A}_h\left(\widehat{u}_h^n; \overline{\mathcal{R}_h u_\tau}^n, v_h\right)\right].
\end{aligned} \tag{71}$$

Consider the complications arising from the error equation (71), we divide the proof into two steps.

Step 1: Error bounds for e_h^1. By taking $v_h = \overline{e}_h^1$ in (71) at $n = 1$ and using the coercivity properties in (21), we obtain that

$$\begin{aligned}
&\frac{1}{2\tau}\left(\mathcal{M}_h(e_h^1, e_h^1) - \mathcal{M}_h(e_h^0, e_h^0)\right) + \beta_* \left\|\nabla \overline{e}_h^1\right\|_{L^2(\Omega)}^2 \\
&\le \left(\Pi_k^0 \overline{f}^1 - \overline{f}^1, \overline{e}_h^1\right) + \left[\mathcal{M}\left(D_\tau u_\tau^1, \overline{e}_h^1\right) - \mathcal{M}_h\left(D_\tau \mathcal{R}_h^1 u_\tau^1, \overline{e}_h^1\right)\right] \\
&+ \left[\mathcal{A}_h\left(\widehat{u}_\tau^1; \mathcal{F}_h^1 \overline{u}_\tau^1, \overline{e}_h^1\right) - \mathcal{A}_h\left(\widehat{u}_\tau^1; \overline{\mathcal{R}_h u_\tau}^1, \overline{e}_h^1\right)\right] \\
&+ \left[\mathcal{A}_h\left(\widehat{u}_\tau^1; \overline{\mathcal{R}_h u_\tau}^1, \overline{e}_h^1\right) - \mathcal{A}_h\left(\widehat{u}_h^1; \overline{\mathcal{R}_h u_\tau}^1, \overline{e}_h^1\right)\right] \\
&:= R_1^1 + R_2^1 + R_3^1 + R_4^1,
\end{aligned} \tag{72}$$

By the approximation properties (14) of Π_k^0, we have

$$R_1^1 \le \left\|\Pi_k^0 \overline{f}^1 - \overline{f}^1\right\|_{L^2(\Omega)} \|\overline{e}_h^1\|_{L^2(\Omega)} \le C_{\mathcal{P}} h^2 \left\|\overline{f}^1\right\|_{H^2(\Omega)} \|\overline{e}_h^1\|_{L^2(\Omega)}.$$

Noting that $\tau \|D_\tau u_\tau^1\|_{H^2(\mathcal{T}_h)} \le C_0'$, and using (10a), (20) and Theorem 4.5, we obtain

$$\begin{aligned}
R_2^1 &= \mathcal{M}\left(D_\tau u_\tau^1 - D_\tau \Pi_k^0 u_\tau^1, \overline{e}_h^1\right) + \mathcal{M}_h\left(D_\tau \Pi_k^0 u_\tau^1 - D_\tau \mathcal{R}_h^1 u_\tau^1, \overline{e}_h^1\right) \\
&\le \left(\left\|(I - \Pi_k^0) D_\tau u_\tau^1\right\|_{L^2(\Omega)} + \alpha^* \left\|(I - \mathcal{R}_h^1) D_\tau u_\tau^1\right\|_{L^2(\Omega)}\right) \left\|\overline{e}_h^1\right\|_{L^2(\Omega)} \\
&\le (1 + \alpha^*)\left(C_{\mathcal{P}} + C_{\mathcal{R}}\right) C_0' \frac{1}{\tau} h^2 \left\|\overline{e}_h^1\right\|_{L^2(\Omega)}.
\end{aligned}$$

Due to the boundedness of $\|u_h^{1**}\|_{L^\infty(\Omega)}$, one applies the continuity properties in (6) and (21), Lemma 4.9 to derive

$$\begin{aligned} R_3^1 = \mathcal{A}_h\left(\widehat{u}_\tau^1; \mathcal{F}_h^1\overline{u}_\tau^1 - \overline{\mathcal{R}_h u_\tau}^1, \overline{e}_h^1\right) &\le \beta^* \left\|\nabla\left(\mathcal{F}_h^1\overline{u}_\tau^1 - \overline{\mathcal{R}_h u_\tau}^1\right)\right\|_{L^2(\Omega)} \left\|\nabla\overline{e}_h^1\right\|_{L^2(\Omega)} \\ &\le \beta^* C_{\mathcal{F}}(h^2+\tau h)\left\|\nabla\overline{e}_h^1\right\|_{L^2(\Omega)}. \end{aligned}$$

To bound term R_4^1, we first note , in view of (70), that

$$\left\|\widehat{u}_\tau^1 - \widehat{u}_h^1\right\|_{L^2(\Omega)} \le \frac{1}{2}\left(\left\|u_\tau^0 - u_h^0\right\|_{L^2(\Omega)} + \left\|u_\tau^{1**} - u_h^{1**}\right\|_{L^2(\Omega)}\right) \le \frac{1}{2}\left(MC_{\mathcal{I}} + \widehat{C}_7\right)\left(h^2+\tau h\right),$$

Further, using the definition of $\mathcal{A}_h(\cdot;\cdot,\cdot)$ in (10b) together with boundedness of $\mathcal{R}_h^1 u_\tau^1$ in (63a), we obtain that

$$\begin{aligned} R_4^1 &= \left(\left(a(\widehat{u}_\tau^1) - a(\widehat{u}_h^1)\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla\overline{\mathcal{R}_h u_\tau}^1\right), \mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla\overline{e}_h^1\right)\right)_\Omega \\ &\quad + \sum_{K\in\mathcal{T}_h}\left(\nu_{\mathcal{A}}^K(\widehat{u}_\tau^1) - \nu_{\mathcal{A}}^K(\widehat{u}_h^1)\right) S_{\mathcal{A}}^K\left((I-\Pi_k^{\nabla,K})\overline{\mathcal{R}_h u_\tau}^1, (I-\Pi_k^{\nabla,K})\overline{e}_h^1\right) \\ &\le \mu^*\left\|\widehat{u}_\tau^1 - \widehat{u}_h^1\right\|_{L^2(\Omega)}\left\|\nabla\overline{\mathcal{R}_h u_\tau}^1\right\|_{L^\infty(\Omega)}\left\|\nabla\overline{e}_h^1\right\|_{L^2(\Omega)} \\ &\quad + C_{\mathcal{P}}\left\|\Pi_0^0\left(a(\Pi_k^0\widehat{u}_\tau^1) - a(\Pi_k^0\widehat{u}_h^1)\right)(I-\Pi_k^{\nabla,K})\nabla\overline{\mathcal{R}_h u_\tau}^1\right\|_{L^2(\Omega)}\left\|\nabla\overline{e}_h^1\right\|_{L^2(\Omega)} \\ &\le \mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left\|\widehat{u}_\tau^1 - \widehat{u}_h^1\right\|_{L^2(\Omega)}\left\|\nabla\overline{e}_h^1\right\|_{L^2(\Omega)} \\ &\le \frac{1}{2}\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left(MC_{\mathcal{I}} + \widehat{C}_7\right)\left(h^2+\tau h\right)\left\|\nabla\overline{e}_h^1\right\|_{L^2(\Omega)}. \end{aligned}$$

Based on the above results and using the fact that $e_h^0 = 0$, we have

$$\frac{\alpha_*}{2\tau}\|e_h^1\|_{L^2(\Omega)}^2 + \frac{\beta_*}{2}\left\|\nabla e_h^1\right\|_{L^2(\Omega)}^2 \le \frac{C_{14}^2}{\tau}\left(h^2+\tau h\right)^2 + \frac{\alpha_*}{4\tau}\|e_h^1\|_{L^2(\Omega)}^2 + \frac{\beta_*}{4}\left\|\nabla e_h^1\right\|_{L^2(\Omega)}^2,$$

which reduces to

$$\|e_h^1\|_{L^2(\Omega)} \le \frac{2C_{14}}{\sqrt{\alpha_*}}\left(h^2+h\tau\right), \tag{73}$$

where the constant C_{18} is defined as

$$\begin{aligned} C_{18} = &\frac{\sqrt{2\alpha_*}}{2}\left(C_{\mathcal{P}}\|\overline{f}^1\|_{H^2(\Omega)} + (1+\alpha^*)(C_{\mathcal{P}}+C_{\mathcal{R}})C_0'\right) \\ &+ \frac{\sqrt{2\beta_*}}{2}\left(\beta^* C_{\mathcal{F}} + \frac{1}{2}\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left(MC_{\mathcal{I}}+\widehat{C}_7\right)\right). \end{aligned}$$

Therefore, there exists the constants h_1 and τ_6 such that

$$\|u_h^1\|_{L^\infty(\Omega)} \le \|R_h u_\tau^1\|_{L^\infty(\Omega)} + h^{-1}C_{\text{Inv}}\|e_h^1\|_{L^2(\Omega)} \le C_1',$$

when $\tau \le \tau_6$ and $h \le h_1$, where $h_1 = \tau_6 := \frac{1}{4}\sqrt{\alpha_*}(C_{\text{Inv}}C_{14})^{-1}$.

Step 2: Error bounds for e_h^n, $n = 1,\ldots,N$. It is obvious that the linear elliptic equation (17) admits a unique bounded solution $u_h^n \in V_h, 2 \le n \le N$ at each time step if u_h^{n-1} and u_h^{n-2} is L^∞ bounded. For this purpose, we invoke a mathematical induction on

$$\max_{1\le n\le N}\|u_h^n\|_{L^\infty(\Omega)} \le M_2. \tag{74}$$

We assume that the above inequality (74) holds for $1 \le n \le m$ (which is already proved for $m = 1$ when $h \le h_1$ and $\tau \le \tau_6$) and try to prove that it also holds for $n = m + 1$. With this assumption, it holds that, for $n = 1, \ldots, m$,

$$\mu_* \le a(u_h^n) \le \mu^*, \quad \text{and} \quad |a_u(u_h^n)| + |a_{uu}(u_h^n)| \le \mu^*, \tag{75}$$

Let $v_h = \bar{e}_h^n$ in (71), and using (21), we obtain that

$$\begin{aligned}
&\frac{1}{2\tau}\left(\mathcal{M}_h(e_h^n, e_h^n) - \mathcal{M}_h(e_h^{n-1}, e_h^{n-1})\right) + \beta_* \|\nabla \bar{e}_h^n\|_{L^2(\Omega)}^2 \\
&\le \left(\Pi_k^0 \overline{f}^n - \overline{f}^n, \bar{e}_h^n\right) + \left[\mathcal{M}\left(D_\tau u_\tau^n, \bar{e}_h^n\right) - \mathcal{M}_h\left(D_\tau \mathcal{R}_h^n u_\tau^n, \bar{e}_h^n\right)\right] \\
&+ \left[\mathcal{A}_h\left(\widehat{u}_\tau^n; \mathcal{F}_h^n \overline{u}_\tau^n, \bar{e}_h^n\right) - \mathcal{A}_h\left(\widehat{u}_\tau^n; \overline{\mathcal{R}_h u_\tau}^n, \bar{e}_h^n\right)\right] \\
&+ \left[\mathcal{A}_h\left(\widehat{u}_\tau^n; \overline{\mathcal{R}_h u_\tau}^n, \bar{e}_h^n\right) - \mathcal{A}_h\left(\widehat{u}_h^n; \overline{\mathcal{R}_h u_\tau}^n, \bar{e}_h^n\right)\right] \\
&:= R_1^n + R_2^n + R_3^n + R_4^n,
\end{aligned}$$

which is equivalent to

$$\mathcal{M}_h(e_h^n, e_h^n) + 2\tau\beta_* \sum_{i=2}^n \left\|\nabla \bar{e}_h^i\right\|_{L^2(\Omega)}^2 \le \alpha^* \|e_h^1\|_{L^2(\Omega)}^2 + 2\tau \sum_{i=2}^n \left(R_1^i + R_2^i + R_3^i + R_4^i\right). \tag{76}$$

From the Lemma 3.2 and using Young's inequality, we have

$$\tau \sum_{i=2}^n R_1^i \le \tau \sum_{i=2}^n \left\|\Pi_k^0 \overline{f}^i - \overline{f}^i\right\|_{L^2(\Omega)} \left\|\bar{e}_h^i\right\|_{L^2(\Omega)} \le T C_{\mathcal{P}}^2 h^4 \|f\|_{L^\infty(0,T;H^2(\Omega))}^2 + \frac{1}{4}\tau \sum_{i=1}^n \left\|e_h^i\right\|_{L^2(\Omega)}^2.$$

To bound R_2^n, we rewrite this term as

$$\tau \sum_{i=2}^n R_2^i = \tau \sum_{i=2}^n \mathcal{M}\left(D_\tau u_\tau^i - D_\tau \Pi_k^0 u_\tau^i, \bar{e}_h^i\right) + \tau \sum_{i=2}^n \mathcal{M}_h\left(D_\tau \Pi_k^0 u_\tau^i - D_\tau \mathcal{R}_h^i u_\tau^i, \bar{e}_h^i\right).$$

Then, applying Lemma 3.2, (27b), and (5), we can prove that

$$\begin{aligned}
&\tau \sum_{i=2}^n \mathcal{M}\left(D_\tau u_\tau^i - D_\tau \Pi_k^0 u_\tau^i, \bar{e}_h^i\right) = \frac{\tau}{2}\mathcal{M}\left(D_\tau u_\tau^2 - D_\tau \Pi_k^0 u_\tau^2, e_h^1\right) \\
&\quad + \tau \sum_{i=2}^{n-1} \mathcal{M}\left(D_\tau \overline{u}_\tau^{i+1} - D_\tau \Pi_k^0 \overline{u}_\tau^{i+1}, e_h^i\right) + \frac{\tau}{2}\mathcal{M}\left(D_\tau u_\tau^n - D_\tau \Pi_k^0 u_\tau^n, e_h^n\right) \\
&\le \frac{\tau}{2}\left\|D_\tau u_\tau^2 - D_\tau \Pi_k^0 u_\tau^2\right\|_{L^2(\Omega)} \left\|e_h^1\right\|_{L^2(\Omega)} + \frac{\tau}{2}\left\|D_\tau u_\tau^n - D_\tau \Pi_k^0 u_\tau^n\right\|_{L^2(\Omega)} \left\|e_h^n\right\|_{L^2(\Omega)} \\
&+ \tau \sum_{i=2}^{n-1} \left\|D_\tau \overline{u}_\tau^{i+1} - D_\tau \Pi_k^0 \overline{u}_\tau^{i+1}\right\|_{L^2(\Omega)} \left\|e_h^i\right\|_{L^2(\Omega)} \\
&\le \frac{\tau}{8} C_{\mathcal{P}}^2 h^4 \left\|D_\tau u_\tau^2\right\|_{H^2(\mathcal{T}_h)}^2 + \frac{\tau^2}{2\alpha_*} C_{\mathcal{P}}^2 h^4 \left\|D_\tau u_\tau^n\right\|_{H^2(\mathcal{T}_h)}^2 + \frac{\alpha_*}{8}\left\|e_h^n\right\|_{L^2(\Omega)}^2 \\
&+ \frac{\tau}{2} C_{\mathcal{P}}^2 h^4 \sum_{i=2}^n \left\|D_\tau \overline{u}_\tau^i\right\|_{H^2(\Omega)}^2 + \frac{\tau}{2}\sum_{i=1}^{n-1} \left\|e_h^i\right\|_{L^2(\Omega)}^2 \\
&\le \frac{1 + \alpha_*(4 + T)}{2\alpha_*}\left(C_{\mathcal{P}} C_0'\right)^2 h^4 + \frac{\alpha_*}{8}\left\|e_h^n\right\|_{L^2(\Omega)}^2 + \frac{\tau}{2}\sum_{i=1}^n \left\|e_h^i\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Similarly, the term $\tau\sum_{i=2}^{n}\mathcal{M}_h(D_\tau\Pi_k^0 u_\tau^i - D_\tau\mathcal{R}_h u_\tau^i,\bar{e}_h^i)$ can be bounded as

$$\begin{aligned}
&\tau\sum_{i=2}^{n}\mathcal{M}_h\left(D_\tau\Pi_k^0 u_\tau^i - D_\tau\mathcal{R}_h^i u_\tau^i,\bar{e}_h^i\right)\\
&=\tau\sum_{i=2}^{n}\mathcal{M}_h\left(D_\tau\Pi_k^0 u_\tau^i - D_\tau u_\tau^i,\bar{e}_h^i\right)+\tau\sum_{i=2}^{n}\mathcal{M}_h\left(D_\tau u_\tau^i - D_\tau\mathcal{R}_h^i u_\tau^i,\bar{e}_h^i\right)\\
&\le\frac{1+\alpha_*(4+T)}{2\alpha_*}\left(C_{\mathcal{P}}^2+C_{\mathcal{R}}^2\right)\left(\alpha^* C_0'\right)^2 h^4+\frac{\alpha_*}{4}\left\|e_h^n\right\|_{L^2(\Omega)}^2+\tau\sum_{i=1}^{n}\left\|e_h^i\right\|_{L^2(\Omega)}^2.
\end{aligned}$$

Combining the above estimates, we obtain that

$$\tau\sum_{i=2}^{n}R_2^i\le C_{15}h^4+\frac{3\alpha_*}{8}\left\|e_h^n\right\|_{L^2(\Omega)}^2+\frac{3}{2}\tau\sum_{i=2}^{n}\left\|e_h^i\right\|_{L^2(\Omega)}^2, \tag{77}$$

where the constant $C_{15}=\frac{1+\alpha_*(4+T)}{\alpha_*}(1+\alpha^*)(C_{\mathcal{P}}^2+C_{\mathcal{R}}^2)(C_0')^2$. Then, taking advantage of the induction condition (74) and following the steps used in deriving R_3^1 and R_4^1, it follows that

$$\begin{aligned}
R_3^n=\mathcal{A}_h\left(\widehat{u}_\tau^n;\mathcal{F}_h^n\overline{u}_\tau^n-\overline{\mathcal{R}_h u_\tau}^n,\bar{e}_h^n\right)&\le\beta^*\left\|\nabla\left(\mathcal{F}_h^n\overline{u}_\tau^n-\overline{\mathcal{R}_h u_\tau}^n\right)\right\|_{L^2(\Omega)}\left\|\nabla\bar{e}_h^n\right\|_{L^2(\Omega)}\\
&\le\beta^* C_{\mathcal{F}}(h^2+\tau h)\left\|\nabla\bar{e}_h^n\right\|_{L^2(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
R_4^n&\le\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left\|\widehat{u}_\tau^n-\widehat{u}_h^n\right\|_{L^2(\Omega)}\left\|\nabla\bar{e}_h^n\right\|_{L^2(\Omega)}\\
&\le\frac{3}{2}\mu^* MC_M\left(1+(C_{\mathcal{P}})^3\right)\left(\sum_{i=1}^{2}\left\|e_h^{n-i}\right\|_{L^2(\Omega)}+2C_{\mathcal{R}}C_0'h^4\right)\left\|\nabla e_h^n\right\|_{L^2(\Omega)}\\
&:=C_{16}\left(\sum_{i=1}^{2}\left\|e_h^{n-i}\right\|_{L^2(\Omega)}+2C_{\mathcal{R}}C_0'h^4\right)\left\|\nabla e_h^n\right\|_{L^2(\Omega)},
\end{aligned}$$

which further implies that

$$\begin{aligned}
\tau\sum_{i=2}^{n}\left(R_3^i+R_4^i\right)\le{}&\frac{T\left(\beta^* C_{\mathcal{F}}\right)^2+4T\left(C_{16}C_{\mathcal{R}}C_0'\right)^2}{2\beta_*}\left(h^4+(\tau h)^2\right)\\
&+\tau\beta_*\sum_{i=1}^{n}\left\|\nabla e_h^i\right\|_{L^2(\Omega)}^2+\frac{\left(C_{16}\right)^2}{4\beta_*}\tau\sum_{i=1}^{n}\left\|e_h^i\right\|_{L^2(\Omega)}^2.
\end{aligned}\tag{78}$$

After plugging the bounds obtained for R_1^i-R_4^i into (76), we have

$$\begin{aligned}
\left\|e_h^n\right\|_{L^2(\Omega)}^2\le{}&\frac{\left(C_{16}\right)^2+14\beta_*}{\alpha_*\beta_*}\tau\sum_{i=1}^{n}\left\|e_h^i\right\|_{L^2(\Omega)}^2+\frac{8C_{15}+8TC_{\mathcal{P}}^2\left\|f\right\|_{L^\infty(0,T;H^2(\Omega))}^2}{\alpha_*}h^4\\
&+\frac{4T\left(\beta^* C_{\mathcal{F}}\right)^2+16T\left(C_{16}C_{\mathcal{R}}C_0'\right)^2}{\alpha_*\beta_*}\left(h^4+(\tau h)^2\right).
\end{aligned}$$

With the Gronwall's inequality, there exists $\tau\le\tau_7=\frac{\alpha_*\beta_*}{(C_{16})^2+14\beta_*}$ such that

$$\left\|e_h^n\right\|_{L^2(\Omega)}\le C_{16}\left(h^2+\tau h\right), \tag{79}$$

where the constant C_{16} only is dependent on $C_{\mathcal{R}}$, $C_{\mathcal{P}}$, C_M, and regularity constant M but independent of discrete parameter τ, h and n.

Therefore, for $h \le h_2$ and $\tau \le \tau_8$, it holds that

$$\|u_h^n\|_{L^\infty(\Omega)} \le \|R_h u_\tau^n\|_{L^\infty(\Omega)} + h^{-1} C_{\mathrm{Inv}} \|e_h^n\|_{L^2(\Omega)} \le C_1',$$

where the constants $h_2 = \tau_8 := \frac{1}{2}(C_{\mathrm{Inv}} C_{16})^{-1}$. This completes the mathematical induction on (74) in the case that $h \le h_2' := \min\{h_1, h_2, \widehat{h}_1\}$ and $\tau \le \tau_1' = \min\{\tau_1', \tau_6, \tau_7, \tau_8, \widehat{\tau}_3\}$. The proof of Theorem 4.10 is finished. $\square$

5. Unconditionally optimal error estimates

Utilizing the established boundedness of the numerical solution as demonstrated in the previous section, one can easily obtain the optimal error estimates of any k-order virtual element method under corresponding regularity assumptions, by following the classical approach of the virtual element method analysis[9,36].

With the help of the L^∞-boundedness of u_h^{1*} and $u_h^n, n = 1, \ldots, N$ derived from Theorem 4.10, it holds that

$$\begin{aligned} &\mu_* \le a(u_h^{1*}), a(u_h^{1**}), a(u_h^n) \le \mu^*, \\ &\left|a_u(u_h^{1*})\right| + \left|a_{uu}(u_h^{1*})\right| + \left|a_u(u_h^{1**})\right| + \left|a_{uu}(u_h^{1**})\right| + |a_u(u_h^n)| + |a_{uu}(u_h^n)| \le \mu^*. \end{aligned} \tag{80}$$

Now, we present the unconditionally optimal error estimates for the fully discrete system (17)–(18) under the L^2- and H^1-norm. For convenience, we will omit the superscript n for elliptic projection $\mathcal{R}_h^n$.

Theorem 5.1. *Suppose that the quasilinear parabolic problem* (2) *has a unique solution u satisfying* (19)*. Then, there exist constants τ_1' and h_1' such that when $\tau \le \tau_3'$ and $h \le h_3'$, the k-degree virtual element system* (17)–(18) *admits a unique solution u_h^{1*}, u_h^{1**} and $u_h^n, n = 0, \ldots, N$, which satisfies*

$$\begin{aligned} &\|u^n - u_h^n\|_{L^2(\Omega)} \le C_3'(\tau^2 + h^{k+1}), \\ &\|u^n - u_h^n\|_{H^1(\mathcal{T}_h)} \le C_3'(\tau^2 + h^k), \end{aligned} \tag{81}$$

where C_3' is a positive constant independent of discretization parameters τ, h and n.

Proof. To start with, we split the error $u^n - u_h^n, n = 0, \ldots, N$ into two terms,

$$\begin{aligned} u_h^{1*} - u^1 &= u_h^{1*} - \mathcal{R}_h u^1 + \mathcal{R}_h u^1 - u^1 := \vartheta^{1*} + \xi^1, \\ u_h^n - u^n &= u_h^n - \mathcal{R}_h u^n + \mathcal{R}_h u^n - u^n := \vartheta^n + \xi^n, \end{aligned}$$

and $\vartheta^{1**} := u_h^{1**} - \mathcal{R}_h u^1$, where the error ξ^n can be controlled by Lemma 4.7, and we only proceed to bound for $\vartheta^{1*}, \vartheta^{1**}$ and ϑ^n. From the fully discrete system (17)–(18) and the variational formulation (4), we have the following error equations, for $v_h \in V_h$,

$$\begin{aligned} &\mathcal{M}_h(D_\tau \vartheta^{1*}, v_h) + \mathcal{A}_h(u_h^0; \vartheta^{1*}, v_h) = \left(\Pi_k^0 f^1 - f^1, v_h\right)_\Omega \\ &\qquad + \left[\mathcal{M}(u_t^1, v_h) - \mathcal{M}_h(D_\tau \mathcal{R}_h u^1, v_h)\right] + \left[\mathcal{A}_h(u^1; \mathcal{R}_h u^1, v_h) - \mathcal{A}_h(u_h^0; \mathcal{R}_h u^1, v_h)\right], \end{aligned} \tag{82a}$$

$$\begin{aligned} &\mathcal{M}_h(D_\tau \vartheta^{1**}, v_h) + \mathcal{A}_h(u_h^{1*}; \vartheta^{1**}, v_h) = \left(\Pi_k^0 f^1 - f^1, v_h\right)_\Omega \\ &\qquad + \left[\mathcal{M}(u_t^1, v_h) - \mathcal{M}_h(D_\tau \mathcal{R}_h u^1, v_h)\right] + \left[\mathcal{A}_h(u^1; \mathcal{R}_h u^1, v_h) - \mathcal{A}_h(u_h^{1*}; \mathcal{R}_h u^1, v_h)\right], \end{aligned} \tag{82b}$$

$$\begin{aligned} &\mathcal{M}_h(D_\tau \vartheta^n, v_h) + \mathcal{A}_h(\widehat{u}_h^n; \overline{\vartheta}^n, v_h) = \left(\Pi_k^0 \overline{f}^n - \overline{f}^n, v_h\right)_\Omega \\ &\qquad + \left[\mathcal{M}(\overline{u}_t^n, v_h) - \mathcal{M}_h(D_\tau \mathcal{R}_h u^n, v_h)\right] + \left[\mathcal{A}_h(\overline{u}^n; \mathcal{R}_h \overline{u}^n, v_h) - \mathcal{A}_h(\widehat{u}_h^n; \mathcal{R}_h \overline{u}^n, v_h)\right]. \end{aligned} \tag{82c}$$

In the following, we divide the proof into three steps.

Step 1: Error bounds for ϑ^{1*} and ϑ^{1**}. By applying the approximation properties in Lemma 3.2 and Hölder's inequality, it holds that

$$\begin{aligned}\left(\Pi_k^0 f^1 - f^1, v_h\right)_\Omega &\le \left\|\Pi_k^0 f^1 - f^1\right\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\le C_{\mathcal{P}} h^{k+1} \left\|f^1\right\|_{H^{k+1}(\mathcal{T}_h)} \|v_h\|_{L^2(\Omega)} =: \gamma_1^{1*} \|v_h\|_{L^2(\Omega)}.\end{aligned} \tag{83}$$

We now use the definition of $\mathcal{M}_h(\cdot,\cdot)$ in (10a), (5), (20), Lemma 3.2 and the Taylor expansion, to deduce

$$\begin{aligned}&\mathcal{M}(u_t^1, v_h) - \mathcal{M}_h(D_\tau \mathcal{R}_h^1 u^1, v_h) \\ &= \mathcal{M}(u_t^1 - D_\tau u^1, v_h) + \mathcal{M}(D_\tau u^1 - D_\tau \Pi_k^0 u^1, v_h) \\ &+ \mathcal{M}_h(D_\tau \Pi_k^0 u^1 - D_\tau \mathcal{R}_h^1 u^1, v_h) \\ &\le \left(\left\|u_t^1 - D_\tau u^1\right\|_{L^2(\Omega)} + \left\|D_\tau u^1 - D_\tau \Pi_k^0 u^1\right\|_{L^2(\Omega)}\right) \|v_h\|_{L^2(\Omega)} \\ &+ \alpha^* \left\|D_\tau \Pi_k^0 u^1 - D_\tau \mathcal{R}_h^1 u^1\right\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \\ &\le \frac{C_{17}}{\tau}\left(\left\|\tau u_t^1 - (u^1 - u^0)\right\|_{L^2(\Omega)} + h^{k+1}\|u^1 - u^0\|_{H^{k+1}(\mathcal{T}_h)}\right) \|v_h\|_{L^2(\Omega)} \\ :&= \frac{1}{\tau}(\gamma_2^{1*} + \gamma_3^{1*}) \|v_h\|_{L^2(\Omega)},\end{aligned} \tag{84}$$

where $C_{17} = (1+\alpha^*)(C_{\mathcal{R}} + C_{\mathcal{P}})$. And applying the definition of $\mathcal{A}_h(\cdot;\cdot,\cdot)$ in (10a) together with boundedness of $\nabla \mathcal{R}_h^1 u^1$ in (61a) yields

$$\begin{aligned}&\mathcal{A}_h\left(u^1; \mathcal{R}_h u^1, v_h\right) - \mathcal{A}_h\left(u_h^0; \mathcal{R}_h u^1, v_h\right) \\ &= \left(\left(a(\Pi_k^0 u^1) - a(\Pi_k^0 u_h^0)\right) \mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla \mathcal{R}_h u^1\right), \mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla v_h\right)\right)_\Omega \\ &+ \sum_{K\in\mathcal{T}_h}\left(\nu_{\mathcal{A}}^K(u^1) - \nu_{\mathcal{A}}^K(u_h^0)\right) S_{\mathcal{A}}^K\left((I - \Pi_k^{\nabla,K})\mathcal{R}_h u^1, (I - \Pi_k^{\nabla,K}) v_h\right) \\ &\le \mu^*\left(1 + (C_{\mathcal{P}})^3\right)\left\|u^1 - u_h^0\right\|_{L^2(\Omega)} \left\|\nabla \mathcal{R}_h u^1\right\|_{L^\infty(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} \\ &\le \mu^* C_M\left(1 + (C_{\mathcal{P}})^3\right)\left\|u^1 - u_h^0\right\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} := \gamma_4^{1*} \|\nabla v_h\|_{L^2(\Omega)}.\end{aligned} \tag{85}$$

Now we substitute (83)–(85) in (82a) and take $v_h = \frac{\vartheta^{1*}}{\tau}$ in (82a) to obtain

$$\begin{aligned}&\frac{1}{\tau^2}\mathcal{M}_h(\vartheta^{1*}, \vartheta^{1*}) + \frac{\beta_*}{\tau}\left\|\nabla \vartheta^{1*}\right\|_{L^2(\Omega)}^2 \\ &\le \frac{1}{\tau^2}(\tau\gamma_1^{1*} + \gamma_2^{1*} + \gamma_3^{1*})\left\|\vartheta^{1*}\right\|_{L^2(\Omega)} + \frac{\gamma_4^{1*}}{\tau}\left\|\nabla \vartheta^{1*}\right\|_{L^2(\Omega)} \\ &\le \frac{\alpha_*}{2\tau^2}\left\|\vartheta^{1*}\right\|_{L^2(\Omega)}^2 + \frac{\beta_*}{2\tau}\left\|\nabla \vartheta^{1*}\right\|_{L^2(\Omega)}^2 + \frac{1}{2\alpha_*}(\tau\gamma_1^{1*} + \gamma_2^{1*} + \gamma_3^{1*})^2 + \frac{1}{2\beta_*\tau}(\gamma_4^{1*})^2,\end{aligned}$$

where we have used the estimate $\vartheta^0 = 0$. And by using coercivity properties of $\mathcal{M}_h(\cdot,\cdot)$ and the fact that $\left\|\nabla \vartheta^{1*}\right\|_{L^2(\Omega)}^2 \ge 0$, the above inequation can be written as

$$\left\|\vartheta^{1*}\right\|_{L^2(\Omega)}^2 \le \frac{\tau^2}{(\alpha_*)^2}(\tau\gamma_1^{1*} + \gamma_2^{1*} + \gamma_3^{1*})^2 + \frac{\tau}{\beta_*\alpha_*}(\gamma_4^{1*})^2, \tag{86}$$

In terms of the Taylor expansion, it follows that

$$\begin{aligned}\gamma_2^{1*} :&= C_{17}\left\|\tau u_t^1 - (u^1 - u^0)\right\|_{L^2(\Omega)} \le C_\kappa C_{17} \tau^2 \left\|u_{tt}(t_*)\right\|_{L^2(\Omega)}, \\ \gamma_3^{1*} :&= C_{17} h^{k+1}\left\|u^1 - u^0\right\|_{H^{k+1}(\mathcal{T}_h)} \le C_\kappa C_{17} \tau h^{k+1} \left\|u_t(t_*)\right\|_{H^{k+1}(\mathcal{T}_h)},\end{aligned} \tag{87}$$

Using the Taylor expansion once again and Lemma 3.2, we have

$$\begin{aligned}\gamma_4^{1*} &\leq \mu^* C_M \left(1+(C_{\mathcal{P}})^3\right)\left(\left\|u^1-u^0\right\|_{L^2(\Omega)}+\left\|u^0-u_h^0\right\|_{L^2(\Omega)}\right)\\ &\leq \mu^* C_M \left(1+(C_{\mathcal{P}})^3\right)\left(C_\kappa \tau \left\|u_t(t_*)\right\|_{L^\infty(\Omega)}+C_{\mathcal{R}} h^{k+1}\left\|u_0\right\|_{H^{k+1}(\mathcal{T}_h)}\right).\end{aligned} \tag{88}$$

Combining (86)–(88), we obtain that

$$\left\|\vartheta^{1*}\right\|_{L^2(\Omega)} \leq C_{18}(\tau^{\frac{3}{2}}+h^{k+1}),$$

which implies that

$$\left\|u^1-u_h^{1*}\right\|_{L^2(\Omega)} \leq \left\|u^1-\mathcal{R}_h^1 u^1\right\|_{L^2(\Omega)}+\left\|\vartheta^{1*}\right\|_{L^2(\Omega)} \leq C_{18}\tau^{\frac{3}{2}}+(C_{18}+C_{\mathcal{R}})h^{k+1},$$

where the constant C_{18} is defined as

$$\begin{aligned}C_{18} = &\frac{1}{\alpha_*}\left(C_{\mathcal{P}}\left\|f^1\right\|_{H^{k+1}(\mathcal{T}_h)}+C_\kappa C_{17}\left(\left\|u_{tt}(t_*)\right\|_{L^2(\Omega)}+\left\|u_t(t_*)\right\|_{H^{k+1}(\mathcal{T}_h)}\right)\right)\\ &+\frac{1}{\sqrt{\alpha_*\beta_*}}\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left(C_\kappa\tau\left\|u_t(t_*)\right\|_{L^\infty(\Omega)}+C_{\mathcal{R}}\left\|u_0\right\|_{H^{k+1}(\mathcal{T}_h)}\right).\end{aligned}$$

Next, we turn our attention to the estimate of $\|\vartheta^{1**}\|_{L^2(\Omega)}$. Selecting $v_h = \frac{\vartheta^{1**}}{\tau}$ in (82b) and using the fact that $\vartheta^0 = 0$, we have

$$\begin{aligned}&\frac{1}{\tau^2}\mathcal{M}_h(\vartheta^{1**},\vartheta^{1**})+\frac{\beta_*}{\tau}\left\|\nabla\vartheta^{1**}\right\|_{L^2(\Omega)}^2\\ &\leq \frac{1}{\tau^2}(\tau\gamma_1^{1*}+\gamma_2^{1*}+\gamma_3^{1*})\left\|\vartheta^{1*}\right\|_{L^2(\Omega)}+\frac{1}{\tau}\left[\mathcal{A}_h(u^1;\mathcal{R}_h u^1,\vartheta^{1**})-\mathcal{A}_h(u_h^{1*};\mathcal{R}_h u^1,\vartheta^{1**})\right]\\ &\leq \frac{\alpha_*}{2\tau^2}\left\|\vartheta^{1*}\right\|_{L^2(\Omega)}^2+\frac{\beta_*}{2\tau}\left\|\nabla\vartheta^{1**}\right\|_{L^2(\Omega)}^2+\frac{1}{2\alpha_*}(\tau\gamma_1^{1*}+\gamma_2^{1*}+\gamma_3^{1*})^2+\frac{1}{2\beta_*\tau}(\gamma_4^{1**})^2,\end{aligned}$$

which leads to

$$\left\|\vartheta^{1**}\right\|_{L^2(\Omega)} \leq C_{19}(\tau^2+h^{k+1}), \tag{89}$$

with the obvious definitions for the constant C_{19}, where we have used the estimate

$$\begin{aligned}\mathcal{A}_h(u^1;\mathcal{R}_h u^1,v_h)-\mathcal{A}_h(u_h^{1*};\mathcal{R}_h u^1,v_h) &\leq \mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left\|u^1-u_h^{1*}\right\|_{L^2(\Omega)}\left\|\nabla v_h\right\|_{L^2(\Omega)}\\ &\leq \mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left(C_{18}\tau^{\frac{3}{2}}+(C_{18}+C_{\mathcal{R}})h^{k+1}\right)\left\|\nabla v_h\right\|_{L^2(\Omega)} := \gamma_4^{1**}.\end{aligned}$$

By applying the triangle inequality and using estimate (55), we obtain that

$$\left\|u^1-u_h^{1**}\right\|_{L^2(\Omega)} \leq \left\|u^1-\mathcal{R}_h^1 u^1\right\|_{L^2(\Omega)}+\left\|\vartheta^{1**}\right\|_{L^2(\Omega)} \leq C_{19}\tau^2+(C_{19}+C_{\mathcal{R}})h^{k+1}. \tag{90}$$

Step 2: Error bounds for ϑ^1. Similar to (83)–(85), we obtain that

$$\left(\Pi_k^0\overline{f}^1-\overline{f}^1,v_h\right)_\Omega \leq C_{\mathcal{P}}h^{k+1}\left\|\overline{f}^1\right\|_{H^{k+1}(\mathcal{T}_h)}\left\|v_h\right\|_{L^2(\Omega)} =: \gamma_1^1\left\|v_h\right\|_{L^2(\Omega)}. \tag{91}$$

and

$$\begin{aligned}&\mathcal{M}(\overline{u}_t^1,v_h)-\mathcal{M}_h(D_\tau\mathcal{R}_h u^1,v_h)\\ &\leq C_{17}\left(\left\|\overline{u}_t^1-\frac{u^1-u^0}{\tau}\right\|_{L^2(\Omega)}+h^{k+1}\left\|\frac{u^1-u^0}{\tau}\right\|_{H^{k+1}(\mathcal{T}_h)}\right)\left\|v_h\right\|_{L^2(\Omega)}\\ &:=(\gamma_2^1+\gamma_3^1)\left\|v_h\right\|_{L^2(\Omega)},\end{aligned} \tag{92}$$

and

$$\begin{aligned}&\mathcal{A}_h(\overline{u}^1;\mathcal{R}_h\overline{u}^1,v_h)-\mathcal{A}_h(\widehat{u}_h^1;\mathcal{R}_h\overline{u}^1,v_h)\\&\le \mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\left\|\overline{u}^1-\widehat{u}_h^1\right\|_{L^2(\Omega)}\left\|\nabla v_h\right\|_{L^2(\Omega)}:=\gamma_4^1\left\|\nabla v_h\right\|_{L^2(\Omega)}.\end{aligned}\tag{93}$$

Let $v_h=\overline{\vartheta}^1$ in (82c) with $n=1$, and plug (91)–(93) into (82a), we have

$$\mathcal{M}_h(D_\tau\vartheta^1,\overline{\vartheta}^1)+\beta_*\left\|\nabla\overline{\vartheta}^1\right\|^2_{L^2(\Omega)}\le(\gamma_1^1+\gamma_2^1+\gamma_3^1)\left\|\overline{\vartheta}^1\right\|_{L^2(\Omega)}+\gamma_4^1\left\|\nabla\overline{\vartheta}^1\right\|_{L^2(\Omega)},$$

which implies that

$$\begin{aligned}\mathcal{M}_h(\vartheta^1,\vartheta^1)+2\beta_*\tau\left\|\nabla\overline{\vartheta}^1\right\|^2_{L^2(\Omega)}\le{}&\frac{\alpha_*}{2}\left\|\vartheta^1\right\|^2_{L^2(\Omega)}+\beta_*\tau\left\|\nabla\overline{\vartheta}^1\right\|^2_{L^2(\Omega)}\\&+\frac{2\tau}{\alpha_*}((\gamma_1^1)^2+(\gamma_2^1)^2+(\gamma_3^1)^2)+\frac{\tau}{\beta_*}(\gamma_4^1)^2,\end{aligned}\tag{94}$$

By applying (20) and the fact that $\left\|\nabla\vartheta^1\right\|^2_{L^2(\Omega)}\ge 0$, the above equation can be reduced to

$$\left\|\vartheta^1\right\|^2_{L^2(\Omega)}\le\frac{4\tau}{\alpha_*^2}((\gamma_1^1)^2+(\gamma_2^1)^2+(\gamma_3^1)^2)+\frac{2\tau}{\alpha_*\beta_*}(\gamma_4^1)^2.\tag{95}$$

From the Taylor expansion and Lemma 3.2, we obtain that

$$\begin{aligned}&(\gamma_2^1)^2:=C_{17}^2\left\|\overline{u}_t^1-\frac{u^1-u^0}{\tau}\right\|^2_{L^2(\Omega)}\le C_{17}^2C_\kappa^2\tau^3\int_{t_0}^{t_1}\left\|u_{ttt}(s)\right\|^2_{L^2(\Omega)}ds,\\&(\gamma_3^1)^2:=C_{17}^2h^{2(k+1)}\left\|\frac{u^1-u^0}{\tau}\right\|^2_{L^2(\Omega)}\le C_{17}^2C_\kappa^2h^{2(k+1)}\left\|u_t(t_{**})\right\|^2_{H^{k+1}(\mathcal{T}_h)},\\&(\gamma_4^1)^2\le\left(\frac12\mu^*C_M\left(1+(C_{\mathcal{P}})^3\right)\right)^2\left(\left\|u^1-u_h^{1*}\right\|_{L^2(\Omega)}+\left\|u^0-u_h^0\right\|_{L^2(\Omega)}\right)^2\\&\le\left(\frac12\mu^*C_M\left(1+(C_{\mathcal{P}})^3\right)\right)^2\left(C_{19}+C_{\mathcal{R}}(1+\left\|u_0\right\|_{H^{k+1}(\mathcal{T}_h)})\right)^2\left(\tau^4+h^{2(k+1)}\right).\end{aligned}\tag{96}$$

With the above results, we conclude that

$$\left\|\vartheta^1\right\|_{L^2(\Omega)}\le C_{20}(\tau^2+h^{k+1}),\tag{97}$$

which leads to

$$\left\|u^1-u_h^1\right\|_{L^2(\Omega)}\le\left\|u^1-\mathcal{R}_h^1u^1\right\|_{L^2(\Omega)}+\left\|\vartheta^1\right\|_{L^2(\Omega)}\le C_{20}\tau^2+(C_{20}+C_{\mathcal{R}})h^{k+1},\tag{98}$$

where $C_{20}=\frac{\mu^*C_M(1+(C_{\mathcal{P}})^3)}{\sqrt{\beta_*\alpha_*}}(C_{19}+C_{\mathcal{R}}(1+M))+\frac{2C_{17}C_\kappa}{\alpha_*}(\int_{t_0}^{t_1}\left\|u_{ttt}(s)\right\|^2_{L^2(\Omega)}ds+\left\|u_t(t_*)\right\|_{H^{k+1}(\mathcal{T}_h)})+\frac{2C_{\mathcal{P}}}{\alpha_*}\left\|\overline{f}^1\right\|_{H^{k+1}(\mathcal{T}_h)}$.

Next, we turn our attention to the estimate of $\left\|\nabla\vartheta^1\right\|_{L^2(\Omega)}$. We now make the choice $v_h=D_\tau\vartheta^1$ in (82c) with $n=1$ and using the fact that $\vartheta^0=0$, to obtain

$$\begin{aligned}\alpha_*\left\|D_\tau\vartheta^1\right\|^2_{L^2(\Omega)}+\frac{\beta_*}{2\tau}\left\|\nabla\vartheta^1\right\|^2_{L^2(\Omega)}&\le(\gamma_1^1+\gamma_2^1+\gamma_3^1)\left\|D_\tau\vartheta^1\right\|_{L^2(\Omega)}+\frac1\tau\gamma_4^1\left\|\nabla\vartheta^1\right\|_{L^2(\Omega)}\\&\le\frac{\alpha_*}{2}\left\|D_\tau\vartheta^1\right\|^2_{L^2(\Omega)}+\frac{\beta_*}{4\tau}\left\|\nabla\vartheta^1\right\|^2_{L^2(\Omega)}\\&+\frac{2}{\alpha_*}\left((\gamma_1^1)^2+(\gamma_2^1)^2+(\gamma_3^1)^2\right)+\frac{1}{\beta_*\tau}(\gamma_4^1)^2.\end{aligned}$$

Due to the fact that $\left\|D_\tau \vartheta^1\right\|_{L^2(\Omega)}^2 \geq 0$ and substitute results (96) into the above inequalities, we have

$$\left\|\nabla \vartheta^1\right\|_{L^2(\Omega)} \leq C_{21}(\tau^2 + h^{k+1}), \tag{99}$$

which implies that

$$\left\|\nabla(u^1 - u_h^1)\right\|_{L^2(\Omega)} \leq \left\|\nabla(u^1 - \mathcal{R}_h^1 u^1)\right\|_{L^2(\Omega)} + \left\|\nabla \vartheta^1\right\|_{L^2(\Omega)} \leq C_{21}\tau^2 + (C_{21} + C_{\mathcal{R}})h^k, \tag{100}$$

with the obvious definitions for the constant C_{21}.

Step 3: Error bounds for $\vartheta^n, n = 1, \ldots, N$**.** Following the steps used in (91) - (93), we have

$$\left(\Pi_k^0 \overline{f}^n - \overline{f}^n, v_h\right)_\Omega \leq C_{\mathcal{P}} h^{k+1} \left\|\overline{f}^n\right\|_{H^{k+1}(\mathcal{T}_h)} \|v_h\|_{L^2(\Omega)} =: \gamma_1^n \|v_h\|_{L^2(\Omega)}. \tag{101}$$

and

$$\begin{aligned}
&\mathcal{M}(\overline{u}_t^n, v_h) - \mathcal{M}_h(D_\tau \mathcal{R}_h u^n, v_h) \\
&= \mathcal{M}(\overline{u}_t^n - D_\tau u^n, v_h) + \mathcal{M}(D_\tau u^n - D_\tau \Pi_k^0 u^n, v_h) \\
&+ \mathcal{M}_h(D_\tau \Pi_k^0 u^n - D_\tau \mathcal{R}_h u^n, v_h) \\
&\leq \left(\|\overline{u}_t^n - D_\tau u^n\|_{L^2(\Omega)} + \left\|D_\tau u^n - D_\tau \Pi_k^0 u^n\right\|_{L^2(\Omega)}\right) \|v_h\|_{L^2(\Omega)} \\
&+ \alpha^* \left\|D_\tau \Pi_k^0 u^n - D_\tau \mathcal{R}_h u^n\right\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \\
&\leq C_{21} \left(\|\overline{u}_t^n - D_\tau u^n\|_{L^2(\Omega)} + h^{k+1}\|D_\tau u^n\|_{H^{k+1}(\mathcal{T}_h)}\right) \|v_h\|_{L^2(\Omega)} \\
:&= (\gamma_2^n + \gamma_3^n) \|v_h\|_{L^2(\Omega)},
\end{aligned} \tag{102}$$

and

$$\begin{aligned}
&\mathcal{A}_h(\overline{u}^n; \mathcal{R}_h \overline{u}^n, v_h) - \mathcal{A}_h(\widehat{u}_h^n; \mathcal{R}_h \overline{u}^n, v_h) \\
&\leq \mu^* C_M \left(1 + (C_{\mathcal{P}})^3\right) \|\overline{u}^n - \widehat{u}_h^n\|_{L^2(\Omega)} \|\nabla v_h\|_{L^2(\Omega)} := \gamma_4^n \|\nabla v_h\|_{L^2(\Omega)}.
\end{aligned} \tag{103}$$

Then, we substitute (101)–(103) into (82c) and take $v_h = \overline{\vartheta}^n$ to obtain

$$\mathcal{M}_h(D_\tau \vartheta^n, \vartheta^n) + \beta_* \left\|\nabla \overline{\vartheta}^n\right\|_{L^2(\Omega)}^2 \leq (\gamma_1^n + \gamma_2^n + \gamma_3^n) \left\|\overline{\vartheta}^n\right\|_{L^2(\Omega)} + \gamma_4^n \left\|\nabla \overline{\vartheta}^n\right\|_{L^2(\Omega)},$$

which leads to

$$\begin{aligned}
&\mathcal{M}_h(\vartheta^n, \vartheta^n) - \mathcal{M}_h(\vartheta^{n-1}, \vartheta^{n-1}) + 2\beta_* \tau \left\|\nabla \overline{\vartheta}^n\right\|_{L^2(\Omega)}^2 \\
&\leq \frac{\tau}{8\alpha_*} \sum_{i=0}^{1} \left\|\vartheta^{n-i}\right\|_{L^2(\Omega)}^2 + \beta_* \tau \left\|\nabla \overline{\vartheta}^n\right\|_{L^2(\Omega)}^2 + \frac{\tau}{2\alpha_*}(\gamma_1^n + \gamma_2^n + \gamma_3^n)^2 + \frac{\tau}{4\alpha_* \beta_*}(\gamma_4^n)^2,
\end{aligned}$$

By using the fact that $\|\nabla \vartheta^n\|_{L^2(\Omega)}^2 \geq 0$ and summing the above inequality up from 2 to n, we arrive at

$$\begin{aligned}
\|\vartheta^n\|_{L^2(\Omega)}^2 \leq& \frac{1}{\alpha_*} \mathcal{M}_h(\vartheta^1, \vartheta^1) + \frac{\tau}{4\alpha_*} \sum_{i=1}^{n} \left\|\vartheta^i\right\|_{L^2(\Omega)}^2 + 2\tau \sum_{i=1}^{n} (\gamma_1^i)^2 + 2\tau \sum_{i=1}^{n} (\gamma_2^i)^2 \\
&+ 2\tau \sum_{i=1}^{n} (\gamma_3^i)^2 + \frac{\tau}{4\beta_*} \sum_{i=1}^{n} (\gamma_4^i)^2.
\end{aligned} \tag{104}$$

Owing to the Taylor expansion and Lemma 3.2, the term on the right side of (104) can be estimated by

$$\begin{aligned}
\tau\sum_{i=1}^{n}(\gamma_2^i)^2 :&= C_{17}^2\tau\sum_{i=1}^{n}\left\|\overline{u}_t^i - D_\tau u^i\right\|_{L^2(\Omega)}^2 \le C_{17}^2 C_\kappa^2\tau^4\int_{t_1}^{t_n}\|u_{ttt}(s)\|_{L^2(\Omega)}^2\,ds,\\
\tau\sum_{i=1}^{n}(\gamma_3^i)^2 :&= C_{17}^2\tau\sum_{i=1}^{n}h^{2(k+1)}\|D_\tau u^i\|_{H^{k+1}(\mathcal{T}_h)}^2 \le C_{17}^2 C_\kappa^2\tau h^{2(k+1)}\int_{t_1}^{t_n}\|u_t(s)\|_{H^{k+1}(\mathcal{T}_h)}^2\,ds,\\
\tau\sum_{i=1}^{n}(\gamma_4^i)^2 :&= \left(\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\right)^2\tau\sum_{i=1}^{n}\left\|\overline{u}^i - \widehat{u}_h^i\right\|_{L^2(\Omega)}^2\\
&\le \left(\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)\right)^2\tau\sum_{i=1}^{n}\left\|\overline{u}^i - \widehat{u}^i\right\|_{L^2(\Omega)}^2 + \left\|\widehat{u}^i - \widehat{u}_h^i\right\|_{L^2(\Omega)}^2\\
&\le \left(\mu^* C_M\left(1+(C_{\mathcal{P}})^3\right)C_\kappa\right)^2\tau^4\int_{t_1}^{t_n}\|u_{tt}(s)\|_{L^2(\Omega)}^2\,ds + 8(\mu^* C_M)^2\tau\sum_{i=1}^{n}\left\|\vartheta^i\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{105}$$

Based on the above results and use the Gronwall's inequality, we conclude that

$$\|\vartheta^n\|_{L^2(\Omega)} \le C_{22}(\tau^2 + h^{k+1}), \tag{106}$$

which leads to

$$\|u^n - u_h^n\|_{L^2(\Omega)} \le \|u^n - \mathcal{R}_h^n u^n\|_{L^2(\Omega)} + \|\vartheta^n\|_{L^2(\Omega)} \le C_{22}\tau^2 + (C_{22}+C_{\mathcal{R}})h^{k+1}, \tag{107}$$

where the constant C_{22} only is dependent on $C_{\mathcal{R}}$, $C_{\mathcal{P}}$, C_κ and regularity constant M.

We now proceed to estimate $\|\nabla\vartheta^n\|_{L^2(\Omega)}$. Let $v_h = D_\tau\vartheta^n$ in (82c) and plug (101)–(103) into (82c), we have

$$\begin{aligned}
&\alpha_*\|D_\tau\vartheta^n\|_{L^2(\Omega)}^2 + \frac{1}{2\tau}\left[\mathcal{A}_h(\widehat{u}_h^n;\nabla\vartheta^n,\nabla\vartheta^n) - \mathcal{A}_h(\widehat{u}_h^n;\nabla\vartheta^{n-1},\nabla\vartheta^{n-1})\right]\\
&\le (\gamma_1^n+\gamma_2^n+\gamma_3^n)\|D_\tau\vartheta^n\|_{L^2(\Omega)} + \frac{1}{2\tau}\gamma_4^n\left(\|\nabla\vartheta^n\|_{L^2(\Omega)} + \left\|\nabla\vartheta^{n-1}\right\|_{L^2(\Omega)}\right)\\
&\le \frac{\alpha_*}{2}\|D_\tau\vartheta^n\|_{L^2(\Omega)}^2 + \frac{1}{2\tau}\sum_{i=0}^{1}\left\|\nabla\vartheta^{n-i}\right\|_{L^2(\Omega)}^2 + \frac{2}{\alpha_*}\left((\gamma_1^n)^2+(\gamma_2^n)^2+(\gamma_3^n)^2\right) + \frac{1}{4\tau}(\gamma_4^n)^2.
\end{aligned}$$

Noticing that $\|\nabla\vartheta^n\|_{L^2(\Omega)}^2 \ge 0$ and summing the above inequality up from 2 to n, we have

$$\beta_*\|\nabla\vartheta^n\|_{L^2(\Omega)}^2 \le (1+\beta^*)\sum_{i=1}^{n}\left\|\nabla\vartheta^i\right\|_{L^2(\Omega)}^2 + \frac{1}{2}\tau\sum_{i=1}^{n}(\gamma_4^n)^2 + \frac{4}{\alpha_*}\tau\sum_{i=1}^{n}\left((\gamma_1^n)^2+(\gamma_2^n)^2+(\gamma_3^n)^2\right).$$

Using the estimates (105) together with the Gronwall's inequality yields

$$\|\nabla\vartheta^n\|_{L^2(\Omega)}^2 \le C_{23}(\tau^2 + h^k), \tag{108}$$

and we further obtain that

$$\|\nabla(u^n - u_h^n)\|_{L^2(\Omega)} \le \|\nabla(u^n - \mathcal{R}_h^n u^n)\|_{L^2(\Omega)} + \|\nabla\vartheta^n\|_{L^2(\Omega)} \le C_{23}\tau^2 + (C_{23}+C_{\mathcal{R}})h^k, \tag{109}$$

where the constant C_{23} only is dependent on $C_{\mathcal{R}}$, $C_{\mathcal{P}}$, C_M, and regularity constant M, etc, but independent of discrete parameter τ, h and n.

The proof is completed by setting $C_3' = \max_{18\le i\le 22}\{C_i\}$. $\square$

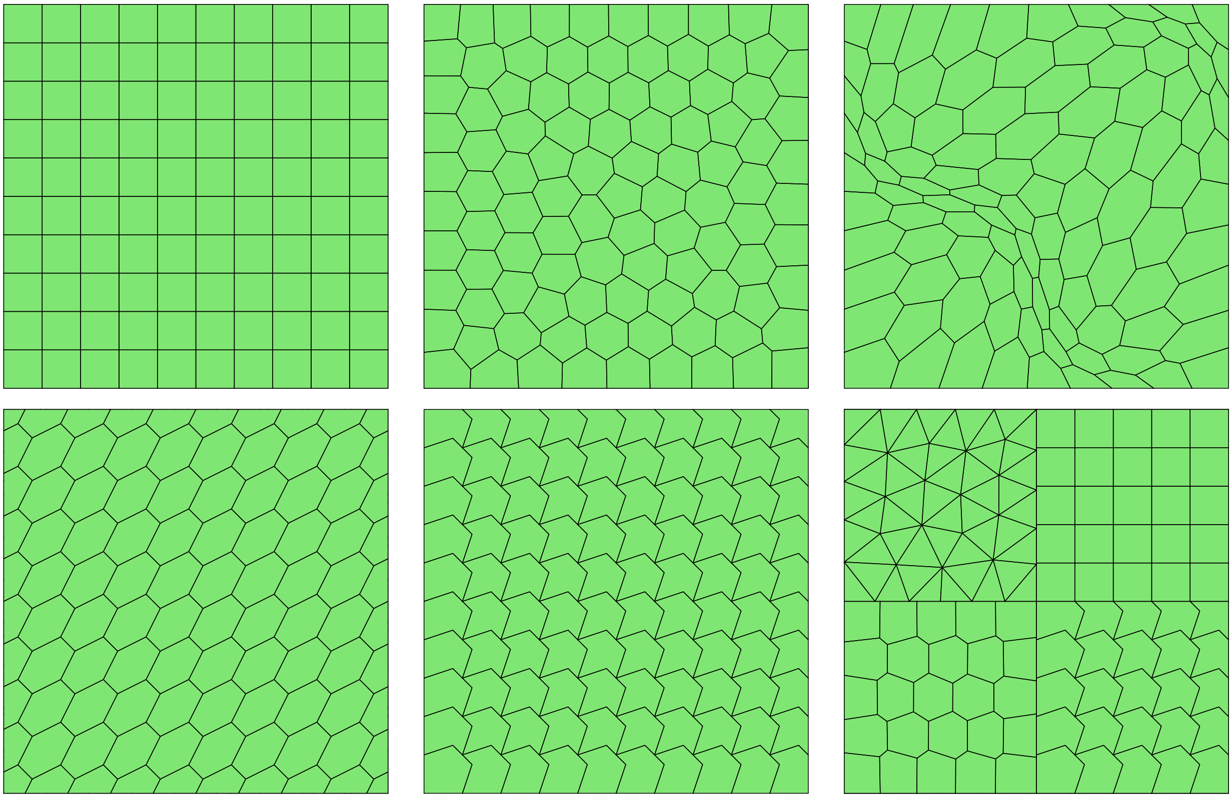

FIGURE 1. An illustration of the six types meshes $\mathcal{T}_h^1, \ldots, \mathcal{T}_h^6$ plotted from left to right and upper to below.

6. Numerical examples

In the section, we provide numerical results for the quasilinear parabolic problems to support our theoretical analysis and show the efficiency of linearized Crank-Nicolson virtual element methods (CN-VEMs). We do mainly tests for the lowest order element, *i.e.*, $k = 1$. To show the flexibility of our proposed methods to mesh discretization, we have considered six types of polygonal meshes, as shown in Figure 1, that is, the rectangular mesh $\mathcal{T}_h^1$, the centroid Voronoi tessellation mesh $\mathcal{T}_h^2$ generated by PolyMesher package[34], the distorted Voronoi mesh $\mathcal{T}_h^3$, the polygonal mesh $\mathcal{T}_h^4$ generated by the dual of the uniform triangle mesh, the non-convex octangle mesh $\mathcal{T}_h^5$, and hybrid mesh $\mathcal{T}_h^6$ (*i.e.* hybridization of triangular mesh, rectangular mesh, the centroid Voronoi tessellation mesh, and non-convex octangle mesh). Throughout this work, all computations are implemented by using mVEM package[40,41].

Since the errors $\|u^n - u_h^n\|_{L^2(\Omega)}$ and $\|u^n - u_h^n\|_{H^1(\mathcal{T}_h)}$, $n = 0, 1, \ldots, N$, are not computable directly, we test the errors between the exact solution u^n and the suitable projection of the virtual element solution u_h^n, that is,

$$\mathcal{E}_{L^2} := \|u^n - u_h^n\|_{L^2(\Omega)} = \left(\sum_{K \in \mathcal{T}_h} \left\| u^n - \Pi_1^{0,K} u_h^n \right\|_{L^2(K)}^2 \right)^{\frac{1}{2}}$$

$$\mathcal{E}_{H^1} := \|u^n - u_h^n\|_{H^1(\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} \left\| \nabla u^n - \Pi_k^{\nabla,K} u_h^n \right\|_{L^2(K)}^2 \right)^{\frac{1}{2}}.$$

TABLE 1. The error profiles and convergence rates for linearized CN-VEM schemes on different meshes (Example 1).

h	Voronoi mesh $\mathcal{T}_h^2$				Hybrid mesh $\mathcal{T}_h^6$			
	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC
1/10	4.98e-3	–	2.60e-1	–	6.06e-3	–	2.69e-1	–
1/20	1.21e-3	2.04	1.37e-1	0.93	1.37e-3	2.15	1.37e-1	0.97
1/30	5.22e-4	2.07	9.27e-2	0.96	5.91e-4	2.07	9.26e-2	0.97
1/40	2.85e-4	2.10	6.96e-2	1.00	3.32e-4	2.00	6.94e-2	1.00
1/50	1.84e-4	1.97	5.53e-2	1.02	2.12e-4	2.01	5.53e-2	1.01

Then, we also provide the following experimental order of convergence (EOC):

$$EOC := \frac{\log\left(\mathcal{E}_1/\mathcal{E}_1\right)}{\log(h_1/h_2)}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the errors within a given norm corresponding to mesh sizes h_1 and h_2, respectively.

6.1. Example 1: the quasilinear parabolic problem with polynomial-type nonlinearities

Consider the quasilinear parabolic problems (2) with polynomial-type nonlinearities $a(u) = 1 + u^2$ in the computational domain $\Omega = (0,1) \times (0,1)$. The force function $f(\mathbf{x}, t)$ is appropriately selected such that its exact solution is $u(\mathbf{x}, t) = xy(x-1)(y-1)e^{\frac{5t}{2}}$.

We solve the above the quasilinear parabolic problem (2) on different meshes by the linearized Crank-Nicolson virtual element schemes (17)–(18). Here, we choose $\tau = h$ and $\tau = h^{\frac{1}{2}}$ to confirm the optimal L^2 and semi-H^1 convergence rates, respectively. All errors are calculated at the final time, *i.e.* $T = 1$.

The error profiles and convergence rates of the L^2 norm and semi-H^1 norm for linearized CN-VEM schemes on different meshes are shown in the Table 1 and Figure 2. The exact solutions and linearized CN-VEM approximate solutions on the 100×100 polygonal meshes are plotted in Figure 3. From the above data, we can observe that the linearized CN-VEM schemes (17)–(18) has optimal approximation properties in L^2 and semi-H^1 norms with respect to various polygonal meshes, and confirming our error estimates (81).

To demonstrate the unconditional stability (convergence), we take several different polygonal meshes $\mathcal{T}_h^2$ and $\mathcal{T}_h^6$ with $h = 1/60, 1/70, 1/80, 1/90, 1/100$ for each $\tau = 1/5, 1/6, 1/7, 1/8$ and present error results in the Tables 2 and 3. The theoretical analysis in Theorem 5.1 (for $k = 1$) shows that, in some cases,

$$\left\|u^N - u_h^N\right\|_{L^2(\Omega)} \leq \mathcal{O}\left(\tau^2 + h^2\right), \quad \left\|u^N - u_h^N\right\|_{H^1(\mathcal{T}_h)} \leq \mathcal{O}\left(\tau^2 + h\right),$$

which tend to $\mathcal{O}(\tau^2)$ as $h \to 0$. Form the Tables 2 and 3, we can observe that for a fixed the time-step size τ, the numerical errors tends to be a constant (*i.e.* tend to $\mathcal{O}(\tau^2)$) as $h/\tau \to 0$, which shows that no time-step condition is needed.

6.2. Example 2: the quasilinear parabolic problem with nonpolynomial-type nonlinearities

In this subsection, we consider the quasilinear parabolic problems (2) with nonpolynomial-type nonlinearities $a(u) = \frac{1}{1+u}$ (referred to as Example 2-1) and $a(u) = cos(u)$ (referred to as Example 2-2) on the computational domain $\Omega = (0,1) \times (0,1)$, respectively. The force function $f(\mathbf{x}, t)$ and the Dirichlet boundary conditions are appropriately selected according to the exact solution $u(\mathbf{x}, t) = t\sin(\pi x)\sin(\pi y)$.

Similarly, the numerical simulations are performed using the virtual element method combined with linearized Crank-Nicolson scheme (17)–(18) treatment of the model (2) on various polygonal meshes $\mathcal{T}_h^1$-$\mathcal{T}_h^6$. To illustrate

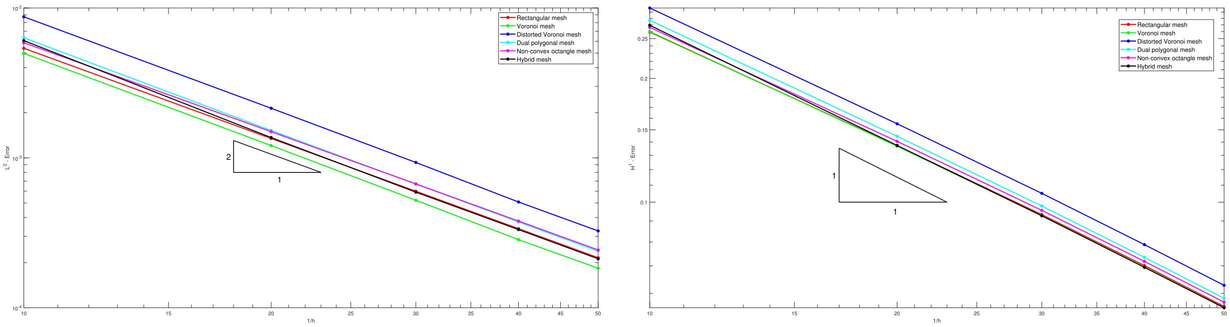

FIGURE 2. Convergence history of linearized CN-VEM schemes on different meshes (Example 1).

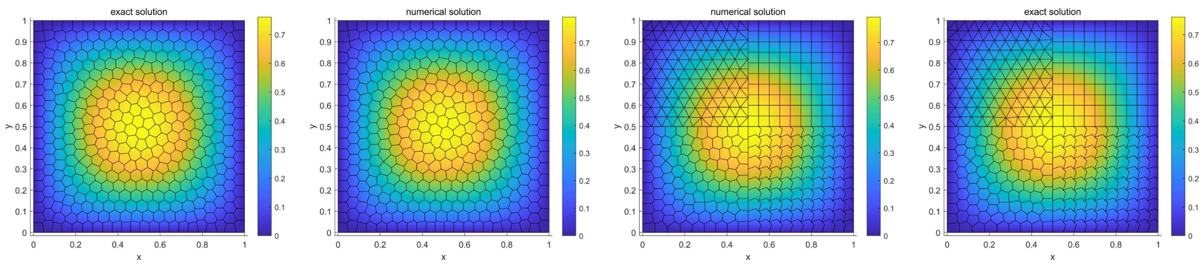

FIGURE 3. Exact solution as well as linearized CN-VEM solutions on the Voronoi mesh $\mathcal{T}_h^2$ (the left two) and the hybrid mesh $\mathcal{T}_h^6$ (the right two) (Example 1).

TABLE 2. The error profiles of linearized CN-VEM schemes with different time-step sizes and space sizes on the Voronoi mesh $\mathcal{T}_h^2$ (Example 1).

	$\|u^N - u_h^N\|_{L^2(\Omega)}$			
	$\tau = 1/5$	$\tau = 1/6$	$\tau = 1/7$	$\tau = 1/8$
$h = 1/60$	1.09e-2	7.91e-3	5.97e-3	4.64e-3
$h = 1/70$	1.09e-2	7.93e-3	5.98e-3	4.66e-3
$h = 1/80$	1.09e-2	7.94e-3	6.00e-3	4.67e-3
$h = 1/90$	1.10e-2	7.95e-3	6.00e-3	4.68e-3
$h = 1/100$	1.10e-2	7.96e-3	6.01e-3	4.69e-3
	$\|u^N - u_h^N\|_{H^1(\mathcal{T}_h^2)}$			
	$\tau = 1/5$	$\tau = 1/6$	$\tau = 1/7$	$\tau = 1/8$
$h = 1/60$	7.39e-2	6.05e-2	5.28e-2	4.84e-2
$h = 1/70$	7.08e-2	5.67e-2	4.85e-2	4.36e-2
$h = 1/80$	6.88e-2	5.41e-2	4.54e-2	4.02e-2
$h = 1/90$	6.74e-2	5.23e-2	4.33e-2	3.78e-2
$h = 1/100$	6.63e-2	5.10e-2	4.16e-2	3.58e-2

TABLE 3. The error profiles of linearized CN-VEM schemes with different time-step sizes and space sizes on the hybrid mesh $\mathcal{T}_h^6$ (Example 1).

	$\|u^N - u_h^N\|_{L^2(\Omega)}$			
	$\tau = 1/5$	$\tau = 1/6$	$\tau = 1/7$	$\tau = 1/8$
$h = 1/60$	1.09e-2	7.88e-3	5.94e-3	4.61e-3
$h = 1/70$	1.09e-2	7.91e-3	5.96e-3	4.64e-3
$h = 1/80$	1.09e-2	7.93e-3	5.98e-3	4.66e-3
$h = 1/90$	1.09e-2	7.94e-3	5.99e-3	4.67e-3
$h = 1/100$	1.10e-2	7.95e-3	6.00e-3	4.67e-3
	$\|u^N - u_h^N\|_{H^1(\mathcal{T}_h^2)}$			
	$\tau = 1/5$	$\tau = 1/6$	$\tau = 1/7$	$\tau = 1/8$
$h = 1/60$	7.38e-2	6.04e-2	5.27e-2	4.83e-2
$h = 1/70$	7.07e-2	5.66e-2	4.84e-2	4.35e-2
$h = 1/80$	6.87e-2	5.40e-2	4.53e-2	4.01e-2
$h = 1/90$	6.73e-2	5.23e-2	4.32e-2	3.77e-2
$h = 1/100$	6.63e-2	5.10e-2	4.16e-2	3.58e-2

TABLE 4. The error profiles and convergence rates of linearized CN-VEM schemes for nonlinearities $a(u) = \frac{1}{1+u}$ on different meshes (Example 2–1).

h	Voronoi mesh $\mathcal{T}_h^2$				Hybrid mesh $\mathcal{T}_h^6$			
	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC
1/10	7.95e-3	–	2.83e-1	–	1.02e-2	–	2.94e-1	–
1/20	1.89e-3	2.07	1.41e-1	1.00	2.35e-3	2.13	1.42e-1	1.05
1/30	8.21e-4	2.06	9.39e-2	1.0	1.01e-3	2.09	9.38e-2	1.02
1/40	4.52e-4	2.07	7.04e-2	1.00	5.66e-4	2.00	7.01e-2	1.01
1/50	2.90e-4	1.99	5.64e-2	1.00	3.66e-4	1.96	5.63e-2	0.98

TABLE 5. The error profiles and convergence rates of linearized CN-VEM schemes for nonlinearities $a(u) = cos(u)$ on different meshes (Example 2–2).

h	Voronoi mesh $\mathcal{T}_h^2$				Hybrid mesh $\mathcal{T}_h^6$			
	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC
1/10	7.51e-3	–	2.83e-1	–	9.84e-3	–	2.94e-1	–
1/20	1.79e-3	2.07	1.42e-1	1.00	2.28e-3	2.11	1.43e-1	1.05
1/30	7.83e-4	2.04	9.43e-2	1.00	9.78e-4	2.08	9.42e-2	1.02
1/40	4.30e-4	2.08	7.07e-2	1.00	5.49e-4	2.00	7.03e-2	1.01
1/50	2.76e-4	1.99	5.65e-2	1.00	3.55e-4	1.96	5.65e-2	0.98

our error estimates, we choose $\tau = h$ for the L^2 errors and $\tau = h^{1/2}$ for semi-H^1 errors. The convergence plots for the propsed scheme are listed in the Table 4–5 and Figures 4–5. From the above data, we can again observe that the linearized CN-VEM scheme (17)–(18) has optimal approximation properties in L^2 and semi-H^1 norms for the quasilinear parabolic problem with nonpolynomial-type nonlinearities.

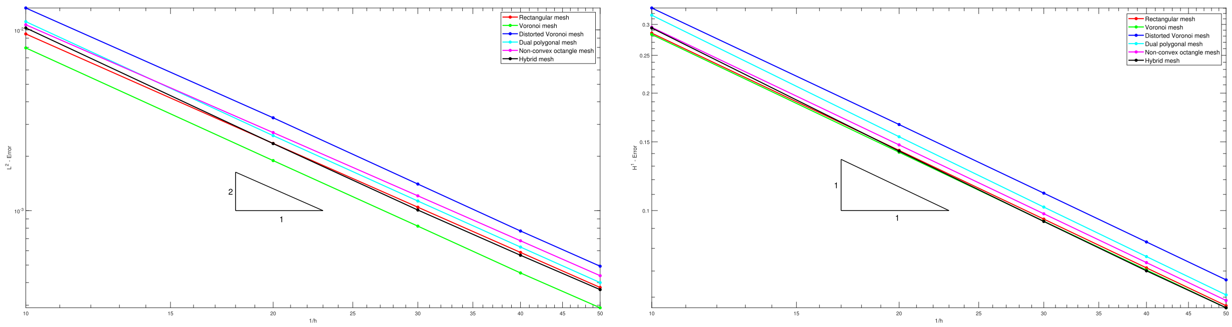

FIGURE 4. Convergence history of linearized CN-VEM schemes for nonlinearities $a(u) = \frac{1}{1+u}$ on different meshes (Example 2–1).

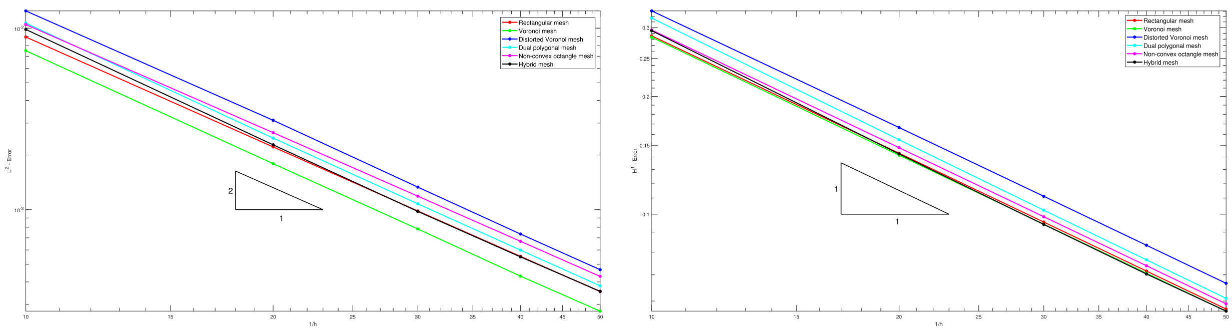

FIGURE 5. Convergence history of linearized CN-VEM schemes for nonlinearities $a(u) = cos(u)$ on different meshes (Example 2–2).

6.3. Example 3: the strongly nonlinear parabolic system

The purpose of the present test is to verify the effectiveness of the proposed linearized CN-VEM scheme for the strongly nonlinear parabolic system. Consider the following problem on the computational domain $\Omega = (-1,1) \times (-1,1)$:

$$\begin{cases} u_t(\mathbf{x},t) - \nabla \cdot (a(u)\nabla u(\mathbf{x},t)) + g(u,\mathbf{x},t) = f(\mathbf{x},t), \text{ in } \Omega \times (0,1], \\ u(\mathbf{x},t) = u_D, \text{ on } \partial\Omega \times (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \text{ in } \Omega. \end{cases} \tag{110}$$

where the nonlinear term $a(u)$ and $g(u)$ was selected

- polynomial-type nonlinearities (Example 3-1): $a(u) = 1 + u^2,\ g(u) = u^3$.
- nonpolynomial-type nonlinearities (Example 3-1): $a(u) = \frac{1}{1+u},\ g(u) = \sin(u)$.

The function $f(\mathbf{x},t)$ and the boundary condition u_D is appropriately selected such that its exact solution is $u(\mathbf{x},t) = (x^2y + sin(2\pi x)sin(2\pi y))t$.

TABLE 6. The error profiles and convergence rates of linearized CN-VEM schemes for nonlinearities $a(u) = 1 + u^2,\ g(u) = u^3$ on different meshes (Example 3–1).

h	Voronoi mesh $\mathcal{T}_h^2$				Hybrid mesh $\mathcal{T}_h^6$			
	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC
1/10	5.42e-3	–	2.47e-1	–	6.86e-3	–	2.54e-1	–
1/20	1.36e-3	2.00	1.24e-1	0.99	1.62e-3	2.08	1.25e-1	1.02
1/30	5.87e-4	2.07	8.28e-2	1.00	7.00e-4	2.08	8.28e-2	1.01
1/40	3.22e-4	2.09	6.19e-2	1.01	3.97e-4	1.97	6.20e-2	1.01
1/50	2.05e-4	2.01	4.96e-2	1.00	2.56e-4	1.96	4.98e-2	0.98

TABLE 7. The error profiles and convergence rates of linearized CN-VEM schemes for nonlinearities $a(u) = \frac{1}{1+u},\ g(u) = \sin(u)$ on different meshes (Example 3–2).

h	Voronoi mesh $\mathcal{T}_h^2$				Hybrid mesh $\mathcal{T}_h^6$			
	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC	$\mathcal{E}_{L^2}$	EOC	$\mathcal{E}_{H^1}$	EOC
1/10	6.10e-3	–	2.47e-1	–	7.89e-3	–	2.54e-1	–
1/20	1.53e-3	2.00	1.24e-1	0.99	1.86e-3	2.08	1.25e-1	1.02
1/30	6.62e-4	2.06	8.28e-2	1.00	7.93e-4	2.10	8.28e-2	1.02
1/40	3.63e-4	2.08	6.19e-2	1.01	4.51e-4	1.96	6.20e-2	1.01
1/50	2.31e-4	2.02	4.95e-2	1.00	2.91e-4	1.96	4.98e-2	0.98

Based on the symbols and definitions introduced in Section 3, the fully discrete system consequently reads as follows:

$$\begin{cases} \mathcal{M}_h\left(D_\tau u_h^{1*}, v_h\right) + \mathcal{A}_h\left(u_h^0; u_h^{1*}, v_h\right) + \left(g\left(\Pi_k^0 u_h^0\right), \Pi_k^0 v_h\right) = (\Pi_k^0 f^1, v_h), \\ \mathcal{M}_h\left(D_\tau u_h^{1**}, v_h\right) + \mathcal{A}_h\left(u_h^{1*}; u_h^{1*}, v_h\right) + \left(g\left(\Pi_k^0 u_h^{1*}\right), \Pi_k^0 v_h\right) = (\Pi_k^0 f^1, v_h), \\ \mathcal{M}_h\left(D_\tau u_h^n, v_h\right) + \mathcal{A}_h\left(\widehat{u}_h^n; \overline{u}_h^n, v_h\right) + \left(g\left(\Pi_k^0 \widehat{u}_h^n\right), \Pi_k^0 v_h\right) = (\Pi_k^0 \overline{g}^n, v_h), \\ u_h^0 := \mathcal{I}_h u_0, \end{cases} \tag{111}$$

for all $v_h \in V_h$ and $n = 1, \ldots, N$.

Similarly, we solve the above the quasilinear parabolic problem (110) on various polygonal meshes $\mathcal{T}_h^1$-$\mathcal{T}_h^6$ by the linearized CN-VEM schemes (111). To illustrate our error estimates, we choose $\tau = h$ for the L^2 errors and $\tau = h^{1/2}$ for semi-H^1 errors. As shown in the Tables 6–7 and Figures 6–7, the linearized CN-VEM schemes achieved the optimal approximation properties both in L^2 norm and semi-H^1 norm for the strongly nonlinear parabolic system (110) on various polygonal meshes. All the above numerical results show that the linearized CN-VEM is a family of efficient algorithms for solving the strongly nonlinear parabolic system.

7. CONCLUSION

In this paper, we have shown that the linearized Crank-Nicolson virtual element method for quasilinear parabolic equations with general nonlinear diffusion coefficient can achieve optimal error estimates without any time-step restriction. Our method can be applied on general polygonal meshes and the analysis is based on a new splitting of the error into the time and spatial directions. This allows us to bound the numerical solution (or its error) in the L^∞ norm without any limitations on the time-step size. A key contribution of this article is the construction of two new elliptical projections and the analysis of the convergence and boundedness of the projection solution. With the VEM projections, we are able to achieve L^∞ boundedness of the fully discrete solution and then prove the optimal error estimates unconditionally. This analysis technique can also potentially

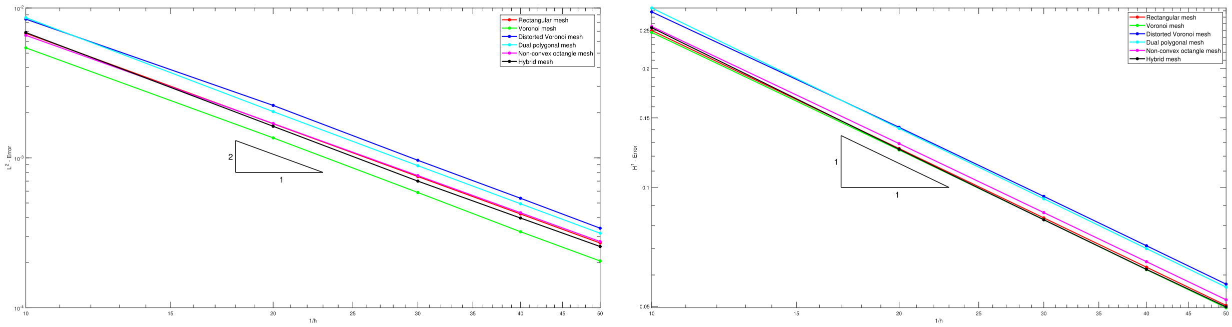

FIGURE 6. Convergence history of linearized CN-VEM schemes for nonlinearities $a(u) = \frac{1}{1+u}$, $g(u) = \sin(u)$ on different meshes (Example 3–1).

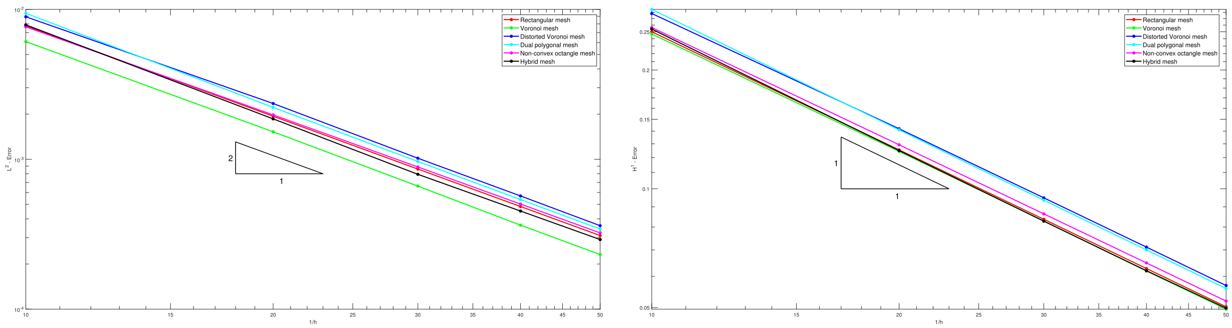

FIGURE 7. Convergence history of linearized CN-VEM schemes for nonlinearities $a(u) = \frac{1}{1+u}$, $g(u) = \sin(u)$ on different meshes (Example 3–2).

be extended to other nonlinear parabolic systems. In future work, we plan to apply this approach to solve other nonlinear physical models on general polygonal meshes, such as the Nernst-Planck-Poisson model, and validate the theoretical results through numerical examples.

APPENDIX A. TECHNICAL RESULTS

In this section, we present the proof of the some technical results.

A.1. Proof of Lemma 4.4

To start with, we denote

$$e_\tau^{1*} := u_\tau^{1*} - u^1, \quad \overline{e}_\tau^{1*} := \frac{1}{2}(e_\tau^{1*} + e_\tau^0), \quad D_\tau e_\tau^{1*} := \frac{1}{\tau}\left(e_\tau^{1*} - e_\tau^0\right)$$

and the corresponding error for u_τ^{1**} can be similarly defined. By taking $t = t_1$ in (2) and using time-discrete system (23), we obtain the following error equations,

$$D_\tau e_\tau^{1*} - \nabla \cdot \left(a(u_\tau^0)\nabla e_\tau^{1*}\right) = E^1 + \nabla \cdot H^{1*}, \tag{A.1a}$$

$$D_\tau e_\tau^{1**} - \nabla \cdot \left(a(u_\tau^{1*})\nabla e_\tau^{1**}\right) = E^1 + \nabla \cdot H^{1**}, \tag{A.1b}$$

with the boundary condition $e_\tau^{1*} = 0$ on $\partial\Omega$, where $E^1 = u_t^1 - D_\tau u^1$, and

$$H^{1*} = \left(a(u_\tau^0) - a(u^1)\right) \nabla u^1, \quad H^{1**} = \left(a(u_\tau^{1*}) - a(u^1)\right) \nabla u^1.$$

By the Taylor expansion, there exists a constant C_κ satisfying

$$\|E^{1*}\|_{L^p(\Omega)}^p \leq (C_\kappa)^p \tau^p \|u_{tt}(t_*)\|_{L^p(\Omega)}^p \tag{A.2}$$

for $2 \leq p \leq 4$, where $t_0 \leq t_* \leq t_1$ and C_κ is a positive constant independent of τ.

For clarity, we divide the proof into two steps.

Step 1: Error bounds for e_τ^{1*}. Due to $u_\tau^0 = u^0 \in H^2(\Omega)$, the boundedness of u_τ^0 follows immediately. Then, (23a) is a linear elliptic problem, and the existence and uniqueness of solution u_τ^{1*} can be obtained from the Lax-Milgram lemma.

Using the fact that $e_\tau^0 = 0$ and taking the inner product of both sides of (A.1a) with $D_\tau e_\tau^{1*}$, we obtain that

$$\begin{aligned}\frac{1}{\tau^2} \left\|e_\tau^{1*}\right\|_{L^2(\Omega)}^2 + \frac{\mu_*}{\tau} \left\|\nabla e_\tau^{1*}\right\|_{L^2(\Omega)}^2 &\leq \frac{1}{2\tau^2} \left\|e_\tau^{1*}\right\|_{L^2(\Omega)}^2 + \|E^1\|_{L^2(\Omega)}^2 + \|\nabla \cdot H^{1*}\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2\tau^2} \left\|e_\tau^{1*}\right\|_{L^2(\Omega)}^2 + (\widehat{C}_1)^2 \tau^2,\end{aligned} \tag{A.3}$$

where we have used the inequality

$$\begin{aligned}\|\nabla \cdot H^{1*}\|_{L^2(\Omega)} &\leq \left\|\nabla \left(a(u_\tau^0) - a(u^1)\right) \cdot \nabla u^1 + \left(a(u_\tau^0) - a(u^1)\right) \Delta u^1\right\|_{L^2(\Omega)} \\ &\leq \left\|\nabla \left(a(u_\tau^0) - a(u^1)\right)\right\|_{L^2(\Omega)} \|\nabla u^1\|_{L^\infty(\Omega)} + \|a(u_\tau^0) - a(u^1)\|_{L^\infty(\Omega)} \left\|u^1\right\|_{H^2(\Omega)} \\ &\leq 2\mu^* M C_\kappa \tau \left(\|u_t(t_*)\|_{H^1(\Omega)} + \|u_t(t_*)\|_{L^\infty(\Omega)}\right),\end{aligned} \tag{A.4}$$

and the constant $\widehat{C}_1 = (2\mu^* M + 1) C_\kappa (\|u_{tt}(t_*)\|_{L^2(\Omega)} + \|u_t(t_*)\|_{H^1(\Omega)} + \|u_t(t_*)\|_{L^\infty(\Omega)})$. Then using the fact that $\|\nabla e_\tau^{1*}\|_{L^2(\Omega)}^2 \geq 0$, it holds that

$$\|e_\tau^{1*}\|_{L^2(\Omega)} \leq \sqrt{2} \widehat{C}_1 \tau^2. \tag{A.5}$$

Next, integrating (A.1a) against $-\nabla \cdot (a(u_\tau^0) \nabla e_\tau^{1*})$ in the over domain Ω, and applying the Cauchy-Schwarz inequality and (A.4), we get

$$\frac{\mu_*}{\tau} \left\|\nabla e_\tau^{1*}\right\|_{L^2(\Omega)}^2 + \|\nabla \cdot (a(u_\tau^0) \nabla e_\tau^{1*})\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|\nabla \cdot (a(u_\tau^0) \nabla e_\tau^{1*})\|_{L^2(\Omega)}^2 + (\widehat{C}_1)^2 \tau^2, \tag{A.6}$$

which implies that

$$\|\nabla e_\tau^{1*}\|_{L^2(\Omega)} \leq \mu_*^{-\frac{1}{2}} \widehat{C}_1 \tau^{\frac{3}{2}}, \quad \|\nabla \cdot (a(u_\tau^0) \nabla e_\tau^{1*})\|_{L^2(\Omega)} \leq \sqrt{2} \widehat{C}_1 \tau. \tag{A.7}$$

And applying the Lemma 4.3 and (A.2), we have

$$\begin{aligned}\|e_\tau^{1*}\|_{H^2(\Omega)} &\leq C_\Omega \|E^{1*} + \nabla \cdot H^{1*} - D_\tau e_\tau^{1*}\|_{L^2(\Omega)} \\ &= C_\Omega \|\nabla \cdot (a(u_\tau^0) \nabla e_\tau^{1*})\|_{L^2(\Omega)} \leq \sqrt{2} C_\Omega \widehat{C}_1 \tau.\end{aligned} \tag{A.8}$$

Taking advantage of the Agmon's inequality (26), we further obtain that

$$\|e_\tau^{1*}\|_{L^\infty(\Omega)} \leq C_\Omega \|e_\tau^{1*}\|_{L^2(\Omega)}^{\frac{1}{2}} \|e_\tau^{1*}\|_{H^2(\Omega)}^{\frac{1}{2}} \leq \sqrt{2} \widehat{C}_1 (C_\Omega)^{\frac{3}{2}} \tau^{\frac{3}{2}}. \tag{A.9}$$

By applying the $W^{2,4}$ estimate to (A.1a) and using Lemma 4.3, we obtain

$$\left\|e_\tau^{1*}\right\|_{W^{2,4}(\Omega)} \leq C_\Omega \left(\left\|E^{1*}\right\|_{L^4(\Omega)} + \left\|\nabla \cdot H^{1*}\right\|_{L^4(\Omega)} + \left\|D_\tau e_\tau^{1*}\right\|_{L^4(\Omega)}\right). \tag{A.10}$$

Then, following the steps used in (A.4) and employing (31), we have

$$
\begin{aligned}
&\left\|E^{1*}\right\|_{L^4(\Omega)} + \left\|\nabla\cdot H^{1*}\right\|_{L^4(\Omega)}\\
&\leq (2\mu^* M + 1)\, C_\kappa \left(\|u_t(t_*)\|_{W^{1,4}(\Omega)} + \|u_t(t_*)\|_{L^\infty(\Omega)} + \|u_{tt}(t_*)\|_{L^4(\Omega)}\right)\tau,
\end{aligned}
$$

and applying the Sobolev embedding theorem and (A.7) to get

$$
\left\|D_\tau e_\tau^{1*}\right\|_{L^4(\Omega)} \leq C_\Omega \left\|D_\tau e_\tau^{1*}\right\|_{H^1(\Omega)} \leq (\mu_*)^{-\frac{1}{2}} C_\Omega \widehat{C}_1 \tau^{\frac{1}{2}}.
$$

which implies that

$$
\left\|\nabla e_\tau^{1*}\right\|_{L^\infty(\Omega)} \leq C_\Omega \left\|e_\tau^{1*}\right\|_{W^{2,4}(\Omega)} \leq \widehat{C}_2 \tau^{\frac{1}{2}}
$$

where $C_2 = C_\kappa C_\Omega^2 (2\mu^* M + 1)\left(\|u_t(t_*)\|_{W^{1,4}(\Omega)} + \|u_t(t_*)\|_{L^\infty(\Omega)} + \|u_{tt}(t_*)\|_{L^4(\Omega)}\right) + \mu_*^{-\frac{1}{2}} \widehat{C}_1 C_\Omega^3$.

Therefore, there exists a positive constant $\widehat{\tau}_1$ such that when $\tau < \widehat{\tau}_1$

$$
\begin{aligned}
&\|u_\tau^{1*}\|_{H^2(\Omega)} \leq \|u^1\|_{H^2(\Omega)} + \|e_\tau^{1*}\|_{H^2(\Omega)} \leq M_1,\\
&\tau\|D_\tau u_\tau^{1*}\|_{H^2(\Omega)} \leq \tau\|D_\tau u^1\|_{H^2(\Omega)} + \tau\|D_\tau e_\tau^{1*}\|_{H^2(\Omega)} \leq M_1,\\
&\|u_\tau^{1*}\|_{L^\infty(\Omega)} \leq \|u^1\|_{L^\infty(\Omega)} + \|e_\tau^{1*}\|_{L^\infty(\Omega)} \leq M_1,\\
&\|\nabla u_\tau^{1*}\|_{L^\infty(\Omega)} \leq \|\nabla u^1\|_{L^\infty(\Omega)} + \|\nabla e_\tau^{1*}\|_{L^\infty(\Omega)} \leq M_1,
\end{aligned}
$$

where the constant $\widehat{\tau}_1 = \min\{(\sqrt{2} C_\Omega \widehat{C}_1)^{-1}, (C_\Omega)^{-1}(\sqrt{2}\widehat{C}_1)^{-\frac{2}{3}}, (\widehat{C}_2)^{-\frac{1}{2}}\}$ independent of τ.

Step 2: Error bounds for e_τ^{1**}. With the boundedness of $\|u_\tau^{1*}\|_{L^\infty(\Omega)}$, the existence and uniqueness of solution to (23b) follow immediately since it is a linear elliptic equation.

Combining the assumption (3) with the estimates (A.5) and (A.9), we easily obtain that

$$
\begin{aligned}
\|\nabla\cdot H^{1**}\|_{L^2(\Omega)} &\leq \left\|\nabla\left(a(u_\tau^{1*}) - a(u^1)\right)\cdot\nabla u^1 + \left(a(u_\tau^{1*}) - a(u^1)\right)\Delta u^1\right\|_{L^2(\Omega)}\\
&\leq \left\|\nabla\left(a(u_\tau^{1*}) - a(u^1)\right)\right\|_{L^2(\Omega)} \|\nabla u^1\|_{L^\infty(\Omega)} + \|a(u_\tau^{1*}) - a(u^1)\|_{L^\infty(\Omega)} \left\|u^1\right\|_{H^2(\Omega)}\\
&\leq \mu^* M \left(\mu_*^{-\frac{1}{2}} \widehat{C}_1 + \sqrt{2}\widehat{C}_1 (C_\Omega)^{\frac{3}{2}}\right)\tau^{\frac{3}{2}},
\end{aligned}
\tag{A.11}
$$

Then, following the steps used in the estimation of e^{1*} in (A.3), (A.6) and using the (A.11), we can see that

$$
\|e_\tau^{1**}\|_{L^2(\Omega)} + \|e_\tau^{1**}\|_{H^1(\Omega)} + \tau\|e_\tau^{1**}\|_{H^2(\Omega)} \leq \widehat{C}_3 \tau^2, \tag{A.12}
$$

where $\widehat{C}_3$ is an appropriate positive constant independent of τ. With the above estimate (A.12) and using the Agmon's inequality (26), we have

$$
\|e_\tau^{1**}\|_{L^\infty(\Omega)} \leq C_\Omega \|e_\tau^{1**}\|_{L^2(\Omega)}^{\frac{1}{2}} \|e_\tau^{1**}\|_{H^2(\Omega)}^{\frac{1}{2}} \leq C_\Omega \widehat{C}_3 \tau^{\frac{3}{2}}. \tag{A.13}
$$

Applying Lemma 4.3 ($p = 4$) to (A.1b) and using (A.8)–(A.9), we have

$$
\begin{aligned}
\left\|e_\tau^{1**}\right\|_{W^{2,4}(\Omega)} &\leq C_\Omega \left(\left\|E^{1**}\right\|_{L^4(\Omega)} + \left\|\nabla\cdot H^{1**}\right\|_{L^4(\Omega)} + \left\|D_\tau e_\tau^{1**}\right\|_{L^4(\Omega)}\right)\\
&\leq C_\Omega \left(\|u_{tt}(t_*)\|_{L^4(\Omega)} + \sqrt{2}\mu^* M C_\Omega \widehat{C}_1 \left(\sqrt{C_\Omega} + 1\right) + \widehat{C}_3\right)\tau,
\end{aligned}
\tag{A.14}
$$

which leads to

$$
\left\|\nabla e_\tau^{1**}\right\|_{L^\infty(\Omega)} \leq C_\Omega \left\|e_\tau^{1**}\right\|_{W^{2,4}(\Omega)} \leq \widehat{C}_4 \tau
$$

where the constant $\widehat{C}_4 = C_\Omega^2 \left(\|u_{tt}(t_*)\|_{L^4(\Omega)} + \sqrt{2}\mu^* M C_\Omega \widehat{C}_1 \left(\sqrt{C_\Omega} + 1\right) + \widehat{C}_3\right)$.

Moreover, when $\tau \le \widehat{\tau}_2 = \min\{(\widehat{C}_3)^{-1}, (\widehat{C}_4)^{-1}\}$, we further have

$$\begin{aligned}
&\|u_\tau^{1**}\|_{H^2(\Omega)} \le \|u^1\|_{H^2(\Omega)} + \|e_\tau^{1**}\|_{H^2(\Omega)} \le M_1,\\
&\tau\|D_\tau u_\tau^{1**}\|_{H^2(\Omega)} \le \tau\|D_\tau u^1\|_{H^2(\Omega)} + \tau\|D_\tau e_\tau^{1**}\|_{H^2(\Omega)} \le M_1,\\
&\|u_\tau^{1**}\|_{L^\infty(\Omega)} \le \|u^1\|_{L^\infty(\Omega)} + \|e_\tau^{1**}\|_{L^\infty(\Omega)} \le M_1,\\
&\|\nabla u_\tau^{1**}\|_{L^\infty(\Omega)} \le \|\nabla u^1\|_{L^\infty(\Omega)} + \|\nabla e_\tau^{1**}\|_{L^\infty(\Omega)} \le M_1.
\end{aligned}$$

Taking $C_0' = \max\{M_1, \widehat{C}_3, C_\Omega \widehat{C}_3\}$ and $\tau_0' = \min\{\widehat{\tau}_1, \widehat{\tau}_2\}$, the proof of Lemma 4.4 is completed.

A.2. Proof of Lemma 4.6

From the definition of $\mathcal{R}_h v$ given in (54) and the definition of L^2-projector, we have

$$\begin{aligned}
&\mathcal{A}_h(w;\mathcal{R}_h v,\mathcal{I}_h\phi) - \mathcal{A}(w;\mathcal{R}_h v,\mathcal{I}_h\phi)\\
&= \left\{\mathcal{A}_h(w;\mathcal{R}_h v - \Pi_k^0 v,\mathcal{I}_h\phi - \Pi_1^0\phi) - \mathcal{A}(w;\mathcal{R}_h v - \Pi_k^0 v,\mathcal{I}_h\phi - \Pi_1^0\phi)\right\}\\
&+ \left\{\mathcal{A}_h(w;\Pi_k^0 v,\mathcal{I}_h\phi - \Pi_1^0\phi) - \mathcal{A}(w;\Pi_k^0 v,\mathcal{I}_h\phi - \Pi_1^0\phi)\right\}\\
&+ \left\{\mathcal{A}_h(w;\mathcal{R}_h v,\Pi_1^0\phi) - \mathcal{A}(w;\mathcal{R}_h v,\Pi_1^0\phi)\right\} =: T_1 + T_2 + T_3.
\end{aligned} \tag{A.15}$$

Since $w \in L^\infty(\Omega)$, we use the continuity of $\mathcal{A}_h(\cdot;\cdot,\cdot)$ and $\mathcal{A}(\cdot;\cdot,\cdot)$ in (21) and (6), together with the interpolation error estimate (13), to derive

$$\begin{aligned}
T_1 &\le (\beta^* + \mu^*)\left\|\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right)\right\|_{L^2(\Omega)}\left\|\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right\|_{L^2(\Omega)}\\
&\le (\beta^* + \mu^*)(C_\mathcal{I} + C_\mathcal{P})\left(C_\mathcal{P} h^{k+1}\|v\|_{H^{k+1}(\mathcal{T}_h)} + h\left\|\nabla\left(\mathcal{R}_h v - v\right)\right\|_{L^2(\Omega)}\right)|\phi|_{H^2(\mathcal{T}_h)}.
\end{aligned} \tag{A.16}$$

As for the term T_2, which can be rewritten as

$$\begin{aligned}
T_2 &= \left(a(\Pi_k^0 w)\nabla\left(\Pi_k^0 v\right), \boldsymbol{\Pi_{k-1}^0}\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega - \left(a(w)\nabla\left(\Pi_k^0 v\right), \nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega\\
&= \left(a(\Pi_k^0 w)\nabla\left(\Pi_k^0 v\right), \left(\boldsymbol{\Pi_{k-1}^0} - \boldsymbol{I}\right)\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega\\
&\quad + \left(\left(a(\Pi_k^0 w) - a(w)\right)\nabla\left(\Pi_k^0 v\right), \nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega := T_2^A + T_2^B.
\end{aligned}$$

Using the fact that $\nabla(\Pi_k^0 v) \in \mathbb{P}_{k-1}(\Omega)$, condition (3), (57) and the Cauchy-Schwarz inequality, the term T_2^A can be estimated by

$$\begin{aligned}
T_2^A &= \left(a(\Pi_k^0 w)\nabla\left(\Pi_k^0 - I\right)v, \left(\boldsymbol{\Pi_{k-1}^0} - \boldsymbol{I}\right)\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega\\
&\quad + \left(a(\Pi_k^0 w)\nabla v, \left(\boldsymbol{\Pi_{k-1}^0} - \boldsymbol{I}\right)\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega\\
&= \left(a(\Pi_k^0 w)\nabla\left(\Pi_k^0 - I\right)v, \left(\boldsymbol{\Pi_{k-1}^0} - \boldsymbol{I}\right)\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega\\
&\quad + \left(\left(\boldsymbol{\Pi_{k-1}^0} - \boldsymbol{I}\right)\left(a(\Pi_k^0 w)\nabla v\right), \nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right)_\Omega\\
&\le 2\mu^* C_\mathcal{P}\left(C_\mathcal{P} + C_\mathcal{I}\right)h^{k+1}\|v\|_{H^{k+1}(\mathcal{T}_h)}|\phi|_{H^2(\mathcal{T}_h)}.
\end{aligned}$$

With the help of the Cauchy-Schwarz inequality and stability property of Π_k^0, we have

$$\begin{aligned}
T_2^B &\le \left\|\left(a(\Pi_k^0 w) - a(w)\right)\right\|_{L^4(\Omega)}\left\|\nabla\left(\Pi_k^0 v\right)\right\|_{L^4(\Omega)}\left\|\nabla\left(\mathcal{I}_h\phi - \Pi_1^0\phi\right)\right\|_{L^2(\Omega)}\\
&\le \mu^*\left(C_\mathcal{P} + C_\mathcal{I}\right)h\left\|\left(\Pi_k^0 - I\right)w\right\|_{L^4(\Omega)}\|\nabla v\|_{L^4(\Omega)}|\phi|_{H^2(\mathcal{T}_h)}.
\end{aligned}$$

Further, by applying the Sobolev embedding theorem, namely $\|\nabla v\|_{L^4(\Omega)} \leq C_\Omega |v|_{H^2(\mathcal{T}_h)}$, and the Gagliardo-Nirenberg inequality (25),

$$\left\|\left(\Pi_k^0 - I\right) w\right\|_{L^4(\Omega)} \leq C_\Omega \left\|\left(\Pi_k^0 - I\right) w\right\|_{L^2(\Omega)}^{\frac{1}{2}} \left\|\nabla\left(\Pi_k^0 - I\right) w\right\|_{L^2(\Omega)}^{\frac{1}{2}},$$

we obtain

$$T_2^B \leq \mu^* C_\Omega C_{\mathcal{P}} \left(C_{\mathcal{P}} + C_{\mathcal{I}}\right) h^{k+1} \|w\|_{H^{k+1}(\mathcal{T}_h)} \|v\|_{H^2(\mathcal{T}_h)} |\phi|_{H^2(\mathcal{T}_h)}.$$

Based on the above estimates, the term T_2 is bounded as

$$T_2 \leq \mu^* C_{\mathcal{P}} \left(2 + C_\Omega\right)\left(C_{\mathcal{P}} + C_{\mathcal{I}}\right) h^{k+1} \left(\|w\|_{H^{k+1}(\mathcal{T}_h)} + \|v\|_{H^{k+1}(\mathcal{T}_h)}\right) |\phi|_{H^2(\mathcal{T}_h)}. \tag{A.17}$$

In view of bounding term T_3, we write

$$\begin{aligned}
T_3 &= \left(a(\Pi_k^0 w)\boldsymbol{\Pi^0_{k-1}}\nabla\mathcal{R}_h v, \nabla\Pi_1^0\phi\right)_\Omega - \left(a(w)\nabla\mathcal{R}_h v, \nabla\Pi_1^0\phi\right)_\Omega \\
&= \left(a(\Pi_k^0 w)\left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\nabla\mathcal{R}_h v, \nabla\Pi_1^0\phi\right)_\Omega + \left(\left(a\left(\Pi_k^0 w\right) - a(w)\right)\nabla\mathcal{R}_h v, \nabla\Pi_1^0\phi\right)_\Omega \\
&= \left(\left(a\left(\Pi_k^0 w\right) - a(w)\right)\left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \nabla\Pi_1^0\phi\right)_\Omega \\
&+ \left(a(w)\left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \nabla\Pi_1^0\phi\right)_\Omega \\
&+ \left(a_w^*\left(\Pi_k^0 w - w\right)\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \nabla\Pi_1^0\phi\right)_\Omega \\
&+ \left(a_w^*\left(\Pi_k^0 w - w\right)\nabla\left(\Pi_k^0 v\right), \nabla\left(\Pi_1^0\phi\right)\right)_\Omega := T_3^A + T_3^B + T_3^C + T_3^D,
\end{aligned}$$

where a_w^* is defined by

$$a\left(\Pi_k^0 w\right) - a(w) = \left(\Pi_k^0 w - w\right)\int_0^1 a_w\left(\Pi_k^0 w - s\left(\Pi_k^0 w - w\right)\right) ds := a_w^*\left(\Pi_k^0 w - w\right).$$

Then, following the steps used in the estimation of T_2^B, we get

$$\begin{aligned}
T_3^A + T_3^C &\leq 2\mu^* \left\|\Pi_k^0 w - w\right\|_{L^4(\Omega)} \left\|\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right)\right\|_{L^2(\Omega)} \left\|\nabla\Pi_1^0\phi\right\|_{L^4(\Omega)} \\
&\leq 2\mu^* C_\Omega C_{\mathcal{P}} \left(C_{\mathcal{P}} h^{k+1}\|v\|_{H^{k+1}(\mathcal{T}_h)} + h\left\|\nabla\left(\mathcal{R}_h v - v\right)\right\|_{L^2(\Omega)}\right) \|w\|_{H^2(\mathcal{T}_h)} |\phi|_{H^2(\mathcal{T}_h)}.
\end{aligned}$$

To estimate T_3^B, we rewrite this term as

$$\begin{aligned}
T_3^B &= \left(a(w)\left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \nabla\left(\Pi_1^0\phi - \phi\right)\right)_\Omega \\
&+ \left(a(w)\left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \nabla\phi\right)_\Omega \\
&= \left(a(w)\left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \nabla\left(\Pi_1^0\phi - \phi\right)\right)_\Omega \\
&+ \left(\nabla\left(\mathcal{R}_h v - \Pi_k^0 v\right), \left(\boldsymbol{\Pi^0_{k-1}} - \boldsymbol{I}\right)\left(a(w)\nabla\phi\right)\right)_\Omega,
\end{aligned}$$

and then using the Lemma 3.2, it immediately following that

$$T_3^B \leq 2\mu^* C_{\mathcal{P}} \left(C_{\mathcal{P}} h^{k+1} \|v\|_{H^{k+1}(\mathcal{T}_h)} + h \left\| \nabla \left(\mathcal{R}_h v - v \right) \right\|_{L^2(\Omega)} \right) |\phi|_{H^2(\mathcal{T}_h)} .$$

In view of the fact that $\nabla \left(\Pi_k^0 v \right) \cdot \nabla \left(\Pi_1^0 \phi \right) \in \mathbb{P}_k(\Omega)$, the term T_3^D can be bounded by

$$\begin{aligned} T_3^D &= \left(\left(\Pi_k^0 w - w \right) \left(a_w^* - \Pi_0^0 a_w^* \right), \nabla \left(\Pi_k^0 v \right) \cdot \nabla \left(\Pi_1^0 \phi \right) \right)_\Omega \\ &\leq \mu^* C_{\mathcal{P}} h \left\| \Pi_k^0 w - w \right\|_{L^4(\Omega)} \left\| \nabla \left(\Pi_k^0 v \right) \right\|_{L^2(\Omega)} \left\| \nabla \left(\Pi_1^0 \phi \right) \right\|_{L^4(\Omega)} \\ &\leq \mu^* C_{\mathcal{P}} h^{k+1} \|w\|_{H^{k+1}(\mathcal{T}_h)} \left\| \nabla \left(\Pi_k^0 v \right) \right\|_{L^2(\Omega)} |\phi|_{H^2(\mathcal{T}_h)} . \end{aligned}$$

Collecting the above bounds yields

$$\begin{aligned} T_3 \leq \widehat{C}_5 &\left(\|w\|_{H^2(\mathcal{T}_h)} + \|v\|_{H^1(\mathcal{T}_h)} \right) \left(h^{k+1} \left(\|v\|_{H^{k+1}(\mathcal{T}_h)} + \|w\|_{H^{k+1}(\mathcal{T}_h)} \right) \right. \\ &\left. + h \left\| \nabla \left(\mathcal{R}_h v - v \right) \right\|_{L^2(\Omega)} \right) |\phi|_{H^2(\mathcal{T}_h)} , \end{aligned} \tag{A.18}$$

where $\widehat{C}_5$ is a h-independent constant but dependent on constants μ^*, C_Ω, $C_{\mathcal{P}}$ and $C_{\mathcal{I}}$.

Finally, by choosing a suitable h-independent positive constant C_ℓ, the claimed estimate follows from (A.15)–(A.18). The proof is completed.

A.3. Proof of (69) for $\|\mathbf{u_h^{1*}}\|_{\mathbf{L^\infty(\Omega)}}$ and $\|\mathbf{u_h^{1**}}\|_{\mathbf{L^\infty(\Omega)}}$

Denoting $e_h^{1*} := u_h^{1*} - \mathcal{R}_h^{1*} u_\tau^{1*}$. From the definition of $\mathcal{R}_h^0$, we obtain that

$$\|e_h^0\|_{L^2(\Omega)} \leq \|\mathcal{R}_h^0 u_\tau^0 - \mathcal{I}_h u_\tau^0\|_{L^2(\Omega)} = 0, \quad \text{and} \quad \|u_h^0\|_{L^\infty(\Omega)} \leq \|\mathcal{R}_h^0 u_\tau^0\|_{L^\infty(\Omega)} \leq C_M. \tag{A.19}$$

The numerical scheme (18) and the time-discrete system (68a) yield the following error equation for e_h^{1*}:

$$\begin{aligned} &\mathcal{M}_h \left(D_\tau e_h^{1*}, v_h \right) + \mathcal{A}_h \left(u_h^0; e_h^{1*}, v_h \right) \\ &= \left(\Pi_k^0 f^1 - f^1, v_h \right) + \left[\mathcal{M} \left(D_\tau u_\tau^{1*}, v_h \right) - \mathcal{M}_h \left(D_\tau \mathcal{R}_h^{1*} u_\tau^{1*}, v_h \right) \right] \\ &+ \left[\mathcal{A}_h \left(u_\tau^0; \mathcal{F}_h^0 u_\tau^{1*}, v_h \right) - \mathcal{A}_h \left(u_\tau^0; \mathcal{R}_h^{1*} u_\tau^{1*}, v_h \right) \right] \\ &+ \left[\mathcal{A}_h \left(u_\tau^0; \mathcal{R}_h^{1*} u_\tau^{1*}, v_h \right) - \mathcal{A}_h \left(u_h^0; \mathcal{R}_h^{1*} u_\tau^{1*}, v_h \right) \right] , \end{aligned} \tag{A.20}$$

Let $v_h = e_h^{1*}$ in (A.20) and using (21), we have

$$\begin{aligned} &\frac{\alpha_*}{\tau} \|e_h^{1*}\|_{L^2(\Omega)}^2 + \beta_* \left\| \nabla e_h^{1*} \right\|_{L^2(\Omega)}^2 \\ &\leq \left(\Pi_k^0 f^1 - f^1, e_h^{1*} \right) + \left[\mathcal{M} \left(D_\tau u_\tau^{1*}, e_h^{1*} \right) - \mathcal{M}_h \left(D_\tau \mathcal{R}_h^{1*} u_\tau^{1*}, e_h^{1*} \right) \right] \\ &+ \left[\mathcal{A}_h \left(u_\tau^0; \mathcal{F}_h^0 u_\tau^{1*}, e_h^{1*} \right) - \mathcal{A}_h \left(u_\tau^0; \mathcal{R}_h^{1*} u_\tau^{1*}, e_h^{1*} \right) \right] \\ &+ \left[\mathcal{A}_h \left(u_\tau^0; \mathcal{R}_h^{1*} u_\tau^{1*}, e_h^{1*} \right) - \mathcal{A}_h \left(u_h^0; \mathcal{R}_h^{1*} u_\tau^{1*}, e_h^{1*} \right) \right] \\ &:= R_1^{1*} + R_2^{1*} + R_3^{1*} + R_4^{1*}. \end{aligned} \tag{A.21}$$

From the approximation properties in Lemma 3.2, the term R_1^{1*} is bounded as follows:

$$R_1^{1*} \leq \left\| \Pi_k^0 f^1 - f^1 \right\|_{L^2(\Omega)} \left\| e_h^{1*} \right\|_{L^2(\Omega)} \leq C_{\mathcal{P}} h^2 \left\| f^1 \right\|_{H^2(\Omega)} \left\| e_h^{1*} \right\|_{L^2(\Omega)} .$$

Using the definition (10a) and continuity (20) of $\mathcal{M}_h(\cdot,\cdot)$ combined with the Theorem 4.5, namely $\tau\|D_\tau u_\tau^{1*}\|_{H^2(\mathcal{T}_h)} \le C_0'$, we have

$$\begin{aligned}
R_2^{1*} &= \mathcal{M}\left(D_\tau u_\tau^{1*} - D_\tau \Pi_k^0 u_\tau^{1*}, e_h^{1*}\right) + \mathcal{M}_h\left(D_\tau \Pi_k^0 u_\tau^{1*} - D_\tau \mathcal{R}_h^{1*} u_\tau^{1*}, e_h^{1*}\right) \\
&\le \left(\left\|(I-\Pi_k^0)D_\tau u_\tau^{1*}\right\|_{L^2(\Omega)} + \alpha^*\left\|(I-\mathcal{R}_h^{1*})D_\tau u_\tau^{1*}\right\|_{L^2(\Omega)}\right)\left\|e_h^{1*}\right\|_{L^2(\Omega)} \\
&\le (1+\alpha^*)\left(C_{\mathcal{P}} + C_{\mathcal{R}}\right) h^2 \left\|D_\tau u_\tau^{1*}\right\|_{H^2(\mathcal{T}_h)}\left\|e_h^{1*}\right\|_{L^2(\Omega)} \\
&\le (1+\alpha^*)\left(C_{\mathcal{P}} + C_{\mathcal{R}}\right) C_0' \frac{1}{\tau} h^2 \left\|e_h^{1*}\right\|_{L^2(\Omega)}.
\end{aligned}$$

Owing to the continuity properties in (6) and Lemma 4.9, term R_3^{1*} can be estimated by

$$\begin{aligned}
R_3^{1*} = \mathcal{A}_h\left(u_\tau^0; \mathcal{F}_h^0 u_\tau^{1*} - \mathcal{R}_h^{1*} u_\tau^{1*}, e_h^{1*}\right) &\le \beta^*\left\|\nabla\left(\mathcal{F}_h^0 u_\tau^{1*} - \mathcal{R}_h^{1*} u_\tau^{1*}\right)\right\|_{L^2(\Omega)}\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)} \\
&\le \beta^* C_{\mathcal{F}}(h^2 + \tau h)\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)}.
\end{aligned}$$

Next, using the definition of $\mathcal{A}_h(\cdot;\cdot,\cdot)$ in (10b) together with boundedness of $\mathcal{R}_h^{1*}u_\tau^{1*}$ in (63a), we obtain that

$$\begin{aligned}
R_4^{1*} &= \left(\left(a(u_\tau^0) - a(u_h^0)\right)\mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla \mathcal{R}_h^{1*} u_\tau^{1*}\right), \mathbf{\Pi}_{k-1}^{\mathbf{0}}\left(\nabla e_h^{1*}\right)\right)_\Omega \\
&+ \sum_{K\in\mathcal{T}_h}\left(\nu_{\mathcal{A}}^K(u_\tau^0) - \nu_{\mathcal{A}}^K(u_h^0)\right) S_{\mathcal{A}}^K\left((I-\Pi_k^{\nabla,K})\mathcal{R}_h^{1*}u_\tau^{1*}, (I-\Pi_k^{\nabla,K})e_h^{1*}\right) \\
&\le \mu^*\left\|u^1 - u_h^0\right\|_{L^2(\Omega)}\left\|\nabla\mathcal{R}_h^{1*}u_\tau^{1*}\right\|_{L^\infty(\Omega)}\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)} \\
&+ C_{\mathcal{P}}\left\|\Pi_0^0\left(a(\Pi_k^0 u_\tau^0) - a(\Pi_k^0 u_h^0)\right)(I-\Pi_k^{\nabla,K})\nabla\mathcal{R}_h^{1*}u_\tau^{1*}\right\|_{L^2(\Omega)}\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)} \\
&\le \mu^* C_M\left(1 + (C_{\mathcal{P}})^3\right)\left\|u_\tau^0 - u_h^0\right\|_{L^2(\Omega)}\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)} \\
&\le \mu^* M C_M C_{\mathcal{I}}\left(1 + (C_{\mathcal{P}})^3\right) h^2\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)}.
\end{aligned}$$

With the estimates above and applying the Young's inequality, we have

$$\frac{\alpha_*}{\tau}\|e_h^{1*}\|_{L^2(\Omega)}^2 + \beta_*\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)}^2 \le \frac{(\widehat{C}_6)^2}{\tau}\left(h^4 + (\tau h)^2\right) + \frac{\alpha_*}{2\tau}\|e_h^{1*}\|_{L^2(\Omega)}^2 + \frac{\beta_*}{2}\left\|\nabla e_h^{1*}\right\|_{L^2(\Omega)}^2,$$

which implies that

$$\|e_h^{1*}\|_{L^2(\Omega)} \le \sqrt{2\alpha_*^{-1}}\,\widehat{C}_6\left(h^2 + \tau h\right), \tag{A.22}$$

where the constant $\widehat{C}_6$ be defined as

$$\widehat{C}_6 = \frac{C_{\mathcal{P}}\|f^1\|_{H^2(\Omega)} + (1+\alpha^*)(C_{\mathcal{P}} + C_{\mathcal{R}})C_0'}{\sqrt{\alpha_*}} + \frac{\beta^* C_{\mathcal{F}} + \mu^* M C_M C_{\mathcal{I}}\left(1 + (C_{\mathcal{P}})^3\right)}{\sqrt{2\beta_*}}.$$

By application of a triangle inequality and Lemma 4.7, we conclude that

$$\left\|u_\tau^{1*} - u_h^{1*}\right\|_{L^2(\Omega)} \le \left\|u_\tau^{1*} - \mathcal{R}_h^{1*}u_\tau^{1*}\right\|_{L^2(\Omega)} + \|e_h^{1*}\|_{L^2(\Omega)} \le \left(C_{\mathcal{R}}C_0' + \sqrt{2\alpha_*^{-1}}\,\widehat{C}_6\right)\left(h^2 + \tau h\right).$$

Finally, for $h \le \widehat{h}_1 := \frac{1}{2}(C_{\text{Inv}}\sqrt{2\alpha_*^{-1}}\,\widehat{C}_6)^{-1}$ and $\tau \le \widehat{\tau}_3 := \frac{1}{2}(C_{\text{Inv}}\sqrt{2\alpha_*^{-1}}\,\widehat{C}_6)^{-1}$, it holds that

$$\|u_h^{1*}\|_{L^\infty(\Omega)} \le \|\mathcal{R}_h^{1*}u_\tau^{1*}\|_{L^\infty(\Omega)} + C_{\text{Inv}}h^{-1}\|e_h^{1*}\|_{L^2(\Omega)} \le C_2'.$$

And the following error and boundedness of u_h^{1**} can be proved similarly,

$$\|u_\tau^{1**} - u_h^{1**}\|_{L^2} \le \widehat{C}_7 \left(h^2 + \tau h\right), \quad \text{and} \quad \|u_h^{1**}\|_{L^\infty} \le C_2'.$$

The proof is completed.

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